Lyapunov Stability

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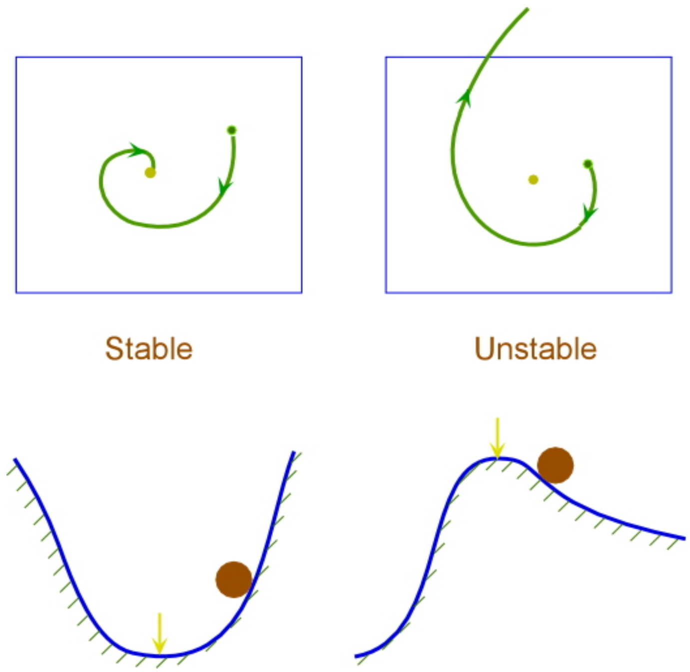
Lyapunov Stability

- If a system is initially in an equilibrium, it remains in the same state thereafter.
- Lyapunov stability is concerned with the behavior of the trajectories of a system when its initial state is near an equilibrium.



- We talk about the stability of an equilibrium.
- There are several notions of stability. We only focus on
  - Stability
  - Instabilility (i.e. absence of stability)
  - Asymptotic stability
  - Exponential stability

Object of Study (I)

- Object of study is the differential equation

$$\boxed{\begin{array}{l} \dot{\mathbf{x}}(t) = f[t, \mathbf{x}(t)] \qquad t \geq 0 \\ \mathbf{x}(t) \in \mathbf{R}^n. \\ f: \ \mathbf{R}_+ \times \mathbf{R}^n \rightarrow \mathbf{R}^n \text{ is continuous} \end{array}} \qquad (\Sigma)$$

- Assume Σ has a unique solution corresponding to each i.c., (true for example when f is globally Lipschitz) Check first!!
- Let $s(t, t_0, x_0)$ ⟺ solution of Σ

evaluated at t initial time Initial conditions

corresponding to $\mathbf{x}(t_0)=\mathbf{x}_0$ evaluated at t.

- s satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t} s(t, t_0, \mathbf{x}_0) = f[t, s(t, t_0, \mathbf{x}_0)] \qquad \forall t \geq t_0$$

$$s(t_0, t_0, \mathbf{x}_0) = \mathbf{x}_0, \qquad \forall \mathbf{x}_0 \in \mathbf{R}^n$$

Object of Study (II)

- Recall: a vector $\mathbf{x}_0 \in \mathbf{R}^n$ is an equilibrium of Σ if

  $f(t, \mathbf{x}_0)=0 \quad \forall\, t \geq 0 \;\Rightarrow\; s(t, t_0, \mathbf{x}_0)=\mathbf{x}_0 \quad \forall\, t \geq t_0 \geq 0$

