PHY 517 / AST 443:
Observational Techniques in Astronomy

Lecture 5:
Statistics, part 1

(A brief intro to)
Statistics

Statistics in Astronomy

- we are almost always working in the low signal-to-noise regime
- have to be very careful to make correct inferences from our data!
- robust (and advanced) statistical techniques play a very important role in astronomy

Measurements

- example: 99.123 ± 0.005

- what is 0.005 called?
  - (measurement) uncertainty
  - NOT “error” (inaccurate, though often used)

- what does this mean?
  - if we repeat the measurement many times, in 68% of the cases the true value would fall within the quoted uncertainty interval
  - not-quite-right interpretation: the quoted interval has a 68% chance of containing the true value

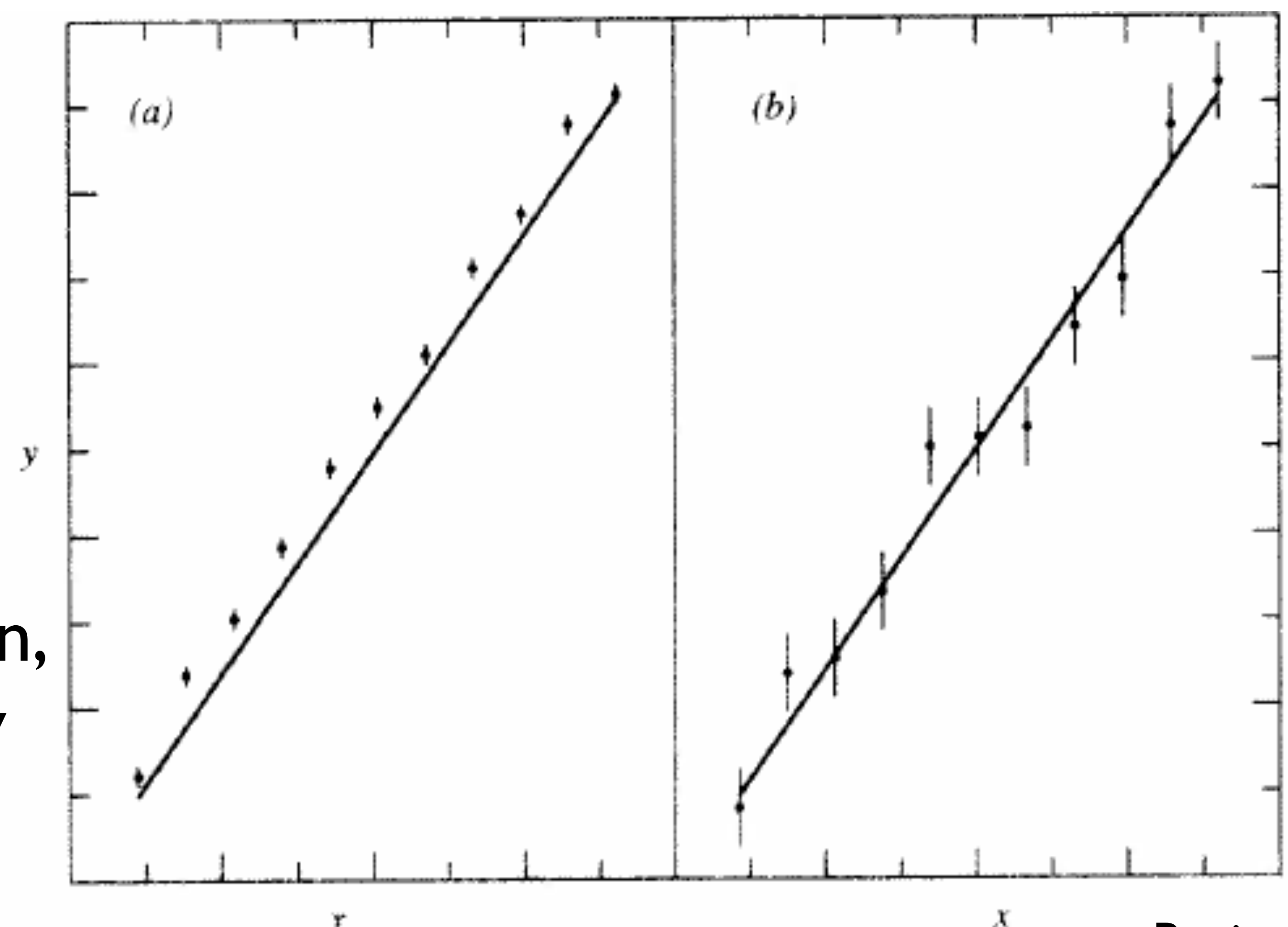
“Error”

- error: difference between *measured* and *true* value
- can be due to:
  - random fluctuations (statistical error)
  - instrumental / algorithmic limitations (systematic error)
  - mistakes (illegitimate error)
- measurements are meaningless if not accompanied by an estimate of the error
- but truth is unknown, have to estimate error indirectly

Accuracy vs. Precision

- accuracy: how close a measurement is to the truth
- precision: size of (statistical) measurement uncertainty

high precision,
low accuracy



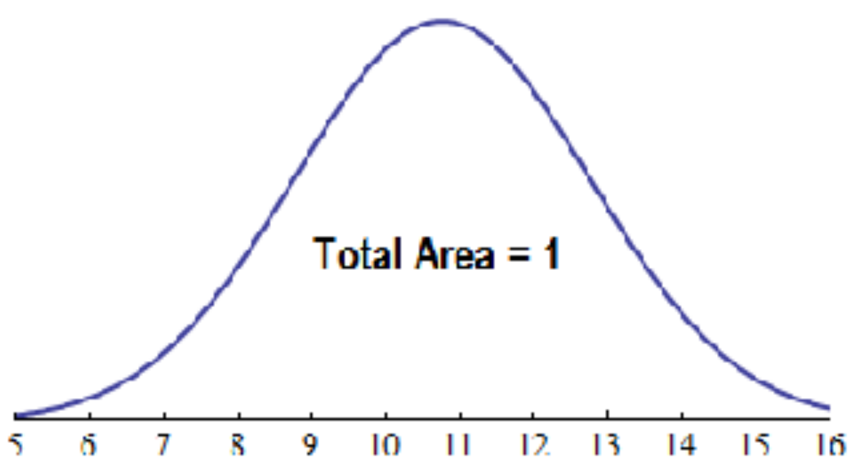
low precision,
high accuracy

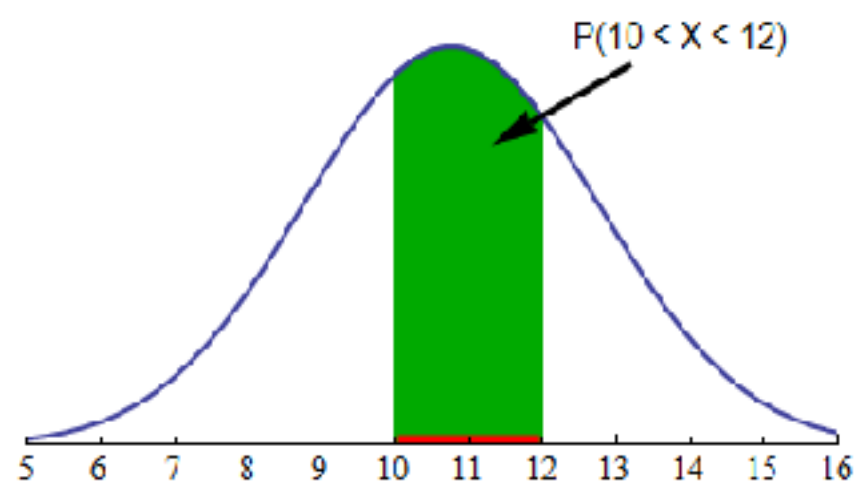
Bevington & Robertson 2003

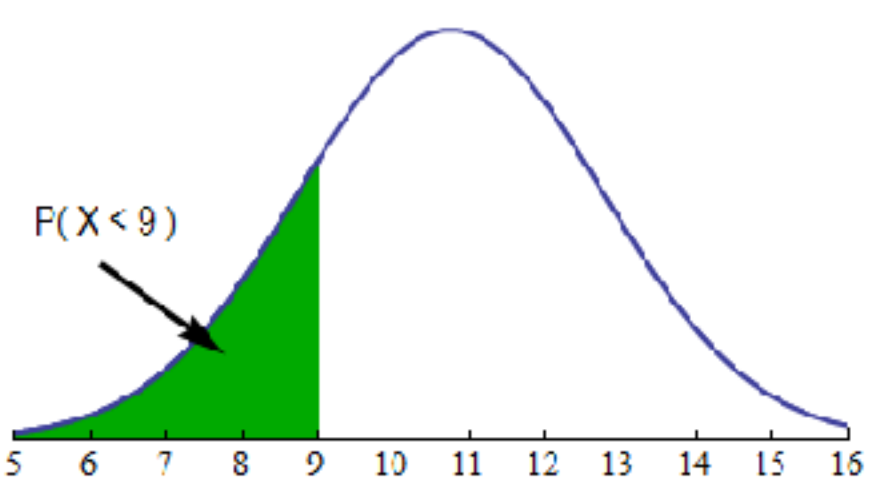
FIGURE 1.1
Illustration of the difference between precision and accuracy. (*a*) Precise but inaccurate data. (*b*) Accurate but imprecise data. True values are represented by the straight lines.

