# Proof of Theorem 1 (II)

We want to show that the above choice of $\delta$ satisfies (*).

2) Suppose $\|\mathbf{x}_0\| \leq \delta$ , then from (2*)

$$V(t_0,\mathbf{x}_0) \leq \beta(t_0, \delta) < \alpha(\varepsilon_1)$$

since $\dot{V}(t,x) \leq 0 \quad \text{whenever} \quad \|x\| < \delta \leq \varepsilon_1 \leq r$

It follows that

$$V[t, s(t,t_0,\mathbf{x}_0)] \leq V(t_0,\mathbf{x}_0) < \alpha(\varepsilon_1) \qquad \forall\ t \geq t_0 \qquad (3*)$$

Now since by definition of lpdf

$$\alpha(\|s(t,t_0,\mathbf{x}_0)\|) \leq V[t, s(t,t_0,\mathbf{x}_0)] \qquad (4*)$$

then (3*) & (4*) imply that

$$\alpha(\|s(t,t_0,\mathbf{x}_0)\|) \leq \alpha(\varepsilon_1) \qquad \forall\ t \geq t_0$$

since $\alpha$ is of class K, hence strictly increasing, it follows

$$\|s(t,t_0,\mathbf{x}_0)\| < \varepsilon_1 \leq \varepsilon \qquad \forall\ t \geq t_0$$

Conclusion:

$$\|\mathbf{x}_0\| < \delta \;\Rightarrow\; \|s\| < \varepsilon$$

# Lyapunov's Direct Method - Theorem 2

**Theorem 2**: [Uniform Stability]

The equilibrium 0 of $\sum$ is uniformly stable if :

$\exists$ a $\mathbf{C}^1$, lpdf, decrescent $V$: $\mathbf{R}^+ \times \mathbf{R}^n \rightarrow \mathbf{R}$ and a constant $r > 0$ s.t.:

$$\dot{V}(t, \mathbf{x}) \leq 0, \quad \forall t \geq 0, \forall \mathbf{x} \in B_r$$

where $\dot{V}$ is evaluated along the solution trajectories of $\Sigma$.

## Proof of Theorem 2

Since $V$ is decrescent, the function

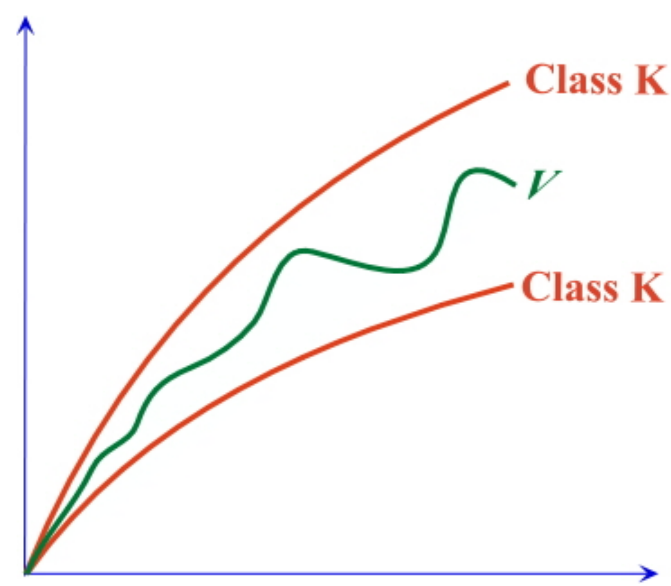
$$\beta(t_0, \delta) \equiv \sup_{\|\mathbf{x}\| < \delta} \sup_{t \geq 0} V(t_0, \mathbf{x})$$

is - finite for all sufficiently small $\delta$;

- non-decreasing in $\delta$.

Now let $\varepsilon_1 = \min\{\, \varepsilon, \ r, \ s \,\}$ and pick $\delta > 0$ s.t. $\beta(\delta) < \alpha(\varepsilon_1)$

Proceed as before. ∎



# Remarks (I)

**(1)**
- Theorem 1 and Theorem2 give sufficient conditions for stability and uniform stability.
- No function $V$, nothing can be said.
- We will see that the converse of these theorems is also true.

  So in fact they are necessary & sufficient.

**(2)**
- The $\varepsilon$-$\delta$ definitions of stability are qualitative:

  $\rightarrow$ given $\varepsilon>0$, required to demonstrate the existence of a suitable $\delta$.
- Thm1 & Thm2 are also qualitative in the sense that they provide conditions under which the existence of a suitable $\delta$ can be concluded.

