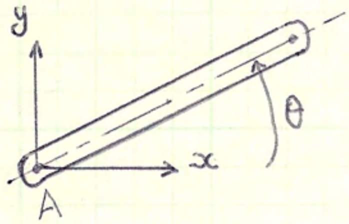
I Introduction

1. Definitions and Examples

Def: The number of degrees-of-freedom (d.o.f) associated with a given mechanical system is equal to the number of coordinates used to describe its configuration minus the number of independent equations of constraints

Ex: Planar One-Link-Robot



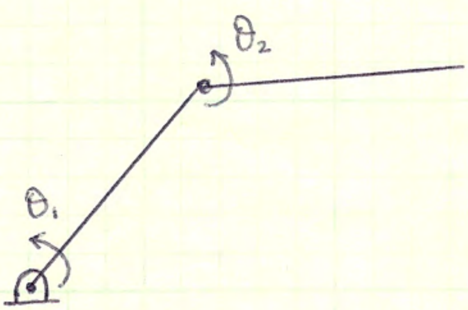
# of coordinates 3

# of constraint eqs 2 $(x=0,\ y=0)$

# of d.o.f $1 = 3-2$

Def: Any set of parameters which gives an unambiguous representation of the configuration of the system will serve as a system of coordinates in a more general sense. These parameters are known as generalized coordinates.

Ex: Planar Two-Link robot



$\begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}$ vector of generalized Coordinates

2. Euler – Lagrange Equations

A very well known method for deriving the dynamic equations of mechanical systems is by using the Euler – Lagrange equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = \tau$$

where $q = \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix}$ is a set of generalized coordinates for the system

L = Lagrangian

$= K - P$ where K is the kinetic energy of the system

P is the potential energy of the system

$\tau = \begin{bmatrix} \tau_1 \\ \vdots \\ \tau_n \end{bmatrix}$ is the vector of generalized forces acting on the system.

3. Dynamic Equations of Robot Manipulators

(n-d.o.f) robot manipulators have two special features:

- The potential energy $P = P(q)$ is independent of $\dot{q}$
- The kinetic energy is a quadratic function of $\dot{q}$

$$K = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{1}{2} d_{ij}(q)\, \dot{q}_i\, \dot{q}_j$$

[big step Kinematics]

$$\triangleq \frac{1}{2} \dot{q}^T D(q) \dot{q}$$

where $D(q)$ is an $n \times n$ matrix called the inertia matrix.

The inertia matrix $D(q)$ has two properties:

- $D(q)$ is symmetric : $d_{ij}(q) = d_{ji}(q) \quad \forall q \in \mathbb{R}^n \quad i,j=1,\dots,n$
- $D(q)$ is positive definite (PD) :

$x^T D(q) x > 0 \quad \forall x \neq 0, \forall q \in \mathbb{R}^n$, or equivalently all eigenvalues of $D(q)$, $\lambda_i(q) > 0 \quad \forall q \in \mathbb{R}^n \quad i=1,\dots,n$

We now derive the Euler-Lagrange equations as follows:

$$L = K - P$$
$$= \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} d_{ij}(q)\, \dot{q}_i \dot{q}_j - P(q)$$

$$\frac{\partial L}{\partial \dot{q}_k} = \sum_{j=1}^{n} d_{kj}(q)\, \dot{q}_j$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_k} = \sum_{j=1}^{n} d_{kj}(q)\, \ddot{q}_j + \sum_{j=1}^{n} \frac{d}{dt} d_{kj}(q)\, \dot{q}_j$$
$$= \sum_{j=1}^{n} d_{kj}(q)\, \ddot{q}_j + \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial d_{kj}(q)}{\partial q_i} \dot{q}_i \dot{q}_j$$

Also $$\frac{\partial L}{\partial q_k} = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial d_{ij}}{\partial q_k} \dot{q}_i \dot{q}_j - \frac{\partial P}{\partial q_k}$$

Hence, the Euler-Lagrange equations can be written as follows

$$\sum_{j=1}^{n} d_{kj}(q)\, \ddot{q}_j + \sum_{i=1}^{n} \sum_{j=1}^{n} \left\{ \frac{\partial d_{kj}(q)}{\partial q_i} - \frac{1}{2} \frac{\partial d_{ij}}{\partial q_k} \right\} \dot{q}_i \dot{q}_j + \frac{\partial P}{\partial q_k} = \tau_k$$

$k = 1, \dots, n$.