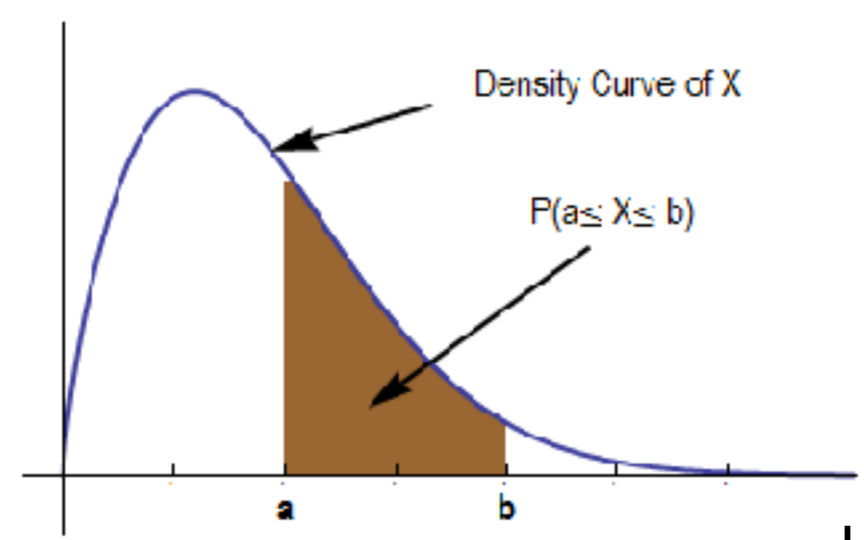
Probability Distributions

- probability distributions: describe expected / measured distributions of measurements
- integrate over range of values to find probability to be in that range









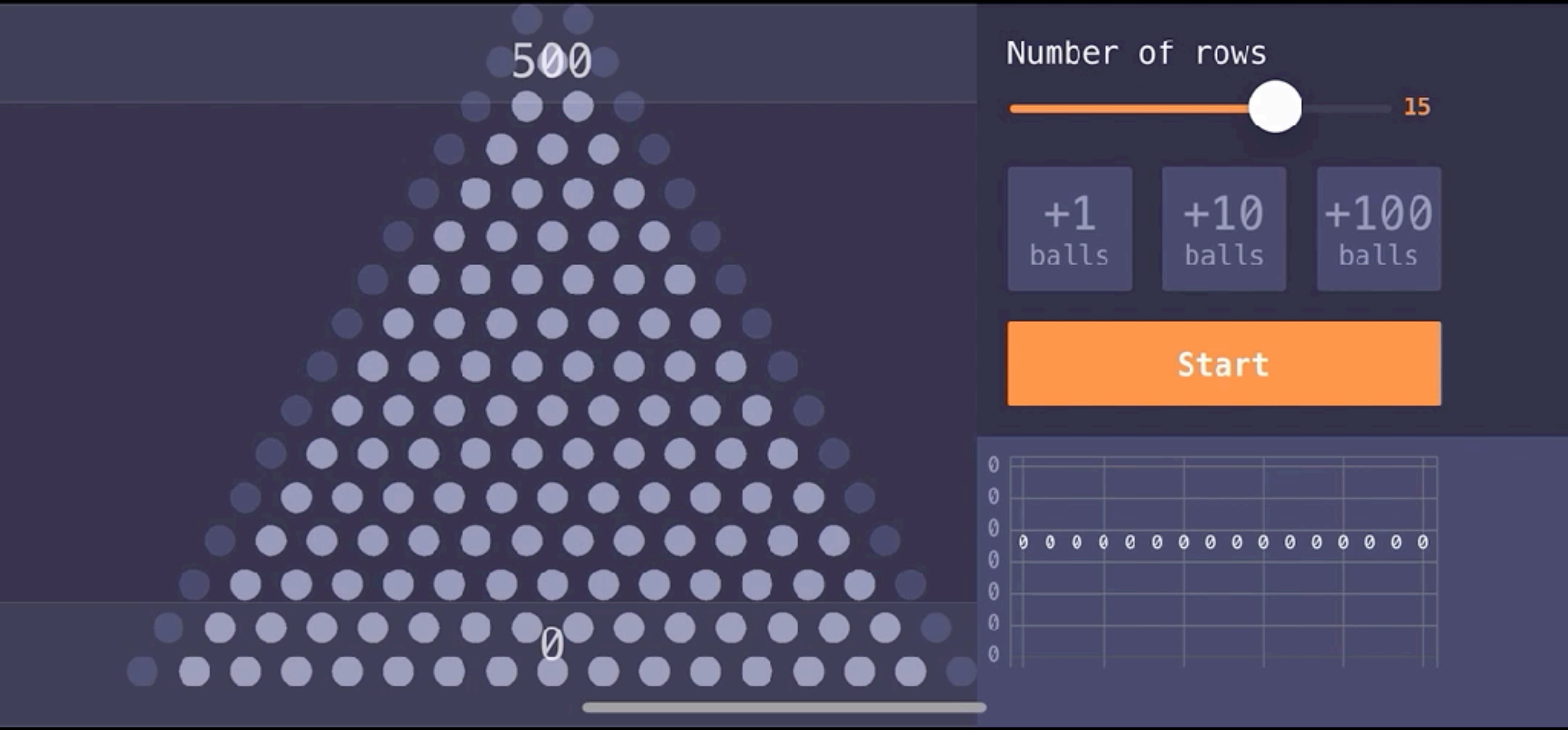
Lumen Learning

Sample vs. Parent Distribution

- measurement x_i of a quantity x:
  - approximates x
  - not necessarily equal to x because of statistical uncertainty
- many measurements x_i:
  - expected to be distributed about true value
  - sample distribution
- parent distribution:
  - probability of particular result from single measurement
  - idealized outcome of infinite number of measurements

the sample distribution *samples* the parent distribution

Galton Board:



Galton Board App for iOS, Edwin Veger

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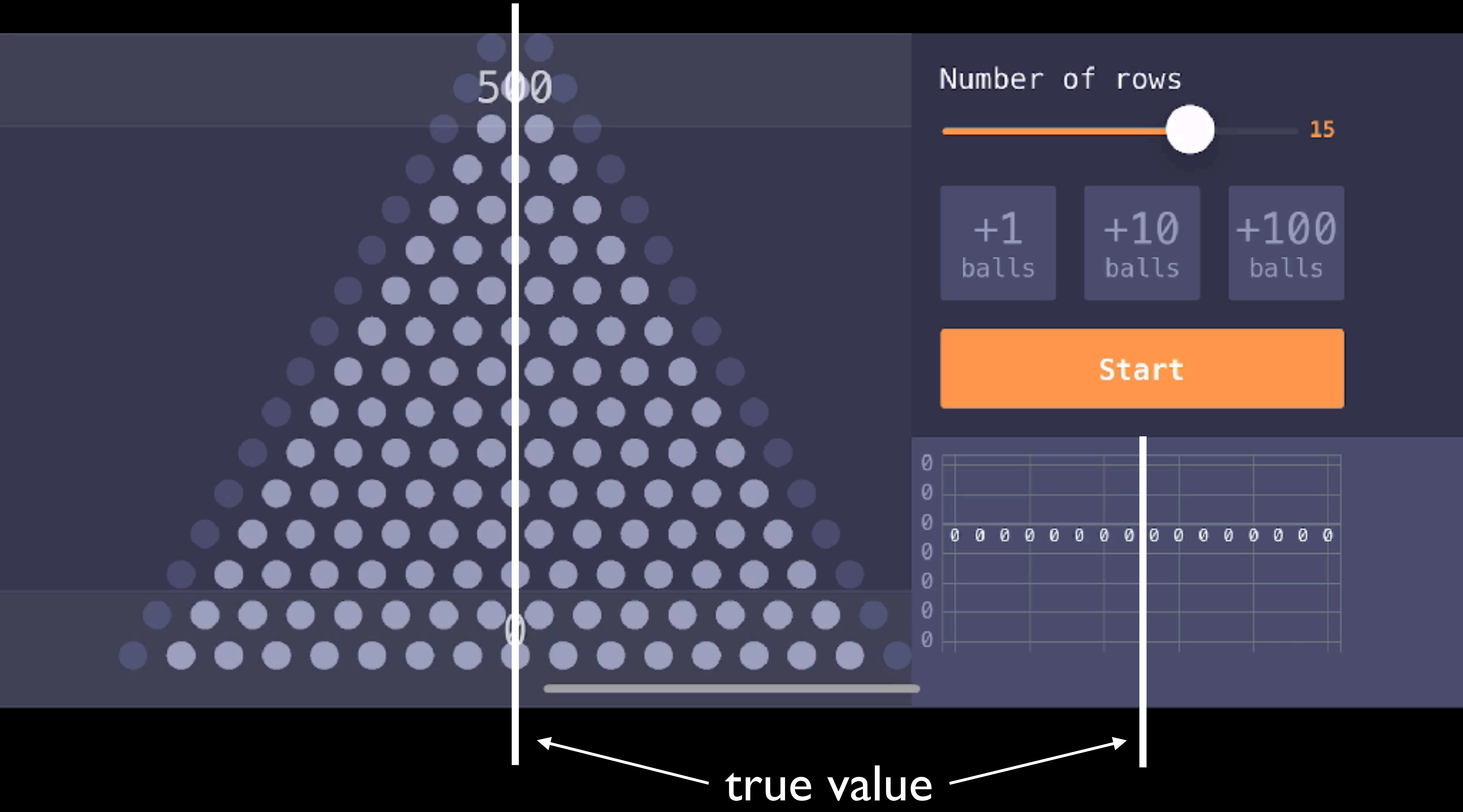

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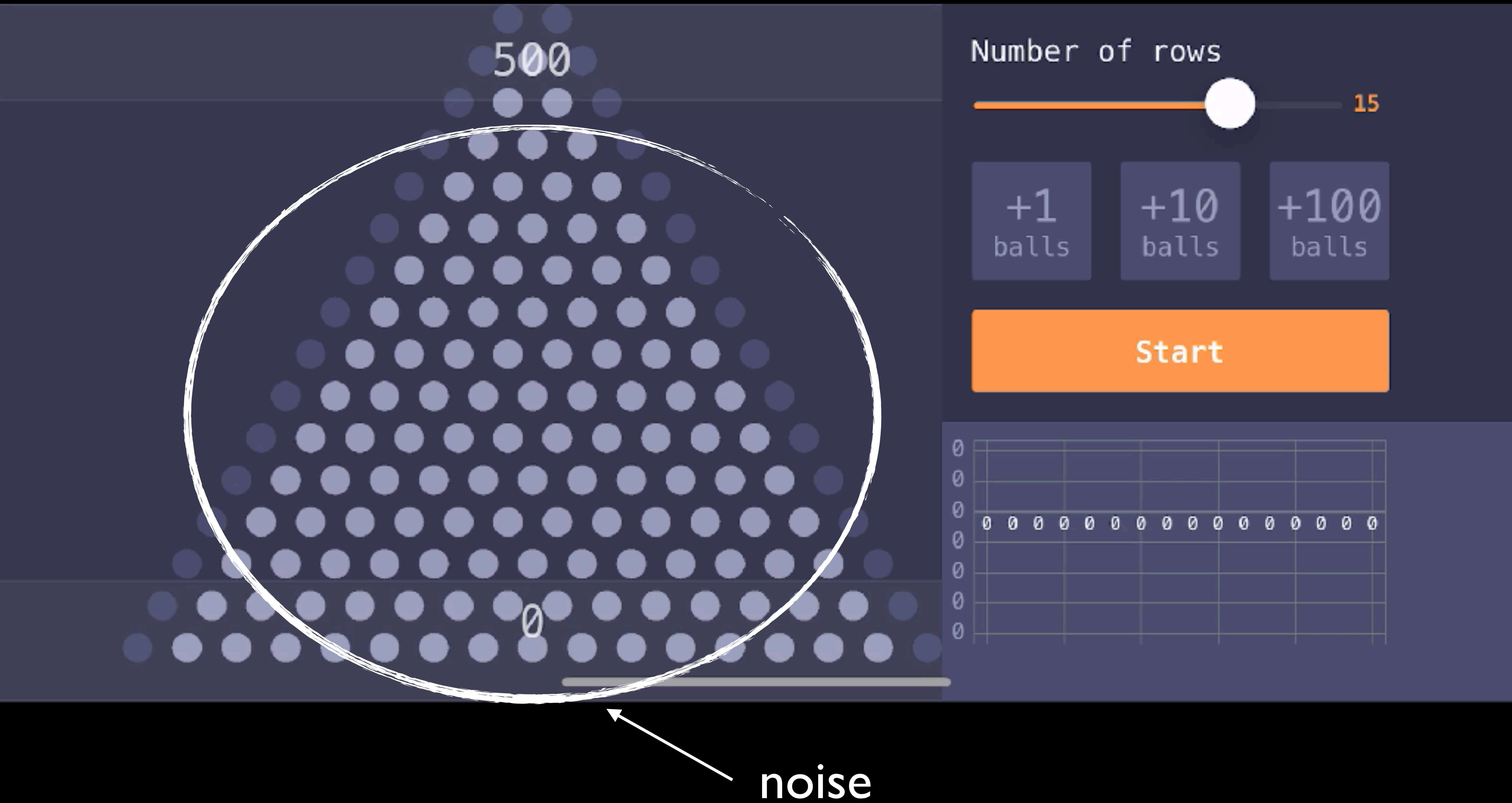

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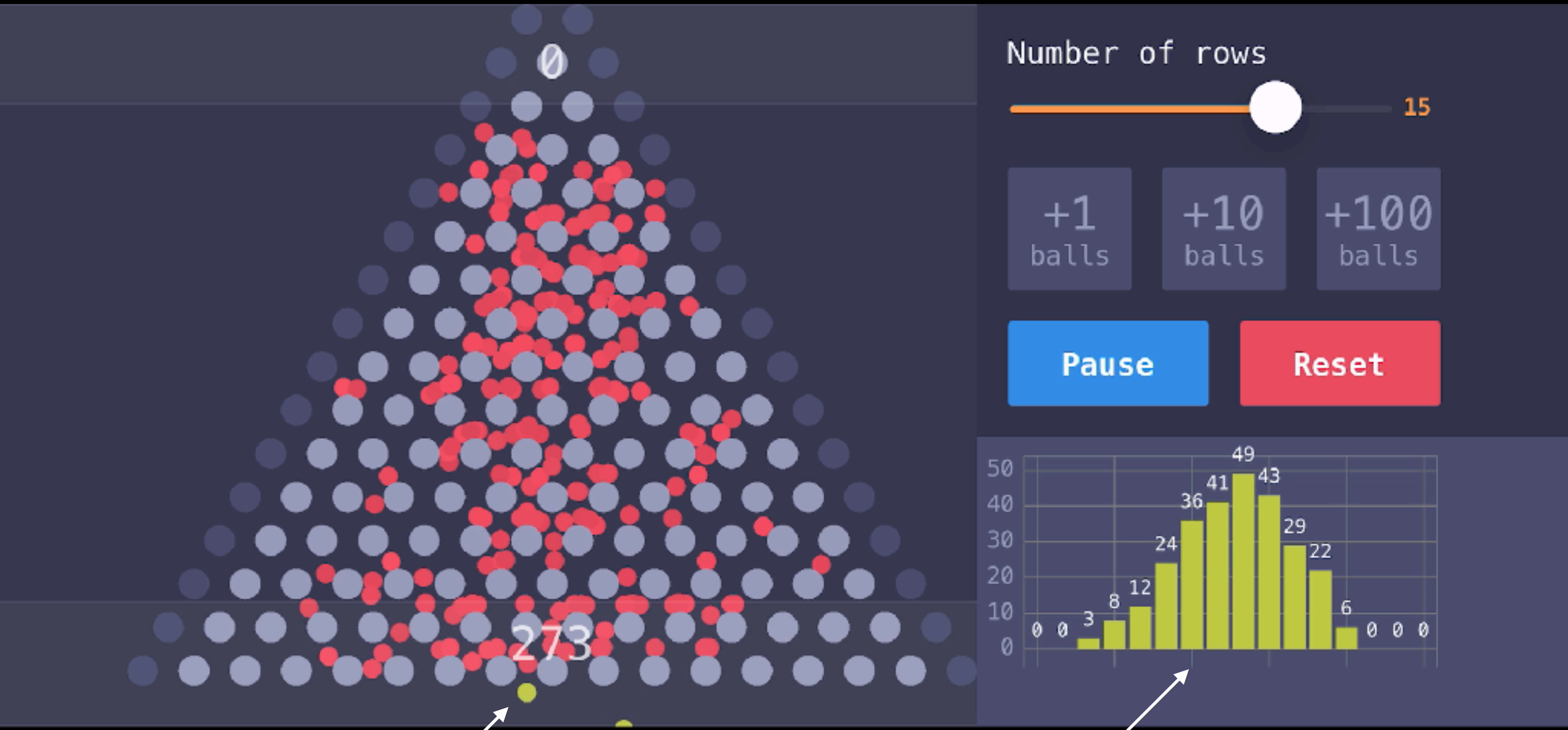

