CHAPTER 6
Dynamics of Rigid Bodies

6.1 INTRODUCTION

Rigid bodies can be regarded as systems of particles of a special type, namely, one in which the distance between any two particles is constant. Hence, all the developments of Chapter 5 for systems of particles are applicable to rigid bodies, provided the velocity of a point in the rigid body relative to another is due only to the angular velocity of the rigid body. This procedure guarantees automatically that the distance between any two points in the body remains constant. In this regard, we must observe that the three-particle system of Example 5.2 was really a rigid body and was treated as such. In general, however, one thinks of a rigid body more in terms of a continuous body rather than a collection of particles, so that many of the definitions of Chapter 5 must be modified by letting the mass of a typical particle m_i approach a differential element of mass dm and by replacing the process of summation over the system of particles by integration over the rigid body.

6.2 LINEAR AND ANGULAR MOMENTUM

First, let us consider the angular momentum of a rigid body rotating with the angular velocity ω about the fixed point 0. We let axes XYZ be an inertial system and envision a set of axes xyz embedded in the body and rotating together with it with the same angular velocity ω, as shown in Fig. 6.1. Axes xyz are known as *body axes*. By analogy with Eq. (5.25), the angular momentum about 0 is defined as

$$\mathbf{H}_0 = \int_m \mathbf{r} \times \mathbf{v}\, dm \tag{6.1}$$

Because the velocity $\mathbf{v}$ of any point in the body is due entirely to rotation about 0, we can write (see Section 2.4)

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r} \tag{6.2}$$

so that Eq. (6.1) becomes

$$\mathbf{H}_0 = \int_m \mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{r})\, dm \tag{6.3}$$

From vector analysis, however, we have for any three vectors

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C} \tag{6.4}$$

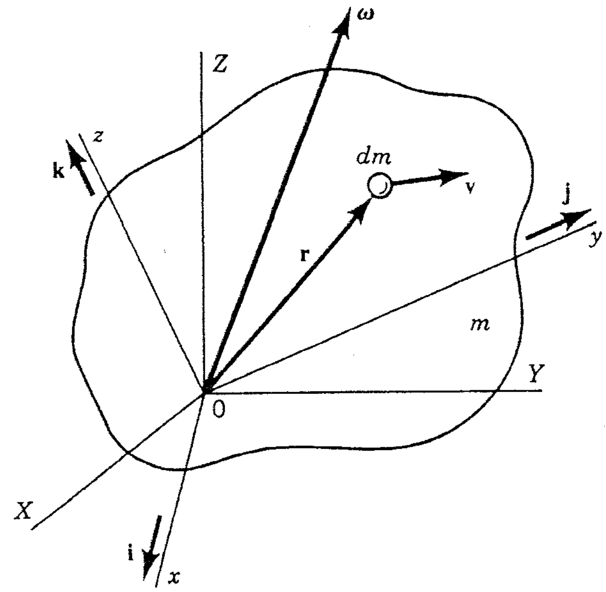

FIGURE 6.1

so that, letting $\mathbf{A}=\mathbf{C}=\mathbf{r}$, $\mathbf{B}=\boldsymbol{\omega}$, we can write Eq. (6.3) in the form

$$\mathbf{H}_0=\int_m [(\mathbf{r}\cdot\mathbf{r})\boldsymbol{\omega}-(\mathbf{r}\cdot\boldsymbol{\omega})\mathbf{r}]\, dm \tag{6.5}$$

To obtain a more detailed expression for $\mathbf{H}_0$, namely, one in terms of the x, y, z components of $\mathbf{r}$ and $\boldsymbol{\omega}$, we let

$$\mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k} \tag{6.6}$$

and

$$\boldsymbol{\omega}=\omega_x\mathbf{i}+\omega_y\mathbf{j}+\omega_z\mathbf{k} \tag{6.7}$$

so that

$$\begin{aligned}(\mathbf{r}\cdot\mathbf{r})\boldsymbol{\omega}-(\mathbf{r}\cdot\boldsymbol{\omega})\mathbf{r}=&[(r^2-x^2)\omega_x-xy\omega_y-xz\omega_z]\mathbf{i}\\ &+[-xy\omega_x+(r^2-y^2)\omega_y-yz\omega_z]\mathbf{j}\\ &+[-xz\omega_x-yz\omega_y+(r^2-z^2)\omega_z]\mathbf{k}\end{aligned} \tag{6.8}$$

where r is the magnitude of $\mathbf{r}$. Introducing Eq. (6.8) into Eq. (6.5), we obtain

$$\begin{aligned}\mathbf{H}_0=&(I_{xx}\omega_x-I_{xy}\omega_y-I_{xz}\omega_z)\mathbf{i}+(-I_{xy}\omega_x+I_{yy}\omega_y-I_{yz}\omega_z)\mathbf{j}\\ &+(-I_{xz}\omega_x-I_{yz}\omega_y+I_{zz}\omega_z)\mathbf{k}\end{aligned} \tag{6.9}$$

where

$$I_{xx}=\int_m (r^2-x^2)\, dm, \quad I_{yy}=\int_m (r^2-y^2)\, dm, \quad I_{zz}=\int_m (r^2-z^2)\, dm \tag{6.10}$$

are mass moments of inertia about the body axes xyz and

$$I_{xy} = I_{yx} = \int_m xy\, dm, \qquad I_{xz} = I_{zx} = \int_m xz\, dm, \qquad I_{yz} = I_{zy} = \int_m yz\, dm \tag{6.11}$$

are mass products of inertia about the same axes.

Next, let us consider a rigid body translating and rotating relative to the inertial space, as shown in Fig. 6.2. The angular momentum about 0 retains the form (6.1), but this time the absolute velocity $\mathbf{v}$ has the expression

$$\mathbf{v} = \mathbf{v}_C + \boldsymbol{\omega} \times \boldsymbol{\rho} \tag{6.12}$$

where $\boldsymbol{\rho}$ is the radius vector from C to the differential element of mass dm, so that

$$\mathbf{r} = \mathbf{r}_C + \boldsymbol{\rho} \tag{6.13}$$

where, by analogy with Eq. (5.9),

$$\mathbf{r}_C = \frac{1}{m} \int_m \mathbf{r}\, dm \tag{6.14}$$

defines the position of the *mass center* C relative to 0. Substituting Eq. (6.13) into Eq. (6.14), we conclude that the alternative definition of the mass center is

$$\int_m \boldsymbol{\rho}\, dm = \mathbf{0} \tag{6.15}$$

Introducing Eqs. (6.12) and (6.13) into Eq. (6.1), we obtain

$$\begin{aligned} \mathbf{H}_0 &= \int_m (\mathbf{r}_C + \boldsymbol{\rho}) \times (\mathbf{v}_C + \boldsymbol{\omega} \times \boldsymbol{\rho})\, dm \\ &= \mathbf{r}_C \times \mathbf{v}_C \int_m dm + \left(\int_m \boldsymbol{\rho}\, dm \right) \times \mathbf{v}_C + \mathbf{r}_C \times \left(\boldsymbol{\omega} \times \int_m \boldsymbol{\rho}\, dm \right) \\ &\quad + \int_m \boldsymbol{\rho} \times (\boldsymbol{\omega} \times \boldsymbol{\rho})\, dm \end{aligned} \tag{6.16}$$

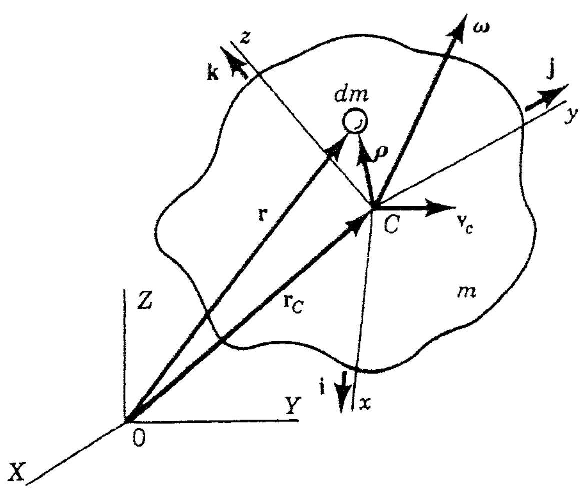

FIGURE 6.2

Moreover, we recognize that

$$\mathbf{v}_C \int_m dm = m\mathbf{v}_C = \mathbf{p} \tag{6.17}$$

is the linear momentum of the body and that

$$\int_m \boldsymbol{\rho} \times (\boldsymbol{\omega} \times \boldsymbol{\rho})\, dm = \mathbf{H}_C = \mathbf{H}'_C \tag{6.18}$$

is the angular momentum of the body about the mass center C, where the actual angular momentum $\mathbf{H}_C$ is equal to the apparent angular momentum $\mathbf{H}'_C$, as for any system of particles. Considering Eqs. (6.17) and (6.18), we can reduce Eq. (6.16) to

$$\mathbf{H}_0 = \mathbf{r}_C \times \mathbf{p} + \mathbf{H}_C \tag{6.19}$$

which states that the angular momentum of the body about 0 is equal to the moment about 0 of the linear momentum obtained if the entire mass were concentrated at the mass center C plus the angular momentum of the body about C. This statement represents a *translation theorem for the angular momentum.* Note that the angular momentum about C has a form similar to that given by (Eq. 6.9), except that here the moments and products of inertia are about body axes through C.

6.3 THE EQUATIONS OF MOTION

Next, we wish to derive the equations of motion for rigid bodies. The equation for the translational motion retains the form (5.21), as for any system of particles, or

$$\mathbf{F} = \dot{\mathbf{p}} = m\mathbf{a}_C \tag{6.20}$$

where $\mathbf{a}_C$ is the acceleration of the mass center. Moreover, for $\mathbf{F} = \mathbf{0}$, the linear momentum of the rigid body is conserved, as indicated by Eq. (5.23).

The equation for the rotational motion of a rigid body retains the same general form (5.29), as for any system of particles. Hence, for pure rotation about the fixed point 0, we have

$$\mathbf{M}_0 = \dot{\mathbf{H}}_0 \tag{6.21}$$

where $\mathbf{H}_0$ is given by Eq. (6.9). But, in taking the time derivative of $\mathbf{H}_0$, we must recall that $\mathbf{H}_0$ is expressed in terms of components about rotating axes, so that the associated unit vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are not constant. Indeed, their time rate of change is

$$\frac{d\mathbf{i}}{dt} = \boldsymbol{\omega} \times \mathbf{i}, \qquad \frac{d\mathbf{j}}{dt} = \boldsymbol{\omega} \times \mathbf{j}, \qquad \frac{d\mathbf{k}}{dt} = \boldsymbol{\omega} \times \mathbf{k} \tag{6.22}$$

Introducing the notation

$$\mathbf{H}_0 = H_{0x}\mathbf{i} + H_{0y}\mathbf{j} + H_{0z}\mathbf{k} \tag{6.23}$$

we can write

$$\dot{\mathbf{H}}_0=\frac{d}{dt}(H_{0x}\mathbf{i}+H_{0y}\mathbf{j}+H_{0z}\mathbf{k})$$

$$=\dot{H}_{0x}\mathbf{i}+H_{0x}\frac{d\mathbf{i}}{dt}+\dot{H}_{0y}\mathbf{j}+H_{0y}\frac{d\mathbf{j}}{dt}+\dot{H}_{0z}\mathbf{k}+H_{0z}\frac{d\mathbf{k}}{dt}$$

$$=\dot{H}_{0x}\mathbf{i}+\dot{H}_{0y}\mathbf{j}+\dot{H}_{0z}\mathbf{k}+H_{0x}\boldsymbol{\omega}\times\mathbf{i}+H_{0y}\boldsymbol{\omega}\times\mathbf{j}+H_{0z}\boldsymbol{\omega}\times\mathbf{k}$$

$$=\dot{H}_{0x}\mathbf{i}+\dot{H}_{0y}\mathbf{j}+\dot{H}_{0z}\mathbf{k}+\boldsymbol{\omega}\times(H_{0x}\mathbf{i}+H_{0y}\mathbf{j}+H_{0z}\mathbf{k})$$

$$=\dot{\mathbf{H}}'_0+\boldsymbol{\omega}\times\mathbf{H}_0 \tag{6.24}$$

where $\dot{\mathbf{H}}'_0$ is the rate of change of $\mathbf{H}_0$ regarding the reference frame as fixed. Incidentally, formula (6.24) is valid for the derivative of any vector expressed in terms of components about rotating axes, as demonstrated in Section 2.6. Inserting Eq. (6.24) into Eq. (6.21), we obtain the moment equations of motion in the vector form

$$\mathbf{M}_0=\dot{\mathbf{H}}'_0+\boldsymbol{\omega}\times\mathbf{H}_0 \tag{6.25}$$

Considering Eqs. (6.7) and (6.9) and introducing the notation

$$\mathbf{M}_0=M_{0x}\mathbf{i}+M_{0y}\mathbf{j}+M_{0z}\mathbf{k} \tag{6.26}$$

we can write Eq. (6.25) by components as follows:

$$\begin{aligned}
M_{0x}&=I_{xx}\dot{\omega}_x-I_{xy}\dot{\omega}_y-I_{xz}\dot{\omega}_z-I_{xz}\omega_x\omega_y+I_{xy}\omega_x\omega_z\\
&\quad+(I_{zz}-I_{yy})\omega_y\omega_z-I_{yz}(\omega_y^2-\omega_z^2)\\
M_{0y}&=-I_{xy}\dot{\omega}_x+I_{yy}\dot{\omega}_y-I_{yz}\dot{\omega}_z+I_{yz}\omega_x\omega_y+(I_{xx}-I_{zz})\omega_x\omega_z\\
&\quad-I_{xy}\omega_y\omega_z-I_{xz}(\omega_z^2-\omega_x^2)\\
M_{0z}&=-I_{xz}\dot{\omega}_x-I_{yz}\dot{\omega}_y+I_{zz}\dot{\omega}_z+(I_{yy}-I_{xx})\omega_x\omega_y-I_{yz}\omega_x\omega_z\\
&\quad+I_{xz}\omega_y\omega_z-I_{xy}(\omega_x^2-\omega_y^2)
\end{aligned} \tag{6.27}$$

Note that the reason for working with body axes is that the moments of inertia about these axes are constant.

