On the PD-Based Control of Robot Manipulators

1. Review

The equations of motion of n-dof robot manipulators are given by

$$D(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = u \tag{1.1}$$

where

- $q \in \mathbb{R}^n$ is the generalized coordinate vector
  $\dot{q}, \ddot{q} \in \mathbb{R}^n$ are the corresponding velocity and acceleration vectors
- $D(q) \in \mathbb{R}^{n\times n}$ is the inertia matrix. $D(q)$ is symmetric and positive definite. For the class of robot manipulators with bounded inertia matrices (see Ghorbel et al. "On the Positive Definiteness and Uniform Boundedness of the Inertia Matrix of Robot Manipulators" Proc. of the 32nd IEEE CDC, San-Antonio, TX, December 15-17, 1993), there exist computable constants σ_1 and σ_2 such that

$$0 < \sigma_1 \leq \| D(q) \| \leq \sigma_2 < \infty \quad \forall q \in \mathbb{R}^n. \tag{1.2}$$

- $C(q,\dot{q})\dot{q}$ is the vector of Coriolis and centrifugal terms.

For a particular choice of the matrix $C(q,\dot{q})$,
$\dot{D} - 2C$ is skew symmetric.

Furthermore, it can be shown that a constant k_c exists such that

$$\|C(q,\dot{q})\| \leq k_c \|\dot{q}\| \tag{1.3}$$

- $g(q)$ is the gravity vector. Note that
$$g(q) = \frac{\partial U(q)}{\partial q}$$
where $U(q)$ is the gravitational potential energy. $U(q)$, $g(q)$ and $\frac{\partial g(q)}{\partial q}$ are all bounded.

<u>Note</u>: The equations of motion (1.1) are linear in parameters. In particular,
$$g(q) = Y_g(q)\,\theta_g \quad \text{where}$$
$\theta_g \in \mathbb{R}^m$ is an unknown parameter vector (in general) with constant elements, and
$Y_g(q) \in \mathbb{R}^{n\times m}$ is a matrix of known components

In the following analysis, we consider the regulation problem in which the joint displacements q are required to follow a <u>constant</u> desired joint displacement vector q_d (hence $\dot{q}_d = 0$)

2. Absence of gravity

Suppose the gravity vector $g(q) = 0$, then the equations of motion become

$$D(q)\ddot{q} + C(q,\dot{q})\dot{q} = u \tag{2.1}$$

An independent joint PD-control scheme can be written as

$$u = K_P(q_d - q) - K_D\dot{q} \tag{2.2}$$

where K_P and K_D are diagonal matrices of (positive) proportional and derivative gains, respectively.

Combine (2.1) and (2.2)

$$D(q)\ddot{q} + C(q,\dot{q})\dot{q} = K_P(q_d - q) - K_D\dot{q} \tag{2.3}$$

At steady state, that is $\ddot{q} = \dot{q} = 0$, (2.3) gives $\qquad \tilde{q} = q - q_d$

$$0 = K_P(q_d - q) \quad \text{or} \quad q = q_d.$$

Hence with the PD control scheme (2.2), the equilibrium point of the system is $q = q_d$ and $\dot{q} = 0$.

We would like now to perform a stability analysis of the equilibrium $q = q_d$ and $\dot{q} = 0$. $\qquad x = \begin{bmatrix} \tilde{q} \\ \dot{\tilde{q}} \end{bmatrix}$

Consider the Lyapunov function candidate

$$V = \frac{1}{2}\dot{q}^T D(q)\dot{q} + \frac{1}{2}(q_d - q)^T K_P (q_d - q) \tag{2.4}$$

The time derivative of V along the solution trajectories of the system (2.3) is given by

$$\begin{aligned}
\dot{V} &= \dot{q}^T\{D\ddot{q}\} + \tfrac{1}{2}\dot{q}^T\dot{D}\dot{q} - \dot{q}^T K_P(q_d - q) \\
&= \dot{q}^T\left\{K_P(q_d-q) - K_D\dot{q} - C\dot{q}\right\} + \tfrac{1}{2}\dot{q}^T\dot{D}\dot{q} - \dot{q}^T K_P(q_d-q) \\
&= -\dot{q}^T K_D\dot{q} + \underbrace{\tfrac{1}{2}\dot{q}^T[\dot{D}-2C]\dot{q}}_{=0 \text{ because of skew symmetry}} \\
&= -\dot{q}^T K_D \dot{q} \leq 0 \qquad (2.5)
\end{aligned}$$

Hence we conclude that the equilibrium $q = q_d$, $\dot{q} = 0$ is stable.

