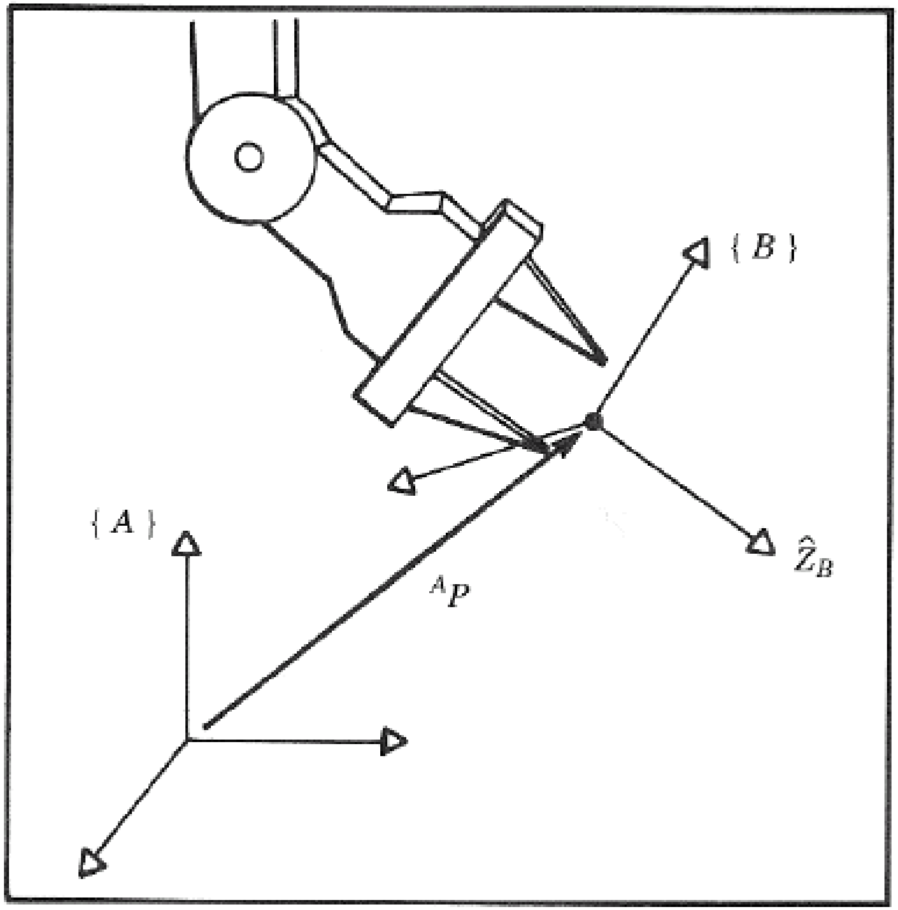
Lecture 2

- Kinematics:
  - Pose (position and orientation) of a Rigid Body

Central Topic

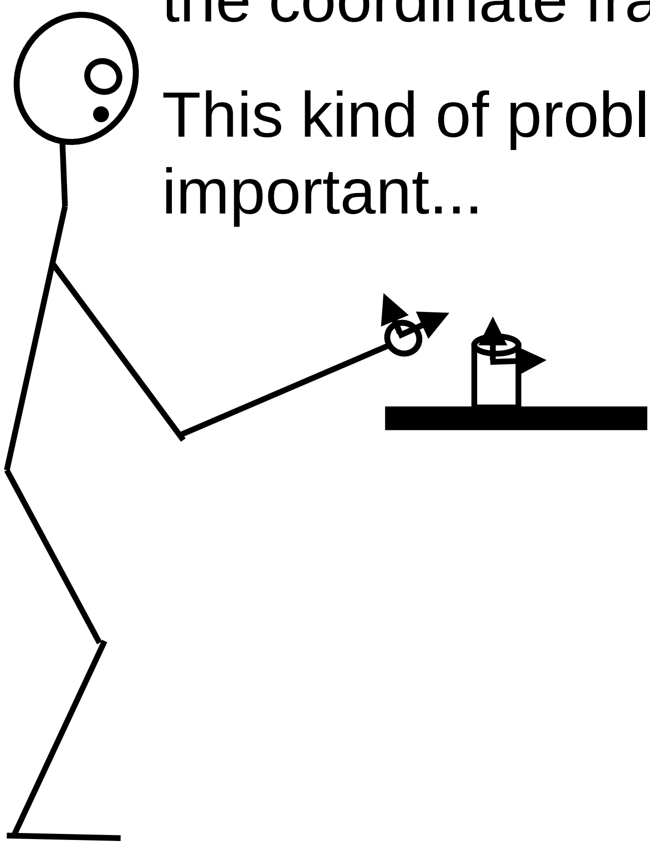
- Problem
  - Robotic manipulation, by definition, implies that parts and tools will be moving around in space by the manipulator mechanism
  - Leads to need of representing positions and orientations of the parts, tools, and mechanism itself
- Solution
  - Mathematical tools for representing position and orientation of objects/frames in 3D space

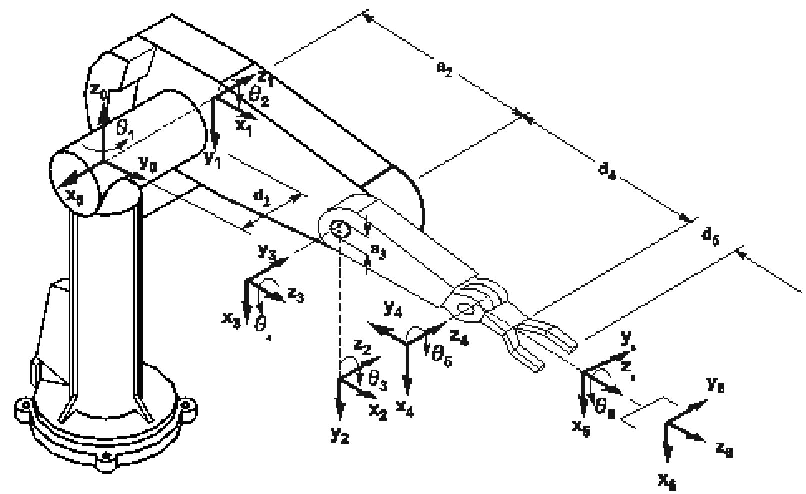


Why are we studying pose?

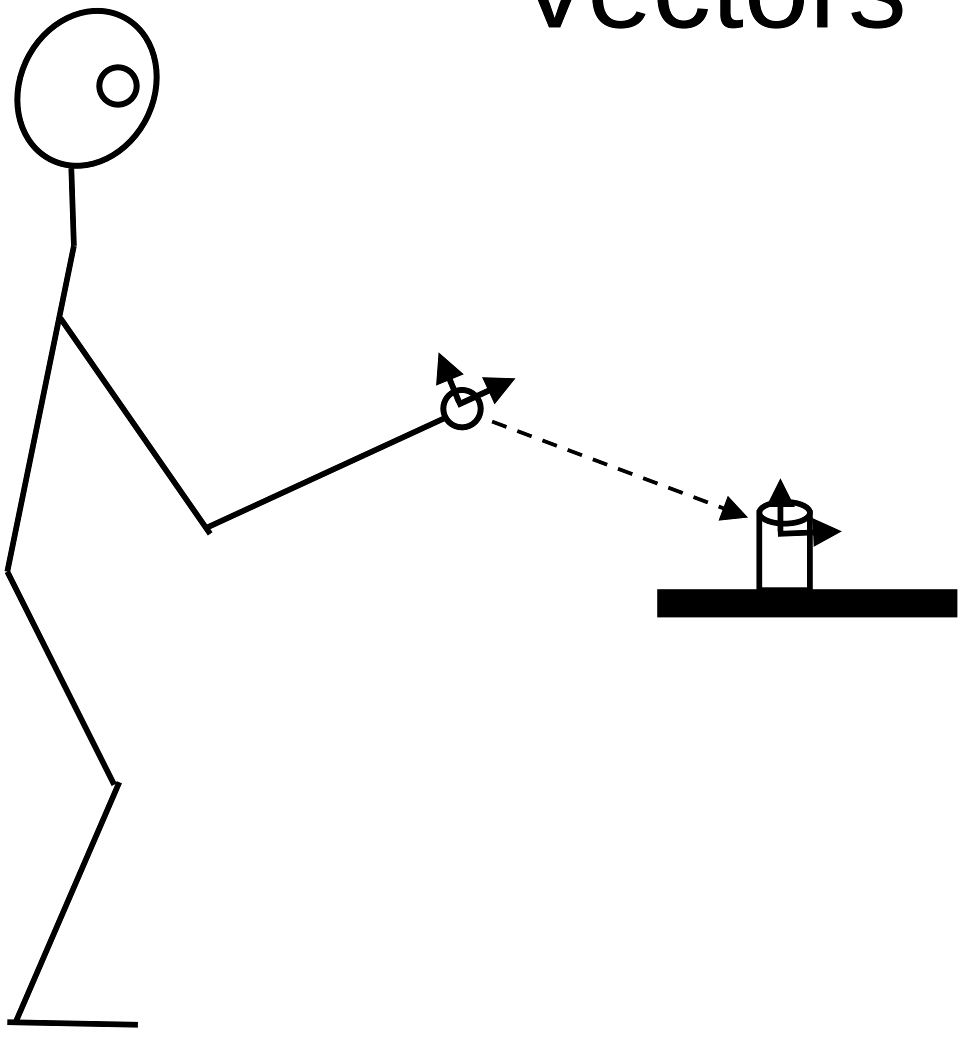
You want to put your hand on the cup…

- Suppose your eyes tell you where the mug is and its orientation in the robot *base frame* (big assumption)
- In order to put your hand on the object, you want to align the coordinate frame of your hand w/ that of the object
- This kind of problem makes representation of pose important...



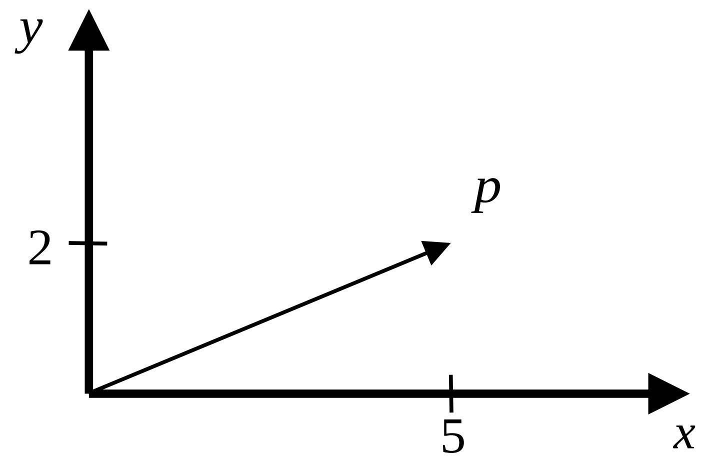


Representing Position: Vectors



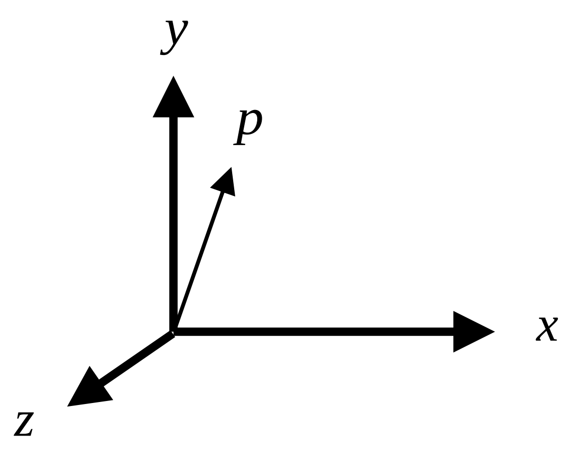
Representing Position: vectors

$$p = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$ ("column" vector)



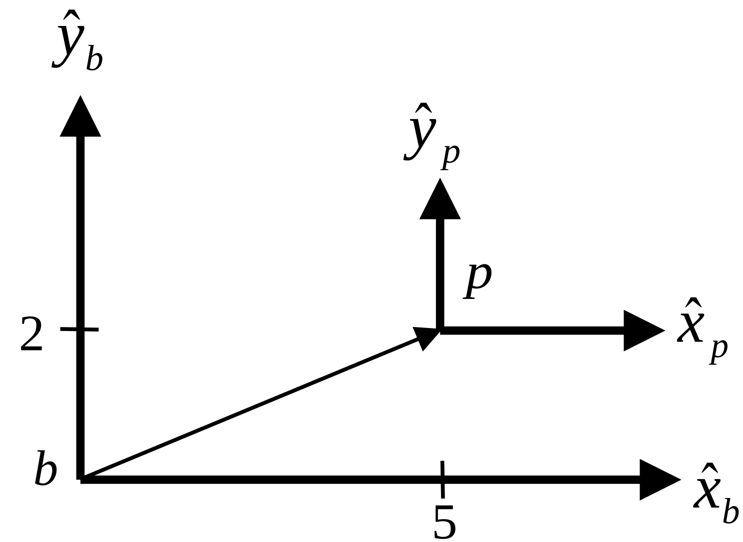
$$p = \begin{bmatrix} 5 & 2 \end{bmatrix}$$ ("row" vector)

$$p = \begin{bmatrix} 2 \\ 5 \\ 2 \end{bmatrix}$$



Representing Position: vectors

- Vectors are a way to transform between two different reference frames w/ the same orientation
- The prefix superscript denotes the reference frame in which the vector should be understood



