Lemma & Theorem

Lemma: The equilibrium 0 of $\sum$ is

– Attractive iff

for each $t_0 \geq 0$, ∃ a number $r(t_0)>0$, and for each $\mathbf{x}_0 \in B_{r(t0)}$ a function $\sigma_{t0,x0}$ of class L s.t. $\|s(t_0+t,t_0,\mathbf{x}_0)\| \leq \sigma_{t0,x0}(t)$, $\forall\ t \geq 0$, $\forall\ \mathbf{x}_0 \in B_{r(t0)}$ (*).

Uniformly attractive iff

∃ a number $r>0$, and a function σ of class L s.t.

$\|s(t_0+t,t_0,\mathbf{x}_0)\| \leq \sigma(t)$, $\forall\ t,\ t_0 \geq 0$, $\forall\ \mathbf{x}_0 \in B_r$.

Note: If (*) holds, then 0 is attractive since $\sigma_{t0,x0} \to 0$ as $t \to \infty$.

Theorem: The equilibrium 0 of $\sum$ is

Uniformly asymptotically stable iff

∃ a number $r>0$, and a function Φ of class K and a function σ of class L s.t. $\|s(t_0+t,t_0,\mathbf{x}_0)\| \leq \Phi(\|\mathbf{x}_0\|)\sigma(t)$, $\forall\ t,\ t_0 \geq 0$, $\forall\ \mathbf{x}_0 \in B_r$.

Exponential stability: $\|s(t_0+t,t_0,\mathbf{x}_0)\| \leq a\|\mathbf{x}_0\|e^{-bt}$, $\forall\ t,\ t_0 \geq 0$, $\forall\ \mathbf{x}_0 \in B_r$.

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Some Preliminaries (I)

Definition:

A function V: $\mathbf{R}^+ \times \mathbf{R}^n \rightarrow \mathbf{R}$

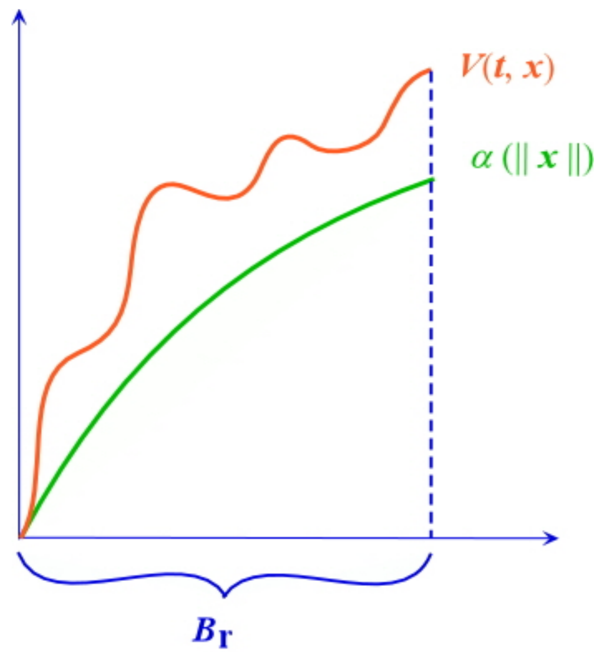
- Is said to be locally positive definite function (l.p.d.f.) if:

  (i) it is continuous;

  (ii) $V(t,0)=0 \;\forall t \geq 0$

  (iii) $\exists$ a constant $r>0$, and a function α of class K s.t.

  $\alpha(\|\mathbf{x}\|) \leq V(t,\mathbf{x}),\ \forall\ t \geq 0,\ \forall\ \mathbf{x} \in B_r$ (* *).



- V is a positive definite function (p.d.f.) if (* *) holds for all $\mathbf{x} \in \mathbf{R}^n$ (i.e. if $r=\infty$).

