1. General Structure of the Algorithm

This method conducts an iterative search over a parameter space that may contain **continuous**, **integer**, and **categorical** variables. Its main components are:

1. **Elite Selection**
   A set of the best-performing trials (based on the objective function) is selected at each iteration. These are the "elites."
2. **Noise Perturbation**
   New candidate solutions are generated by perturbing the parameters of these elite trials with a noise term that adapts over time.
3. **Noise Annealing**
   The noise level is decreased as the number of iterations increases, often using a cosine-annealing schedule. This ensures broader exploration at the beginning and more focused exploitation later on.
4. **Categorical Handling**
   Categorical parameters are internally represented via one-hot encoding. A softmax function (with a temperature parameter) is used to stochastically choose among possible categories based on the (perturbed) mean of elite vectors.
5. **Integer Handling**
   Integer parameters are sampled as continuous values and then probabilistically rounded to the nearest integers.

Let:

- N be the total number of iterations (trials).
- t be the index of the current iteration, with $0 \leq t < N$.
- $p_t = \frac{t}{N}$ be the **progress ratio**.

2. Number of Elite Trials n_{elite}

At each iteration t, the number of elite trials selected to guide the next sample can be defined by a function that depends on the progress ratio p_t. One commonly used form is:

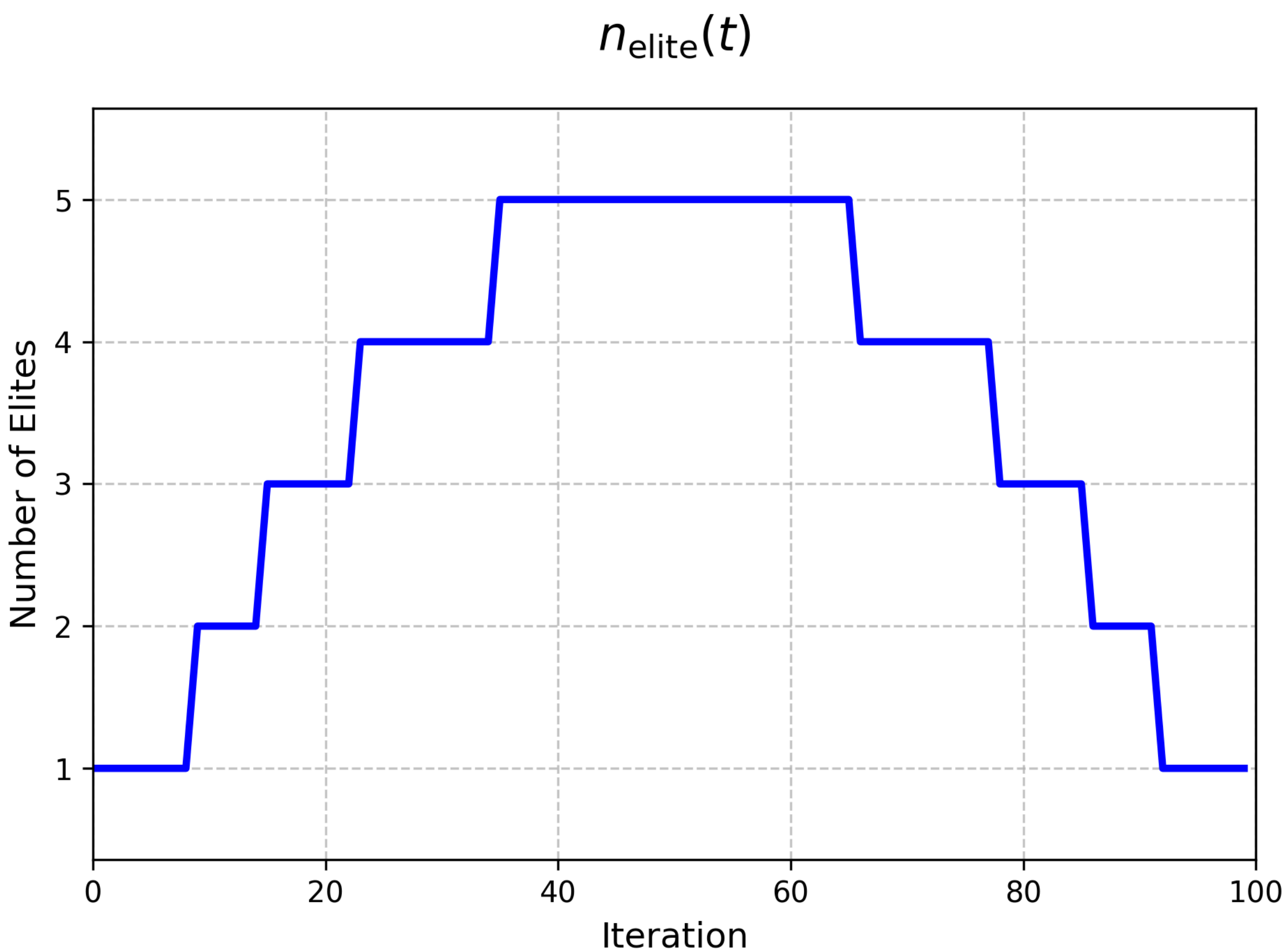
$$n_{\text{elite}}(t) \;=\; \max\Big(1,\; \text{round}\big(\,\alpha\,\sqrt{N}\,\cdot\,p_t\cdot\big(1-p_t\big)\big)\Big),$$

