Fundamentals of Control Systems

MECH 420 / ELEC 436
Department of Mechanical Engineering
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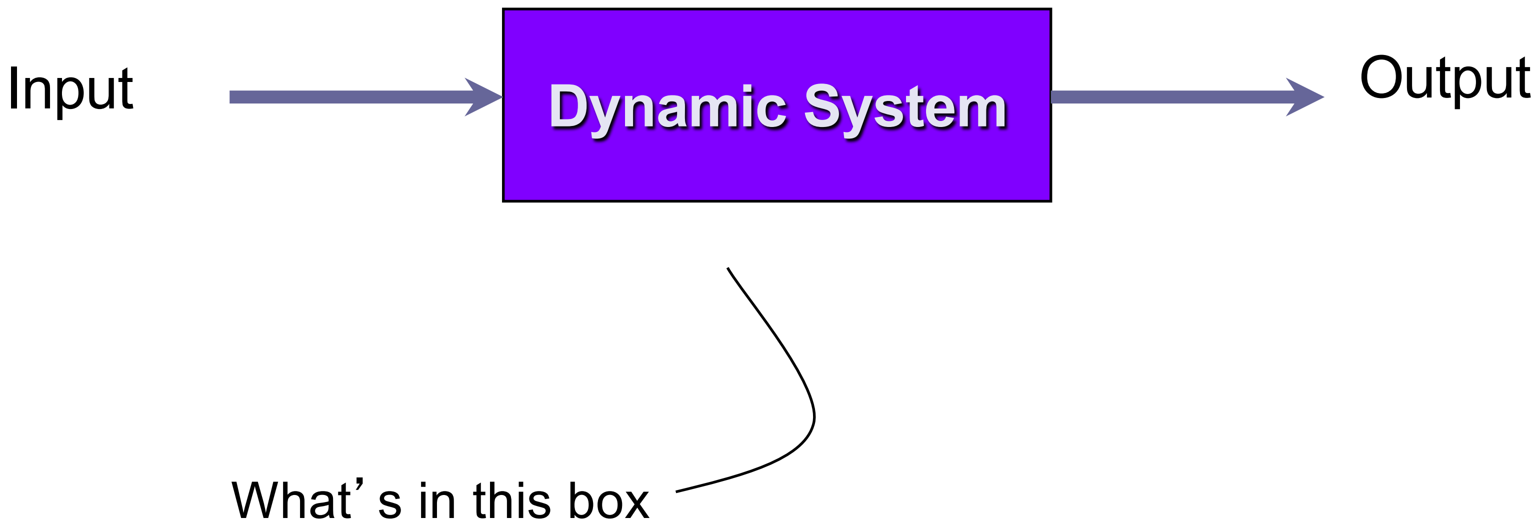
4. Transfer Function

Objective:

Given ODE/state space representation of dynamic systems, characterize our preferred abstract control system representation



ODE System Interpretation

A generic linear system is described by an input/output relationship

$$D\ \underbrace{y(t)}_{\text{Output}} = \underbrace{u(t)}_{\text{Input}}$$

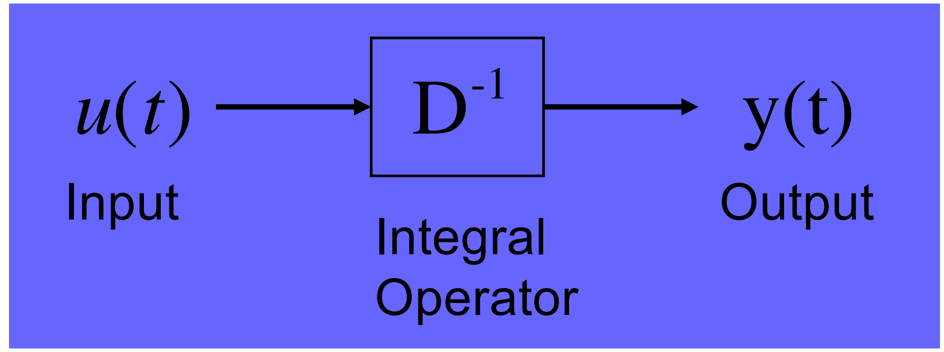
Linear (homogeneous) differential operator D is