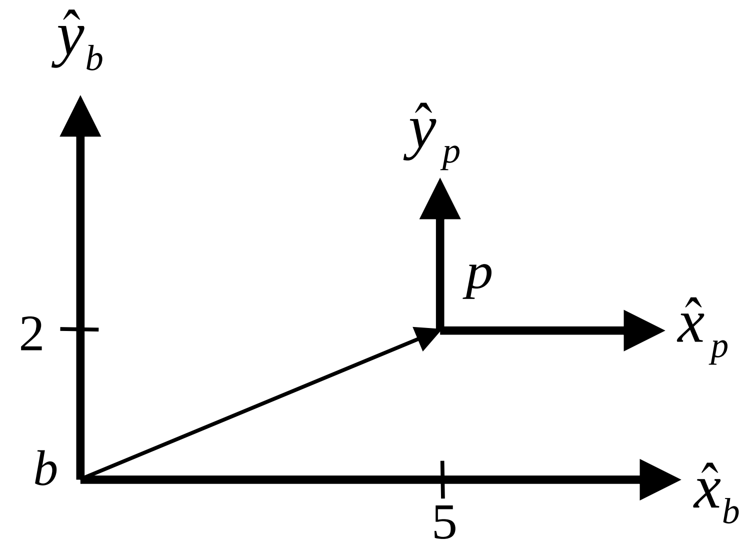
$${}^{b}p = \begin{bmatrix} 5 \\ 2 \end{bmatrix} \qquad {}^{p}p = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Same point, two different reference frames

Representing Position: vectors

- Note that I am denoting the axes as *orthogonal* unit basis vectors

This means “perpendicular”



$\hat{x}_b$ ← A vector of length one pointing in the direction of the base frame *x* axis

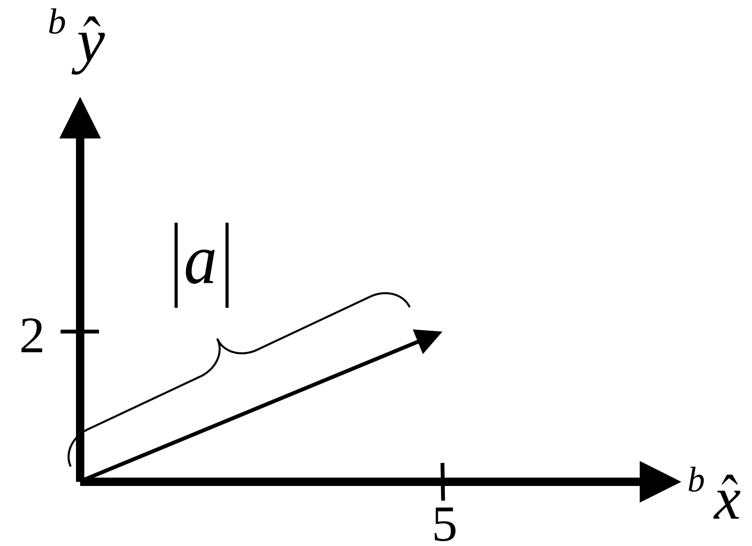
$\hat{y}_b$ ← *y* axis

$\hat{y}_p$ ← *p* frame y axis

What is this unit vector you speak of?

These are the elements of *a*:

$$a = \begin{bmatrix} a_x \\ a_y \end{bmatrix}$$



Vector length/magnitude: $$|a| = \sqrt{a_x^2 + a_y^2}$$

Definition of unit vector: $$|\hat{a}| = 1$$

You can turn *a* into a unit vector of the same direction this way:

$$\hat{a} = \frac{a}{\sqrt{a_x^2 + a_y^2}}$$

And what does orthogonal mean?

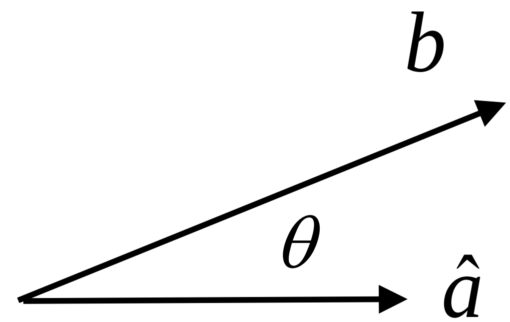
First, define the dot product: $a \cdot b = a_x b_x + a_y b_y$

$$= |a||b|\cos(\theta)$$

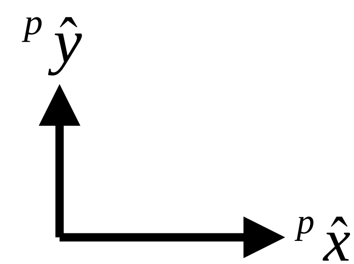
$a \cdot b = 0$ when: $a = 0$

or, $b = 0$

or, $\cos(\theta) = 0$



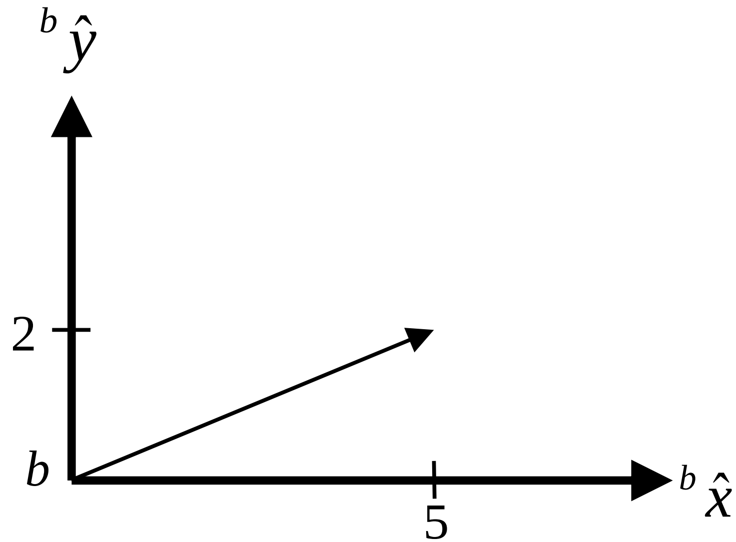
Unit vectors are orthogonal *iff* (if and only if) the dot product is zero:



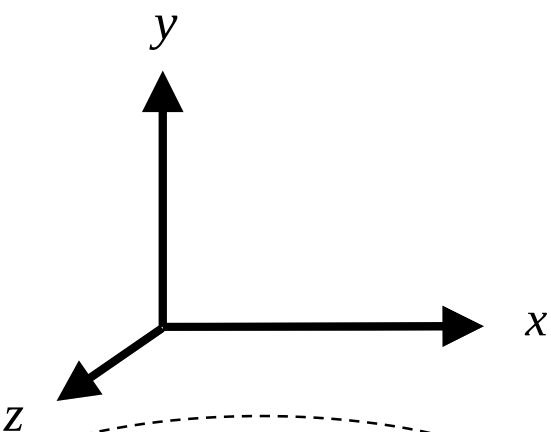
${}^{p}\hat{x}$ is orthogonal to ${}^{p}\hat{y}$ iff ${}^{p}\hat{x} \cdot {}^{p}\hat{y} = 0$