The moment equations about the mass center C have an entirely analogous form, or

$$\mathbf{M}_C=\dot{\mathbf{H}}_C=\dot{\mathbf{H}}'_C+\boldsymbol{\omega}\times\mathbf{H}_C \tag{6.28}$$

where the components of $\mathbf{H}_C$ have the same expressions as in Eqs. (6.9), *the only difference being that the moments and products of inertia are about body axes through C.*

Equations (6.27) are very complicated, and they are seldom used. Fortunately, there exists a much simpler and more useful form. Indeed, letting axes x, y, and z be the principal axes 1, 2, and 3, respectively, we can write

$$\begin{gathered}
I_{xx}=I_1,\qquad I_{yy}=I_2,\qquad I_{zz}=I_3\\
I_{xy}=I_{xz}=I_{yz}=0
\end{gathered} \tag{6.29}$$

where I_1, I_2, and I_3 are the principal mass moments of inertia. Introducing Eqs. (6.29) into Eqs. (6.27) and replacing x, y, and z by 1, 2, and 3, respectively, we obtain

$$\begin{aligned} M_1 &= I_1\dot{\omega}_1 + (I_3 - I_2)\omega_2\omega_3 \\ M_2 &= I_2\dot{\omega}_2 + (I_1 - I_3)\omega_1\omega_3 \\ M_3 &= I_3\dot{\omega}_3 + (I_2 - I_1)\omega_1\omega_2 \end{aligned} \tag{6.30}$$

which are known as *Euler's moment equations.* They are valid for two important cases: (1) when the moment and the mass moments of inertia are about principal axes with the origin at the fixed point 0, and (2) when they are about principal axes with the origin at the mass center C. We note, in passing, that Eqs. (6.30) are nonlinear in the angular velocities ω_1, ω_2, and ω_3.

6.4 KINETIC ENERGY

Using the analogy with the kinetic energy for a system of particles, Eq. (5.45), the kinetic energy for a rigid body can be written in the form

$$T = \frac{1}{2}\int_m \mathbf{v}\cdot\mathbf{v}\, dm \tag{6.31}$$

where $\mathbf{v}$ is the velocity of the differential element dm relative to an inertial space. But, according to Eq. (6.12), the velocity $\mathbf{v}$ can be expressed as the sum of the translational velocity $\mathbf{v}_C$ of the mass center C and the velocity $\boldsymbol{\omega}\times\boldsymbol{\rho}$ of dm relative to C, where the latter is due entirely to the rotation of the body about C. Hence, introducing Eq. (6.12) into Eq. (6.31), we obtain

$$\begin{aligned} T &= \frac{1}{2}\int_m (\mathbf{v}_C + \boldsymbol{\omega}\times\boldsymbol{\rho})\cdot(\mathbf{v}_C + \boldsymbol{\omega}\times\boldsymbol{\rho})\, dm \\ &= \frac{1}{2}\mathbf{v}_C\cdot\mathbf{v}_C\int_m dm + \mathbf{v}_C\cdot\boldsymbol{\omega}\times\int_m \boldsymbol{\rho}\, dm + \frac{1}{2}\int_m (\boldsymbol{\omega}\times\boldsymbol{\rho})\cdot(\boldsymbol{\omega}\times\boldsymbol{\rho})\, dm \end{aligned} \tag{6.32}$$

But, according to Eq. (6.15), $\int_m \boldsymbol{\rho}\, dm = \mathbf{0}$. It follows that Eq. (6.32) reduces to

$$T = T_{\text{tr}} + T_{\text{rot}} \tag{6.33}$$

where

$$T_{\text{tr}} = \tfrac{1}{2}m\mathbf{v}_C\cdot\mathbf{v}_C \tag{6.34a}$$

is the kinetic energy that would be obtained if the rigid body were undergoing pure translation with the velocity $\mathbf{v}_C$ and

$$T_{\text{rot}} = \frac{1}{2}\int_m (\boldsymbol{\omega}\times\boldsymbol{\rho})\cdot(\boldsymbol{\omega}\times\boldsymbol{\rho})\, dm \tag{6.34b}$$

is the kinetic energy of rotation of the rigid body about the mass center C. Denoting the Cartesian components of ρ by x, y, and z and recalling Eq. (6.7), we can write

$$\omega \times \rho = (\omega_y z - \omega_z y)\mathbf{i} + (\omega_z x - \omega_x z)\mathbf{j} + (\omega_x y - \omega_y x)\mathbf{k} \tag{6.35}$$

so that, introducing Eq. (6.35) into Eq. (6.34b), we obtain

$$\begin{aligned} T_{\text{rot}} &= \frac{1}{2}\int_m [(\omega_y z - \omega_z y)^2 + (\omega_z x - \omega_x z)^2 + (\omega_x y - \omega_y x)^2]\, dm \\ &= \frac{1}{2}(I_{xx}\omega_x^2 + I_{yy}\omega_y^2 + I_{zz}\omega_z^2 - 2I_{xy}\omega_x\omega_y - 2I_{xz}\omega_x\omega_z - 2I_{yz}\omega_y\omega_z) \end{aligned} \tag{6.36}$$

In the special case in which x, y, and z are the principal axes 1, 2, and 3, the rotational kinetic energy reduces to

$$T_{\text{rot}} = \tfrac{1}{2}(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2) \tag{6.37}$$

It should be reiterated that the separation of the kinetic energy into two terms, one due to translation and the other due to rotation, is possible only if the mass center C is used as the origin of the body axes.

6.5 PLANAR MOTION OF A RIGID BODY

A case of particular interest is the one in which the motion takes place in a given plane. Assuming that the motion takes place in the xy plane, we have

$$v_z = a_z = 0 \tag{6.38a}$$

$$\omega_x = \dot{\omega}_x = \omega_y = \dot{\omega}_y = 0, \qquad \omega_z = \omega \tag{6.38b}$$

Moreover, we shall assume that one of the dimensions of the body is infinitesimally small and that this small dimension is in the z-direction. Hence, we have

$$I_{xz} = I_{yz} = 0 \tag{6.39}$$

The various quantities derived in Sections 6.2–6.4, such as momentum and kinetic energy, as well as the equations of motion can be given a simpler form by considering Eqs. (6.38) and (6.39). Certain special cases of motion involve further simplifications. Hence, before proceeding to the more general case, we propose to study these special cases.

i. Pure Translation

In this case, the rotation of the body is zero

$$\omega = \dot{\omega} = 0 \tag{6.40}$$

so that every point of the body undergoes the same translational motion. Using Eq. (6.20) and considering Eqs. (6.38a), we can write the force equations in the form

$$F_x = ma_{Cx} \tag{6.41a}$$

$$F_y = ma_{Cy} \tag{6.41b}$$

Moreover, the only moment equation is about the z-axis. Considering Eqs. (6.38b) and (6.39), the moment equation about the mass center C is simply

$$M_{Cz} = 0 \tag{6.42}$$

The kinetic energy is due to translation alone, and from Eq. (6.33) it has the explicit form

$$T = \tfrac{1}{2} m \mathbf{v}_C \cdot \mathbf{v}_C = \tfrac{1}{2} m (v_{Cx}^2 + v_{Cy}^2) \tag{6.43}$$

where v_{Cx} and v_{Cy} are the Cartesian components of the velocity vector of the mass center.

Example 6.1

Derive the equations of motion for the horizontal translation of the body shown in Fig. 6.3*a*. The horizontal reactions at the points A and B are proportional to the vertical reactions at the same points, where the proportionality constant is the coefficient of friction μ. Then, determine the magnitude of the force $\mathbf{F}$ when the body is on the verge of tipping over.

Figure 6.3*b* shows the corresponding free-body diagram. Using Eqs. (6.41) and (6.42) we obtain simply

$$F_x = F \cos \beta - \mu(N_A + N_B) = ma_{Cx} \tag{a}$$

$$F_y = F \sin \beta + N_A + N_B - mg = 0 \tag{b}$$

$$M_{Cz} = -F \cos \beta \frac{H}{2} + (N_B - N_A - F \sin \beta) \frac{L}{2} - \mu(N_A + N_B)\left(D + \frac{H}{2}\right) = 0 \tag{c}$$

Just before tipping over, the reaction N_A reduces to zero. Then, from Eqs. (b) and (c), we obtain

$$F = \frac{mg(L - 2\mu D - \mu H)}{H \cos \beta + (2L - 2\mu D - \mu H) \sin \beta} \tag{d}$$

ii. Rotation about a Fixed Point

Let us consider the case of planar motion about a fixed point 0, insert Eqs. (6.38b) and (6.39) into Eq. (6.9), recognize that there is only one angular momentum

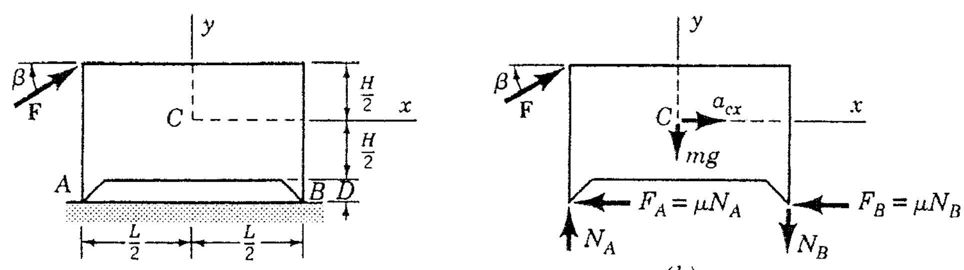

FIGURE 6.3

component, and write

$$H_0 = I_0 \omega \tag{6.44}$$

where we introduced the notation $I_{zz} = I_0$ for the mass moment of inertia of the body about 0. Similarly, using Eq. (6.21), we can write

$$M_0 = I_0 \alpha \tag{6.45}$$

where $\alpha = \dot{\omega}$ is the angular acceleration. If the interest lies in the rotational motion alone, then Eq. (6.45) is the only one needed. On the other hand, if the reactions at point 0 are also of interest, then Eqs. (6.41) must also be considered.

The kinetic energy can be attributed entirely to rotation about 0, and, from Eq. (6.36), it has the form

$$T = \tfrac{1}{2} I_0 \omega^2 \tag{6.46}$$

Example 6.2

A uniform bar of total mass m is hinged at point 0, as shown in Fig. 6.4*a*. If the bar is released from rest in the horizontal position, determine: (i) the angular acceleration of the bar immediately after release, (ii) the reaction at point 0 at the same time, and (iii) the angular velocity of the bar when it passes through the vertical position.