# Remarks (II)

**(3)** Convention:

- Lyapunov function candidate:

  is a Lyapunov function that satisfies the conditions imposed on by the stability theorem. (ex: $V$ is $\mathbf{C}^1$, lpdf)

- Lyapunov function:

  if in addition the conditions on its $\dot{V}$ imposed by the theorem are satisfied.

**(4)** Since $\dot{V}$ is allowed to be zero for $\|\mathbf{x}\|\neq 0$, all coordinates need not to show up in expression of $\dot{V}$ .

Nevertheless, all coordinates NEED to show up in expression of $V$. $\Leftarrow$ This is crucial!

# Example (I)

## (1) Simple Pendulum

$$\Sigma: \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\sin x_1 \end{cases}$$

Total energy:

$$V(x_1,x_2) = \underbrace{(1-\cos x_1)}_{\text{P.E.}} + \underbrace{\frac{1}{2}x_2^{\ 2}}_{\text{K.E.}} \ , \qquad V(x_1,x_2) \text{ is } C^1, \ \text{lpdf}$$

$\Rightarrow V(x_1,x_2)$ is a suitable Lyapunov function candidate for applying Theorem 1.

$$\begin{aligned} \dot{V}(x_1,x_2) &= \sin x_1 \dot{x}_1 + x_2 \dot{x}_2 \\ &= x_2 \sin x_1 - x_2 \sin x_1 \\ &= 0 \end{aligned}$$

$\Rightarrow \dot{V}$ also satisfies the requirements of Thm 1. Hence $V$ is actually a Lyapunov function, and the equilibrium $O$ is stable by Thm 1.

$\rightarrow$ Furthermore, since $\sum$ is autonomous, *0* is a uniformly stable equilibrium.

# Example (II)

## (2) Damped Mathiew Equation

$$\Sigma : \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_2 - (2+\sin t)x_1 \end{cases}$$

- No physical intuituion readily available to guide us in the choices of a suitable $V$.
- After a great deal of trial & error $V(t, x_1, x_2) = x_1^{\;2} + \dfrac{x_2^{\;2}}{2+\sin t}$.

Note: $V$ is $\mathrm{C}^1$, and

$$W_1 = x_1^{\;2} + \frac{x_2^{\;2}}{3} \le V(t, x_1, x_2) \le x_1^{\;2} + x_2^{\;2} = W_2$$

So $V(t, x_1, x_2)$ is pdf & decrescent and $V$ is a suitable Lyapunov function candidate for applying Theorem2.

# Example (III)

Now

$$\dot{V}(t,x_1,x_2) = 2x_1\dot{x}_1 - x_2{}^2 \frac{\cos t}{(2+\sin t)^2} + \frac{2x_2\dot{x}_2}{2+\sin t}$$

$$= 2x_1x_2 - x_2{}^2 \frac{\cos t}{(2+\sin t)^2} + \frac{2x_2[-x_2-(2+\sin t)x_1]}{2+\sin t}$$

$$= -\frac{4+2\sin t+\cos t}{(2+\sin t)^2} x_2{}^2$$

$$\leq 0 \qquad \forall t \geq 0, \quad \forall x_1, x_2$$

Thus requirements on $\dot{V}$ in Thm2 are also met.

Hence $V$ is a Lyapunov function

and 0 is a uniformly stable equilibrium.

# Example (IV)

**(3) Using Lyapunov theory to obtain stability conditions involving parameters of systems**

$$\Sigma: \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -p(t)x_2 - e^{-t}x_1 \end{cases}$$

- Want to find conditions on $p(t)$ that insure stability of equilibrium 0.
- Choose $V(t,x_1,x_2) = x_1^{\ 2} + e^t x_2^{\ 2}$.
- $V$ is $C^1$
- $V(t,x_1,x_2) \geq W \equiv x_1^{\ 2} + x_2^{\ 2}$, hence $V$ is p.d.f.

- *V* is a suitable Lyapunov function candidate for applying Theorem1.
- Note that *V* is NOT decrescent (why?), hence *V* is NOT a suitable Lyapunov function candidate for applying Theorem2.
- $\Rightarrow$ Using this particular *V*, we can not hope to establish uniform stability.