A couple of other random things

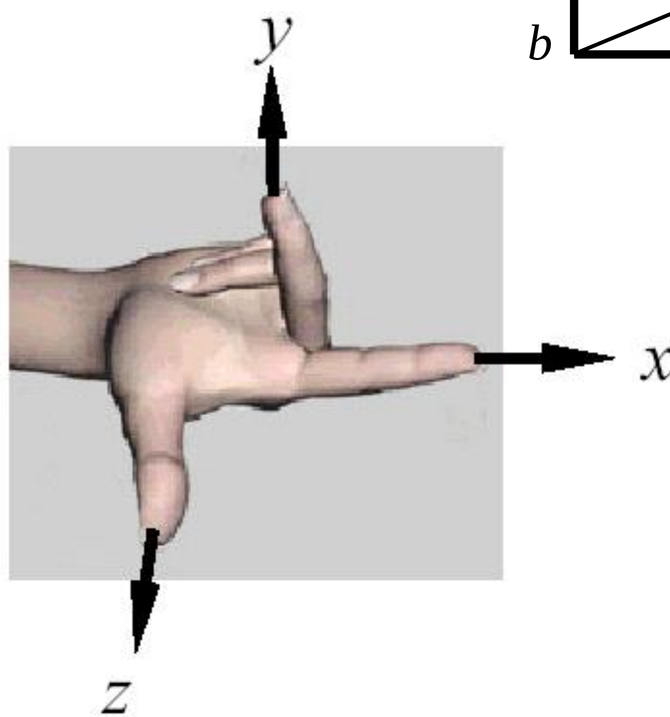
$$p_b = 5\hat{x}_b + 2\hat{y}_b$$

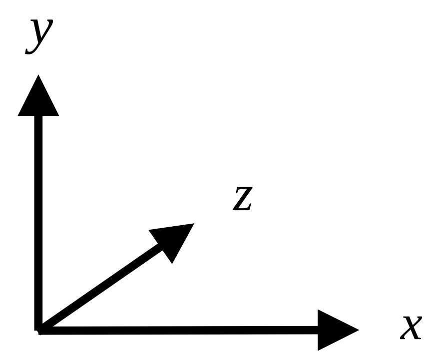


Vectors are elements of $\boldsymbol{R}^n$



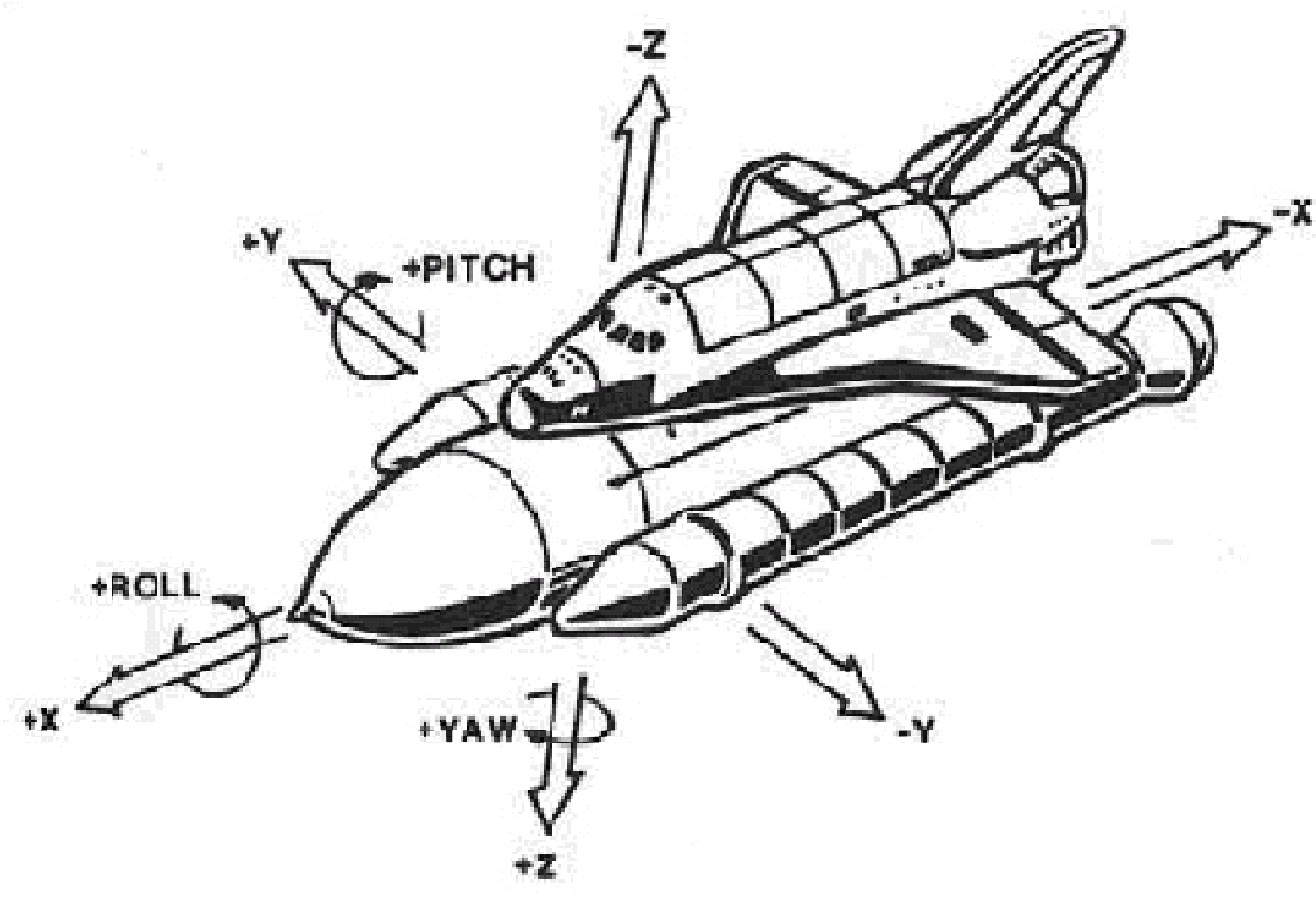
right-handed
coordinate frame





left-handed
coordinate frame

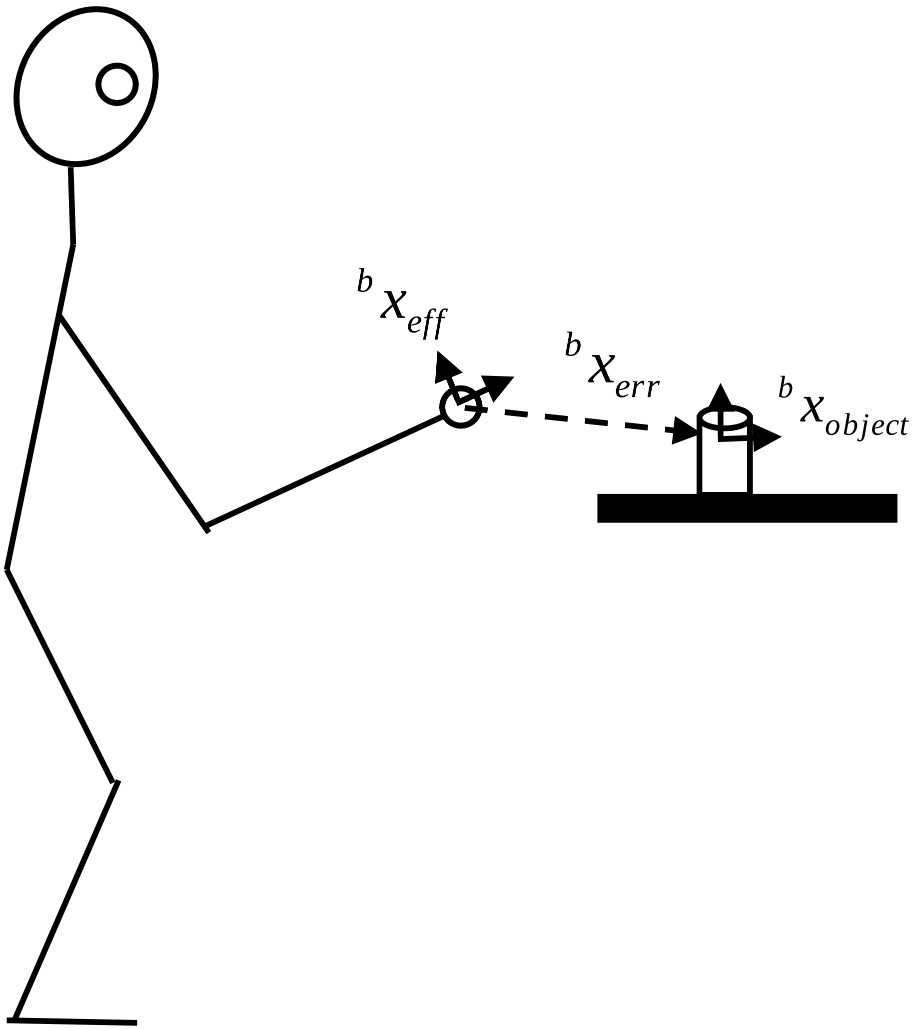
Coordinate System



The importance of differencing two vectors

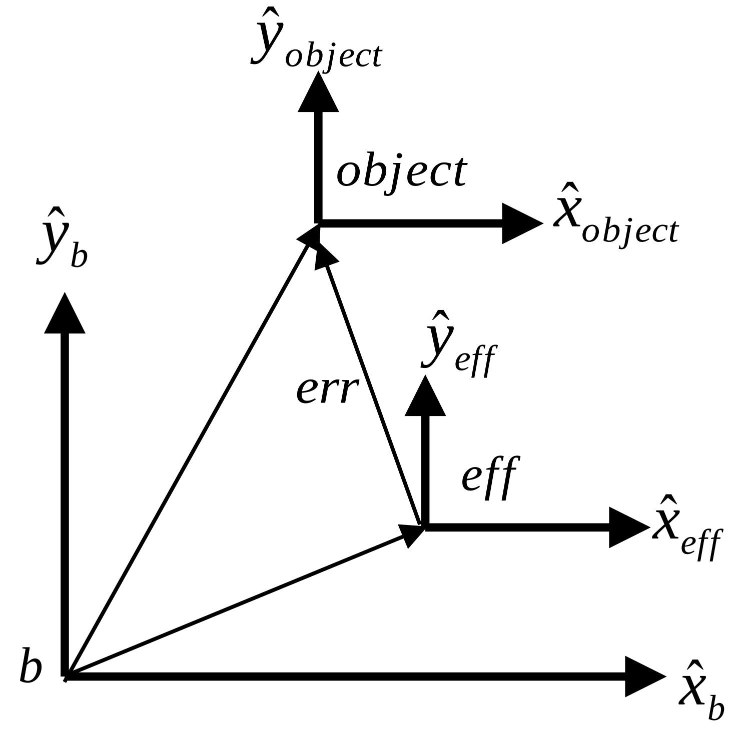
$$^{b}x_{object} - {}^{b}x_{eff} = {}^{b}x_{err}$$

The *eff* needs to make a Cartesian displacement of this much to reach the object



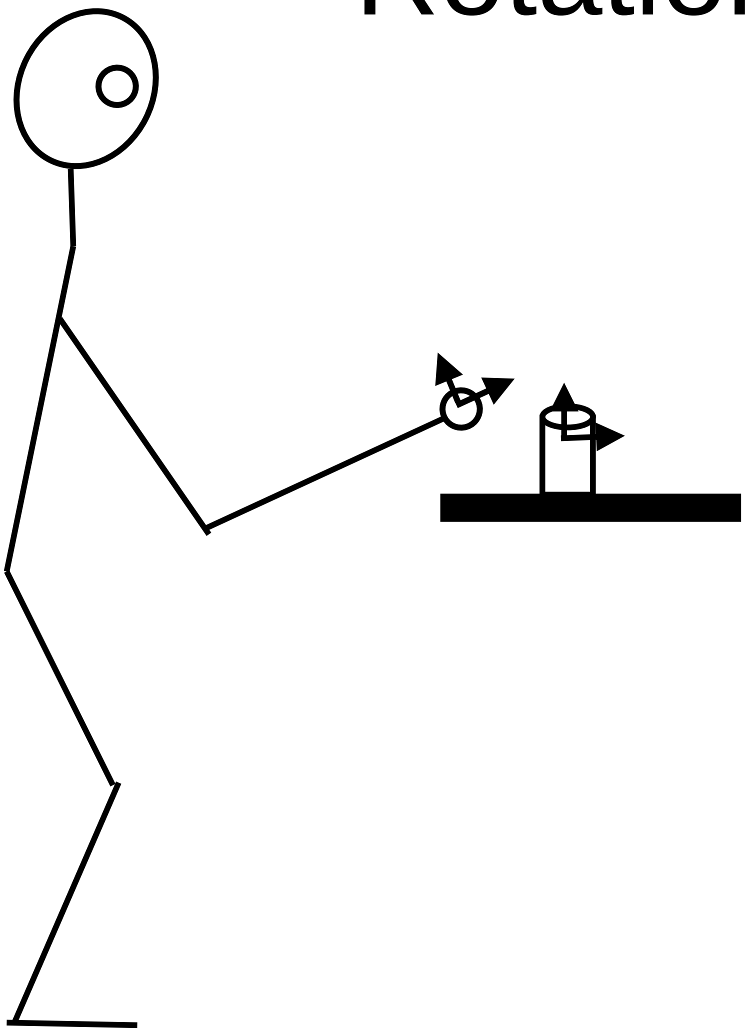
The importance of differencing two vectors

$$ {}^{b}x_{object} - {}^{b}x_{eff} = {}^{b}x_{err} $$



The *eff* needs to make a Cartesian displacement
of this much to reach the object

Representing Orientation: Rotation Matrices



- The reference frame of the hand and the object have different orientations
- We want to represent and difference orientations just like we did for positions…

Before we go there – review of matrix transpose

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\mathbf{A}^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$$

$$p = \begin{bmatrix} 5 \\ 2 \end{bmatrix} \longrightarrow p^T = \begin{bmatrix} 5 & 2 \end{bmatrix}$$

Important property: $\mathbf{A}^T\mathbf{B}^T = (\mathbf{BA})^T$

and matrix multiplication…

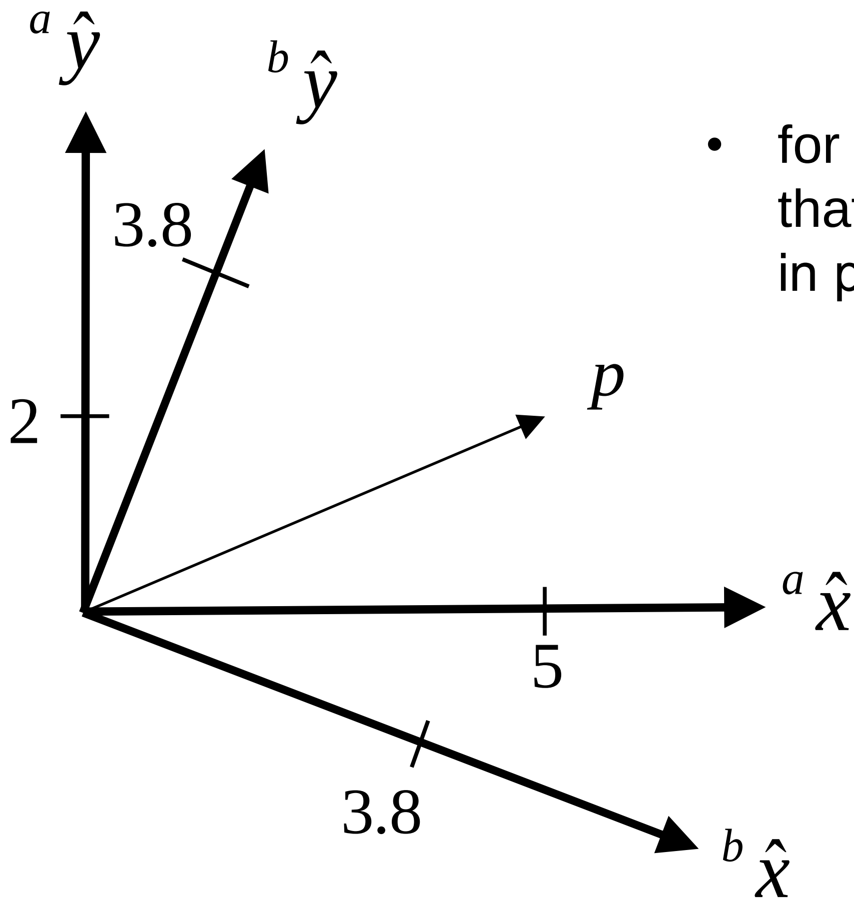
$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

$$\mathbf{AB} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix}$$

Can represent dot product as a matrix multiply:

$$a \cdot b = a_x b_x + a_y b_y = \begin{bmatrix} a_x & a_y \end{bmatrix} \begin{bmatrix} b_x \\ b_y \end{bmatrix} = a^T b$$

