CHAPTER 3
Dynamics of a Particle

3.1 INTRODUCTION

In Chapter 2, we considered the motion of a particle without regard to the forces causing the motion. In this chapter, we propose to relate the motion to the forces causing it. This relation is commonly known as the equations of motion.

The beginnings of dynamics can be traced to the development of geometry by the ancient Greeks, such as Euclid and Pythagoras. Indeed, Euclidean geometry was later accepted as the framework for Newtonian mechanics. The Greeks were not successful in explaining the motion of bodies, however. For example, Aristotle believed that wherever there is motion there must be a force, although he was puzzled by the ability of bodies to move even in the absence of forces. The first to place the study of dynamics on a sound scientific foundation was Galileo. His *law of inertia* states that force causes a change in velocity but that no force is necessary to maintain motion in which the magnitude and direction of the velocity do not change, thus anticipating Newton's first law and perhaps the second law as well. Galileo postulated the existence of an *inertial space* or *Galilean reference frame*, where an inertial system is either at rest or translating with uniform velocity relative to a fixed space. It was Newton, however, who formulated the laws of motion in a clear and concise manner. Moreover, Newton's correct interpretation of Kepler's planetary laws resulted in his *law of gravitation*. Based on observations of the motion of planets made by Tycho Brahe, Kepler enunciated three laws of planetary motion. These laws were strictly geometric in nature, and it remained for Newton to give them physical content by demonstrating that the motion of planets is governed by the so-called *inverse square law*. Of course, the same law governs the motion of man-made satellites.

Newton's laws dominated mechanics through most of the nineteenth century, although in the latter part of the nineteenth century cracks began to appear. These discrepancies were finally resolved by Einstein's special and general relativity theories. The incidents in which Newtonian mechanics fails to provide the correct answers are not very common in engineering practice, so that Newtonian mechanics can be accepted with confidence, unless relativistic effects are present. Relativistic effects become important when the bodies under consideration move with velocities approaching in magnitude the speed of light. Cases in which relativistic mechanics must be invoked lie beyond the scope of this text.

3.2 NEWTON'S LAWS

Newton's laws were formulated for a single particle and can be extended to systems of particles and rigid bodies. There are three laws of motion, although the first law is merely a special case of the second law. Together with Newton's gravitational law, they form the basis for the so-called *Newtonian mechanics*. Newton referred to his laws as *axioms*.

Newtonian mechanics postulates the existence of *inertial systems of reference*, that is, systems of reference that are either at rest or moving with uniform velocity relative to a fixed reference frame. The motion of any particle is measured relative to such an inertial system and is said to be *absolute*. Newton's laws can be stated as follows:

First Law. *If there are no forces acting on a particle, then the particle will move in a straight line with constant velocity.*

A particle is defined as an idealization of a material body whose dimensions are very small when compared with the distance to other bodies, so that in essence a particle can be regarded as a point mass. Denoting the *resultant force vector* by $\mathbf{F}$ and the *absolute velocity vector* by $\mathbf{v}$, where we recall that an absolute velocity is measured relative to an inertial frame of reference, we can state the first law mathematically as

$$\text{If} \quad \mathbf{F}=\mathbf{0}, \quad \text{then} \quad \mathbf{v}=\text{const} \tag{3.1}$$

Second Law. *A particle acted on by a force moves so that the force vector is equal to the time rate of change of the linear momentum vector.*

The *linear momentum vector*, denoted $\mathbf{p}$, is defined as the product of the mass m of the particle and the absolute velocity vector $\mathbf{v}$, or

$$\mathbf{p}=m\mathbf{v} \tag{3.2}$$

Then, the second law can be stated mathematically as

$$\mathbf{F}=\frac{d\mathbf{p}}{dt}=\frac{d}{dt}(m\mathbf{v}) \tag{3.3}$$

In SI units, the unit of mass is the kilogram (kg) and the unit of force is the newton (N). The kilogram is a basic unit and the newton is a derived unit and is such that $1\,\text{N}=1\ \text{kg}\cdot\text{m/s}^2$. The mass of the particle is defined as a positive quantity whose value does not depend on time, so that Eq. (3.3) can be rewritten in the familiar form

$$\mathbf{F}=m\mathbf{a} \tag{3.4}$$

where

$$\mathbf{a}=d\mathbf{v}/dt \tag{3.5}$$

is the *absolute acceleration* of m. Clearly, the first law is a special case of the second law. Equation (3.4) represents the *equations of motion* for a particle.

Third Law. *When two particles exert forces on one another, the forces act along the line joining the particles, and they are equal in magnitude but opposite in directions.*

This law is also known as the *law of action and reaction.* Denoting by $\mathbf{F}_{12}$ the force exerted by particle m_2 on particle m_1, and vice versa, the third law can be stated mathematically as

$$\mathbf{F}_{12} = -\mathbf{F}_{21} \tag{3.6}$$

where the vectors $\mathbf{F}_{12}$ and $\mathbf{F}_{21}$ are *collinear* (Fig. 3.1). Because of this, there are no moments acting on the particles m_1 and m_2. There are cases, however, when moments do act on particles, in which cases the forces are no longer collinear. This is the case when there are electromagnetic forces between moving particles. Such exceptions to the third law are very rare indeed and will be excluded from further discussion.

In addition to the three laws of motion, Newton formulated the *law of universal gravitation.* The law states that gravity produces mutual attraction forces of the type shown in Fig. 3.1 and that the magnitude of these forces is given by

$$F(r) = \frac{Gm_1m_2}{r^2} \tag{3.7}$$

where r is the distance between the two particles and G is the *universal gravitational constant.* Later in this chapter we shall show how to establish the value of G. The gravitational law, Eq. (3.7), is commonly known as the *inverse square law.*

Newton's second law relates the forces acting on a particle with the acceleration of the particle. Hence, if some of these quantities are given, then one can use Newton's second law to solve for the remaining quantities. Of course, the number of unknowns must be equal to the number of equations of motion. An indispensable tool in the application of Newton's second law is the *free-body diagram,* a diagram containing all the forces acting on the particle. In drawing a free-body diagram, one must isolate the particle from any other particles. If the particle is attached to a massless member, such as a spring or a string, then in isolating the particle any force internal to the system, such as the force in the spring or string, becomes external to the system and must be treated as such. The concept of free-body diagram is equally useful for rigid bodies. We shall have ample opportunity to use the concept in this text.

