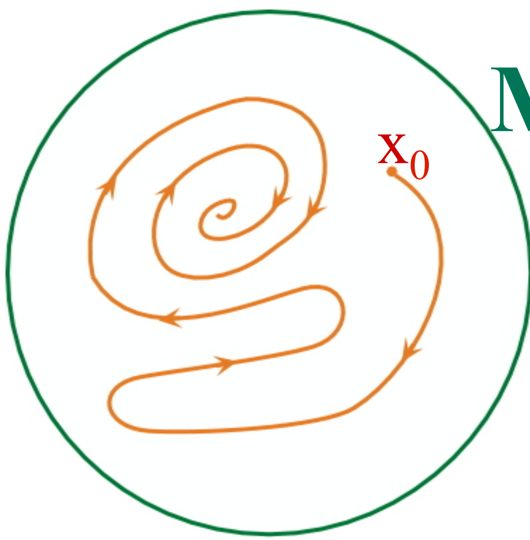
Invariance & Domain of Attraction

Definition: Invariant Set

A set $\mathbf{M} \subseteq \mathbf{R}^n$ is called an invariant set of the differential equation $\sum$ if for each $\mathbf{x}_0 \in \mathbf{M}$, $\exists$ a $t_0 \in \mathbf{R}^+$ s.t. $s(t, t_0, \mathbf{x}_0) \in \mathbf{M}, \forall\ t \geq t_0$.

In words:

A set is invariant if, for every initial state in the set, a suitable initial time can be found s.t. the resulting trajectory stays in the set at all future times.



$\mathbf{M} \subseteq \mathbf{R}^n$

Start there
$\Rightarrow$ Stay there

Examples

- Let $\mathbf{x}_0 \in \mathbf{R}^n$, $t_0 \in \mathbf{R}^+$ be arbitrary, define $S(t_0,\mathbf{x}_0)$ to be the resulting trajectory viewed as a subset of $\mathbf{R}^n$, i.e.

$$S(t_0,\mathbf{x}_0) = \bigcup_{t \geq t_0} s(t,t_0,\mathbf{x}_0)$$

  Then $S(t_0,\mathbf{x}_0)$ is invariant.
- An equilibrium is an invariant set.
- A periodic solution is an invariant set.

Limit Point & Limit Set

Definition: Limit Point & Limit Set

Suppose $\mathbf{x}_0 \in \mathbf{R}^n$, $t_0 \in \mathbf{R}^+$.

Then a point $P \in \mathbf{R}^n$ is called a limit point of the trajectory $S(t, t_0, \mathbf{x}_0)$ if $\exists$ a sequence $\{t_i\}$ of real numbers in $[t_0, \infty)$ s.t.

$$t_i \to \infty \quad \text{and} \quad \lim_{t_i \to \infty} \| P - S(t_i, t_0, \mathbf{x}_0) \| = 0$$

The set of all limit points of $S(\bullet, t_0, \mathbf{x}_0)$ is called the limit set of $S(\bullet, t_0, \mathbf{x}_0)$ and is denoted by $\Omega(t_0, \mathbf{x}_0)$.

Lemma:

Let $\mathbf{x}_0 \in \mathbf{R}^n$, $t_0 \in \mathbf{R}^+$ and suppose $S(\bullet, t_0, \mathbf{x}_0)$ is bounded. Then the limit set $\Omega(t_0, \mathbf{x}_0)$ is

Nonempty

Closed

Bounded

More Definition and Special Result for Periodic (hence Autonomous) Systems

Definition:

The distance $d(\mathbf{x}, \Omega)$ between a point $\mathbf{x}$ and a nonempty set Ω is defined as

$$d(\mathbf{x}, \Omega) = \min_{\mathbf{y} \in \Omega} \| \mathbf{x} - \mathbf{y} \|$$

Lemma:

Let $\mathbf{x}_0 \in \mathbf{R}^n$, $t_0 \in \mathbf{R}^+$ and suppose $S(t, t_0, \mathbf{x}_0)$ is bounded.

Then

$$d[s(t, t_0, \mathbf{x}_0), \Omega(t_0, \mathbf{x})] \to 0 \text{ as } t \to \infty.$$

Special Result for Periodic (hence Autonomous) Systems

Lemma:

Suppose the system $\sum$ is periodic and let $\mathbf{x}_0 \in \mathbf{R}^n$, $t_0 \in \mathbf{R}^+$.