Considering counterclockwise moments and angular motions as positive, Eq. (6.45) in conjunction with the free-body diagram of Fig. 6.4*b* yields

$$M_0 = -\tfrac{1}{6} Lmg = I_0 \alpha \tag{a}$$

where the mass moment of inertia of the bar about point 0 can be obtained through integration as follows:

$$I_0 = \int_m x^2\, dm = \frac{m}{L} \int_{-L/3}^{2L/3} x^2\, dx = \frac{m}{L} \tfrac{1}{3} x^3 \Big|_{-L/3}^{2L/3} = \tfrac{1}{9} m L^2 \tag{b}$$

Hence, from Eqs. (a) and (b), the angular acceleration immediately after release is

$$\alpha = -\tfrac{3}{2} \frac{g}{L} \tag{c}$$

To obtain the reaction at point 0, we use Eq. (6.41b) in conjunction with the

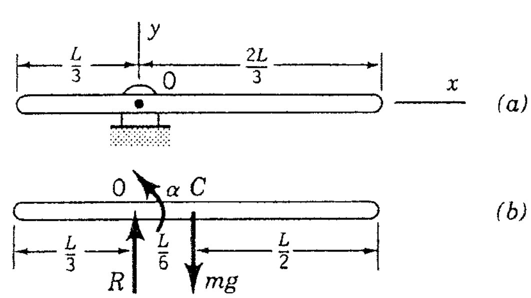

FIGURE 6.4

free-body diagram of Fig. 6.4*b*, or

$$F_y = R - mg = ma_{Cy} = m\tfrac{1}{6}L\alpha = m\tfrac{1}{6}L\left(-\tfrac{3}{2}\frac{g}{L}\right) = -\tfrac{1}{4}mg \tag{d}$$

from which we obtain

$$R = \tfrac{3}{4}mg \tag{e}$$

Finally, to determine the angular velocity of the bar when it passes through the vertical position, we invoke the conservation of energy

$$E = T + V = \text{const} \tag{f}$$

where E is the total energy, T is the kinetic energy, and V is the potential energy. Denoting the horizontal configuration by the subscript 1 and the vertical one by the subscript 2, letting the potential energy in the configuration 1 be equal to zero, and recalling that the angular velocity is zero when the bar is horizontal, we can write

$$E = T_1 + V_1 = 0 \tag{g}$$

In the configuration 2, the potential energy is due to the lowering of the mass center C, or

$$V_2 = -\tfrac{1}{6}Lmg \tag{h}$$

and the kinetic energy is

$$T_2 = \tfrac{1}{2}I_O\omega_2^2 = \tfrac{1}{2}\tfrac{1}{9}mL^2\omega_2^2 \tag{i}$$

Because $T_2 + V_2 = 0$, we obtain

$$\omega_2 = -\sqrt{\frac{3g}{L}} \tag{j}$$

where the negative sign was chosen in recognition of the fact that the angular velocity is in the clockwise sense.

iii. General Case of Planar Motion

In this case it is often convenient to express the equations of motion in terms of the mass center C. The force equations remain in the form (6.41) and the moment equation has the scalar form

$$M_C = I_C\alpha \tag{6.47}$$

where I_C is the mass moment of inertia about an axis normal to the plane of motion and passing through C.

The kinetic energy consists of two parts, one due to the translation of C and one due to rotation about C, or

$$T = \tfrac{1}{2}m(v_{Cx}^2 + v_{Cy}^2) + \tfrac{1}{2}I_C\omega^2 \tag{6.48}$$

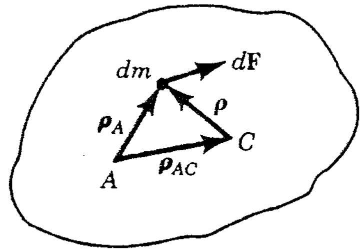

FIGURE 6.5

On occasions it is more convenient to express the moment equation about a point other than the mass center C. Let us consider the system of Fig. 6.5 and write the moment about the arbitrary point A in the form

$$\mathbf{M}_A = \int \boldsymbol{\rho}_A \times d\mathbf{F} = \int (\boldsymbol{\rho}_{AC} + \boldsymbol{\rho}) \times d\mathbf{F} = \boldsymbol{\rho}_{AC} \times \mathbf{F} + \mathbf{M}_C \tag{6.49}$$

In view of Eq. (6.20), however, Eq. (6.49) becomes

$$\mathbf{M}_A = \boldsymbol{\rho}_{AC} \times m\mathbf{a}_C + \mathbf{M}_C \tag{6.50}$$

where we note that for planar motion

$$\mathbf{M}_C = I_C \boldsymbol{\alpha} \tag{6.51}$$

in which the vectors $\mathbf{M}_C$ and $\boldsymbol{\alpha}$ are both normal to the plane of motion.

Next, let us consider the kinematical equation

$$\mathbf{a}_C = \mathbf{a}_A + \mathbf{a}_{C/A} \tag{6.52}$$

which for planar motion reduces to (see Section 2.5)

$$\mathbf{a}_C = \mathbf{a}_A - \omega^2 \boldsymbol{\rho}_{AC} + \boldsymbol{\alpha} \times \boldsymbol{\rho}_{AC} \tag{6.53}$$

Introducing Eq. (6.53) into Eq. (6.50) and considering Eq. (6.51), we obtain

$$\mathbf{M}_A = \boldsymbol{\rho}_{AC} \times m\mathbf{a}_A + I_A \boldsymbol{\alpha} \tag{6.54}$$

where

$$I_A = I_C + m\rho_{AC}^2 \tag{6.55}$$

is recognized as the mass moment of inertia of the body about point A.

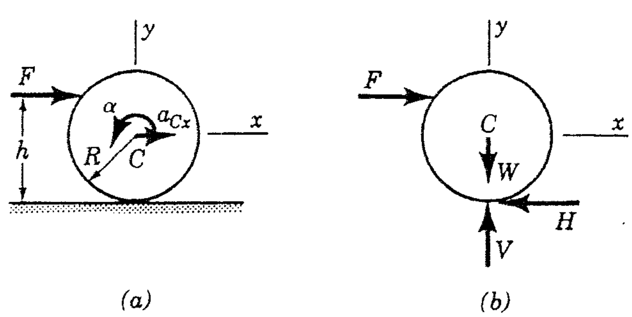

FIGURE 6.6

Example 6.3

The disk shown in Fig. 6.6*a* is traveling on a rough surface while acted upon by a force F. Determine the angular acceleration of the disk for the two cases: (1) the disk rolls without slipping and (2) the disk rolls and slips. The coefficient of friction between the disk and the surface is μ.

From the free-body diagram depicted in Fig. 6.6*b*, we can write the equations of motion

$$
\begin{aligned}
F_x &= F - H = ma_{Cx} \\
F_y &= V - W = 0 \\
M_C &= -F(h-R) - HR = I_C\alpha
\end{aligned}
\tag{a}
$$

In the case of roll without slip the friction force must satisfy the inequality $|H| < \mu V = \mu W$. On the other hand, the translational and rotational accelerations are related by

$$a_{Cx} = -\alpha R \tag{b}$$

Eliminating H from the first and third of Eqs. (a), using Eq. (b), and recalling that for a disk $I_C = mR^2/2$, we obtain

$$a_{Cx} = \frac{FRh}{I_C + mR^2} = \frac{2Fh}{3mR} \tag{c}$$

from which it follows that

$$\alpha = -\frac{2Fh}{3mR^2} \tag{d}$$

Note that, from the first of Eqs. (a), we can write

$$H = F - ma_{Cx} = F\left(1 - \frac{2h}{3R}\right) \tag{e}$$

so that, for no slip, F must satisfy the inequality

$$F < \frac{\mu W}{1 - 2h/3R} \tag{f}$$

In the case of rolling and slipping, the kinematical relation (b) is no longer valid. On the other hand, the friction force is

$$H = \mu W \tag{g}$$

Hence, solving the first and third of Eqs. (a), we obtain

$$a_{Cx} = \frac{F - \mu W}{m} \tag{h}$$

and

$$\alpha = -\frac{2[F(h-R) + \mu WR]}{mR^2} \tag{i}$$

Example 6.4
Solve the problem of Example 3.6 but, instead of a particle as shown in Fig. 3.11, the body consists of a disk of radius R. Assume that the disk rolls without slipping.

Because no energy is lost in roll without slip, the energy is conserved. Hence, the only difference between the solution of Example 3.6 and the one here is that here we must include the kinetic energy of rotation. It follows that the kinetic energy in position 2 is

$$T_2 = \tfrac{1}{2}mv_2^2 + \tfrac{1}{2}I_C\omega_2^2 \tag{a}$$

Recalling that $I_C = mR^2/2$ and recognizing that $v_2 = -R\omega_2$, we obtain

$$T_2 = \tfrac{3}{4}mv_2^2 \tag{b}$$

The conservation of energy yields

$$E = T_2 + V_2 = -mgH + \tfrac{3}{4}mv_2^2 = 0 \tag{c}$$

so that the velocity in position 2 is

$$v_2 = \sqrt{4gH/3} \tag{d}$$

which is lower than the value obtained in Example 3.6. This is to be expected, because in the case of this example part of the kinetic energy is due to rotation.

6.6 ROTATION OF A RIGID BODY ABOUT A FIXED AXIS

Let us consider the body of Fig. 6.1 and assume that the motion takes place about a fixed axis. Hence, the fixed origin 0 must lie on this axis. For convenience, we take the axis of rotation as the z-axis. Rotation about the fixed axis z is characterized by

$$\omega_x = \omega_y = 0, \qquad \omega_z = \omega \tag{6.56}$$

Introducing Eqs. (6.56) into Eqs. (6.27), we obtain the moment equations

$$\begin{aligned} M_{0x} &= -I_{xz}\dot{\omega} + I_{yz}\omega^2 \\ M_{0y} &= -I_{yz}\dot{\omega} - I_{xz}\omega^2 \\ M_{0z} &= I_{zz}\dot{\omega} \end{aligned} \tag{6.57}$$

The force equations remain in the form (6.41).

Various pieces of machinery involve rotors in the form of rigid masses of relatively large moments of inertia mounted on a shaft and spinning rapidly about the axis coinciding with the shaft. The shaft is supported at both ends by bearings. Generally the rotor is symmetric. In this case the products of inertia I_{xz} and I_{yz} are zero, and the first two of Eqs. (6.57) reduce to zero identically, so that there are no torques M_{0x} and M_{0y}. Moreover, the mass center C lies on the z-axis, and hence a_{Cx} and a_{Cy} are zero. It follows that for a symmetric rotor, the only forces present are the bearing reactions, which are simply static forces due to the weight of the rotor. On the other hand, if there is some asymmetry in the rotor, so that not all products of inertia are zero, then this asymmetry will generate some dynamic bearing reactions.

As an illustration, let us consider the disk shown in Fig. 6.7*a*, where the normal to the disk makes an angle β with respect to the shaft. The disk rotates uniformly about z, so that $\omega=\dot{\theta}=\text{const}$, $\dot{\omega}=0$. Figure 6.7*b* depicts the free-body diagram. Axes XYZ are inertial and axes xyz are body axes, where z is along the shaft, x is embedded in the disk and normal to z, and y is normal to both. Note that in the position shown, x makes an angle θ with respect to X. The object is to determine the reactions in that position. The products of inertia entering into Eqs. (6.57) are I_{xz} and I_{yz}. But, xyz are not principal axes, which is likely to cause some difficulty in determining I_{xz} and I_{yz} by direct integration. Indeed, it is easier to determine I_{xz} and I_{yz} by means of a coordinate transformation from axes xyz to the principal axes $x'y'z'$ (Fig. 6.7*c*). From Fig. 6.7*c*, we can write the relation between the two sets of axes as follows:

$$\begin{aligned} x &= x' \\ y &= y' \cos\beta + z' \sin\beta \\ z &= -y' \sin\beta + z' \cos\beta \end{aligned} \tag{6.58}$$

Inserting Eqs. (6.58) into the second and third of Eqs. (6.11), we obtain

$$\begin{aligned} I_{xz} &= \int_m xz\,dm = \int_m x'(-y' \sin\beta + z' \cos\beta)\,dm \\ &= -\sin\beta \int_m x'y'\,dm + \cos\beta \int_m x'z'\,dm \\ &= -\sin\beta\, I_{x'y'} + \cos\beta\, I_{x'z'} \end{aligned} \tag{6.59a}$$

