# Theorems on Stability

1. Stability
2. Uniform Stability
3. Uniform Asymptotic Stability
4. Global Uniform Asymptotic Stability
5. Exponential Stability
6. Global Exponential Stability
7. An Instability Theorem
8. LaSalle's Theorem
9. LaSalle's Theorem, Global Version

# Summary (I)

| Thm. | Type of Stability | Conditions on $V$ | Conditions on $\dot{V}$ |
|---|---|---|---|
| 1 | Stability | $C^1$, lpdf | $\dot{V} \leq 0$, in a ball $B_r$, $\forall\ t \geq 0$. |
| 2 | Uniform Stability | $C^1$, lpdf, decrescent | $\dot{V} \leq 0$, in a ball $B_r$, $\forall\ t \geq 0$. |
| 3 | Uniform Asymptotic Stability | $C^1$, lpdf, decrescent | $-\dot{V}$ is lpdf. |
| 4 | Global Uniform Asymptotic Stability | $C^1$, pdf, decrescent, radially unbounded | $-\dot{V}$ is pdf. |
| 5 | Exponential Stability | $\exists\ a,b,\ r>0,\ P\geq 1$, s.t. $a\|\mathbf{x}\|^P \leq V \leq b\|\mathbf{x}\|^P$, in a ball $B_r$, $\forall\ t \geq 0$ | $\exists\ c>0$, s.t. $\dot{V} \leq -c\|\mathbf{x}\|^P$, in a ball $B_r$, $\forall\ t \geq 0$ |

## Summary (II)

| Thm. | Type of Stability | Conditions on $V$ | Conditions on $\dot{V}$ |
|---|---|---|---|
| 6 | Global Exponential Stability | $\exists\ a,b>0,\ P\geq 1$, s.t. $a\|\|\mathbf{x}\|\|^P \leq V \leq b\|\|\mathbf{x}\|\|^P$, $\forall\ t\geq 0,\ \forall\ \mathbf{x}\in\mathbf{R}^n$ | $\exists\ c>0$, s.t. $\dot{V}\leq -c\|\|\mathbf{x}\|\|^P$, $\forall\ t\geq 0,\ \forall\ \mathbf{x}\in\mathbf{R}^n$ |
| 7 | An Instability Theorem | $V$: $C^1$, decrescent $V(t,0)=0$, and $V(t_0,x_0)\geq 0$ close to $0$. | $\dot{V}$ is lpdf. |
| 8 | (LaSalle's Thm) Uniform Asmptotic Stability for Autonomous $\sum_A$ | $V$: $C^1$, lpdf | $\dot{V}\leq 0$ in neighborhood N $L_v(c)$ bounded & $\subset$ N $S=\{x\in L_v(c): \dot{V}(\mathbf{x})=0\}=\{0\}$ |
| 9 | (LaSalle's Thm, Global) Global Uniform Asymptotic Stability | $V$: $C^1$, pdf, radially unbounded | $\dot{V}\leq 0,\ \forall\ \mathbf{x}\in\mathbf{R}^n$, $R=\{x\in\mathbf{R}^n: \dot{V}(\mathbf{x})=0\}=\{0\}$ |

## Remarks

Theorems on Stability:

+ Enable one to draw conclusions about the stability status of an equilibrium without solving the system equations.
+ Lyapunov function V sometimes has an intuitive appeal as the total energy of the system.

− Represent only sufficient conditions.

  No V that satisfy hypothesis on $\dot{V}$

  $\rightarrow$ No conclusion.

− No systematic procedure for generating Lyapunov function candidates.

One is justified in asking what is the role of Lyapunov theory today?

$\rightarrow$ Originally Lyapunov stability theory was advanced as a means of testing the stability status of a given system. Nowadays increasingly being used to guarantee stability.

$\rightarrow$ Converse Lyapunov Theorems.

## Converse Theorems

- Present only converse theorems on:
  – uniform asymptotic stability (converse. Of Thm 3)
  – exponential stability (converse of Thm 5)
  – global exponential stability (converse of Thm 6)
- In essence, these theorem state that the conditions of
  Theorem 3
  Theorem 5, &
  Theorem 6
  are necessary and sufficient conditions.

  **Notation**

  If $f$ is a function of several arguments, then $D_i$ denotes the partial derivative of $f$ with respect to the i-th argument. For example, if $f=f(x,y,z)$, then

$$D_1 f = \frac{\partial f}{\partial x}; \qquad D_2 f = \frac{\partial f}{\partial y}; \qquad D_3 f = \frac{\partial f}{\partial z}$$

# Theorem 10 (I)

**Theorem 10**: [Converse of Thm 3 on Uniform Asymptotic Stability]

Consider the system $\sum$ defined in $[0,\infty)\times B_h \rightarrow \mathbf{R}^n$ and suppose that $f(t, 0)=0, \quad \forall\, t \geq 0$ and that $f$ is $C^k$ for some integer $k \geq 1$.

Suppose $\exists$ a constant $h \geq r > 0$;

a function $\phi$ of class K;

a function $\sigma$ of class L s.t.

the solution trajectories of $\sum$ satisfy

$$\|s(t,t_0,\mathbf{x}_0)\| \leq \Phi(\|\mathbf{x}_0\|)\sigma(t - t_0)\ ,\ \forall\, t \geq t_0 \geq 0,\ \forall\, \mathbf{x}_0 \in B_r \qquad (1)$$

Finally, suppose in addition that for some finite $\lambda$

$$\|D_2 f(t, \mathbf{x})\| \leq \lambda,\ \forall\, t \geq 0,\ \forall\, \mathbf{x} \in B_h \qquad (2)$$

# Theorem 10 (II)

Under these conditions, ∃ a $C^k$ function

$V: \mathbf{R}^+ \times \mathbf{R}^n \to \mathbf{R}$ and $C^\infty$ functions $\alpha$, $\beta$, $\gamma$ of class K s.t.