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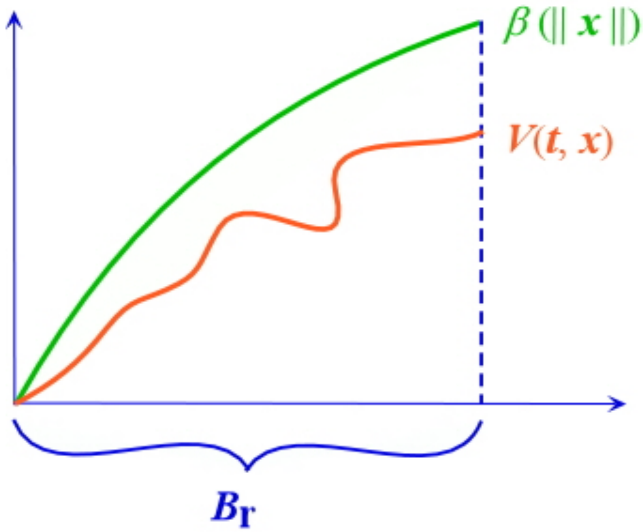
Some Preliminaries (II)

- V is <u>decrescent</u> if ∃ a constant $r>0$, and a function β of class K s.t.

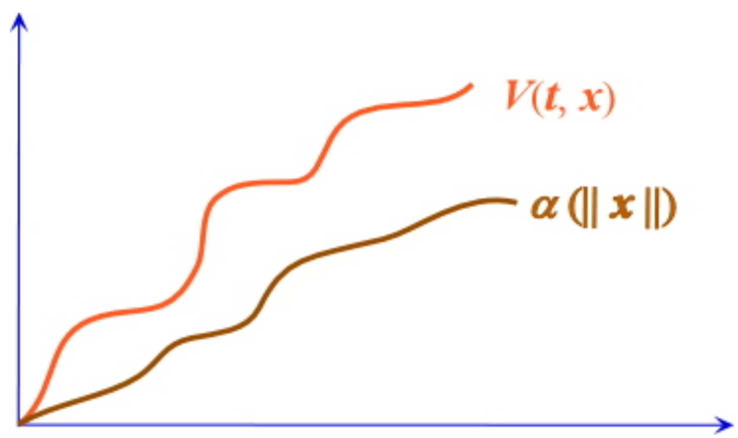
  $$V(t,\mathbf{x}) \leq \beta(\|\mathbf{x}\|),\ \forall\ t\geq 0,\ \forall\ \mathbf{x}\in B_r.$$

This means

$$\sup_{|x|\leq p}\ \sup_{t\geq 0} V(t,x) < \infty \quad \forall\ p \in (0,r)$$



- V is <u>radially unbounded</u> if (**) is satisfied for all $\mathbf{x}\in\mathbf{R}^n$ (not necessarily of class K) with the additional property that $\alpha(r) \to \infty$ as $r \to \infty$.



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Some Preliminaries (III)

- V is a <u>locally negative definite function</u> if $-V$ is an l.p.d.f.
- V is a <u>negative definite function</u> if $-V$ is a p.d.f.

<u>Remark</u>:

Using these definitions, it is rather difficult to determine whether or not a given continuous function V: $\mathbf{R}^{+}\times\mathbf{R}^{n}\rightarrow\mathbf{R}$ is p.d.f. or l.p.d.f..

The source of difficulty comes from the need to exhibit the function $\alpha(\bullet)$.

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Conditions More Verifiable (I)

Lemma:

- A continuous function W: $\mathbf{R}^n \rightarrow \mathbf{R}$ is an l.p.d.f. iff it satisfies the following 2 conditions:
  (i) $W(0) = 0$;
  (ii) $\exists$ a constant $r > 0$ s.t. $W(\mathbf{x}) > 0$, $\forall\ \mathbf{x} \in B_r - \{0\}$.

- W is a p.d.f. iff it satisfies the following 3 conditions:
  (iii) $W(0) = 0$;
  (iv) $W(\mathbf{x}) > 0$, $\forall\ \mathbf{x} \in \mathbf{R}^n - \{0\}$;
  (v) $\exists$ a constant $r > 0$ s.t. $\inf\limits_{\|\mathbf{x}\| \geq r} = W(x) > 0$

- W is radially unbounded iff
  (vi) $W(\mathbf{x}) \rightarrow \infty$ as $||\mathbf{x}|| \rightarrow \infty$ uniformly in $\mathbf{x}$.

Note: Condition (v) is important!

Why? Consider: $W(\mathbf{x}) = \dfrac{x^2}{1 + x^4}$ l.p.d.f but not p.d.f.