many measurements

one measurement x_i

Galton Board App for iOS, Edwin Veger

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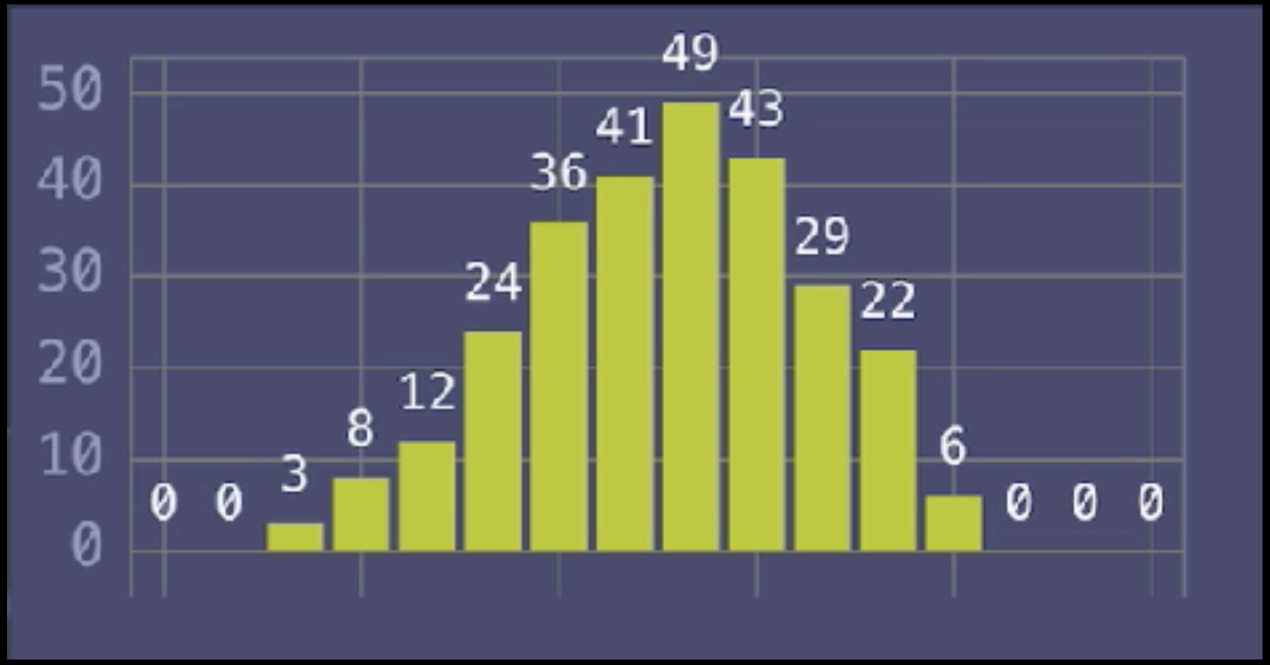
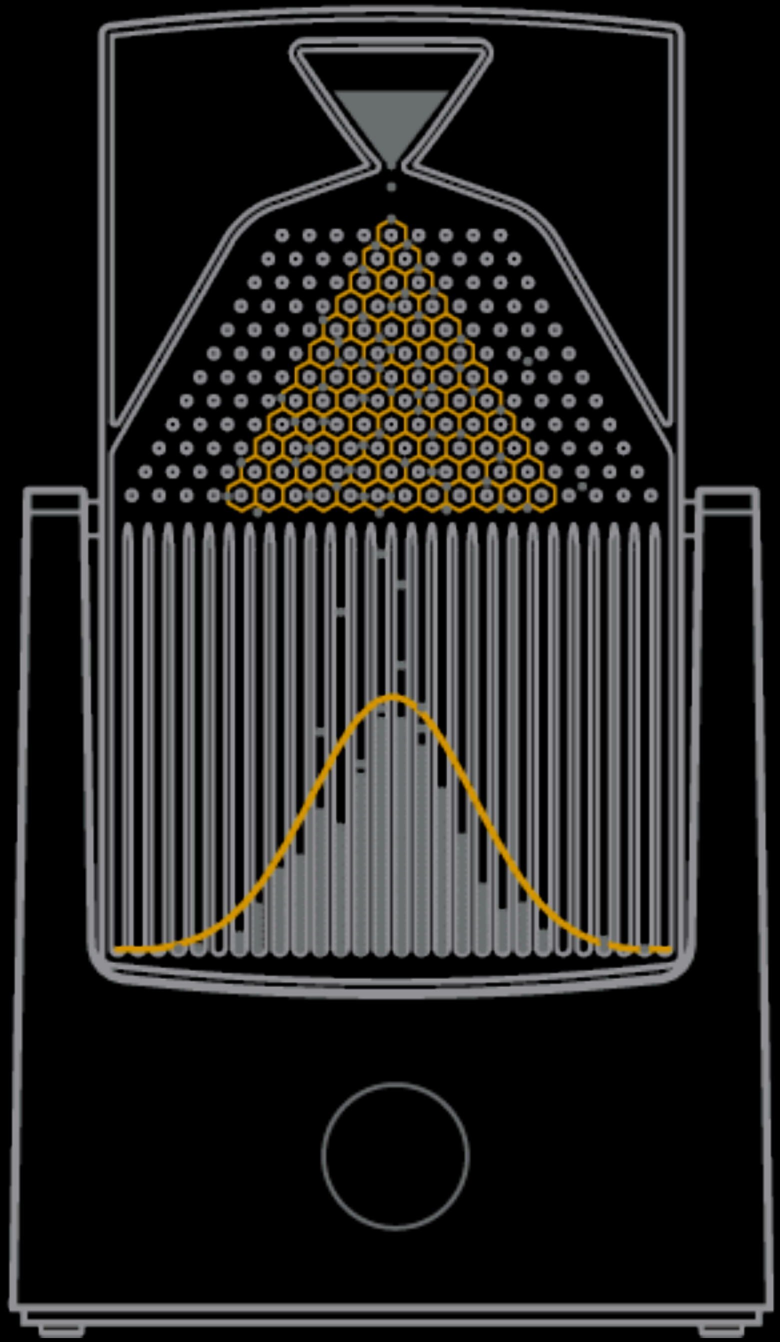




sample population: distribution of many measurements

parent population: Gaussian / normal distribution

Galton Board App for iOS, Edwin Veger

Summary statistics

- only the full sample distribution is the full description of your data
- but usually, it is helpful to describe the sample distribution with a few numbers → summary statistics

Q: can you think of examples?