where:

- α is a constant (for example, $\alpha = 2$) that scales according to the total number of trials N.
- The factor $p_t\,(1 - p_t)$ creates a bell-shaped curve over $t \in [0, N]$, reaching its maximum around $t \approx \frac{N}{2}$.
- The use of $\max(1, \ldots)$ ensures that at least one trial is always considered elite.

Visualizing $n_{\text{elite}}(t)$

If desired, a plot of $n_{\text{elite}}(t)$ against t can show how the number of elite trials starts near 0 or 1 at $t = 0$, grows to a maximum in the middle iterations, and then decreases again near $t = N$.



3. Noise Scheduling with Cosine Annealing

Let η_{init} be the **initial noise** (e.g., 0.2) and $\eta_{\text{final}} = \frac{1}{N}$ be the **final noise** (or another chosen small value). At iteration t, define a **cosine annealing** factor:

$$\text{cos\_anneal}(t) \;=\; 0.5\,\bigl(1 + \cos\bigl(\pi\,p_t\bigr)\bigr),$$

where $p_t = \frac{t}{N}$.

Then, the noise level $\eta(t)$ can be updated as:

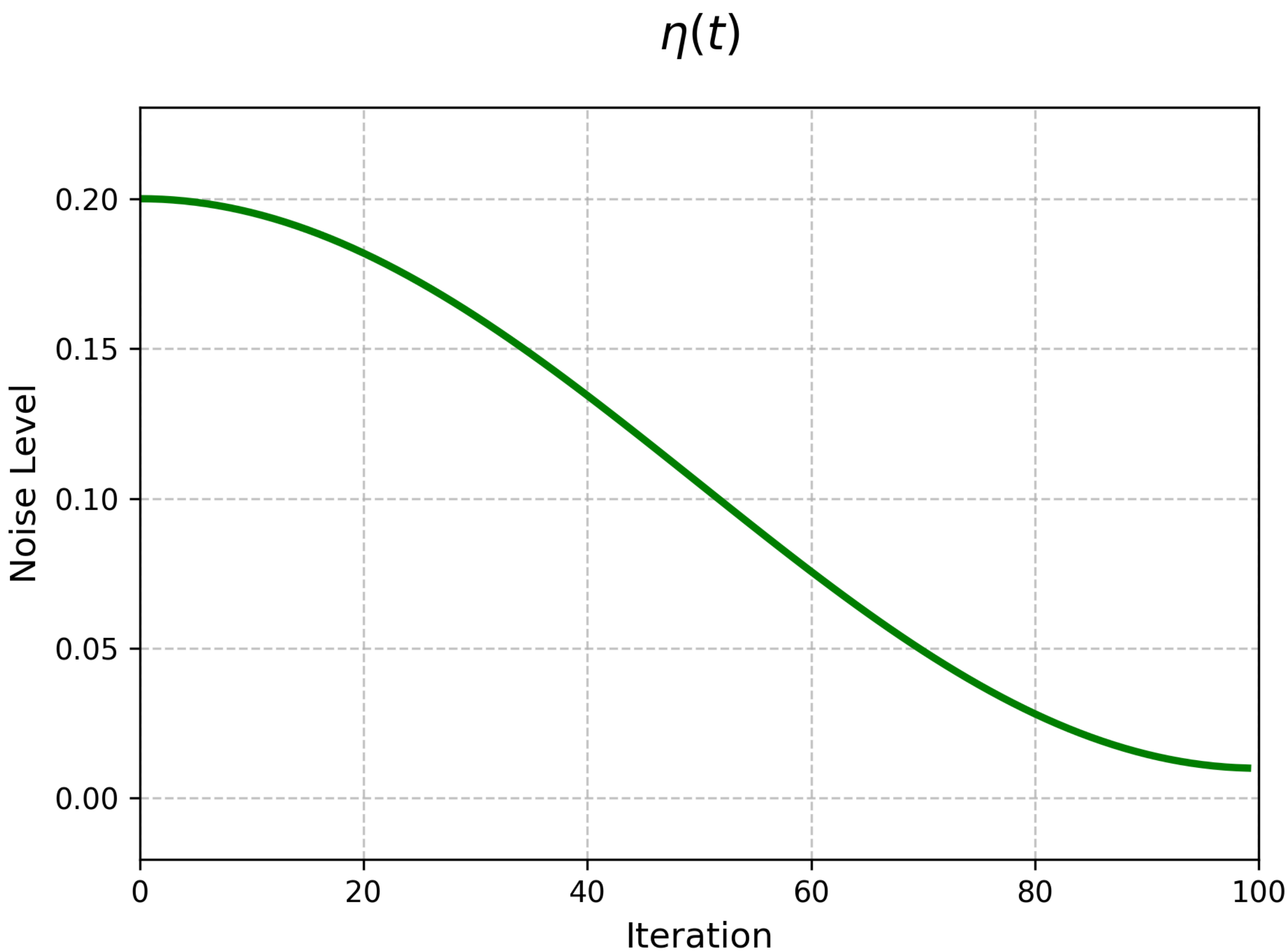
$$\eta(t) \;=\; \eta_{\text{final}} \;+\; \bigl(\eta_{\text{init}} \;-\; \eta_{\text{final}}\bigr)\,\text{cos\_anneal}(t).$$