- $\alpha(\|\mathbf{x}\|) \le V(t,\mathbf{x}) \le \beta(\|\mathbf{x}\|)$ , $\forall\ t \ge 0$, $\forall\ \mathbf{x} \in B_r$ (3)
- $\dot{V}(t,x) \le -\gamma(\|\mathbf{x}\|)$, $\forall\ t \ge 0$, $\forall\ \mathbf{x} \in B_r$ (4)
- $\sup_{x \in B_r} \| D_2 V(t,x) \| < \infty$ (5)

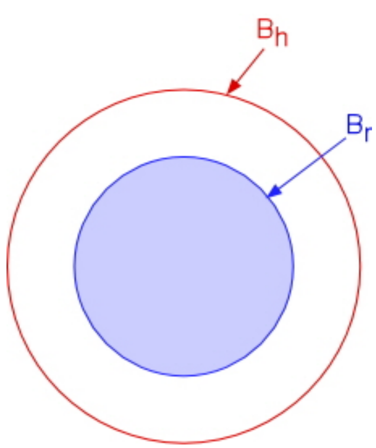
**Note:**

(2) → $D_2 f(t, \mathbf{x})$ bounded uniformly in t;

(3) → $V$ lpdf & decrescent;

(4) → $-\dot{V}$ lpdf.

(5) → Extra condition



## Remarks

1. Condition (1) is equivalent to the requirement that 0 be uniformly asymptotically stable equilibrium.
2. If $f$ does not depend explicitly on $t$, then (2) is automatically satisfied since $f$ is $C^1$ and the closure of $B_r$ is compact . Hence, this condition only comes into the picture for non-autonomous systems[1].
3. Compare conditions (3) & (4) in Theorem 3. Condition (5) is an added bonus, so to speak.
4. The Lyapunov function $V$ is differentiable as many times as is the function $f$ of $\sum$, Thus, if $f$ is $C^\infty$, then so is $V$.

[1]. For autonomous systems, condition (2) is automatically satisfied.

# Theorem 11

**Theorem 11**: [Converse of Thm 5 on Exponential Stability]

Consider the system $\sum$ defined in $[0,\infty)\times B_h \rightarrow \mathbf{R}^n$ and suppose $f$ is $C^k$ and that $f(t, 0)=0, \quad \forall\, t \geq 0$ for some integer $k \geq 1$

Suppose $\exists$ constants $\mu, \delta, r>0$, s.t.

$$||s(t_0+t,t_0,\mathbf{x}_0)|| \leq \mu||\mathbf{x}_0|| \exp[-\delta t],\ \forall\, t,\ t_0\geq 0,\ \forall\, \mathbf{x}_0 \in B_r \text{ , s.t. } r<h/\mu$$

Finally, suppose that, for some finite $\lambda$

$$||D_2 f(t, \mathbf{x})|| \leq \lambda,\ \forall\, t \geq 0,\ \forall\, \mathbf{x} \in B_h \supset B_r$$

Under these conditions, $\exists$ a $C^k$ function

$V: \mathbf{R}^+ \times \mathbf{R}^n \rightarrow \mathbf{R}$ and constants $a,b,c,m>0,\ p>1$ s.t.

- $a||\mathbf{x}||^p \leq V(t,\mathbf{x}) \leq b||\mathbf{x}||^p ,\ \forall\, t\geq 0,\ \forall\, \mathbf{x}\in B_r$
- $\dot{V}(t,x) \leq -c||\mathbf{x}||^p, \qquad \forall\, t\geq 0,\ \forall\, \mathbf{x}\in B_r$
- $||D_2 V(t, \mathbf{x})|| \leq m||\mathbf{x}||^{p-1}, \quad \forall\, t\geq 0,\ \forall\, \mathbf{x}\in B_r$

# Corollary of Theorem 11

The next Corollary shows that, for an exponentially stable equilibrium, it is possible to construct a "quadratic type" Lyapunov function:

**Corollary**:

Suppose all hypothesis of Thm 11 are satisfied. Then ∃ a $C^k$ function $W$: $\mathbf{R}^+ \times \mathbf{R}^n \rightarrow \mathbf{R}$ and constants $\alpha, \beta, \gamma, \mu > 0$ s.t.

- $\alpha \|\mathbf{x}\|^2 \leq W(t,\mathbf{x}) \leq \beta \|\mathbf{x}\|^2$, $\forall\ t \geq 0$, $\forall\ \mathbf{x} \in B_r$
- $\dot{W}(t,x) \leq -\gamma \|\mathbf{x}\|^2$, $\forall\ t \geq 0$, $\forall\ \mathbf{x} \in B_r$
- $\|D_2 V(t, \mathbf{x})\| \leq \mu \|\mathbf{x}\|$, $\forall\ t \geq 0$, $\forall\ \mathbf{x} \in B_r$

# Theorem 12

**Theorem 12**: [Converse of Thm 6 on Global Exponential Stability]

Consider the system $\sum$ defined in $[0,\infty)\times B_h \rightarrow \mathbf{R}^n$

Suppose $f$ is $C^k$ and that $f(t, 0)=0$, $\forall\, t \geq 0$ for some integer $k \geq 1$

Suppose $\exists$ constants $\mu, \delta, \lambda > 0$, s.t.

$$\|s(t_0+t, t_0, \mathbf{x}_0)\| \leq \mu\|\mathbf{x}_0\| \exp[-\delta t],\ \forall\, t, t_0 \geq 0,\ \forall\, \mathbf{x}_0 \in \mathbf{R}^n$$

$$\|D_2 f(t, \mathbf{x})\| \leq \lambda,\ \forall\, t \geq 0,\ \forall\, \mathbf{x} \in \mathbf{R}^n$$

Under these conditions, $\exists$ a $C^k$ function

$W: \mathbf{R}^+ \times \mathbf{R}^n \rightarrow \mathbf{R}$ and constants $\alpha, \beta, \gamma, \mu > 0$, s.t.