Same point - different reference frames



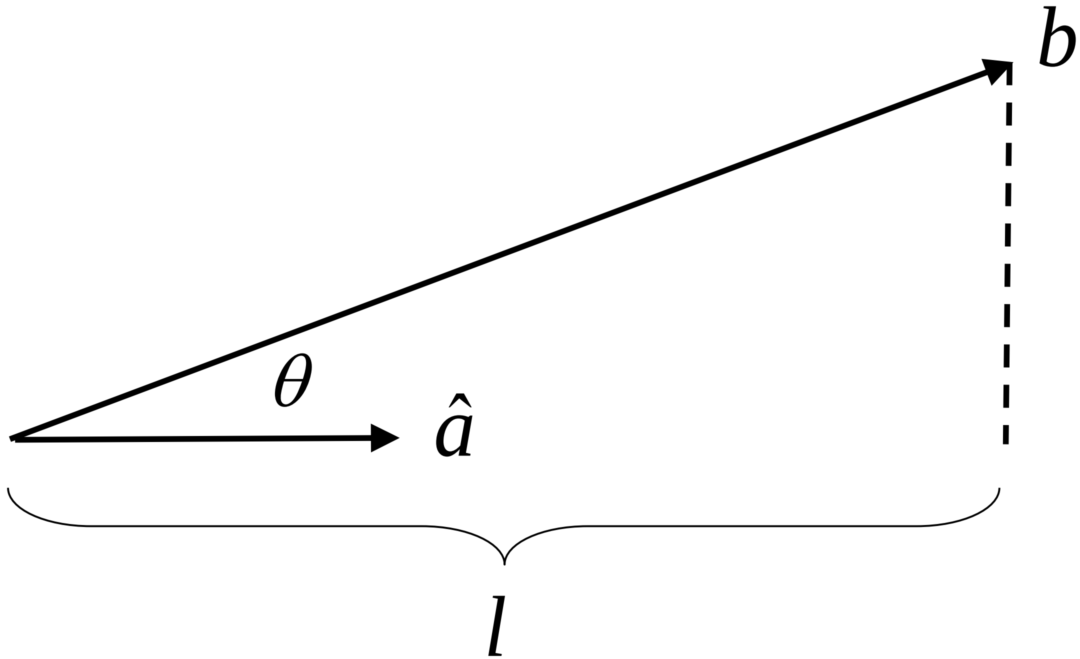
- for the moment, assume that there is no difference in position…

$$
{}^a p = \begin{bmatrix} 5 \\ 2 \end{bmatrix} \qquad {}^b p = \begin{bmatrix} 3.8 \\ 3.8 \end{bmatrix}
$$

Another important use of the dot product: projection

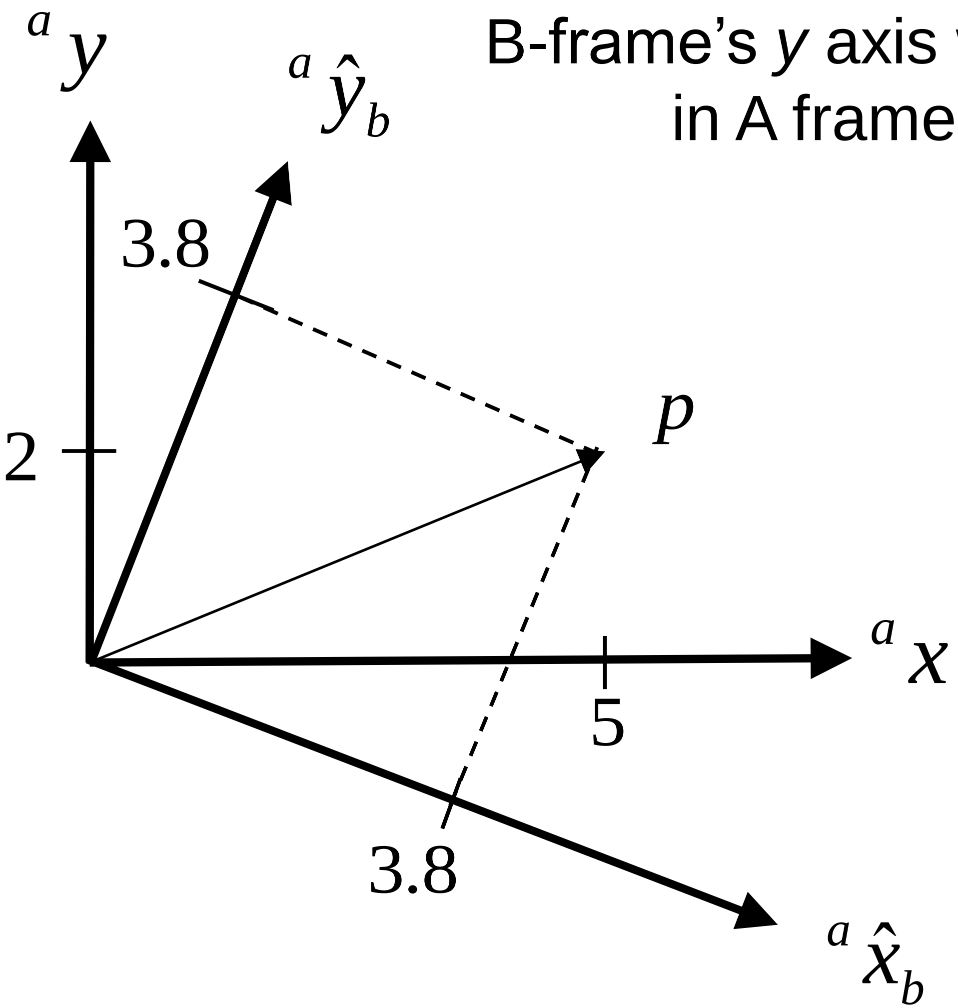
$$a \cdot b = a_x b_x + a_y b_y$$

$$= |a||b|\cos(\theta)$$



$$l = \hat{a} \cdot b = |\hat{a}||b|\cos(\theta) = |b|\cos(\theta)$$

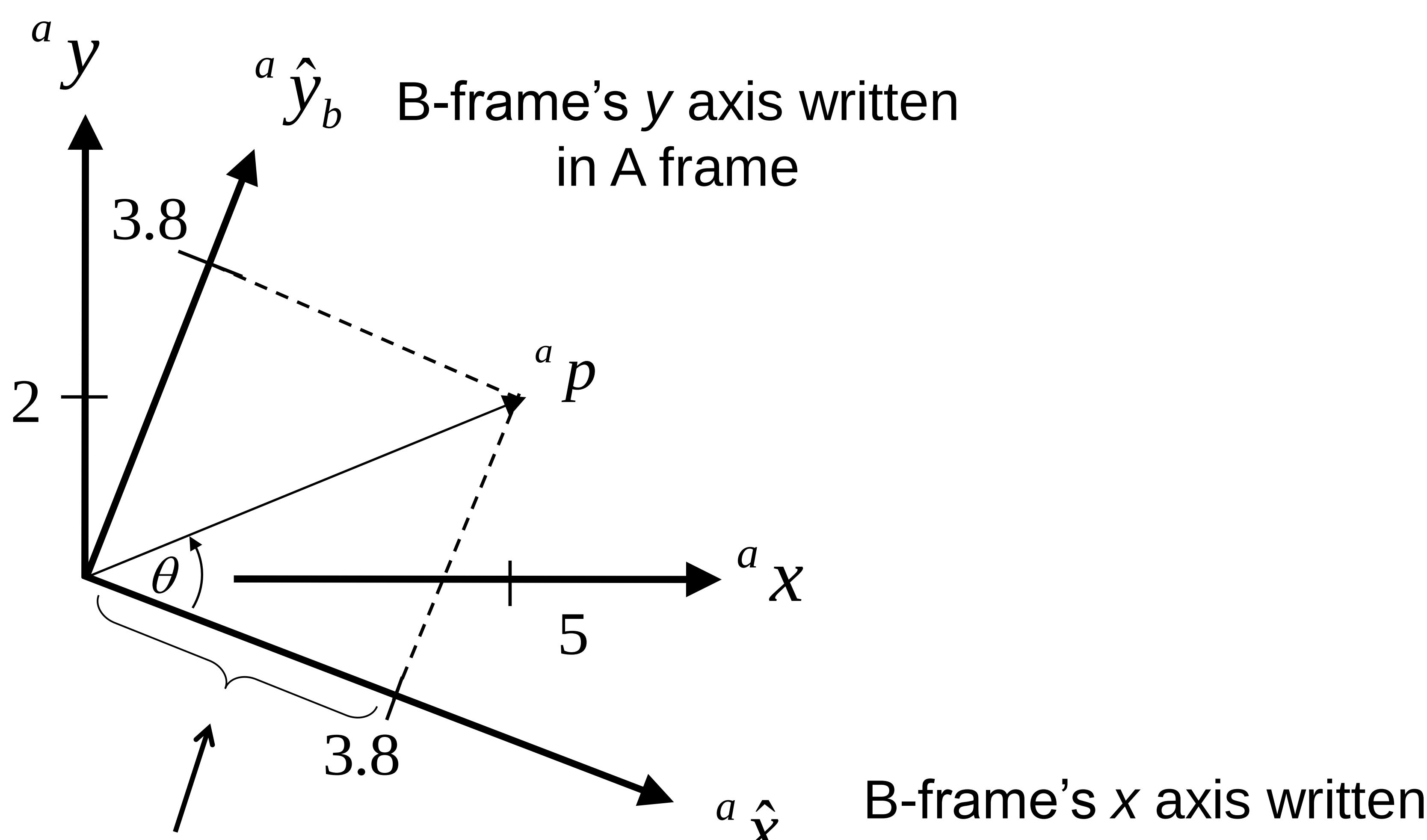
Same point - different reference frames



B-frame's *y* axis written in A frame

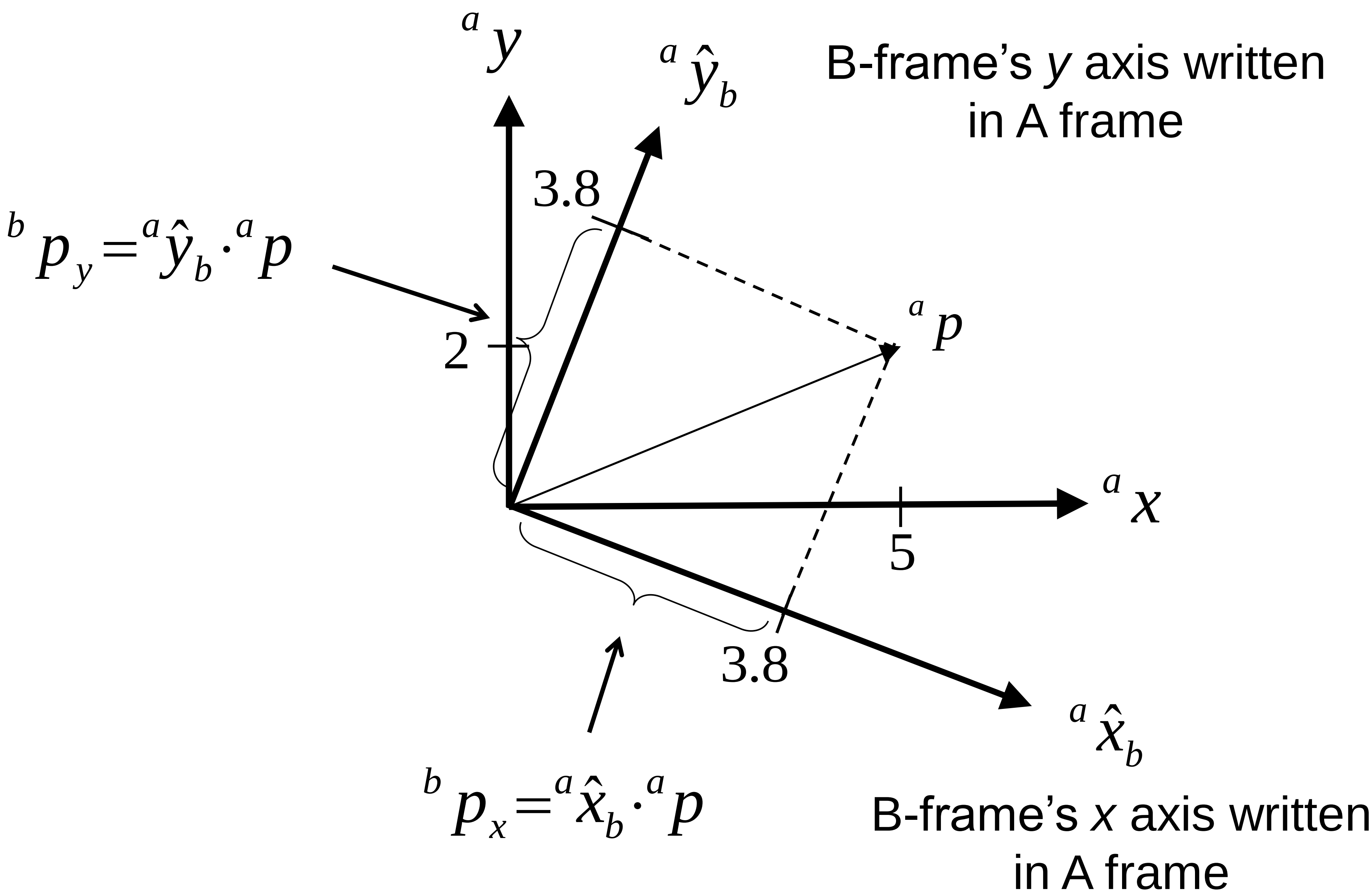
B-frame's *x* axis written in A frame

Same point - different reference frames



$$^a\hat{x}_b \cdot {}^a p = {}^a p \cos(\theta) = {}^b p_x$$

Same point - different reference frames



Same point - different reference frames

$$^{B}p = \begin{pmatrix} {}^{A}\hat{x}_B \cdot {}^{A}p \\ {}^{A}\hat{y}_B \cdot {}^{A}p \end{pmatrix} = \begin{pmatrix} {}^{A}\hat{x}_B{}^{T}\,{}^{A}p \\ {}^{A}\hat{y}_B{}^{T}\,{}^{A}p \end{pmatrix} = \begin{pmatrix} {}^{A}\hat{x}_B{}^{T} \\ {}^{A}\hat{y}_B{}^{T} \end{pmatrix} {}^{A}p$$