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Conditions More Verifiable (II)

Lemma:

- A continuous function V: $\mathbf{R}^{+}\times\mathbf{R}^{n}\rightarrow\mathbf{R}$ is an l.p.d.f. iff:
  (i) $V(t,0)=0$, $\forall\ t$, and
  (ii) $\exists$ an l.p.d.f. W: $\mathbf{R}^{n}\rightarrow\mathbf{R}$ and a constant $r>0$ s.t.
  $V(t,\mathbf{x})\geq W(\mathbf{x})$, $\forall\ t\geq 0$, $\forall\ \mathbf{x}\in B_r$.
- V is a p.d.f. iff:
  (i) $V(t,0)=0$, $\forall\ t$, and
  (ii) $\exists$ a p.d.f. W: $\mathbf{R}^{n}\rightarrow\mathbf{R}$ s.t.
  $V(t,\mathbf{x})\geq W(\mathbf{x})$, $\forall\ t\geq 0$, $\forall\ \mathbf{x}\in\mathbf{R}^{n}$ $(*)$.
- V is radially unbounded iff
  $\exists$ an radially unbounded function W: $\mathbf{R}^{n}\rightarrow\mathbf{R}$ s.t $(*)$ is satisfied.

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Conditions More Verifiable (III)

<u>**Remarks**</u>:

- Basically, lemma shows that

  - A continuous function of t and $\mathbf{x}$ is an <u>l.p.d.f.</u> iff it dominates at each instant of time and over some ball in $\mathbf{R}^n$, an <u>l.p.d.f.</u> of $\mathbf{x}$ alone.

  Similarly,

  - A continuous function of t and $\mathbf{x}$ is a <u>p.d.f.</u> iff it dominates for all t and $\mathbf{x}$, a <u>p.d.f.</u> of $\mathbf{x}$ alone.

- The conditions given in lemma are easier to verify.

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Examples (I)

→ $W_1(x_1, x_2) = x_1^2+x_2^2$

- W_1 is p.d.f. because
  - $W_1=0$;
  - $W_1(x_1, x_2)>0, \ \forall\ \mathbf{x}\neq 0$;
  - For $r > 0$, $\inf_{\|\mathbf{x}\|\geq r} = W(\mathbf{x}) > 0$
- W_1 is radially unbounded because

  $W_1(x_1, x_2)= \|\mathbf{x}\|_2^2 \to \infty$ as $\|\mathbf{x}\|_2 \to \infty$ uniformly in $\mathbf{x}$.

→ $V_1(t, x_1, x_2) = (1+t)(x_1^2+x_2^2) \geq W_1$

Since W_1 is p.d.f. $\Rightarrow V_1$ is p.d.f.;

Since W_1 is r.u. $\Rightarrow V_1$ is radially unbounded.

V_1 decrescent? No.

Because for each $\mathbf{x}\neq 0$, the function $V_1(t, \mathbf{x})$ is unbounded as a function of t.

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Examples (II)

$\rightarrow V_2(t, x_1, x_2) = e^{-t}(x_1^2+x_2^2)$

- is NOT a p.d.f. : there doesn't exist a p.d.f. W: $\mathbf{R}^n \rightarrow \mathbf{R}$ which is dominated by V_2 as for each $\mathbf{x}$, $V_2(t, \mathbf{x}) \rightarrow 0$ as $t \rightarrow \infty$.
- is decrescent as $V_2(t, x_1, x_2) \leq \|\mathbf{x}\|_2$

$\rightarrow W_2(x_1, x_2) = x_1^2+\sin^2 x_2$

- is an l.p.d.f
  - $W_2(0,0)=0$
  - $W_2(x_1, x_2)>0$, whenever $\mathbf{x}\neq 0$ and $|x_2| < \pi$;
- is NOT a p.d.f as it vanishes at points other than 0, for example, at $(0, \pi)$.