- $\alpha \|\mathbf{x}\|^2 \leq W(t,\mathbf{x}) \leq \beta \|\mathbf{x}\|^2$, $\forall\, t \geq 0$, $\forall\, \mathbf{x} \in \mathbf{R}^n$
- $\dot{W}(t,x) \leq -\gamma \|\mathbf{x}\|^2$, $\forall\, t \geq 0$, $\forall\, \mathbf{x} \in \mathbf{R}^n$
- $\|D_2 V(t, \mathbf{x})\| \leq \mu\|\mathbf{x}\|$, $\forall\, t \geq 0$, $\forall\, \mathbf{x} \in \mathbf{R}^n$

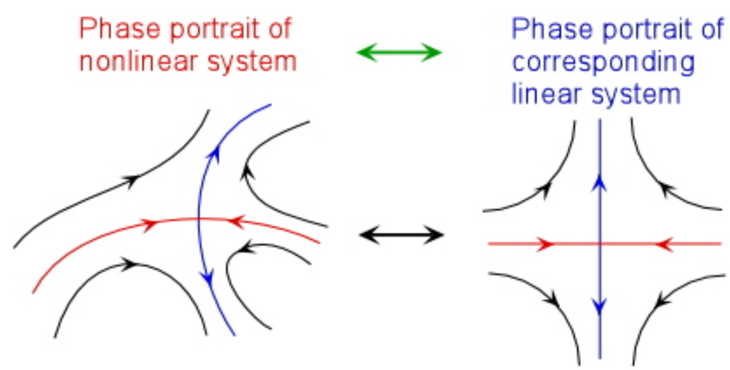
# Stability of Linear Systems (I)

- We study the Lyapunov stability of systems described by linear vector differential equations:

$$\dot{x}(t) = A(t)x(t), \qquad t \geq 0 \qquad \sum^{L}$$

$$\dot{x}(t) = Ax(t) \qquad \sum^{L}_{A}$$

- We will obtain necessary and sufficient conditions for the stability of linear systems.
- And pave the way to deriving Lyapunov's linearization method (a way to draw conclusions about a nonlinear system by studying the behavior of a linear system).




**Recall**: For 2nd order nonlinear systems
Hortman-Grobman Theorem:
∃ a homeomorhpism (continuous map with a continuous inverse)

# Stability of Linear Systems (II)

- Autonomous Linear systems $\sum_{A}^{L} : \dot{x} = Ax$

  Stability conclusions can be determined from location of eigenvalue of $A$.

- Non-autonomous (time-varying) Linear systems

  $$\sum^{L} : \dot{x} = A(t)x$$

  Stability conclusions can not be characterized by the location of the eigenvalues of the matrix A.

Ex : Consider $\dot{x} = A(t)x$

$$A(t) = \begin{bmatrix} -1+1.5\cos^2 t & 1-1.5\sin t\cos t \\ -1-1.5\sin t\cos t & -1+1.5\sin^2 t \end{bmatrix}$$

For each $t$, eigenvalues of $A(t)$ are

$$-0.25 \pm 0.25\sqrt{7}\,j$$

Yet! $x(t) = \Phi(t,0)x(0)$ where

$$\Phi(t,0) = \begin{bmatrix} e^{0.5t}\cos t & e^{-t}\sin t \\ -e^{0.5t}\sin t & e^{-t}\cos t \end{bmatrix}$$

and the equilibrium O is unstable.

# Stability & the State Transition Matrix (I)

Consider $\sum^{\mathrm{L}} : \dot{x}(t) = A(t)x(t) \qquad t \geq 0$

- 0 of $\sum^{\mathrm{L}}$ is always an equilibrium of $\sum^{\mathrm{L}}$.
- 0 is an isolated equilibrium if $A(t)$ is nonsingular for $t \geq 0$.
- General solution of $\sum^{\mathrm{L}}$:

$$x(t) = \underbrace{\Phi(t, t_0)}_{\text{State Transition Matrix}} x(t_0) \qquad t \geq 0$$

$\Phi(\cdot, \cdot)$ is the unique solution of the equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi(t, t_0) = A(t)\Phi(t, t_0), \quad \forall t \geq t_0 \geq 0$$

$$\Phi(t_0, t_0) = I, \qquad \forall t_0 \geq 0$$

# Stability & the State Transition Matrix (II)

**Fact**:

- If A(t) has the following commutative property

$$A(t)\left[\int_{t_0}^{t} A(\tau)\,\mathrm{d}\tau\right] = \left[\int_{t_0}^{t} A(\tau)\,\mathrm{d}\tau\right]A(t), \quad \forall t \quad \& \quad t_0$$

$$\Rightarrow \Phi(t,t_0) = e^{\left[\int_{t_0}^{t} A(\tau)\,\mathrm{d}\tau\right]}$$

  This is the case when $A(t)$ is diagonal or $A(t)= A$ =Constant.

