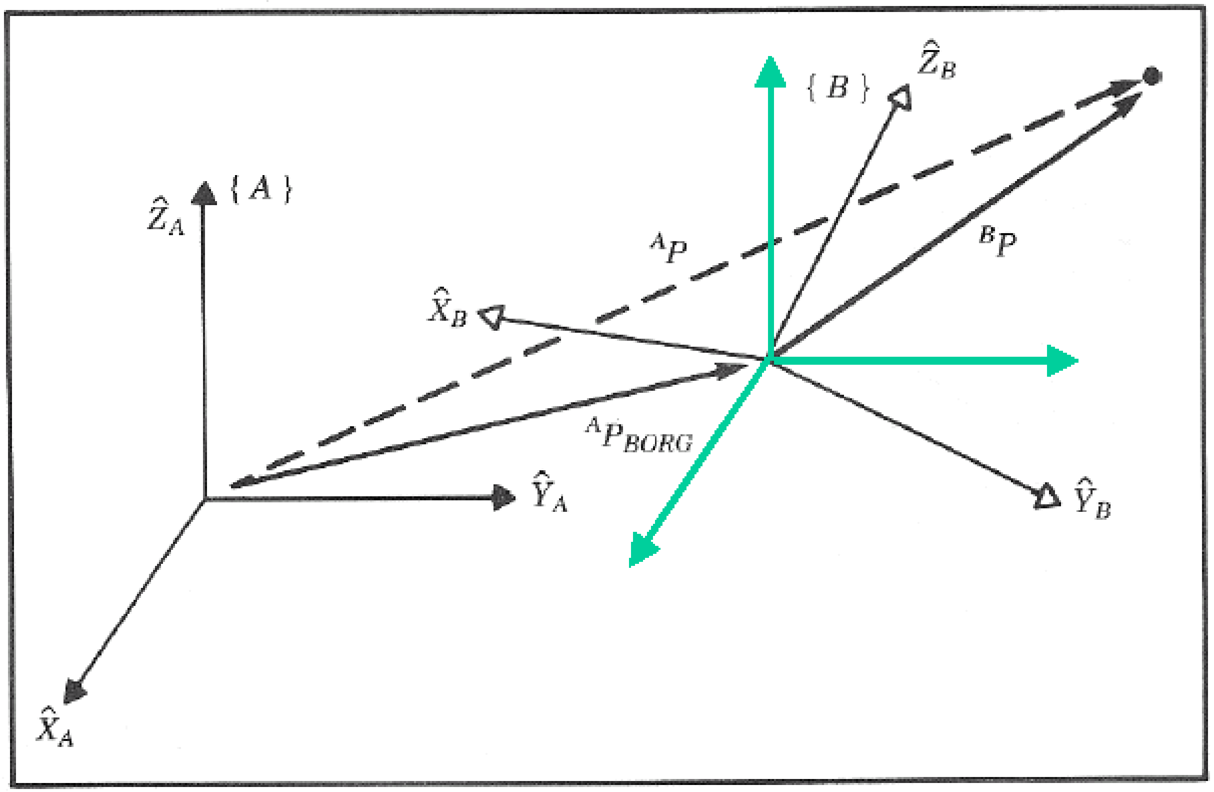
Lecture 3

Homogeneous Transformations
and Forward Kinematics

Mapping – General Frames

- Assuming that frame {B} is both ***translated*** and ***rotated*** with respect frame {A},
- The position of the point expressed in frame {B} can be expressed in frame {A} as follows



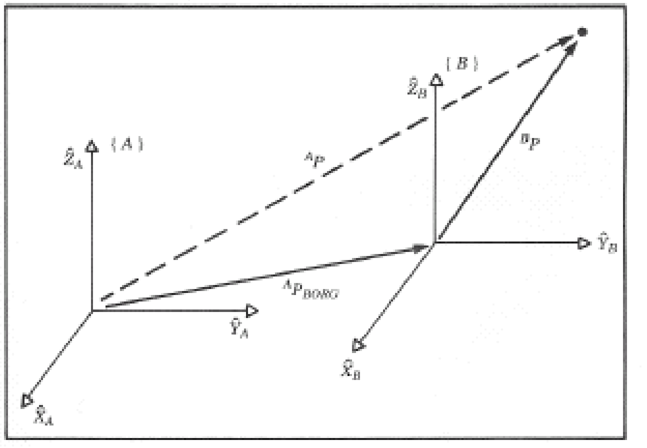
Mapping – Homogeneous Transform

- The homogeneous transform is a 4x4 matrix casting the ***rotation*** and ***translation*** of a general transform into a single matrix
- In other fields of study it can be used to compute perspective and scaling operations when the last row is other than [0001] or the rotation matrix is not orthonormal.

Homogeneous Transform – Special Cases

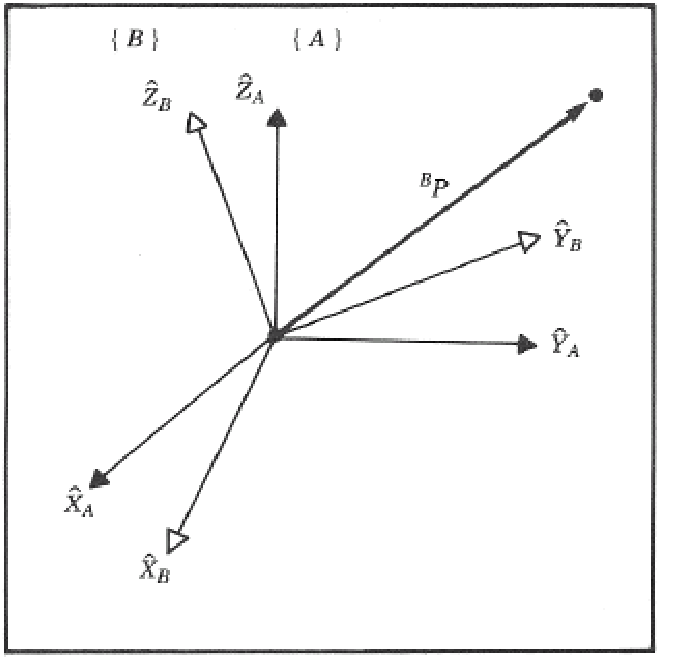
Translation

$$ {}^{A}_{B}T = \begin{bmatrix} 1 & 0 & 0 & {}^{A}P_{BORGx} \\ 0 & 1 & 0 & {}^{A}P_{BORGy} \\ 0 & 0 & 1 & {}^{A}P_{BORGz} \\ 0 & 0 & 0 & 1 \end{bmatrix} $$