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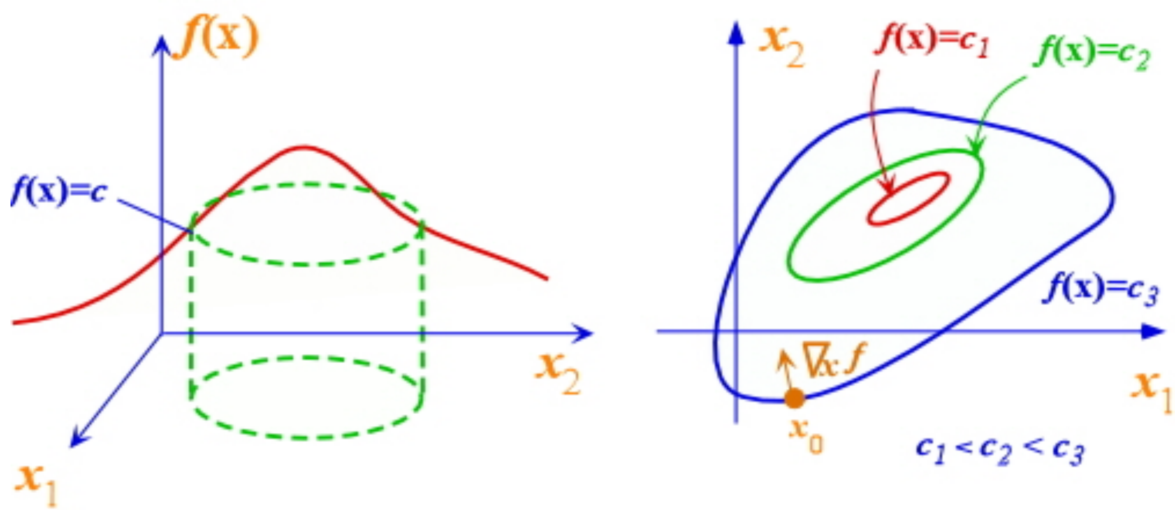
The Gradient Operator & Differentiation w.r.t. a vector (I)

- Let $f(x_1, x_2, \ldots, x_n)$ be a scalar-valued function of n variables x_i.

The n partial derivatives of $f(x)$, $\dfrac{\partial f}{\partial x_i}$ are defined by $\nabla_x f$ or simply ∇f :

$$\nabla f = \begin{bmatrix} \dfrac{\partial f}{\partial x_1} & \dfrac{\partial f}{\partial x_2} & \cdots & \dfrac{\partial f}{\partial x_n} \end{bmatrix}$$

- 2-dim. example:



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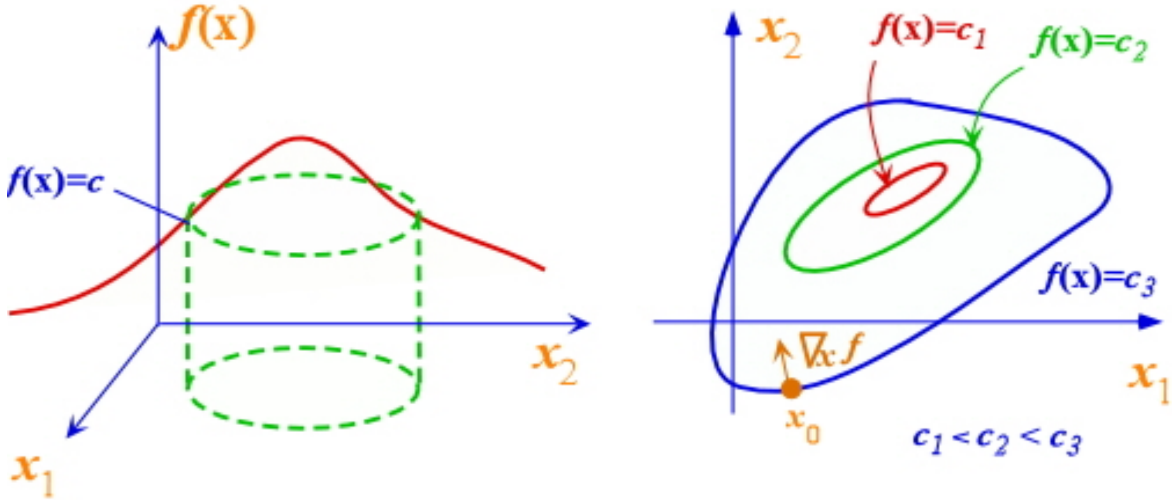
The Gradient Operator & Differentiation w.r.t. a vector (II)



→ → Equation $f(\mathbf{x})=c$, with c constant, specifies a locus of points in the plane.