$$D[\cdot] = a_0 \frac{d^n}{dt^n}[\cdot] + a_1 \frac{d^{n-1}}{dt^{n-1}}[\cdot] + \ldots\ldots a_{n-1}\frac{d}{dt}[\cdot] + a_n$$

$$Ex : D[.] = a_0 \frac{d^2}{dt^2}[.] + a_1 \frac{d}{dt}[.] + a_2 \ \text{ so that } a_0\ddot{y} + a_1\dot{y} + a_2 y = u$$

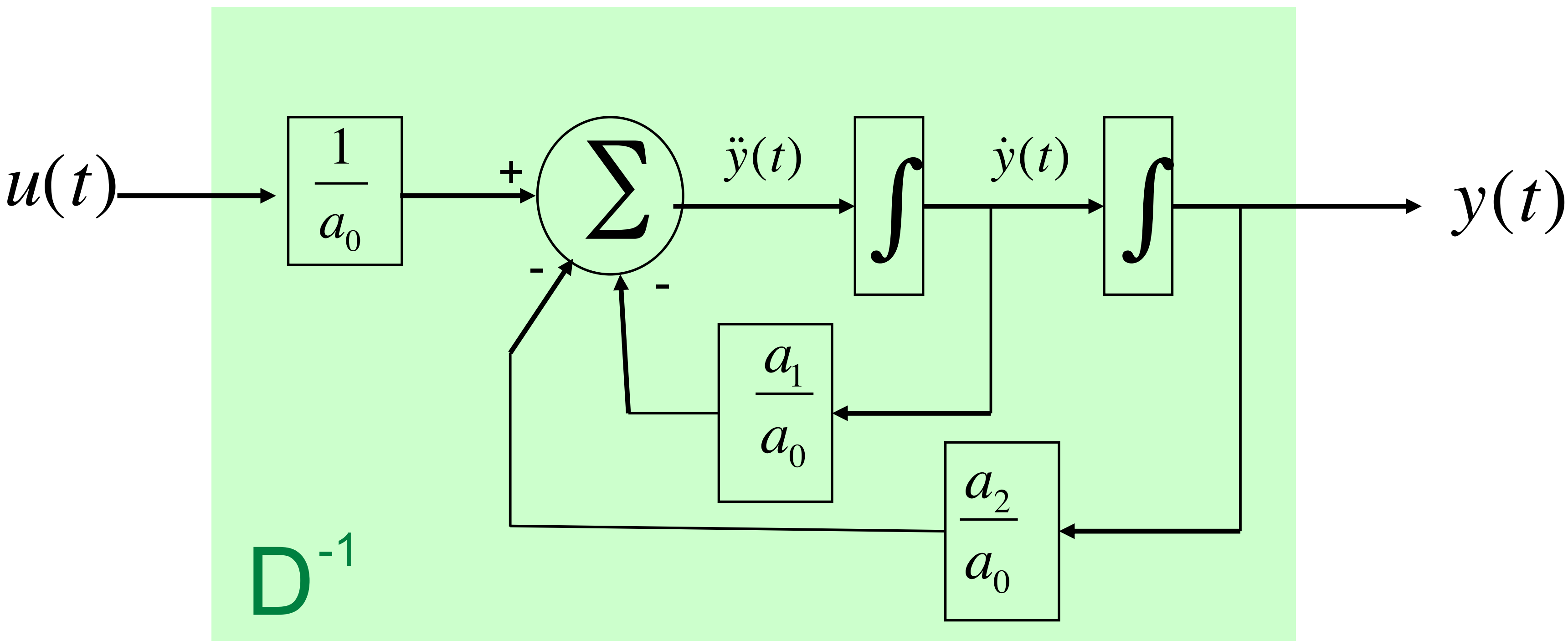
The following representation is of interest