Rotation

$$ {}^{A}_{B}T = \begin{bmatrix} r_{11} & r_{12} & r_{13} & 0 \\ r_{21} & r_{22} & r_{23} & 0 \\ r_{31} & r_{32} & r_{33} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$



Homogeneous Transform
Example

Given:

$$ {}^{B}P = \begin{bmatrix} \boxed{} \end{bmatrix} = \begin{bmatrix} 3.0 \\ 7.0 \\ 0.0 \end{bmatrix} $$

Frame {B} is rotated relative to frame {A} about $\hat{Z}$ by 30 degrees, and translated 10 units in $\hat{X}_A$ and 5 units in $\hat{Y}_A$

Calculate: The vector ${}^{A}P$ expressed in frame {A}.

Homogeneous Transform

Example

Transformation Arithmetic - Compound Transformations

Given: Vector ${}^{C}P$

Frame {C} is known relative to frame {B} - ${}^{B}_{C}T$

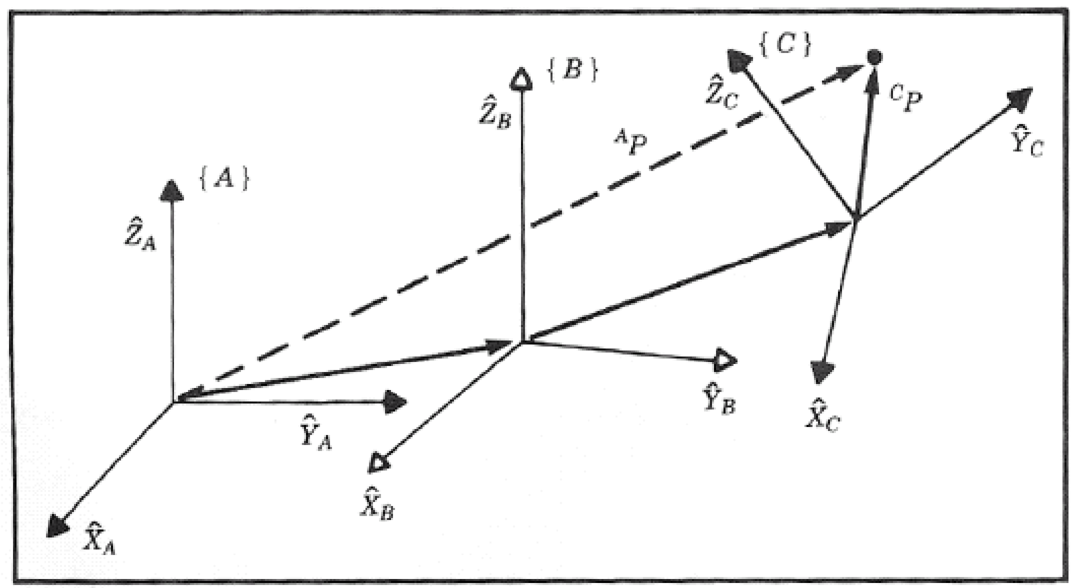
Frame {B} is known relative to frame {A} - ${}^{A}_{B}T$

Calculate: Vector ${}^{A}P$

$$ {}^{B}P = {}^{B}_{C}T \, {}^{C}P $$

$$ {}^{A}P = {}^{A}_{B}T \, {}^{B}P $$

$$ {}^{A}P = {}^{A}_{B}T \, {}^{B}_{C}T \, {}^{C}P $$

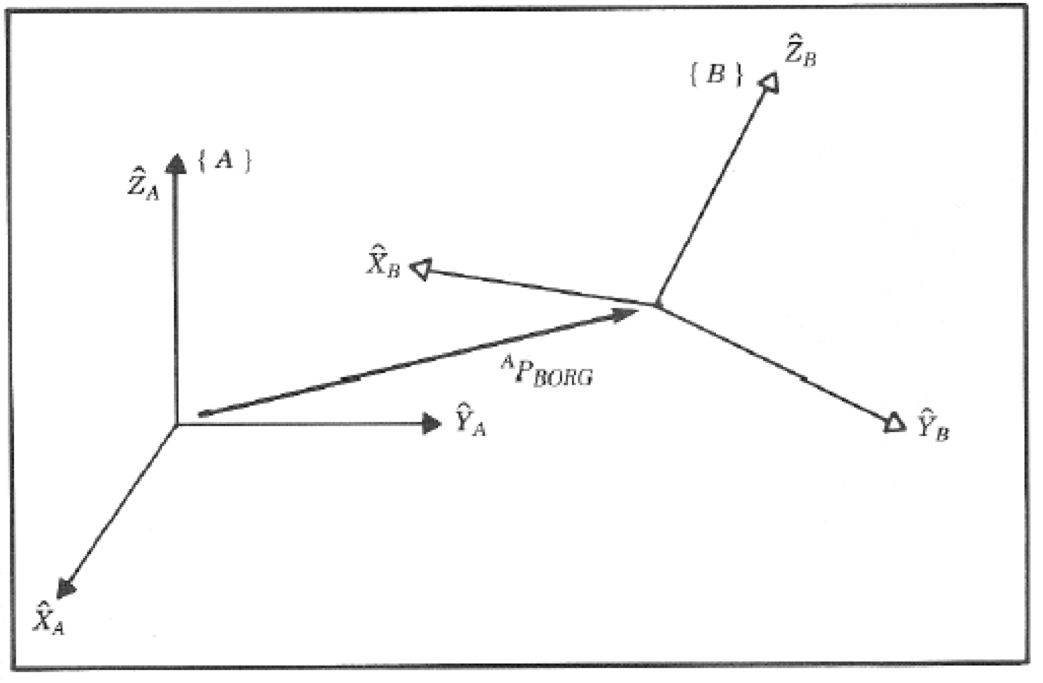


Transformation Arithmetic – Inverted Transformation

Given: Description of frame {B} relative to frame {A} - ${}^{A}_{B}T \quad ({}^{A}_{B}R, {}^{A}P_{BORG})$