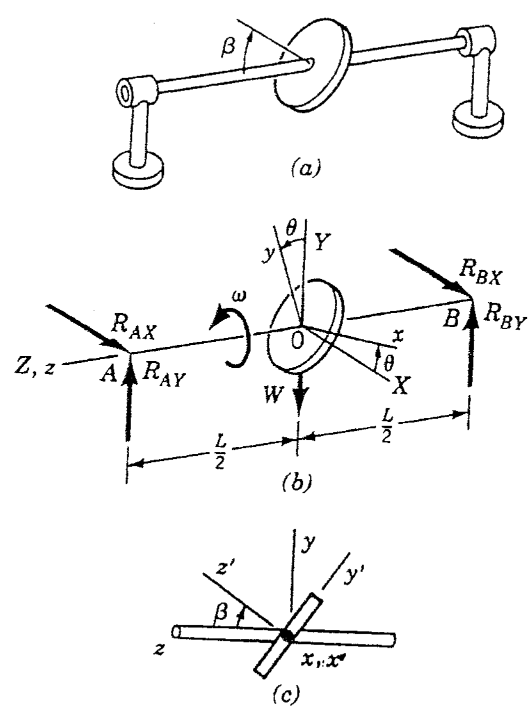

FIGURE 6.7

$$I_{yz}=\int_m yz\,dm=\int_m (y'\cos\beta+z'\sin\beta)(-y'\sin\beta+z'\cos\beta)\,dm$$

$$=\sin\beta\cos\beta\int_m (z'^2-y'^2)\,dm+(\cos^2\beta-\sin^2\beta)\int_m y'z'\,dm$$

$$=\sin\beta\cos\beta\int_m [(r'^2-y'^2)-(r'^2-z'^2)]\,dm$$

$$+(\cos^2\beta-\sin^2\beta)\int_m y'z'\,dm$$

$$=\sin\beta\cos\beta(I_{y'y'}-I_{z'z'})+(\cos^2\beta-\sin^2\beta)I_{y'z'} \tag{6.59b}$$

But, because $x'y'z'$ are principal axes, the products of inertia are zero. Moreover, the moments of inertia of a thin disk are

$$I_{x'x'}=I_{y'y'}=\tfrac{1}{4}mR^2, \qquad I_{z'z'}=\tfrac{1}{2}mR^2 \tag{6.60}$$

where m is the total mass and R is the radius of the disk. Hence, Eqs. (6.59) yield

$$I_{xz}=0, \qquad I_{yz}=-\tfrac{1}{4}mR^2\sin\beta\cos\beta \tag{6.61}$$

Letting $\dot{\omega}=0$ in Eqs. (6.57) and using Eqs. (6.61), we obtain

$$\begin{aligned} M_{0x}&=I_{yz}\omega^2=-\tfrac{1}{4}mR^2\omega^2\sin\beta\cos\beta \\ M_{0y}&=0 \\ M_{0z}&=0 \end{aligned} \tag{6.62}$$

For convenience, we resolve M_{0x} in terms of components along X and Y as follows:

$$M_{0X}=M_{0x}\cos\theta,\ M_{0Y}=M_{0x}\sin\theta \tag{6.63}$$

Because the acceleration of the mass center is zero, the force equations yield

$$\begin{aligned} F_X&=R_{AX}+R_{BX}=0 \\ F_Y&=R_{AY}+R_{BY}-W=0 \end{aligned} \tag{6.64}$$

On the other hand, the moment equations about 0 are

$$\begin{aligned} M_{0X}&=(R_{BY}-R_{AY})\frac{L}{2}=-\frac{1}{4}mR^2\omega^2\sin\beta\cos\beta\cos\theta \\ M_{0Y}&=(R_{AX}-R_{BX})\frac{L}{2}=-\frac{1}{4}mR^2\omega^2\sin\beta\cos\beta\sin\theta \end{aligned} \tag{6.65}$$

Because $\theta=\omega t$, Eqs. (6.64) and (6.65) yield

$$R_{BX} = -R_{AX} = \frac{1}{4}\frac{mR^2\omega^2}{L}\sin\beta\cos\beta\sin\omega t$$

$$R_{AY} = \frac{1}{2}W - \frac{1}{4}\frac{mR^2\omega^2}{L}\sin\beta\cos\beta\cos\omega t \tag{6.66}$$

$$R_{BY} = \frac{1}{2}W + \frac{1}{4}\frac{mR^2\omega^2}{L}\sin\beta\cos\beta\cos\omega t$$

Hence, in addition to the static reactions equal to half the weight, there are dynamic reactions which vary harmonically with a frequency equal to the spin frequency ω. These dynamic reactions tend to wear out the bearings.

6.7 SYSTEMS OF RIGID BODIES

At times, one is faced with the problem not of a single rigid body but of a number of rigid bodies. The bodies are connected at given points through hinges, thus ensuring that the set of rigid bodies acts as a system. This, of course, gives rise to reaction forces at the hinges, so that at a given point connecting two of the bodies there are forces equal in magnitude and opposite in directions acting on the two bodies. One such example was encountered earlier in connection with the rotating unbalanced masses depicted in Fig. 4.23.

Consider a system of N rigid bodies. Then, using Eqs. (6.20) and (6.28), we can write the force and moment equations for each of the bodies in the form

$$\mathbf{F}_i = m_i \mathbf{a}_{Ci}, \qquad i = 1, 2, \ldots, N \tag{6.67a}$$

$$\mathbf{M}_{Ci} = \dot{\mathbf{H}}_{Ci}, \qquad i = 1, 2, \ldots, N \tag{6.67b}$$

where $\mathbf{F}_i$ is the resultant of the force vectors, m_i is the mass, $\mathbf{a}_{Ci}$ is the acceleration of the mass center C_i, $\mathbf{M}_{Ci}$ is the resultant of the moment vectors about the mass center, and $\mathbf{H}_{Ci}$ is the angular momentum vector about the mass center, all quantities pertaining to a typical body $i\,(i = 1, 2, \ldots, N)$. Equations (6.67) represent a system of simultaneous equations of motion. In writing Eqs. (6.67), we must draw a free-body diagram for each of the bodies and be sure to include in the forces the reaction forces between any two bodies, as shown in Figs. 4.23*b* and 4.23*c*. In this regard we observe that, although the reaction forces are internal and they cancel out in pairs if the system of bodies is considered as a whole, when the bodies are considered separately the reaction forces become external and must be treated as any other external force acting on a given body. Of course, the same forces act in opposite directions on the adjacent body. Note that the reaction forces are not known in advance and must be treated as unknowns, together with the motions of the bodies. Quite often they can be eliminated by combining several equations.

In deriving explicit expressions for Eqs. (6.67), we must choose a consistent notation for the motions of the various bodies in the system, as the motions of the individual bodies are not free but constrained kinematically. To arrive at a kinematically consistent set of accelerations for the mass centers of the bodies in the system, we are likely to use equations similar to Eq. (2.57).

In the case of planar motion, the vector Eq. (6.67a) reduces to two scalar equations, and Eq. (6.67b) reduces to a single scalar equation. Moreover, because the angular velocity vectors are all normal to the plane of motion, Eq. (6.67b) can be reduced to

$$M_{Ci}=I_{Ci}\ddot{\theta}_i, \qquad i=1, 2, \ldots, N \tag{6.68}$$

where I_{Ci} is the mass moment of inertia of the body i about an axis normal to the plane of motion and passing through the mass center and $\ddot{\theta}_i$ is the angular acceleration of the body.

Example 6.5

Derive the equations for the planar motion of the system of two bodies shown in Fig. 6.8*a*. Note that one of the bodies can be treated as a particle and the other as a thin, slender rigid bar.

Figures 6.8*b* and 6.8*c* show the free-body diagrams for the two bodies. Because the body of Fig. 6.8*b* can be treated as a particle, there are only two scalar force equations and no moment equation. On the other hand, for the bar of Fig. 6.8*c* there are two force equations and one moment equation. From Figs. 6.8*b* and 6.8*c*, the equations can be written as follows:

$$F_x - kx = Ma_{0x} \tag{a}$$

$$f + F_y - Mg = Ma_{0y} \tag{b}$$

$$F - F_x = ma_{Cx} \tag{c}$$

$$-F_y - mg = ma_{Cy} \tag{d}$$

$$(F+F_x)a\cos\theta + F_y a\sin\theta = I_C\ddot{\theta} \tag{e}$$

where $I_C=m(2a)^2/12=ma^2/3$ is the mass moment of inertia of the bar about the mass center C. Equations (a) and (c) on the one hand and Eqs. (c), (d), and (e) on the

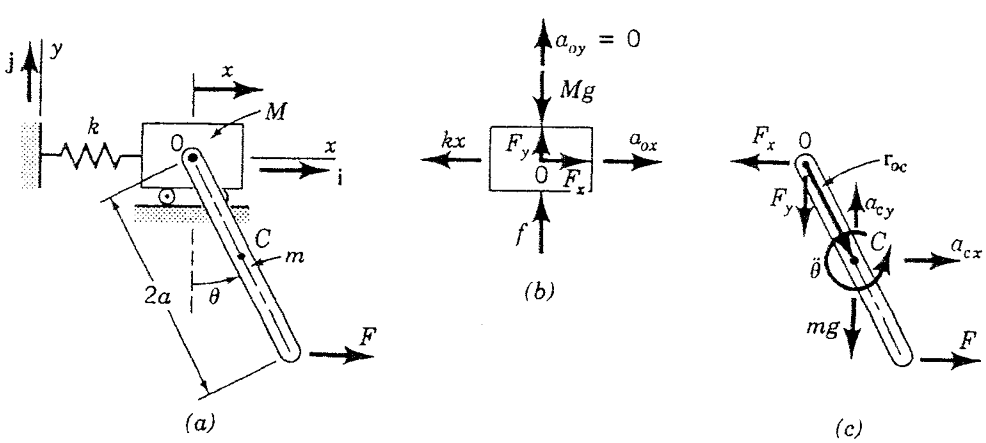

FIGURE 6.8

other hand can be combined into

$$
\begin{aligned}
&Ma_{0x} + ma_{Cx} + kx = F \\
&I_C\ddot{\theta} + ma(a_{Cx}\cos\theta + a_{Cy}\sin\theta) + mga\sin\theta = 2Fa\cos\theta
\end{aligned} \tag{f}
$$

which are entirely free of F_x and F_y.

The system possesses only two degrees of freedom, and the generalized coordinates x and θ determine the motion of the system fully. Indeed, the accelerations a_{Cx} and a_{Cy} can all be described in terms of these coordinates. To this end, we observe that

$$\mathbf{a}_0 = a_{0x}\mathbf{i} + a_{0y}\mathbf{j} = \ddot{x}\mathbf{i} \tag{g}$$

so that

$$a_{0x} = \ddot{x}, \qquad a_{0y} = 0 \tag{h}$$

Moreover, Eq. (2.57) can be written in the form

$$\mathbf{a}_C = \mathbf{a}_{Cx}\mathbf{i} + \mathbf{a}_{Cy}\mathbf{j} = \mathbf{a}_0 + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}_{0C}) + \dot{\boldsymbol{\omega}}\times\mathbf{r}_{0C} \tag{i}$$

where

$$\mathbf{r}_{0C} = a(\sin\theta\ \mathbf{i} - \cos\theta\ \mathbf{j}), \qquad \boldsymbol{\omega} = \dot{\theta}\mathbf{k}, \qquad \dot{\boldsymbol{\omega}} = \ddot{\theta}\mathbf{k} \tag{j}$$

in which $\mathbf{k}$ is a unit vector normal to the plane of motion. Inserting Eqs. (g), (h), and (j) into Eq. (i), we obtain simply

$$\mathbf{a}_C = [\ddot{x} + a(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta)]\mathbf{i} + a(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta)\mathbf{j} \tag{k}$$

so that

$$a_{Cx} = \ddot{x} + a(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta), \qquad a_{Cy} = a(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta) \tag{l}$$

Introducing Eqs. (h) and (l) into Eqs. (f), we obtain the equations of motion

$$
\begin{aligned}
&(M+m)\ddot{x} + ma(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta) + kx = F \\
&\tfrac{4}{3}ma^2\ddot{\theta} + ma\ddot{x}\cos\theta + mga\sin\theta = 2Fa\cos\theta
\end{aligned} \tag{m}
$$

which can be identified as two nonlinear ordinary differential equations in x and θ.

Note that for any $x = x(t)$ and $\theta = \theta(t)$, one can use Eqs. (a) and (d) in conjunction with Eqs. (h) and (l) to determine the hinge reactions F_x and F_y. Moreover, introducing the latter into Eq. (b) and recalling that $a_{0y} = 0$, we can determine the ground reaction f.