→ → At a given point such as $\mathbf{x}_0$, ∇f is a vector normal to the curve $f(\mathbf{x})=c$, and it points in the direction of increasing values of $f(\mathbf{x})$.

→ → The gradient defines the direction of maximum increase of the function $f(\mathbf{x})$.

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Motivation to Lyapunov Stability (I)

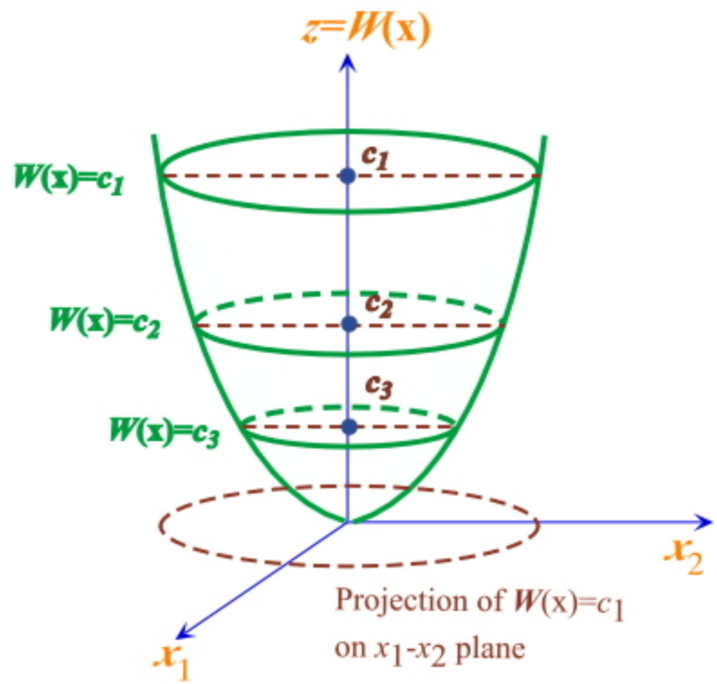
- $\dot{\mathbf{x}} = f(\mathbf{x}),\ \ \mathbf{x}(t_0) = \mathbf{x}_0 \quad \Sigma \text{ in } \mathbf{R}^2$

  $s(t, t_0, x_0) = \text{ solution trajectory}$

- $W : \mathbf{R}^2 \to \mathbf{R}; \quad W = x_1^2 + x_2^2$

- A set of constant $c_i \geq 0,\ i = 1, 2, \ldots$ will define a set of closed curves (or level curves).

  $J_i = \{\mathbf{x} \in \mathbf{R}^2 : W(\mathbf{x}) = c_i\}$



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Motivation to Lyapunov Stability (II)

- A continium of constants will define a continium of level curves s.t.

  $c_i < c_j \Rightarrow J_i$ is in the interior of J_j.

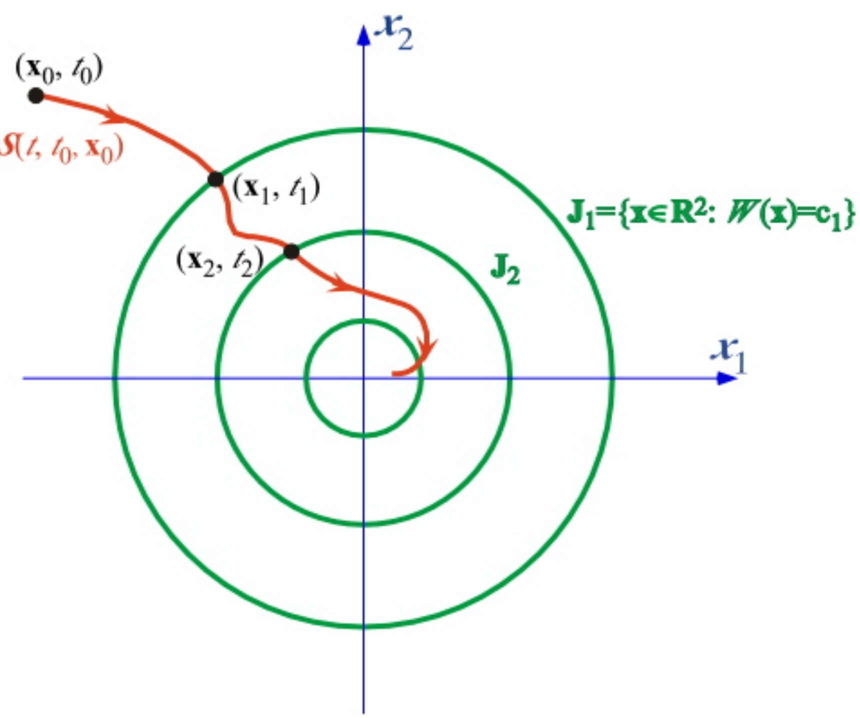
- Let's evaluate $W(\mathbf{x})$ along a trajectory $s(t, t_0, \mathbf{x}_0)$:

  $W(\mathbf{x}) = W(s(t, t_0, \mathbf{x}_0)) \equiv c(t).$

  where $c(t)$ defines at a fixed t a closed curve J_t whose interior is $\sqrt{c(t)}$ neighborhood of zero.

- When $c(t)$ as a function of time t decreases,
  i.e. $\dfrac{\mathrm{d}\,c(t)}{\mathrm{d}t} < 0, \Rightarrow J_t$ shrinks.
  But since $s(t, t_0, \mathbf{x}_0) \in J_t, \Rightarrow s(t, t_0, \mathbf{x}_0)$ moves toward 0.

- If $\dfrac{\mathrm{d}\,c(t)}{\mathrm{d}t} < 0$ for any trajectory that originates in neighborhood B_r, then $s(t, t_0, \mathbf{x}_0) \rightarrow 0$ as $t \rightarrow \infty$



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Motivation to Lyapunov Stability (III)

- Let's evaluate $W(\mathbf{x})$ along a trajectory $s(t, t_0, \mathbf{x}_0)$:
  $W(\mathbf{x}) = W(s(t, t_0, \mathbf{x}_0)) \equiv c(t)$.
  where $c(t)$ defines at a fixed t a closed curve J_t whose interior is $\sqrt{c(t)}$ neighborhood of zero.

- When $c(t)$ as a function of time t decreases,
  i.e. $\frac{\mathrm{d}\, c(t)}{\mathrm{d}t} < 0$, $\Rightarrow$ J_t shrinks.
  But since $s(t, t_0, \mathbf{x}_0) \in J_t$, $\Rightarrow$ $s(t, t_0, \mathbf{x}_0)$ moves toward 0.

- If $\frac{\mathrm{d}\, c(t)}{\mathrm{d}t} < 0$ for any trajectory that originates in neighborhood B_r ,
  then $s(t, t_0, \mathbf{x}_0) \to 0$ as $t \to \infty$

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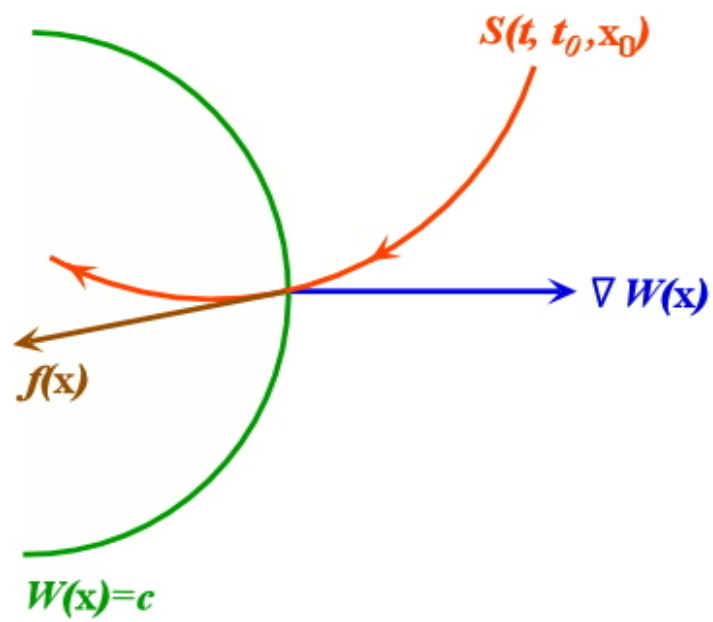
Motivation to Lyapunov Stability (IV)

- Evaluate

$$\frac{\mathrm{d}c(t)}{\mathrm{d}t} = \frac{\mathrm{d}W(s(t,t_0,\mathbf{x}_0))}{\mathrm{d}t} = \frac{\mathrm{d}W(\mathbf{x})}{\mathrm{d}t}$$
$$= \nabla_x W(\mathbf{x}) \bullet \dot{\mathbf{x}}$$
$$= \nabla W(\mathbf{x}) \bullet f(\mathbf{x})$$



Hence,
if $\nabla W \bullet f < 0$,
$\Rightarrow s(t,\ t_0,\ \mathbf{x}_0)$ crosses level curves $W(\mathbf{x})=c$ from outside
$\rightarrow 0$ as $t \rightarrow \infty$.