Calculate: Description of frame {A} relative to frame {B} -

Homogeneous Transform ${}^{B}_{A}T \quad ({}^{B}_{A}R, {}^{B}P_{AORG})$



Inverted Transformation
Example

Given: Description of frame {B} relative to frame {A} - ${}^{A}_{B}T \quad ({}^{A}_{B}R, {}^{A}P_{BORG})$

Frame {B} is rotated relative to frame {A} about $\hat{Z}$ by 30 degrees, and translated 4 units in $\hat{X}$, and 3 units in $\hat{Y}$

Calculate: Homogeneous Transform ${}^{B}_{A}T \quad ({}^{B}_{A}R, {}^{B}P_{AORG})$

Inverted Transformation

Example

Operator – Transforming Vector

Transformation Operator - Operates on a a vector $^{A}P_1$ and changes that vector to a new vector $^{B}P_1$, by means of a rotation by R and translation by Q

Note: The matrix of the transform operator T which rotates vectors by R and translation by Q, is the same as the transformation matrix which describes a frame rotated by R and translated by Q relative to the reference frame

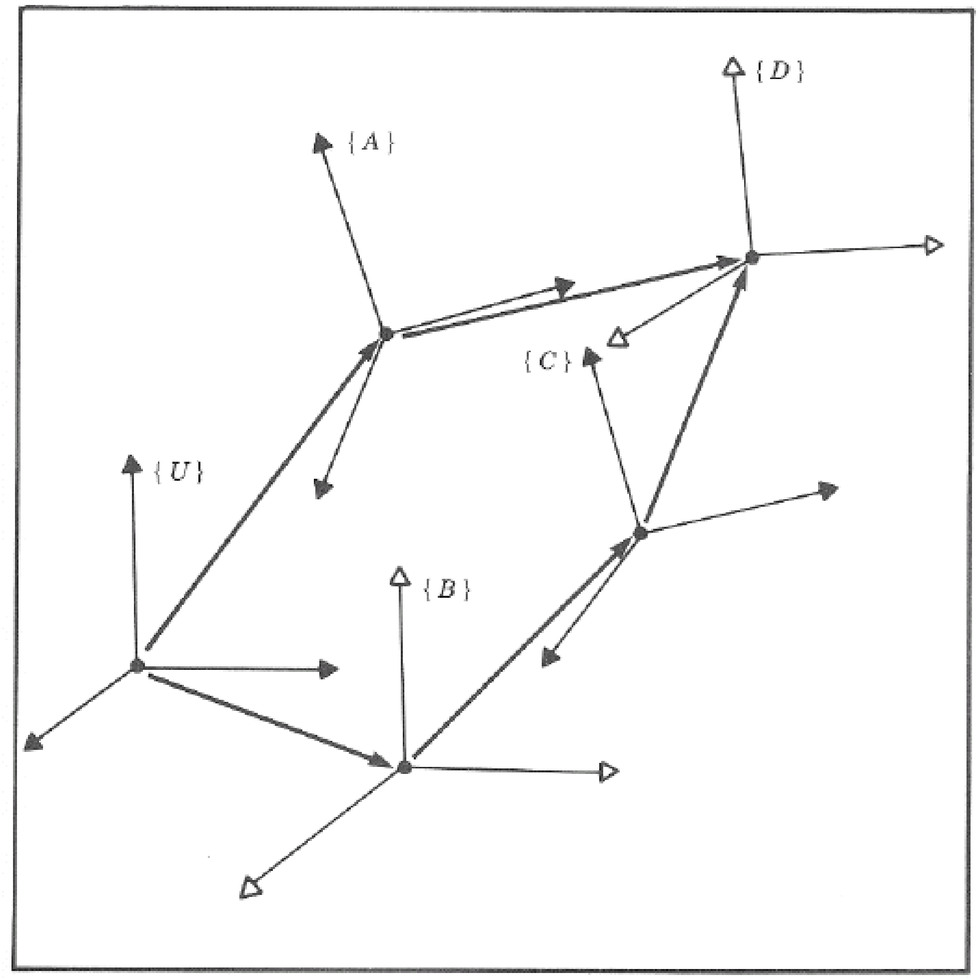
Homogeneous Transform - Summary of Interpretation

- As a general tool to represent a frame we have introduced the ***homogeneous transformation***, a 4x4 matrix containing orientation and position information.
- ***Three interpretations of the homogeneous transformation:***

