CHAPTER 7
Elements of Analytical Dynamics

7.1 INTRODUCTION

Newton's laws were formulated for a single particle and can be extended to systems of particles and rigid bodies, as shown in Chapters 5 and 6. A basic tool in the application of Newton's second law is the free-body diagram, which is a diagram of a given mass and all the forces acting upon it. For a system consisting of several bodies, we must use one free-body diagram for each of the bodies. In drawing these free-body diagrams, interacting forces between the bodies, which result from kinematical constraints and are internal to the system, must be regarded as external to the individual bodies and must be included in the free-body diagrams. Quite often, the object is to determine the motion of a system only, and the interacting forces present no particular interest. In describing the motion of a system, Newtonian mechanics uses physical coordinates and forces, which are generally vector quantities. For this reason, Newtonian mechanics is often referred to as *vectorial mechanics*. The inclusion of interacting forces between bodies in free-body diagrams and the use of physical coordinates and forces, both vector quantities, can be regarded as two disadvantages of Newtonian mechanics when compared with other approaches.

D'Alembert's principle is a variational principle permitting the derivation of the equations of motion without considering explicitly the interacting forces. It is still basically a vectorial approach, so that it goes only part way toward removing the objections to Newtonian mechanics.

Another variational approach to mechanics is known as Lagrangian mechanics, or analytical mechanics, and it removes both objections to Newtonian mechanics. Lagrangian mechanics permits the derivation of the equations of motion from two scalar functions, the kinetic energy and the potential energy, as well as from a differential expression known as virtual work in the case of nonconservative systems. The equations of motion can be derived without the need of free-body diagrams. The kinetic energy and the potential energy can be expressed in terms of so-called generalized displacements and generalized velocities, and the virtual work can be expressed in terms of generalized virtual displacements and generalized forces, all scalar quantities not necessarily representing physical quantities. The basic tool for deriving the equations of motion consists of a set of general differential equations known as Lagrange's equations. Explicit equations of motions for a given system can be produced as soon as the kinetic energy, the potential energy, and the virtual work are given for the system. The advantage of Lagrange's equations becomes more and more evident as the number of degrees of freedom of the system increases.

In this chapter we develop the framework of the variational approach to mechanics, including the principle of virtual work, d'Alembert's principle, and Lagrange's equations.

7.2 GENERALIZED COORDINATES

In Chapter 5, we considered the dynamics of a system of N particles, each particle having the mass m_i. The position of a typical particle is given by the radius vector $\mathbf{r}_i(t)(i=1, 2, \ldots, N)$, where $\mathbf{r}_i$ can be written in terms of the cartesian components x_i, y_i, z_i, as follows:

$$\mathbf{r}_i = x_i\mathbf{i} + y_i\mathbf{j} + z_i\mathbf{k}, \qquad i=1, 2, \ldots, N \tag{7.1}$$

The motion of the system is defined completely if the rectangular coordinates of all the particles are known functions of time,

$$x_i = x_i(t), \qquad y_i = y_i(t), \qquad z_i = z_i(t), \qquad i=1, 2, \ldots, N \tag{7.2}$$

Quite often it is more convenient to express the motion not in terms of the rectangular coordinates x_i, y_i, z_i but in terms of a different set of coordinates, say, $q_1, q_2, \ldots, q_n$, where $n=3N$. The relation between the rectangular coordinates x_i, y_i, z_i $(i=1, 2, \ldots, N)$ and the new coordinates q_k $(k=1, 2, \ldots, n)$ can be written in the general form

$$\begin{aligned}
x_1 &= x_1(q_1, q_2, \ldots, q_n)\\
y_1 &= y_1(q_1, q_2, \ldots, q_n)\\
z_1 &= z_1(q_1, q_2, \ldots, q_n)\\
x_2 &= x_2(q_1, q_2, \ldots, q_n)\\
&\;\vdots\\
z_N &= z_N(q_1, q_2, \ldots, q_n)
\end{aligned} \tag{7.3}$$

Equations (7.3) represent a *coordinate transformation* whose purpose is to facilitate the treatment of dynamical problems. For example, in Section 3.7 we found it convenient to describe the planar motion of a particle in terms of the polar coordinates r and θ, related to the cartesian coordinates x and y by

$$x = r\cos\theta, \qquad y = r\sin\theta \tag{7.4}$$

Hence, letting $r=q_1$, $\theta=q_2$, we can write

$$x = x(q_1, q_2) = q_1\cos q_2, \qquad y = y(q_1, q_2) = q_1\sin q_2 \tag{7.5}$$

Clearly, Eqs. (7.5) represent a special case of the general coordinate transformation (7.3)

Equations (7.3) represent a transformation between two sets of $n=3N$ coordinates. Implied in this is the assumption that the N particles are free to move unrestricted in a three-dimensional space. In many physical systems, however, the particles are not free but subject to constraints restricting their freedom of motion. The particle in planar motion mentioned above is an example of constrained motion. Indeed, because the particle is confined to planar motion, Eqs. (7.5) should be completed by writing

$$x=q_1 \cos q_2, \qquad y=q_1 \sin q_2, \qquad z=0 \tag{7.6}$$

where $z=0$ is to be interpreted as a *constraint equation*. Another simple example of a constrained system is a dumbbell, consisting of two mass particles connected by a rigid rod. Denoting the cartesian coordinates of the two particles by x_1, y_1, z_1 and x_2, y_2, z_2 and the length of the rod by L, the fact that the length of the rod does not change can be expressed in the form

$$(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2=L^2=\text{const} \tag{7.7}$$

Equation (7.7) represents a constraint equation implying that not all six coordinates $x_1, y_1, z_1, x_2, y_2, z_2$ are independent. Indeed, the motion of the dumbbell is fully determined by only five of these coordinates, as Eq. (7.7) can be used to determine the sixth.