No that

$$\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial d_{kj}}{\partial q_i}\dot{q}_i\dot{q}_j = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\left\{\frac{\partial d_{kj}}{\partial q_i} + \frac{\partial d_{ki}}{\partial q_j}\right\}\dot{q}_i\dot{q}_j \quad \text{why?}$$

It follows that

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\left\{\frac{\partial d_{kj}}{\partial q_i} - \frac{1}{2}\frac{\partial d_{ij}}{\partial q_k}\right\}\dot{q}_i\dot{q}_j$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} \underbrace{\frac{1}{2}\left\{\frac{\partial d_{kj}}{\partial q_i} + \frac{\partial d_{ki}}{\partial q_j} - \frac{\partial d_{ij}}{\partial q_k}\right\}}\dot{q}_i\dot{q}_j$$

The coefficients

$$c_{ijk} \triangleq \frac{1}{2}\left\{\frac{\partial d_{kj}}{\partial q_i} + \frac{\partial d_{ki}}{\partial q_j} - \frac{\partial d_{ij}}{\partial q_k}\right\}$$

are known as Christoffel symbols.

Now set $\phi_k = \dfrac{\partial P}{\partial q_k}$

Then the equations of motion, the Euler-Lagrange equations, can be written as

$$\sum_{j=1}^{n} d_{kj}(q)\ddot{q}_j + \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ijk}(q)\dot{q}_i\dot{q}_j + \phi_k(q) = \tau_k \;, \quad k=1,2,\dots,n$$

There are three types of terms in this equation

- Terms involving the second derivative of the generalized coordinates
- Quadratic terms in the first derivatives of q where the coefficients may depend on q. These are further classified into two types

- Terms involving a product of the type $\dot{q}_i^2$ are called <u>centrifugal</u>
- Terms involving a product of the type $\dot{q}_i \dot{q}_j$ where $i \neq j$ are called <u>Coriolis terms</u>

- Terms involving only q but not $\dot{q}$. These arise from differentiating the potential energy.

We commonly write the equations of motion in matrix form as follows

$$\boxed{D(q)\ddot{q} + C(q,\ddot{q})\dot{q} + g(q) = \tau}$$

where the elements c_{kj} of the matrix $C(q,\ddot{q})$ are defined <u>here</u> as

$$\begin{aligned} c_{kj} &= \sum_{i=1}^{n} c_{ijk}(q)\,\dot{q}_i \\ &= \sum_{i=1}^{n} \frac{1}{2}\left\{\frac{\partial d_{kj}}{\partial q_i} + \frac{\partial d_{ki}}{\partial q_j} - \frac{\partial d_{ij}}{\partial q_k}\right\}\dot{q}_i \end{aligned}$$

<u>Note:</u> Other choices of the $C(q,\dot{q})$ matrix are possible.

4. Fundamental Properties of the Equations of Motion

The equations of motion are generally complex nonlinear equations but they have several fundamental properties which are exploited to facilitate control system design.

Property 1:

The inertia matrix $D(q)$ is symmetric, positive definite, and both $D(q)$ and $D^{-1}(q)$ are bounded as a function of $q \in \mathbb{R}^n$.

Property 2:

In general
There is an independent control input for each d.o.f.

Property 3:

All of the constant parameters of interest such as link masses, moments of inertia, etc., appear as coefficients of known functions of the generalized coordinates. By defining each coefficient as a separate parameter, a linear relationship results so that we may write the dynamic equations as

$$D(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = Y(q,\dot{q},\ddot{q})\,\theta = \tau$$

where

$Y(q,\dot{q},\ddot{q})$ is an $n \times r$ matrix of known functions, known as the <u>regressor</u> and θ is an r-dimensional vector of parameters.

Aside:

Def: A matrix S is said to be skew symmetric iff:

$$S^T + S = 0$$

Ex: $S = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ is skew symmetric

S is skew symmetric with components s_{ij}, iff

$$s_{ij} + s_{ji} = 0 \qquad \text{i.e. } s_{ij} = -s_{ji}$$

$$\Rightarrow \quad s_{ii} = 0$$

Note that: S is skew symmetric $\Longrightarrow x^T S x = 0 \;\; \forall x$ (*)

$\cancel{\Longleftarrow}$!

HW: Prove (*)

Property 4:

Define the matrix $N(q,\dot{q}) = \dot{D}(q) - 2C(q,\dot{q})$.

Then $N(q,\dot{q})$ is skew symmetric, i.e. the components n_{jk} of $N(q,\dot{q})$ satisfy $n_{jk} = -n_{kj}$.