# Example (IV) - continued

- $$\dot{V}(t,x_1,x_2) = 2x_1x_2 + e^t x_2^{\ 2} + 2e^t x_2[-p(t)x_2 - e^{-t}x_1]$$
  $$= e^t x_2^{\ 2}[-2\,p(t) + 1]$$

  if $p(t) \geq \dfrac{1}{2}\ , \quad \forall t \geq 0 \qquad (*)$

  $\Rightarrow \quad \dot{V}(t,x_1,x_2) \leq 0.$

  Thus equilibrium $O$ is stable provided condition $(*)$ holds .

- Note:

By employing a different Lyapunov function candidate, we might be able to obtain entirely different stability condition involving $p(\bullet)$.

# Theorems on Asymptotic Stability - Theorem 3 (I)

**Theorem 3**: [ Uniform Asymptotic Stability ]

The equilibrium 0 of $\sum$ is uniformly asymptotically stable if $\exists$ a $C^1$ decrescent lpdf $V$ s.t.

$-\dot{V}$ is an lpdf.

**Remarks:**

- Compare with Thm 2. ($-\dot{V} \geq \alpha(\|\mathbf{x}\|)$ )
  all coordinates need to show up in expression of $\dot{V}$ !
- Uniform asymptotic stability means uniform stability & uniform attractivity, hence $s(t, t_0, \mathbf{x}_0) \to 0$ as $t \to \infty$.
- Intuitively, since $-\dot{V}$ is an lpdf (i.e. $\dot{V}<0$ and $\dot{V}=0$ only when $\mathbf{x}=0$. Then indeed $s(t, t_0, \mathbf{x}_0) \to 0$ as $t \to \infty$ !)

# Theorems on Asymptotic Stability - Theorem 3 (II)

**Proof of Theorem 3:**

If $-\dot{V}$ is an lpdf, then clearly $\dot{V}$ satisfies the hypothesis of Thm2, so that O is a uniformly stable equilibrium.

Thus, it only remains to prove that O is uniformly attractive (Why?)

Precisely, it is necessary to show the existence of a $\delta_1>0$ s.t. for each $\varepsilon>0$ $\exists$ a $T(\varepsilon)<\infty$ s.t.

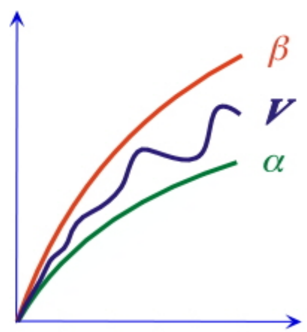
$$\|\mathbf{x}\|<\delta_1,\ t\geq 0 \Rightarrow \|s(t_0+t,t_0,\mathbf{x}_0)\|\leq\varepsilon,\quad \forall\ t\geq T(\varepsilon).$$

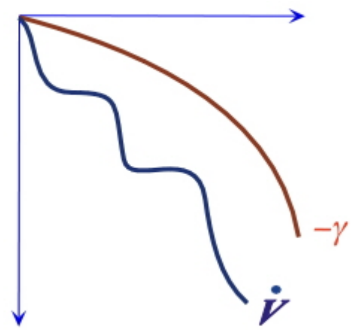
- The hypothesis on $V$ and $\dot{V}$ imply that there are functions $\alpha(\bullet)$, $\beta(\bullet)$, $\gamma(\bullet)$ of class K and a constant $r>0$ s.t.

$$\alpha(\|\mathbf{x}\|)\leq V(t,x)\leq\beta(\|\mathbf{x}\|)\ \forall\ t\geq t_0,\ \forall\ \mathbf{x}\in B_r.$$

$$\dot{V}\quad(t,x)\leq-\gamma(\|\mathbf{x}\|),\ \forall\ t\geq t_0,\ \forall\ \mathbf{x}\in B_r$$

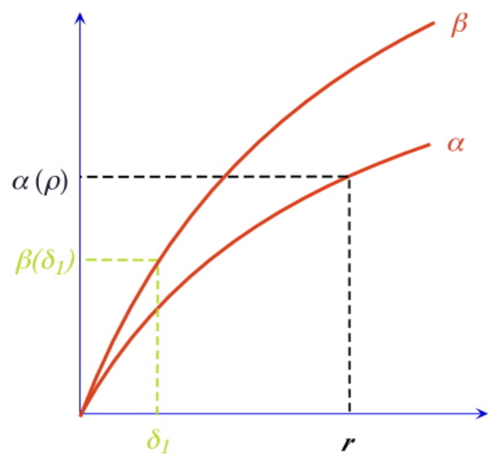
# Theorems on Asymptotic Stability - Theorem 3 (III)