In general, if a system of N particles moving in a three-dimensional space is subject to c kinematical constraints, such as that expressed by Eq. (7.7), the motion of the system can be described completely by n coordinates $q_1, q_2, \ldots, q_n$, where

$$n=3N-c \tag{7.8}$$

is the *number of degrees of freedom* of the system. Hence, the number of degrees of freedom of a system can be defined as the minimum number of coordinates required to describe the motion of a system completely. The n coordinates $q_1, q_2, \ldots, q_n$ describing the motion of the system are known as *generalized coordinates.*

The generalized coordinates $q_1, q_2, \ldots, q_n$ may not always have physical meaning, nor are they necessarily unique. For example, any set of five of the six cartesian coordinates used to describe the motion of the dumbbell discussed above can serve as generalized coordinates, although they are not the most convenient ones. In all likelihood, the most convenient set of generalized coordinates consists of three coordinates describing the translation of the mass center of the dumbbell relative to an inertial space and two angular coordinates describing the orientation of the rigid rod relative to the same inertial space.

7.3 THE PRINCIPLE OF VIRTUAL WORK

The principle of virtual work is a first variational principle of mechanics, and it represents a statement of the static equilibrium of a mechanical system. Although we shall use this principle to illustrate how to calculate the position of static equilibrium of a system, our real interest lies in its ability to facilitate the transition from Newtonian to Lagrangian mechanics. Before the principle can be discussed, it is necessary to introduce a new class of displacements, known as *virtual displacements*.

Let us consider once again the system of N particles of Section 7.2 and define the virtual displacements $\delta x_1, \delta y_1, \delta z_1, \ldots, \delta z_N$ as infinitesimal changes in the coordinates $x_1, y_1, z_1, x_2, \ldots, z_N$. The virtual displacements must be consistent with the constraints of the system, but they are otherwise arbitrary. They are not true displacements but small variations in the system coordinates resulting from imagining the system in a slightly displaced position, a process taking place without any change in time, so that the forces and constraints do not change during this process. This is in direct contrast with actual displacements, which require a certain amount of time to evolve, during which time the forces and constraints may change. The virtual displacements obey the rules of differential calculus and in the event the system is subject to a constraint of the form

$$f(x_1, y_1, z_1, x_2, \ldots, z_N, t) = C \tag{7.9}$$

they must be such that the equation

$$f(x_1 + \delta x_1, y_1 + \delta y_1, z_1 + \delta z_1, x_2 + \delta x_2, \ldots, z_N + \delta z_N, t) = C \tag{7.10}$$

is also satisfied, and note that the time t has not been varied in Eq. (7.10). Expanding Eq. (7.10) in a Taylor series about the position $x_1, y_1, z_1, x_2, \ldots, z_N$, we obtain

$$f(x_1, y_1, z_1, x_2, \ldots, z_N, t) + \sum_{i=1}^{N} \left(\frac{\partial f}{\partial x_i} \delta x_i + \frac{\partial f}{\partial y_i} \delta y_i + \frac{\partial f}{\partial z_i} \delta z_i \right) + 0(\delta^2) = C \tag{7.11}$$

where $0(\delta^2)$ denotes nonlinear terms in the virtual displacements. Considering Eq. (7.10) and ignoring the nonlinear terms as being of higher order in magnitude, we conclude that the virtual displacements must satisfy the relation

$$\sum_{i=1}^{N} \left(\frac{\partial f}{\partial x_i} \delta x_i + \frac{\partial f}{\partial y_i} \delta y_i + \frac{\partial f}{\partial z_i} \delta z_i \right) = 0 \tag{7.12}$$

so that only $3N-1$ of the virtual displacements are arbitrary. As an example, in the case of the dumbbell, which is subject to the constraint equation (7.7), the virtual displacements must satisfy

$$(x_2 - x_1)(\delta x_2 - \delta x_1) + (y_2 - y_1)(\delta y_2 - \delta y_1) + (z_2 - z_1)(\delta z_2 - \delta z_1) = 0 \tag{7.13}$$

so that, if five of the virtual displacements $\delta x_1, \delta y_1, \delta z_1, \delta x_2, \delta y_2, \delta z_2$ are given, the sixth can be determined by means of Eq. (7.13). In general, for a system of N particles subject to c constraints, i.e., for a system possessing $n = 3N - c$ degrees of freedom, only n virtual displacements are arbitrary, so that there are as many arbitrary virtual displacements as the number of independent coordinates.

Next, let us assume that each of the particles in the system is acted upon by a set of forces with resultant $\mathbf{R}_i$ ($i = 1, 2, \ldots, N$). For a system in equilibrium the resultant force is zero, $\mathbf{R}_i = \mathbf{0}$, so that

$$\overline{\delta W}_i = \mathbf{R}_i \cdot \delta \mathbf{r}_i = 0, \qquad i = 1, 2, \ldots, N \tag{7.14}$$

where $\overline{\delta W_i}$ is the *virtual work* performed by the resultant force on the ith particle over the virtual displacement vector $\delta \mathbf{r}_i$. Note that the overbar in $\overline{\delta W_i}$ indicates that $\overline{\delta W_i}$ is in general a mere infinitesimal expression and not the variation of a function W_i, because a work function W_i exists only when the system is conservative. Summing over the system, we have simply

$$\overline{\delta W} = \sum_{i=1}^{N} \overline{\delta W_i} = \sum_{i=1}^{N} \mathbf{R}_i \cdot \delta \mathbf{r}_i = 0 \tag{7.15}$$

where $\overline{\delta W}$ is the virtual work for the entire system.