Example 3.1

A stuntman rides a motorcycle inside a vertical cylindrical wall of radius $R = 15$ m (Fig. 3.2*a*). If the minimum velocity required to perform the stunt is $v = 54$ km/h,

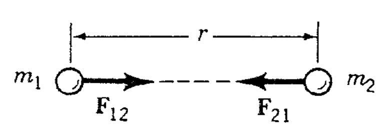


FIGURE 3.1

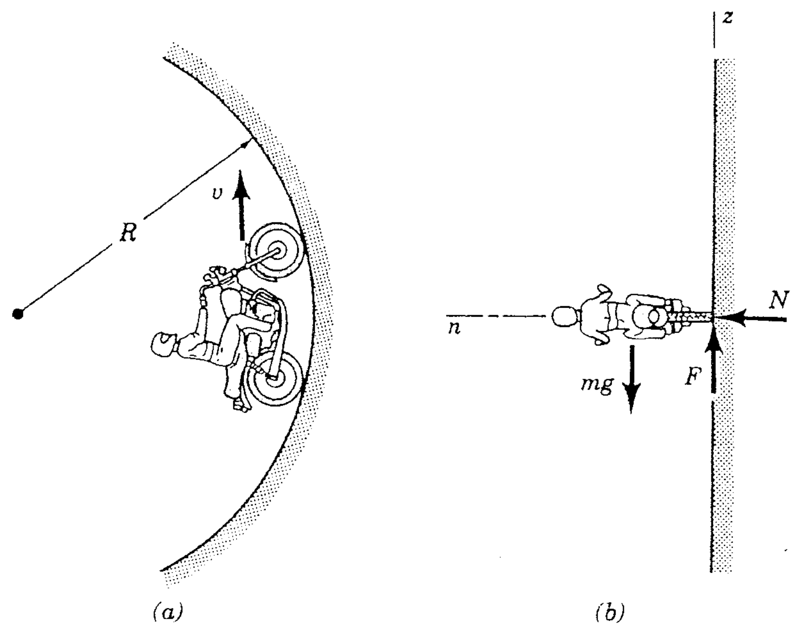


FIGURE 3.2

determine the coefficient of friction μ between the tires and the wall. Treat the system as a particle.

The equations of motion in the normal and vertical directions (Fig. 3.2b) are

$$F_n = N = ma_n \\ F_z = F - mg = ma_z = 0 \qquad \text{(a)}$$

where

$$F = \mu N \qquad \text{(b)}$$

From Eqs. (2.41), the acceleration component in the normal direction is

$$a_n = \frac{v^2}{R} \qquad \text{(c)}$$

Solving Eqs. (a), (b), and (c), we obtain

$$\mu = \frac{mg}{N} = \frac{gR}{v^2} = \frac{9.81 \times 15}{(54 \times 1000/3600)^2} = 0.654 \qquad \text{(d)}$$

3.3 INTEGRATION OF THE EQUATIONS OF MOTION

Let us consider the particle m shown in Fig. 3.3 and assume that the rectangular axes xyz represent an inertial frame. Then, the absolute position of m can be expressed in terms of the cartesian components x, y, and z by the radius vector

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \qquad (3.8)$$

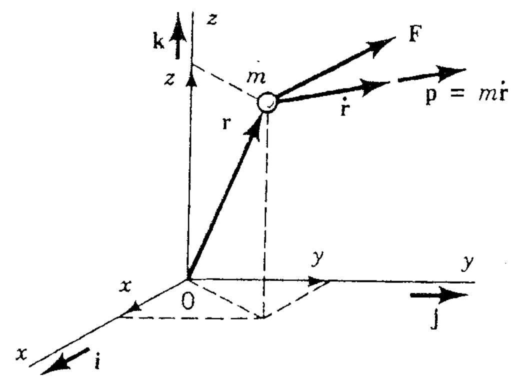


FIGURE 3.3

where **i**, **j**, and **k** are associated constant unit vectors. From Section 2.2, we can write the velocity and acceleration of m in the form

$$\mathbf{v} = \dot{\mathbf{r}} = v_x\mathbf{i} + v_y\mathbf{j} + v_z\mathbf{k} \tag{3.9a}$$

$$\mathbf{a} = \ddot{\mathbf{r}} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k} \tag{3.9b}$$

where

$$v_x = \dot{x}, \qquad v_y = \dot{y}, \qquad v_z = \dot{z} \tag{3.10a}$$

$$a_x = \ddot{x}, \qquad a_y = \ddot{y}, \qquad a_z = \ddot{z} \tag{3.10b}$$

are the cartesian components of the velocity and acceleration vectors, respectively.

The force **F** depends in general on the position **r**, the velocity **v**, and possibly the time t, or

$$\mathbf{F} = \mathbf{F}(\mathbf{r}, \mathbf{v}, t) \tag{3.11}$$

The force vector can be expressed as follows:

$$\mathbf{F} = F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k} \tag{3.12}$$

where

$$F_x = F_x(x, y, z, \dot{x}, \dot{y}, \dot{z}, t), \quad F_y = F_y(x, y, z, \dot{x}, \dot{y}, \dot{z}, t), \quad F_z = F_z(x, y, z, \dot{x}, \dot{y}, \dot{z}, t) \tag{3.13}$$

are the cartesian components of the vector.