  If $A(t)= A$ =Constant $\Rightarrow \Phi(t,t_0)=e^{A(t-t0)}$

- With the aid of this explicit characterization of the solution of $\sum^{L}$, it is possible to derive some useful conditions for the stability of $\sum^{L}$.

$\rightarrow$ Since in general impossible to compute $\Phi$, results are of conceptual (not computational) value.

$\rightarrow$ Enable one to understand stability & instability mechanism.

# Theorem on Transition Matrix of Non-Autonomous System (I)

<u>Claim</u> : If $\int_{t_0}^{t} A(\tau)\,\mathrm{d}\tau$ and $A(\tau)$ commute for all $t$, i.e.,

$$A(t)\left[\int_{t_0}^{t} A(\tau)\,\mathrm{d}\tau\right] = \left[\int_{t_0}^{t} A(\tau)\,\mathrm{d}\tau\right]A(t), \quad \text{for all } t \qquad (*)$$

then the unique solution of

$$\frac{\partial}{\partial t}\Phi(t,t_0) = A(t)\Phi(t,t_0), \qquad \Phi(t_0,t_0) = I \qquad (2*)$$

is $\Phi(t,t_0) = e^{\left[\int_{t_0}^{t} A(\tau)\,\mathrm{d}\tau\right]}$ (3*)

<u>Proof</u> :

Let $B = \int_{t_0}^{t} A(\tau)\,\mathrm{d}\tau$ Denote $\dot{B} \equiv \dfrac{\partial B}{\partial t} = A(t)$

If (3*) is a solution, i.e. $\Phi(t,t_0) = e^{B}$ is a solution,

then (2*) must be satisfied.

We will show that (2*) is satisfied if (*) is satisfied, i.e., if

$$\dot{B}B = B\dot{B}$$

# Theorem on Transition Matrix of Non-Autonomous System (II)

Let $\Phi = e^{B}$

$$\frac{\partial \Phi}{\partial t} = \frac{\partial}{\partial t}\left[I + B + \frac{1}{2!}B^{2} + \frac{1}{3!}B^{3} + \ldots\right]$$

$$= \frac{\partial}{\partial t}\left[I + B + \frac{1}{2!}BB + \frac{1}{3!}BBB + \ldots\right]$$

$$= 0 + B + \frac{1}{2!}[\dot{B}B + B\dot{B}] + \frac{1}{3!}[\dot{B}BB + B\dot{B}B + BB\dot{B}] + \ldots$$

$$\underset{\text{if } (*) \text{ satisfied}}{=}$$

$$= B + \frac{1}{2}[2\dot{B}B] + \frac{1}{3!}[3\dot{B}B^{2}] + \ldots$$

$$= \dot{B}\left[I + B + \frac{1}{2!}B^{2} + \ldots\right]$$

$$= A(t)e^{B}$$

$$= A(t)\Phi(t,t_0) \qquad \text{Q.E.D.}$$

# Stability & the State Transition Matrix (III)

**Theorem**: [stability]

The equilibrium 0 of $\sum^L$ is stable iff for each $t_0 \geq 0$, it is true that

$$\sup_{t \geq t_0} \| \Phi(t, t_0) \|_i \equiv m(t_0) < \infty$$

Where $\|\bullet\|_i$ denote the induced norm of a matrix.

Proof :

- $\rightarrow \sup_{t \geq t_0} \| \Phi(t, t_0) \|_i \equiv m(t_0) < \infty$

  Let $\varepsilon > 0,\ t_0 \geq 0$ be specified. Let $\delta(\varepsilon, t_0) = \dfrac{\varepsilon}{m(t_0)}$

  $$\|\mathbf{x}(t_0)\| = \delta \Rightarrow \quad \|\mathbf{x}(t)\| = \|\Phi(t, t_0)\mathbf{x}(t_0)\|$$

  $$\leq \|\Phi(t, t_0)\| \ \|\mathbf{x}(t_0)\| = m(t_0)\frac{\varepsilon}{m(t_0)} = \varepsilon$$

  $\Rightarrow$ O is stable.

- $\leftarrow$ (see textbook )

  $\| \quad \|$ unbounded

  $\Rightarrow$ O is unstable.

# Stability & the State Transition Matrix (IV)

**Remark**:

In the case of linear systems, the instability of the equilibrium 0 does indeed imply that some solution trajectories actually "blow up".

**Theorem**: [Uniform Stability]

The equilibrium O of $\sum^{L}$ is uniformly stable iff

$$m_0 \equiv \sup_{t_0 \geq 0} m(t_0) = \sup_{t_0 \geq 0} \sup_{t \geq t_0} \| \Phi(t, t_0) \|_i < \infty$$

# Stability & the State Transition Matrix (V)

**Theorem**: [Global Uniform Asymptotic Stability]

The equilibrium O of $\sum^{L}$ is (globally) uniformly asymptotically stable iff

- $\sup_{t_0 \geq 0} \sup_{t \geq t_0} \| \Phi(t, t_0) \|_i < \infty$ (1)
- $\| \Phi(t, t_0) \|_i \to 0 \quad \text{as} \quad t \to \infty, \text{uniformly in } t_0$ (2)

**Theorem:**

The equilibrium O of $\sum^{L}$ is uniformly asymptotically stable iff

∃ constants $m, \lambda > 0$ s.t.

$$\| \Phi(t, t_0) \|_i \leq m \exp[-\lambda(t - t_0)] \quad \forall\, t \geq t_0 \geq 0 \qquad (3)$$

$\Rightarrow$ For linear systems:

$$\begin{Bmatrix} \text{uniform asymptotic} \\ \text{stability} \end{Bmatrix} \Leftrightarrow \begin{Bmatrix} \text{exponential} \\ \text{stability} \end{Bmatrix}$$

# Notes

**<u>Question</u>**:

If the origin of is (locally) uniformly asymptotically stable

$\Longrightarrow$ it is globally so.