6.8 MOTION OF A TORQUE-FREE SYMMETRIC BODY

Let us consider a torque-free symmetric rigid body rotating about the mass center C. Letting $M_1 = M_2 = M_3 = 0$ and $I_2 = I_1$ in Eqs. (6.30), we obtain

$$\begin{aligned} I_1\dot{\omega}_1+(I_3-I_1)\omega_2\omega_3&=0 \\ I_1\dot{\omega}_2+(I_1-I_3)\omega_1\omega_3&=0 \\ I_3\dot{\omega}_3&=0 \end{aligned} \tag{6.69}$$

The third of Eqs. (6.69) can be integrated immediately to yield

$$\omega_3=\text{const} \tag{6.70}$$

Introducing the notation

$$(I_3-I_1)\omega_3/I_1=\Omega=\text{const} \tag{6.71}$$

the first two of Eqs. (6.69) become

$$\begin{aligned} \dot{\omega}_1+\Omega\omega_2&=0 \\ \dot{\omega}_2-\Omega\omega_1&=0 \end{aligned} \tag{6.72}$$

Multiplying the first of Eqs. (6.72) by ω_1 and the second by ω_2 and adding the results, we obtain

$$\omega_1\dot{\omega}_1+\omega_2\dot{\omega}_2=0 \tag{6.73}$$

which, upon integration, yields

$$\omega_1^2+\omega_2^2=\omega_{12}^2=\text{const} \tag{6.74}$$

where ω_{12} is the projection of the angular velocity $\boldsymbol{\omega}$ on the 1, 2-plane. Combining Eqs. (6.70) and (6.74), we conclude that

$$|\boldsymbol{\omega}|=(\omega_1^2+\omega_2^2+\omega_3^2)^{1/2}=\text{const} \tag{6.75}$$

or, the magnitude of the angular velocity vector $\boldsymbol{\omega}$ is constant.

Because $\mathbf{M}_C=\mathbf{0}$, the angular momentum about C is conserved

$$\mathbf{H}_C=\text{const} \tag{6.76}$$

which means that both the magnitude and the direction of $\mathbf{H}_C$ are constant. The projection of $\mathbf{H}_C$ on the 1, 2 plane is

$$H_{12}=(I_1^2\omega_1^2+I_1^2\omega_2^2)^{1/2}=I_1(\omega_1^2+\omega_2^2)^{1/2}=I_1\omega_{12} \tag{6.77}$$

so that the projections of $\boldsymbol{\omega}$ and $\mathbf{H}_C$ on the 1, 2-plane are along the same line. Hence, axes 3, $\mathbf{H}_C$, and $\boldsymbol{\omega}$ are in the same plane, so that the motion can be interpreted geometrically as the rotation of the plane defined by 3, $\mathbf{H}_C$, and $\boldsymbol{\omega}$ about $\mathbf{H}_C$, as shown in Fig. 6.9. Letting α be the angle between 3 and $\mathbf{H}_C$, we have

$$\tan\alpha=\frac{H_{12}}{H_3}=\frac{I_1\omega_{12}}{I_3\omega_3}=\text{const} \tag{6.78}$$

Moreover, letting β be the angle between 3 and $\boldsymbol{\omega}$, we can write

$$\tan\beta=\frac{\omega_{12}}{\omega_3}=\text{const} \tag{6.79}$$

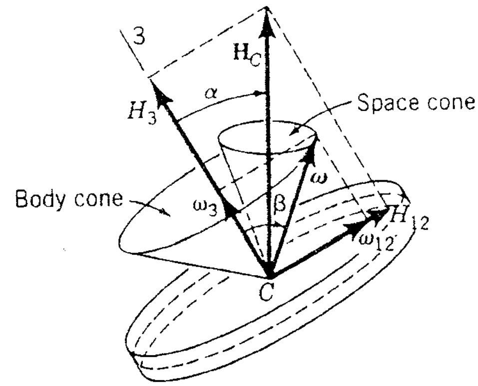

FIGURE 6.9

so that Eq. (6.78) can be rewritten in the form

$$\tan\alpha = \frac{I_1}{I_3}\tan\beta \tag{6.80}$$

There are two cases:

i. $I_3 > I_1$, which implies that $\alpha < \beta$.
ii. $I_3 < I_1$, which implies that $\alpha > \beta$.

The first case represents a flat body, such as a disk, and the second one represents a slender body, such as a rod. Figure 6.9 shows the relative position of the vectors ω and $\mathbf{H}_C$ for the first case.

Because the angle $\beta - \alpha$ is constant and $\mathbf{H}_C$ is fixed in space, the angular velocity vector ω rotates about $\mathbf{H}_C$. In the process, it generates a cone fixed in space called the *space cone*. The component ω_{12} of the velocity vector moves relative to axes 1, 2. To study this motion, let us introduce the complex notation

$$\omega_{12} = \omega_1 + i\omega_2 \tag{6.81}$$

Then, multiplying the second of Eqs. (6.72) by i and adding the result to the first, we obtain

$$\dot{\omega}_{12} - i\Omega\omega_{12} = 0 \tag{6.82}$$

which has the solution

$$\omega_{12}(t) = \omega_{12}(0)e^{i\Omega t} \tag{6.83}$$

where $\omega_{12}(0)$ is some initial angular velocity, so that the vector ω_{12} rotates in the plane 1, 2 with the angular velocity Ω. This is equivalent to saying that the vector ω rotates relative to the body with the same angular velocity Ω. In the process, the vector ω generates a cone in the body frame known as the *body cone*. Because the vector ω is a generatrix for both the space cone and the body cone, the two cones are tangent to one another. Hence, the motion of the body can be visualized as the rolling of the body cone on the space cone. For $I_3 > I_1$, $\alpha < \beta$, and the space cone lies inside the body cone. For $I_3 < I_1$, $\alpha > \beta$, and the space cone lies outside the body cone. To an observer in an inertial space, the motion of the body appears as a wobbling, spinning motion, such as in imperfect throwing of a discus or a football. As a crude approximation, the earth can be regarded as an example of a symmetric torque-free body. It is common to assume that the earth is an oblate spheroid with the polar axis coinciding with the symmetry axis for which $(I_3 - I_1)/I_1 \cong 0.0033$. Moreover, assuming that $\omega_3 \cong |\omega| = 1$ rotation per day, we obtain from Eq. (6.71)

$$\Omega = (I_3 - I_1)\omega_3/I_1 \cong 0.0033 \quad \text{rotation per day} \tag{6.84}$$

so that the vector ω completes one sweep around the body cone approximately every 300 days, and, in the process, it traces a circle on the earth around the North Pole. A phenomenon resembling this wobbling motion has actually been observed and is known as the *variation in latitude.* The radius of the circle is about 3.1 m and the period is approximately 433 days instead of 300 days. The discrepancy is attributed to factors unaccounted for, such as the earth not being perfectly rigid.

6.9 ROTATION ABOUT A FIXED POINT. GYROSCOPES

An important class of dynamical systems is characterized by symmetric rigid bodies spinning about the symmetry axis. This is the class of gyroscopic systems, and it includes the rotating earth discussed in Section 6.8 and the spinning top. It also includes a number of devices in which a symmetric rotor spins rapidly relative to the shaft. Such systems are known as *gyroscopes*, and they are used as devices measuring angular motions or as torquing devices. They are essential components in inertial navigation, which is a self-contained system of navigation based only on Newton's laws. In this section we present some basic features of gyroscopic motion.

Quite often, the treatment of gyroscopic systems is facilitated by referring the motion to a set of axes other than the body axes. We recall that the advantage of referring the motion to body axes is that the moments of inertia about these axes are constant. But, in the case of bodies exhibiting axial symmetry, the body axes are not the only axes about which the moments of inertia are constant. Considering an axisymmetric body and letting $\xi\eta\zeta$ be a set of principal axes, with ζ coinciding with the symmetry axis and ξ and η coinciding with any two orthogonal transverse axes, it is clear that the moments of inertia about ξ and η are equal to one another, $I_\xi = I_\eta$. Moreover, the moments of inertia I_ξ, I_η, and I_ζ about axes ξ, η, and ζ, respectively, remain the same even when the body rotates relative to axes $\xi\eta\zeta$, provided the only component of relative rotation is about the symmetry axis, which is assumed to be the case here. Note that, for the moments of inertia about $\xi\eta\zeta$ to remain unchanged, it is not necessary that the body be cylindrical but only that it possess the inertial symmetry defined by $I_\xi = I_\eta$. It is easy to verify that this inertial symmetry is possessed by a parallelepiped with a square cross-sectional area. In this case, if we let axes ξ and η lie in the plane of the square cross section, then $I_\xi = I_\eta$ regardless of the directions of ξ and η. In the following, we wish to produce a set of moment equations of motion in terms of axes $\xi\eta\zeta$.

The derivation of the moment equations of motion in terms of axes $\xi\eta\zeta$ is based on Eq. (6.25). In carrying out this derivation, we distinguish between the angular velocity $\boldsymbol{\omega}$ of axes $\xi\eta\zeta$ and the angular velocity $\boldsymbol{\Omega}$ of the body about axes $\xi\eta\zeta$, where $\boldsymbol{\Omega}$ differs from $\boldsymbol{\omega}$ only in the ζ-component. Moreover, it is important to recognize that the symbol $\boldsymbol{\omega}$ appearing in Eq. (6.25) represents the angular velocity of the reference axes and that the angular velocity entering into the angular momentum $\mathbf{H}_0$ is the angular velocity $\boldsymbol{\Omega}$ of the body. Because the only difference between $\boldsymbol{\omega}$ and $\boldsymbol{\Omega}$ is in the ζ-component, we can write

$$\boldsymbol{\omega} = \omega_\xi \mathbf{i} + \omega_\eta \mathbf{j} + \omega_\zeta \mathbf{k} \tag{6.85a}$$

$$\boldsymbol{\Omega} = \Omega_\xi \mathbf{i} + \Omega_\eta \mathbf{j} + \Omega_\zeta \mathbf{k} = \omega_\xi \mathbf{i} + \omega_\eta \mathbf{j} + \Omega_\zeta \mathbf{k} \tag{6.85b}$$

The angular momentum vector has the general expression

$$\mathbf{H}_0 = H_\xi \mathbf{i} + H_\eta \mathbf{j} + H_\zeta \mathbf{k} \tag{6.86}$$

Similarly, the moment about 0 can be expressed in the form

$$\mathbf{M}_0 = M_\xi \mathbf{i} + M_\eta \mathbf{j} + M_\zeta \mathbf{k} \tag{6.87}$$

Hence, inserting Eqs. (6.85a), (6.86), and (6.87) into Eq. (6.25), we obtain the equations

$$\begin{aligned} M_\xi &= \dot{H}_\xi + \omega_\eta H_\zeta - \omega_\zeta H_\eta \\ M_\eta &= \dot{H}_\eta + \omega_\zeta H_\xi - \omega_\xi H_\zeta \\ M_\zeta &= \dot{H}_\zeta + \omega_\xi H_\eta - \omega_\eta H_\xi \end{aligned} \tag{6.88}$$

which are known as the *modified Euler equations*. But, recalling the symmetry of the rotor, we can write the angular momentum components about axes ξ, η, and ζ

$$H_\xi = I_\xi \Omega_\xi = I_\xi \omega_\xi, \qquad H_\eta = I_\eta \Omega_\eta = I_\xi \omega_\eta, \qquad H_\zeta = I_\zeta \Omega_\zeta \tag{6.89}$$

where I_ξ, $I_\eta = I_\xi$, and I_ζ are the associated principal moments of inertia. It follows that Eqs. (6.88) can be written in the more explicit form

$$\begin{aligned} M_\xi &= I_\xi \dot{\omega}_\xi + \omega_\eta (I_\zeta \Omega_\zeta - I_\xi \omega_\zeta) \\ M_\eta &= I_\xi \dot{\omega}_\eta + \omega_\xi (I_\xi \omega_\zeta - I_\zeta \Omega_\zeta) \\ M_\zeta &= I_\zeta \dot{\Omega}_\zeta \end{aligned} \tag{6.90}$$

Next, we propose to use Eqs. (6.90) to develop the theory explaining the phenomenon of gyroscopic motion. To this end, we consider the mathematical model shown in Fig. 6.10. The model consists of a shaft pivoted at the fixed point and a wheel spinning rapidly relative to the shaft. The wheel, also known as a rotor, is at a distance L from point 0. Before we can derive the equations of motion, it is necessary to choose a set of angles so as to be able to describe the motion of the rotor. A set which enjoys wide acceptance consists of *Euler's angles*. These are