Transform Equations

Given: ${}^{U}_{A}T,\ {}^{A}_{D}T,\ {}^{U}_{B}T,\ {}^{C}_{D}T$

Calculate: ${}^{B}_{C}T$

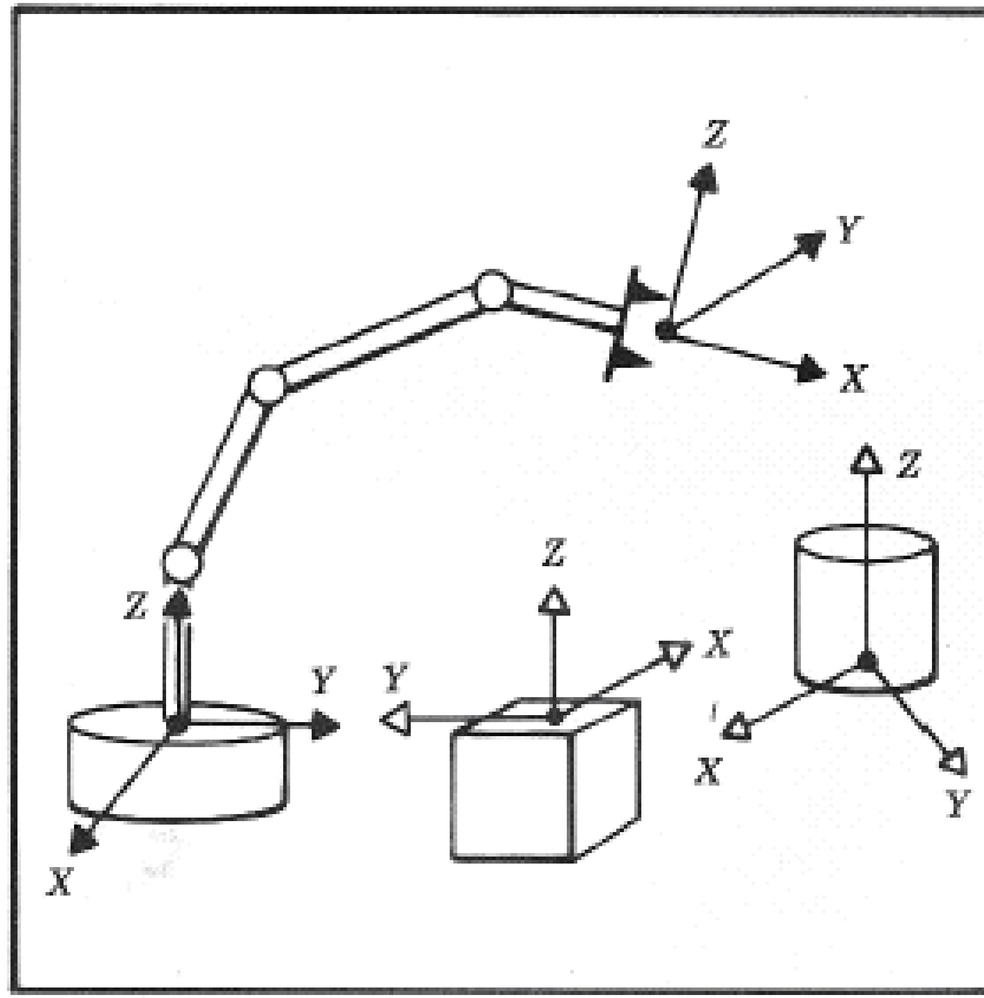


Kinematics - Introduction

- ***Kinematics*** - the science of motion which treat motions without regard to the forces that cause them
  - e.g. position, velocity, acceleration, higher derivatives of the position
- ***Kinematics of Manipulators*** - All the geometrical and time based properties of the motion

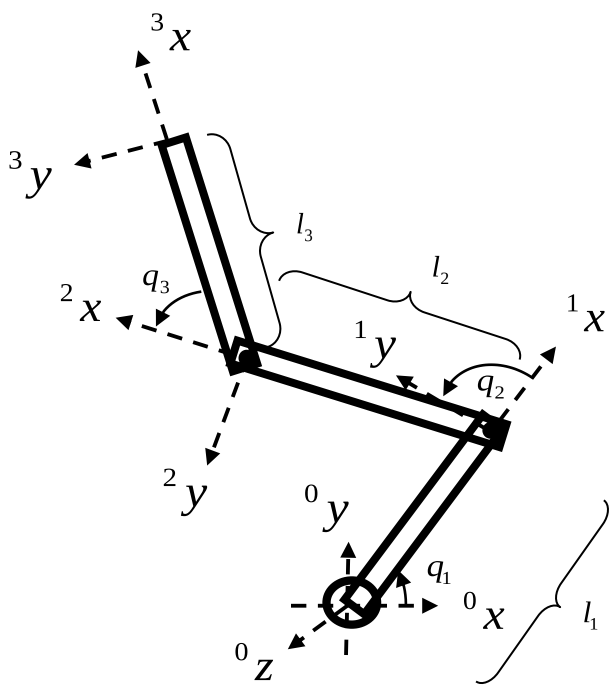
Central Topic

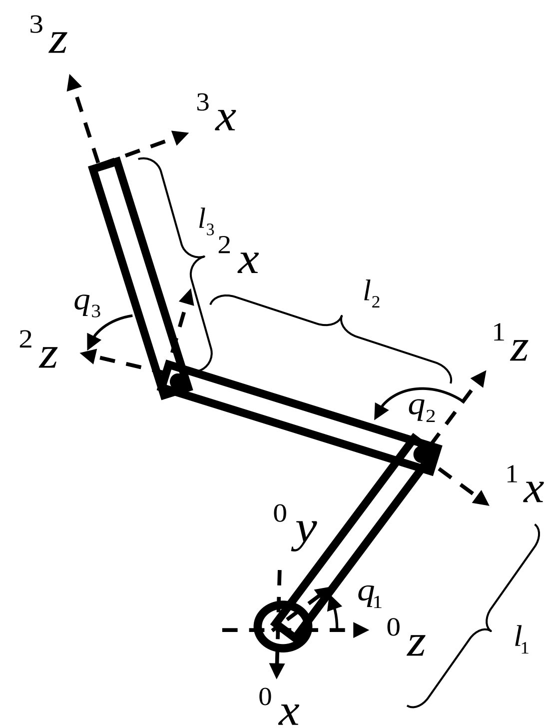
- **Problem**
  - Given: The manipulator geometrical parameters
  - Specify: The position and orientation of manipulator
- **Solution**
  - Coordinate system or “Frames” are attached to the manipulator and objects in the environment following the Denenvit-Hartenberg notation.



DH parameters

- There are a large number of ways that homogeneous transforms can encode the kinematics of a manipulator
- We will sacrifice some of this flexibility for a more systematic approach: DH (Denavit-Hartenberg) parameters.
- DH parameters is a standard for describing a series of transforms for arbitrary mechanisms.





Forward kinematics: DH parameters

These four DH parameters,

$$\begin{pmatrix} a_i & \alpha_i & d_i & \theta_i \end{pmatrix}$$

represent the following homogeneous matrix:

$$T = \underbrace{\begin{pmatrix} c_{\theta_i} & -s_{\theta_i} & 0 & 0 \\ s_{\theta_i} & c_{\theta_i} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{\text{First, translate by } d_i \text{ along } z \text{ axis and rotate by } \theta_i \text{ about } z \text{ axis}} \underbrace{\begin{pmatrix} 1 & 0 & 0 & a_i \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c_{\alpha_i} & -s_{\alpha_i} & 0 \\ 0 & s_{\alpha_i} & c_{\alpha_i} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{\text{Then, translate by } a_i \text{ along } x \text{ axis and rotate by } \alpha_i \text{ about } x \text{ axis}}$$