Why?

**<u>Answer</u>**:

Due to the linear dependence of $\mathbf{x}(t)$ on $\mathbf{x}(t_0)$.

# Stability & the State Transition Matrix (VI)

**Proof**:

$\rightarrow$ If (3) is satisfied, then clearly (1) & (2) are also true and hence O is uniformly asymptotically stable.

$\leftarrow$ Suppose (1) & (2) are true.

Then $\exists$ finite constants $\mu$ and $T$ s.t.

(i) $\|\Phi(t,t_0)\|_i \leq \mu, \ \forall\ t \geq t_0 \geq 0$ (4)

(ii) $\|\Phi(t_0+t,t_0)\|_i \leq 1/2\ \forall\ t \geq T,\ \forall\ t_0 \geq 0$

in particular, (ii) implies that

$$\|\Phi(t_0+T,t_0)\|_i \leq 1/2,\ \forall\ t_0 \geq 0 \qquad (5)$$

Now given any $t_0$ and any $t \geq t_0$, pick an integer $k$ s.t.

$$t_0+kT \leq t \leq t_0+(k+1)T$$

then

$$\Phi(t,t_0)=\Phi(t, t_0+kT)\,\Phi(t_0+kT, t_0+kT-T)\ldots\,\Phi(t_0+T, t_0)$$

(semi group property of the transition matrix)

# Stability & the State Transition Matrix (VII)

Hence

$$\|\Phi(t,t_0)\|_i \le \|\Phi(t,t_0+kT)\|_i \prod_{j=1}^{k} \|\Phi(t_0+jT, t_0+jT-T)\|_i$$

where the empty products is takens as one.

Repeated application of (4) & (5) gives

$$\|\Phi(t,t_0)\|_i \le \mu 2^{-k} = (2\mu)2^{-(k+1)}$$

$$\le 2\mu\, 2^{-\frac{(t-t_0)}{T}} \qquad [\because t < t_0 + (t+1)T \rightarrow \frac{(t-t_0)}{T} < (k+1)]$$

Hence (3) is satisfied if we define

$$m = 2\mu$$

$$\lambda = \frac{\ln 2}{T}$$

## Autonomous Systems (I)

$$\sum\nolimits_A^L : \dot{x}(t) = Ax(t)$$

det($A$)≠0 ⇒O unique equilibrium

In this case, Lyapunov theory is very complete.

**Theorem**:

- The equilibrium O of $\sum^L_A$ is (globally) exponentially stable iff all eigenvalues of $A$ have negative real parts.
- The equilibrium O of $\sum^L_A$ is stable iff

  all eigenvalues of $A$ have nonpositive real part and in addition,

  every eigenvalue of $A$ having a zero real part is a simple zero of the minimal polynomial of $A$.

**Note**:

$$x(t) = \exp(At)x(0)$$

$$\exp(At) = \sum_{i=1}^{r}\sum_{j=1}^{m_i} P_{ij}(A)t^{j-1}\exp(\lambda_i t)$$

r: # of distinctive eigenvalue of $A$;
$\lambda_1, \lambda_2, \ldots \lambda_r$: distinctive eigenvalues;
$m_i$: multiplicity of the eigenvalue $\lambda_i$;
$P_{ij}$: interpolating polynomials.

# The Lyapunov Approach (I)

$$\textstyle\sum_{\mathrm{A}}^{\mathrm{L}}: \qquad \dot{x}(t) = Ax(t)$$

Choose a Lyapunov function candidate of the form

$$V(\mathbf{x}) = \mathbf{x}^T P\mathbf{x}, \qquad P: \text{ real, symmetric}$$

$$\Rightarrow \quad \dot{V}(\mathbf{x}) = \dot{\mathbf{x}}^T P\mathbf{x} + \mathbf{x}^T P\dot{\mathbf{x}} = \mathbf{x}^T A^T P\mathbf{x} + \mathbf{x}^T PA\mathbf{x}$$

$$= \mathbf{x}^T (A^T P + PA)\mathbf{x} = -\mathbf{x}^T Q\mathbf{x}$$

$$(*) \quad A^T P + PA = -Q \qquad \text{Lyapunov Matrix Equation}$$

- If $(P,Q)$ satisfying (*) can be found s.t. both $P$ & $Q$ are positive definite $\Rightarrow$ both $V$ & $-\dot{V}$ are p.d.f. and $V$ is radially unbounded.

  $\Rightarrow$ O is globally exponentially stable.
- If $(P,Q)$ can be found s.t. $Q$ is p.d. and $P$ has at least one nonpositive eigenvalues, then $-\dot{V}$ is p.d.f. and V assumes non-positive values arbitraily close to 0

  $\Rightarrow$ O is an unstable equilibrium.

# The Lyapunov Approach (II)

$$A^{\mathrm{T}}P + PA = -Q \qquad\qquad (*)$$

Two possible ways to tackle (*)

1) Given a particular matrix $A$, pick a particular matrix $P$ and study the properties of the matrix $Q$ resulting from (*).
2) Given $A$, pick $Q$ and study $P$ resulting from (*).

We adopt 2) because:

Pick a p.d. $Q$.

If resulting $P$ is p.d. $\Rightarrow$ O is exponentially stable.

If $P$ turns out to have at least one nonpositive eigenvalues

$\Rightarrow$ O is unstable.

Difficulty with 2):

Given $Q$, (*) may not have a unique solution for $P$!