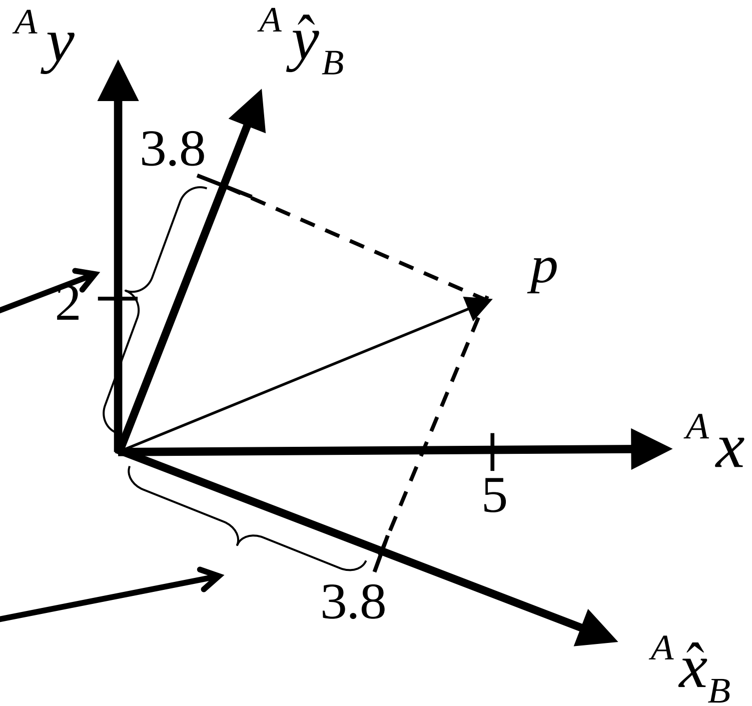
$$^{B}p = \begin{pmatrix} {}^{A}\hat{x}_B{}^{T} \\ {}^{A}\hat{y}_B{}^{T} \end{pmatrix} {}^{A}p$$

$$^{B}p = {}^{A}R_B{}^{T}\,{}^{A}p$$

Rotation matrix

$$^{B}p_y = {}^{A}\hat{y}_B \cdot {}^{A}p$$

$$^{B}p_x = {}^{A}\hat{x}_B \cdot {}^{A}p$$



The rotation matrix

From last page: $${}^{B}p = \begin{pmatrix} {}^{A}\hat{x}_B{}^{T} \\ {}^{A}\hat{y}_B{}^{T} \end{pmatrix} {}^{A}p \longrightarrow {}^{B}p = {}^{A}R_B{}^{T}\,{}^{A}p$$

By the same reasoning: $${}^{A}p = \begin{pmatrix} {}^{B}\hat{x}_A{}^{T} \\ {}^{B}\hat{y}_A{}^{T} \end{pmatrix} {}^{B}p \longrightarrow {}^{A}p = {}^{B}R_A{}^{T}\,{}^{B}p$$

The rotation matrix

$${}^{A}R_{B} = \begin{pmatrix} {}^{A}\hat{x}_{B} & {}^{A}\hat{y}_{B} \end{pmatrix} \quad \text{and} \quad {}^{A}R_{B} = {}^{B}R_{A}^{\;T} = \begin{pmatrix} {}^{B}\hat{x}_{A}^{\;T} \\ {}^{B}\hat{y}_{A}^{\;T} \end{pmatrix}$$

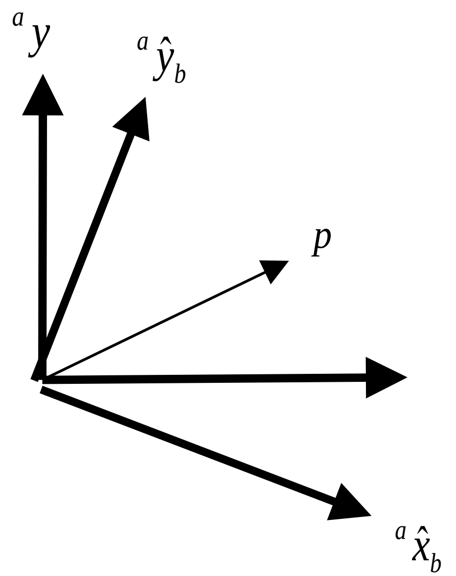
$${}^{A}R_{B} = \begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix} \qquad {}^{A}R_{B} = \begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix}$$

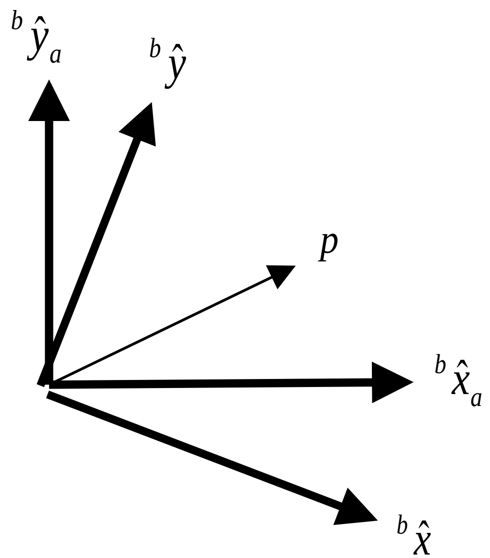
$${}^{A}\hat{x}_{B} \qquad {}^{A}\hat{y}_{B} \qquad {}^{B}\hat{x}_{A}^{\;T} \qquad {}^{B}\hat{y}_{A}^{\;T}$$

The rotation matrix can be understood as:

1. Columns of vectors of *B* in *A* reference frame, OR
2. Rows of column vectors *A* in *B* reference frame

The rotation matrix

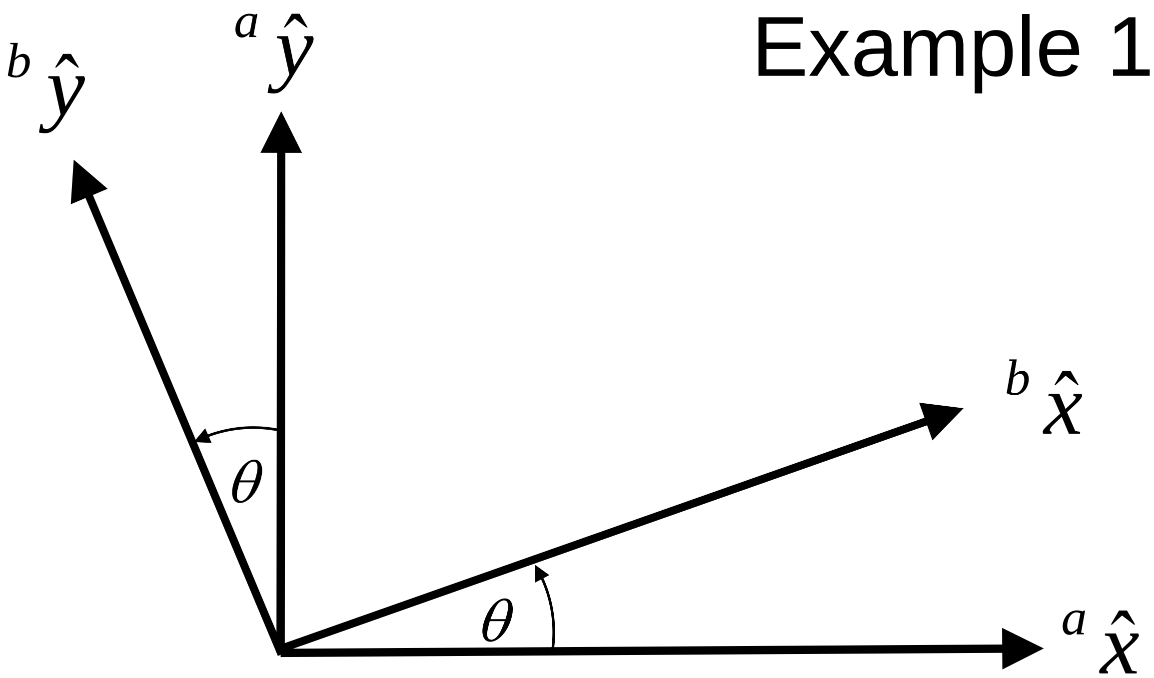




$${}^{A}R_{B} = \begin{pmatrix} {}^{A}\hat{x}_{B} & {}^{A}\hat{y}_{B} \end{pmatrix}$$

$${}^{A}R_{B} = \begin{pmatrix} {}^{B}\hat{x}_{A}{}^{T} \\ {}^{B}\hat{y}_{A}{}^{T} \end{pmatrix}$$

Example 1: rotation matrix



$$^a\hat{x}_b = \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix} \qquad {}^aR_b = \begin{pmatrix} {}^a\hat{x}_b & {}^a\hat{y}_b \end{pmatrix} = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$$

$$^a\hat{y}_b = \begin{pmatrix} -\sin(\theta) \\ \cos(\theta) \end{pmatrix} \qquad {}^bR_a = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}$$

Rotations about x, y, z

These rotation matrices encode the basis vectors of the after-rotation reference frame in terms of the before-rotation reference frame

$$R_z(\alpha) = \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) & 0 \\ \sin(\alpha) & \cos(\alpha) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$R_y(\beta) = \begin{pmatrix} \cos(\beta) & 0 & \sin(\beta) \\ 0 & 1 & 0 \\ -\sin(\beta) & 0 & \cos(\beta) \end{pmatrix}$$

$$R_x(\gamma) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\gamma) & -\sin(\gamma) \\ 0 & \sin(\gamma) & \cos(\gamma) \end{pmatrix}$$

Remember those double-angle formulas…

$$\sin(\theta \pm \phi) = \sin(\theta)\cos(\phi) \pm \cos(\theta)\sin(\phi)$$

$$\cos(\theta \pm \phi) = \cos(\theta)\cos(\phi) \mp \sin(\theta)\sin(\phi)$$

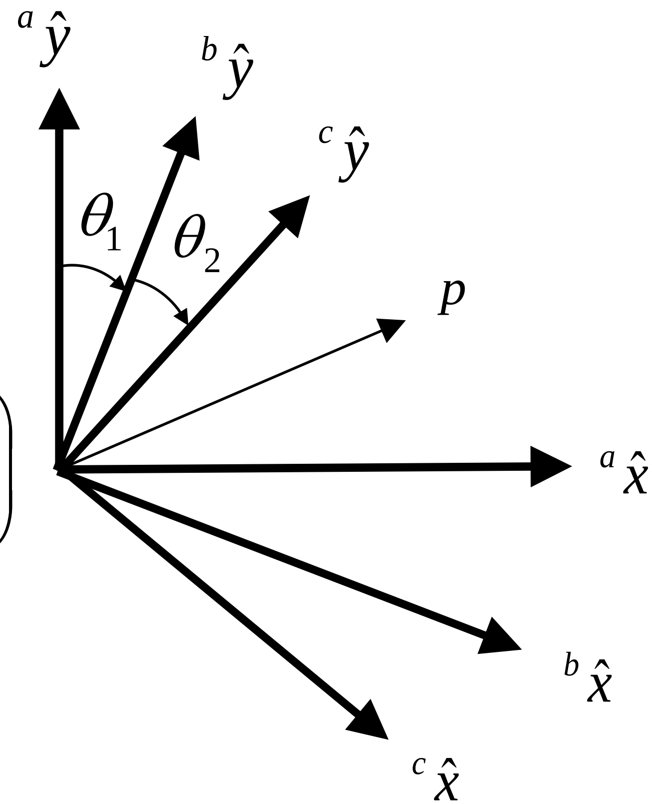
Example 1: composition of rotation matrices