First, translate by d_i along *z* axis

and rotate by θ_i about *z* axis

Then, translate by a_i along *x* axis

and rotate by α_i about *x* axis

Forward kinematics: DH parameters

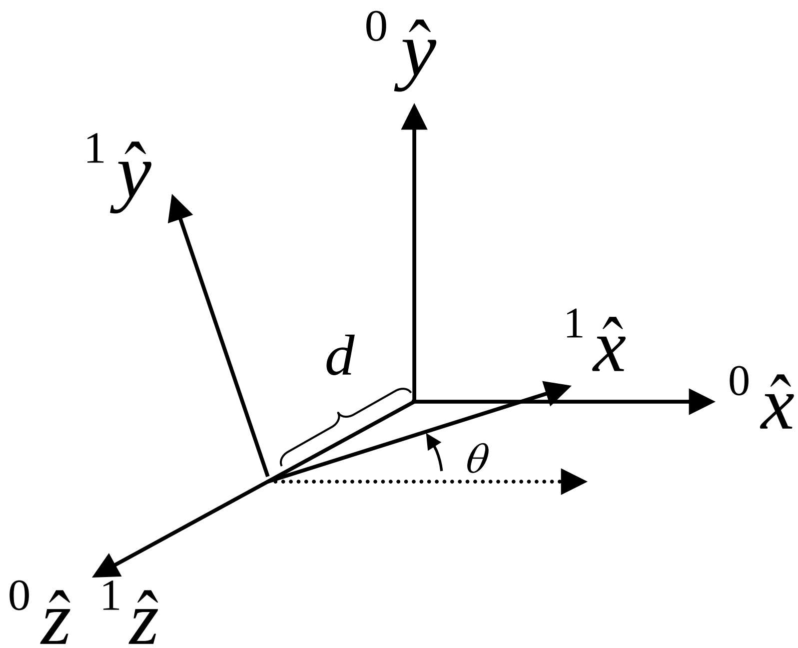
$$T = \begin{pmatrix} c_{\theta_i} & -s_{\theta_i} & 0 & 0 \\ s_{\theta_i} & c_{\theta_i} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & a_i \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c_{\alpha_i} & -s_{\alpha_i} & 0 \\ 0 & s_{\alpha_i} & c_{\alpha_i} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

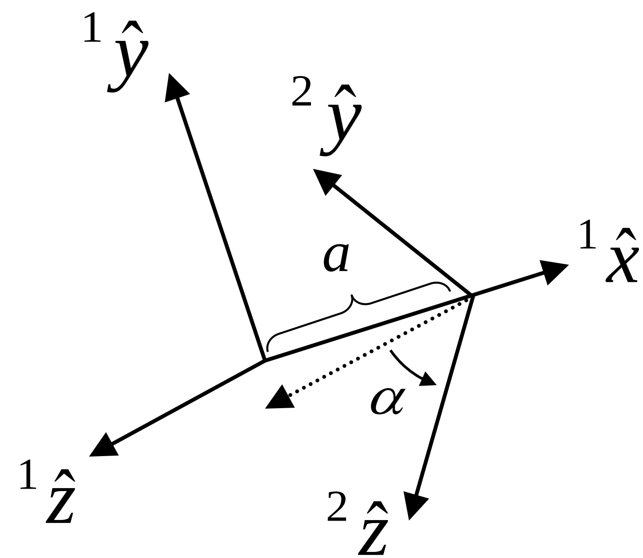
$$= \begin{pmatrix} c_{\theta_i} & -s_{\theta_i} c_{\alpha_i} & s_{\theta_i} s_{\alpha_i} & a_i c_{\theta_i} \\ s_{\theta_i} & c_{\theta_i} c_{\alpha_i} & -c_{\theta_i} s_{\alpha_i} & a_i s_{\theta_i} \\ 0 & s_{\alpha_i} & c_{\alpha_i} & d_i \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Forward kinematics: DH parameters

Four DH parameters: $\begin{pmatrix} a_i & \alpha_i & d_i & \theta_i \end{pmatrix}$

$$T = T_{rot(z,\theta_i)} T_{trans(z,d_i)} T_{rot(x,\alpha_i)} T_{trans(x,a_i)}$$





First, translate by d_i along *z* axis

and rotate by θ_i about *z* axis

Then, translate by a_i along *x* axis

and rotate by α_i about *x* axis

Joint/Link Description

- ***Lower pair*** - The connection between a pair of bodies when the relative motion is characterize by two surfaces sliding over one another.