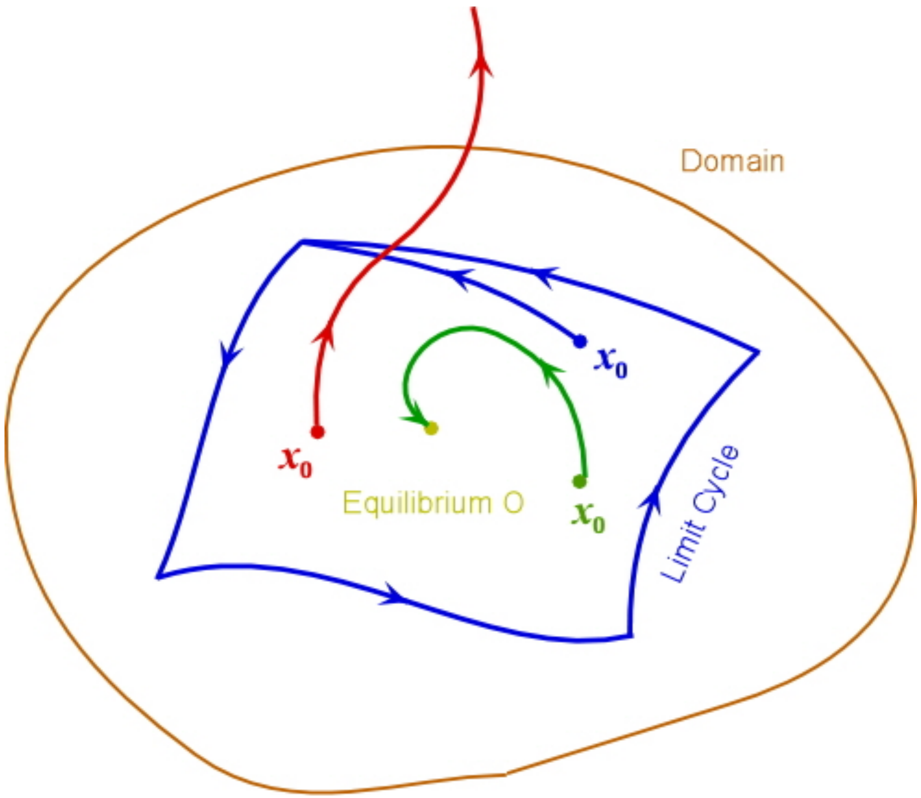
- WLG, assume that $\mathbf{x}_0=0$ is an equilibrium of Σ. Hence ,

  $f(t, 0)=0 \quad \forall\, t \geq 0 \;\Rightarrow\; s(t, t_0, \mathbf{0})=\mathbf{0} \quad \forall\, t \geq t_0$

- If $\mathbf{x}_0 \neq 0$ is an equilibrium of Σ, define

$$
\begin{aligned}
& z \equiv x - x_0 \\
& \Rightarrow \dot{z}(t) = \dot{x} = f(t,x) \\
& \qquad\qquad\quad = f(t, z + x_0) \\
& \qquad\qquad\quad \equiv f_1(t,z)
\end{aligned}
$$

Object of Study (III)



- Lyapunov theory is concerned with the behavior of the solution $s(t, t_0, \mathbf{x}_0)=\mathbf{x}_0$ when $\mathbf{x}_0 \neq 0$ but

  $\mathbf{x}_0$ is "close" to 0: local behavior

  $\mathbf{x}_0$ is "far" from 0: global behavior

- We talk of

  local stability

  global stability

Stability Definitions (I)

- **Definition**: The equilibrium 0 is
  - Stable if

    For each $\varepsilon > 0$ and each $t_0 \in \mathbf{R}_+$

    $\exists$ a $\delta = \delta(\varepsilon, t_0)$ $s.t.$

    $$\| \mathbf{x}_0 \| < \delta(\varepsilon, t_0) \quad \Rightarrow \quad \| s(t, t_0, x_0) \| < \varepsilon \quad \forall t \geq t_0$$

  - Uniformly Stable if

    For each $\varepsilon > 0$

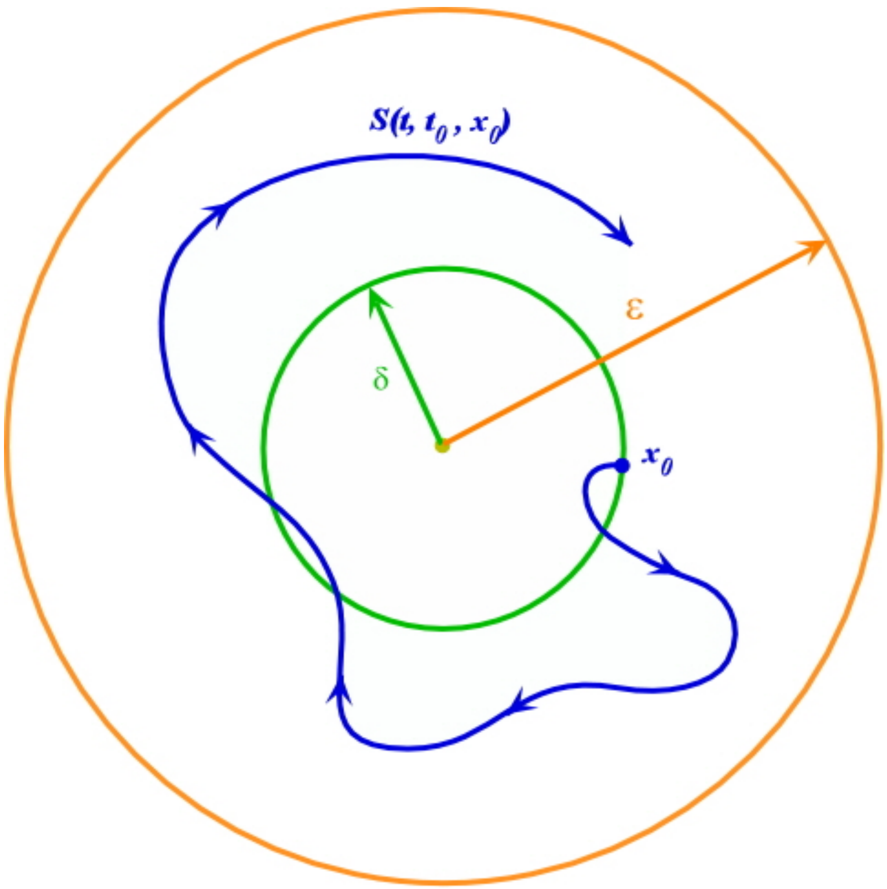
    $\exists$ a $\delta = \delta(\varepsilon)$ $s.t.$