If $S(\bullet, t_0, \mathbf{x}_0)$ is bounded, then

the limit set $\Omega(t_0, \mathbf{x}_0)$ is an invariant set of $\sum$.

Further Result for Autonomous Systems

$$\sum_{\mathrm{A}} : \dot{\mathbf{x}} = f[\mathbf{x}(t)] \qquad \text{Autonomous System}$$

Suppose 0 is an attractive equilibrium of $\sum_{\mathrm{A}}$. This implies by definition that ∃ a ball B_r s.t. every trajectory starting inside B_r approaches O as $t \rightarrow \infty$.

Definition: [Domain of Attraction]

Suppose 0 is an attractive equilibrium of $\sum_{\mathrm{A}}$. The domain of attraction $\boldsymbol{D}(0)$ is defined as

$$D(0) = \{x_0 \in \mathbf{R}^n : s(t,0,x_0) \rightarrow 0 \ \text{ as } \ t \rightarrow \infty\}$$

Remarks:

- Definition implies that O is an interior point of $\boldsymbol{D}(0)$.
- Other terms for domain of attraction: *"region of attraction","basin".*
- We identify the region of $\boldsymbol{D}(0)$ with the equilibrium 0, since $\sum_{\mathrm{A}}$ may have more that one attractive equilibrium, in which case each equilibrium will have its own domain of attraction.

Lemma and Notes

Lemma:

Suppose 0 is an attractive equilibrium of the autonomous system $\sum_A$. Then, the domain of attraction $\boldsymbol{D}(0)$ is

- Open
- Connected, and
- Invariant

Moreover, the boundary of $\boldsymbol{D}(0)$ is an invariant set, hence, is formed by trajectories.

Notes:

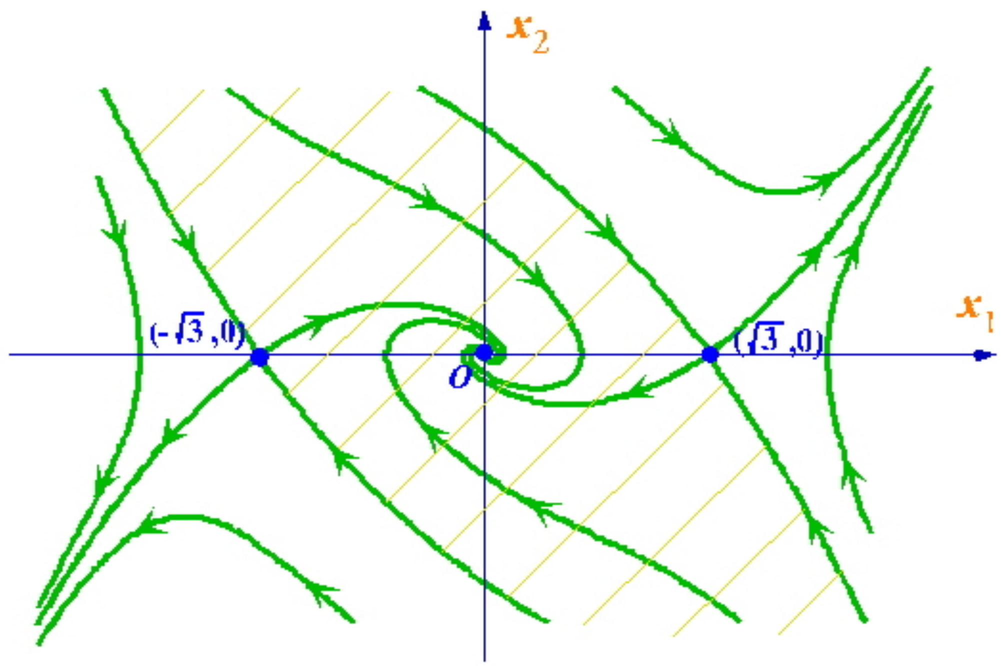
- In $\mathbf{R}^n$, a set is connected iff it cannot be contained in the union of two disjoint open sets.
- It is not an easy task to exactly determine the domain of attraction $\boldsymbol{D}(0)$. We can estimate it though!