The above result appears quite trivial, as nothing new has been gained. When the system is subject to constraints, however, Eq. (7.15) leads to some interesting conclusions. Hence, let us assume that the resultant forces $\mathbf{R}_i$ consist of *impressed*, or *applied, forces* $\mathbf{F}_i$ and *constraint forces* $\mathbf{f}_i$, or

$$\mathbf{R}_i = \mathbf{F}_i + \mathbf{f}_i, \qquad i = 1, 2, \ldots, N \tag{7.16}$$

Introducing Eqs. (7.16) into Eq. (7.15), we can write

$$\overline{\delta W} = \sum_{i=1}^{N} \mathbf{F}_i \cdot \delta \mathbf{r}_i + \sum_{i=1}^{N} \mathbf{f}_i \cdot \delta \mathbf{r}_i = 0 \tag{7.17}$$

We shall confine ourselves to systems for which the virtual work performed by the constraint forces is zero, or

$$\sum_{i=1}^{N} \mathbf{f}_i \cdot \delta \mathbf{r}_i = 0 \tag{7.18}$$

As an example, consider a particle on a smooth surface. The constraint force in this case is normal to the surface and any virtual displacement consistent with this constraint must be parallel to the surface. Clearly, the constraint force vector and the virtual displacement vector are normal to one another, so that their scalar product is zero. It follows that the work performed by the constraint force over a virtual displacement consistent with the constraint is zero. Hence, Eq. (7.17) reduces to

$$\sum_{i=1}^{N} \mathbf{F}_i \cdot \delta \mathbf{r}_i = 0 \tag{7.19}$$

Equation (7.19) represents the mathematical expression of the *principle of virtual work*, which can be stated as follows: *The work performed by the applied forces through virtual displacements compatible with the system constraints is zero.*

In the case of a conservative system (Section 3.6), we can write

$$\sum_{i=1}^{N} \mathbf{F}_i \cdot \delta \mathbf{r}_i = -\delta V = 0 \tag{7.20}$$

where δV is the variation in the potential energy, which has the form

$$\delta V = \sum_{i=1}^{N} \left(\frac{\partial V}{\partial x_i} \delta x_i + \frac{\partial V}{\partial y_i} \delta y_i + \frac{\partial V}{\partial z_i} \delta z_i \right) \tag{7.21}$$

so that Eq. (7.20) can be rewritten

$$\sum_{i=1}^{N}(F_{xi}\delta x_i+F_{yi}\delta y_i+F_{zi}\delta z_i)=-\sum_{i=1}^{N}\left(\frac{\partial V}{\partial x_i}\,\delta x_i+\frac{\partial V}{\partial y_i}\,\delta y_i+\frac{\partial V}{\partial z_i}\,\delta z_i\right)=0 \quad (7.22)$$

where F_{xi}, F_{yi}, F_{zi} are the cartesian components of $\mathbf{F}_i$. For no constraints acting on the system, the virtual displacements δx_i, δy_i, δz_i $(i=1, 2, \ldots, N)$ are all independent. Moreover, because the virtual displacements are arbitrary and can be assigned values at will, the only way Eq. (7.22) can remain valid for all possible values of the virtual displacements is for the quantities multiplying the virtual displacements to be zero independently. Hence we must have

$$F_{xi}=-\frac{\partial V}{\partial x_i}=0, \qquad F_{yi}=-\frac{\partial V}{\partial y_i}=0, \qquad F_{zi}=-\frac{\partial V}{\partial z_i}=0, \qquad i=1,2,\ldots,N \quad (7.23)$$

Equations (7.23) represent the equilibrium conditions for a system of N free particles, and they state that the components of the applied force acting on every particle must be zero. In terms of the potential energy function V, these are the conditions for the function V to have a *stationary value*. Hence, *the potential energy has a stationary value when the system is in static equilibrium*. Equations (7.23) are written in terms of cartesian components, but they are not so restricted. Indeed, the equilibrium conditions can be expressed in terms of any other set of coordinates, including curvilinear coordinates.

Example 7.1

Consider a mass on a smooth inclined plane attached to a spring of stiffness k. Use the principle of virtual work to determine the equilibrium position.

Figure 7.1 shows the mass and the forces acting upon it. We denote the displacement of the spring in the equilibrium position by the vector $\mathbf{r}$ as measured parallel to the smooth surface from the unstretched position. There are two applied forces: the first due to the weight of the mass and equal to $\mathbf{W}=-mg\mathbf{j}$ and the second the restoring force $-k\mathbf{r}$ due to the elasticity of the spring. In addition,

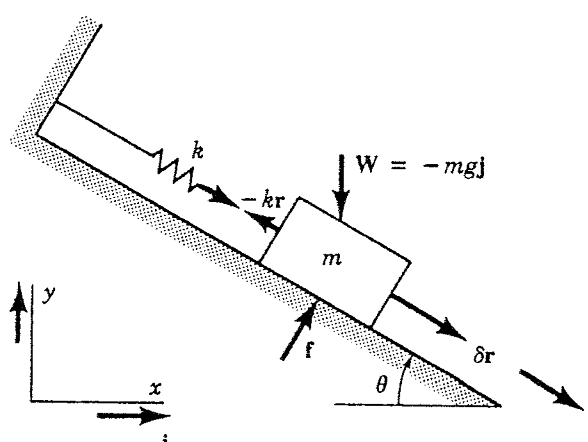

FIGURE 7.1

there is the constraint force $\mathbf{f}$ acting in a direction normal to the surface. Letting $\delta\mathbf{r}$ be the virtual displacement vector, as shown in Fig. 7.1, we can write the virtual work expression

