RICE

Department of Mechanical Engineering
MECH 411/501 Fall 17
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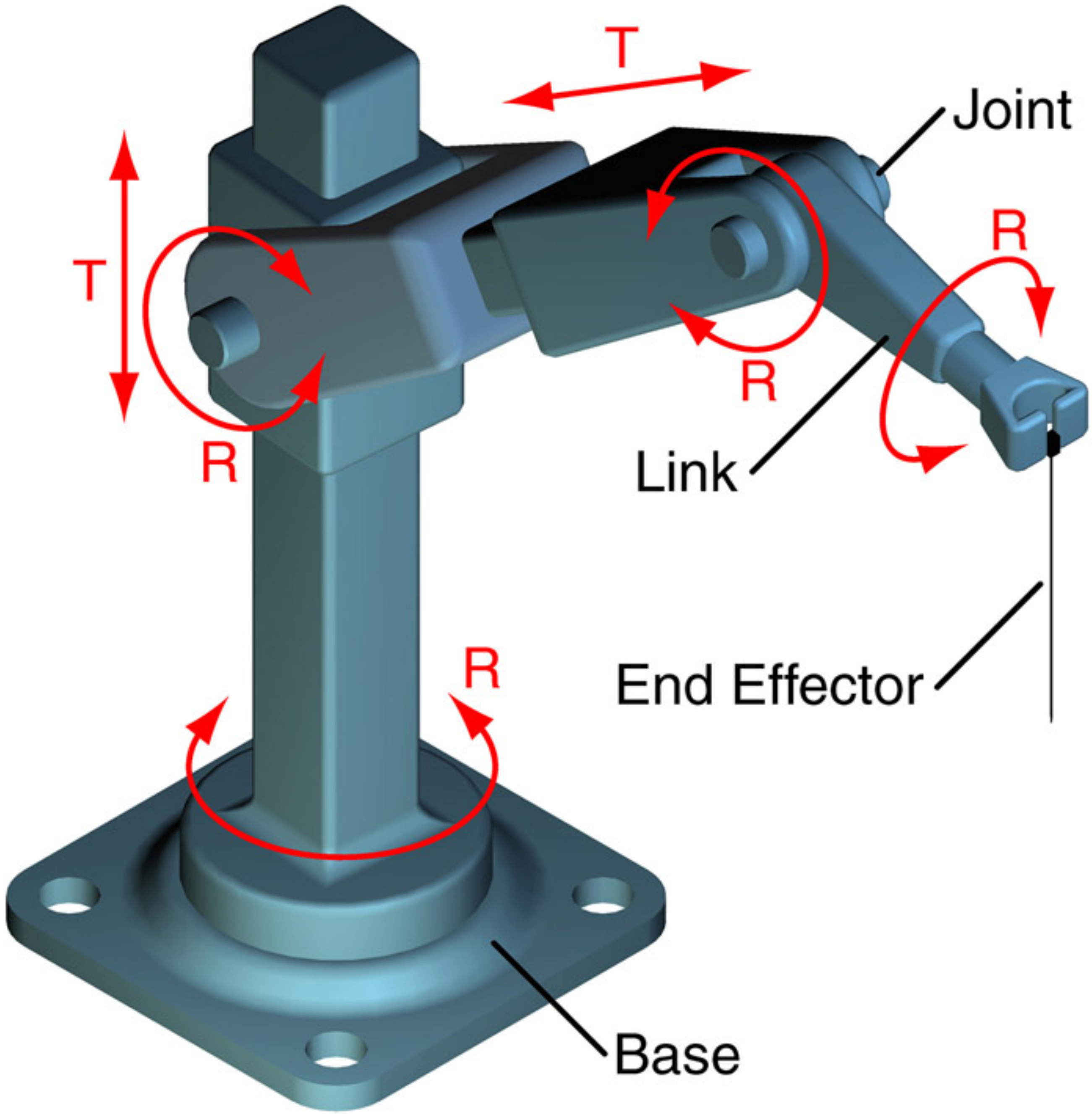
Dynamics & Control of Mechanical Systems

Kinematics


T
Joint
T
R
R
R
Link
R
R
End Effector
Base

Kinematics

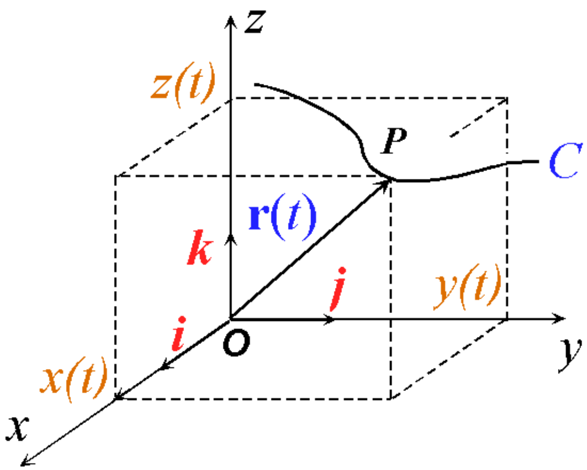
"The study of the motion of a body without regard to the forces causing the motion".

- We describe motion using a reference frame.
- An inertial reference frame may be thought of as a frame fixed in some ideal space relative to which motion of stars, planets, and so on, can be described.
- In most engineering applications:

  Reference frame fixed to the surface of earth
  - Technically not an inertial frame
  - However acceptable for engineering purposes.
- Sometimes it's convenient to describe motion in moving reference frame.

1. Fixed Cartesian Reference Frame

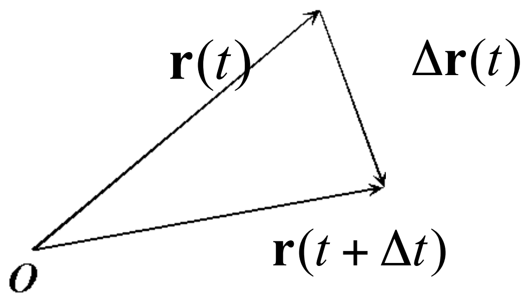
Consider motion of a particle P along curve ***C*** in 3-D space.