## The Lyapunov Approach (III)

**Lemma**:

[Necessary & Sufficient Conditions for Unique solution of $P$ given $Q$]

Let $A \in \mathbf{R}^{n\times n}$

And let $\{\lambda_1, \ldots, \lambda_n\}$ denote the (not necessarily distinct) eigenvalues of $A$. Then

$$(*) \qquad A^T P + PA = -Q$$

has a unique solution for P corresponding to each $Q \in \mathbf{R}^{n\times n}$ iff

$$\lambda_i + \lambda_j \neq 0, \quad \forall i, j$$

**Proof**: (see Chen)

# The Lyapunov Approach (IV)

**Corollary**:

If for some choice of $Q \in \mathbf{R}^{n\times n}$ , equation (*) does not have a unique solution for $P$, then 0 is not an exponentially stable equilibrium.

*Discussion:*

(*) does not have unique solution

$\Longrightarrow \exists\, k, l$ s.t. $\lambda_k + \lambda_l = 0$

two cases:

1) $\lambda_k = \lambda_l = 0 \Rightarrow$ not exponentially stable.

2) $\mathrm{Re}(\lambda_k) = -\mathrm{Re}(\lambda_k) \Rightarrow$ unstable.

**Lemma**:

Let $A$ be a Hurwitz (all eigenvalues are <0) matrix. Then for each $Q \in \mathbf{R}^{n\times n}$, the corresponding solution of (*) is

$$P = \int_0^\infty e^{A^T t} Q e^{At} \mathrm{d}t$$

# Theorem

**Theorem**:

Given $A \in \mathbf{R}^{n \times n}$, the following three statements are equivalent:

(1) $A$ is a Hurwitz matrix.

(2) $\exists$ some p.d. $Q \in \mathbf{R}^{n \times n}$ s.t.

$$(*) \qquad A^{T}P + PA = -Q$$

has a corresponding unique solution for $P$, and this $P$ is p.d..

(3) For every p.d. $Q \in \mathbf{R}^{n \times n}$, (*) has a unique solution for $P$, and this solution is p.d..

## Remarks

1. This Thm. enables one to determine the stability status of 0 unambiguously:
   - Given $A \in \mathbf{R}^{n \times n}$, pick $Q \in \mathbf{R}^{n \times n}$ (pick $Q=I$).
   - Attempt to solve (*) for $P$, i.e. $A^T P + PA = -Q$ (*)
   - If (*) has non-unique or no solution, then 0 is not exponentially stable.
   - If $P$ is unique but not p.d., then 0 is not exponentially stable.
   - If $P$ is uniquely determined and p.d., then 0 is an exponentially stable equilibrium.
2. This Thm. States that if $A$ is Hurwitz, then whenever $Q$ is p.d. $\Rightarrow P$ is p.d.. It does NOT say that whenever $P$ is p.d., then $Q$ is p.d. (false in general).

# Examples (I)

$$\dot{x} = Ax$$

$$A = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} \quad \text{choose } Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

let $P = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}$, where due to symmetry $p_{12} = p_{21}$.

The Lyapunov equation $PA + A^T P = -Q$ can be written as

$$\left.\begin{aligned} 2p_{12} &= -1 \\ -p_{11} - p_{12} + p_{22} &= 0 \\ -2p_{12} - 2p_{22} &= -1 \end{aligned}\right\} \quad \text{or} \quad \begin{bmatrix} 0 & 2 & 0 \\ -1 & -1 & 1 \\ 0 & -2 & -2 \end{bmatrix} \begin{bmatrix} p_{11} \\ p_{12} \\ p_{22} \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix}$$

$P = \begin{bmatrix} 1.5 & -0.5 \\ -0.5 & 1.0 \end{bmatrix}$ is p.d. $(\because 1.5 > 0;\ 1.5 - (0.5)^2 > 0)$

$\Rightarrow$ all eigenvalues of $A$ are in the open left-hand complex plane.

# Examples (II)

$$\dot{x} = Ax \,, \qquad \text{where } A = \begin{bmatrix} -3 & -2 \\ -1 & -1 \end{bmatrix}$$

- eigenvalues : $\begin{cases} \lambda_1 = -2 + j \\ \lambda_1 = -2 - j \end{cases}$
- let $P = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}$, $\quad \Rightarrow A^T P + PA = \begin{bmatrix} -6p_{11} - 2p_{12} & -4p_{12} - p_{22} + 2p_{11} \\ -4p_{12} - p_{22} + 2p_{11} & 4p_{12} - 2p_{22} \end{bmatrix}$

using $Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and solving $A^T P + PA = -Q$ gives

$$P = \frac{1}{40}\begin{bmatrix} 7 & -1 \\ -1 & 18 \end{bmatrix}$$

Principal minors : $\quad \Delta_1 = 7/40 > 0; \quad \Delta_1 = \; |P| = 5/64 > 0 \quad \Rightarrow \; P$ is p.d..

$\Rightarrow \quad O$ is exponentially stable.

# Lemmas on Nonautonomous Linear Systems (I)

$$\sum^{L} : \dot{x} = A(t)x$$

**Lemma** (*)

Suppose $Q$: $\mathbf{R}^{+} \rightarrow \mathbf{R}^{n\times n}$ is continuous and bounded, and that the equilibrium 0 of $\sum^{L}$ is exponentially stable.
Then, for each $t \geq 0$, the matrix

$$P(t) = \int_0^{\infty} \Phi^T(\tau,t)Q(\tau)\Phi(\tau,t)\mathrm{d}\tau$$

is well defined; moreover, $P(t)$ is bounded as a function of $t$.