- When t is close to 0, $p_t = 0$, so $\cos(\pi\,p_t) = 1$ and $\eta(t) \approx \eta_{\text{init}}$.
- Near $t = N$, $\cos(\pi\,p_t) = -1$, so $\eta(t) \approx \eta_{\text{final}}$.

Hence, the noise transitions gradually from a larger initial value down to a smaller final value.

Visualizing $\eta(t)$

A plot of $\eta(t)$ across iterations t typically shows a smooth curve descending from η_{init} at $t = 0$ to η_{final} at $t = N$.



4. Continuous and Integer Parameters

4.1. Continuous Variables

For a continuous variable x in the range $[\,\text{low},\ \text{high}\,]$, new samples may first be drawn randomly (uniformly or log-uniformly) during early iterations. Once enough iterations have passed, the algorithm exploits the elite solutions:

1. **Select an Elite Value**
   One of the elite trials (in terms of objective value) is chosen at random. Let its parameter be x_{elite}.
2. **Add Noise**
   Draw a random value $\delta \sim \mathcal{N}(0,\ \sigma)$, where σ depends on $\eta(t)$ and possibly the range $\text{high} - \text{low}$. A typical approach is:

$$x_{\text{new}} = x_{\text{elite}} + \delta \cdot \big(\text{high} - \text{low}\big) \cdot \eta(t).$$

3. **Reflect at Boundaries**
   If x_{new} goes below low or above high, it is reflected back into the valid range, for instance by:

$$\text{while } x_{\text{new}} < \text{low or } x_{\text{new}} > \text{high:} \quad \begin{cases} x_{\text{new}} = \text{high} - (\, x_{\text{new}} - \text{high}\,)/2 & \text{if } x_{\text{new}} > \text{high}, \\ x_{\text{new}} = \text{low} + (\, \text{low} - x_{\text{new}}\,)/2 & \text{if } x_{\text{new}} < \text{low}. \end{cases}$$

4.2. Integer Variables

To handle an integer parameter in $\{\text{low}, \ldots, \text{high}\}$, one can:

1. Sample a **continuous** value as above, obtaining v.
2. Let $\lfloor v \rfloor$ be the floor of v and $f = v - \lfloor v \rfloor$ be its fractional part.
3. Draw u from a uniform distribution $\mathrm{U}(0, 1)$.
4. If $u < f$, set the integer value to $\lceil v \rceil$. Otherwise, set it to $\lfloor v \rfloor$.

Thus, a value close to 10.7 is more likely to become 11 than 10, while a value close to 10.2 is more likely to become 10 than 11.

5. Categorical Parameters: One-Hot and Softmax

Categorical parameters are represented as **one-hot vectors**. Suppose there are k possible categories $c_1, c_2, \ldots, c_k$. Each trial stores a vector of length k, e.g., $[1, 0, 0]$ if category c_1 is chosen, $[0, 1, 0]$ if c_2 is chosen, etc.