- x-y-z frame: Fixed origin O.
  x-y-z assumed to maintain fixed orientation.
  i, **j**, **k** unity vectors, constant.
- Position Vector:

$$\boxed{\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k} \quad [\mathrm{m}] \qquad (\text{SI units})}$$

- Velocity of particle P in [m/s]:



$$\boxed{\mathbf{v}(t) = \lim_{\Delta t \to 0} \frac{\mathbf{r}(t+\Delta t) - \mathbf{r}(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta \mathbf{r}(t)}{\Delta t} = \frac{\mathrm{d}\mathbf{r}(t)}{\mathrm{d}t} = \dot{\mathbf{r}}(t)}$$

$$\boxed{\text{So}: \quad \mathbf{v}(t) = \dot{\mathbf{r}}(t) = \dot{x}(t)\mathbf{i} + \dot{y}(t)\mathbf{j} + \dot{z}(t)\mathbf{k}}$$

- Cartesian components of the velocity vector:

$$\mathbf{v}(t) = \dot{\mathbf{r}}(t) = v_x(t)\mathbf{i} + v_y(t)\mathbf{j} + v_z(t)\mathbf{k}$$
$$v_x(t) = \dot{x}(t)$$
$$v_y(t) = \dot{y}(t)$$
$$v_z(t) = \dot{z}(t)$$

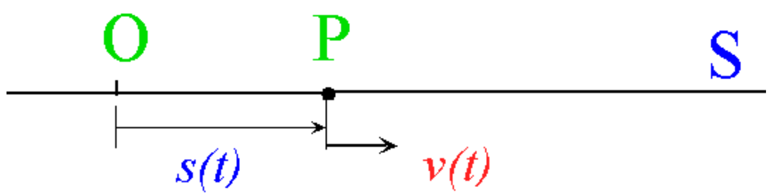
- Note velocity vector v(t) is tangent to the curve C at all times.

Figure12

- Cartesian components of the acceleration vector:

$$\boxed{\begin{gathered}\mathbf{a}(t) = \dot{\mathbf{v}}(t) = \ddot{\mathbf{r}}(t) = a_x(t)\mathbf{i} + a_y(t)\mathbf{j} + a_z(t)\mathbf{k} \\ a_x(t) = \dot{v}_x(t) = \ddot{x}(t) \\ a_y(t) = \dot{v}_y(t) = \ddot{y}(t) \\ a_z(t) = \dot{v}_z(t) = \ddot{z}(t)\end{gathered}}$$

- Example: Rectilinear Motion:



$$\boxed{v(t) = \dot{s}(t), \qquad a(t) = \ddot{s}(t)}$$

$$a = \frac{dv}{dt} = \frac{dv}{ds} \times \frac{ds}{dt} = \frac{dv}{ds} \times v$$

$$\boxed{\Rightarrow a\, ds = v\, dv}$$

$$\int_{s_1}^{s_2} a\, ds = \int_{v_1}^{v_2} v\, dv = \frac{1}{2} v^2 \Big|_{v_1}^{v_2} = \frac{1}{2}(v_2^2 - v_1^2)$$

$$\boxed{\Rightarrow \int_{s_1}^{s_2} a\, ds = \frac{1}{2}(v_2^2 - v_1^2)}$$

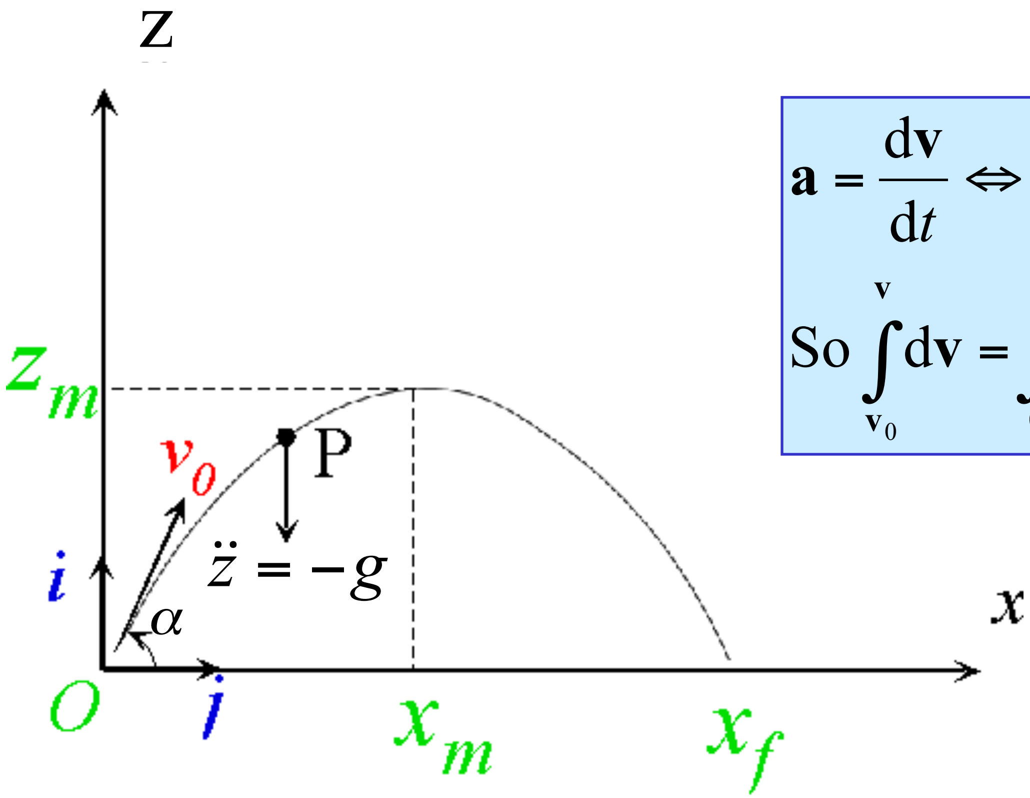
Figure13

- Example: Planar Motion, Trajectories

Particle traveling in the plane x-z with the constant acceleration

$$a = \ddot{z}\mathbf{k} = -g\mathbf{k}, \quad \text{g : acceleration due to gravity.}$$

after being propelled with initial velocity v_0 from the origin O



$$\mathbf{a} = \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} \Leftrightarrow \mathrm{d}\mathbf{v} = \mathbf{a}\,\mathrm{d}t$$

$$\text{So } \int_{\mathbf{v}_0}^{\mathbf{v}} \mathrm{d}\mathbf{v} = \int_0^t \mathbf{a}\,\mathrm{d}t = \int_0^t \mathbf{a}\,\mathrm{d}t = \int_0^t -g\mathbf{k}\,\mathrm{d}t$$