The motion of m is governed by Newton's second law, as expressed by Eq. (3.4). Introducing Eqs. (3.9b), (3.10b), (3.12), and (3.13) into Eq. (3.4), we can write the three cartesian components of the equations of motion in the form

$$m\ddot{x} = F_x(x, y, z, \dot{x}, \dot{y}, \dot{z}, t) \tag{3.14a}$$

$$m\ddot{y} = F_y(x, y, z, \dot{x}, \dot{y}, \dot{z}, t) \tag{3.14b}$$

$$m\ddot{z} = F_z(x, y, z, \dot{x}, \dot{y}, \dot{z}, t) \tag{3.14c}$$

Equations (3.14) represent a set of three second-order ordinary differential equations of motion. At least in principle, they can be integrated to determine the three components $x(t)$, $y(t)$, and $z(t)$ of the motion. This requires the explicit form of the components F_x, F_y, and F_z of the force vector **F**, as well as six conditions permitting the evaluation of the six constants of integration involved. The six conditions are normally the initial displacements $x(0)$, $y(0)$, $z(0)$ and the initial velocities $\dot{x}(0)$, $\dot{y}(0)$, $\dot{z}(0)$. In practice, however, straightforward integration of the equations of motion is possible only when **F** is a simple function of **r**, **v**, and t, or in some special cases. In more complicated cases, it may be necessary to integrate the equations of motion numerically.

Example 3.2

Let us consider a particle m moving in the vicinity of a point 0 on the earth's surface (Fig. 3.4*a*) and determine the motion of the particle if the initial conditions are $x(0)=z(0)=0$, $\dot{x}(0)=v_{x0}$, and $\dot{z}(0)=v_{z0}$.

We assume that the motion involves distances that are much smaller than the radius of the earth. Letting $m_1=m$, $m_2=M$, and $r\cong R$ in Eq. (3.7), where M and R are the mass and the radius of the earth, respectively, the force vector can be written in the form (Fig. 3.4*b*)

$$\mathbf{F}=-\frac{mMG}{R^2}\mathbf{k}=-mg\mathbf{k}=\text{const} \tag{a}$$

where $g=GM/R^2$ is the gravitational acceleration. Equation (a) implies that in

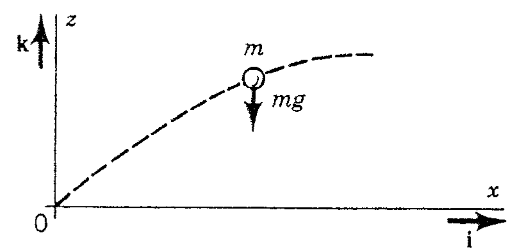
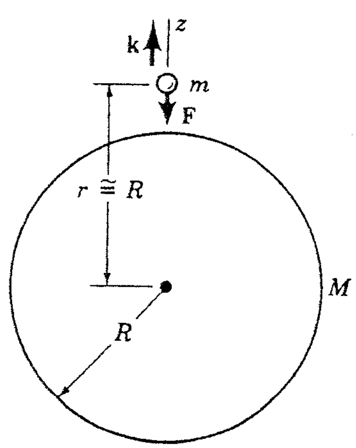


FIGURE 3.4*a*



b.

FIGURE 3.4*b*

the neighborhood of a given point on the earth's surface the gravitational force field can be assumed to be uniform.

Equations (3.14) in conjunction with Eq. (a) yield the differential equations of motion

$$m\ddot{x}=0, \qquad m\ddot{z}=-mg \tag{b}$$

Integrating Eqs. (b) twice with respect to time and considering the initial conditions, we obtain

$$x=v_{x0}t, \qquad z=v_{z0}t-\tfrac{1}{2}gt^2 \tag{c}$$

This problem has already been solved in Section 2.2 by regarding it simply as a kinematical problem. Of course, the solution is the same, as can be observed from Eqs. (2.18), where we recognize that $v_0 \cos\alpha = v_{x0}$ and $v_0 \sin\alpha = v_{z0}$. Eliminating t from Eqs. (c), one can derive the trajectory equation, that is, an equation relating x and z alone, with the time t acting as an implicit parameter. The trajectory equation has the form

$$z=x\frac{v_{z0}}{v_{x0}}-\frac{1}{2}g\frac{x^2}{v_{x0}^2} \tag{d}$$

and can be verified to be identical to Eq. (2.20) in Section 2.2.

Example 3.3

A rock of mass m is thrown vertically upward from the earth's surface with the initial velocity $\mathbf{v}_0$. Assume that there is a force proportional to the velocity resisting the motion (Fig. 3.5), and derive an expression for the velocity of the rock when it hits the ground upon its return.

Because the motion is one dimensional, we can dispense with the vector notation and write the differential equation of motion

$$\sum F_z=-mg-c\dot{z}=m\ddot{z} \tag{a}$$

where c is a proportionality constant. Equation (a) can be rewritten as

$$m\ddot{z}+c\dot{z}=-mg \tag{b}$$

Equation (b) is of second order, so that its solution requires two initial conditions.

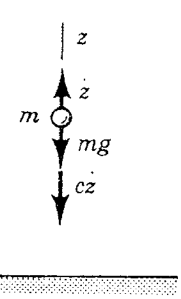


FIGURE 3.5

These conditions are

$$z(0)=0, \qquad \dot{z}(0)=v_0 \tag{c}$$

The solution of Eq. (b) can be obtained conveniently by the Laplace transformation method. Transforming both sides of Eq. (b), we obtain

$$m[s^2Z(s)-\dot{z}(0)-sz(0)]+c[sZ(s)-z(0)]=-\frac{mg}{s} \tag{d}$$

where $Z(s)$ is the Laplace transform of $z(t)$. Considering Eqs. (c), we can solve Eq. (d) for $Z(s)$ as follows:

$$Z(s)=\frac{v_0}{s(s+c/m)}-\frac{g}{s^2(s+c/m)} \tag{e}$$

The solution of Eq. (b) is obtained by carrying out the inverse Laplace transformation on Eq. (e). To this end, we recall from Example 1.4 that

$$\mathscr{L}^{-1}\frac{1}{s(s+c/m)}=\frac{m}{c}(1-e^{-ct/m}) \tag{f}$$

Moreover, let us consider the partial fractions expansion

$$\frac{1}{s^2(s+c/m)}=\frac{A+Bs}{s^2}+\frac{C}{s+c/m}=\frac{(A+Bs)(s+c/m)+Cs^2}{s^2(s+c/m)} \tag{g}$$