Mean, median, and mode

- (unweighted) mean of the sample distribution:

$$\bar{x} = \frac{1}{N}\sum_i x_i$$

- IF there are no systematic errors, the mean of the parent population is:

$$\mu = \lim_{N\to\infty} \frac{1}{N}\sum_i x_i$$

Mean, median, and mode

- median: 50th percentile of distribution (half the measurements are smaller, half are greater)

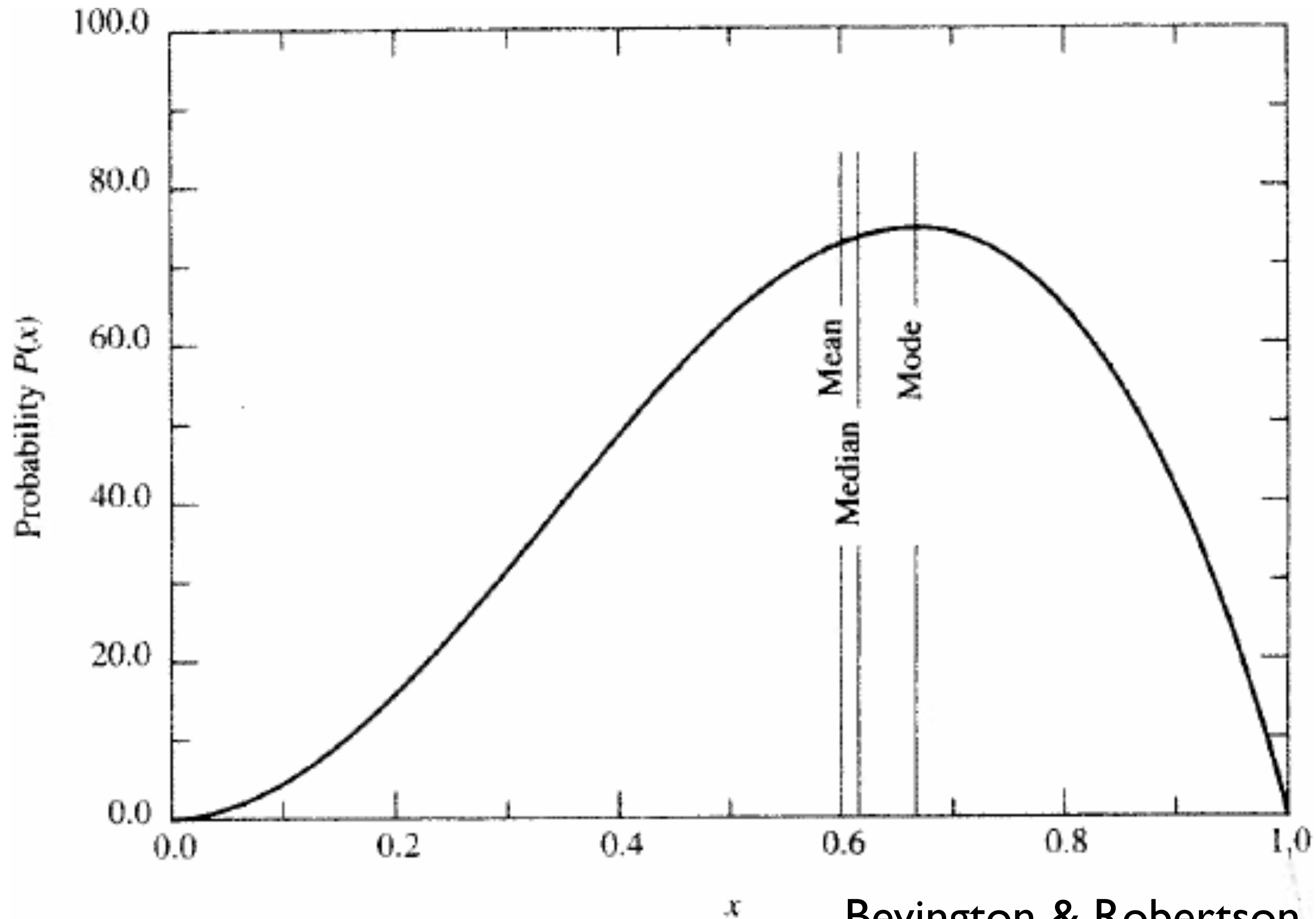
  motivation: less susceptible to "outliers" than the mean

- mode: the most common measurement value

  motivation: the most likely value

Mean, median, and mode

mean, median and mode for an example distribution:



Bevington & Robertson 2003

- generally not equal to each other
- all 3 are useful; which to quote depends on the problem (and personal preference)

Deviation / variance / std. deviation

- deviation of one measurement: $d_i = x_i - \mu$
- sample variance: average of the squares of the deviations

$$\sigma^2 = \frac{1}{N} \sum_i (x_i - \mu)^2$$

- when computing from sample population:

$$s^2 = \frac{1}{N-1} \sum_i (x_i - \bar{x})^2$$

- standard deviation: $\sigma = \sqrt{\text{variance}}$

indicates how much the measurements typically deviate from the mean

Weighted mean

- previously, all measurements had equal weight

- some measurements are more precise than others; can assign weight w_i to each measurement x_i

- weighted mean:

$$\bar{x} = \frac{\sum_i w_i x_i}{\sum_i w_i} \qquad \sigma_{\bar{x}}^2 = \frac{1}{\sum_i w_i}$$

- for reasonable (Gaussian) distributions, optimal weight is the inverse of the variance of each measurement:

$$\bar{x} = \frac{\sum_i x_i/\sigma_i^2}{\sum_i 1/\sigma_i^2} \qquad \sigma_{\bar{x}}^2 = \frac{1}{\sum_i 1/\sigma_i^2}$$

How well can we measure the mean?

- width of sample distribution (std. dev.) remains the same
- the more measurements, the closer the mean to the true value

Galton Board App for iOS, Edwin Veger

Uncertainty on the mean

- variance and std. deviation are measures of the *width* of the sample distribution

- with increasing number N of measurements, the typical deviation of measured mean and true mean decreases

- measurement uncertainty on the mean:

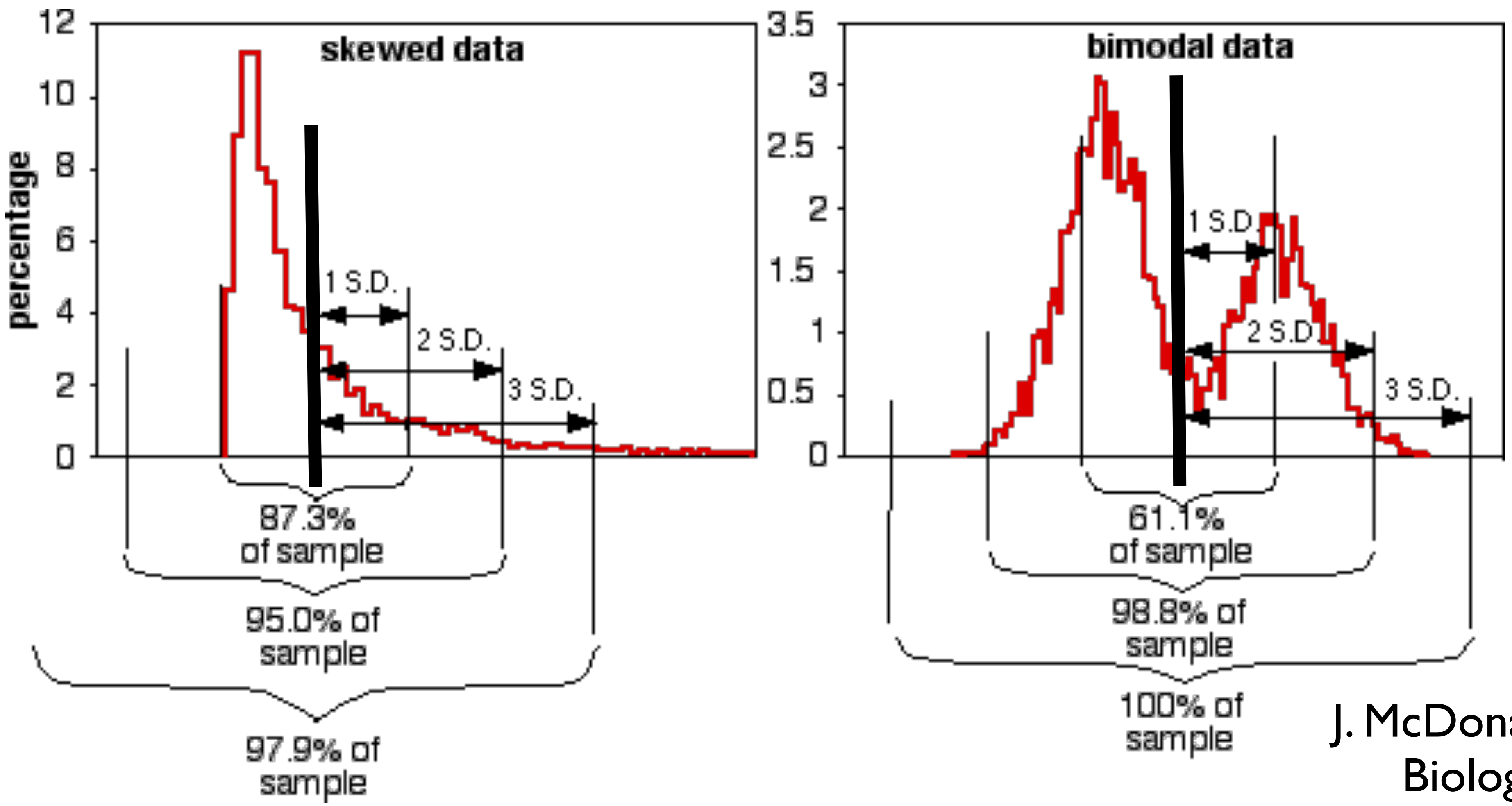
$$\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{N}}$$