$$y(t) = \mathrm{D}^{-1}\, u(t)$$

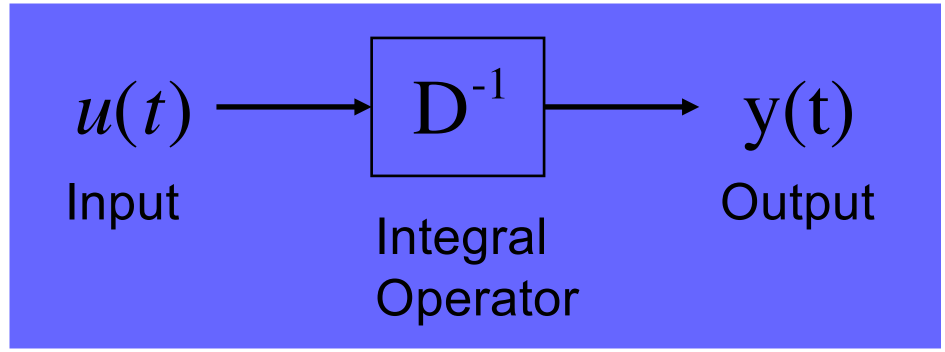


D^{-1} is the reciprocal of the operator D

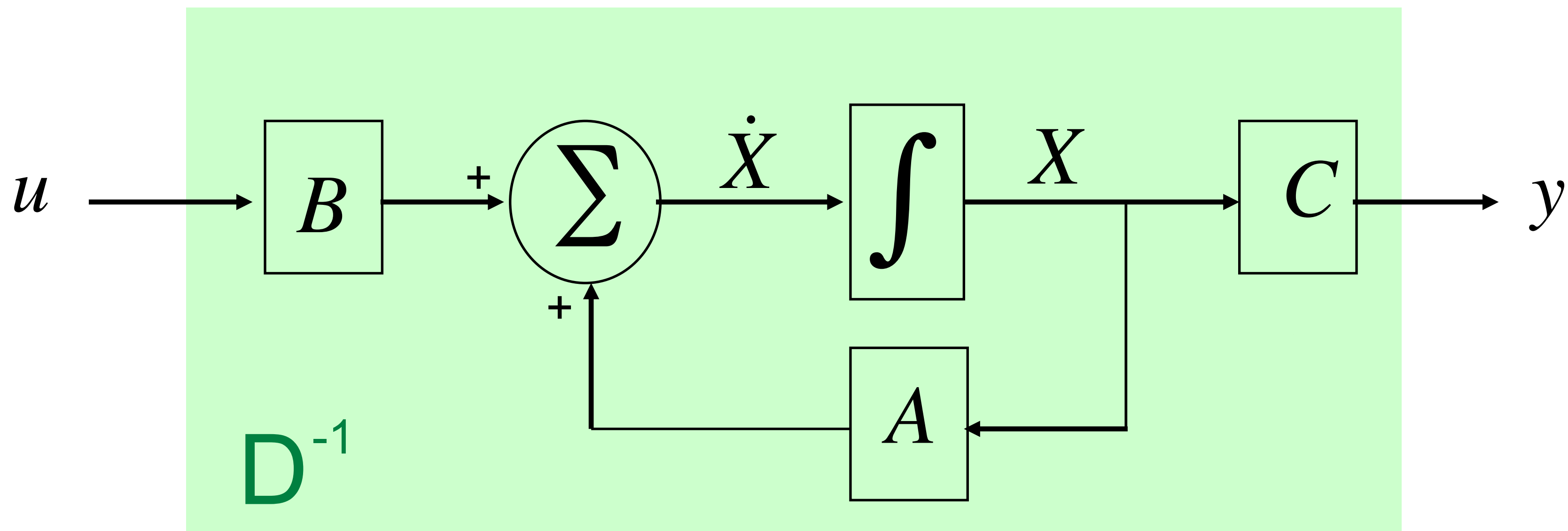
Ex. $a_0\ddot{y} + a_1\dot{y} + a_2 y = u$