Equating the coefficients of s^2 and s at the numerator of the last fraction in Eq. (g) to zero and the remaining term to 1, we obtain the equations

$$B+C=0, \qquad A+Bc/m=0, \qquad Ac/m=1 \tag{h}$$

which have the solution

$$A=m/c, \qquad B=-C=-(m/c)^2 \tag{i}$$

so that the partial fractions expansion has the form

$$\frac{1}{s^2(s+c/m)}=\frac{m}{c}\frac{1}{s^2}-\left(\frac{m}{c}\right)^2\left(\frac{1}{s}-\frac{1}{s+c/m}\right) \tag{j}$$

Using the table of Laplace transform pairs in Section A.7, we can write

$$\mathscr{L}^{-1}\frac{1}{s^2(s+c/m)}=\frac{m}{c}t-\left(\frac{m}{c}\right)^2(1-e^{-ct/m}) \tag{k}$$

so that, considering Eqs. (f) and (k), we conclude that the inverse transformation of Eq. (e), and hence the solution of Eq. (b), is

$$\begin{aligned} z(t)&=\frac{mv_0}{c}(1-e^{-ct/m})-\frac{mg}{c}\left[t-\frac{m}{c}(1-e^{-ct/m})\right] \\ &=-\frac{m}{c}\left[gt-\left(v_0+\frac{mg}{c}\right)(1-e^{-ct/m})\right] \end{aligned} \tag{l}$$

To obtain the velocity of the rock when it strikes the ground, we must first produce an expression for the velocity. Differentiating Eq. (1) with respect to time, we obtain simply

$$\dot{z}(t) = -\frac{mg}{c} + \left(v_0 + \frac{mg}{c}\right) e^{-ct/m} \tag{m}$$

The rock strikes the ground at the instant $t = t_f$, where t_f denotes the final time, at which time $z(t_f) = 0$. Hence, from Eq. (1), t_f is the solution of the equation

$$gt_f - \left(v_0 + \frac{mg}{c}\right)(1 - e^{-ct_f/m}) = 0 \tag{n}$$

so that, letting $\dot{z}(t_f) = v_f$ in Eq. (m), where v_f is the final velocity of the rock, and considering Eq. (n), we obtain

$$v_f = v_0 - gt_f \tag{o}$$

where t_f is the solution of Eq. (n).

In general, the rock increases its velocity continuously as it falls toward the ground. However, under certain circumstances, the rock can reach constant velocity before it strikes the ground. Indeed, if the ratio c/m is relatively large, then for a certain value of time such that $ct \gg m$, we conclude from Eq. (m) that the velocity reaches the constant value

$$\dot{z} = -\frac{mg}{c} = \text{const} \tag{p}$$

Because the velocity is constant for any subsequent time, it must be the same velocity with which the rock strikes the ground, or

$$v_f = -\frac{mg}{c} \tag{q}$$

The negative sign can be explained by the fact that, when the rock strikes the ground, its velocity is directed along the negative direction of the z-axis.

3.4 IMPULSE AND MOMENTUM

The motion of a particle can be determined by integrating the equations of motion, as shown in Section 3.3. This task can be made easier at times by one of several principles based on Newton's second law. One such principle is the impulse-momentum principle, which is simply an integral with respect to time of Newton's second law.

The *linear impulse vector* is defined as the time integral of the force vector between two distinct times. Letting these times be t_1 and t_2, we can write the mathematical definition of the linear impulse vector

$$\hat{\mathbf{F}} = \int_{t_1}^{t_2} \mathbf{F}\, dt \tag{3.15}$$

where $\hat{\mathbf{F}}$ has unit newton·seconds. Multiplying Eq. (3.3) by dt and integrating between the times t_1 and t_2, we obtain

$$\hat{\mathbf{F}} = \int_{t_1}^{t_2} \mathbf{F}\, dt = \int_{t_1}^{t_2} \frac{d\mathbf{p}}{dt}\, dt = \mathbf{p}_2 - \mathbf{p}_1 = m\mathbf{v}_2 - m\mathbf{v}_1 \tag{3.16}$$

where

$$\mathbf{p}_i = m\mathbf{v}_i = m\mathbf{v}(t_i), \qquad i = 1, 2 \tag{3.17}$$

is the linear momentum vector at $t = t_i$. Letting

$$\Delta\mathbf{p} = \mathbf{p}_2 - \mathbf{p}_1 \tag{3.18}$$

be the change in the linear momentum vector between the times t_1 and t_2, we can rewrite Eq. (3.16) in the form

$$\hat{\mathbf{F}} = \Delta\mathbf{p} \tag{3.19}$$

or, *the linear impulse vector corresponding to the times t_1 and t_2 is equal to the change in the linear momentum vector between the same two instants.* It should be stressed that both the impulse and the linear momentum are vector quantities. Equation (3.19) is sometimes referred to as a first integral of motion with respect to time.

If there are no forces acting on the particle ($\mathbf{F} = \mathbf{0}$), Eq. (3.16) yields

$$\mathbf{p}_2 = \mathbf{p}_1 = \mathbf{p} = \text{const} \tag{3.20}$$

or *in the absence of forces the linear momentum does not change.* Equation (3.20) is the mathematical statement of the principle of *conservation of linear momentum.* It is also a restatement of Newton's first law.

Example 3.4

A ball of mass $m = 10^{-2}$ kg approaches a rigid wall with the velocity $v_1 = 40$ m/s normal to the wall (Fig. 3.6*a*), hits the rigid wall (Fig. 3.6*b*), and returns with the velocity $v_2 = 30$ m/s also normal to the wall (Fig. 3.6*c*). It is known that the force between the ball and the wall during impact is a triangular function of time (Fig. 3.7) and that the ball is in contact with the wall for a time interval $\Delta t = 0.01$ s. Calculate the maximum force on the ball.