$$(-k\mathbf{r}-mg\,\mathbf{j})\cdot\delta\mathbf{r}=0 \tag{a}$$

The vectors $\mathbf{r}$ and $\delta\mathbf{r}$ can be written in the form

$$\mathbf{r}=r\mathbf{u}_r, \qquad \delta\mathbf{r}=\delta r\,\mathbf{u}_r \tag{b}$$

where r and δr are the corresponding magnitudes and $\mathbf{u}_r$ is a unit vector in the direction of the vector $\mathbf{r}$, as shown in Fig. 7.1. Moreover, we have

$$\mathbf{j}\cdot\delta\mathbf{r}=-\delta r\sin\theta \tag{c}$$

where θ is the angle between the plane and the horizontal, so that inserting Eqs. (b) and (c) into Eq. (a), we obtain

$$(-kr+mg\sin\theta)\,\delta r=0 \tag{d}$$

Due to the arbitrariness of the virtual displacement δr, Eq. (d) can be satisfied only if the coefficient of δr is zero. Letting the coefficient of δr in Eq. (d) be equal to zero, we obtain the equilibrium condition

$$-kr+mg\sin\theta=0 \tag{e}$$

which can be solved for the equilibrium position, with the result

$$r=\frac{mg\sin\theta}{k} \tag{f}$$

***Example* 7.2**

Determine the equilibrium position for the system of Example 7.1 by differentiating the potential energy expression.

The potential energy consists of two parts: the strain energy due to the elongation of the spring and the gravitational potential energy. Hence, using the unstretched spring position as a reference position, we can write

$$V=\tfrac{1}{2}kr^2-mgr\sin\theta=0 \tag{a}$$

so that

$$F_r=-\frac{\partial V}{\partial r}=-kr+mg\sin\theta=0 \tag{b}$$

where F_r is the force in the direction of the vector $\mathbf{r}$. Equation (b) is identical to Eq. (e) of Example 7.1. Hence, the coefficient of δr in Eq. (d) of Example 7.1 can be identified as F_r, so that Eq. (d) of Example 7.1 represents the virtual work expression

$$F_r\,\delta r=0 \tag{c}$$

7.4 D'ALEMBERT'S PRINCIPLE

As pointed out in Section 7.3, the principle of virtual work represents the statement of static equilibrium of a system. However, our interest lies not in problems of static equilibrium but in problems of dynamics. As shown in Section 5.2, the equations of motion for a system of particles can be written by invoking Newton's second law. In writing Newton's equations of motion, one must consider any constraint forces present. But, constraint forces are seldom known explicitly. Indeed, more often than not constraints appear only in the form of kinematical relations, such as Eq. (7.7). As a result, the constraint forces must be carried along as unknowns, which requires that the equations of motion be supplemented by the constraint equations. This increases the complexity of the problem, so that an approach permitting the elimination of the constraint forces from the problem formulation is highly desirable. One such approach is known as d'Alembert's principle.

Newton's equations of motion for a system of particles $m_i\ (i=1, 2, \ldots, N)$ are

$$\mathbf{F}_i+\mathbf{f}_i=m_i\ddot{\mathbf{r}}_i, \qquad i=1, 2, \ldots, N \tag{7.24}$$

where $\mathbf{F}_i$ are applied forces, $\mathbf{f}_i$ are constraint forces, and $\ddot{\mathbf{r}}_i$ is the acceleration of the mass m_i. Equations (7.24) can be rewritten in the form

$$\mathbf{F}_i+\mathbf{f}_i-m_i\ddot{\mathbf{r}}_i=\mathbf{0} \qquad i=1, 2, \ldots, N \tag{7.25}$$

where $-m_i\ddot{\mathbf{r}}_i$ is referred to as an *inertia force*. Equations (7.25) can be regarded as representing the *dynamic equilibrium* of the system of particles. If we add the inertia force to the forces acting on m_i, we can use the analogy with the virtual work for the static case to write the virtual work for the individual particles

$$(\mathbf{F}_i+\mathbf{f}_i-m_i\ddot{\mathbf{r}}_i)\cdot\delta\mathbf{r}_i=0, \qquad i=1, 2, \ldots, N \tag{7.26}$$

Summing over the entire system, we obtain

$$\sum_{i=1}^{N}(\mathbf{F}_i+\mathbf{f}_i-m_i\ddot{\mathbf{r}}_i)\cdot\delta\mathbf{r}_i=0 \tag{7.27}$$

But, as indicated by Eq. (7.18), the virtual work performed by the constraint forces over virtual displacements $\delta\mathbf{r}_i$ compatible with the system constraints is zero. Hence, Eq. (7.27) reduces to

$$\sum_{i=1}^{N}(\mathbf{F}_i-m_i\ddot{\mathbf{r}}_i)\cdot\delta\mathbf{r}_i=0 \tag{7.28}$$

Equation (7.28) represents the mathematical statement of *d'Alembert's principle*. The sum of the applied force $\mathbf{F}_i$ and the inertia force $-m_i\ddot{\mathbf{r}}_i$ is known as the *effective force*. Hence, d'Alembert's principle can be enunciated as follows: *The virtual work performed by the effective forces through virtual displacements compatible with the system constraints is zero.*

Equation (7.28) is suitable for a system of particles. On occasions, one is faced with a system of rigid bodies instead of a system of particles. If the motion is planar, then use of Eqs. (6.20) and (6.47) permits us to write d'Alembert's principle for a system of rigid bodies in the form

$$\sum_{i=1}^{N} [(\mathbf{F}_i - m_i \ddot{\mathbf{r}}_{Ci}) \cdot \delta \mathbf{r}_{Ci} + (M_{Ci} - I_{Ci}\ddot{\theta}_i)\, \delta\theta_i] = 0 \tag{7.29}$$

where the subscript i identifies now a rigid body in the system. Otherwise, the notation is consistent with that in Eqs. (6.20) and (6.47). Once again, the virtual displacements $\delta \mathbf{r}_{Ci}$ and $\delta\theta_i$ $(i = 1, 2, \ldots, N)$ must be consistent with the system constraints.