$$y(t) = D^{-1}\, u(t)$$

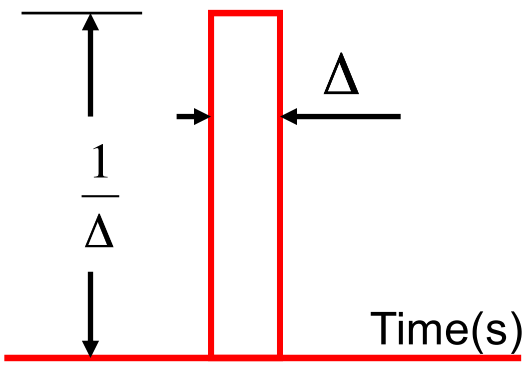


For State Space Representation $\begin{cases} \dot{X} = AX + Bu \\ y = CX \end{cases}$



System Representation in terms of Impulse Response

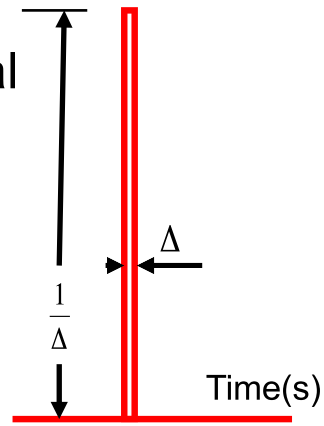
Pulse signal

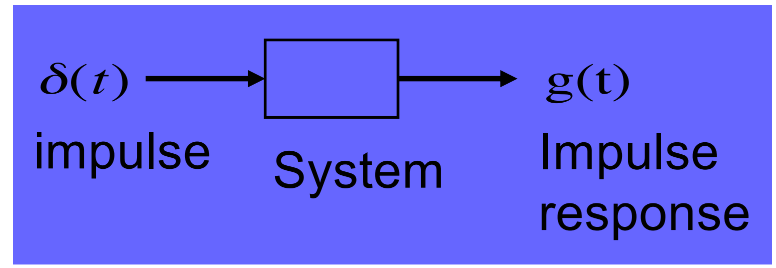