Proof: Given the inertia matrix $D(q)$, the kj^{th} component of $\dot{D}(q)$ is given by the chain rule as

$$\dot{d}_{kj} = \sum_{i=1}^{n} \frac{\partial d_{kj}}{\partial q_i} \dot{q}_i .$$

Therefore, the kj^{th} component of $N = \dot{D} - 2C$ is given by

$$n_{kj} = \dot{d}_{kj} - 2c_{kj}$$

$$= \sum_{i=1}^{n} \left[\frac{\partial d_{kj}}{\partial q_i} - \left\{ \frac{\partial d_{kj}}{\partial q_i} + \frac{\partial d_{ki}}{\partial q_j} - \frac{\partial d_{ij}}{\partial q_k} \right\} \right] \dot{q}_i$$

$$= \sum_{i=1}^{n} \left[\frac{\partial d_{ij}}{\partial q_k} - \frac{\partial d_{ki}}{\partial q_j} \right] \dot{q}_i .$$

It follows that $n_{jk} = \sum_{i=1}^{n} \left[\frac{\partial d_{ik}}{\partial q_j} - \frac{\partial d_{ji}}{\partial q_k} \right] \dot{q}_i$

$$= \sum_{i=1}^{n} \left[\frac{\partial d_{ki}}{\partial q_j} - \frac{\partial d_{ij}}{\partial q_k} \right] \dot{q}_i \quad \text{(used symmetry of } D(q)\text{)}$$

$$= -n_{kj}$$

Remark 1:

$N(q,\dot{q}) = \dot{D}(q) - 2C(q,\dot{q})$ is skew symmetric

$\Rightarrow$

$$\dot{q}^T \left[\dot{D}(q) - 2C(q,\dot{q}) \right] \dot{q} = 0$$

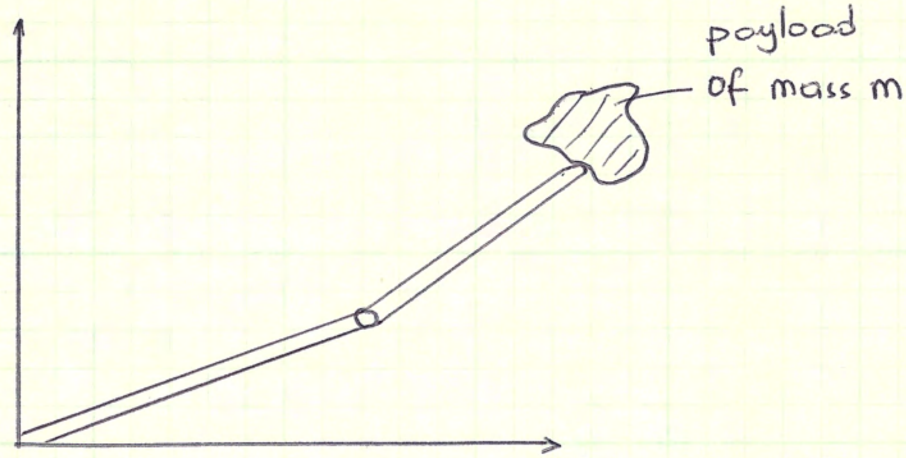
Remark 2:

It turns out that it is <u>always true</u> that

$$\dot{q}^T [\dot{D} - 2C] \dot{q} = 0$$

no matter how C is chosen. However $[\dot{D}-2C]$ is skew-symmetric only in the case the C matrix is as defined earlier.

HW: Suppose <u>all</u> the system parameters in the 2 d.o.f planar manipulator are completely known. Suppose now that we attach an unknown payload of mass m to the tip of the second link



Write the equations of motions in the following form

$$D\ddot{q} + C\dot{q} + g = Y_1\theta_1 + Y_2\theta_2$$

where Y_1, θ_1, and Y_2 are known and θ_2 is a vector of unknown parameters

Example :
A two d.o.f Planar Manipulator

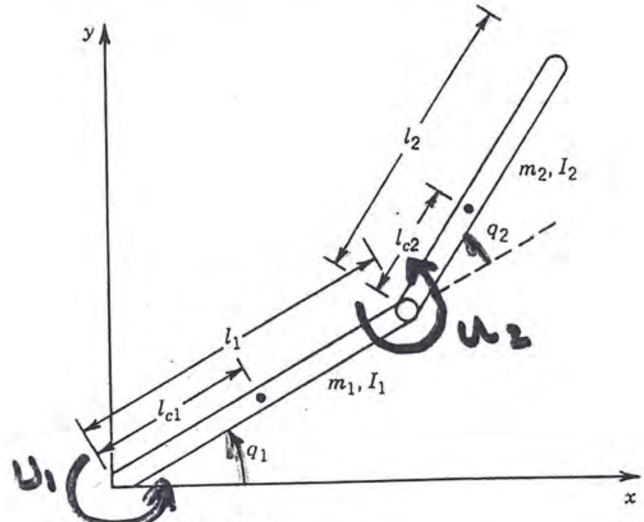


FIGURE 6-3
Two-link revolute joint arm.