- width of distribution of repeated measurements of the mean

- σ: distribution of single measurements around the true value

- σ_{x}: distribution of means of N measurements around the true value

- can calculate mean, variance, etc. for any set of data points
- *that does not guarantee that they are useful descriptions of the distribution !*



J. McDonald, Handbook of Biological Statistics

- if we know the shape of the parent distribution, we know which summary statistics to use

Common Probability Distributions

- many, many possible distributions have been quantified; here, consider 3 particularly important ones:
  - Binomial distribution: for experiments with only two possible final states (e.g. coin toss)
  - Poisson distribution: counting experiments for discrete events (e.g. photon counts)
  - Gaussian (or Normal) distribution: distribution of events about the mean for a wide variety of processes; limiting case of binomial and poisson distributions

Binomial Distribution

- experiment with only two possible outcomes:
  - state 0: probability p
  - state 1: probability q = (1-p)
- n realizations
- probability that x of the n realizations are in state 0:

$$P_{\mathrm{B}}(x|n,p) = \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x}$$

- x: positive integers; $0 \leq x \leq n$
- $0 < p < 1$
- $\sum_{x=0}^{n} P_{\mathrm{B}}(x|n,p) = 1$

Binomial Distribution

- mean of the binomial distribution:

$$\mu = \sum_{x=0}^{n} (x \cdot P_{\mathrm{B}}(x|n,p)) = np$$

(agrees with intuition!)

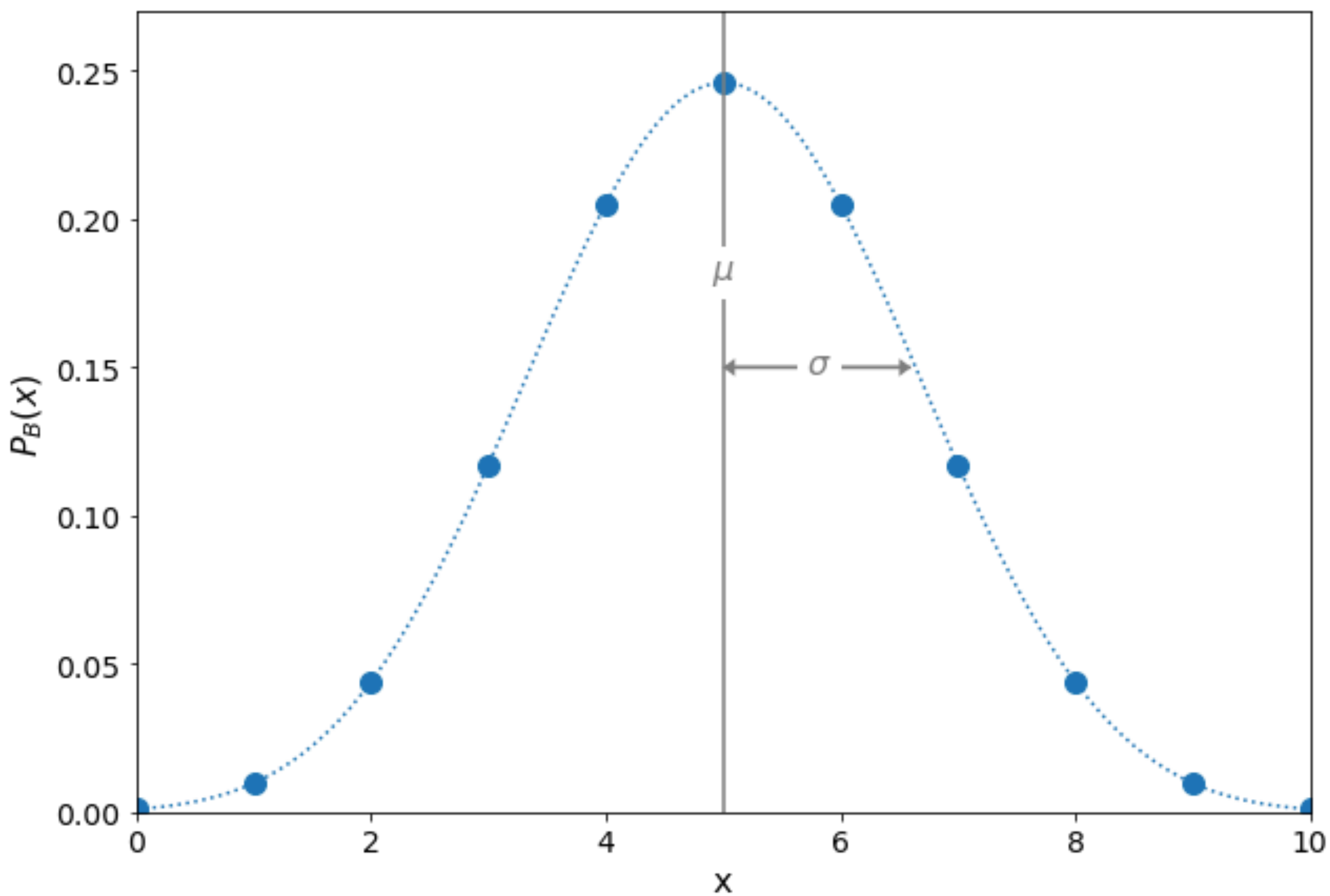
- variance of the binomial distribution:

$$\sigma^2 = \sum_{x=0}^{n} \left((x-\mu)^2 \cdot P_{\mathrm{B}}(x|n,p)\right) = np(1-p)$$

Binomial Distribution - Example (1)