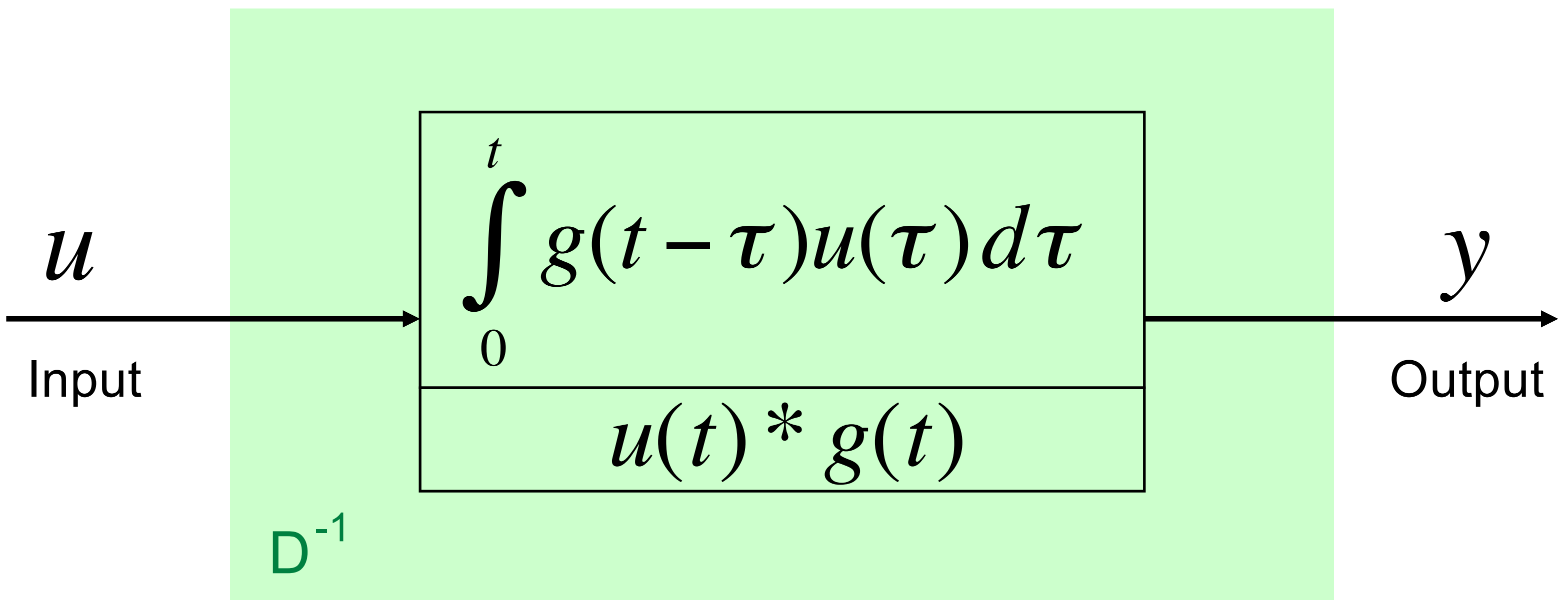
Impulse signal

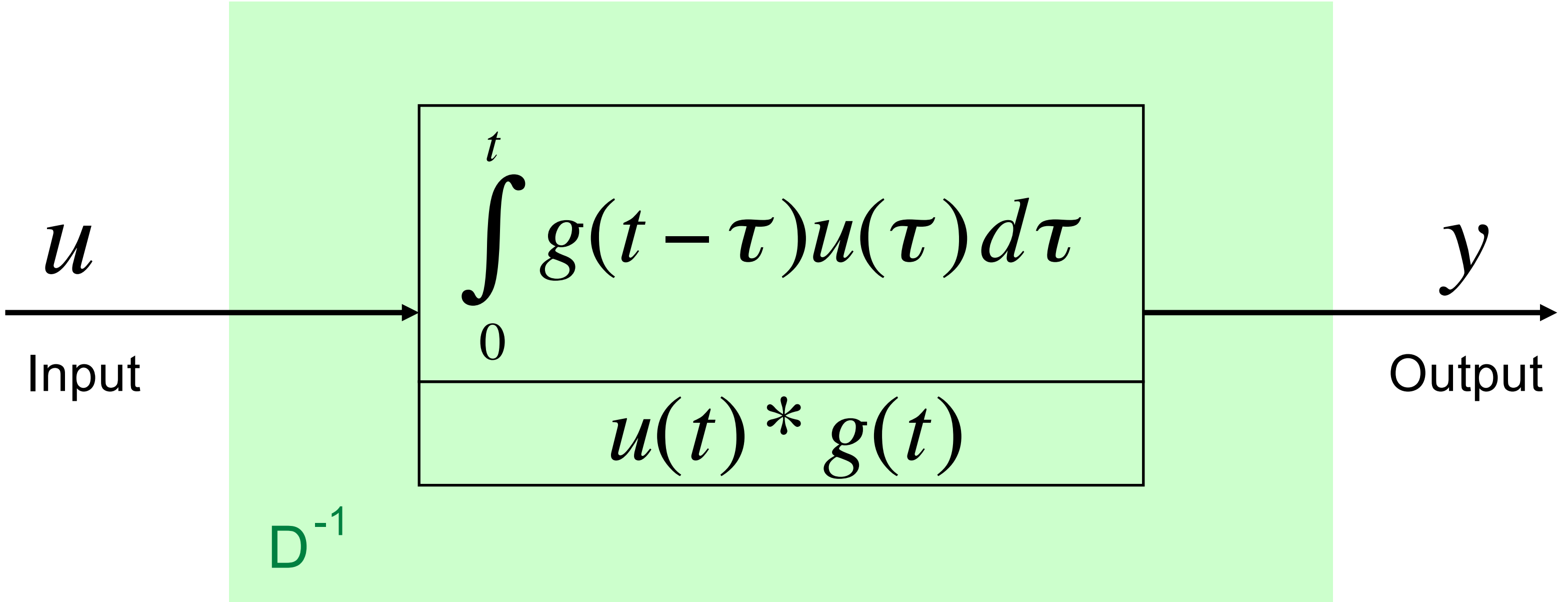
$$\delta(t)$$

$$\delta(t) = \lim_{\Delta \to 0}$$



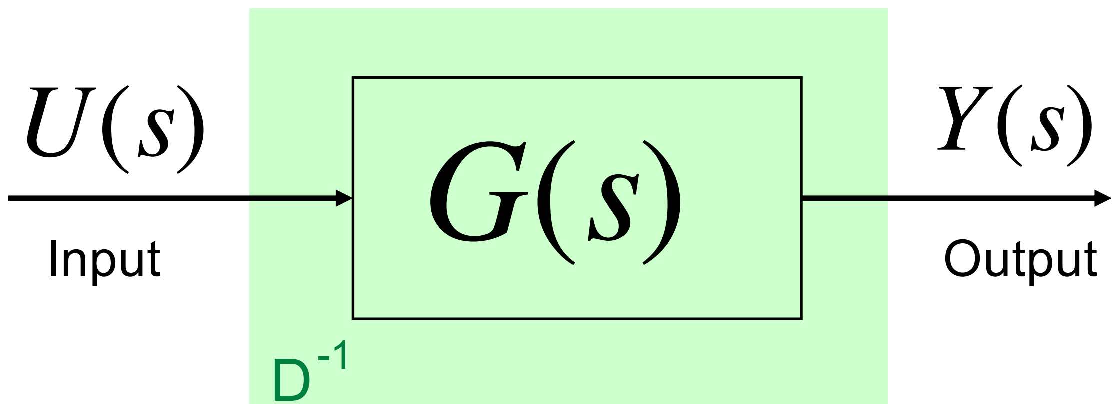






$$y(t) = u(t) * g(t) \quad \xrightarrow{\mathcal{L}} \quad Y(s) = U(s)G(s)$$

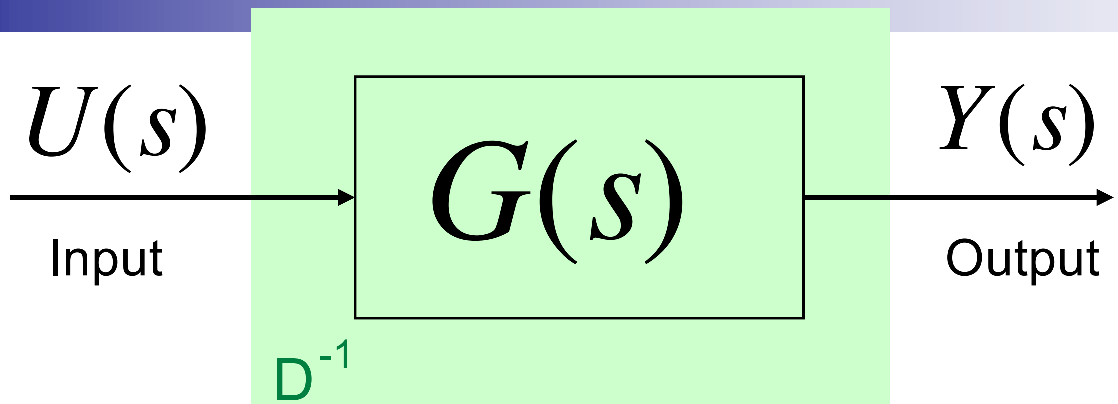
$$\Big\downarrow \mathcal{L}$$



Definition:

Transfer function

$$G(s) = \mathcal{L}\left[g(t)\right] = \frac{Y(s)}{U(s)} = \frac{Output}{Input}$$



Note 1:

$$D\,y(t) = u(t) \; or \; y(t) = D^{-1}u(t) \leftarrow \quad \text{Symbolic}$$

$$Y(s) = G(s)U(s) \leftarrow \quad \text{True multiplication, hence } G(s) = \frac{Y(s)}{U(s)}$$