The motion is one dimensional, so that, choosing the return direction as positive, we can write the impulse-momentum relation

$$\tfrac{1}{2}F_{\max}\,\Delta t = mv_2 - (-mv_1) = m(v_2 + v_1) \tag{a}$$

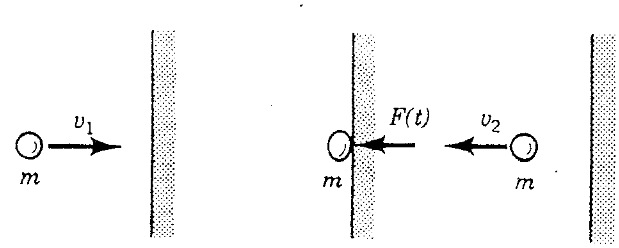


FIGURE 3.6

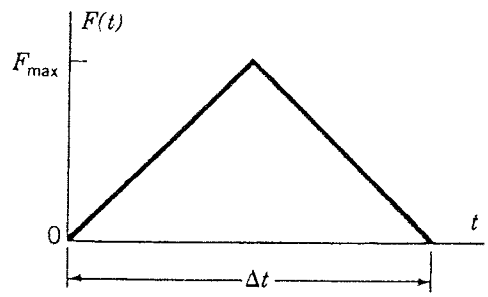


FIGURE 3.7

or

$$F_{max} = \frac{2m(v_2 + v_1)}{\Delta t} = \frac{2 \times 10^{-2}(40+30)}{0.01} = 140 \quad \text{N} \tag{b}$$

3.5 MOMENT OF A FORCE AND ANGULAR MOMENTUM ABOUT A FIXED POINT

Let us consider a particle of mass m moving under the action of a force $\mathbf{F}$. We shall denote the position of m relative to the origin 0 of the inertial frame xyz by $\mathbf{r}$ and the absolute velocity of m by $\dot{\mathbf{r}}$ (Fig. 3.3). The *moment of momentum* or *angular momentum* of m with respect to point 0 is defined as the moment of the linear momentum $\mathbf{p}$ about 0 and is represented mathematically by the cross product (vector product) of the radius vector $\mathbf{r}$ and the linear momentum vector $\mathbf{p}$. Denoting the angular momentum about 0 by $\mathbf{H}_0$, we can write

$$\mathbf{H}_0 = \mathbf{r} \times \mathbf{p} = \mathbf{r} \times m\dot{\mathbf{r}} \tag{3.21}$$

Let us consider now the rate of change of $\mathbf{H}_0$. Assuming that m is constant, we have

$$\dot{\mathbf{H}}_0 = \dot{\mathbf{r}} \times m\dot{\mathbf{r}} + \mathbf{r} \times m\ddot{\mathbf{r}} = \mathbf{r} \times m\ddot{\mathbf{r}} \tag{3.22}$$

where $\dot{\mathbf{r}} \times m\dot{\mathbf{r}} = m(\dot{\mathbf{r}} \times \dot{\mathbf{r}}) = \mathbf{0}$. By Newton's second law, however,

$$m\ddot{\mathbf{r}} = \mathbf{F} \tag{3.23}$$

Moreover, recognizing that

$$\mathbf{r} \times \mathbf{F} = \mathbf{M}_0 \tag{3.24}$$

is the moment of the force $\mathbf{F}$ about point 0, we obtain

$$\mathbf{M}_0 = \dot{\mathbf{H}}_0 \tag{3.25}$$

or, *the moment of a force about a fixed point* 0 *is equal to the time rate of change of the moment of momentum about* 0.

By definition, the cross product of two vectors is a vector normal to the plane defined by the two vectors involved in the product. Hence, $\mathbf{H}_0$ is a vector normal to both $\mathbf{r}$ and $\dot{\mathbf{r}}$. On the other hand, $\mathbf{M}_0$ is a vector normal to $\mathbf{r}$ and $\mathbf{F}$. Because in general $\mathbf{r}$, $\dot{\mathbf{r}}$, and $\mathbf{F}$ are not in the same plane, $\mathbf{M}_0$ and $\mathbf{H}_0$ are not in the same direction. However, from Eq. (3.25) we conclude that $\mathbf{M}_0$ and $\dot{\mathbf{H}}_0$ are in the same direction. In the special case in which the motion is planar, the vectors $\mathbf{M}_0$, $\mathbf{H}_0$, and $\dot{\mathbf{H}}_0$ are all in the same direction, namely, in the direction normal to the plane of motion.

Next, let us define the angular impulse about 0 between the times t_1 and t_2 as the vector

$$\hat{\mathbf{M}}_0 = \int_{t_1}^{t_2} \mathbf{M}_0 \, dt \tag{3.26}$$

where $\hat{\mathbf{M}}_0$ has the unit newton·meter·seconds. Multiplying Eq. (3.25) by dt and integrating with respect to time between t_1 and t_2, we obtain

$$\hat{\mathbf{M}}_0 = \int_{t_1}^{t_2} \mathbf{M}_0 \, dt = \int_{t_1}^{t_2} \frac{d\mathbf{H}_0}{dt} \, dt = \mathbf{H}_{02} - \mathbf{H}_{01} \tag{3.27}$$

where

$$\mathbf{H}_{0i} = \mathbf{H}_0(t_i) = \mathbf{r}(t_i) \times m\dot{\mathbf{r}}(t_i) = \mathbf{r}(t_i) \times m\mathbf{v}(t_i), \qquad i = 1, 2 \tag{3.28}$$

is the angular momentum at $t = t_i$. Letting

$$\Delta \mathbf{H}_0 = \mathbf{H}_{02} - \mathbf{H}_{01} \tag{3.29}$$

be the change in the angular momentum vector between the times t_1 and t_2, Eq. (3.27) becomes

$$\hat{\mathbf{M}}_0 = \Delta \mathbf{H}_0 \tag{3.30}$$

or, *the angular impulse vector about* 0 *between the times* t_1 *and* t_2 *is equal to the change in the angular momentum vector about* 0 *between the same two instants.*

If there are no torques about 0, $\mathbf{M}_0 = \mathbf{0}$, Eq. (3.27) yields

$$\mathbf{H}_{02} = \mathbf{H}_{01} = \mathbf{H}_0 = \text{const} \tag{3.31}$$

or, *in the absence of torques about* 0 *the angular momentum about* 0 *is constant*, which is the statement of the principle of *conservation of angular momentum.* Note that the conservation of the angular momentum does not require that the force be zero. Indeed, the angular momentum is conserved if the force passes through 0. Such a force is known as a *central force* and plays an important role in the motion of planets and satellites.