The above analysis shows that V is decreasing as long as $\dot{q}$ is not zero. This, by itself is not enough to prove asymptotic stability since it conceivable, using the above Lyapunov analysis, that the manipulator can reach a position where $\dot{q} = 0$ but $q \neq q_d$. To show that this cannot happen we suppose that $\dot{V} \equiv 0$. Then (2.5) implies that $\dot{q} \equiv 0$ and hence $\ddot{q} \equiv 0$. From the equations of motion with PD-control, namely (2.3), we must then have $0 = -K_P(q_d - q)$ which implies that $q_d - q = 0$, $\dot{q} = 0$. Hence (invoking the so-called La-Salle's Theorem) we conclude that the equilibrium $q = q_d$, $\dot{q} = 0$ is (globally) asymptotically stable, that is

$$\begin{aligned} q &\to q_d \\ \dot{q} &\to 0 \end{aligned} \quad \text{as } t \to \infty$$

The results of this section are summarized as follows:

Theorem 1:

For the manipulator equations (2.1) with no gravity and with the PD-control (2.2), the equilibrium $q = q_d$, $\dot{q} = 0$ is globally asymptotically stable, that is for any initial conditions $q = q_0$, $\dot{q} = \dot{q}_0$, we have

$$\begin{matrix} q \longrightarrow q_d \\ \dot{q} \longrightarrow 0 \end{matrix} \quad \text{as } t \to \infty$$

3. Presence of Gravity, No Gravity Compensation

Consider now the general (and more realistic) case in which the gravity vector $g(q) \neq 0$ so that the equations of motion are (1.1), or restated here

$$D(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = u \tag{3.1}$$

Suppose we use the same PD-control (2.2) without gravity compensation, then combining (2.2) and (3.1) we get

$$\begin{aligned} D(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) &= u \\ &= K_P(q_d - q) - K_D\dot{q} \end{aligned}$$

Assuming the closed loop system is stable then at steady state we have

$$K_P(q_d - q) = g(q) \Leftrightarrow q_d - q = K_P^{-1} g(q) \tag{3.2}$$

Hence the steady state error (3.2) can be made smaller by choosing large K_P gains!

4. PD-control ⊕ Full Gravity Compensation

Given

$$D(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = u \qquad (4.1)$$

Choose the control

$$u = K_P(q_d - q) - K_D\dot{q} + g(q) \qquad (4.2)$$

The control law (4.2), in effect, cancels the effect of the gravitational terms and we get the same closed loop equation (2.3) and consequently Theorem 1 applies and the equilibrium $q_d = q$, $\dot{q} = 0$ is globally asymptotically stable.

The control law (4.2) requires microprocessor implementation to compute at each instant the gravitational terms $g(q)$ from the Lagrangian equations. In the case that these terms are unknown, the control law (4.2) is unachievable.

5. PD-control ⊕ Simple Gravity Compensation

Again, consider the equations of motion

$$D(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = u \qquad (5.1)$$

We will show in this section that it may not be necessary to compensate $g(q)$ for all values of q. Indeed, the computationally simpler control law

$$u = K_P(q_d - q) - K_D\dot{q} + g(q_d) \qquad (5.2)$$

suffices, under suitable assumptions, to achieve global asymptotic stability

Note that $\frac{\partial g(q)}{\partial q}$ is bounded, hene $\exists$ a positive constant M_1 s.t.

$$\left\| \frac{\partial g(q)}{\partial q} \right\| \leq M_1 \qquad \forall q \in \mathbb{R}^n \tag{5.3}$$

which implies (using the Mean Value Theorem)

$$\| g(q_1) - g(q_2) \| \leq M_1 \| q_1 - q_2 \| \qquad \forall q_1, q_2 \in \mathbb{R}^n \tag{5.4}$$

We now have the following result

Theorem 2:

Consider the manipulator equations (5.1) with the control law (5.2). If the elements of the diagonal matrix K_P are chosen such that $k_{P_i} > M_1 \quad 1 \leq i \leq n$, then the equilibrium $q = q_d$, $\dot{q} = 0$ is globally asymptotically stable, that is

$$\begin{aligned} q &\longrightarrow q_d \\ \dot{q} &\longrightarrow 0 \end{aligned} \quad \text{as } t \to \infty$$

Proof:

Combine (5.1) & (5.2)

$$D\ddot{q} + C\dot{q} + g(q) = K_P(q_d - q) - K_D\dot{q} + g(q_d)$$

At steady state

$$K_P(q_d - q) = g(q) - g(q_d) \tag{5.5}$$

The hypothesis $k_{P_i} > M_1$ and (5.4) enable us to write

$$\| K_P(q_d - q) \| \geq \min(k_{P_i}) \| q_d - q \| > M_1 \| q_d - q \| \geq \| g(q_d) - g(q) \|$$

$$\forall q \neq q_d$$

The previous chain of inequalities implies that the left-hand side of (5.5) is different from the right hand side for every $q \neq q_d$. Therefore $q = q_d$ is the unique solution of (5.5).