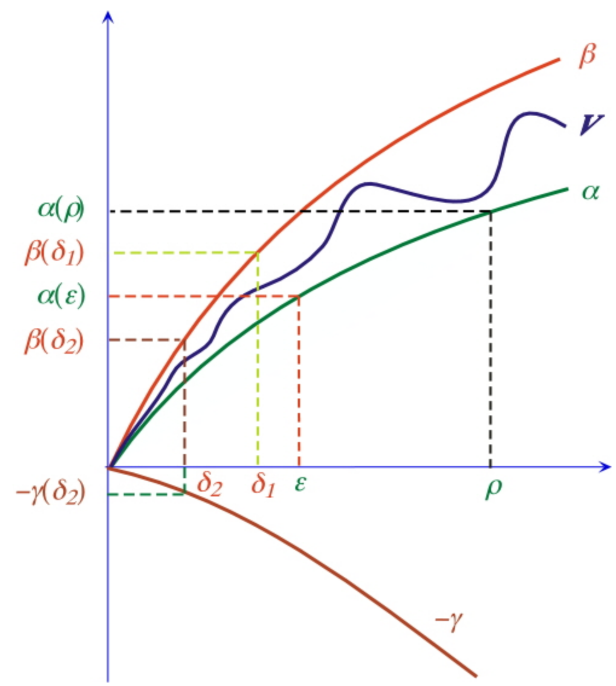




- Now choose $\varepsilon > 0$ and define

(a) $\delta_1 > 0$ s.t. $\beta(\delta_1) < \alpha(r)$




(b) $\delta_2 > 0$ s.t. $\beta(\delta_2) < \min\{\alpha(\varepsilon), \beta(\delta_1)\}$




$\delta_2 < \delta_1$

since $\alpha(\varepsilon) < \beta(\varepsilon) \Rightarrow \delta_2 < \varepsilon$

# Theorems on Asymptotic Stability - Theorem 3 (IV)

(C) Define $T \equiv \dfrac{\beta(\delta_1)}{\gamma(\delta_2)}$ (*)

We now show that these are the required constants.

- Next, it is shown that:

$$\|\mathbf{x}_0\| < \delta_2 \Rightarrow \|S(t_1, t_0, \mathbf{x}_0)\| \leq \delta_2 \text{, for some } t_1 \in [t_0,\ t_0 + T] \text{ (*)}$$

To prove , (*) suppose by way of contradiction that (*) is false, so that

$$\|\mathbf{x}_0\| < \delta_2 \Rightarrow \|s(t_1, t_0, \mathbf{x}_0)\| \leq \delta_2 \text{, } \forall t_1 \in [t_0,\ t_0 + T]$$

$$\Rightarrow 0 < \alpha(\delta_2) \leq V(t_0 + T, s(t_0 + T, t_0, x_0))$$

$$= \underbrace{V(t_0, x_0)}_{\substack{\leq \beta(\|\mathbf{x}_0\|) \\ \leq \beta(\delta_1) \text{ if } \|\mathbf{x}_0\| < \delta_1}} + \underbrace{\int_{t_0}^{t_0+T} \dot{V}[\tau, s(\tau, t_0, x_0)]\mathrm{d}\tau}_{\substack{\leq \int_{t_0}^{t_0+T} -\gamma(\|S(\tau, t_0, x_0\|)\mathrm{d}\tau \\ \leq \int_{t_0}^{t_0+T} -\gamma(\delta_2)\mathrm{d}\tau = -T\gamma(\delta_2)}}$$

$$\leq \beta(\delta_1) - T\gamma(\delta_2)$$

$$= 0 \quad \text{by} \quad (*)$$

This contradiction shows that (*) is true !

# Theorems on Asymptotic Stability - Theorem 3 (V)

- To complete the proof, suppose $t \geq t_0+T$.

  then, with $t_1 \in [t_0,\, t_0+T]$ defined in (*), we have

  $\alpha[\ \|s(t,t_0,\mathbf{x}_0)\|\ ] \leq V[\ t,\, s(t,t_0,\mathbf{x}_0)\ ] \leq V[\ t_1,\, s(t_1,t_0,\mathbf{x}_0)]$ since $\dot{V} < 0$ for $x \neq 0$.