$$\boxed{\mathbf{v} = \mathbf{v}_0 - gt\,\mathbf{k}}$$

Figure14

- $$\mathbf{v} = \frac{d\mathbf{r}}{dt} \Rightarrow \int_0^{\mathbf{r}} d\mathbf{r} = \int_0^t \mathbf{v}\, dt = \int_0^t (\mathbf{v}_0 - gt\,\mathbf{k})\, dt = \left(\mathbf{v}_0 t - \frac{1}{2} g t^2\, \mathbf{k}\right)\Big|_0^t$$

$$\Rightarrow \mathbf{r} = \mathbf{v_0} t - \frac{1}{2} g t^2\, \mathbf{k}$$

- Let $\mathbf{v_0} = \mathrm{v}_0 \cos\alpha \mathbf{i} + \mathrm{v}_0 \sin\alpha \mathbf{k}$

$$x_m = \frac{v_0^2}{g} \sin\alpha \cos\alpha; \quad x_f = 2x_m$$

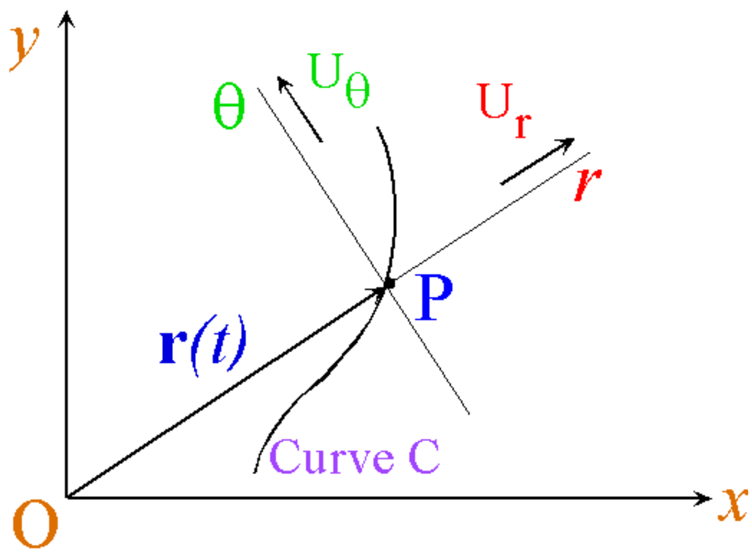
$$z_m = \frac{1}{2} \frac{v_0^2}{g} \sin^2\alpha$$

$$\mathbf{v}_f = v_0 \cos\alpha \mathbf{i} - v_0 \sin\alpha \mathbf{k}$$

- Trajectory is symmetric w.r.t. vertical through $x = x_m$.

2. Planar Curvilinear Coordinates

- Radial & Transverse Coordinates (Polar Coordinates)

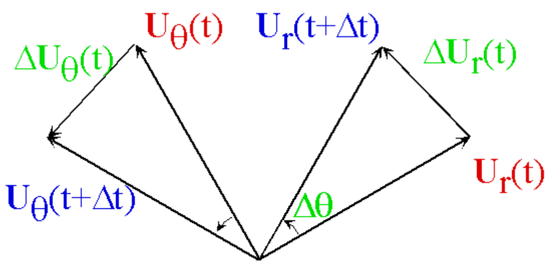


- Particle P travels along Curve C;
- Radial axis r coincides with the direction of the radius vector **r**(t) from O to P;
- Transverse axis θ is perpendicular to Radial axis r

- $U_r(t)$: unit vector in the radial direction;
- $U_\theta(t)$: unit vector in the transverse direction;
- $U_r(t)$ and $U_\theta(t)$ constant magnitude but continuously change direction $\rightarrow$ time varying vector!!

Figure15

- Position vector: $\mathbf{r}(t) = r(t)\mathbf{u}_r(t)$
- Velocity vector: $\mathbf{v}(t) = \dot{\mathbf{r}}(t) = \dot{r}(t)\mathbf{u}_r(t) + r(t)\dot{\mathbf{u}}_r(t)$



$$\dot{\mathbf{u}}_r = \dot{\theta}(t)\mathbf{u}_\theta(t)$$
$$\dot{\mathbf{u}}_\theta = -\dot{\theta}(t)\mathbf{u}_r(t)$$

Where $\dot{\theta}(t)$: angular rate of change of the position vector $\mathbf{r}$(t) as its tip moves along the curve C.

$$\text{So : } \mathbf{v}(t) = \dot{r}(t)\mathbf{u}_r(t) + r(t)\dot{\theta}(t)\mathbf{u}_\theta(t)$$

$$\mathbf{v}(t) = v_r(t)\mathbf{u}_r(t) + v_\theta(t)\mathbf{u}_\theta(t)$$

Where

$v_r(t) = \dot{r}(t)$ ← Radial Component

$v_\theta(t) = r(t)\dot{\theta}(t)$ ← Transverse Component

- Acceleration vector

Recall $$\boxed{\mathbf{v} = \dot{r}\,\mathbf{u}_r + r\dot{\theta}\,\mathbf{u}_\theta}$$

$$\Rightarrow \mathbf{a} = \dot{\mathbf{v}} \quad = \ddot{r}\,\mathbf{u}_r + \dot{r}\dot{\mathbf{u}}_r + \dot{r}\dot{\theta}\,\mathbf{u}_\theta + r\ddot{\theta}\mathbf{u}_\theta + r\dot{\theta}\dot{\mathbf{u}}_\theta$$

$$= (\ddot{r} - r\dot{\theta}^2)\mathbf{u}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\,\mathbf{u}_\theta$$