    $$\| \mathbf{x}_0 \| < \delta(\varepsilon), \quad t_0 \geq 0 \quad \Rightarrow \quad \| s(t, t_0, x_0) \| < \varepsilon \quad \forall t \geq t_0$$

  - Unstable if it is NOT stable.

Stability Definitions (II)

- The ε-δ requirement for stability is a challenge-answer problem.
  - To show that the equilibrium is stable, then,



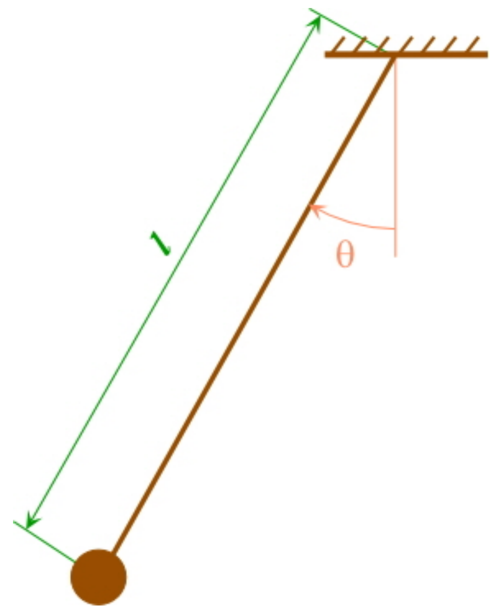
for every ε that a challenger may care to designate, we must produce a value δ (possibly $\delta(\varepsilon)$) s.t. a trajectory starting from a δ-neighborhood of the equilibrium will never leave the ε-neighborhood.

Remarks

1. ||•|| is any norm on $\mathbf{R}^n$.
   All norms on $\mathbf{R}^n$ are topologically equivalent.
2. If system $\sum$ is autonomous
   (i.e. f doesn't depend explicitly on t, $\dot{\mathbf{x}} = f(\mathbf{x})$),
   then there is no distinction between stability and uniform stability, though in general, there is a difference between the two notions of stability (see example 18 in V. book).

Example: Motion of a Pendulum (I)