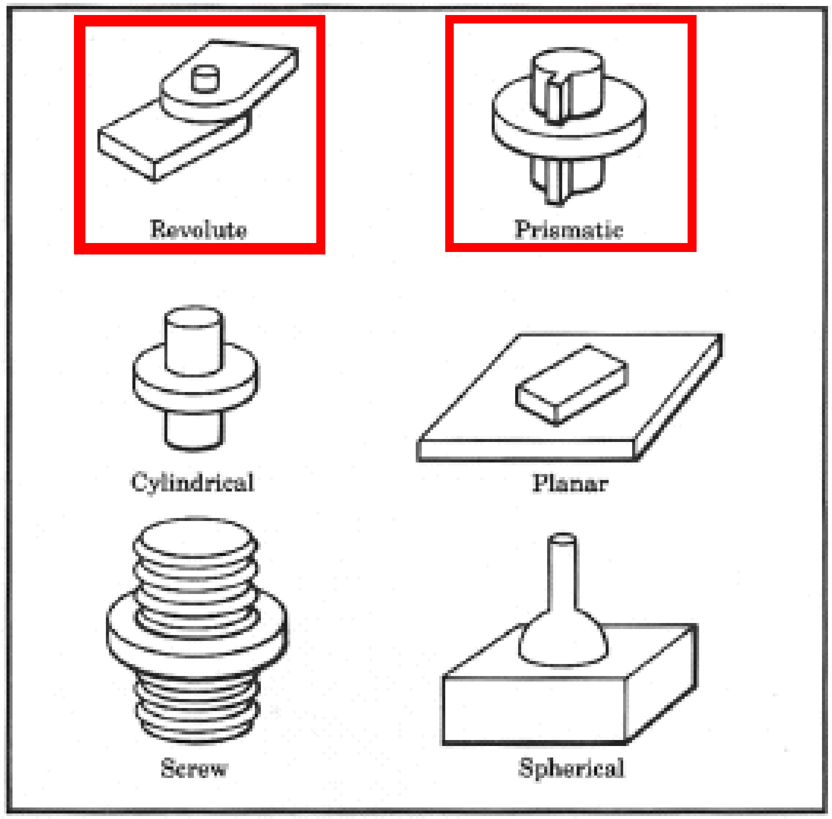
Mechanical Design Constraints

1 DOF Joint

Revolute Joint

Prismatic Joint

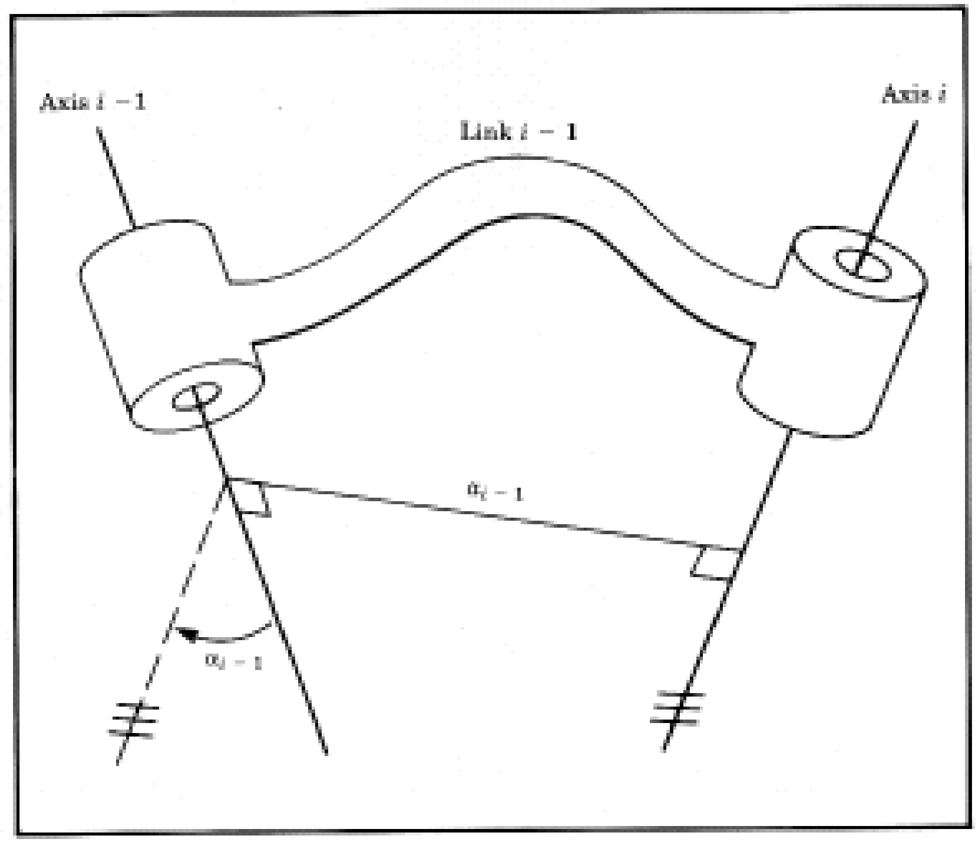
- ***Link*** - A rigid body which defines the relationship between two neighboring joint axes of the manipulator



Link Parameters (Denevit-Hartenberg) – Length & Twist)

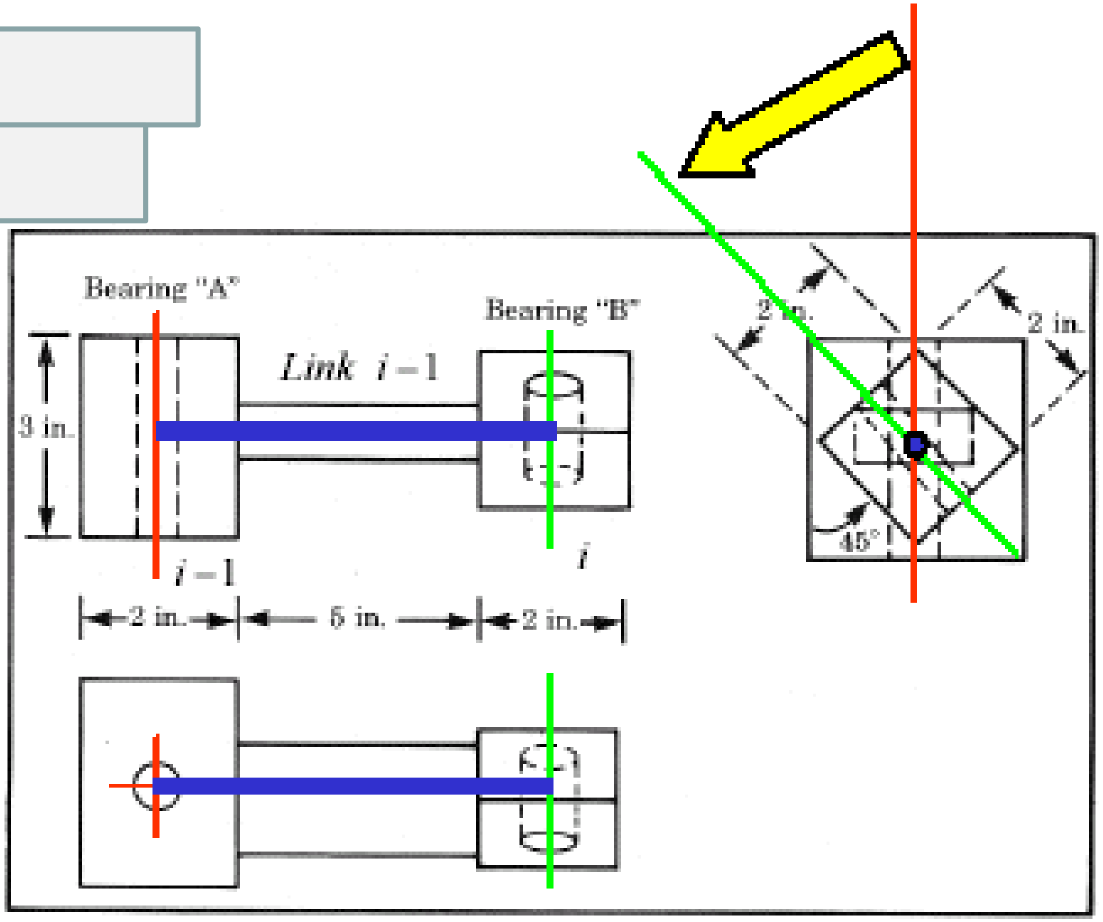
- ***Joint Axis -*** A line in space (or a vector direction) about which link ***i*** rotates relative to link ***i-1***
- ***Link Length – a_{i-1}***
  - The distance between axis ***i*** and axis ***i-1***

  Notes:
  - Expanding cylinder analogy
  - Distance
    - Parallel axes → ∞
    - Non-Parallel axes → 1
  - Sign → $a_{i-1} \geq 0$
- ***Link Twist – α_{i-1}***
  - The angle measured from axis ***i-1*** to axis ***i***
- ***Note :*** *Sign α_{i-1} by right hand rule*