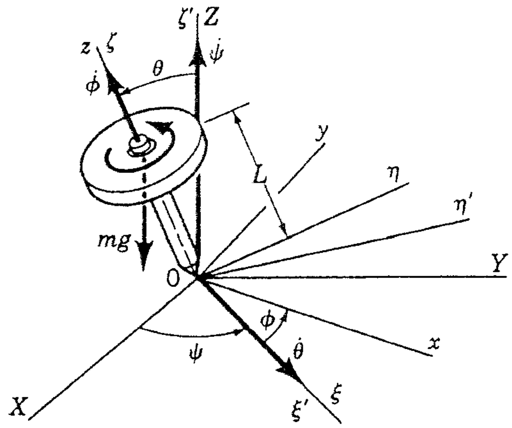

FIGURE 6.10

three successive angular displacements, ψ, θ, and ϕ, defining the angular position of the rotor relative to an inertial space. To demonstrate how Euler's angles provide a description of the body orientation, we envision a body-axes triad, initially coincident with a set of inertial axes XYZ. A rotation through an angle ψ about axis Z brings the triad into coincidence with the set of axes $\xi'\eta'\zeta'$. A further rotation θ about axis ξ' brings the triad to the set of axes $\xi\eta\zeta$, where ξ, sometimes referred to as the *nodal axis*, remains in the horizontal plane at all times. In fact, ξ is the intersection of the horizontal plane defined by axes X and Y and the plane defined by axes ξ and η, where the angle between the two planes is θ. Finally, a rotation ϕ about ζ causes the triad to coincide with the body axes xyz. The sequence of rotations is shown in Fig. 6.10. It is easy to see from Fig. 6.10 that axes $\xi\eta\zeta$ undergo the angular velocities $\dot\psi$ about $Z=\zeta'$ and $\dot\theta$ about $\xi'=\xi$. Hence, *the angular velocity components of the reference frame* $\xi\eta\zeta$ *along* ξ, η, *and* ζ are

$$\omega_\xi=\dot\theta, \qquad \omega_\eta=\dot\psi \sin\theta, \qquad \omega_\zeta=\dot\psi\cos\theta \tag{6.91}$$

But the wheel rotates relative to axes $\xi\eta\zeta$ with the angular velocity $\dot\phi$ about $\zeta=z$. Hence, *the angular velocity components of the body along* ξ, η, *and* ζ are

$$\Omega_\xi=\omega_\xi=\dot\theta, \qquad \Omega_\eta=\omega_\eta=\dot\psi\sin\theta, \qquad \Omega_\zeta=\dot\phi+\omega_\zeta=\dot\phi+\dot\psi\cos\theta \tag{6.92}$$

Moreover, the components of the moment along ξ, η, and ζ are

$$M_\xi=mgL\sin\theta, \qquad M_\eta=0, \qquad M_\zeta=0 \tag{6.93}$$

Introducing Eqs. (6.91)–(6.93) into Eqs. (6.90), we obtain

$$\begin{gathered}
I_\xi\ddot\theta+\dot\psi\sin\theta[I_\zeta(\dot\phi+\dot\psi\cos\theta)-I_\xi\dot\psi\cos\theta]=mgL\sin\theta\\
I_\xi\frac{d}{dt}(\dot\psi\sin\theta)+\dot\theta[I_\xi\dot\psi\cos\theta-I_\zeta(\dot\phi+\dot\psi\cos\theta)]=0\\
I_\zeta\frac{d}{dt}(\dot\phi+\dot\psi\cos\theta)=0
\end{gathered} \tag{6.94}$$

which represent the differential equations of motion describing the motion of a gyroscope. Incidentally, the same equations govern the behavior of a spinning top, a child's toy, where L is the distance from 0 to the mass center of the top.

Equations (6.94) can be simplified somewhat by observing that the third equation can be integrated immediately to yield

$$\dot{\phi}+\dot{\psi}\cos\theta=\Omega_\zeta=\text{const} \tag{6.95}$$

so that the first two equations can be reduced to

$$\begin{gathered} I_\xi\ddot{\theta}+\dot{\psi}\sin\theta(I_\zeta\Omega_\zeta-I_\xi\dot{\psi}\cos\theta)=mgL\sin\theta \\ I_\xi\frac{d}{dt}(\dot{\psi}\sin\theta)+\dot{\theta}(I_\xi\dot{\psi}\cos\theta-I_\zeta\Omega_\zeta)=0 \end{gathered} \tag{6.96}$$

Equations (6.96) represent two simultaneous differential equations for the angular motions $\theta(t)$ and $\psi(t)$, where the motions are known as *nutation* and *precession*, respectively. The equations are highly nonlinear and admit no closed-form solution. Hence, we wish to explore a special solution known as *steady precession* of the spin axis and defined by

$$\theta=\theta_0=\text{const},\qquad \dot{\psi}=\dot{\psi}_0=\text{const} \tag{6.97}$$

It is easy to verify that the second of Eqs. (6.96) is satisfied identically. On the other hand, the first of Eqs. (6.96) yields the quadratic equation

$$\dot{\psi}_0^2-\frac{I_\zeta\Omega_\zeta}{I_\xi\cos\theta_0}\dot{\psi}_0+\frac{mgL}{I_\xi\cos\theta}=0 \tag{6.98}$$

which has the solutions

$$\left.\begin{matrix}\dot{\psi}_{01}\\ \dot{\psi}_{02}\end{matrix}\right\}=\frac{I_\zeta\Omega_\zeta}{2I_\xi\cos\theta_0}\pm\frac{I_\zeta\Omega_\zeta}{2I_\xi\cos\theta_0}\left(1-\frac{4mgLI_\xi\cos\theta_0}{I_\zeta^2\Omega_\zeta^2}\right)^{1/2} \tag{6.99}$$

where the roots $\dot{\psi}_{01}$ and $\dot{\psi}_{02}$ are called *fast precession* and *slow precession*, respectively. Because the roots must be real, steady precession is possible only if the spin is sufficiently high that the discriminant in Eq. (6.99) is positive. This leads to the condition

$$\Omega_\zeta^2>\frac{4mgLI_\xi\cos\theta_0}{I_\zeta^2} \tag{6.100}$$

For relatively large values of Ω_ζ, we can retain only the first two terms in a binomial expansion of the type $(1-\varepsilon)^{1/2}=1-\frac{1}{2}\varepsilon-\frac{1}{2}\cdot\frac{1}{4}\varepsilon^2-\ldots$, where ε is a small quantity, and write the approximation

$$\left(1-\frac{4mgLI_\xi\cos\theta_0}{I_\zeta^2\Omega_\zeta^2}\right)^{1/2}\cong 1-\frac{2mgLI_\xi\cos\theta_0}{I_\zeta^2\Omega_\zeta^2} \tag{6.101}$$

Hence, inserting Eq. (6.101) into Eq. (6.99), we obtain

$$\dot{\psi}_{01}\cong\frac{I_\zeta\Omega_\zeta}{I_\xi\cos\theta_0},\qquad \dot{\psi}_{02}\cong\frac{mgL}{I_\zeta\Omega_\zeta} \tag{6.102}$$

Because of the high-energy requirement, the fast precession is generally unattainable, so that in referring to steady precession we really mean $\dot{\psi}_{02}=\dot{\psi}_0=mgL/I_\zeta\Omega_\zeta$.

Note that the same value for the steady precession can be obtained from the first of Eqs. (6.96) by assuming that the term $I_\xi \dot{\psi}_0 \cos\theta_0$ is small relative to the term $I_\zeta \Omega_\zeta$.

The steady precession can be demonstrated in a simpler fashion by referring directly to Eq. (6.25). In the case of steady precession, we have

$$\mathbf{H}_0 = I_\zeta \Omega_\zeta \mathbf{k}, \qquad \dot{\mathbf{H}}'_0 = \mathbf{0} \tag{6.103}$$

Moreover, the torque has the vector form

$$\mathbf{M}_0 = mgL \sin\theta \; \mathbf{i} \tag{6.104}$$

Introducing Eqs. (6.103) and (6.104) into Eq. (6.25), we obtain

$$\begin{aligned} \mathbf{M}_0 &= \boldsymbol{\omega} \times \mathbf{H}_0 = (\omega_\xi \mathbf{i} + \omega_\eta \mathbf{j} + \omega_\zeta \mathbf{k}) \times I_\zeta \Omega_\zeta \mathbf{k} \\ &= -\omega_\xi I_\zeta \Omega_\zeta \mathbf{j} + \omega_\eta I_\zeta \Omega_\zeta \mathbf{i} = mgL \sin\theta \; \mathbf{i} \end{aligned} \tag{6.105}$$

from which we conclude that $\omega_\xi = 0$ and, recalling the second of Eqs. (6.91), that

$$\dot{\psi} = \frac{mgL}{I_\zeta \Omega_\zeta} \tag{6.106}$$

where $\Omega_\zeta = \text{const}$ by virtue of the third component of $\dot{\mathbf{H}}'_0 = \mathbf{0}$.

A physical interpretation of the above result should prove most rewarding. Assume first that the shaft is held fixed in the position $\theta = \theta_0$ while the rotor is at rest. Then, if the support is removed, the torque $mgL \sin\theta$ about the ξ-axis will cause the shaft to drop in the $\eta\zeta$-plane, thus rotating with the velocity $\dot{\theta}$ and acceleration $\ddot{\theta}$ about the ξ-axis. On the other hand, if the rotor is spinning with the angular velocity Ω_ζ when the support is removed, where Ω_ζ satisfied inequality (6.100), then the shaft begins to precess uniformly about the Z-axis. This behavior is at odds with intuition, according to which one might expect the shaft to fail. This behavior has prompted the statement that a gyroscope defies gravity, which is not quite correct, since it is the moment due to gravity that causes the precession. We note that the steady precession is proportional to the product of the mass m and the distance L and inversely proportional to the magnitude of the angular momentum. We also note that if the rotor is spinning originally in the opposite sense, then the precession is also in the opposite sense.

6.10 SMALL OSCILLATIONS ABOUT STEADY PRECESSION

Next, let us examine how the uniformly precessing gyroscope behaves if the spin axis is imparted some small initial angular velocities $\omega_\xi(0)$ and $\omega_\eta(0)$, where $\omega_\eta(0)$ is in addition to the steady precession. To this end, we assume that the solution of Eqs. (6.96) can be written in the form

$$\theta(t) = \theta_0 + \theta_1(t), \qquad \dot{\psi}(t) = \dot{\psi}_0 + \dot{\psi}_1(t) \tag{6.107}$$

where θ_0 and $\dot{\psi}_0$ define the steady precession and $\theta_1(t)$ and $\dot{\psi}_1(t)$ are small perturbations from the steady precession. Introducing Eqs. (6.107) into Eqs. (6.96), making the approximation

$$\begin{aligned}\sin\theta &= \sin(\theta_0+\theta_1) \cong \sin\theta_0 + \theta_1\cos\theta_0 \\ \cos\theta &= \cos(\theta_0+\theta_1) \cong \cos\theta_0 - \theta_1\sin\theta_0\end{aligned} \tag{6.108}$$

and ignoring nonlinear terms in θ_1 and $\dot{\psi}_1$, we obtain

$$\begin{aligned}&I_\xi\ddot{\theta}_1 + [\dot{\psi}_0(I_\zeta\Omega_\zeta - I_\xi\dot{\psi}_0\cos\theta_0) - mgL]\sin\theta_0 \\ &\quad + \dot{\psi}_1(I_\zeta\Omega_\zeta - I_\xi\dot{\psi}_0\cos\theta_0)\sin\theta_0 - \dot{\psi}_0 I_\xi(\dot{\psi}_1\cos\theta_0 - \theta_1\dot{\psi}_0\sin\theta_0)\sin\theta_0 \\ &\quad + [\dot{\psi}_0(I_\zeta\Omega_\zeta - I_\xi\dot{\psi}_0\cos\theta_0) - mgL]\theta_1\cos\theta_0 = 0 \\ &I_\xi(\ddot{\psi}_1\sin\theta_0 + \dot{\theta}_1\dot{\psi}_0\cos\theta_0) + \dot{\theta}_1(I_\xi\dot{\psi}_0\cos\theta_0 - I_\zeta\Omega_\zeta) = 0\end{aligned} \tag{6.109}$$

The terms in Eqs. (6.109) not containing the perturbations are called *zero-order terms* and those linear in the perturbations are called *first-order terms*. Because these terms are of different orders of magnitude, according to the perturbation theory,* the zero-order terms and the first-order terms must be set equal to zero separately. There is only one equation for the zero-order terms, namely,

$$[\dot{\psi}_0(I_\zeta\Omega_\zeta - I_\xi\dot{\psi}_0\cos\theta_0) - mgL]\sin\theta_0 = 0 \tag{6.110}$$

Equation (6.110) admits two solutions: (1) the trivial one given by $\sin\theta_0$, which implies $\theta_0 = 0$, and (2) the nontrivial one. In the trivial case, the rotor spins in the upright position, appearing motionless. Note that in the case of a spinning top, $\theta_0 = 0$ corresponds to the so-called *sleeping top*. In the nontrivial case, Eq. (6.110) yields

$$\dot{\psi}_0(I_\zeta\Omega_\zeta - I_\xi\dot{\psi}_0\cos\theta_0) - mgL = 0 \tag{6.111}$$

which can be recognized as being identical to the equation for the steady precession, Eq. (6.98). Then, considering Eq. (6.111), we can reduce the equations for the first-order terms to

$$\begin{aligned}I_\xi\ddot{\theta}_1 + (\dot{\psi}_0\sin\theta_0)^2\theta_1 + (I_\zeta\Omega_\zeta - 2I_\xi\dot{\psi}_0\cos\theta_0)\sin\theta_0\,\dot{\psi}_1 &= 0 \\ -(I_\zeta\Omega_\zeta - 2I_\xi\dot{\psi}_0\cos\theta_0)\dot{\theta}_1 + I_\xi\sin\theta_0\ddot{\psi}_1 &= 0\end{aligned} \tag{6.112}$$

Equations (6.112) are known as *perturbation equations*, and they represent the linearized equations of motion about the steady precession. Introducing the notation

$$\frac{I_\zeta\Omega_\zeta - 2I_\xi\dot{\psi}_0\cos\theta_0}{I_\xi} = \sigma, \qquad \dot{\psi}_0\sin\theta_0 = \omega_{\eta 0} \tag{6.113}$$

we can rewrite Eqs. (6.112) in the simple form

$$\begin{aligned}\ddot{\theta}_1 + \omega_{\eta 0}^2\theta_1 + \sigma\sin\theta_0\,\dot{\psi}_1 &= 0 \\ -\sigma\dot{\theta}_1 + \sin\theta_0\,\ddot{\psi}_1 &= 0\end{aligned} \tag{6.114}$$

*See L. Meirovitch, *Methods of Analytical Dynamics*, McGraw-Hill, New York, 1970, Chapter 8.