tossing 10 coins, x = number of tails

$$n = 10\,,\ p = 0.5 \quad \rightarrow \quad P(x) = P_{\mathrm{B}}(x|10, 0.5)$$



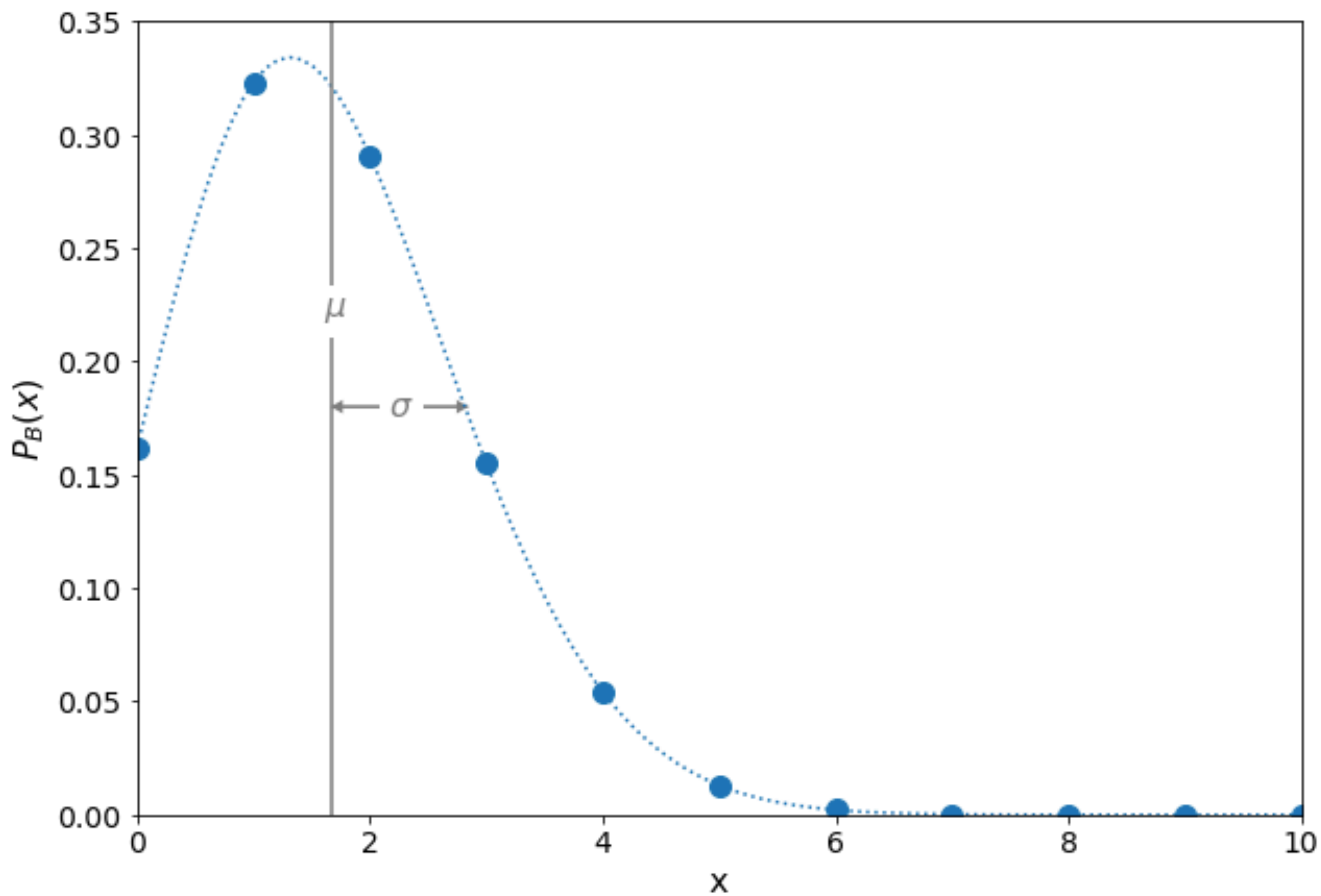
$$\mu = np = 5$$
$$\sigma^2 = np(1-p)$$
$$= 2.5$$

since q=p: distribution is symmetric

Binomial Distribution - Example (2)

roll 10 dice, x = number of rolls with 6 eyes

$$n = 10\,,\; p = 1/6 \quad \rightarrow \quad P(x) = P_{\mathrm{B}}(x|10, 1/6)$$



$$\mu = np = 5/3$$

$$\sigma^2 = np(1-p)$$
$$= 1.39$$

note: mean is not the mode

since q≠p: distribution is not symmetric

Poisson Distribution

- limit of binomial distribution if number of trials is large, and probability of "success" in a given trial is small, while the mean $\mu = np$ remains finite

$$P_{\mathrm{P}}(x|\mu) = \lim_{\substack{n \to \infty \\ p \to 0}} P_{\mathrm{B}}(x|n,p) = \frac{\mu^x}{x!} e^{-\mu}$$

- for experiments where a mean number of events, μ, can be measured
- neither the number of realizations n, nor the probability of "success" p need to be known
- example: "counting" experiments, e.g. flux measurements

Poisson Distribution

- mean of the Poisson distribution: μ
- variance of the Poisson distribution: $\sigma^2 = \mu$
- standard deviation: $\sigma = \sqrt{\mu}$

- a flux measurement typically consists of measuring a number of events, N, per time interval Δt, with $\mu = N/\Delta t$
- assuming that the time interval is precisely known, we can assign an uncertainty to a single measurement from $\sqrt{N}$

Poisson Distribution - Example

On your CCD, you count a flux of 100 photons from a star.

Q: Report an estimate of the flux (in number of photons) and the uncertainty.

Only 1 measurement → flux estimate (μ) is 100 photons.

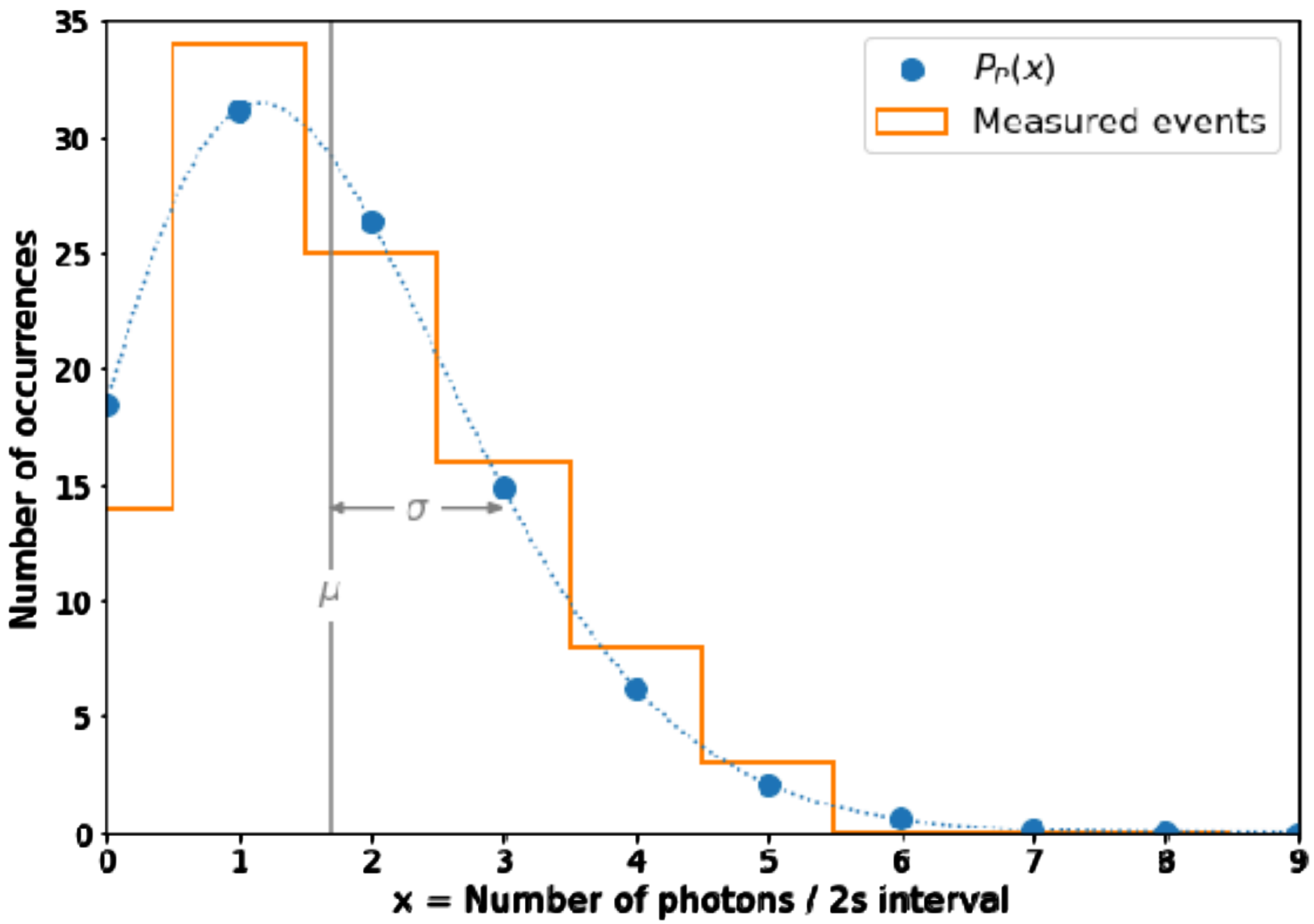
Counting experiment: standard deviation is sqrt(μ).

Measurement: F = 100 +/- 10

Q: How can multiple measurements improve on this?