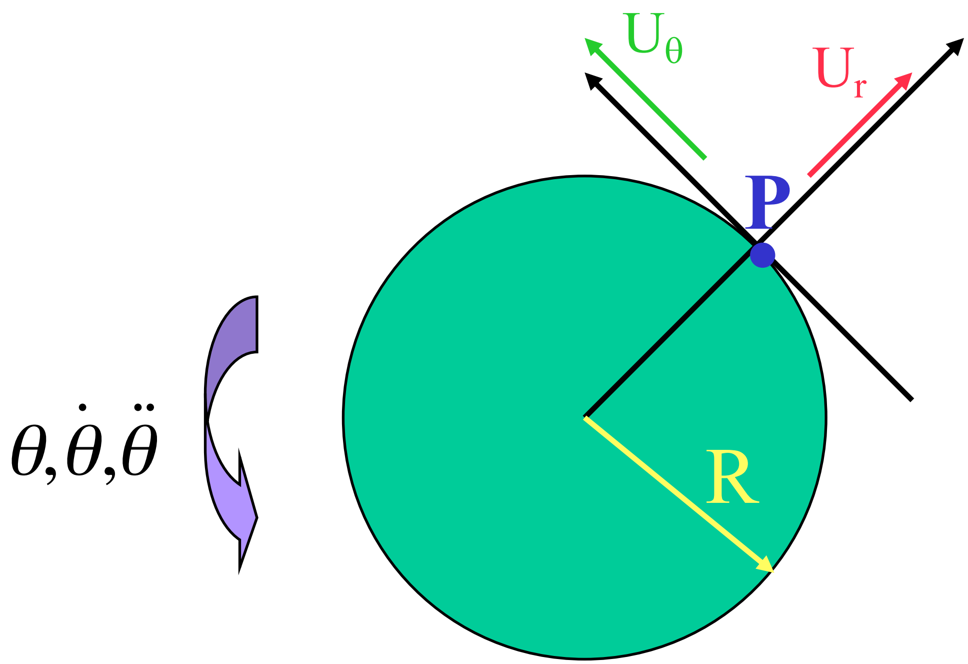
(Note : $\dot{\mathbf{u}}_r = \dot{\theta}\mathbf{u}_\theta\,;\quad \dot{\mathbf{u}}_\theta = -\dot{\theta}\mathbf{u}_r$)

$$\mathbf{a}(t) = a_r(t)\mathbf{u}_r + a_\theta(t)\,\mathbf{u}_\theta$$
$$a_r(t) = \ddot{r}(t) - r(t)\dot{\theta}^2(t)$$
$$a_\theta(t) = r(t)\ddot{\theta}(t) + 2\dot{r}(t)\dot{\theta}(t)$$

$\xrightarrow{\text{or}}$

$$\mathbf{a} = a_r\,\mathbf{u}_r + a_\theta\,\mathbf{u}_\theta$$
$$a_r = \ddot{r} - r(t)\dot{\theta}^2$$
$$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}$$

Example: Particle P moving on a circular path

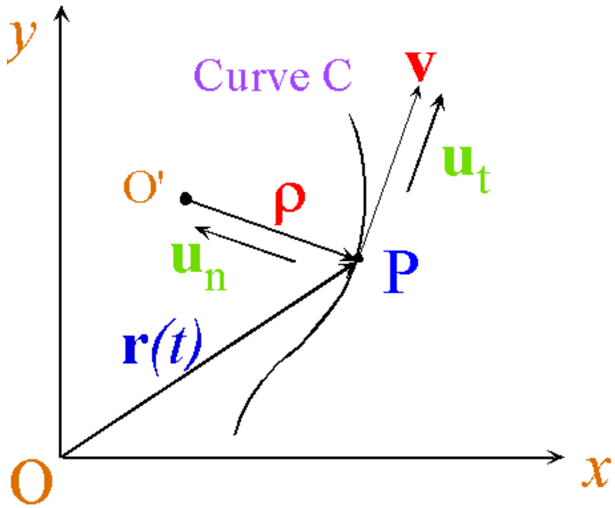


$r_P(t) = ?$

$v_P(t) = ?$

$a_P(t) = ?$

- Tangential and Normal Coordinates



- Tangential & normal coordinates: set of axes tangent & normal to curve C at point P and time t;

- O' : center of curvature of C;

  ρ : radius of curvature corresponding to the instantaneous position of P on curve C

Both O' and ρ change as P moves along curve C !!

- $$\mathbf{v} = \dot{s}\mathbf{u}_t$$

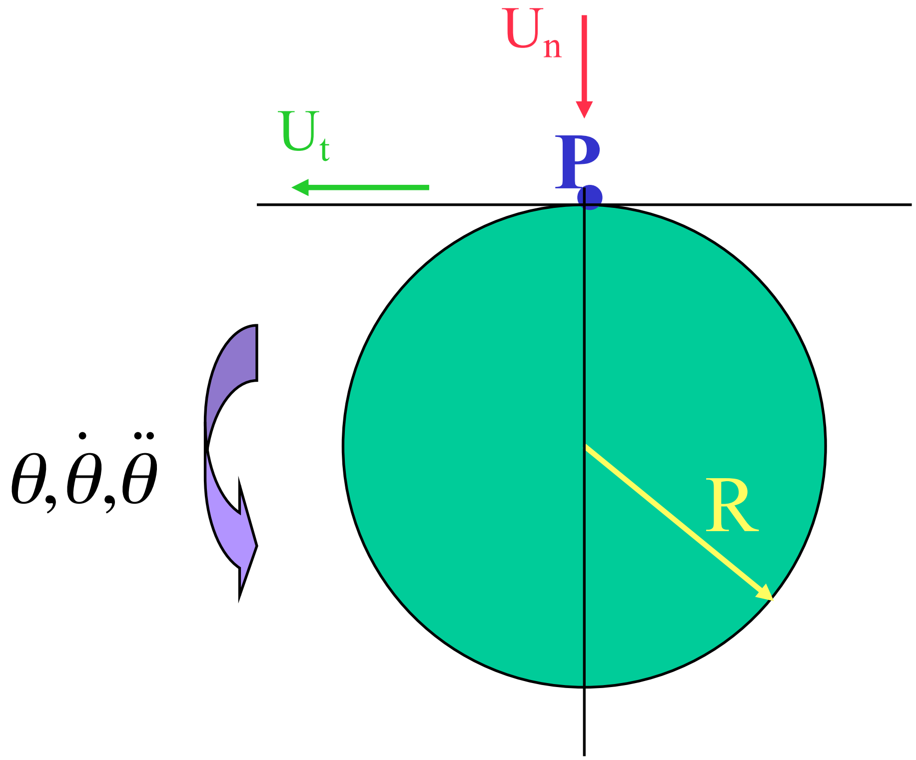
  $$\mathbf{a} = \ddot{s}\mathbf{u}_t + \frac{\dot{s}^2}{\rho}\mathbf{u}_n = a_t\mathbf{u}_t + a_n\mathbf{u}_n$$

where $a_t = \ddot{s}$; $a_n = \dfrac{\dot{s}^2}{\rho}$;

Δs : distance along C traveled by P;

$\dot{s}$: magnitude of velocity vector.