Example 7.3

Derive the equations of motion for the system of Fig. 6.8a by means of d'Alembert's principle. Use x and θ as independent coordinates.

The system of Fig. 6.8a consists of two rigid bodies, where the first behaves like a particle, so that it is subjected to no moments and possesses no moment of inertia. Hence, Eq. (7.29) reduces to

$$(\mathbf{F}_1 - m_1\ddot{\mathbf{r}}_{C1}) \cdot \delta \mathbf{r}_{C1} + (\mathbf{F}_2 - m_2\ddot{\mathbf{r}}_{C2}) \cdot \delta \mathbf{r}_{C2} + (M_{C2} - I_{C2}\ddot{\theta}_2)\, \delta\theta_2 = 0 \tag{a}$$

In our case,

$$m_1 = M, \qquad m_2 = m, \qquad I_{C2} = \tfrac{1}{12} m(2a)^2 = \tfrac{1}{3} ma^2 \tag{b}$$

Moreover,

$$\mathbf{F}_1 = -kx\mathbf{i} - Mg\mathbf{j}, \qquad \mathbf{F}_2 = F\mathbf{i} - mg\mathbf{j}, \qquad M_{C2} = Fa\cos\theta \tag{c}$$

and

$$\begin{aligned} \mathbf{r}_{C1} &= \mathbf{r}_0 = (L + x)\mathbf{i} \\ \mathbf{r}_{C2} &= \mathbf{r}_C = \mathbf{r}_0 + \mathbf{r}_{0C} = (L + x + a\sin\theta)\mathbf{i} - a\cos\theta\, \mathbf{j} \end{aligned} \tag{d}$$

where $\theta = \theta_2$ and L is the unstretched length of the spring. From Eqs. (d), we can write the virtual displacements

$$\delta \mathbf{r}_{C1} = \delta x \mathbf{i}, \qquad \delta \mathbf{r}_{C2} = (\delta x + a\,\delta\theta\cos\theta)\mathbf{i} + a\,\delta\theta\sin\theta\,\mathbf{j} \tag{e}$$

and, of course, $\delta\theta_2 = \delta\theta$. Also, differentiating Eqs. (d) twice with respect to time, we obtain the accelerations

$$\begin{aligned} \ddot{\mathbf{r}}_{C1} &= \ddot{x}\mathbf{i} \\ \ddot{\mathbf{r}}_{C2} &= [\ddot{x} + a(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta)]\mathbf{i} + a(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta)\mathbf{j} \end{aligned} \tag{f}$$

Inserting Eqs. (b)–(f) into Eq. (a), we can write

$$\begin{aligned} &(-kx\mathbf{i} - Mg\mathbf{j} - M\ddot{x}\mathbf{i}) \cdot \delta x\mathbf{i} + \{F\mathbf{i} - mg\mathbf{j} - m[\ddot{x} + a(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta)]\mathbf{i} \\ &\quad - ma(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta)\mathbf{j}\} \cdot [(\delta x + a\,\delta\theta\cos\theta)\mathbf{i} + a\,\delta\theta\sin\theta\mathbf{j}] \\ &\quad + (Fa\cos\theta - \tfrac{1}{3}ma^2\ddot{\theta})\,\delta\theta = \{-kx - M\ddot{x} + F - m[\ddot{x} + a(\ddot{\theta}\cos\theta \\ &\quad - \dot{\theta}^2\sin\theta)]\}\,\delta x + \langle\{F - m[\ddot{x} + a(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta)]\}a\cos\theta \\ &\quad - [mg + ma(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta)]a\sin\theta + Fa\cos\theta - \tfrac{1}{3}ma^2\ddot{\theta}\rangle\,\delta\theta = 0 \end{aligned} \tag{g}$$

Because the generalized displacements x and θ are independent, the generalized virtual displacements δx and $\delta\theta$ are arbitrary. Hence, Eq. (g) can be satisfied only if the coefficients of δx and $\delta\theta$ are each equal to zero. Setting these coefficients equal to zero and collecting terms, we obtain the equations of motion

$$\begin{gathered}(M+m)\ddot{x}+ma(\ddot{\theta}\cos\theta-\dot{\theta}^2\sin\theta)+kx=F \\ ma\ddot{x}\cos\theta+\tfrac{4}{3}ma^2\ddot{\theta}+mga\sin\theta=2Fa\cos\theta\end{gathered} \tag{h}$$

Equations (h) are identical to Eqs. (m) obtained in Example 6.5 by means of Newton's second law (as extended to systems of rigid bodies).

7.5 LAGRANGE'S EQUATIONS OF MOTION

In deriving the equations of motion, d'Alembert's principle has an advantage over Newton's second law in that the constraint forces do not come into play. However, d'Alembert's principle is still basically a vectorial approach involving accelerations. As a result, the derivation of the equations of motion remains a relatively cumbersome process. Hence, a simpler approach to the derivation of the equations of motion is highly desirable. Such an approach is provided by the Lagrangian mechanics, and it permits the derivation of the equations of motion from two scalar functions, the kinetic energy and the potential energy, as well as the virtual work due to nonconservative forces. It is here that d'Alembert's principle proves its worth since it can be used to facilitate the transition from the Newtonian mechanics to Lagrangian mechanics. Lagrangian mechanics works with the generalized coordinates q_k $(k=1,2,\ldots,n)$ instead of the physical coordinates $\mathbf{r}_i$ $(i=1,2,\ldots,N)$, and the equations of motion in terms of the generalized coordinates are known as Lagrange's equations.