Poisson Distribution - Example

a detector measures the number of gamma-ray photons per 2 second interval, making 100 measurements:



measured mean:

$$\bar{x} = 1.69$$

blue points:

$$P_{\mathrm{P}}(x|1.69)$$

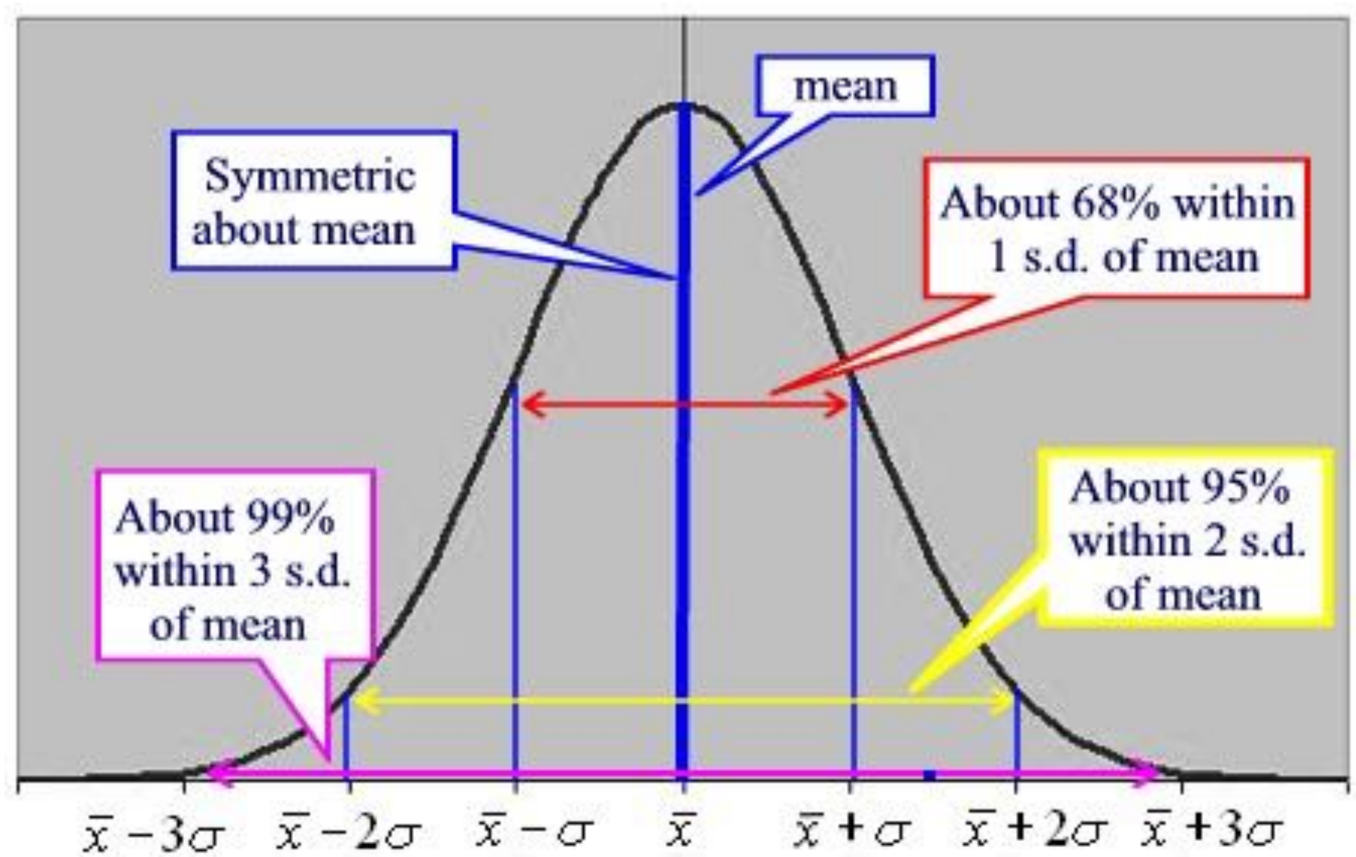
note: Poisson distribution is defined for positive, integer values of x

Gaussian / Normal Distribution

- the most commonly encountered probability distribution

$$P_{\mathrm{G}}(x|\mu,\sigma) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

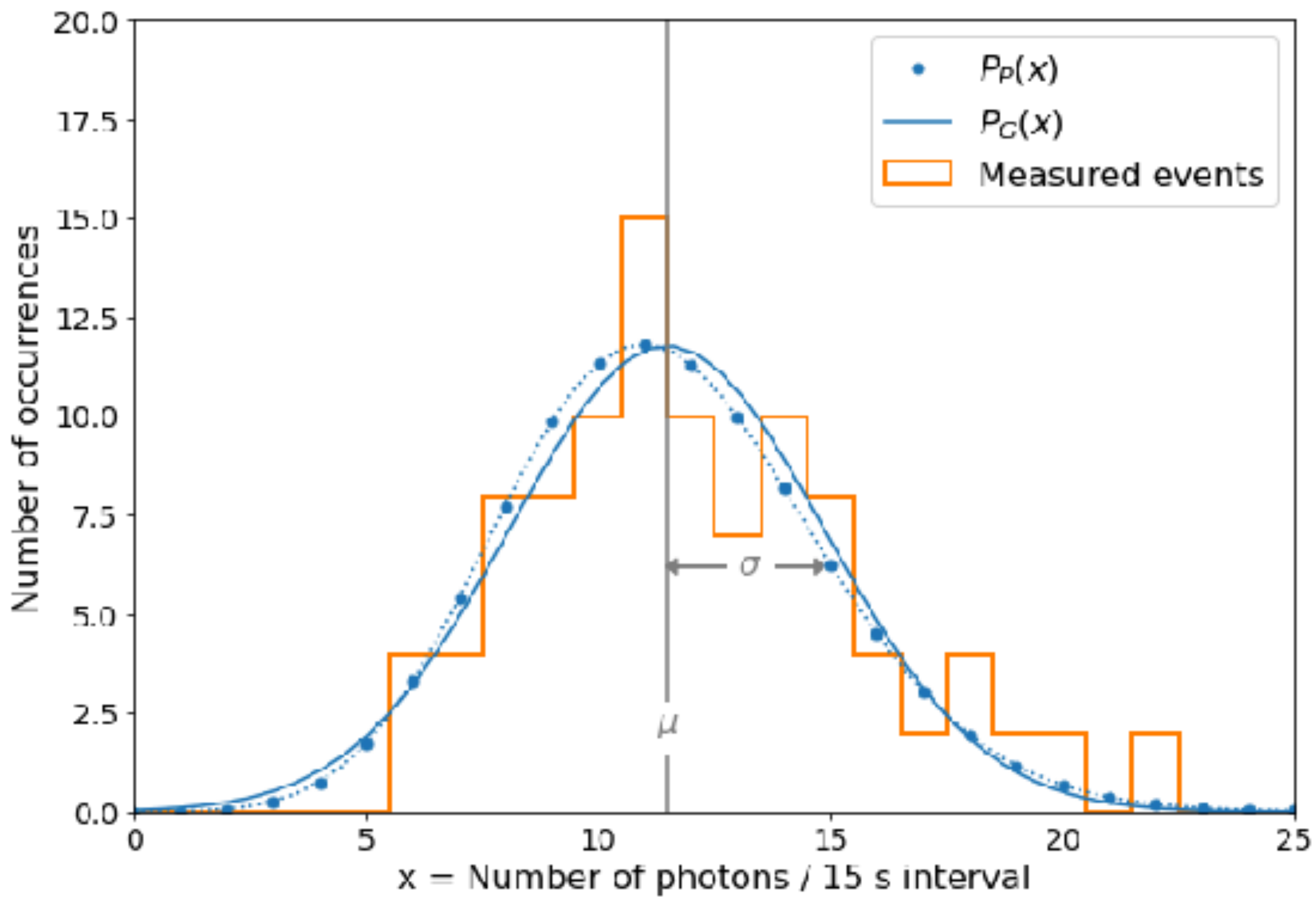
- mean: μ , standard deviation: σ
- can be derived as limit of the Poisson distribution for large values of the mean, $\mu \gtrsim 30$
- can also be derived as limit of many other distributions



Regents Prep

Gauss Distribution - Example

a detector measures the number of gamma-ray photons per 15 second interval, making 60 measurements:



measured mean:

$$\bar{x} = 11.48$$

blue points:

$$P_{\mathrm{P}}(x|11.48)$$

blue curve:

$$P_{\mathrm{G}}(x|\bar{x}, \sqrt{\bar{x}})$$

note: unlike Poisson distribution, Gaussian is continuous and defined for all x

Rest of today

Data Analysis Help Session: work on your data!

- Lab 1 analysis / check-ins
- Lab 2 preparation - which lab / which targets
- Homework 4