Example: The Van der Pol Equation

$$\begin{cases} \dot{x}_1 = -x_2 \\ \dot{x}_2 = x_1 + (x_1^2 - 1)x_2 \end{cases}$$

Example

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1 + \dfrac{1}{3}x_1^3 - x_2 \end{cases}$$



$(-\sqrt{3},0)$, $(\sqrt{3},0)$ } Saddle Points $(0,0)$ Stable Point

Estimation of Domain of Attraction for Autonomous Systems

Suppose system is $\sum_A$ (autonomous) and that the Lyapunov function candidate is independent of time, then $\dot{V}$ is also independent of time. Suppose now that we have succeeded in finding a domain (i.e. an open connected set) $\mathbf{S} \subset \mathbf{R}^n$ containing O with the property that

$$V(\mathbf{x}) > 0, \quad \dot{V}(\mathbf{x}) < 0 \quad \forall x \neq 0 \text{ in } S \qquad (*)$$

Then, Theorem on Asymptotic Stability applies and O is an asymptotically stable equilibrium.

Question:

Does (*) imply that $\boldsymbol{S}$ is contained in the domain of attraction $\boldsymbol{D}(0)$? That is, does (*) imply that

$$s(t,0,\mathbf{x}_0) \to 0 \text{ as } t \to 0 \text{ whenever } \mathbf{x}_0 \in \boldsymbol{S}?$$

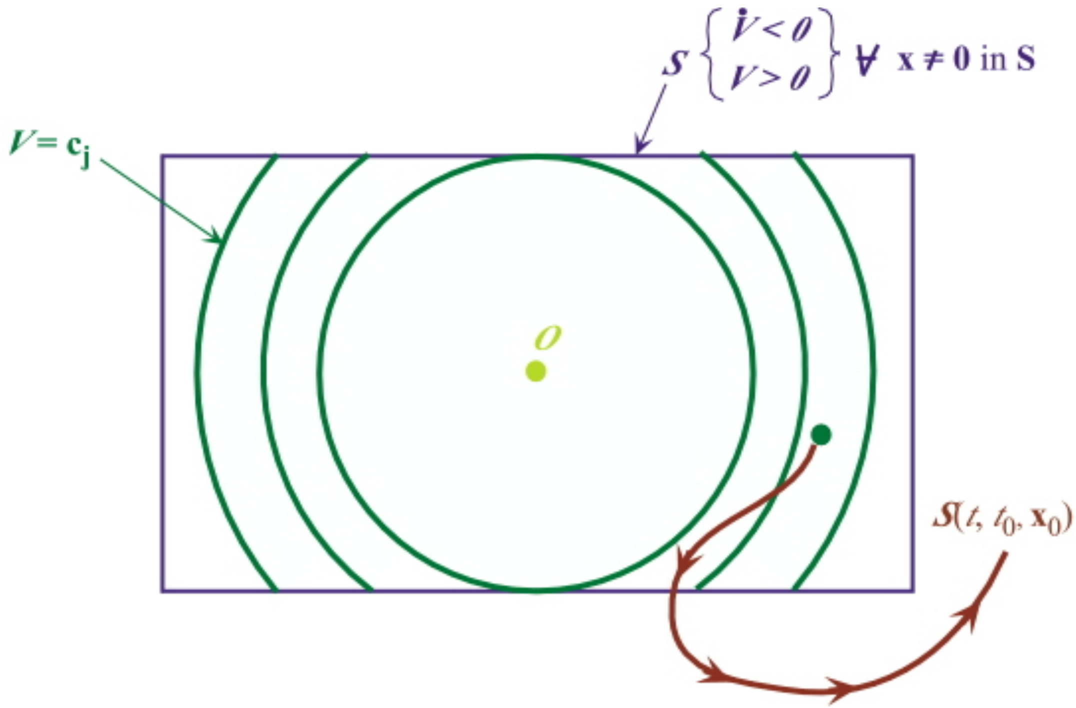
Further Discussion

Answer: No!

(*) alone does no guarantee that every solution starts in $\boldsymbol{S}$ stays in S.

What is true though:

If (*) holds, then every invariant set of $\sum_A$ contained in $\boldsymbol{S}$ is also contained in $\boldsymbol{D}(0)$, but S itself need not be contained in $\boldsymbol{D}(0)$!