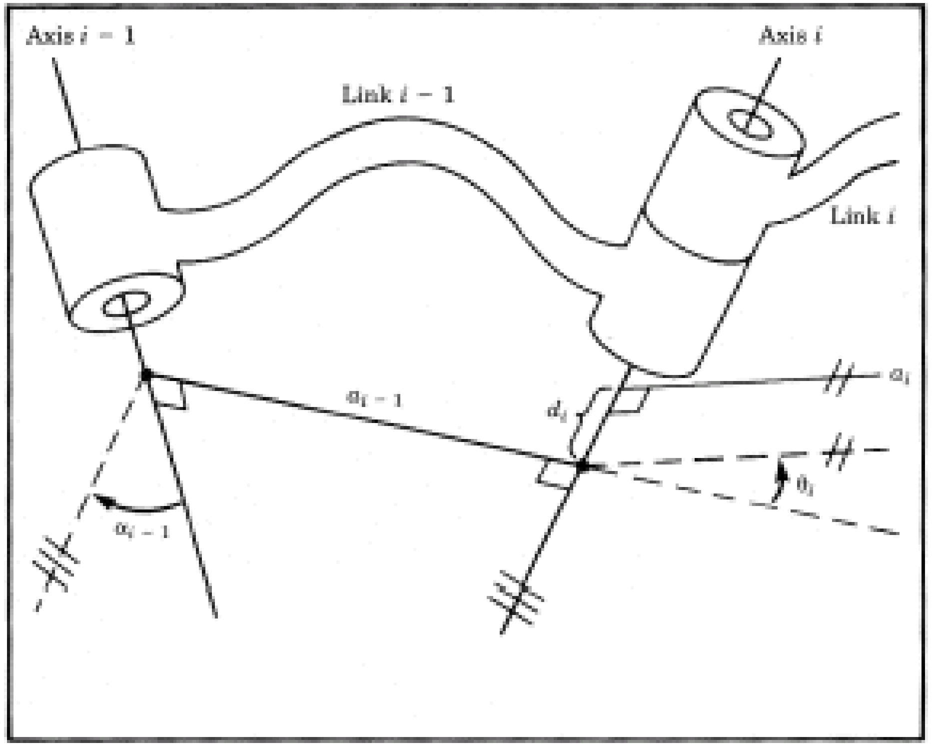
Link Parameters - Example

- Axes
- Link Length →
- Link Twist →

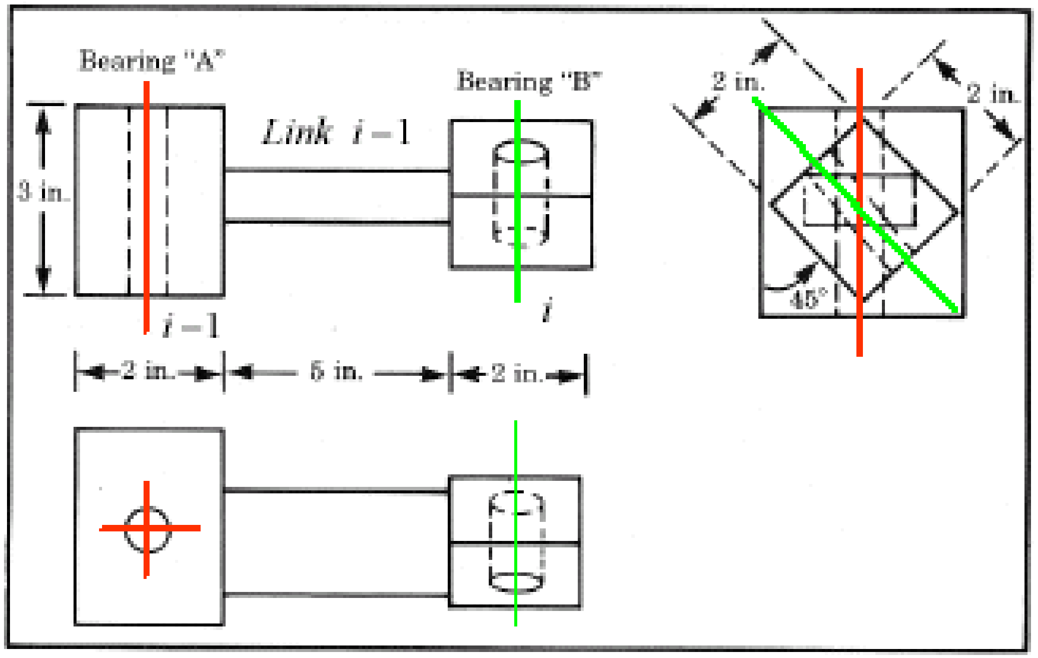


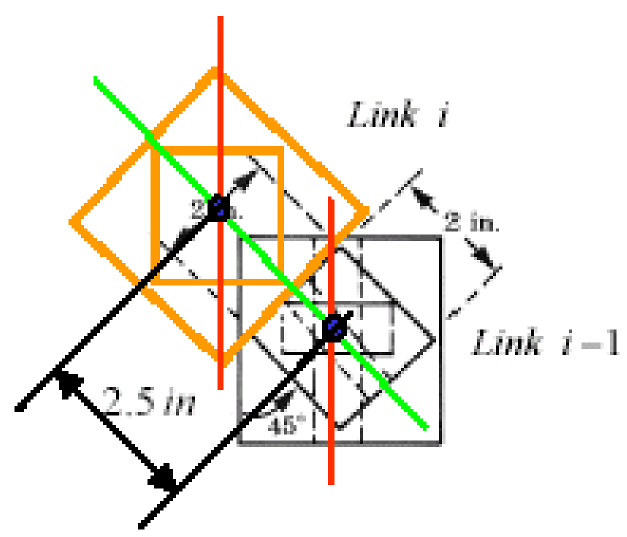
Joint Variables (Denevit-Hartenberg) – Angle & Offset

- **Link Offset – d_i**
  - The signed distance measured along the axis of joint **i** from the point where $\mathbf{a_{i-1}}$ intersects the axis to the point where $\mathbf{a_i}$ intersects the axis
    - The link offset $\mathbf{d_i}$ is variable if joint **i** is prismatic
    - Sign of $\mathbf{d_i}$
- **Joint Angle – θ_i**
  - The signed angle made between an extension of $\mathbf{a_{i-1}}$ and $\mathbf{a_i}$ measured about the the axis of the joint **i**
- **Note:**
  - The joint angle θ_i is variable if the joint **i** is revolute
- Sign - θ_i → Right hand rule