$$M\ddot{\mathbf{q}} + C\dot{\mathbf{q}} + \mathbf{g} = \mathbf{u}$$

$C_i \triangleq \text{Cos}(q_i)$
$S_i \triangleq \text{Sin}(q_i)$
$C_{ij} \triangleq \text{Cos}(q_1 + q_2)$

$$\mathbf{q} = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$M = \left[\begin{array}{c|c} m_1 l_{c_1}^2 + m2(l_1^2 + l_{c_2}^2 + 2l_1 l_{c_2} C_2) + I_1 + I_2 & m_2(l_{c_2}^2 + l_1 l_{c_2} C_2) + I_2 \\ m_2(l_{c_2}^2 + l_1 l_{c_2} C_2) \cancel{+ I_2 m_2 l_{c_2}^2} + I_2 & \cancel{0}\ m_2 l_{c_2}^2 + I_2 \end{array} \right]$$

$$C = \begin{bmatrix} -m_2 l_1 l_{c_2} S_2 \dot{q}_2 & -m_2 l_1 l_{c_2} S_2 (\dot{q}_2 + \dot{q}_1) \\ m_2 l_1 l_{c_2} S_2 \dot{q}_1 & 0 \end{bmatrix}$$

$$\mathbf{g} = \begin{bmatrix} (m_1 l_{c_1} + m2 l_1) g C1 + m_2 l_{c_2} g C_{12} \\ m_2 l_{c_2} C_{12} \end{bmatrix}.$$

- Note that

$$\dot{M} - 2C = \begin{bmatrix} 0 & m_2 l_1 l_{c_2} S_2 (\dot{q}_2 + 2\dot{q}_1) \\ -m_2 l_1 l_{c_2} S_2 (\dot{q}_2 + 2\dot{q}_1) & 0 \end{bmatrix}$$

which is skew symmetric

Example : A two d.o.f Planar Manipulator (Continued)

Define

$$
\begin{array}{lll}
\theta_1 = m_1 l_{c_1}^2 & \theta_4 = m_2 l_1 l_{c_2} & \theta_7 = m_1 l_{c_1} g \\
\theta_2 = m_2 l_1^2 & \theta_5 = I_1 & \theta_8 = m_2 l_1 g \\
\theta_3 = m_2 l_{c_2}^2 & \theta_6 = I_2 & \theta_9 = m_2 l_{c_2} g
\end{array}
$$

$\Rightarrow$

$$M(\mathbf{q})\ddot{\mathbf{q}} + C(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} + g(\mathbf{q}) = Y(\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}})\theta = \mathbf{u}$$

$\theta = \begin{bmatrix} \theta_1 & \theta_2 & \theta_3 & \theta_4 & \theta_5 & \theta_6 & \theta_7 & \theta_8 & \theta_9 \end{bmatrix}^T$

$Y(\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}}) =$

$$
\begin{bmatrix}
\ddot{q}_1 & \dot{q}_2 & \dot{q}_1 + \dot{q}_2 & 2C_2\ddot{q}_1 + C_2\ddot{q}_2 - 2S_2\dot{q}_1\dot{q}_2 - S_2\dot{q}_2^2 & \ddot{q}_1 & \dot{q}_1 + \dot{q}_2 & C_1 & C_1 & C_{12} \\
0 & 0 & \ddot{q}_1 + \ddot{q}_2 & C_2\ddot{q}_1 + S_2 & \ddot{q}_2 & \ddot{q}_2 & 0 & 0 & C_{12}
\end{bmatrix}
$$

Notes

- choice of parameters in above is not unique and dimension of parameter space may depend on particular choice of parameters
- some of the parameters may be known

Example: Inverted Pendulum

$$(M+m)\ddot{x} + ml\cos\theta\,\ddot{\theta} - ml\dot{\theta}^2\sin\theta = u$$

$$ml\cos\theta\,\ddot{x} + ml^2\ddot{\theta} - mgl\sin\theta = 0$$

$$\underbrace{\begin{bmatrix} M+m & ml\cos\theta \\ ml\cos\theta & ml^2 \end{bmatrix}}_{D(q)} \underbrace{\begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix}}_{\ddot{q}} + \underbrace{\begin{bmatrix} 0 & -ml\dot{\theta}\sin\theta \\ 0 & 0 \end{bmatrix}}_{C(q,\dot{q})} \underbrace{\begin{bmatrix} \dot{x} \\ \dot{\theta} \end{bmatrix}}_{\dot{q}} + \underbrace{\begin{bmatrix} 0 \\ -mgl\sin\theta \end{bmatrix}}_{g(q)} = \begin{bmatrix} u \\ 0 \end{bmatrix}$$