  Finally,

  $$V[\ t_1,\, s(t_1,t_0,\mathbf{x}_0)] \leq \beta[\ \|s(t_1,t_0,\mathbf{x}_0)\|\ ] \leq \beta(\delta_2) \text{ by (*)}.$$

  Hence,

  $$\alpha[\ \|s(t_1,t_0,\mathbf{x}_0)\|\ ] \leq \beta(\delta_2) \leq \alpha(\varepsilon).$$

  that is

  $$\|s(t,t_0,\mathbf{x}_0)\| \leq \varepsilon.$$

  So we have shown uniform attractivity, i.e.

  For each $\varepsilon>0$, $\exists\ \delta_1>0$ and $T(\varepsilon)<\infty$ s.t.

  $\|\mathbf{x}\|<\delta_1,\ t\geq 0 \Rightarrow \|s(t_0+t,t_0,\mathbf{x}_0)\| \leq \varepsilon,\ \ \forall\ t \geq T(\varepsilon).$ $\square$

# Theorems on Asymptotic Stability - Theorem 4

**Theorem 4**: [ Global Uniform Asymptotic Stability ]

The equilibrium 0 of $\sum$ is globally uniformly asymptotically stable if $\exists$ a function $V$: $\mathbf{R}^+ \times \mathbf{R}^n \rightarrow \mathbf{R}$ s.t.

(i) $V$ is $C^1$, decrescent, pdf & radially unbounded;

(ii) $-\dot{V}$ is a pdf.

**Remarks:**

- The assumption that $V$ is radially unbounded is indispensable. Without this assumption, the theorem is not valid.
- $V$ is required to be radially unbounded but $-\dot{V}$ is not.

# Theorems on Asymptotic Stability

**Theorem 5**: [ Exponential Stability ]

Suppose ∃ constants $a,b,c,r>0,\ p\geq 1$, and a $C^1$ function $V: \mathbf{R}^+\times\mathbf{R}^n\rightarrow\mathbf{R}$ s.t.

(i) $a\|\mathbf{x}\|^p \leq V(t,x) \leq b\|\mathbf{x}\|^p$, $\forall\ t \geq 0$, $\forall\ \mathbf{x}\in B_r$

(ii) $\dot{V}(t,x) \leq -c\|\mathbf{x}\|^p$, $\forall\ t \geq 0$, $\forall\ \mathbf{x}\in B_r$ .

Then, the equilibrium O of $\sum$ is exponentially stable.

**Proof of Theorem 5**:

Define $\eta = r\left[\frac{a}{b}\right]^{\frac{1}{p}} \leq r$

And suppose $\forall\ \mathbf{x}_0\in B_r$ , $t_0 \geq 0$.

Let $\mathbf{x}(t)$ denote the solution $s(t, t_0, \mathbf{x}_0)$,

## Proof of Theorem 5 - continued

$$\frac{\mathrm{d}}{\mathrm{d}t}V[t,\mathbf{x}(t)] \le -c\,\|\,\mathbf{x}(t)\,\|^{P} \le -\frac{c}{b}V[t,\mathbf{x}(t)]$$

i.e. $\dot{V} + \frac{c}{b}V \le 0 \qquad \Rightarrow (\dot{V} + \frac{c}{b}V)e^{\frac{c}{b}t} \le 0$

$$\frac{\mathrm{d}}{\mathrm{d}t}[Ve^{\frac{c}{b}t}] \le 0$$

$\therefore$ for any $t_0 \ge 0$ and $\forall t \ge t_0$

$$V(t)e^{\frac{c}{b}t} \le V(t_0)e^{\frac{c}{b}t_0} \qquad \Rightarrow V(t) \le V(t_0)e^{-\frac{c}{b}(t-t_0)} \qquad \forall t \ge t_0 \ge 0$$

or $\quad V(t+t_0) \le V(t_0)e^{-\frac{c}{b}t}$

i.e $\quad V[t+t_0, x(t+t_0)] \le V[t_0, x_0]e^{-\frac{c}{b}t} \quad \forall t \ge 0$

# Proof of Theorem 5 - continued

But since

$$V[t_0, x_0] \leq b \| x_0 \|^P, \text{ and}$$

$$a \| x(t_0 + t) \|^P \leq V[t + t_0, x(t + t_0)]$$

it follows that

$$a \| x(t_0 + t) \|^P \leq b \| x_0 \|^P \, e^{-\frac{c}{b}t} \quad \forall t \geq 0$$

Finally,

$$\| x(t_0 + t) \| \leq \left[ \frac{b}{a} \right]^{\frac{1}{p}} \| x_0 \| e^{-\frac{c}{bP}t} \quad \forall t \geq 0$$

Note that :

$$\| x(t_0 + t) \| = S(t_0 + t, t_0, x_0)$$

and that $\| S(t_0 + t, t_0, x_0) \| \leq r$ (why?)