Example 3.5

A particle of mass m moves on a smooth surface while attached to a string rotating with the angular velocity ω. At the same time, the string is being pulled through a small hole, as shown in Fig. 3.8. Let the distance between the hole and the particle when the angular velocity is ω_1 be equal to R, and calculate the angular velocity ω_2 when the distance is reduced to $R/2$.

The only force acting on the mass m is the tension in the string. Because this

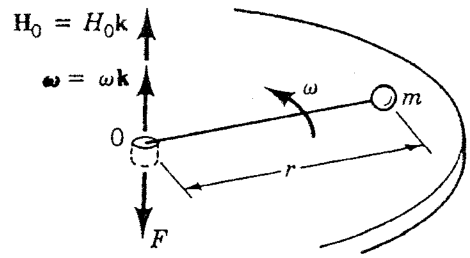


FIGURE 3.8

force goes through the center of rotation 0, as shown in Fig. 3.8, there is no torque about 0, so that the angular momentum about 0 is conserved. This being a planar problem, we can dispense with the vector notation and write

$$H_0 = mr^2\omega = \text{const} \tag{a}$$

Letting $r=R$, $\omega=\omega_1$ and $r=R/2$, $\omega=\omega_2$ in Eq. (a), in sequence, we obtain

$$mR^2\omega_1 = m(R/2)^2\omega_2 \tag{b}$$

from which it follows that

$$\omega_2 = 4\omega_1 \tag{c}$$

3.6 WORK AND ENERGY

Let us consider a particle m moving along a curve s under the action of a given force $\mathbf{F}$ (Fig. 3.9). By definition, the *increment of work* corresponding to the displacement of m from position $\mathbf{r}$ to position $\mathbf{r}+d\mathbf{r}$ is given by the dot product (scalar product)

$$dW = \mathbf{F}\cdot d\mathbf{r} \tag{3.32}$$

Clearly, dW is a scalar quantity. By Newton's second law, however, we have $\mathbf{F}=m\ddot{\mathbf{r}}$ and from kinematics $d\mathbf{r}=\dot{\mathbf{r}}\,dt$, so that

$$dW = m\ddot{\mathbf{r}}\cdot\dot{\mathbf{r}}\,dt = m\frac{d\dot{\mathbf{r}}}{dt}\cdot\dot{\mathbf{r}}\,dt = m\dot{\mathbf{r}}\cdot d\dot{\mathbf{r}} = d(\tfrac{1}{2}m\dot{\mathbf{r}}\cdot\dot{\mathbf{r}}) \tag{3.33}$$

But the *kinetic energy* of the particle m is defined as

$$T = \tfrac{1}{2}m\dot{\mathbf{r}}\cdot\dot{\mathbf{r}} \tag{3.34}$$

and we note that T is a scalar function. Hence, inserting Eq. (3.34) into Eq. (3.33), we obtain

$$dW = dT \tag{3.35}$$

Next, let us consider the work performed by $\mathbf{F}$ in moving the particle m from position $\mathbf{r}_1$ to position $\mathbf{r}_2$ (Fig. 3.9). Considering Eqs. (3.32) and (3.35), we can write

$$W_{1-2} = \int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{F}\cdot d\mathbf{r} = \int_{T_1}^{T_2} dT = T_2 - T_1 \tag{3.36}$$

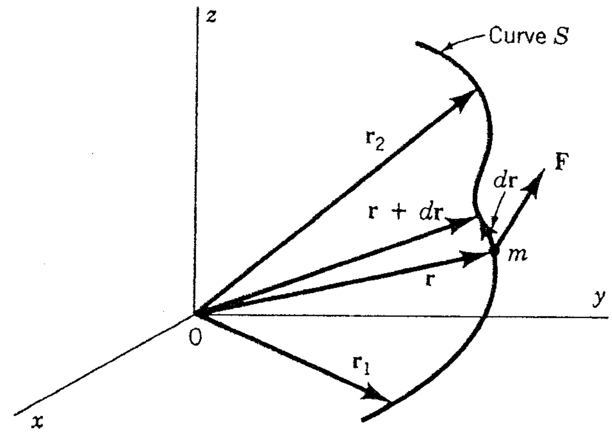


FIGURE 3.9

so that *the work performed by the force* $\mathbf{F}$ *in moving the particle m from position* $\mathbf{r}_1$ *to position* $\mathbf{r}_2$ *is equal to the change in the kinetic energy from* T_1 *to* T_2.

There exists one class of forces for which *the work performed depends only on the terminal positions* $\mathbf{r}_1$ *and* $\mathbf{r}_2$, and not on the path taken to travel from $\mathbf{r}_1$ to $\mathbf{r}_2$. Considering two distinct paths, I and II, as shown in Fig. 3.10, we conclude that the above statement implies that

$$\underset{\text{path I}}{\int_{\mathbf{r}_1}^{\mathbf{r}_2}} \mathbf{F}\cdot d\mathbf{r} = \underset{\text{path II}}{\int_{\mathbf{r}_1}^{\mathbf{r}_2}} \mathbf{F}\cdot d\mathbf{r} \tag{3.37}$$

Equation (3.37) yields

$$\underset{\text{path I}}{\int_{\mathbf{r}_1}^{\mathbf{r}_2}} \mathbf{F}\cdot d\mathbf{r} - \underset{\text{path II}}{\int_{\mathbf{r}_1}^{\mathbf{r}_2}} \mathbf{F}\cdot d\mathbf{r} = \underset{\text{path I}}{\int_{\mathbf{r}_1}^{\mathbf{r}_2}} \mathbf{F}\cdot d\mathbf{r} + \underset{\text{path II}}{\int_{\mathbf{r}_2}^{\mathbf{r}_1}} \mathbf{F}\cdot d\mathbf{r} = \oint \mathbf{F}\cdot d\mathbf{r} = 0 \tag{3.38}$$

where $\oint$ denotes an integral over a closed path. Hence, the statement that the work performed does not depend on the path is equivalent to the statement that *the work performed in traveling over a closed path* (starting at a given point and returning to the same point) *is zero.* Forces for which the above statements are true for all possible paths are said to be *conservative* and will be identified by the subscript c.