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Motivation to Lyapunov Stability - An Example

Let $\Sigma: \quad \begin{cases} \dot{x}_1 = -x_1 \\ \dot{x}_2 = -x_2 \end{cases}$

[verify 0 is a stable node]

$$W(\mathbf{x}) = \frac{1}{2}(x_1^{\ 2} + x_2^{\ 2}).$$

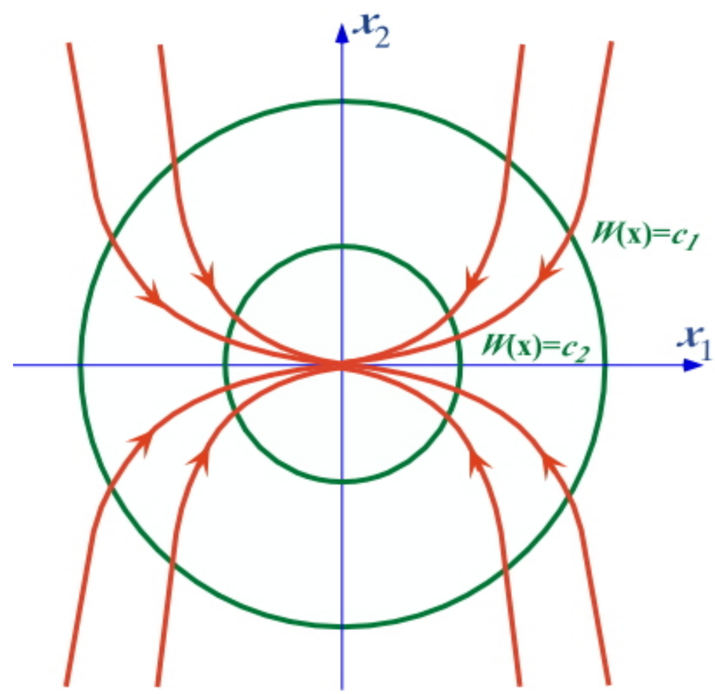
$$\frac{\mathrm{d}W}{\mathrm{d}t} = \nabla W \bullet f$$

$$= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} -x_1 \\ -x_2 \end{bmatrix}$$

$$= -(x_1^{\ 2} + x_2^{\ 2}) < 0$$

for any $x_1 \neq 0, x_2 \neq 0.$

$\Rightarrow$ Any trajectory of $\Sigma \to 0$ as $t \to 0.$



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Derivative of a Function along the Trajectory of an o.d.e.

Suppose V: $\mathbf{R}^{+}\times\mathbf{R}^{n}\rightarrow\mathbf{R}$ has continuous partial derivatives, and suppose $\mathbf{x}(\bullet)$ satisfies the d.e. $\sum$.

$$\Rightarrow \quad \frac{\mathrm{d}}{\mathrm{d}t}V[t,\mathbf{x}(t)] = \frac{\partial V}{\partial t}[t,\mathbf{x}(t)] + \nabla V[t,\mathbf{x}(t)] \bullet \underbrace{f[t,\mathbf{x}(t)]}_{\dot{x}}$$

Definition:

Let V: $\mathbf{R}^{+}\times\mathbf{R}^{n}\rightarrow\mathbf{R}$ be continuously differentiable w.r.t. all of its arguments, and let ∇V denote the gradient of V w.r.t. $\mathbf{x}$ (written as a row vector).

Then the function $\dot{V}:\mathbf{R}^{+}\times\mathbf{R}^{n}\rightarrow\mathbf{R}$ is defined by

$$\dot{V}(t,\mathbf{x}) = \frac{\partial V}{\partial t}(t,\mathbf{x}) + \nabla V(t,\mathbf{x}) \bullet f(t,\mathbf{x})$$

and is called the derivative of V along the solution trajectories of $\sum$.