The solution of Eqs. (6.114) can be obtained conveniently by the Laplace transformation method. Transforming Eqs. (6.114), we obtain

$$\begin{aligned} s^2\Theta_1 - s\theta_1(0) - \dot{\theta}_1(0) + \omega_{\eta 0}^2\Theta_1 + \sigma \sin\theta_0[s\Psi_1 - \psi_1(0)] &= 0 \\ -\sigma[s\Theta_1 - \theta_1(0)] + \sin\theta_0[s^2\Psi_1 - s\psi_1(0) - \dot{\psi}_1(0)] &= 0 \end{aligned} \tag{6.115}$$

where Θ_1 and Ψ_1 are the Laplace transforms of θ_1 and ψ_1, respectively. Moreover, the initial conditions are

$$\theta_1(0) = \psi_1(0) = 0, \qquad \dot{\theta}_1(0) = \omega_\xi(0), \qquad \sin\theta_0\,\dot{\psi}_1 = \omega_\eta(0) \tag{6.116}$$

so that Eqs. (6.115) become

$$\begin{aligned} (s^2 + \omega_{\eta 0}^2)\Theta_1 + \sigma \sin\theta_0\, s\Psi_1 &= \omega_\xi(0) \\ -\sigma s\Theta_1 + \sin\theta_0\, s^2\Psi_1 &= \omega_\eta(0) \end{aligned} \tag{6.117}$$

Equations (6.117) have the solution

$$\begin{aligned} \Theta_1 &= \frac{\begin{vmatrix} \omega_\xi(0) & s\sigma\sin\theta_0 \\ \omega_\eta(0) & s^2\sin\theta_0 \end{vmatrix}}{\begin{vmatrix} s^2+\omega_{\eta 0}^2 & s\sigma\sin\theta_0 \\ -s\sigma & s^2\sin\theta_0 \end{vmatrix}} = \frac{s[s\omega_\xi(0) - \omega_\eta(0)]\sin\theta_0}{s^2(s^2+\omega_{\eta 0}^2+\sigma^2)\sin\theta_0} = \frac{\omega_\xi(0)}{s^2+\Omega^2} - \frac{\sigma\omega_\eta(0)}{s(s^2+\Omega^2)} \\ \Psi_1 &= \frac{\begin{vmatrix} s^2+\omega_{\eta 0}^2 & \omega_\xi(0) \\ -s\sigma & \omega_\eta(0) \end{vmatrix}}{\begin{vmatrix} s^2+\omega_{\eta 0}^2 & s\sigma\sin\theta_0 \\ -s\sigma & s^2\sin\theta_0 \end{vmatrix}} = \frac{\sigma s\omega_\xi(0) + (s^2+\omega_{\eta 0}^2)\omega_\eta(0)}{s^2(s^2+\omega_{\eta 0}^2+\sigma^2)\sin\theta_0} \\ &= \frac{\sigma\omega_\xi(0)}{s(s^2+\Omega^2)\sin\theta_0} + \left[\frac{1}{s^2} - \frac{\sigma^2}{s^2(s^2+\Omega^2)}\frac{\omega_\eta(0)}{\sin\theta_0}\right] \end{aligned} \tag{6.118}$$

where

$$\Omega^2 = \omega_{\eta 0}^2 + \sigma^2 \tag{6.119}$$

Hence, using the table of Laplace transforms in Section A.7, we can write the inverse transformations

$$\begin{aligned} \theta_1(t) &= \frac{\omega_\xi(0)}{\Omega}\sin\Omega t - \frac{\sigma\omega_\eta(0)}{\Omega^2}(1-\cos\Omega t) \\ \psi_1(t) &= \frac{\sigma\omega_\xi(0)}{\Omega^2\sin\theta_0}(1-\cos\Omega t) + \frac{\omega_\eta(0)}{\sin\theta_0}\left[t - \frac{\sigma^2}{\Omega^2}(\Omega t - \sin\Omega t)\right] \end{aligned} \tag{6.120}$$

But, in general the spin velocity is much larger than the precessional velocity, so that

$$\Omega \cong \sigma \cong \frac{I_\zeta \Omega_\zeta}{I_\xi} \tag{6.121}$$

in which case Eqs. (6.120) reduce to

$$\theta_1(t)=\frac{\omega_\xi(0)}{\Omega}\sin\Omega t-\frac{\omega_\eta(0)}{\Omega}(1-\cos\Omega t)$$
$$\psi_1(t)=\frac{\omega_\xi(0)}{\Omega\sin\theta_0}(1-\cos\Omega t)+\frac{\omega_\eta(0)}{\Omega\sin\theta_0}\sin\Omega t \tag{6.122}$$

Finally, considering Eqs. (6.107), we obtain

$$\theta(t)=\theta_0+\frac{\omega_\xi(0)}{\Omega}\sin\Omega t-\frac{\omega_\eta(0)}{\Omega}(1-\cos\Omega t)$$
$$\psi(t)=\dot\psi_0 t+\frac{\omega_\xi(0)}{\Omega\sin\theta_0}(1-\cos\Omega t)+\frac{\omega_{\xi\eta}(0)}{\Omega\sin\theta_0}\sin\Omega t \tag{6.123}$$

where $\dot\psi_0$ is the steady precession given by Eq. (6.106).

To interpret the motion geometrically, let us introduce the notation

$$\omega_\xi(0)=\omega_{\xi\eta}(0)\cos\gamma,\qquad \omega_\eta(0)=\omega_{\xi\eta}(0)\sin\gamma \tag{6.124}$$

Then, Eqs. (6.123) can be rewritten as

$$\theta(t)=\theta_0-\frac{\omega_\eta(0)}{\Omega}+\frac{\omega_{\xi\eta}(0)}{\Omega}\sin(\Omega t+\gamma)$$
$$\psi(t)=\dot\psi_0 t+\frac{\omega_\xi(0)}{\Omega\sin\theta_0}-\frac{\omega_{\xi\eta}(0)}{\Omega}\cos(\Omega t+\gamma) \tag{6.125}$$

It is convenient to envision the spin axis as tracing a path on an imaginary unit sphere centered at 0, so that the angles $\theta(t)$ and $\psi(t)$ can be regarded as being equivalent to arcs on the unit sphere. Ignoring the steady precession for the moment, we conclude from Eqs. (6.125) that the spin axis traces a circle centered at $\theta=\theta_0-\omega_\eta(0)/\Omega$, $\psi=\omega_\xi(0)/\Omega\sin\theta_0$ and of radius $\omega_{\xi\eta}(0)/\Omega$, as shown in Fig. 6.11, where the circular velocity is Ω. The steady precession causes the circle to advance horizontally at the constant rate $\dot\psi_0$. Of course, because of this precession, the

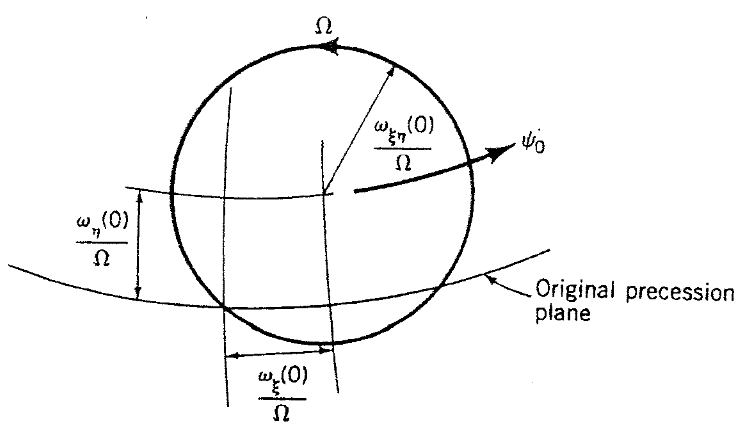

FIGURE 6.11

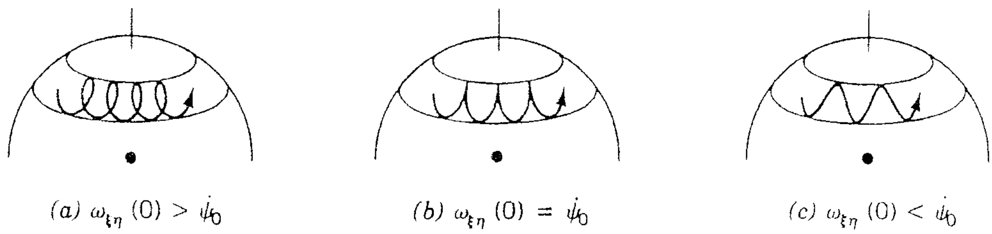

FIGURE 6.12

circle does not close, and the spin axis follows one of the trajectories shown in Fig. 6.12, where the type of trajectory depends on the ratio $\omega_{\xi\eta}(0)/\dot{\psi}_0$, as indicated in the figure.

PROBLEMS

6.1 A truck carries a board leaning against the cab at an angle β, as shown in Fig. 6.13. If the coefficient of friction between the truck floor and the board is $\mu = 0.3$, determine the safe acceleration of the truck to prevent the board from sliding and the angle β to prevent the board from tipping.

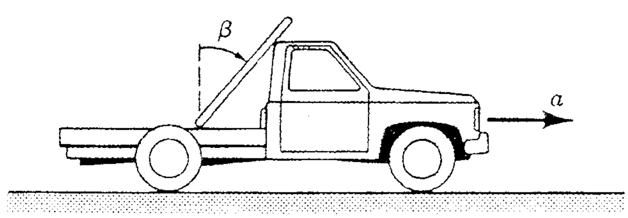

FIGURE 6.13

6.2 Upon applying the brakes, the automobile shown in Fig. 6.14 was observed to decelerate uniformly at 6 m/s^2. Assume that all four wheels are locked, and determine the reactions on the wheels, as well as the coefficient of friction between the wheels and the road.

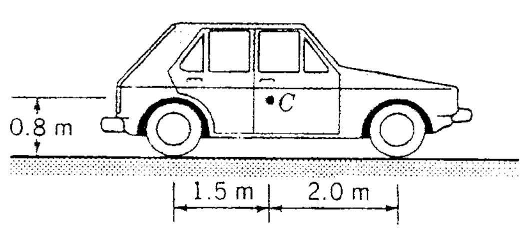

FIGURE 6.14

6.3 A monorail system consists of a car suspended on a horizontal cable (Fig. 6.15). The car is driven by one of the wheels. If the coefficient of friction between the driving wheel and the cable is $\mu = 0.5$, determine the following: (1) which wheel should be the driving wheel and (2) the maximum acceleration of the car.