Example: Particle P moving on a circular path

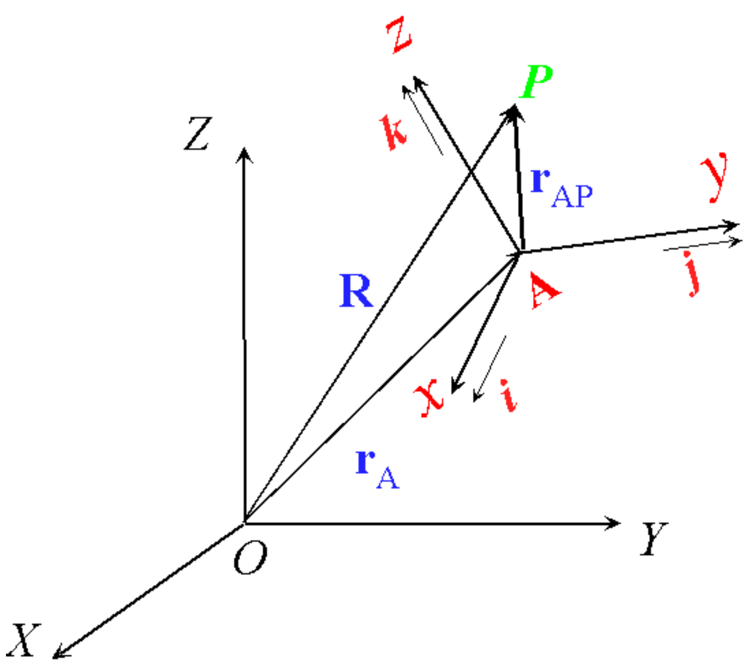


$v_P(t) = ?$

$a_P(t) = ?$

3. Moving Reference Frame

Consider a reference frame *xyz* moving relative to a fixed reference frame *XYZ*.



xyz: with unit vectors ijk

XYZ: with unit vectors IJK

Three Cases:

(i) xyz *translates* relative to XYZ

(ii) xyz *rotates* relative to XYZ

(iii) xyz *both translates & rotates* relative to XYZ

Figure18

- In three cases:

$$\mathbf{R} = \mathbf{r}_A + \mathbf{r}_{AP} \quad \text{where} \begin{cases} \mathbf{r}_A : \text{position vector from O to A;} \\ \mathbf{r}_{AP} : \text{ position vector from A to P.} \end{cases}$$

$$\mathbf{V} = \dot{\mathbf{R}} = \mathbf{v}_A + \mathbf{v}_{AP} \quad \text{where} \begin{cases} \mathbf{v}_A = \dot{\mathbf{r}}_A : \text{velocity of A;} \\ \mathbf{v}_{AP} = \dot{\mathbf{r}}_{AP} : \text{ velocity of P relative to A.} \end{cases}$$

$$\mathbf{a} = \dot{\mathbf{v}} = \ddot{\mathbf{R}} = \mathbf{a}_A + \mathbf{a}_{AP} \quad \text{where} \begin{cases} \mathbf{a}_A = \dot{\mathbf{v}}_A = \ddot{\mathbf{r}}_A ; \\ \mathbf{a}_{AP} = \dot{\mathbf{v}}_{AP} = \ddot{\mathbf{r}}_{AP} . \end{cases}$$

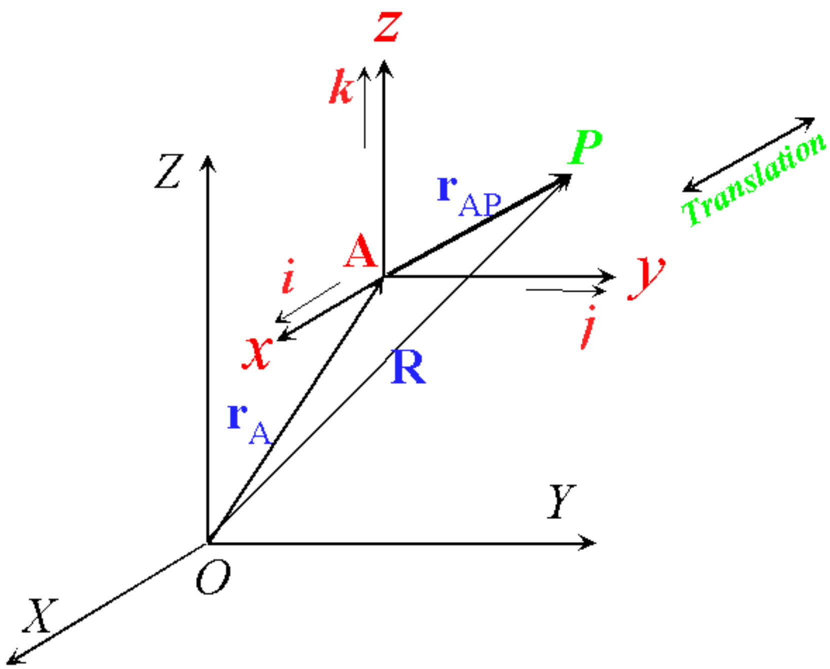
All expressed **w.r.t** XYZ (inertial reference frame)

But either **in terms of** IJK or ijk

Evaluation of velocity and acceleration

Case 1 : *xyz* translates relative to *XYZ*.

let $x || X$, $y || Y$, $z || Z$,

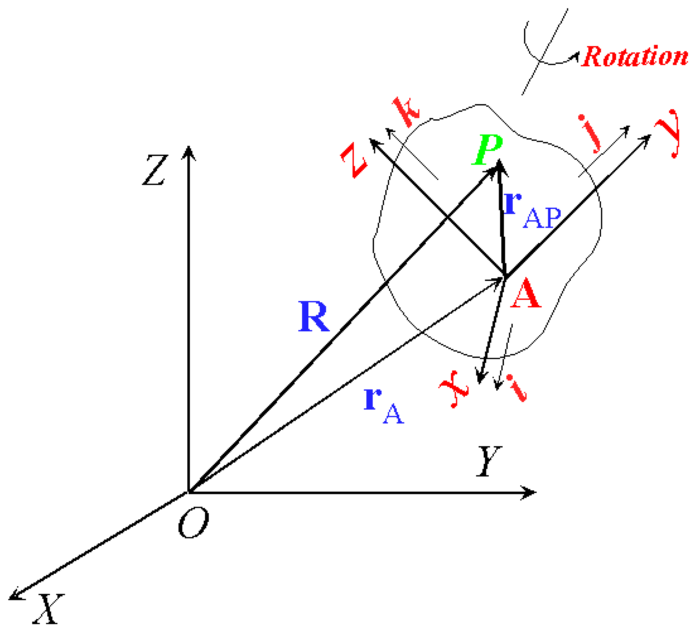


$$\mathbf{R} = r_A{}^x\mathbf{i} + r_A{}^y\mathbf{j} + r_A{}^z\mathbf{k} + r_{AP}{}^x\mathbf{i} + r_{AP}{}^y\mathbf{j} + r_{AP}{}^z\mathbf{k}$$
$$= (r_A{}^x + r_{AP}{}^x)\mathbf{i} + (r_A{}^y + r_{AP}{}^y)\mathbf{j} + (r_A{}^z + r_{AP}{}^z)\mathbf{k}$$