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Remarks

- $\dot{V}$ depends not only on the function V, but also on the system $\sum$.
- If we keep the same V but change the system $\sum$, the resulting $\dot{V}$ will in general be different.

- The quantity $\dot{V}(t,\mathbf{x})$ can be interpreted as follows:

  Suppose a solution trajectory of $\sum$ passes through $\mathbf{x}_0$ at time t_0. Then,at the instant t_0, the rate of change of the quantity $V[t,\mathbf{x}(t)]$ is $\dot{V}(t_0,\mathbf{x}_0)$.

- If V=$V[\mathbf{x}]$ (V independent of t) and the system $\sum$ is autonomous, then $\dot{V} = \dot{V}(\mathbf{x}) \quad [\dot{V} \text{ independent of } t]$.

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Lyapunov's Direct Method

Notation:

$\mathbf{C}^1$: continuously differentiable;

lpdf: locally positive definite function;

pdf: positive definite function.

Theorems on Stability:

Theorem 1: [Stability]

The equilibrium 0 of $\sum$ is stable if :

$\exists$ a C^1 lpdf V: $\mathbf{R}^+ \times \mathbf{R}^n \rightarrow \mathbf{R}$ and a constant $r > 0$ s.t.:

$$\dot{V}(t, \mathbf{x}) \leq 0, \quad \forall t \geq t_0, \forall \mathbf{x} \in B_r$$

where $\dot{V}$ is evaluated along the solution trajectories of Σ.

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Proof of Theorem 1 (I)

Since V is an lpdf, ∃ a function α of class K and a constant $s > 0$ s.t.

$\alpha(\|\mathbf{x}\|) \leq V(t,\mathbf{x}),\ \forall\ t \geq 0,\ \forall\ \mathbf{x}_0 \in B_s$

→ want to show that 0 is a stable equilibrium point, i.e.

$\forall \varepsilon > 0, \exists\ \delta(t_0, \varepsilon) > 0$ s.t. $\|\mathbf{x}_0\| < \delta(t_0, \varepsilon) \Rightarrow \|s(t, t_0, \mathbf{x}_0)\| \leq \varepsilon,\ \forall\ t \geq t_0$ (*)

1) Let $\varepsilon > 0$ and t_0 be given.

Define $\varepsilon_1 = \min\{\ \varepsilon,\ \ r,\ \ s\ \}$

given

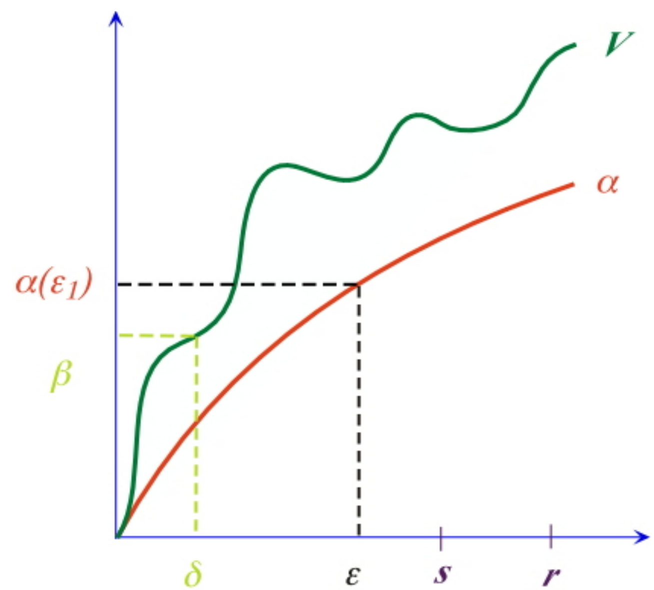
Size of ball where $\dot{V} \leq 0$

Size of ball where α bounds from below V

Choose s.t.

$$\sup_{\|\mathbf{x}\| < \delta} V(t_0, \mathbf{x}) \equiv \beta(t_0, \delta) < \alpha(\varepsilon_1) \quad (2*)$$

such a δ can always be found because $\alpha(\varepsilon_1) > 0$ & $\beta(t_0, \delta) \to 0$ as $\delta \to 0$.



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