$$^{A}R_{C} = {}^{A}R_{B}\,{}^{B}R_{C}$$

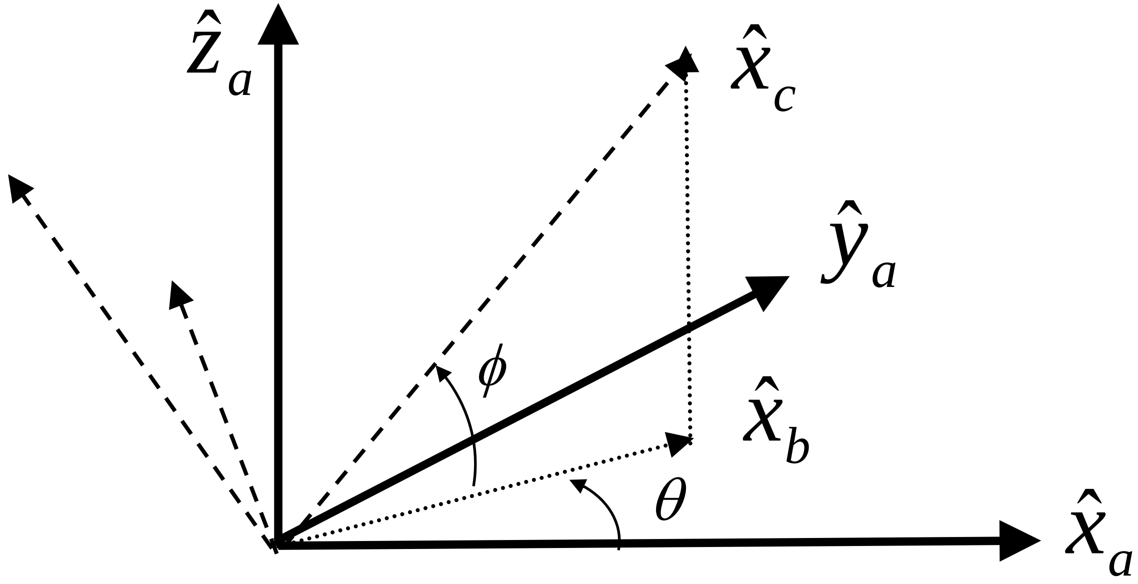
$$^{a}R_{c} = \begin{pmatrix} \cos(\theta_1) & -\sin(\theta_1) \\ \sin(\theta_1) & \cos(\theta_1) \end{pmatrix} \begin{pmatrix} \cos(\theta_2) & -\sin(\theta_2) \\ \sin(\theta_2) & \cos(\theta_2) \end{pmatrix}$$

$$^{a}R_{c} = \begin{pmatrix} c_1 c_2 - s_1 s_2 & -c_1 s_2 - s_1 c_2 \\ s_1 c_2 + c_1 s_2 & c_1 c_2 - s_1 s_2 \end{pmatrix}$$

$$^{a}R_{c} = \begin{pmatrix} c_{12} & -s_{12} \\ s_{12} & c_{12} \end{pmatrix}$$



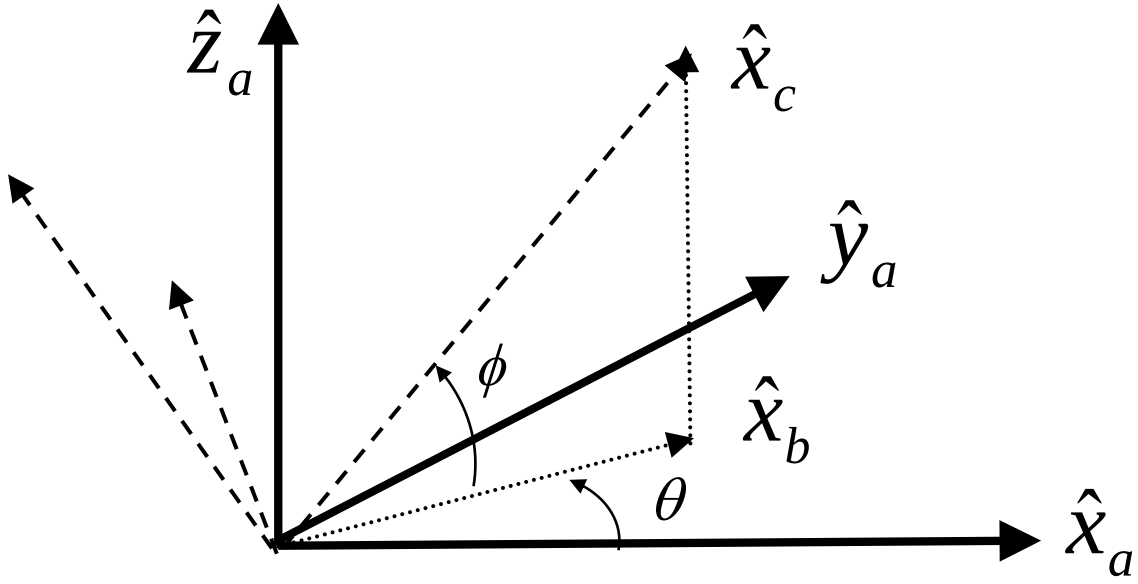
Example 2: composition of rotation matrices



$$ {}^{a}R_{b} = \begin{pmatrix} c_\theta & -s_\theta & 0 \\ s_\theta & c_\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} $$

$$ {}^{b}R_{c} = \begin{pmatrix} c_{-\phi} & 0 & s_{-\phi} \\ 0 & 1 & 0 \\ -s_{-\phi} & 0 & c_{-\phi} \end{pmatrix} = \begin{pmatrix} c_\phi & 0 & -s_\phi \\ 0 & 1 & 0 \\ s_\phi & 0 & c_\phi \end{pmatrix} $$

Example 2: composition of rotation matrices



$$ {}^{a}R_{c} = {}^{a}R_{b}\,{}^{b}R_{c} = \begin{pmatrix} c_\theta & -s_\theta & 0 \\ s_\theta & c_\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c_\phi & 0 & -s_\phi \\ 0 & 1 & 0 \\ s_\phi & 0 & c_\phi \end{pmatrix} = \begin{pmatrix} c_\theta c_\phi & -s_\theta & -c_\theta s_\phi \\ s_\theta c_\phi & c_\theta & -s_\theta s_\phi \\ s_\phi & 0 & c_\phi \end{pmatrix} $$

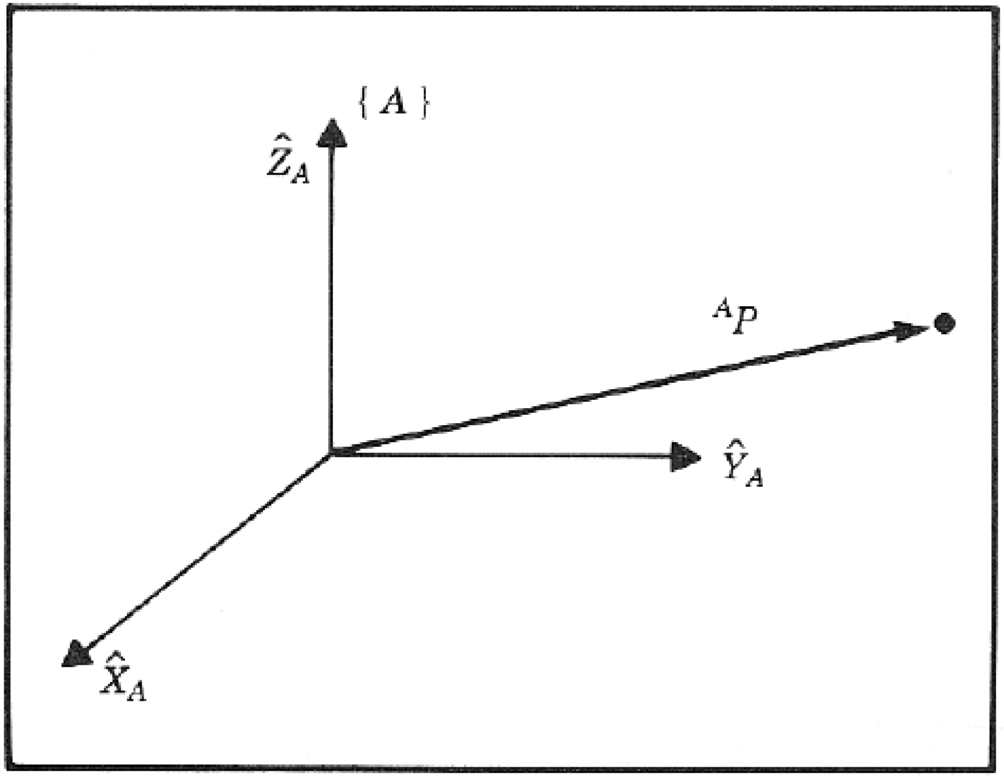
Recap of rotation matrices

$$ {}^{A}R_{B} = \begin{pmatrix} {}^{A}\hat{x}_{B} & {}^{A}\hat{y}_{B} \end{pmatrix} = \begin{pmatrix} {}^{b}\hat{x}_{a}{}^{T} \\ {}^{b}\hat{y}_{a}{}^{T} \end{pmatrix} $$

$$ {}^{b}R_{a}{}^{-1} = {}^{b}R_{a}{}^{T} = {}^{a}R_{b} $$

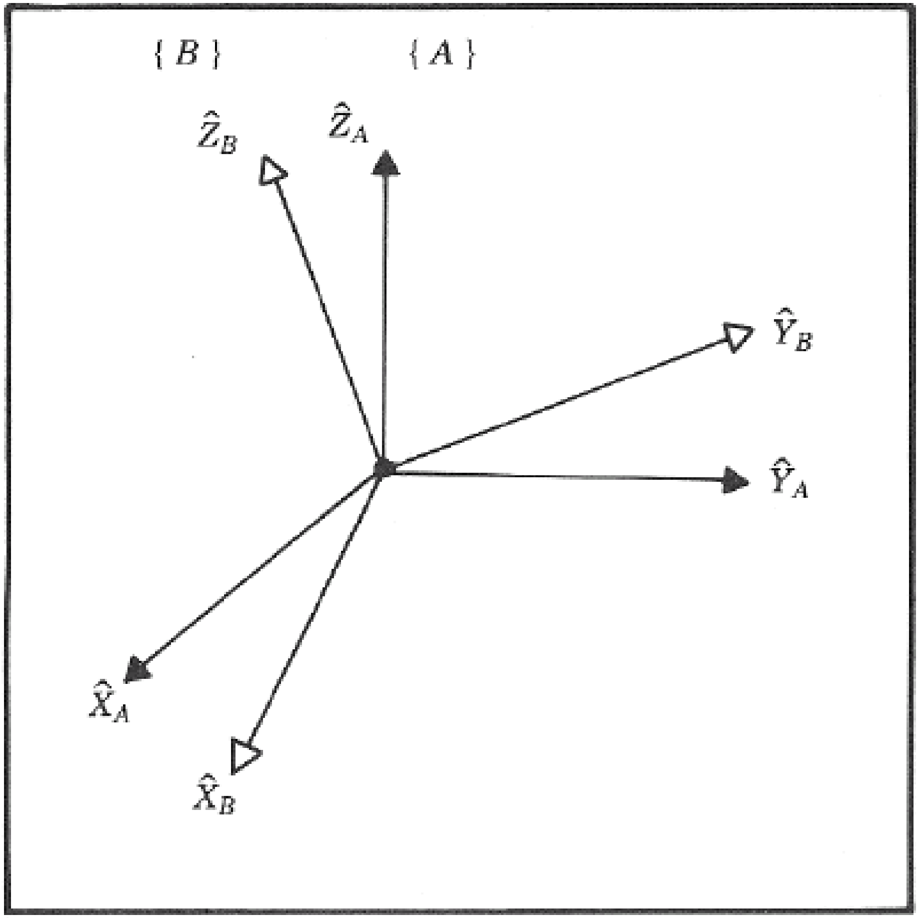
Description of a Position

- The location of any point in space can be described as a 3x1 ***position vector*** in a reference coordinate system