$$\boxed{\ddot{\theta} + \frac{g}{l}\sin\theta = 0}$$



l : length of pendulum

θ : angle of pendulum measured from a vertical line

g : gravitational acceleration

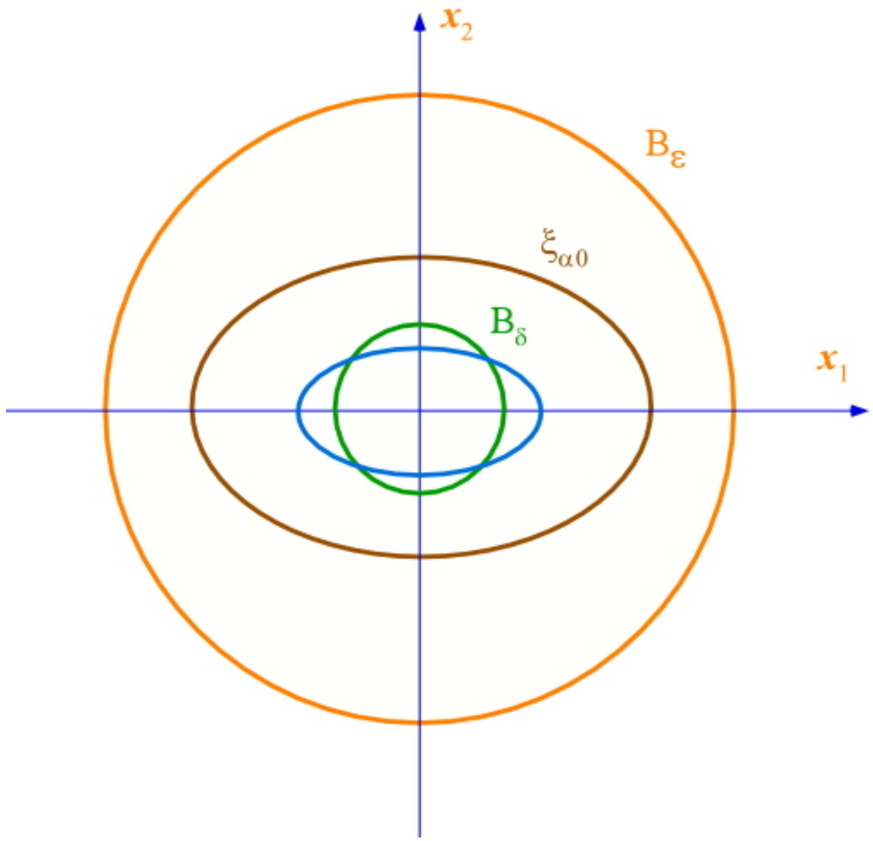
Let $x_1 \equiv \theta, \quad x_2 \equiv \dot{\theta}$ $\quad \Rightarrow \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\dfrac{g}{l}\sin x_1 \end{cases}$

Trajectories of this system are described by:

$$\frac{x_2^{\,2}}{2} - \frac{g}{l}\cos x_1 = \frac{x_{20}^{\,2}}{2} - \frac{g}{l}\cos x_{10} \equiv a_0 \qquad (*)$$

Example: Motion of a Pendulum (II)

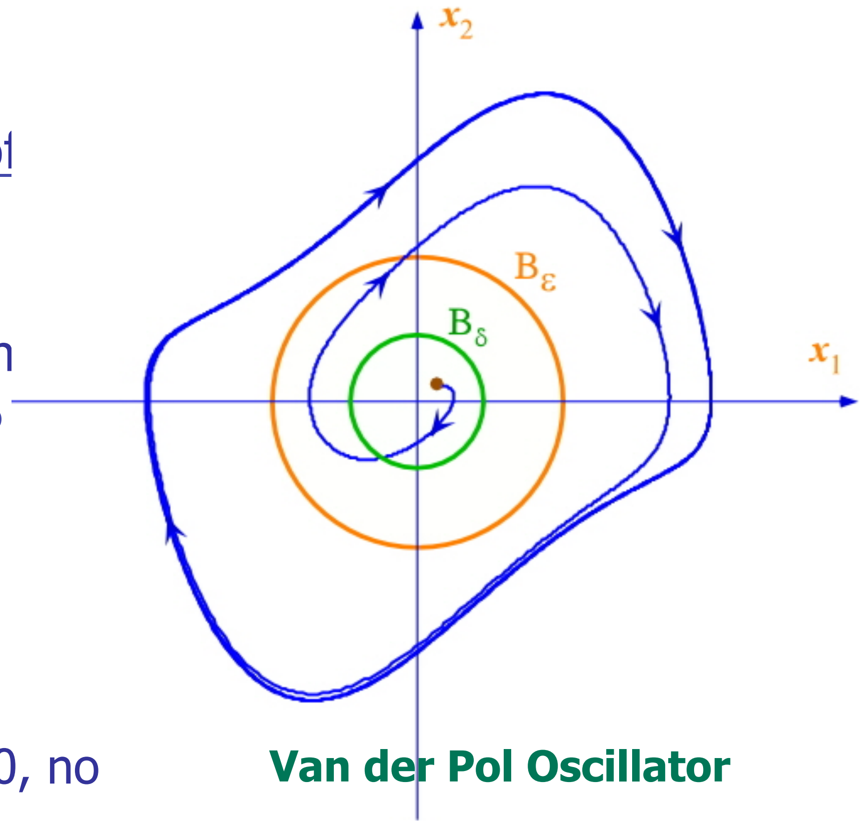
- ε>0 given.
- Possible to choose a number a_0>0 s.t. the curve (*) lies entirely within the ball $B\varepsilon$.
- Now choose a δ>0 s.t. the ball **$B\delta$** lies entirely within the curve.
- $\Rightarrow$ definition of stability satisfied
- Since this procedure can be carried out for any ε>0
- $\Rightarrow$ 0 is a stable equilibrium.