$$\mathbf{V} = \underbrace{(\dot r_A{}^x + \dot r_{AP}{}^x)}_{v_x}\mathbf{i} + \underbrace{(\dot r_A{}^y + \dot r_{AP}{}^y)}_{v_y}\mathbf{j} + \underbrace{(\dot r_A{}^z + \dot r_{AP}{}^z)}_{v_z}\mathbf{k}$$

$$\mathbf{a} = \underbrace{(\ddot r_A{}^x + \ddot r_{AP}{}^x)}_{a_x}\mathbf{i} + \underbrace{(\ddot r_A{}^y + \ddot r_{AP}{}^y)}_{a_y}\mathbf{j} + \underbrace{(\ddot r_A{}^z + \ddot r_{AP}{}^z)}_{a_z}\mathbf{k}$$

<u>Case 2</u> : *xyz* rotates relative to *XYZ*, about axis AB.



Kinematics of Point P

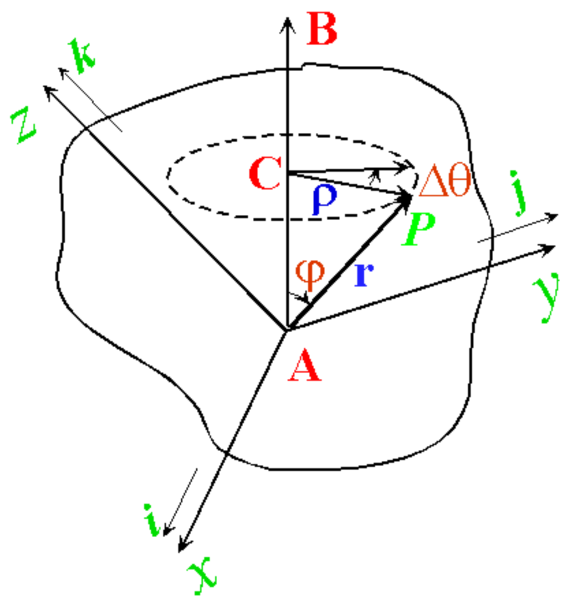
$$\mathbf{R} = \overbrace{\mathbf{r}_A}^{\text{fixed}} + \mathbf{r}_{AP},$$

$$\mathbf{v} = \dot{\mathbf{R}} = \dot{\mathbf{r}}_{AP},$$

$$\mathbf{a} = \ddot{\mathrm{R}} = \ddot{\mathbf{r}}_{AP}.$$

- Motion of P relative to A due to rotation of the frame xyz can be visualized by imagining the xyz embedded in a rigid body.
- So rotation of frame is identical to that of the rigid body.

- Angular Velocity:



- vector ρ rotates in a plane normal to AB with angular rate $\boxed{\dot{\theta}}$.

$\Rightarrow$ rigid body and hence xyz rotates about AB at $\boxed{\dot{\theta}}$ also.

- Want to represent the angular rate as a vector ω with magnitude

$$\boxed{|\omega| = \dot{\theta}}$$

- Note

$$\boxed{|\dot{\rho}| = \rho\dot{\theta}}$$

why?

- In vector form

$$\boxed{\dot{\rho} = \omega \times \rho}$$

- Note $\rho = r\sin\varphi$

$$\boxed{\Rightarrow \dot{\mathbf{r}} = \omega \times \mathbf{r}}$$

- So:

$$\mathbf{R} = \mathbf{r}_A + \mathbf{r}_{AP},$$

$$\boxed{\Rightarrow \mathbf{v} = \dot{\mathbf{r}}_{AP} = \omega \times \mathbf{r}_{AP}}$$ velocity

- Let vector angular acceleration [rad/sec²]: $\alpha = \dot{\omega}$

$$\mathbf{a} = \dot{\mathbf{v}} = \dot{\omega} \times \mathbf{r}_{AP} + \omega \times \dot{\mathbf{r}}_{AP}$$

$$\boxed{\Rightarrow \mathbf{a} = \alpha \times \mathbf{r}_{AP} + \omega \times (\omega \times \mathbf{r}_{AP})}$$ acceleration

- Above expressions are valid when:

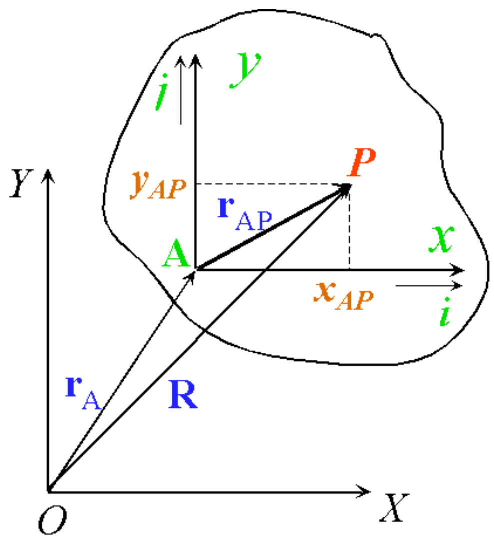
xyz rotates relative to *XYZ*.