Thus O of $\sum$ is an exponentially stable equilibrium. ∎

# Theorems on Asymptotic Stability - Theorem 6

**Theorem 6**: [ Global Exponential Stability ]

The equilibrium 0 of $\sum$ is globally exponentially stable

if $\exists$ constants $a,b,c>0,\ p\geq 1$, and a $C^1$ function $V: \mathbf{R}^+ \times \mathbf{R}^n \to \mathbf{R}$ s.t.

(i) $a\|\mathbf{x}\|^p \leq V(t,x) \leq b\|\mathbf{x}\|^p,\ \forall\ t \geq 0,\ \forall\ \mathbf{x} \in \mathbf{R}^n$

(ii) $\dot{V}(t,x) \leq -c\|\mathbf{x}\|^p,\ \forall\ t \geq 0,\ \forall\ \mathbf{x} \in \mathbf{R}^n$

**Proof of Theorem 6**:

Entirely analogous to that of Theorem 5.

## Example

$$\Sigma: \begin{cases} \dot{x}_1 = -a(t)x_1 - bx_2 \\ \dot{x}_2 = bx_1 - c(t)x_2 \end{cases}$$

- $b =$ constant $> 0$
- $a(t)$, $c(t)$ continuous $\forall\ t \geq 0$ s.t.

  $a(t) \geq \delta > 0$

  and

  $c(t) \geq \delta > 0$

  $\delta =$ constant.

# Example- Continued

- 0 is the only equilibrium point. WHY?
- Choose $W(x) = \frac{1}{2}(x_1^2 + x_2^2)$

$$\begin{aligned}\dot{W} &= x_1[-a(t)x_1 - bx_2] + x_2[bx_1 - c(t)x_2] \\ &= -a(t)x_1^2 - c(t)x_2^2 \\ &< -\delta(x_1^2 + x_2^2)\end{aligned}$$

- So in Theorem 6, $a = \frac{1}{2}, \; b = 1, \; p = 2, c = \delta$

and 0 of $\sum$ is globally exponentially stable.

## An Instability Theorem

**Theorem 7**: [ Instability ]

The equilibrium 0 of $\sum$ is unstable

if ∃ a $C^1$ decrescent function $V$: $\mathbf{R}^+ \times \mathbf{R}^n \rightarrow \mathbf{R}$ and a time $t_0$ s.t.

(i) $\dot{V}(t, x)$ is an lpdf

(ii) $V(t,0) = 0 \quad \forall\ t \geq t_0$, and

(iii) ∃ points $x_o \neq 0$ arbitrarily close to 0 s.t. $V(t_o, x_o) \geq 0.$

**Remark**:

In contrast to previous stability theorems, the Lyapunov function V in Theorem 7 above can assume both positive as well as negative values.

## Example

$$\Sigma: \begin{cases} \dot{x}_1 = x_1 - x_2 + x_1 x_2 \\ \dot{x}_2 = -x_2 - x_2^2 \end{cases}$$

Choose the Lyapunov function candidate

$$V(x_1, x_2) = (2x_1 - x_2)^2 - x_2^2$$

- $V$ assumes both positive and negative values .

  but $V$ assumes nonnegative values arbitrarily close to the origin

  as required by (iii) in Thm 7.

  Hence $V$ is a suitable Lyapunov function candidate.

- $$\dot{V}(x_1, x_2) = 2(2x_1 - x_2)(2\dot{x}_1 - \dot{x}_2) - 2x_2\dot{x}_2^2$$
  $$= [(2x_1 - x_2)^2 + x_2^2](1 + x_2)$$

$\Rightarrow$ $\dot{V}$ is lpdf over the ball $B_{1\text{-}d}$, $d \in (0,1)$

and all conditions of Thm 7 are satisfied.

It follows that O of $\sum$ is an unstable equilibrium. ∎