Further Remarks

- Instability is basically the absence of stability.

  → not necessarily a situation where some trajectory of the system "blows up" in the sense that $\|\mathbf{x}\| \to \infty$ as $t \to \infty$, although this is one way instability can occur.



Van der Pol Oscillator

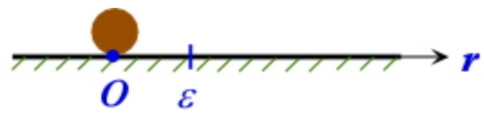
$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1 + (1 - x_1^{\ 2}) x_2 \end{cases}$$

Definition of instability

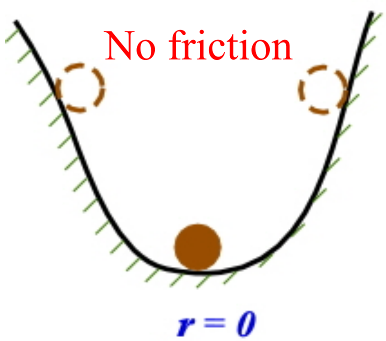
O is an unstable equilibrium if for some $\varepsilon > 0$, no δ can be found s.t. (definition) holds.

⇔ There is a ball B_ε s.t. for every $\delta > 0$, no matter how small, there is a nonzero initial state $\mathbf{x}(t_0)$ in B_δ s.t. the corresponding trajectory eventually leaves B_ε.

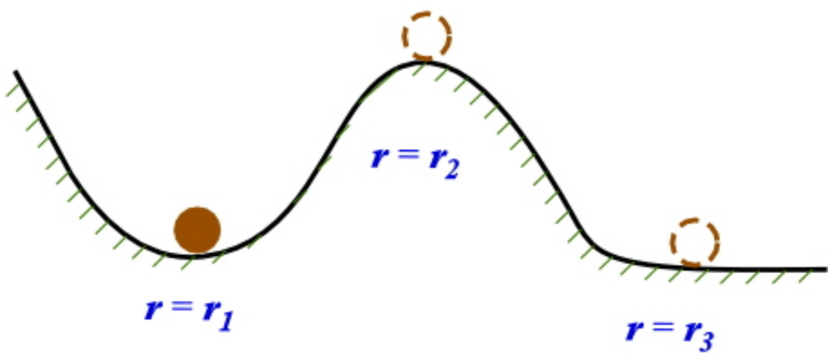
Examples



$$\mathbf{x} = \begin{bmatrix} r = 0 \\ \dot{r} = 0 \end{bmatrix} \text{ is stable;} \qquad \mathbf{x} = \begin{bmatrix} r_0 \\ 0 \end{bmatrix} \text{ is stable.}$$



$\mathbf{x} = 0$ is stable



$$\mathbf{x} = \begin{bmatrix} r_1 \\ 0 \end{bmatrix} \text{ is asymptotically stable;}$$

$$\mathbf{x} = \begin{bmatrix} r_2 \\ 0 \end{bmatrix} \text{ is unstable;}$$

$$\mathbf{x} = \begin{bmatrix} r_3 \\ 0 \end{bmatrix} \text{ is stable.}$$

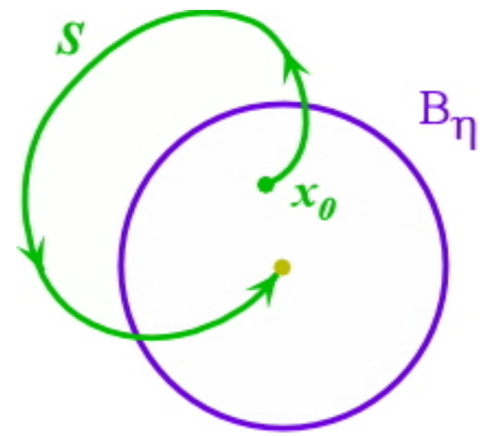
There is frictional force proportional to $\dot{r}$

More Definitions - Attractivity (I)

Definition:

The equilibrium 0 is:

- Attractive if :



for each $t_0 \in \mathbf{R}^+$, there is an $\eta(t_0) > 0$ s.t.

$$\|\mathbf{x}_0\| < \eta(t_0) \;\Rightarrow\; s(t_0 + t, t_0, \mathbf{x}_0) \to 0 \text{ as } t \to \infty$$

- Uniformly attractive if :

there is a number η>0 s.t.

$$\|\mathbf{x}_0\| < \eta \;\Rightarrow\; s(t_0 + t, t_0, \mathbf{x}_0) \to 0 \text{ as } t \to \infty \text{ uniformly in } \mathbf{x}_0, t_0$$

- Attractivity means:

at each initial time $t_0 \in \mathbf{R}^+$, every solution trajectory starting sufficiently close to 0 actually approaches 0 as $t_0 + t \to \infty$.