Consider now a conservative force $\mathbf{F}_c$, choose a path passing through the

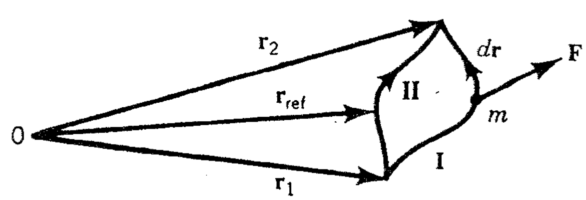


FIGURE 3.10

reference position $\mathbf{r}_{\text{ref}}$ (Fig. 3.10) and write

$$W_{1-2c} = \int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{F}_c \cdot d\mathbf{r} = \int_{\mathbf{r}_1}^{\mathbf{r}_{\text{ref}}} \mathbf{F}_c \cdot d\mathbf{r} + \int_{\mathbf{r}_{\text{ref}}}^{\mathbf{r}_2} \mathbf{F}_c \cdot d\mathbf{r}$$

$$= \int_{\mathbf{r}_1}^{\mathbf{r}_{\text{ref}}} \mathbf{F}_c \cdot d\mathbf{r} - \int_{\mathbf{r}_2}^{\mathbf{r}_{\text{ref}}} \mathbf{F}_c \cdot d\mathbf{r} \tag{3.39}$$

Then, defining the *potential energy* as *the work performed by a conservative force in moving a particle from the position* $\mathbf{r}$ *to the reference position* $\mathbf{r}_{\text{ref}}$, or

$$V(\mathbf{r}) = \int_{\mathbf{r}}^{\mathbf{r}_{\text{ref}}} \mathbf{F}_c \cdot d\mathbf{r} \tag{3.40}$$

where we note that V is a scalar function, and using definition (3.40), we can rewrite Eq. (3.39)

$$W_{1-2c} = V_1 - V_2 = -(V_2 - V_1) \tag{3.41}$$

where

$$V_i = V(\mathbf{r}_i), \qquad i = 1, 2 \tag{3.42}$$

Equation (3.41) states that *the work performed by a conservative force in moving a particle from* $\mathbf{r}_1$ *to* $\mathbf{r}_2$ *is equal to the negative of the change in the potential energy from* V_1 *to* V_2. Note that, because our interest lies mainly in changes in the potential energy rather than in the potential energy itself, the reference position is arbitrary, i.e., it can be selected at will. It should be pointed out that, in contrast with the potential energy, the kinetic energy represents an absolute quantity, because it is expressed in terms of velocities relative to an inertial space.

In general, we can distinguish between conservative and nonconservative forces, so that the work can be separated accordingly, or

$$W_{1-2} = W_{1-2c} + W_{1-2nc} \tag{3.43}$$

where

$$W_{1-2nc} = \int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{F}_{nc} \cdot d\mathbf{r} \tag{3.44}$$

in which W_{1-2nc} is the nonconservative work and $\mathbf{F}_{nc}$ the nonconservative force. Inserting Eqs. (3.36) and (3.41) into Eq. (3.43), we obtain

$$T_2 - T_1 = -(V_2 - V_1) + W_{1-2nc} \tag{3.45}$$

Introducing the *total energy* as *the sum of the kinetic energy and potential energy*

$$E = T + V \tag{3.46}$$

we can rearrange Eq. (3.45) into

$$W_{1-2nc} = E_2 - E_1 \tag{3.47}$$

or, *the work performed by the nonconservative force* $\mathbf{F}_{nc}$ *in moving a particle from* $\mathbf{r}_1$ *to* $\mathbf{r}_2$ *is equal to the change in the total energy from* E_1 *to* E_2. The work–energy relation in any of its forms, Eq. (3.36), or Eq. (3.41), or Eq. (3.47), is sometimes referred to as a first integral of motion with respect to displacement.

Equation (3.41) can also be written in the form

$$dW_c = -dV \tag{3.48}$$

and Eq. (3.43) in the form

$$dW = dW_c + dW_{nc} \tag{3.49}$$

so that, introducing Eqs. (3.35) and (3.48) into Eq. (3.49) and recalling Eq. (3.46), we obtain

$$dW_{nc} = dE \tag{3.50}$$

But, from Eq. (3.44), we have

$$dW_{nc} = \mathbf{F}_{nc} \cdot d\mathbf{r} \tag{3.51}$$

so that

$$\mathbf{F}_{nc} \cdot d\mathbf{r} = dE \tag{3.52}$$

Dividing both sides of Eq. (3.52) by dt, we obtain

$$\mathbf{F}_{nc} \cdot \dot{\mathbf{r}} = \dot{E} \tag{3.53}$$

But, for any general force $\mathbf{F}$, the expression

$$P = \mathbf{F} \cdot \dot{\mathbf{r}} \tag{3.54}$$

represents the rate of work performed by the force $\mathbf{F}$ and is known as the *power*. Hence, Eq. (3.53) states that the power associated with the nonconservative force $\mathbf{F}_{nc}$ is equal to the time rate of change of the total energy E. Note that nonconservative forces can either add energy to a system, as in the case of applied forces, or dissipate energy, as in the case of damping forces.