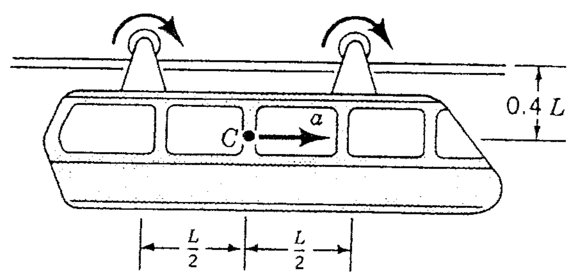

FIGURE 6.15

6.4 A uniform bar hinged at point 0 is at rest in the position shown (Fig. 6.16). A force F strikes horizontally at a point P a distance h from 0. Determine h so that the horizontal reaction at 0 is zero. Note that such a point is called the center of percussion.

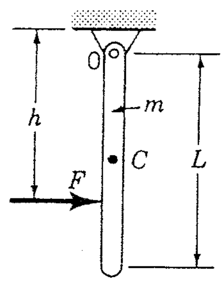

FIGURE 6.16

6.5 A uniform rectangular door hangs at an angle α with respect to the vertical (Fig. 6.17). If disturbed from equilibrium, the door will oscillate. Derive the equation of motion and then assume small motions and calculate the natural frequency of oscillation.

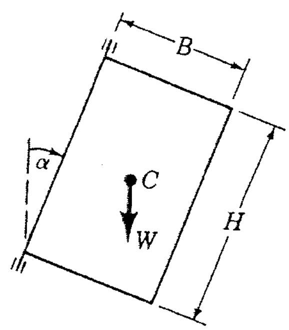

FIGURE 6.17

6.6 A pendulum consists of a massless rigid rod of length L and a thin disk of mass m and radius R (Fig. 6.18). Assume that the pendulum is released from rest in the position $\theta = \pi/2$ and calculate the reactions at 0 when the pendulum reaches an arbitrary angle θ. Derive the differential equation of motion, linearize the equation by assuming small oscillations about $\theta = 0$, and calculate the natural frequency.

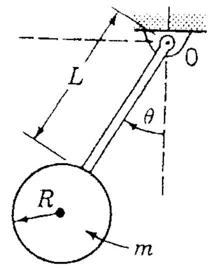

FIGURE 6.18

6.7 A uniform bar of mass m and length $\sqrt{2}R$ slides inside a smooth cylindrical surface (Fig. 6.19). Derive the equation of motion when the bar is in the position shown, and determine the forces exerted by the surface on the bar.

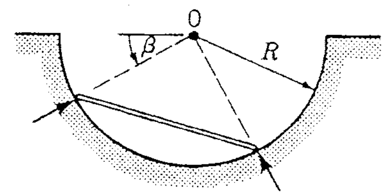

FIGURE 6.19

6.8 A thin disk of mass m and radius R is at rest at the end of a ledge (Fig. 6.20) when nudged slightly to the left. Assume that the friction is such that the disk rotates initially about 0 and then leaves the ledge. Find the angle θ, the angular velocity ω, and the angular acceleration α when the disk leaves the ledge.

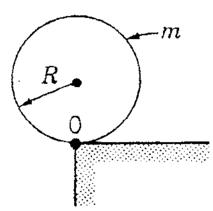

FIGURE 6.20

6.9 A thin uniform bar of mass m and length L is suspended by a string at one end, as shown in Fig. 6.21, when the string is cut. Initially the bar rotates about 0, and then it begins to slip. If the coefficient of friction between the corner of the ledge and the bar is μ, determine the angle between the bar and the horizontal when slip begins.

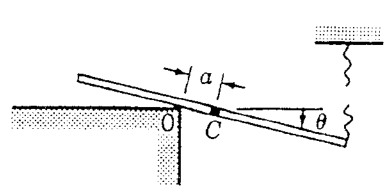

FIGURE 6.21

6.10 A thin disk of mass m and radius r rolls without slip inside a rough cylindrical surface of radius R, as shown in Fig. 6.22. Derive the differential equation for the angular motion θ of the mass center C in two ways: (1) by writing two force equations and one moment equation about C and then eliminating the constraint forces at the point of contact 0 and (2) by writing a moment equation about the point of contact. Compare results and draw conclusions. Then, linearize the differential equation by assuming small oscillations about $\theta = 0$, and calculate the natural frequency.

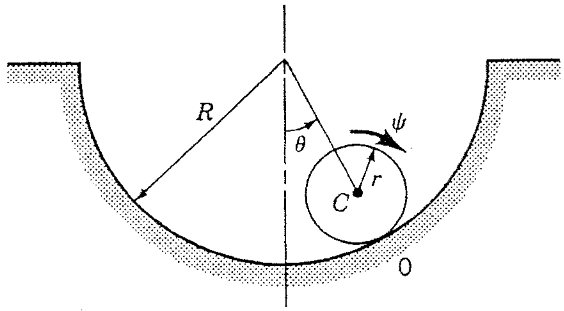

FIGURE 6.22

6.11 The uniform thin bar shown in Fig. 6.23 is initially at rest in the vertical position on a rough horizontal surface when it begins to fall under gravity. When the angle between the bar and the surface reaches 60°, the bar begins to slip. Determine the coefficient of friction between the bar and the surface.

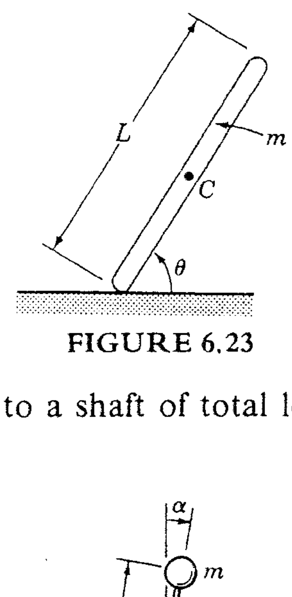

FIGURE 6.23

6.12 A dumbbell is attached to a shaft of total length L, as shown in Fig. 6.24.

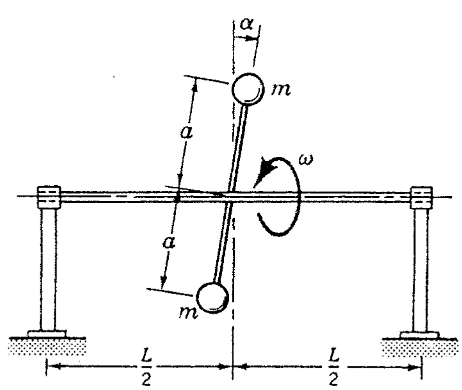

FIGURE 6.24

Determine the bearing reactions if the system rotates with the uniform angular velocity ω about an horizontal axis passing through the bearings.

6.13 A cylinder of total mass m and radius R is attached to a shaft of total length L, as shown in Fig. 6.25. Determine the bearing reactions if the system rotates with the angular velocity $\omega(t)$ about an horizontal axis passing through the bearings.

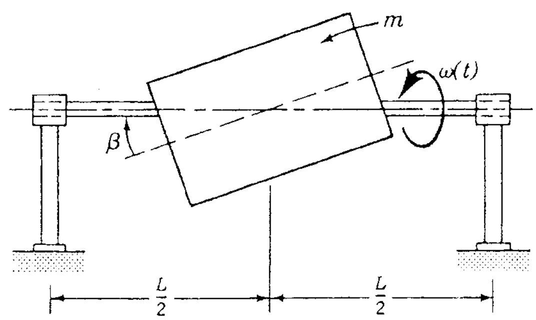

FIGURE 6.25

6.14 Figure 6.26 shows a simplified model of an airfoil section regarded as a rigid body of mass m and mass moment of inertia I_E about point E (representing the elastic axis of the aircraft wing). Point C denotes the center of mass of the section, and the distance between E and C is denoted by s. The elasticity of the structure is simulated by a spring k_x restraining the vertical motion x (known as plunge) and by a torsional spring k_θ restraining the rotational motion θ (known as pitch). Derive the equations of motion under the assumption that the angle θ is small.

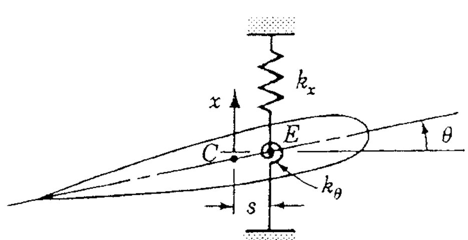

FIGURE 6.26

6.15 The system depicted in Fig. 6.27 consists of two uniform rigid links of masses m_1 and m_2 and lengths L_1 and L_2. Use Eqs. (6.67) and derive the system

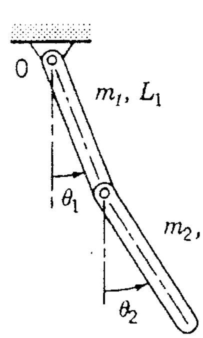

FIGURE 6.27

equations of motion. Then, eliminate the constraint forces, and reduce the equations to two equations in terms of θ_1 and θ_2.

6.16 The system shown in Fig. 6.28 consists of two identical uniform rigid links of mass m and length L, a mass M, and two springs k_1 and k_2. The pulley at point B is massless. Use Eqs. (6.67) and derive the system equations of motion. Then, eliminate the constraint forces at A, and reduce the equations to two equations in terms of x and θ. Assume that $x=\theta=0$ when the springs are unstretched.

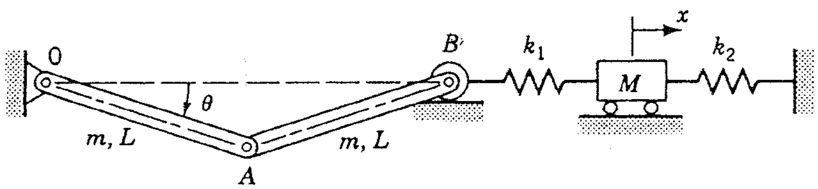

FIGURE 6.28

6.17 A symmetric communications satellite is designed to spin about the symmetry axis at 5π rad/s. The satellite has the form of a drum of radius 2 m, and its moments of inertia about the symmetry axis and a transverse axis are $2000\ \mathrm{kg\cdot m^2}$ and $1600\ \mathrm{kg\cdot m^2}$, respectively. During deployment, the satellite is hit accidentally by an impulsive force of magnitude 300 N·s, where the force is parallel to the symmetry axis and at a distance of 0.32 m from the rim. Determine the subsequent rotational motion of the satellite, and give it a geometric interpretation. *Hints:* (1) The impulsive force imparts an initial angular velocity to the satellite, and (2) at the termination of the impulse, the satellite is torque-free once again.

6.18 A motorcycle is rounding a curve of radius R with the velocity v. Calculate the torque necessary to overcome the gyroscopic effect of the wheels, thus permitting the motorcycle to round the curve. How is this torque produced?

6.19 A single-engine turboprop aircraft has a propeller of moment of inertia I_p rotating clockwise (as seen by the pilot) with the angular velocity ω_p and an engine rotor of moment of inertia I_e rotating counterclockwise with the angular velocity ω_e. The pilot has the elevator and rudder set to make a right turn of radius R at a speed v. If the pilot did not take into consideration the gyroscopic effect in attempting to make the turn, determine whether the nose of the aircraft tends to rise or fall.

6.20 Projectiles are stabilized in flight by means of a system of spiral grooves cut in the gun barrel in a process known as rifling. As a result, the projectile leaves the gun barrel with a given spin velocity Ω_ζ. If the spin axis deviates from the velocity vector $\mathbf{v}$ by a given angle α, then the drag force $\mathbf{D}$ acting at the center of pressure P, which lies ahead of the mass center C at a distance h, will cause the projectile to wobble about the vector $\mathbf{v}$ at the constant angular rate ω (Fig. 6.29). Determine the minimum spin Ω_ζ necessary to prevent the projectile from tumbling. Neglect gravity effects and assume that the pro-

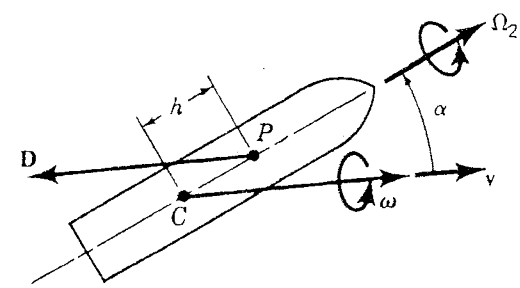

FIGURE 6.29

jectile is a rigid cylinder with centroidal mass moments of inertia I_S and I_T about the symmetry and a transverse axes, respectively.

6.21 The shaft of the gyroscope shown in Fig. 6.10 is precessing steadily when struck by the impulsive force $\mathbf{F}(t)=\hat{F}\delta(t)\mathbf{i}$ at a point a distance $0.8L$ from the pivot 0. Determine the subsequent motion.