More Definitions - Attractivity (II)

- Attractivity:

  no uniformity
  - Size of "ball of attraction" $\mathbf{B}_\eta$ is dependent on t_0.
  - fix t_0, vary $\mathbf{x}_0$, approach 0 at different rates.

- Uniform attractivity:

  uniformity
  - Size of "ball of attraction" $\mathbf{B}_\eta$ is independent of t_0.
  - Solution trajectories starting inside $\mathbf{B}_\eta$, all approach 0 at a uniform rate.

Note:

- Definition of uniform attractivity

$$\boxed{\|\mathbf{x}_0\| < \eta,\ t_0 \geq 0 \quad \Rightarrow \quad s(t_0 + t, t_0, \mathbf{x}_0) \to 0 \ \text{ as } \ t \to \infty \text{ uniformly in } \mathbf{x}_0, t_0}$$

$\Leftrightarrow$

- $$\boxed{\begin{array}{l} \text{for each } \varepsilon > 0, \exists \ \text{ a } \ T = T(\varepsilon) \text{ s.t.} \\ \|\mathbf{x}_0\| < \eta, \quad t_0 \geq 0 \quad \Rightarrow \quad \|s(t_0 + t, t_0, \mathbf{x}_0)\| < \eta \ , \ \ \forall \ t \geq T(\varepsilon) \end{array}}$$

More Definitions - Asymptotical Stability and Exponential Stability

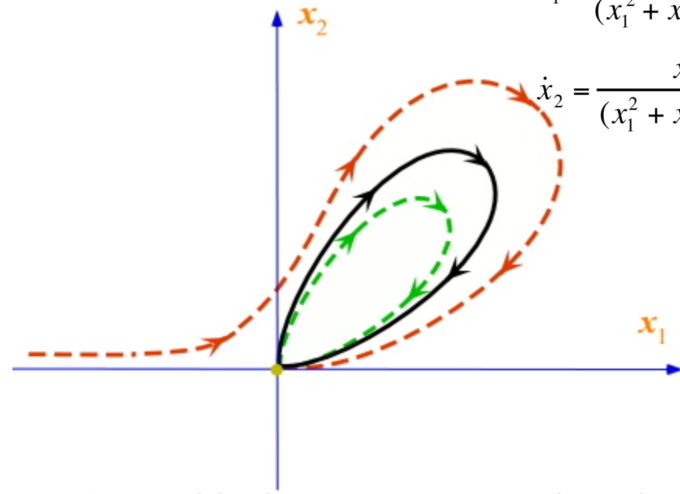
$$\dot{x}_1 = \frac{x_1^2(x_2 - x_1) + x_2^5}{(x_1^2 + x_2^2)\left[1 + (x_1^2 + x_2^2)^2\right]}$$

$$\dot{x}_2 = \frac{x_{2\,1}^2(x_2 - 2x_1)}{(x_1^2 + x_2^2)\left[1 + (x_1^2 + x_2^2)^2\right]}$$

Definition:

The equilibrium 0 is:

- Asymptotically stable if :
  it is stable and attractive.
- Uniformly asymptotically stable (u.a.s.) if:
  it is uniformly stable and uniformly attractive.



An equilibrium can be attractive without being stable.

Definition:

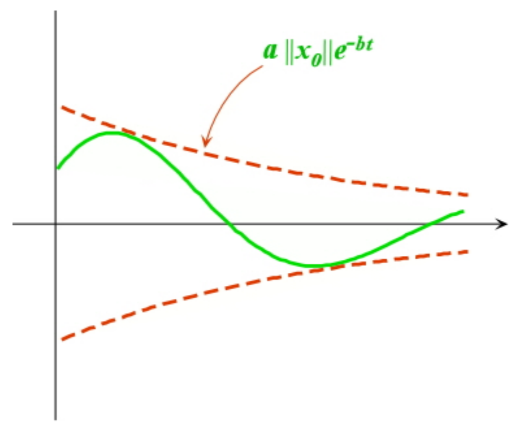
The equilibrium 0 is

Exponentially stable if :

$$\exists \text{ constants } r, a, b > 0 \text{ s.t.}$$

$$\| s(t_0 + t, t_0, \mathbf{x}_0) \| < a \| \mathbf{x}_0 \| \; e^{-bt},$$

$$\forall \; t, t_0 \geq 0, \forall \mathbf{x}_0 \in B_r$$



→ **Exponential stability is a stronger property than uniform asymptotic stability.**

Remarks and Definition of Global Stability

Remarks:

All concepts of stability introduced thus far are local in nature. Following definition pertains to the global behavior of solution trajectories.