Example

$$\Sigma_A : \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1 + \frac{1}{3}x_1^2 - x_2 \end{cases}$$

- Let

$$V(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1 \end{bmatrix} \mathbf{x} + \int_0^{x_1} (y - \frac{1}{3}y^3)\,dy$$

$$= \frac{3}{4}x_1^2 - \frac{1}{12}x_1^4 + \frac{1}{2}x_1 x_2 + \frac{1}{2}x_2^2$$

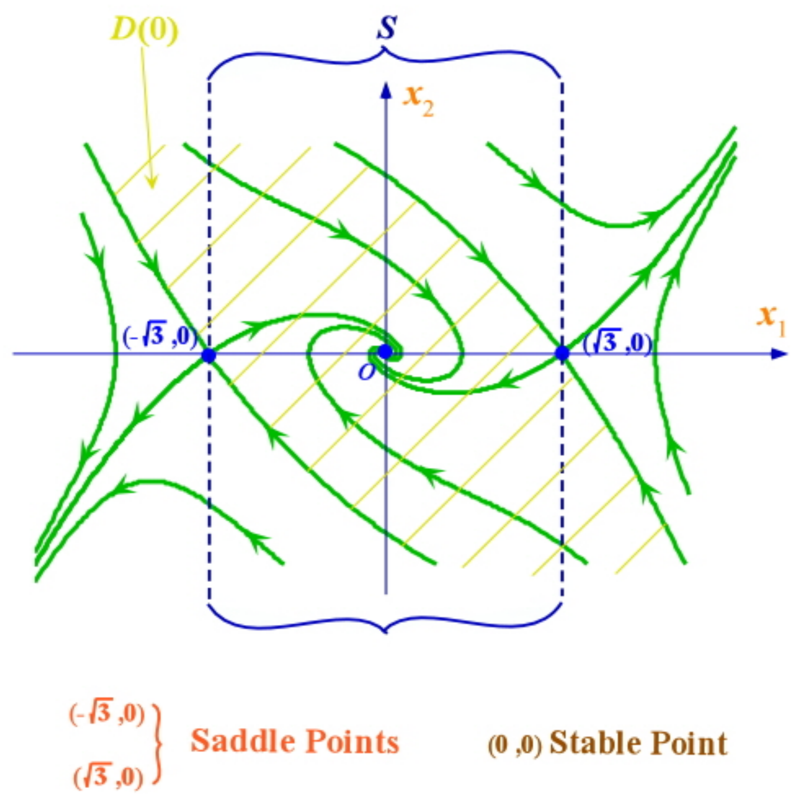
$$\Rightarrow \dot{V}(\mathbf{x}) = -\frac{1}{2}x_1^2(1 - \frac{1}{3}x_1^2) - \frac{1}{2}x_2^2$$

- Define a domain S by

$$S = \{\mathbf{x} \in \mathbf{R}^2 : -\sqrt{3} < x_1 < \sqrt{3}\}$$

then $V(\mathbf{x}) > 0$ and $\dot{V}(x) < 0$ in $S - \{0\}$.

S is not a subset of $D(0)$.

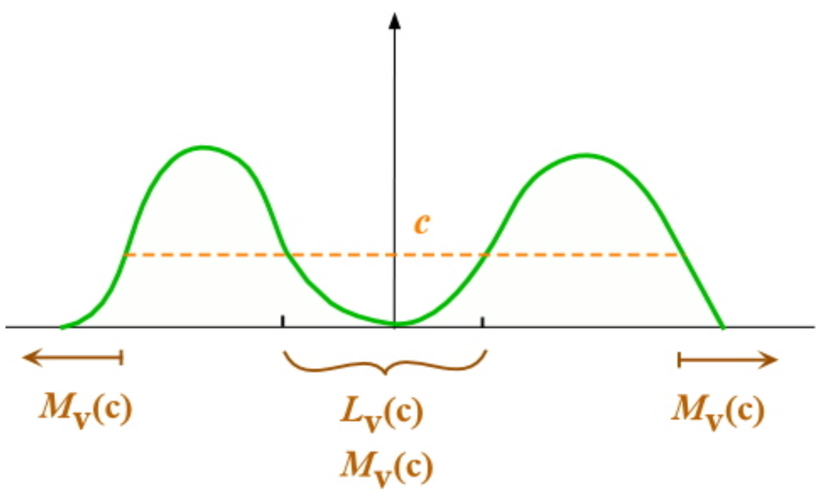


Discussion

Question: How does one go about finding an estimate of the domain of attraction?

Answer: One way is to use so-called level sets of the Lyapunov function V. Let $c \in \mathbf{R}^{+}$ and consider the set

$$M_V(c)=\{\mathbf{x}\in \mathbf{R}^n: V(\mathbf{x}) \leq c\}$$



Depending on nature of V, $M_V(c)$ need not be connected.