$\Rightarrow$ equilibrium $q = q_d$ & $\dot{q} = 0$ or $\begin{cases} \tilde{q} = 0 \\ \dot{\tilde{q}} = 0 \end{cases}$

Now consider the function

$$P_1(q) = U(q) - q^T g(q_d) + \frac{1}{2} q^T K_P q - q^T K_P q_d \qquad (5.6)$$

The stationary values of (5.6) are given by the solutions of

$$\frac{\partial P_1(q)}{\partial q} = 0 \Leftrightarrow 0 = g(q) - g(q_d) + K_P(q - q_d)$$

which coincides with (5.5). Consequently, (5.6) has the stationary values $q = q_d$ as discussed above.

Moreover

$$\frac{\partial^2 P_1(q)}{\partial q^2} = K_P + \frac{\partial g(q)}{\partial q} \qquad (5.7)$$

Owing to (5.3) and to the assumption $k_{P_i} > M_1$, the matrix (5.7) is positive definite. Hence $P_1(q)$ given by (5.6) has an absolute minimum at $q = q_0$.

We now introduce the Lyapunov function candidate

$$V = \frac{1}{2} \dot{q}^T D(q) \dot{q} + P_1(q) - P_1(q_0) \qquad (5.8)$$

The time derivative of V along the solution trajectories of the system is given by

$$\begin{aligned}
\dot{V} &= \cancel{\tfrac{1}{2}} \dot{q}^T \{D \ddot{q}\} + \frac{1}{2} \dot{q}^T \dot{D} \dot{q} + \dot{q}^T \frac{\partial P_1(q)}{\partial q} \\
&= \frac{1}{2} \dot{q}^T \dot{D} \dot{q} + \dot{q}^T \{-C\dot{q} - g(q) + K_P(q_d - q) - K_D \dot{q} + g(q_d)\} \\
&\quad + \dot{q}^T g(q) - \dot{q}^T g(q_d) + \dot{q}^T K_P (q - q_d) \\
&= \underbrace{\frac{1}{2} \dot{q}^T [\dot{D} - 2C] \dot{q}}_{=0 \text{ why?}} - \dot{q}^T K_D \dot{q} = -\dot{q}^T K_D \dot{q} \leq 0 \qquad (5.9)
\end{aligned}$$

Using the Lyapunov function V given by (5.8), and the fact from (5.9) that $\dot{V} \leq 0$ we conclude that the equilibrium $q = q_d$, $\dot{q} = 0$ is <u>stable</u>.
Using a similar argument as in the proof of Theorem 1, we can show that the equilibrium $q = q_d$, $\dot{q} = 0$ is indeed (globally) <u>asymptotically stable</u>.

6. Adaptive PD Control

Consider one more time our famous equations of motion

$$D(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = u \tag{6.1}$$

and consider the following adaptive control law

$$u = k_P(q_d - q) - K_D\dot{q} + Y_g(q)\hat{\theta}_g \tag{6.2}$$

with the parameter adaptation law

$$\dot{\hat{\theta}} = -\beta\, Y_g^T(q)\left[\gamma\dot{q} + \frac{2(q-q_d)}{1 + 2(q-q_d)^T(q-q_d)}\right] \tag{6.3}$$

where β is a positive constant, k_P and K_D are diagonal positive matrices, and γ satisfies

$$\gamma > \max\left\{\left(\frac{2\sigma_2}{\sqrt{\sigma_1 \min(k_{p_i})}}\right), \left(\frac{1}{\min(k_{d_i})}\frac{(\max(k_{d_i}))^2}{2\min(k_{p_i})} + 4\sigma_2 + \frac{k_c}{\sqrt{2}}\right)\right\}$$

where σ_1 and σ_2 are given by (1.2), and k_c is given by (1.3)

We summarize the adaptive PD control in the following

Theorem 3:

Consider the dynamics equation (6.1) with the control law (6.2) and the parameter update law (6.3). If γ satisfies (6.3), then $\{q_d - q(t)\}$, $\dot{q}(t)$ and $\hat{\theta}_g(t)$ are bounded for $t \geq 0$. Moreover

$$\lim_{t \to \infty} \left\| \begin{matrix} q_d - q(t) \\ \dot{q}(t) \end{matrix} \right\| = 0.$$