Definition:

The equilibrium O is

– Globally uniformly asymptotically stable (g.u.a.s) if

(i) it is uniformly stable, and

(ii) for each pair of positive numbers M, ε with M arbitrarily large and ε arbitrarily small, a finite number $T=T(M,\varepsilon)$ s.t.

$$\| \mathbf{x}_0 \| < M, \quad t \geq 0 \quad \Rightarrow \quad \| s(t_0 + t, t_0, \mathbf{x}_0) \| < \varepsilon \,, \;\; \forall \; t \geq T(M,\varepsilon)$$

– Globally exponentially stable (g.e.s) if

$$\exists \;\; \text{constants } a,b > 0 \;\; \text{s.t.}$$

$$\| s(t_0 + t, t_0, \mathbf{x}_0) \| < a \| \mathbf{x}_0 \| \; e^{-bt}, \;\; \forall \; t, t_0 \geq 0, \forall \mathbf{x}_0 \in \mathbf{R}^n$$

Periodic & Autonomous Systems

- System $\sum$ is periodic with period T if

$$f(t+T,\mathbf{x}) = f(t,\mathbf{x}), \;\; \forall t \geq 0, \;\; \forall \mathbf{x} \in \mathbf{R}^n$$

- If system $\sum$ is autonomous, i.e., if f does not depend explicitly on t, then we can think of it as periodic with arbitrary period.

Theorem:

Suppose system $\sum$ is periodic. Then the equilibrium O is uniformly stable iff it is stable (obvious for autonomous system).

Theorem:

Suppose system $\sum$ is periodic. Then the equilibrium O is uniformly asymptotically stable iff it is asymptotically stable.

Class K & Class L Functions

To simplify considerably the statement and proofs of subsequent stability theorems, we recast the various stability definitions in terms of so-called functions of class K & class L.

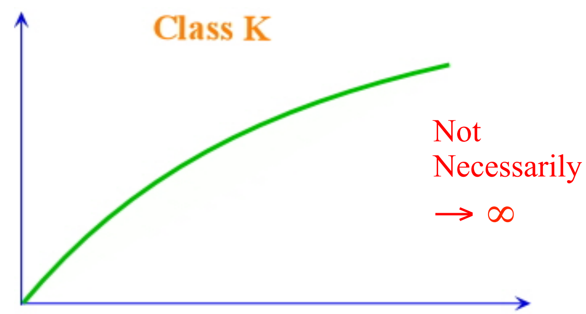
Definition: A function

$\Phi : \mathbf{R}^{+} \rightarrow \mathbf{R}^{+}$ is of class K if it is

– continuous

– strictly increasing

– $\Phi(0)=0$.



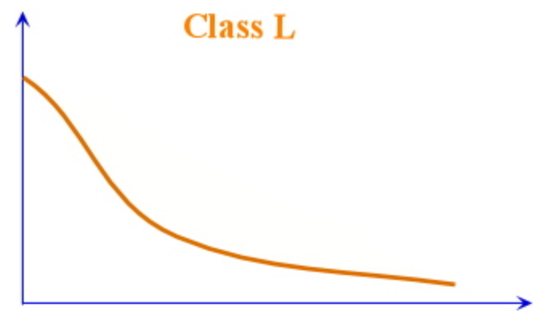
It is of class L if it is

– continuous

– strictly decreasing

– $\Phi(0)<\infty$

– $\Phi(r) \rightarrow 0$ as $r \rightarrow \infty$



Lemma: let $\alpha_1(\bullet)$ and $\alpha_1(\bullet)$ be of class K $\Rightarrow$ - α_1^{-1} is of class K;

- $\alpha_1 \bullet \alpha_2$ is of class K.