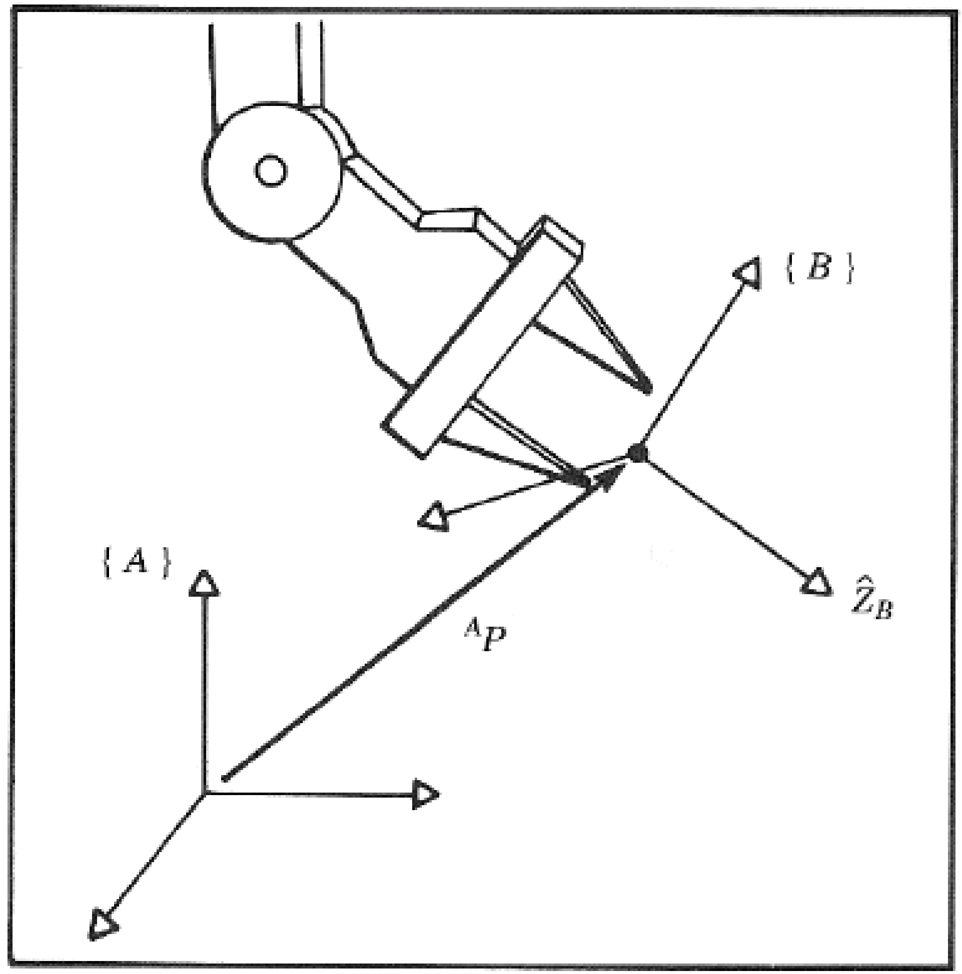
Description of an Orientation

- The orientation of a body is described by attaching a coordinate system to the body {B} and then defining the relationship between the body frame and the reference frame {A} using the rotation matrix.



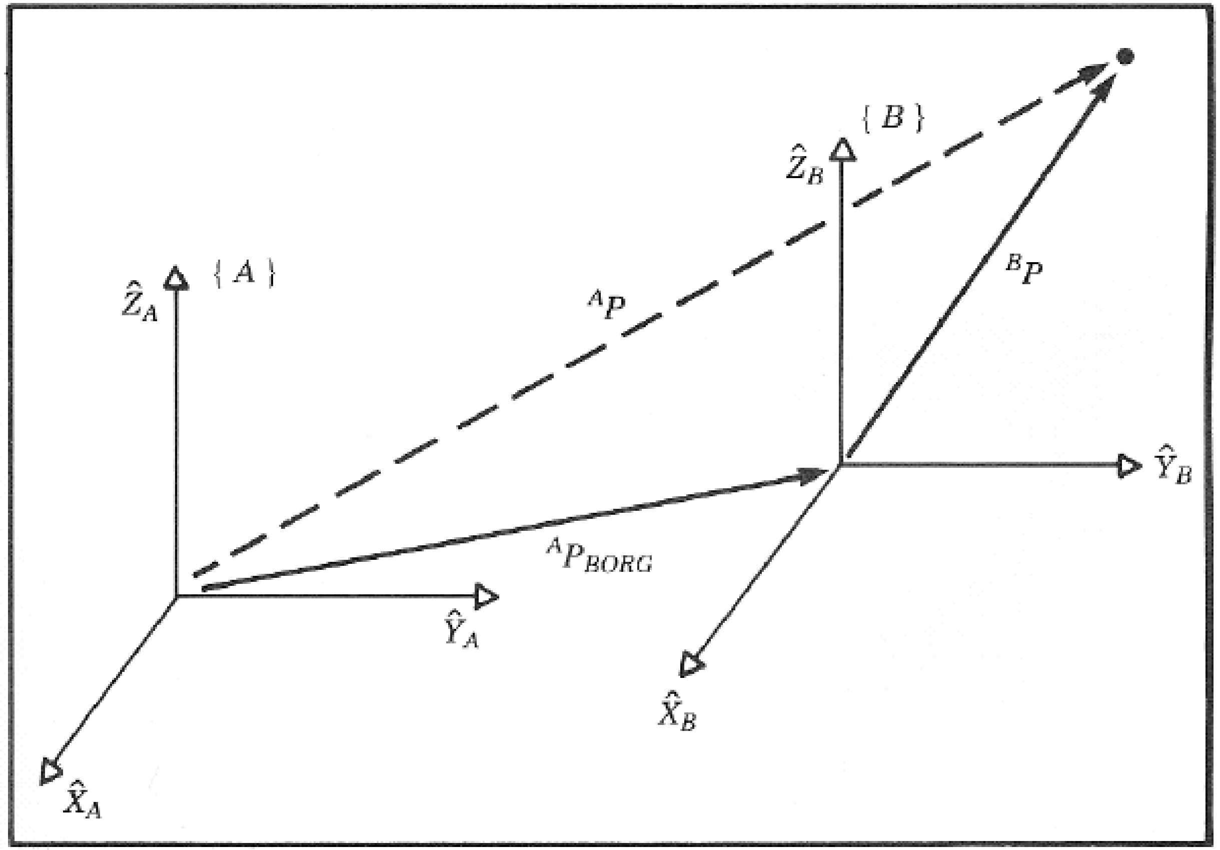
Description of a Frame

- The information needed to completely specify where is the manipulator hand is
- a position and an orientation.



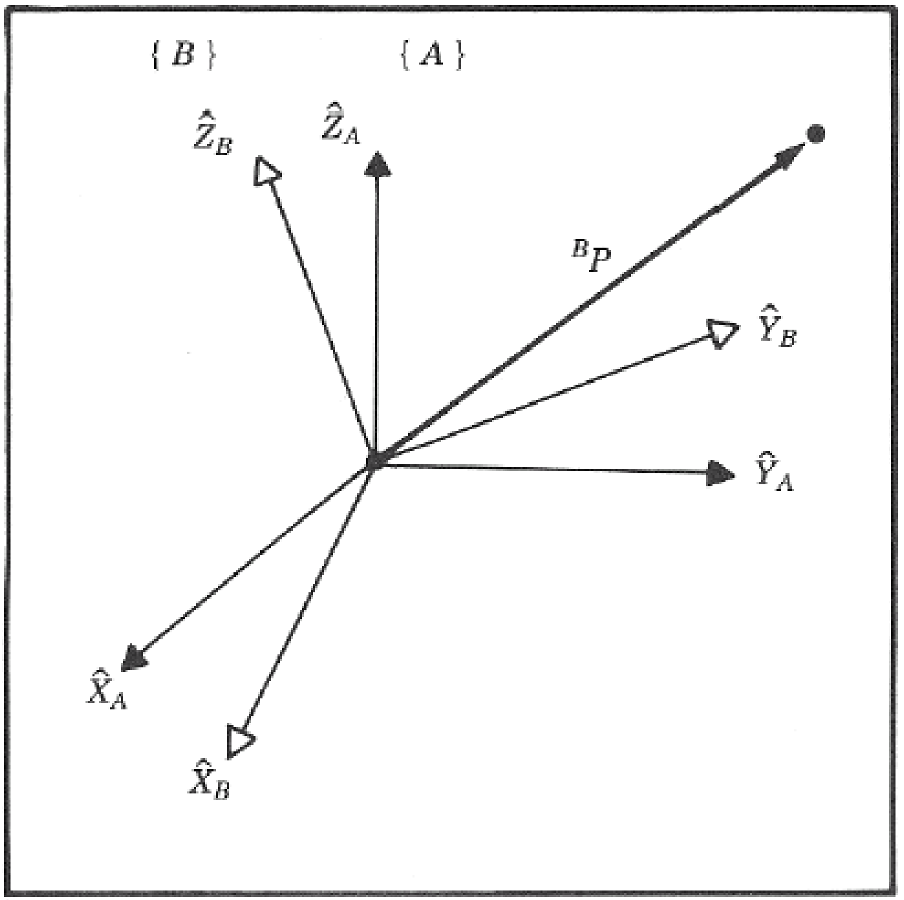
Mapping – Translated Frames

- Assuming that frame {B} is only ***translated*** (not rotated) with respect frame {A}.
- The position of the point can be expressed in frame {A} as follows.



Mapping – Rotated Frames

- Assuming that frame {B} is only ***rotated*** (not translated) with respect frame {A} (the origins of the two frames are located at the same point) the position of the point in frame {B} can be expressed in frame {A} using the rotation matrix as follows



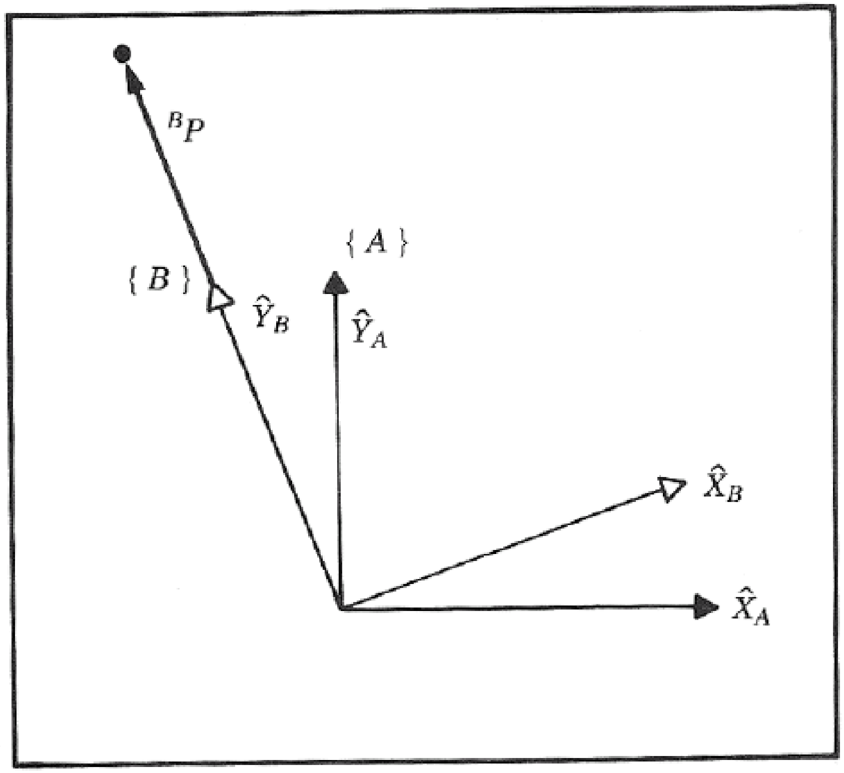
Mapping - Rotated Frames - Inversion

- Given: The rotation matrix from frame {A} to frame {B}

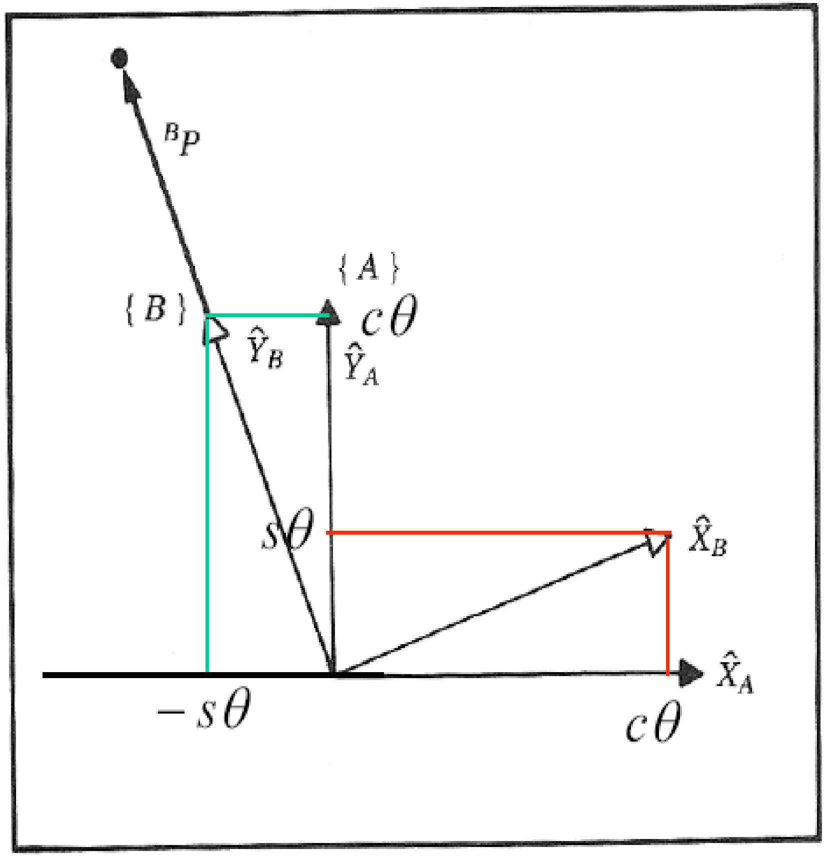
- Calculate: The rotation matrix from frame {B} to frame {A}