5.1. Averaging and Noise

Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{n_{\text{elite}}}$ be the one-hot vectors of the best $n_{\text{elite}}(t)$ trials. Compute the component-wise mean:

$$\overline{\mathbf{v}} \;=\; \frac{1}{n_{\text{elite}}(t)} \sum_{i=1}^{n_{\text{elite}}(t)} \mathbf{v}_i.$$

Then add Gaussian noise $\mathbf{z}$ with scale $\eta(t)$, typically ensuring the result stays within $[0,\ 1]$ by reflection if necessary.

5.2. Temperature and Softmax

A temperature parameter $T_{\text{cat}}(t)$ is introduced to control how sharply categories are chosen. One approach is to define

$$T_{\text{cat}}(t) \;=\; \frac{1}{\eta_{\text{final}} + (1 - \eta_{\text{final}})\,\text{cos\_anneal}(t)},$$

where

$$\text{cos\_anneal}(t) \;=\; 0.5\,\bigl(1 + \cos(\pi p_t)\bigr).$$

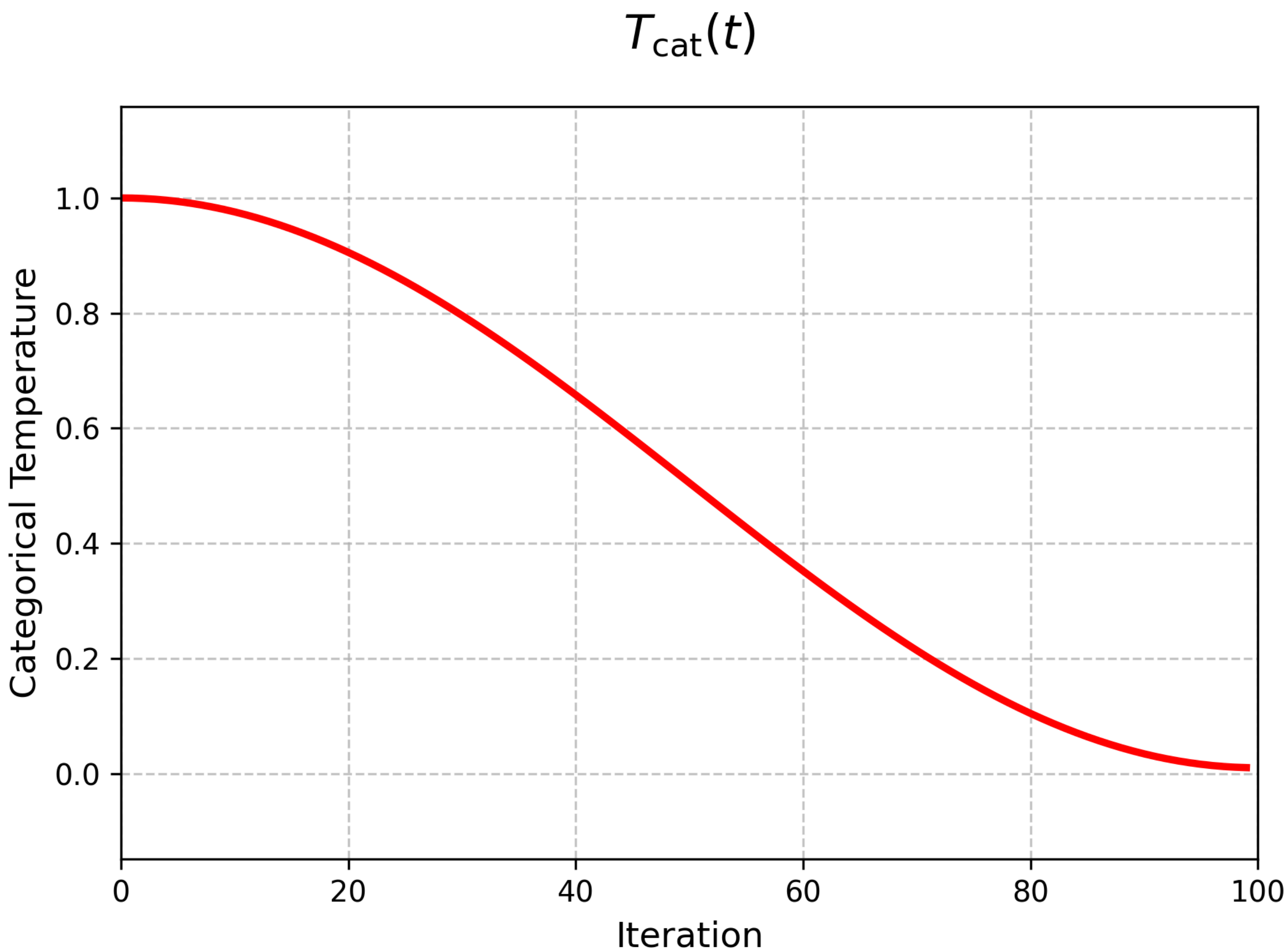
After adding noise to $\overline{\mathbf{v}}$, each component m_j represents the "score" for category j. These scores are converted to probabilities $\{\pi_1, \ldots, \pi_k\}$ via a softmax scaled by $T_{\text{cat}}(t)$:

$$\pi_j \;=\; \frac{\exp\bigl(m_j\,T_{\text{cat}}(t)\bigr)}{\sum_{r=1}^{k} \exp\bigl(m_r\,T_{\text{cat}}(t)\bigr)}, \quad j = 1, \ldots, k.$$

Finally, a category c_j is sampled with probability π_j, and the corresponding one-hot vector is set to $[0, \ldots, 1, \ldots, 0]$ with 1 at position j.

Visualizing $T_{\text{cat}}(t)$

If desired, a plot of $T_{\text{cat}}(t)$ against t can show how the categorical temperature starts high at $t = 0$, allowing broad exploration, then gradually decreases, focusing more on the best categories over time.



6. Iterative Procedure

Let:

- N be the total number of iterations (trials).
- $n_{\text{init\_points}}$ be the number of initial trials that are sampled purely at random (commonly $\text{round}(\sqrt{N})$ if not specified).
- t be the index of the current iteration, with $0 \leq t < N$.
- $p_t = \frac{t}{N}$ be the **progress ratio**.

At **each iteration** t (from 0 up to $N - 1$):

If $t < n_{\text{init\_points}}$**:**

- **Randomly sample** all parameters (continuous, integer, and categorical) within their valid ranges.

- Skip steps 2, 3, and 4 below (since no elite-based adaptation is used yet).

Otherwise ($t \geq n_{\text{init\_points}}$):

1. **Compute Progress**:

$$p_t \ = \ \frac{t}{N}.$$

2. **Determine Elite Count**:

$$n_{\text{elite}}(t) \ = \ \max\Big(1,\ \text{round}\big(\alpha\sqrt{N}\ \cdot\ p_t\,(1-p_t)\big)\Big),$$

3. **Update Noise (Cosine Annealing)**:

$$\eta(t) \ = \ \eta_{\text{final}} \ + \ \left(\eta_{\text{init}} - \eta_{\text{final}}\right) \times 0.5\,\left(1 + \cos(\pi\,p_t)\right).$$

4. **Handle Parameters**:
   - **Continuous**: Select an elite value, add $\mathcal{N}(0, \sigma)$ noise scaled by $\eta(t)$ and reflect if out of bounds.
   - **Integer**: Same as continuous, but use *probabilistic rounding* (fractional part decides rounding up/down).
   - **Categorical**: Form an average one-hot vector from the elites, add noise, apply a temperature-based softmax, then pick a category.
5. **Evaluate Objective**:
   - Pass the newly sampled parameter set to the objective function for a score.
6. **Update Ranking**:
   - Keep track of the best $n_{\text{elite}}(t)$ trials (“elites”) for the next iteration.

This process repeats until $t = N$. Early in the search ($t < n_{\text{init\_points}}$), the algorithm explores broadly by drawing random samples. Once $t \geq n_{\text{init\_points}}$, it transitions to the adaptive phase: higher noise in the beginning encourages wide exploration, whereas lower noise in later iterations focuses the search around the most promising solutions found so far.

Additional Notes

- **Reflections at Boundaries**
  Ensuring that samples do not remain outside a valid range often involves a “mirror” or “reflect” step.

- **Log-Scale Sampling**
  If a parameter is specified as log-scaled $\in [\mathrm{low}, \mathrm{high}]$, sampling can be done in $\log$-space, i.e., $\exp\big(\mathrm{Uniform}(\log(\mathrm{low}),\ \log(\mathrm{high}))\big)$.
- **Temperature**
  When $\eta(t)$ becomes small, $T_{\mathrm{cat}}(t)$ becomes large, so the softmax distribution becomes more "peaked" around the best categories discovered.