Link Parameters - Example





Link offset $d_i = 2.5in$

Joint/Link Parameters & Values – First and last links in chain

	$\begin{cases} a_1 \to a_{n-1} \\ a_0 = a_n = 0 \end{cases}$	See Definition Convention
	$\begin{cases} \alpha_1 \to \alpha_{n-1} \\ \alpha_0 = \alpha_n = 0 \end{cases}$	See Definition Convention
	$\begin{cases} d_2 \to d_{n-1} \\ \theta_2 \to \theta_{n-1} \end{cases}$	See Definition
Joint 1 - Revolute Joint	$\begin{cases} \theta_1 = 0 \\ d_1 = 0 \end{cases}$	Arbitrary Convention
Joint 1 - Prismatic Joint	$\begin{cases} \theta_1 = 0 \\ d_1 = 0 \end{cases}$	Convention Arbitrary

Affixing Frames to Links – Intermediate Links in the Chain

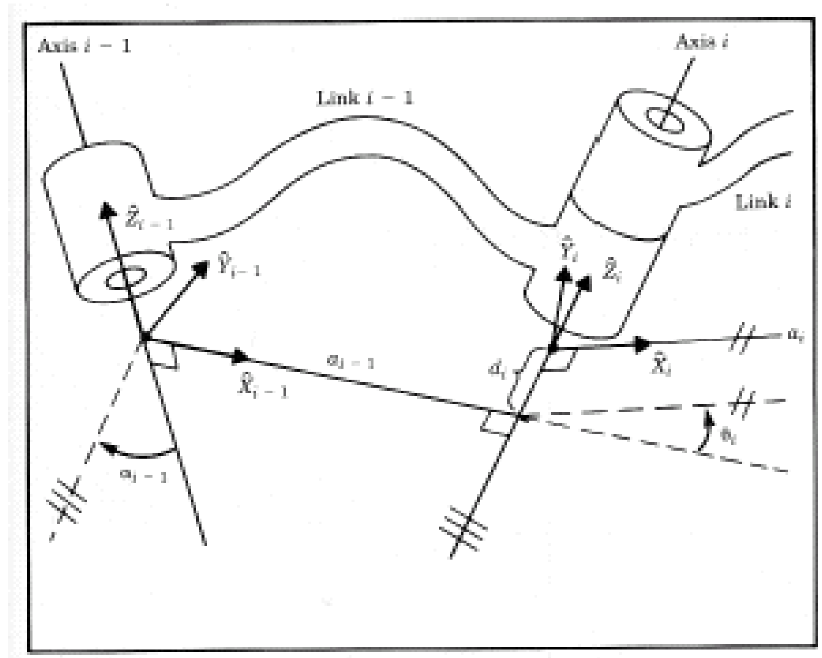
- ***Origin of Frame {i}*** –
  - The origin of frame ***{i}*** is located where the $\boldsymbol{a_i}$ perpendicular intersects the joint ***i*** axis

- ***Z Axis -***
  - The $\hat{Z}_i$ axis of frame ***{i}*** is coincident with the joint axis ***I***

- ***X Axis -***
  - The $\hat{X}_i$ axis points along the distance $\boldsymbol{a_i}$ in the direction from joint ***i*** to joint ***i+1***

  Note:
  - For $\boldsymbol{a_i} = 0$, $\hat{X}_i$ is normal to the plane of $\hat{Z}_i$ and $\hat{Z}_{i+1}$
  - The link twist angle $\boldsymbol{\alpha_i}$ is measured in a right hand sense about $\hat{X}_i$

- ***Y Axis-***
  - The $\hat{Y}_i$ axis completes frame *{**i**}* following the right hand rule



Affixing Frames to Links – First & Last Links in the Chain

- ***Frame {0}*** - The frame attached to the base of the robot or link 0 called frame {0} This frame does not move and for the problem of arm kinematics can be considered as the ***reference frame.***

- ***Frame {0} coincides with Frame {1}*** - $\begin{cases} \alpha_0 = 0 \\ a_0 = 0 \end{cases}$

$$\text{Joint 1 - Revolute Joint} \begin{cases} \theta_1 = 0 & \textbf{Arbitrary} \\ d_1 = 0 & \textbf{Convention} \end{cases}$$

$$\text{Joint 1 - Prismatic Joint} \begin{cases} \theta_1 = 0 & \textbf{Convention} \\ d_1 = 0 & \textbf{Arbitrary} \end{cases}$$

Link Frame Attachment Procedure - Summary

1. Identify the joint axes and imagine (or draw) infinite lines along them. For step 2 through step 5 below, consider two of these neighboring lines (at axes ***i*** and ***i+1***)
2. Identify the common perpendicular between them, or point of intersection. At the point of intersection, or at the point where the common perpendicular meets the ***i*** th axis, assign the link frame origin.
3. Assign the $\hat{Z}_i$ axis pointing along the ***i*** th joint axis.
4. Assign the $\hat{X}_i$ axis pointing along the common perpendicular, or if the axes intersect, assign $\hat{X}_i$ to be normal to the plane containing the two axes
5. Assign the $\hat{Y}_i$ axis to the complete a right hand coordinate system.
6. Assign {0} to match {1} when the first joint veritable is zero. For {N}, choose an origin location and $\hat{X}_N$ direction freely, but generally so as to cause as many linkage parameters as possible to be zero

DH Parameters - Summary

- If the link frame have been attached to the links according to our convention, the following definitions of the DH parameters are valid:

a_i - The distance from $\hat{Z}_i$ to $\hat{Z}_{i+1}$ measured along $\hat{X}_i$

α_i - The angle between $\hat{Z}_i$ and $\hat{Z}_{i+1}$ measured about $\hat{X}_i$

d_i - The distance from $\hat{X}_{i-1}$ to $\hat{X}_i$ measured along $\hat{Z}_i$

θ_i - The angle between $\hat{X}_{i-1}$ and $\hat{X}_i$ measured about $\hat{Z}_i$

- ***Note:***
  - $a_i \geq 0$, and α_i , d_i , and θ_i are signed quantities