Considering an n-degree-of-freedom system and assuming that the coordinates $\mathbf{r}_i$ do not depend explicitly on time, we can write the coordinate transformation in the general form

$$\mathbf{r}_i=\mathbf{r}_i(q_1,q_2,\ldots,q_n),\qquad i=1,2,\ldots,N \tag{7.30}$$

The velocities $\dot{\mathbf{r}}_i$ are obtained by simply taking the total time derivative of Eqs. (7.30), yielding

$$\dot{\mathbf{r}}_i=\frac{\partial\mathbf{r}_i}{\partial q_1}\dot{q}_1+\frac{\partial\mathbf{r}_i}{\partial q_2}\dot{q}_2+\cdots+\frac{\partial\mathbf{r}_i}{\partial q_n}\dot{q}_n=\sum_{k=1}^{n}\frac{\partial\mathbf{r}_i}{\partial q_k}\dot{q}_k,\qquad i=1,2,\ldots,N \tag{7.31}$$

Because the quantities $\partial\mathbf{r}_i/\partial q_k$ do not depend explicitly on the generalized velocities $\dot{q}_k$, Eqs. (7.31) yield

$$\frac{\partial\dot{\mathbf{r}}_i}{\partial\dot{q}_k}=\frac{\partial\mathbf{r}_i}{\partial q_k},\qquad i=1,2,\ldots,N;\qquad k=1,2,\ldots,n \tag{7.32}$$

Moreover, by analogy with Eqs. (7.31), we can write

$$\delta\mathbf{r}_i=\frac{\partial\mathbf{r}_i}{\partial q_1}\delta q_1+\frac{\partial\mathbf{r}_i}{\partial q_2}\delta q_2+\cdots+\frac{\partial\mathbf{r}_i}{\partial q_n}\delta q_n=\sum_{i=1}^{n}\frac{\partial\mathbf{r}_i}{\partial q_k}\delta q_k,\quad k=1,2,\ldots,n \tag{7.33}$$

In view of Eqs. (7.33), the second term in Eq. (7.28) becomes

$$\sum_{i=1}^{N} m_i \ddot{\mathbf{r}}_i \cdot \delta \mathbf{r}_i = \sum_{i=1}^{N} m_i \ddot{\mathbf{r}}_i \cdot \sum_{k=1}^{n} \frac{\partial \mathbf{r}_i}{\partial q_k} \delta q_k$$
$$= \sum_{k=1}^{n} \sum_{i=1}^{N} \left(m_i \ddot{\mathbf{r}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_k} \right) \delta q_k \tag{7.34}$$

Concentrating on a typical term on the right side of (7.34), we observe that

$$m_i \ddot{\mathbf{r}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_k} = \frac{d}{dt}\left(m_i \dot{\mathbf{r}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_k} \right) - m_i \dot{\mathbf{r}}_i \cdot \frac{d}{dt}\left(\frac{\partial \mathbf{r}_i}{\partial q_k} \right) \tag{7.35}$$

Considering Eqs. (7.32) and assuming that the order of the total derivatives with respect to time and partial derivatives with respect to q_k is interchangeable, we can write Eq. (7.35) in the form

$$m_i \ddot{\mathbf{r}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_k} = \frac{d}{dt}\left(m_i \dot{\mathbf{r}}_i \cdot \frac{\partial \dot{\mathbf{r}}_i}{\partial \dot{q}_k} \right) - m_i \dot{\mathbf{r}}_i \cdot \frac{\partial \dot{\mathbf{r}}_i}{\partial q_k}$$
$$= \left[\frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}_k} \right) - \frac{\partial}{\partial q_k} \right] \left(\tfrac{1}{2} m_i \dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}_i \right) \tag{7.36}$$

But the second term in parentheses on the right side of Eq. (7.36) is recognized as the kinetic energy of particle i (see Section 5.6). Hence, insertion of Eq. (7.36) into Eq. (7.34) yields

$$\sum_{i=1}^{N} m_i \ddot{\mathbf{r}}_i \cdot \delta \mathbf{r}_i = \sum_{k=1}^{n} \left\{ \left[\frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}_k} \right) - \frac{\partial}{\partial q_k} \right] \left(\sum_{i=1}^{N} \frac{1}{2} m_i \dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}_i \right) \right\} \delta q_k$$
$$= \sum_{k=1}^{n} \left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k} \right) - \frac{\partial T}{\partial q_k} \right] \delta q_k \tag{7.37}$$

where, in view of transformation (7.31),

$$T = \frac{1}{2} \sum_{i=1}^{N} m_i \dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}_i = T(q_1, q_2, \ldots, q_n, \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n) \tag{7.38}$$

is the kinetic energy of the entire system expressed as a function of the generalized coordinates and velocities.