**Proof**:

If O is exponentially stable, then ∃ constants $m, \lambda > 0$ s.t.

$$\|\Phi(\tau,t)\|_i \leq m\exp[-\lambda(\tau - t)] , \quad \forall \tau \geq t_0 \geq 0$$

This with the boundedness of $Q(\bullet)$ proves the lemma.

# Lemmas on Nonautonomous Linear Systems (II)

**Lemma** (2*)

Suppose that in addition to the hypotheses of Lemma (*), the following conditions also hold:

(1) $Q(t)$ is symmetric & p.d. for each $t \geq 0$; moreover, ∃ a constant $\alpha > 0$ s.t. $Q(t)$ uniformly p.d., i.e.

$$\alpha \mathbf{x}^{\mathrm{T}} \mathbf{x} \leq \mathbf{x}^{\mathrm{T}} Q(t) \mathbf{x}, \ \forall \ t \geq 0, \ \forall \ \mathbf{x} \in \mathbf{R}^n.$$

(2) The matrix $A(\bullet)$ is bounded, i.e.

$$m_0 \equiv \sup_{t \geq 0} \| A(t) \|_i < \infty$$

Under these conditions, the matrix $P(t)$ defined in Lemma (*) is p.d for each $t \geq 0$, moreover, ∃ a constant $\beta > 0$ s.t.

$$\beta \, \mathbf{x}^{\mathrm{T}} \mathbf{x} \leq \mathbf{x}^{\mathrm{T}} P(t) \mathbf{x}, \ \forall \ t \geq 0, \ \forall \ \mathbf{x} \in \mathbf{R}^n.$$

# Existence of Quadratic Lyapunov Function

**Theorem**: [Existence of Quadratic Lyapunov Function]

Suppose $Q(\bullet)$ & $A(\bullet)$ satisfy the hypotheses of Lemmas (*) & (2*). Then for each function $Q(\bullet)$ satisfying the hypotheses, the function

$$V(t, \mathbf{x}) = \mathbf{x}^{\mathrm{T}} P(t) \mathbf{x}$$

is a Lyapunov function in the sense of exponential stability theorem (Thm6) for establishing the exponential stability of the equilibrium O.

**Proof** :

with $V(t,\mathbf{x}) = \mathbf{x}^T P(t)\mathbf{x}$, we have

$$\dot{V}(t,\mathbf{x}) = \dot{\mathbf{x}}^T P(t)\mathbf{x} + \mathbf{x}^T \dot{P}(t)\mathbf{x} + \mathbf{x}^T P(t)\dot{\mathbf{x}} = \mathbf{x}^T[\dot{P}(t) + A^T P(t) + P(t)A(t)]\mathbf{x}$$

It is easy to verify that (see Khalil page177)

$$\dot{P}(t) = -A^T P(t) - P(t)A(t) - Q(t)$$

Hence $\dot{V}(t,\mathbf{x}) = -\mathbf{x}^T Q(t)\mathbf{x}.$

Thus $V$ and $\dot{V}$ satisfy exponential stability Thm.

# Lyapunov Linearization Method (Indirect/First Method) (I)

With this method, we draw conclusions about a nonlinear system by studying the behavior of a linear system.

**Linearization of a nonlinear system around an equilibrium:**

- Consider first autonomous case:

$$\Sigma_A : \dot{x} = f(x)$$

- Suppose $f(0)=0$, so that 0 is an equilibrium.

  Suppose also that $f \in C^1$.

- Define $A = \left[ \frac{\partial f}{\partial \mathbf{x}} \right]_{x=0} \leftarrow$ Jacobian of $f$ at $\mathbf{x} = 0$.

- Define $f_1(\mathbf{x}) \equiv f(\mathbf{x}) - A\mathbf{x}$

$$\Rightarrow \lim_{\|\mathbf{x}\|_2 \to 0} \frac{\| f_1(\mathbf{x}) \|_2}{\| \mathbf{x} \|} = 0$$

  why?

Can think of $f(\mathbf{x}) = A\mathbf{x} + f_1(\mathbf{x})$
As the Taylor's series expansion of $f(\bullet)$ around equilibrium $\mathbf{x}=0$.

# Lyapunov Linearization Method (Indirect/First Method) (II)

write (Mean Value Thm)

$$f(\mathbf{x}) - f(0) = \frac{\partial f}{\partial \mathbf{x}}(z)[\mathbf{x} - 0], \quad z \in \text{line segment of } \mathbf{x} \,\&\, 0.$$

or

$$f(\mathbf{x}) = \frac{\partial f}{\partial \mathbf{x}}(z)\mathbf{x} = \underbrace{\frac{\partial f}{\partial \mathbf{x}}(0)\mathbf{x}}_{Ax} + \underbrace{\left[\frac{\partial f}{\partial \mathbf{x}}(z) - \frac{\partial f}{\partial \mathbf{x}}(0)\right]\mathbf{x}}_{f_1(\mathbf{x})}$$

$$\frac{f_1(\mathbf{x})}{\mathbf{x}} = \frac{\partial f}{\partial \mathbf{x}}(z) - \frac{\partial f}{\partial \mathbf{x}}(0)$$

$$\Rightarrow \frac{\| f_1(\mathbf{x}) \|}{\| \mathbf{x} \|} \to 0 \quad (\text{by continuity of } \frac{\partial f}{\partial \mathbf{x}}) \ \text{ as } \| \mathbf{x} \| \to 0 \ \text{ or } z \to 0.$$

- With this notation

$$\dot{z} = Az \quad \leftarrow$$

linearization or linearized system of $\sum_A$ around equilibrium O.