Mapping – Rotating Frames

Example



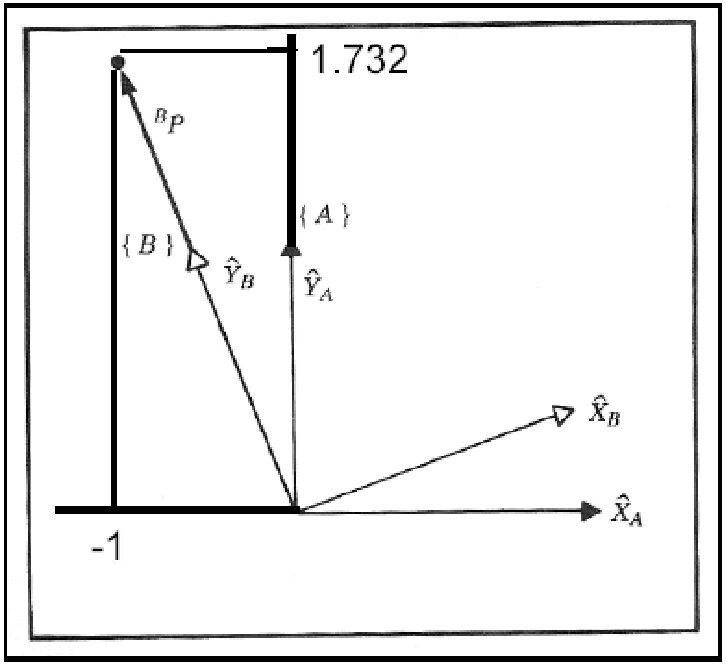
Mapping – Rotating Frames
Example



Mapping – Rotating Frames

Example

$$
{}^{A}P = {}^{A}_{B}R\ {}^{B}P = \begin{bmatrix} c\theta & -s\theta & 0 \\ s\theta & c\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ {}^{B}p_y \\ 0 \end{bmatrix} = \begin{bmatrix} 0.866 & -0.500 & 0.000 \\ 0.500 & 0.866 & 0.000 \\ 0.000 & 0.000 & 1.000 \end{bmatrix} \begin{bmatrix} 0.000 \\ 2.000 \\ 0.000 \end{bmatrix} = \begin{bmatrix} -1.000 \\ 1.732 \\ 0.000 \end{bmatrix}
$$



Mapping – Rotated Frames – General Notation

- The rotation matrices with respect to the reference frame are defined as follows:

Mapping - Rotated Frames - Methods

- X-Y-Z Fixed Angles
  - ***The rotations performed about an axis of a fixed reference frame***
- Z-Y-X Euler Angles
  - ***The rotations performed about an axis of a moving reference frame***

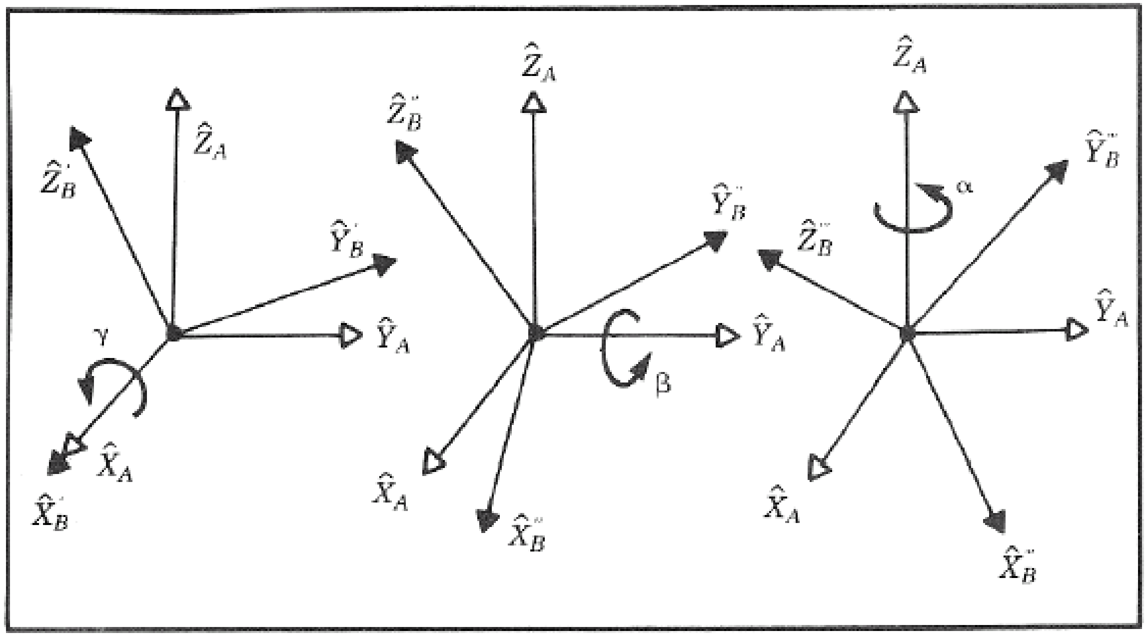
Mapping - Rotated Frames – X-Y-Z Fixed Angles

Start with frame {B} coincident with a known reference frame {A}.

- Rotate frame {B} about $\hat{X}_A$ by an angle γ
- Rotate frame {B} about $\hat{Y}_A$ by an angle β
- Rotate frame {B} about $\hat{Z}_A$ by an angle α

$\left.\begin{matrix} \\ \\ \\ \end{matrix}\right\}$ **Fixed Angles**

Note - *Each of the three rotations takes place about an axis in the fixed reference frame {A}*



Mapping - Rotated Frames – X-Y-Z Fixed Angles

$$ {}^{A}_{B}R_{XYZ}(\gamma,\beta,\alpha) = R_Z(\alpha)R_Y(\beta)R_X(\gamma) = \begin{bmatrix} c\alpha & -s\alpha & 0 \\ s\alpha & c\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\beta & 0 & s\beta \\ 0 & 1 & 0 \\ -s\beta & 0 & c\beta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\gamma & -s\gamma \\ 0 & s\gamma & c\gamma \end{bmatrix} $$

$$ {}^{A}_{B}R_{XYZ}(\gamma,\beta,\alpha) = \begin{bmatrix} c\alpha c\beta & c\alpha s\beta s\gamma - s\alpha c\gamma & c\alpha s\beta c\gamma + s\alpha s\gamma \\ s\alpha c\beta & s\alpha s\beta s\gamma + c\alpha c\gamma & s\alpha s\beta c\gamma - c\alpha s\gamma \\ -s\beta & c\beta s\gamma & c\beta c\gamma \end{bmatrix} $$

Mapping - Rotated Frames – X-Y-Z Fixed Angles

$$
{}^{A}_{B}R_{XYZ}(\gamma,\beta,\alpha)=\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}=\begin{bmatrix} c\alpha c\beta & c\alpha s\beta s\gamma - s\alpha c\gamma & c\alpha s\beta c\gamma + s\alpha s\gamma \\ s\alpha c\beta & s\alpha s\beta s\gamma + c\alpha c\gamma & s\alpha s\beta c\gamma - c\alpha s\gamma \\ -s\beta & c\beta s\gamma & c\beta c\gamma \end{bmatrix}
$$

$$\beta = \text{Atan2}(-r_{31}, \sqrt{r_{11}^2 + r_{21}^2}) \qquad \text{for} \quad -90^{\circ} \le \beta \le 90^{\circ}$$

$$\alpha = \text{Atan2}(r_{21}/c\text{â}, r_{11}/c\text{â})$$

$$\gamma = \text{Atan2}(r_{32}/c\text{â}, r_{33}/c\text{â})$$

$$\beta = \pm 90^{\circ}$$

$$\alpha = 0$$

$$\gamma = \text{Atan2}(r_{12}, r_{22})$$

Atan2 - Definition

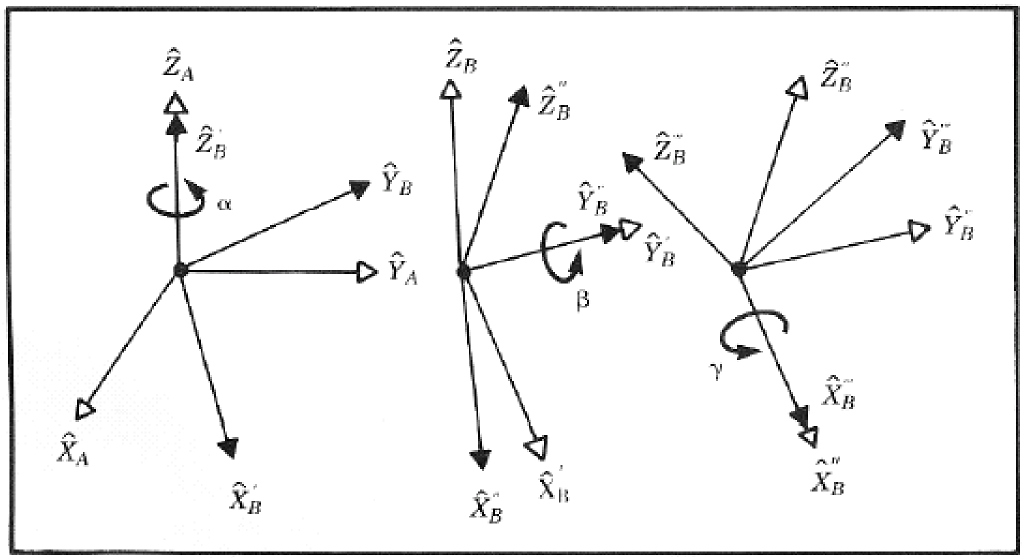
- Four-quadrant inverse tangent (arctangent) in the range of
- For example

Mapping - Rotated Frames – Z-Y-X Euler Angles

- Start with frame {B} coincident with a known reference frame {A}.
  - Rotate frame {B} about $\hat{Z}_A$ by an angle α
  - Rotate frame {B} about $\hat{Y}_B$ by an angle β
  - Rotate frame {B} about $\hat{X}_B$ by an angle γ

  Euler Angles

- **Note** - Each rotation is performed about an axis of the moving reference frame {B}, rather then a fixed reference frame {A}.



Mapping - Rotated Frames – Z-Y-X Euler Angles

$$= R_Z(\alpha)R_Y(\beta)R_X(\gamma) = \begin{bmatrix} c\alpha & -s\alpha & 0 \\ s\alpha & c\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\beta & 0 & s\beta \\ 0 & 1 & 0 \\ -s\beta & 0 & c\beta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\gamma & -s\gamma \\ 0 & s\gamma & c\gamma \end{bmatrix}$$

$$= \begin{bmatrix} c\alpha c\beta & c\alpha s\beta s\gamma - s\alpha c\gamma & c\alpha s\beta c\gamma + s\alpha s\gamma \\ s\alpha c\beta & s\alpha s\beta s\gamma + c\alpha c\gamma & s\alpha s\beta c\gamma - c\alpha s\gamma \\ -s\beta & c\beta s\gamma & c\beta c\gamma \end{bmatrix}$$

Mapping - Rotated Frames

- **Fixed Angles Versus Euler Angles**

- Three rotations taken about fixed axes (Fixed Angles) yield the same final orientation as the same three rotations taken in an opposite order about the axes of the moving frame (Euler Angles)

Operator - Rotating Vector

- Rotational Operator - Operates on a vector ${}^{A}P_1$ and changes that vector to a new vector ${}^{B}P_1$, by means of a rotation R

- Note: The rotation matrix which rotates vectors through the same rotation R , is the same as the rotation which describes a frame rotated by R relative to the reference frame

Operator - Rotating Vector

Example

Mapping – General Frames

- Assuming that frame {B} is both ***translated*** and ***rotated*** with respect frame {A},
- The position of the point expressed in frame {B} can be expressed in frame {A} as follows