Theorem

Theorem:

The equilibrium O of $\sum$ is

- Stable iff

  for each $t_0 \in \mathbf{R}^+$, $\exists$ a number $d(t_0) > 0$

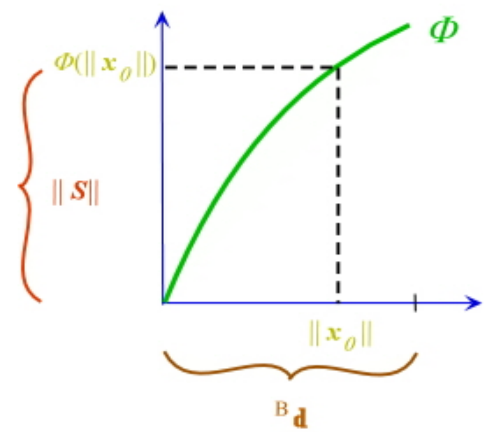
  and a function Φ_{t_0} of class K s.t.

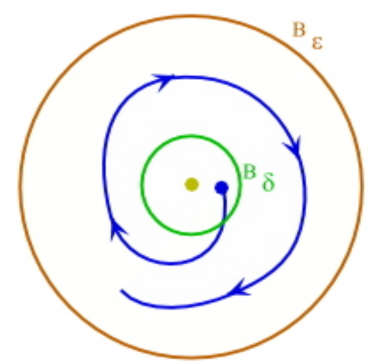
  $\|s(t, t_0, \mathbf{x}_0)\| \leq \Phi_{t_0}(\|\mathbf{x}_0\|), \forall t \geq t_0, \forall \mathbf{x}_0 \in B_{d(t_0)}$

- Uniformly Stable iff

  $\exists$ a number $d > 0$ and a function Φ of class K s.t.

  $\|s(t, t_0, \mathbf{x}_0)\| \leq \Phi(\|\mathbf{x}_0\|), \forall t \geq t_0 \geq 0, \forall \mathbf{x}_0 \in B_d$





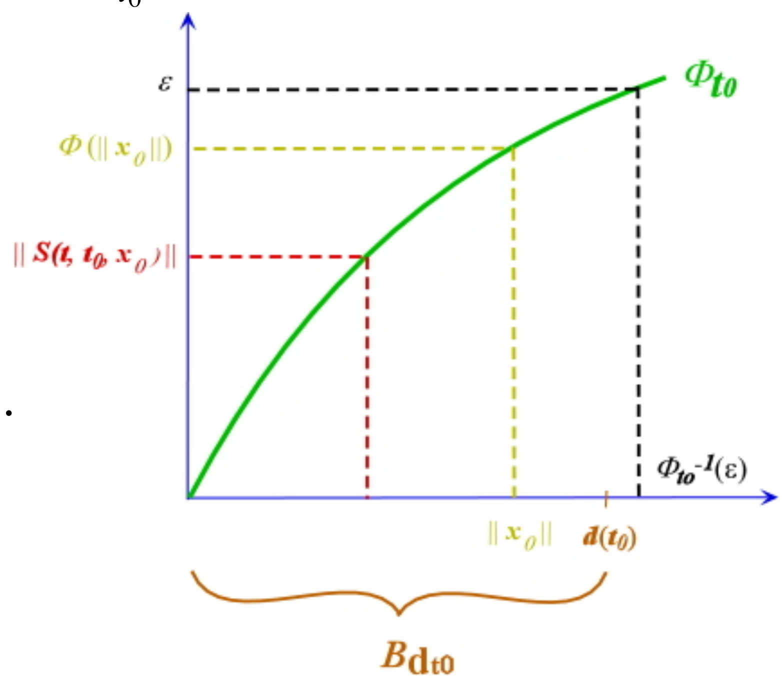
Proof (Stability)

→ Suppose there is a class K function Φ_{t_0} s.t.

$$\|s(t,t_0,\mathbf{x}_0)\| \le \Phi_{t_0}(\|\mathbf{x}_0\|)$$

$$\forall t \ge t_0 \ge 0,$$

$$\forall \mathbf{x}_0 \in B_{d_{t_0}} \text{ (i.e. } \|\mathbf{x}_0\| \le d(t_0))$$



Given $\varepsilon > 0$, let $\delta = \min\{d(t_0), \Phi_{t_0}^{-1}(\varepsilon)\}$.

Then for

$$\|\mathbf{x}_0\| < \delta(\varepsilon, t_0) (\Rightarrow \|\mathbf{x}_0\| \in B_{d_{t_0}}),$$

we have

$$\|s(t,t_0,\mathbf{x}_0)\| \le \Phi_{t_0}(\|\mathbf{x}_0\|) \le \Phi_{t_0}(\delta(\varepsilon_0, t_0))$$

$$\le \Phi_{t_0}(\Phi_{t_0}^{-1}(\varepsilon)) = \varepsilon$$

See book