- Case 3: *xyz* translates & rotates relative to *XYZ*.

$$\boxed{\begin{aligned} \mathbf{R} &= \mathbf{r}_{\mathrm{A}} + \mathbf{r}_{\mathrm{AP}}, \\ \mathbf{v} &= \dot{\mathbf{r}}_{\mathrm{A}} + \omega \times \mathbf{r}_{\mathrm{AP}}, \\ \mathbf{a} &= \ddot{\mathbf{r}}_{\mathrm{A}} + \alpha \times \mathbf{r}_{\mathrm{AP}} + \omega \times (\omega \times \mathbf{r}_{\mathrm{AP}}). \end{aligned}}$$

4. Planar Motion of Rigid Bodies

- Specialize previous results to 2D



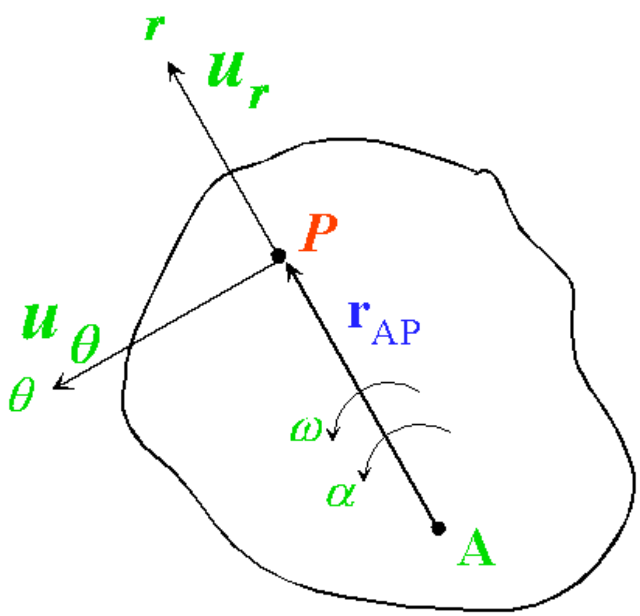
Let $\mathbf{r}_{AP} = x_{AP}\mathbf{i} + y_{AP}\mathbf{j}$, since motion is planar, $\boldsymbol{\omega} = \omega\mathbf{k}$, $\boldsymbol{\alpha} = \alpha\mathbf{k}$.

$$\mathbf{v} = \mathbf{v}_A + \mathbf{v}_{AP} = \mathbf{v}_A + \omega \times \mathbf{r}_{AP}$$
$$= \mathbf{v}_A - \omega(y_{AP}\mathbf{i} - x_{AP}\mathbf{j})$$

- $\mathbf{v}_A$ is the velocity due to translation of A
- $-\omega(y_{AP}\mathbf{i} - x_{AP}\mathbf{j})$ is the velocity due to rotation about A

$$\mathbf{a} = \mathbf{a}_A + \alpha \times \mathbf{r}_{AP} + \omega \times (\omega \times \mathbf{r}_{AP})$$
$$= \mathbf{a}_A + \alpha\mathbf{k}(x_{AP}\mathbf{i} + y_{AP}\mathbf{j}) + \omega\mathbf{k} \times [\omega\mathbf{k} \times (x_{AP}\mathbf{i} + y_{AP}\mathbf{j})]$$
$$= \mathbf{a}_A - \alpha(y_{AP}\mathbf{i} - x_{AP}\mathbf{j}) - \omega^2\mathbf{r}_{AP}$$

Planar motion using radial & transverse coordinates



Position vector

$$\mathbf{r}_{AP} = r_{AP}\mathbf{u}_r$$

Velocity :

$$\mathbf{v} = \mathbf{v}_A + \overbrace{\omega \times \mathbf{r}_{AP}}^{\dot{\mathbf{r}}_{AP}}$$

$$= \mathbf{v}_A + \omega r_{AP}\mathbf{u}_\theta$$

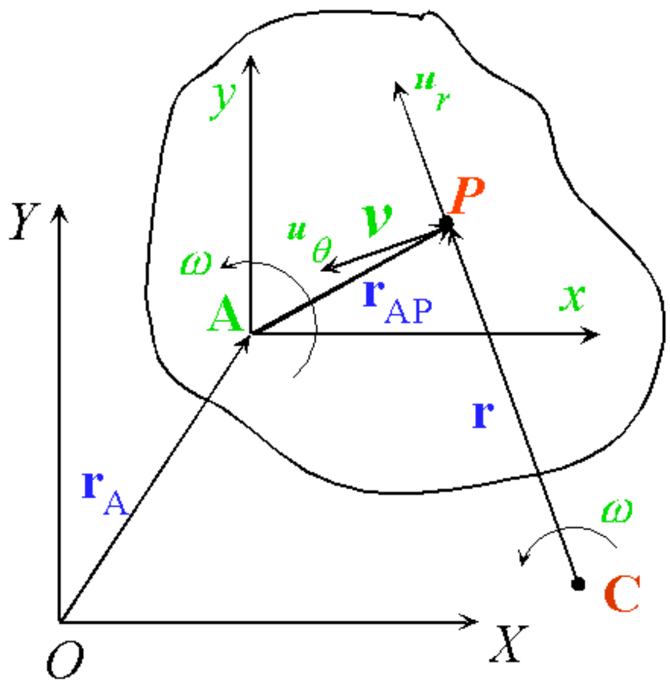
Acceleration :

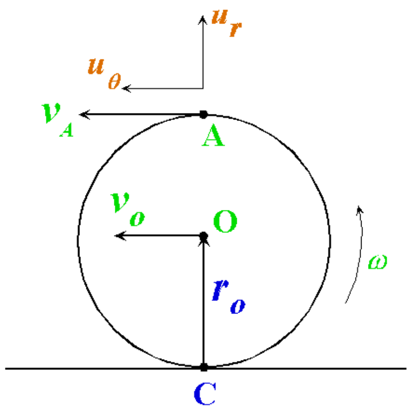
$$\mathbf{a} = \mathbf{a}_A + \alpha \times \mathbf{r}_{AP} + \omega \times \overbrace{(\omega \times \mathbf{r}_{AP})}^{\omega r_{AP}\mathbf{u}_\theta}$$

$$= \mathbf{a}_A + \alpha r_{AP}\mathbf{u}_\theta - \omega^2 r_{AP}\mathbf{u}_r$$

Instantaneous Center of Rotation

- At a given instance,the motion of P can be thought of as consisting entirely of rotation about a center C called instantaneous center of rotation (not necessarily inside the body).
- Say P rotates with angular velocity ω