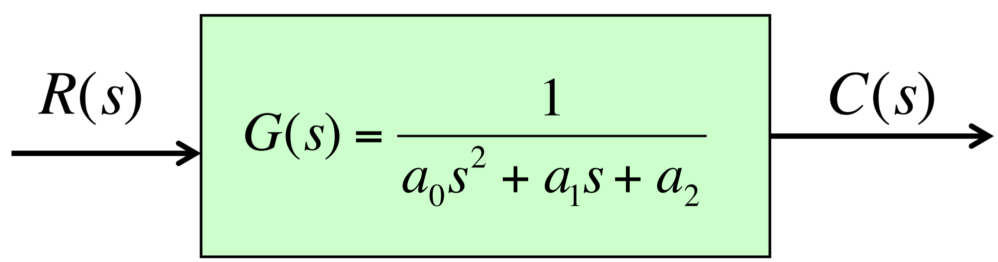
Note 2:

In general, we compute the transfer function directly from the o.d.e <u>assuming zero initial conditions</u>

$$a_0\ddot{c} + a_1\dot{c} + a_2 c = r$$

$$[a_0 s^2 + a_1 s + a_2]C(s) = R(s)$$

$$\Rightarrow G(s) = \frac{C(s)}{R(s)} = \frac{1}{a_0 s^2 + a_1 s + a_2}$$



Transfer Function from State Space Representation

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases}$$

$$sX(s) = AX(s) + BU(s)$$

$$(sI - A)X(s) = BU(s)$$

$$Y(s) = CX(s)$$

$$Y(s) = C(sI - A)^{-1}BU(s)$$

$$\frac{Y(s)}{U(s)} = G(s) = C(sI - A)^{-1}B$$

Definition: Characteristic Equation

The characteristic equation (C.E.) of a linear system is defined as the equation obtained by setting the denominator polynomial of the transfer function to zero.

$$G(s) = \frac{Q(\mathrm{s})}{P(\mathrm{s})}$$

$$\mathrm{C.E}: \quad Solution\ of\ \ P(\mathrm{s}) = 0$$

Multi-Input Multi-Output (MIMO) Control Systems

Single-Input Single-Output (SISO) & Multi-Input Multi-Output (MIMO)

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases}$$

$$x \in R^n \Rightarrow A \in R^{nxn}$$

$SISO$ (1 $input$ & 1 $output$)

$$u \in R^1 \Rightarrow B \in R^{nx1}$$

$$y \in R^1 \Rightarrow C \in R^{1xn}$$

$\Rightarrow$ Transfer Function

$$G(s) = \frac{Y(s)}{U(s)} \in R^1$$

$MIMO$ (m $inputs$ & p $outputs$)

$$u \in R^m \Rightarrow B \in R^{nxm}$$

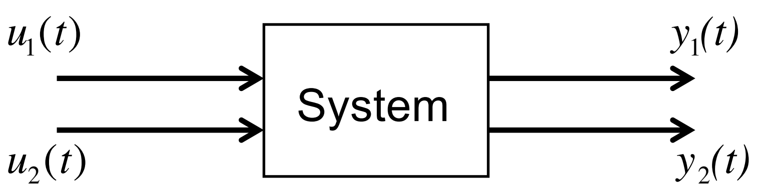
$$y \in R^p \Rightarrow C \in R^{pxn}$$

$\Rightarrow$ Transfer Function Matrix

$$Y(s) = G(s)U(s)$$

$$G(s) \in R^{mxp}$$

Ex:



$$Y_1(s) = G_{11}(s)U_1(s) + G_{12}(s)U_2(s)$$

$$Y_2(s) = G_{21}(s)U_1(s) + G_{22}(s)U_2(s)$$

$$\underbrace{\begin{bmatrix} Y_1(s) \\ Y_2(s) \end{bmatrix}}_{Y(s)} = \underbrace{\begin{bmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{bmatrix}}_{G(s)} \underbrace{\begin{bmatrix} U_1(s) \\ U_2(s) \end{bmatrix}}_{U(s)}$$