Definition: Level Set

The level set $L_V(c)$ is the connected component of $M_V(c)$ containing O.

Lemma

Lemma:

Consider the autonomous system $\sum_A$. Suppose $\exists$ a C^1 function V: $\mathbf{R}^n \rightarrow \mathbf{R}$ and a domain S containing 0 s.t.

(i) $V(0) = 0$;

(ii) $V(x) > 0, \quad \dot{V}(x) < 0 \quad \forall x \neq 0 \text{ in } S$

Let c be any positive constant s.t. the level set $L_V(c)$ is contained in S and is bounded.

Then $L_V(c)$ is a subset of $D(0)$.

Note:

The largest $L_V(c)$ is obtained by using the largest c.

Example

$$\Sigma_A : \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -4(x_1 + x_2) - h(x_1 + x_2) \end{cases}$$

where $h : \mathbf{R} \to \mathbf{R}$, $h(0) = 0$, $uh(u) \geq 0 \quad \forall |u| \leq 1$

- Consider the Lyapunov function candidate

$$V(\mathbf{x}) = \mathbf{x}^{\mathrm{T}} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \mathbf{x} = 2x_1^2 + 2x_1x_2 + x_2^2$$

$$\begin{aligned} \Rightarrow \dot{V}(\mathbf{x}) &= (4x_1 + 2x_2)\dot{x}_1 + 2(x_1 + x_2)\dot{x}_2 \\ &= -2x_1^2 - 6(x_1 + x_2)^2 - 2(x_1 + x_2)h(x_1 + x_2) \\ &\leq -2x_1^2 - 6(x_1 + x_2)^2 \qquad \forall |x_1 + x_2| \leq 1 \\ &= -\mathbf{x}^{\mathrm{T}} \begin{bmatrix} 8 & 6 \\ 6 & 6 \end{bmatrix} \mathbf{x} \end{aligned}$$

$\therefore \dot{V}(\mathbf{x})$ is negative definite in the set

$$S = \{\mathbf{x} \in \mathbf{R}^2 : |x_1 + x_2| \leq 1\} \qquad \leftarrow \text{here } V(0) = 0; V(\mathbf{x}) > 0; \dot{V} < 0, \forall \mathbf{x} \neq 0$$

$\Rightarrow O$ is asymptotically stable.