# Lyapunov Linearization Method (Indirect/First Method) (III)

- Nonautonomous case

$$\sum: \quad \dot{\mathbf{x}} = f(t, \mathbf{x})$$

Suppose $f(t,0) = 0, \forall t \geq 0;\ f$ is $C^1$.

Define $A(t) = \left[\dfrac{\partial f}{\partial \mathbf{x}}(t, \mathbf{x})\right]_{\mathbf{x}=0}$

$$f_1(t, \mathbf{x}) = f(t, \mathbf{x}) - A(t)\mathbf{x},$$

Note $\displaystyle\lim_{\|\mathrm{x}\|_2 \to 0} \frac{\| f_1(t, \mathbf{x}) \|_2}{\| \mathbf{x} \|_2} = 0$

However, it may or may not be ture that

$$\lim_{\|\mathrm{x}\|_2 \to 0} \sup_{t \geq 0} \frac{\| f_1(t, \mathbf{x}) \|_2}{\| \mathbf{x} \|_2} = 0 \qquad (*)$$

Provided (*) holds,

$$\dot{z} = A(t)z \quad \leftarrow \text{ Lineariztion or Linearized system of } \sum.$$

# Example

$$\begin{cases}\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}-x_1+tx_2^2\\ x_1-x_2\end{bmatrix}=\underbrace{\begin{bmatrix}-1 & 0\\ 1 & -1\end{bmatrix}}_{A}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\underbrace{\begin{bmatrix}tx_2^2\\ 0\end{bmatrix}}_{f_1(t,\mathbf{x})}\end{cases}$$

Here $f$ is $C^1$ and $A(t)=\begin{bmatrix}-1 & 0\\ 1 & -1\end{bmatrix}, \forall t\geq 0.$

$$f_1(t,\mathbf{x})=\begin{bmatrix}tx_2^2 & 0\end{bmatrix}^{\mathrm{T}},$$

$$\lim_{\|\mathrm{x}\|_2\to 0}\frac{\|f_1\|_2}{\|\mathbf{x}\|_2}=\lim_{\|\mathrm{x}\|_2\to 0}\frac{|t|x_2^2}{\sqrt{x_1^2+x_2^2}}=0$$

$$\lim_{\|\mathrm{x}\|_2\to 0}\sup_{t\geq 0}\frac{\|f_1\|_2}{\|\mathbf{x}\|_2}\neq 0$$

Hence,

$$\begin{cases}\dot{z}_1=-z_1\\ \dot{z}_2=z_1-z_2\end{cases}$$ is NOT a lineariztion of the original system.

$$\sum: \quad \boxed{\begin{array}{l} \dot{\mathbf{x}}(t) = f[t, \mathbf{x}(t)] \qquad t \geq 0 \\ \mathbf{x}(t) \in \mathbf{R}^n. \\ f: \ \mathbf{R}_+ \times \mathbf{R}^n \to \mathbf{R}^n \end{array}}$$

**Theorem 13**: [Lyapunov's Linearization Method]

Consider system $\sum$.

Suppose $f(t,0) = 0, \forall t \geq 0$; $f$ is $C^1$.

Define $A(t) = \left[\dfrac{\partial f}{\partial \mathbf{x}}(t,\mathbf{x})\right]_{\mathbf{x}=0}$

$$f_1(t,\mathbf{x}) = f(t,\mathbf{x}) - A(t)\mathbf{x},$$

Assume

(1) $\quad \displaystyle \lim_{\|\mathbf{x}\| \to 0} \sup_{t \geq 0} \frac{\| f_1(t,\mathbf{x}) \|}{\| \mathbf{x} \|} = 0,$

(2) $\quad$ A($\bullet$) is bounded.

Under these conditions,

if 0 is an exponentially stable equilibrium of the linear system

$$\dot{z} = A(t)z$$

then it is also an exponentially stable equilibrium of system $\sum$.

# Corollary of Theorem 13

**Corollary:**

Consider $\sum_A : \quad \dot{\mathbf{x}} = f(\mathbf{x})$

Suppose that $f(0) = 0$, and that $f$ is $C^1$.

Define $A = \left[\dfrac{\partial f}{\partial \mathbf{x}}\right]_{\mathbf{x}=0}$.

Under these conditions, 0 is an exponentially stable equilibrium of $\sum_A$ if all eigenvalues of $A$ have negative real parts.

# Theorem 14

**Theorem 14:**

Consider the system $\sum$: $\quad \dot{\mathbf{x}} = f(t,\mathbf{x})$

Suppose $f(t,0) = 0, \forall t \geq 0$; $f$ is $C^1$.

and suppose in addition that

$$A(t) = \left[\frac{\partial f(t,\mathbf{x})}{\partial \mathbf{x}}\right] \equiv A_0 \quad (\underline{\text{a constant matrix}}), \quad \forall t \geq 0$$

and that

$$\lim_{\|x\| \to 0} \sup_{t \geq 0} \frac{\| f_1 \|}{\| \mathbf{x} \|} = 0 \ .$$

Under these conditions, the equilibrium 0 is unstable if $A_0$ has at least one eigenvalue with a positive real part.

## Remarks on Theorem 14

- Advantages of Thm 13 & Thm 14 are self evident.
- Limitations of Thm 13 & Thm 14 :

i. Conclusions are purely local in nature. To study global results, it is still necessary to resort to Lyapunov's direct method.

ii. In the case where the linearized system is autonomous, if some eigenvalues of $A$ have zero real parts and the remainder have negative real parts, then linearization techniques are inconclusive.

iii. In the case where the linearized system is nonautonomous, if O is asymptotically stable but not uniformly asymptotically stable, then linearization is inconclusive.