It remains to write the forces $\mathbf{F}_i(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N, \dot{\mathbf{r}}_1, \dot{\mathbf{r}}_2, \ldots, \dot{\mathbf{r}}_N, t)$ in terms of the generalized coordinates q_k $(k = 1, 2, \ldots, n)$. This is done by using the virtual work expression, in conjunction with transformation (7.33), as follows:

$$\overline{\delta W} = \sum_{i=1}^{N} \mathbf{F}_i \cdot \delta \mathbf{r}_i = \sum_{i=1}^{N} \mathbf{F}_i \cdot \sum_{k=1}^{n} \frac{\partial \mathbf{r}_i}{\partial q_k} \delta q_k$$
$$= \sum_{k=1}^{n} \left(\sum_{i=1}^{N} \mathbf{F}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_k} \right) \delta q_k \tag{7.39}$$

The virtual work, however, can be regarded as the product of n *generalized forces* Q_k acting over the virtual displacements δq_k,

$$\overline{\delta W} = \sum_{k=1}^{n} Q_k \, \delta q_k \tag{7.40}$$

so that, comparing Eqs. (7.39) and (7.40), we conclude that the generalized forces have the form

$$Q_k = \sum_{i=1}^{N} \mathbf{F}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_k}, \qquad k = 1, 2, \ldots, n \tag{7.41}$$

In actual situations, the generalized forces are derived by identifying physically a set of generalized coordinates and writing the virtual work directly in the form (7.40), rather than by using formula (7.41) (see Example 7.4). We note that the generalized forces are not necessarily forces. They can be moments or any other quantities such that the product $Q_k \delta q_k$ has units of work.

If the forces acting on the system can be divided into conservative forces, which are derivable from the potential energy $V = V(q_1, q_2, \ldots, q_n)$, and nonconservative forces, which are not, then the first term in Eq. (7.28) becomes

$$\begin{aligned} \sum_{i=1}^{N} \mathbf{F}_i \cdot \delta \mathbf{r}_i &= \overline{\delta W} = \delta W_c + \overline{\delta W}_{nc} = -\delta V + \sum_{k=1}^{n} Q_{knc} \, \delta q_k \\ &= -\left(\frac{\partial V}{\partial q_1} \delta q_1 + \frac{\partial V}{\partial q_2} \delta q_2 + \cdots + \frac{\partial V}{\partial q_n} \delta q_n \right) + \sum_{k=1}^{n} Q_{knc} \, \delta q_k \\ &= -\sum_{k=1}^{n} \left(\frac{\partial V}{\partial q_k} - Q_{knc} \right) \delta q_k \end{aligned} \tag{7.42}$$

where Q_{knc} $(k = 1, 2, \ldots, n)$ are nonconservative generalized forces. Note that the overbar was omitted from δW_c, because δW_c represents the variation of the work function W_c due to conservative forces, where W_c is the negative of the potential energy V. Introducing Eqs. (7.37) and (7.42) into Eq. (7.28), we obtain

$$-\sum_{k=1}^{n} \left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k} \right) - \frac{\partial T}{\partial q_k} + \frac{\partial V}{\partial q_k} - Q_{knc} \right] \delta q_k = 0 \tag{7.43}$$

However, by definition, the generalized virtual displacements δq_k are both arbitrary and independent. Hence, letting $\delta q_k = 0$, $k \neq j$, and $\delta q_j \neq 0$, we conclude that Eq. (7.43) can be satisfied if and only if the coefficient of δq_j is zero. The procedure can be repeated n times for $j = 1, 2, \ldots, n$. Moreover, with the understanding that Q_j represents nonconservative forces, we can drop the subscript nc in Q_{jnc} and arrive at the set of equations

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} + \frac{\partial V}{\partial q_j} = Q_j, \qquad j = 1, 2, \ldots, n \tag{7.44}$$

which are the famous *Lagrange's equations of motion.* In general, the potential energy does not depend on the generalized velocities $\dot{q}_j$ ($j=1, 2, \ldots, n$). In view of this, we can introduce the *Lagrangian* defined by

$$L=T-V \tag{7.45}$$

and reduce Eqs. (7.44) to the more compact form

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)-\frac{\partial L}{\partial q_j}=Q_j, \qquad j=1, 2, \ldots, n \tag{7.46}$$

In many problems there are no nonconservative forces involved, in which cases $Q_j=0$ ($j=1, 2, \ldots, n$). Hence, *Lagrange's equations for conservative systems* are simply

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)-\frac{\partial L}{\partial q_j}=0, \qquad j=1, 2, \ldots, n \tag{7.47}$$

The Lagrangian approach is very efficient for deriving the system equations of motion, especially when the number of degrees of freedom of the system is large. All the differential equations of motion are derived from two scalar functions, namely, the kinetic energy T and the potential energy V, and the virtual work δW_{nc} associated with the nonconservative forces. The equations apply to linear as well as nonlinear systems. Although it appears that the identification of the generalized coordinates and generalized forces is a major stumbling block in using this approach, this is actually not the case. Indeed, in most physical systems this aspect presents no particular difficulty, as illustrated in Example 7.4.

Example 7.4

Derive Lagrange's equations of motion for the system of Example 7.3.

As in Example 7.3, we propose to use x and θ as generalized coordinates, so that $q_1=x$, $q_2=\theta$. Accordingly, the nonconservative generalized forces are denoted by $Q_1=X$, $Q_2=\Theta$. To calculate the kinetic energy, we must express the velocities of the two bodies in terms of these coordinates. From Eqs. (d) of Example 7.3, we can write

$$\mathbf{v}_0=\dot{\mathbf{r}}_0=\dot{x}\,\mathbf{i}, \qquad \mathbf{v}_C=\dot{\mathbf{r}}_C=(\dot{x}+a\dot{\theta}\cos\theta)\mathbf{i}+a\dot{\theta}\sin\theta\,\mathbf{j} \tag{a}$$

so that the kinetic energy can be written in the form

$$\begin{aligned} T&=\tfrac{1}{2}M\,\mathbf{v}_0\cdot\mathbf{v}_0+\tfrac{1}{2}m\,\mathbf{v}_C\cdot\mathbf{v}_C+\tfrac{1}{2}I_C\dot{\theta}^2 \\ &=\tfrac{1}{2}M\dot{x}^2+\tfrac{1}{2}m[(\dot{x}+a\dot{\theta}\cos\theta)^2+(a\dot{\theta}\sin\theta)^2]+\tfrac{1}{2}\tfrac{1}{12}m(2a)^2\dot{\theta}^2 \\ &=\tfrac{1}{2}M\dot{x}^2+ma\dot{x}\dot{\theta}\cos\theta+\tfrac{2}{3}ma^2\dot{\theta}^2 \end{aligned} \tag{b}$$