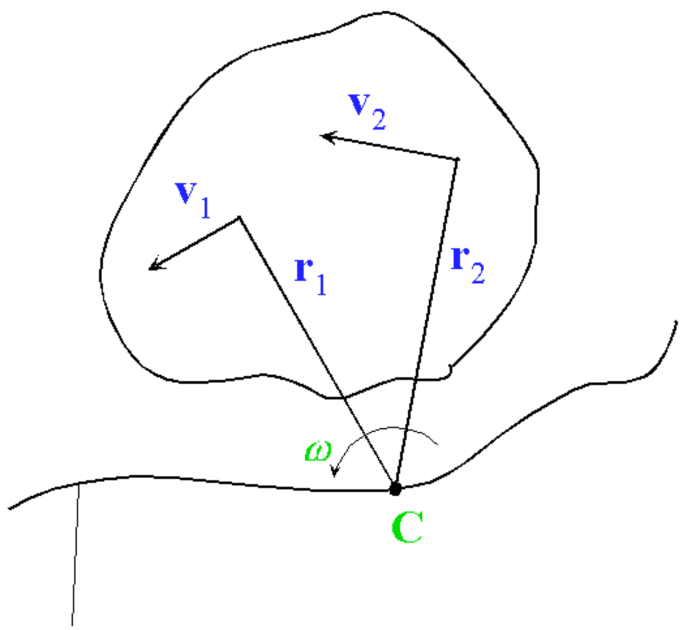
$$\mathbf{v} = \omega \times \mathbf{r} = \omega\mathbf{k} \times r\mathbf{u}_r = \omega r\mathbf{u}_\theta$$

$$\text{So } \mathbf{v} = v\mathbf{u}_\theta , v = \omega r$$

$$\Rightarrow r = \frac{v}{\omega}$$

Figure24-25

- Now solve problems such as



Space centrode: is the locus of the instantaneous centers.

$$\mathbf{v}_1 = \omega \times \mathbf{r}_1$$

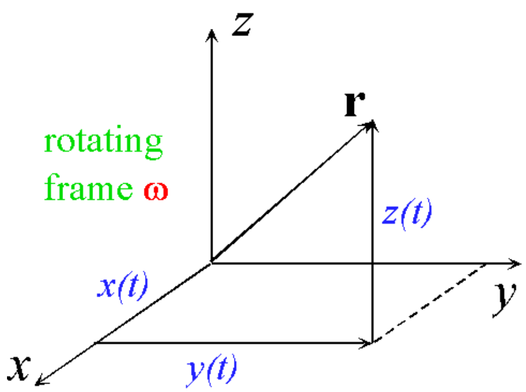
$$\Rightarrow \underbrace{v_1}_{\text{known}} = \omega \underbrace{r_1}_{\text{known}}$$

$$\mathbf{v}_2 = \omega \times \mathbf{r}_2$$

$$\Rightarrow \underbrace{v_2}_{\text{unknown}} = \omega \underbrace{r_2}_{\text{known}}$$

5. General Cases of Motion

- **Already considered:**
  - *xyz* translates & rotates with angular velocity ω
- **Now in addition:**
  - P can move relative to *xyz*
- **Aside:**
  - Consider a vector with time dependent magnitude and embedded in a rotating reference frame.



- $\boxed{\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}}$

- $x(t), y(t), z(t) \rightarrow$ time varying
  $\mathbf{i},\mathbf{j},\mathbf{k} \rightarrow$ not constant, rotate with ω

$$\dot{\mathbf{r}} = \dot{x}\mathbf{i} + \dot{y}\mathbf{j} + \dot{z}\mathbf{k} + x\dot{\mathbf{i}} + y\dot{\mathbf{j}} + z\dot{\mathbf{k}}$$

recall that $\dot{\mathbf{r}}_{AP} = \omega \times \mathbf{r}_{AP}$

$$\dot{\mathbf{i}} = \omega \times \mathbf{i}, \dot{\mathbf{j}} = \omega \times \mathbf{j}, \dot{\mathbf{k}} = \omega \times \mathbf{k}$$

$$\dot{\mathbf{r}} = \dot{\mathbf{r}}' + \omega \times \mathbf{r}$$

$$\dot{\mathbf{r}}' = \underbrace{\dot{x}\mathbf{i} + \dot{y}\mathbf{j} + \dot{z}\mathbf{k}}$$

time rate of change of **r** regarding *xyz* inertial

Figure27

$$\mathbf{R} = \mathbf{r}_A + \mathbf{r}_{AP},$$

$$\mathbf{v} = \dot{\mathbf{R}} = \dot{\mathbf{r}}_A + \mathbf{v}'_{AP} + \underbrace{\omega \times \mathbf{r}_{AP}}_{\substack{\text{velocity of P due} \\ \text{entirely to rotation} \\ \text{of frame } xyz}}, \text{ where } \mathbf{v}'_{AP} = \underbrace{\dot{x}_{AP}\mathbf{i} + \dot{y}_{AP}\mathbf{j} + \dot{z}_{AP}\mathbf{k}}_{\substack{\text{velocity of P} \\ \text{relative to the} \\ \text{moving frame } xyz}}$$

$$\begin{aligned}
\mathbf{a} = \dot{\mathbf{v}} &= \dot{\mathbf{v}}_A + \frac{d}{dt}\mathbf{v}'_{AP} + \dot{\omega} \times \mathbf{r}_{AP} + \omega \times \dot{\mathbf{r}}_{AP} \\
&= \mathbf{a}_A + (\mathbf{a}'_{AP} + \omega \times \mathbf{v}'_{AP}) + \alpha \times \mathbf{r}_{AP} + \omega \times (\mathbf{v}'_{AP} + \omega \times \mathbf{r}_{AP}) \\
&= \underbrace{\mathbf{a}_A}_{\substack{\text{acceleration of} \\ \text{A relative to O} \\ \text{(inertial) frame}}} + \underbrace{\mathbf{a}'_{AP}}_{\substack{\text{acceleration} \\ \text{of P relative} \\ \text{to rotating} \\ \text{frame } xyz}} + \underbrace{2\omega \times \mathbf{v}'_{AP}}_{\substack{\text{Coriolis} \\ \text{acceleration}}} + \underbrace{\alpha \times \mathbf{r}_{AP} + \omega \times (\omega \times \mathbf{r}_{AP})}_{\substack{\text{acceleration of P due entirely to} \\ \text{rotation of frame } xyz}}
\end{aligned}$$

$$\text{Note:} \quad \mathbf{a}'_{AP} = \ddot{x}_{AP}\mathbf{i} + \ddot{y}_{AP}\mathbf{j} + \ddot{z}_{AP}\mathbf{k}$$