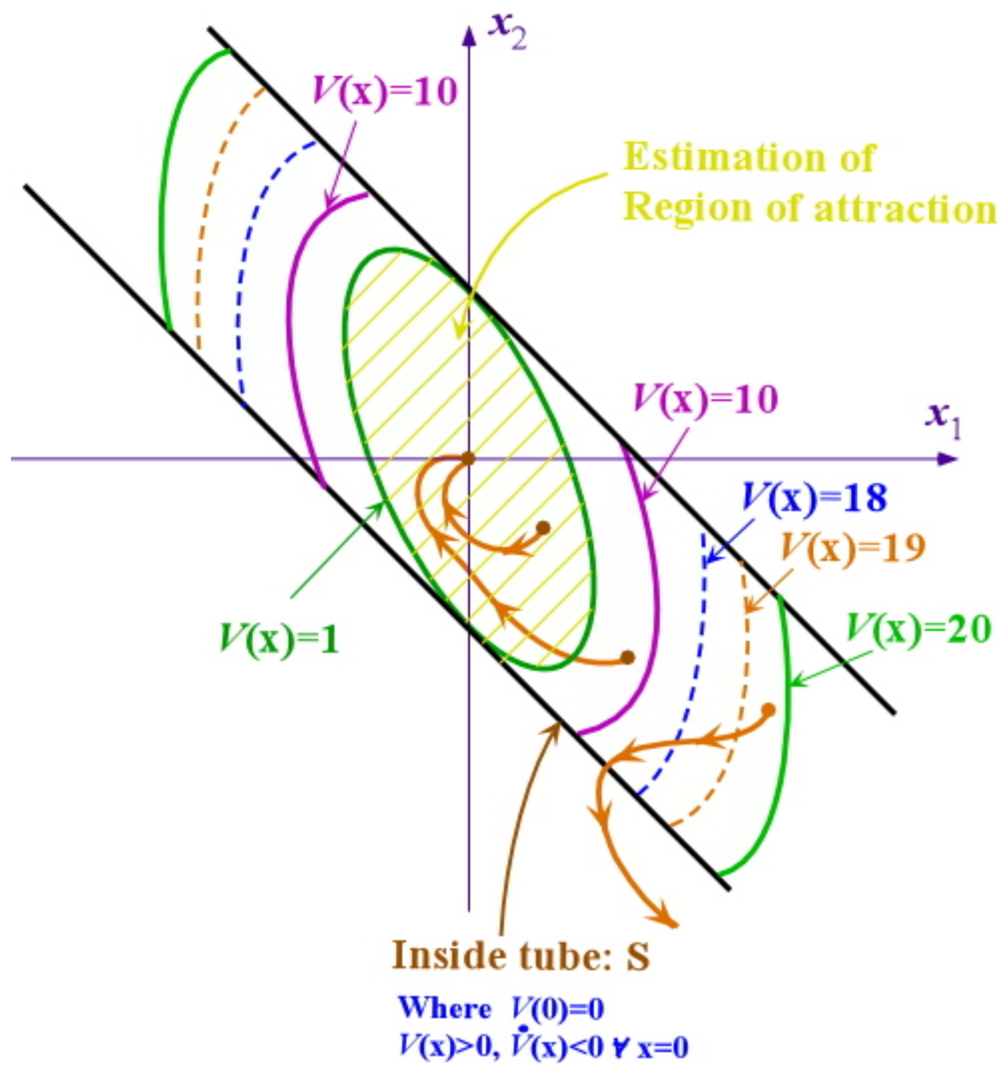
Example-continued

- We now find largest c s.t. the level set $L_v(c)$ is contained in S

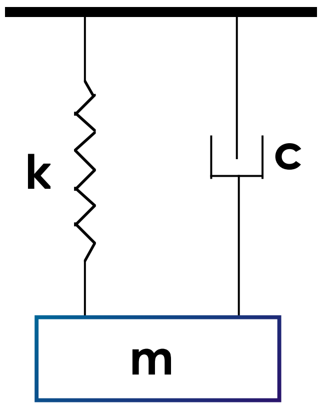
$$c = \min_{|x_1 + x_2| = 1} V(x) = 1$$

Hence largest is

$L_v(1) = \{\mathbf{x} \in \mathbf{R}^2 : V(\mathbf{x}) \leq 1\}$

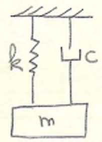


Motivation to Invariance Principle



Motivation to Invariance Principle

Reconsider (see pg 5.39)



$$m\ddot{x} + c\dot{x} + kx = 0$$

Let $\left.\begin{matrix} x_1 = x \\ x_2 = \dot{x} \end{matrix}\right\} \Rightarrow \left\{\begin{matrix} \dot{x}_1 = x_2 & (a) \\ \dot{x}_2 = -\frac{b}{m}x_2 - \frac{k}{m}x_1 & (b) \end{matrix}\right.$ | equilibrium $x_1 = 0$, $x_2 = 0$

$$V = \tfrac{1}{2} m x_2^2 + \tfrac{1}{2} k x_1^2$$

$$\dot{V} = m x_2 \dot{x}_2 + k x_1 \dot{x}_1$$

$$= m x_2\left[-\tfrac{b}{m}x_2 - \tfrac{k}{m}x_1\right] + k x_1[x_2]$$

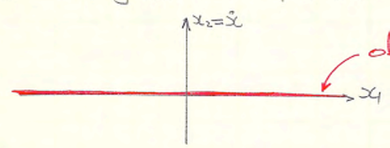
$$= -b x_2^2$$ why ≤ 0 because $= 0$ for $x_2 = 0$ & $x_1 \neq 0$

$\leq 0 \implies (0,0)$ is stable

[Though in reality (0,0) is asymptotically stable]

Observation

$\dot{V}$ is < 0 everywhere except on the line $x_2 = 0$ where $\dot{V} = 0$



along this line $\dot{V} = 0$

But : $\dot{V} = 0$ when $x_2 \equiv 0 \overset{(a)}{\Longrightarrow} \dot{x}_1 \equiv 0 \Rightarrow \underline{x_1(t) = \text{constant}}$

Also $x_2 \equiv 0 \Longrightarrow \dot{x}_2 = 0 \overset{(b)}{\Longrightarrow} \underline{x_1 = 0}$