- ✓ symmetric
  ✓ p.d: $|M| = (M+m)ml^2 - m^2l^2\cos^2\theta \leq Mml^2$

- $$(\dot{D} - 2C) = \begin{bmatrix} 0 & -ml\dot{\theta}\sin\theta \\ -ml\dot{\theta}\sin\theta & 0 \end{bmatrix} - 2\begin{bmatrix} 0 & -ml\dot{\theta}\sin\theta \\ 0 & 0 \end{bmatrix}$$
  $$= \begin{bmatrix} 0 & ml\dot{\theta}\sin\theta \\ -ml\dot{\theta}\sin\theta & 0 \end{bmatrix}$$ Skew Symmetric

- $$D(q)\ddot{q} = Y_D(q,\ddot{q})\,\theta_D$$

  $$D(q)\ddot{q} = \begin{bmatrix} (M+m)\ddot{x} + ml\cos\theta\,\ddot{\theta} \\ ml\cos\theta\,\ddot{x} + ml^2\ddot{\theta} \end{bmatrix} = \underbrace{\begin{bmatrix} \ddot{x} & \cos\theta\,\ddot{\theta} & 0 \\ 0 & \cos\theta\,\ddot{x} & \ddot{\theta} \end{bmatrix}}_{Y_D} \underbrace{\begin{bmatrix} M+m \\ ml \\ ml^2 \end{bmatrix}}_{\theta_D}$$

- $$C(q,\dot{q})\dot{q} = Y_c\,\theta_c$$
  $$= \begin{bmatrix} -ml\dot{\theta}^2\sin\theta \\ 0 \end{bmatrix} = \underbrace{\begin{bmatrix} -\dot{\theta}^2\sin\theta \\ 0 \end{bmatrix}}_{Y_c} \underbrace{ml}_{\theta_c}$$

- $$g(q) = Y_g\,\theta_g = \begin{bmatrix} 0 \\ -mgl\sin\theta \end{bmatrix} = \underbrace{\begin{bmatrix} 0 \\ -\sin\theta \end{bmatrix}}_{Y_g} \underbrace{mgl}_{\theta_g}$$

Hence

D

Hence,

$$D(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) =$$
$$Y_d(q,\ddot{q})\theta_D + Y_c(q,\dot{q})\theta_c + Y_g(q)\theta_g = Y(q,\dot{q},\ddot{q})\underset{\sim}{\theta}$$

$$\underbrace{\begin{bmatrix} \ddot{x} & \cos\theta\,\ddot{\theta} - \dot{\theta}^2\sin\theta & 0 & 0 \\ 0 & \cos\theta\,\ddot{x} & \ddot{\theta} & -\sin\theta \end{bmatrix}}_{Y(q,\dot{q},\ddot{\theta})} \underbrace{\begin{bmatrix} M+m \\ ml \\ ml^2 \\ mgl \end{bmatrix}}_{\theta}$$

Given a <u>constant</u> matrix $A \in \mathbb{R}^n$,

The norm of a matrix, denoted $\|A\|$ is a measure of the size of A.

We can define many norms. In particular the so-called induced 2-norm

$$\lambda_{min}[A] \leq \|A\|_2 = \sqrt{\lambda_{max}[A^T A]} \leq \lambda_{max}[A]$$

For a positive definite matrix (i.e $\lambda_i > 0$ or all minors are > 0)

i) $0 < \lambda_{min}(A)$

$\lambda_{max}(A) < \infty$

If $A = A(x)$ where $x \in \mathbb{R}$, then

i) $\lambda_{min} = \lambda_{min}(x)$

$\lambda_{max} = \lambda_{max}(x)$

ii) $\lambda_{max}(x) \leq \infty$

iii) if $A(x)$ is positive definite then

$$0 < \lambda_{min}(x) \leq \|A\|_2 \qquad \forall x \in \mathbb{R}$$

<u>Def</u>: $A(x)$ is bounded if all elements of A are bounded $\Rightarrow \lambda_{max}(x) < \infty$

<u>Def</u>: $A(x)$ is <u>uniformly</u> positive definite if $\exists$ a constant $c > 0$ s.t.

$$0 < c \leq \sigma_1(x) \leq \|A\|_2 \qquad \forall x \in \mathbb{R}$$