The potential energy is due to the deformation of the spring and the rise of the mass center of the bar. Its expression is

$$V=\tfrac{1}{2}kx^2+mga(1-\cos\theta) \tag{c}$$

Hence, the Lagrangian of the system is

$$L = T - V = \tfrac{1}{2}(M+m)\dot{x}^2 + ma\dot{x}\dot{\theta}\cos\theta + \tfrac{2}{3}ma^2\dot{\theta}^2$$
$$-\tfrac{1}{2}kx^2 - mga(1-\cos\theta) \tag{d}$$

It remains to calculate the virtual work associated with the nonconservative force F. Denoting the point of application of the force by B, we can write the position vector of B in terms of cartesian components as follows:

$$\mathbf{r}_B = (L + x + 2a\sin\theta)\mathbf{i} - 2a\cos\theta\,\mathbf{j} \tag{e}$$

so that the virtual displacement of B is

$$\delta\mathbf{r}_B = (\delta x + 2a\,\delta\theta\cos\theta)\mathbf{i} + 2a\,\delta\theta\sin\theta\,\mathbf{j} \tag{f}$$

Moreover, the force F can be written in the vector form

$$\mathbf{F} = F\mathbf{i} \tag{g}$$

Hence, the nonconservative virtual work is

$$\overline{\delta W}_{nc} = \mathbf{F}\cdot\delta\mathbf{r}_B = F\mathbf{i}\cdot[(\delta x + 2a\,\delta\theta\cos\theta)\mathbf{i} + 2a\,\delta\theta\sin\theta\,\mathbf{j}]$$
$$= F(\delta x + 2a\,\delta\theta\cos\theta) \tag{h}$$

Equation (h) is in the form (7.40), so that the coefficients of δx and $\delta\theta$ are the nonconservative generalized forces, or

$$X = F, \qquad \Theta = 2Fa\cos\theta \tag{i}$$

The derivatives entering into Lagrange's equations are as follows:

$$\begin{aligned}
\frac{\partial L}{\partial \dot{x}} &= (M+m)\dot{x} + ma\dot{\theta}\cos\theta \\
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) &= (M+m)\ddot{x} + ma(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta) \\
\frac{\partial L}{\partial x} &= -kx \\
\frac{\partial L}{\partial \dot{\theta}} &= ma\dot{x}\cos\theta + \tfrac{4}{3}ma^2\dot{\theta} \\
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) &= ma(\ddot{x}\cos\theta - \dot{x}\dot{\theta}\sin\theta) + \tfrac{4}{3}ma^2\ddot{\theta} \\
\frac{\partial L}{\partial \theta} &= -ma\dot{x}\dot{\theta}\sin\theta - mga\sin\theta
\end{aligned} \tag{j}$$

Introducing Eqs. (i) and (j) into Eqs. (7.46) with $q_1 = x$, $q_2 = \theta$, $Q_1 = X$ and $Q_2 = \Theta$, we obtain Lagrange's equations of motion for the system in the explicit form

$$\begin{aligned}
(M+m)\ddot{x} + ma(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta) + kx &= F \\
ma\ddot{x}\cos\theta + \tfrac{4}{3}ma^2\ddot{\theta} + mga\sin\theta &= 2Fa\cos\theta
\end{aligned} \tag{k}$$

Equations (k) are identical to Eqs. (h) obtained in Example 7.3 by means of d'Alembert's principle and Eqs. (m) obtained in Example 6.5 by means of the Newtonian approach.

Comparing the amount of work involved in Examples 6.5, 7.3, and 7.4, it appears that the Lagrangian approach permits the derivation of the equations of motion in the most expeditious manner, at least in this example. Actually this is true in most cases and not merely in this particular example. In particular, we observe that equations of motion having the same form as Eqs. (k) were obtained in Example 6.5 only after the elimination of the internal forces at the hinge connecting the rigid bar to the lumped mass. In other cases the equations derived by means of Newton's second law do not appear to correspond exactly to those derived by means of Lagrange's equations. In such cases, to show that the equations derived by the two approaches are indeed identical, it is necessary to combine linearly several equations in one set or the other.

PROBLEMS

7.1 Determine the position of static equilibrium of the system of Fig. 6.28 by means of the virtual work principle.

7.2 Determine the position of static equilibrium of the airfoil section of Fig. 6.26 by the virtual work principle.

7.3 Derive the equations of motion for the system of Fig. 6.26 by means of d'Alembert's principle.

7.4 Derive the equations of motion for the system of Fig. 6.28 by means of d'Alembert's principle.

7.5 Derive Lagrange's equation of motion for the system of Fig. 6.18.

7.6 Derive Lagrange's equation of motion for the system of Fig. 6.19.

7.7 Derive Lagrange's equation of motion for the system of Fig. 6.22.

7.8 Derive Lagrange's equations of motion for the system of Fig. 6.26.

7.9 Derive Lagrange's equations of motion for the system of Fig. 6.27.

7.10 Derive Lagrange's equations of motion for the system of Fig. 6.28.

7.11 Derive Lagrange's equations of motion for the gyroscope shown in Fig. 6.10.