Hence $\dot{V} = 0$ is possible only when $\boxed{x_2 = 0 \ \& \ x_1 = 0}$

Hence indeed $\left.\begin{matrix} x_1(t) \to 0 \\ x_2(t) \to 0 \end{matrix}\right\}$ as $t \to \infty$ as expected

Conclusion : Concluded uniform asymptotic stability using only $\dot{V} \leq 0$ & NOT $-\dot{V} = \text{lpdf}$

LaSalle's Theorem for Autonomous Systems

Definition:

The distance $d(\mathbf{x}, \Omega)$ between a point $\mathbf{x}$ and a nonempty closed set Ω is defined as

$$d(\mathbf{x}, \Omega) = \min_{\mathbf{y} \in \Omega} \| \mathbf{x} - \mathbf{y} \|$$

Lemma:

Suppose the system $\sum_A$ is autonomous.

Suppose $\exists$ a C^1 lpdf function V: $\mathbf{R}^n \rightarrow \mathbf{R}$.

Suppose also $\exists$ an open neighborhood N of O s.t.

$$\dot{V} \leq 0, \quad \forall \mathbf{x} \in N$$

Choose a constant $c>0$ s.t. the level $L_v(c)$ set is bounded and contained in N.

Finally, let

$$S = \{\mathbf{x} \in L_v(c) : \quad \dot{V} = 0\}$$

and let M denote the largest invariant set of $\sum_A$ contained in S.

Then,

$$\mathbf{x} \in L_v(c) \Rightarrow d[s(t,0,\mathbf{x}_0), M] \rightarrow 0 \ \text{ as } \ t \rightarrow \infty$$

(i.e. the solution $s(t,0,\mathbf{x}_0)$ of $\sum_A$ approaches M as $t \rightarrow \infty$)

LaSalle's Theorem for Autonomous Systems

Theorem 8: [LaSalle's Theorem]

Suppose the system $\sum_A$ is autonomous.

Suppose ∃ a C^1 lpdf function V: $\mathbf{R}^n \rightarrow \mathbf{R}$.

Suppose also ∃ an open neighborhood N of O s.t.

$$\dot{V} \leq 0, \quad \forall \mathbf{x} \in N$$

Choose a constant $c>0$ s.t. the level $L_v(c)$ set is bounded and contained in N.

Finally, let

$$S = \{\mathbf{x} \in L_v(c) : \quad \dot{V} = 0\}$$

Under these conditions, if S contains no trajectories of the system other than the trivial trajectory $\mathbf{x}(t)=0$, then the equilibrium 0 of $\sum_A$ is

Uniformly asymptotically stable.

LaSalle's Theorem for Autonomous Systems

Theorem 9: [LaSalle's Theorem, Global Version]

Suppose the system $\sum_A$ is autonomous.

Suppose ∃ a C^1 lpdf function V: $\mathbf{R}^n \to \mathbf{R}$ s.t.

(i) V is pdf and is radially unbounded,

(ii) $\dot{V} \leq 0, \quad \forall \mathbf{x} \in \mathbf{R}^n$

Define

$$R = \{\mathbf{x} \in \mathbf{R}^n : \quad \dot{V} = 0\}$$

and suppose R does not contain any trajectories of the system other than the trivial trajectory $\mathbf{x}(t)=0$.

Then the equilibrium 0 of $\sum_A$ is

globally uniformly asymptotically stable.

Example (I)

Consider a unit mass constrained by a nonlinear spring and nonlinear friction

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -g(x_1) - f(x_2) \end{cases}$$

$g(\bullet)$ is the restoring force of the spring;

$f(\bullet)$ is the force due to friction.

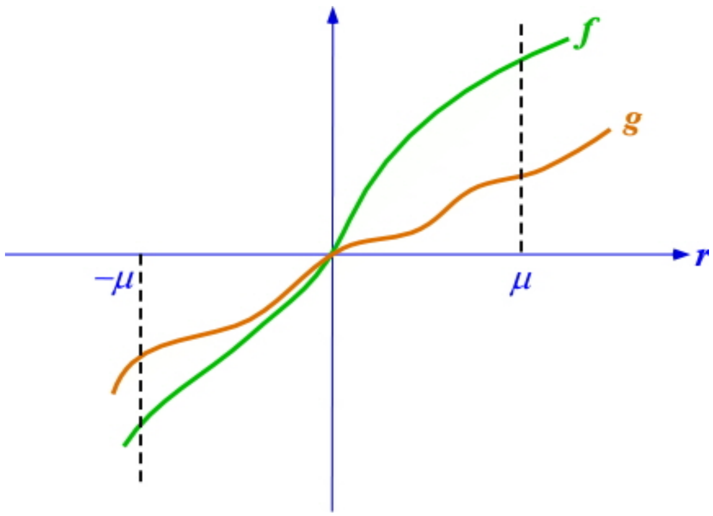
Suppose $f(\bullet)$, $g(\bullet)$ are continuous s.t. $f(0) = g(0) = 0$

$\Rightarrow$ O is an equilibrium.

Suppose in addition, ∃ a constant $\mu > 0$ s.t.

$rf(r) > 0$, $\forall \neq 0$, $r \in [-\mu, \mu]$;

$rf(r) > 0$, $\forall \neq 0$, $r \in [-\mu, \mu]$;



Example (II)

Define the function $\Phi(r) = \int_0^r g(\sigma)d\sigma.$

The level set $L_\Phi(c) = \{r \in \mathbf{R} : \Phi(r) \le c\}$

Suppose there is a constant $c>0$ s.t.

the level set $L_\Phi(c)$ is bounded

Under these conditions, it is claimed that O is an asymptotically stable equilibrium.

This means:

If the restoring spring force and the friction force are first and third quadrant function in some neighborhood of the origin, then the origin is asymptotically stable.

Example (III)

Proof of claim:

$$V(x_1,x_2)=\Phi(x_1)+\frac{1}{2}x_2^2=\underbrace{\int_0^{x_1} g(\sigma)d\sigma}_{P.E.}+\underbrace{\frac{1}{2}x_2^2}_{K.E.}$$

$$\Rightarrow \dot{V}(x_1,x_2)=g(x_1)\dot{x}_1+x_2\dot{x}_2$$

$$=x_2 g(x_1)+x_2[-g(x_1)-f(x_2)]$$

$$=-x_2 f(x_2)$$

Example (IV)

<u>Properties of V & $\dot{V}$:</u>

(i) V is lpdf because :

suppose $\mathbf{x} \neq 0$ and let $|x_1|,|x_2| \leq \mu.$

$\Rightarrow V(\mathbf{x}) > 0$ by virtue of conditions on f & g.

(ii) $\dot{V} \leq 0$ whenever $|x_1|,|x_2| \leq \mu.$

(iii) The level set $L_v(c)$ is bounded. To see this :

when $\Phi(x_1) \leq \dfrac{1}{2}c$ & $|x_2| < \sqrt{c} \equiv d$

$$\Rightarrow V(\mathbf{x}) \leq \frac{1}{2}c + \frac{1}{2}c = c.$$

Hence

$L_v(c)$ is contained in the bounded set $L_\Phi(\dfrac{1}{2}c) \times [-d, d]$

Example (V)

To apply LaSalle's Thm (Thm 8),

necessary to determine set S

$$S = \{(x_1, x_2) : \dot{V}(x_1, x_2) = 0\}$$

Suppose $\mathbf{x} \in L_v(c)$ and $\dot{V} = 0$, then

$x_2 f(x_2) = 0 \Rightarrow x_2 = 0$. Hence

$$S = \{(x_1, x_2) : x_2 = 0\}.$$

To apply Thm, it only remains to verify that S contains

no nontrivial system trajectories.

Suppose $\mathbf{x}(t), t \geq 0$ is a trajectory that lies entirely in S. Then

$$x_2(t) \equiv 0, \quad \forall t \geq 0 \Rightarrow f[x_2(t)] \equiv 0, \quad \forall t \geq 0$$

Since $\dot{x}_2 = -g(x_1) - f(x_2)$, it follows that

$$\dot{x}_2(t) \equiv 0 \Rightarrow g[x_1(t)] \equiv 0 \Rightarrow x_1(t) \equiv 0$$

That is, $\mathbf{x}(t)$ is the trivial solution.

Example (VI)

Hence by LaSalle's Thm, O is
uniformly asymptotically stable.

If condition on f and g are strengthened to

$$rf(r) > 0, \quad rg(r) > 0, \quad \forall r \neq 0$$

$$\Phi(r) \to \infty \text{ as } |r| \to \infty,$$

then we conclude that the origin is
globally uniformly asymptotically stable.