In the case in which there are no nonconservative forces ($\mathbf{F}_{nc} = \mathbf{0}$), Eq. (3.53) yields

$$E = \text{const} \tag{3.55}$$

or, *in the absence of nonconservative forces the total energy remains constant*. This statement is known as the principle of *conservation of energy*. Conservative forces are characterized by the fact that they depend on the position $\mathbf{r}$ alone and not on the velocity $\dot{\mathbf{r}}$ or time t.

Example 3.6

A particle m is originally at rest at point 1 when it begins to slide on a smooth surface (Fig. 3.11). Calculate the velocity v_2 with which the particle leaves the slide.

The only forces acting on m are the reaction force $\mathbf{N}$ exerted by the surface and the gravitational force. The reaction force is normal to the path, so that it performs no work. The gravitational force can be regarded as conservative (although it is not necessary to do so), so that we can solve the problem by using the conservation of energy principle, Eq. (3.55).

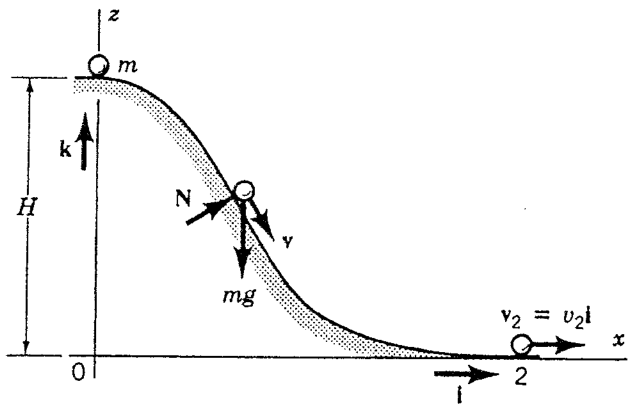


FIGURE 3.11

Letting point 1 be the reference position, Eq. (3.40) yields the potential energy in position 2 in the form

$$V_2 = V(\mathbf{r}_2) = \int_{\mathbf{r}_2}^{\mathbf{r}_1} (-mg\mathbf{k})\cdot d\mathbf{r} = \int_{x_2,0}^{0,H} (-mg\mathbf{k})\cdot(dx\mathbf{i} + dz\mathbf{k})$$

$$= -mg\int_0^H dz = -mgH \tag{a}$$

so that the potential energy depends only on the weight mg and the difference in heights H between the reference position 1 and point 2. Of course, because point 1 represents the reference position, $V_1 = 0$. In the reference position the particle is at rest, so that $T_1 = 0$. On the other hand, at point 2 the kinetic energy is

$$T_2 = \tfrac{1}{2}mv_2^2 \tag{b}$$

Hence, using Eq. (3.50), we can write

$$E = T_1 + V_1 = T_2 + V_2 \tag{c}$$

or

$$0 = -mgH + \tfrac{1}{2}mv_2^2 \tag{d}$$

which yields the exit velocity

$$v_2 = \sqrt{2gH} \tag{e}$$

Example 3.7

Consider the particle m of Example 3.5 and determine the work performed in moving the particle from the position $r = R$ to the position $r = R/2$.

Because the particle moves on a horizontal table, there is no change in the potential energy, so that Eq. (3.47) yields

$$W_{1-2nc} = T_2 - T_1 \tag{a}$$

where, recalling from Example 3.5 that $\omega_2 = 4\omega_1$,

$$T_1 = \tfrac{1}{2}mv_1^2 = \tfrac{1}{2}mR^2\omega_1^2$$
$$T_2 = \tfrac{1}{2}mv_2^2 = \tfrac{1}{2}m[(R/2)\omega_2)]^2 = 2mR^2\omega_1^2 \tag{b}$$

Hence,

$$W_{1-2\text{nc}} = \tfrac{3}{2} m R^2 \omega_1^2 \tag{c}$$

3.7 MOTION IN A CENTRAL-FORCE FIELD

Let us consider a particle m moving under the influence of a force **F** passing through a fixed point 0 at all times (Fig. 3.12). Because the moment of **F** about 0 is zero, the angular momentum about 0 is conserved (Section 3.5), so that

$$\mathbf{H}_0 = \mathbf{r} \times m\mathbf{v} = \text{const} \tag{3.56}$$

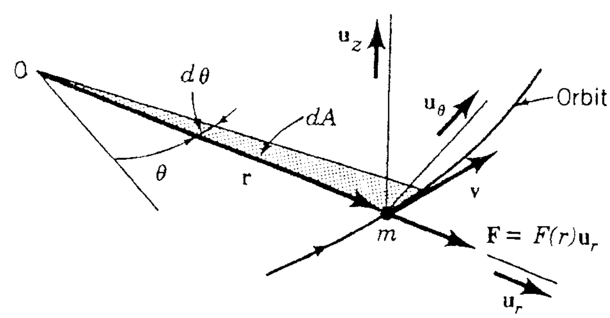


FIGURE 3.12

The constancy of $\mathbf{H}_0$ implies that *the magnitude H_0 and the direction in space of the angular momentum vector $\mathbf{H}_0$ are both constant.* Moreover, because the vector $\mathbf{H}_0$ is normal to both the radius vector **r** and the velocity vector **v**, it follows that *the motion takes place in a plane fixed in space.* It will prove convenient to refer the motion to a set of cylindrical axes defined by the unit vectors $\mathbf{u}_r$, $\mathbf{u}_\theta$, and $\mathbf{u}_z$, where $\mathbf{u}_r$ is in the radial direction, $\mathbf{u}_\theta$ is in the transverse direction and in the plane of motion, and $\mathbf{u}_z$ is in the direction normal to the plane of motion. Whereas $\mathbf{u}_z$ is a constant unit vector, $\mathbf{u}_r$ and $\mathbf{u}_\theta$ are not constant because their directions change with the position of m. In essence, we propose to describe the motion in terms of the polar coordinates r and θ, which depend on the instantaneous position of the particle m.

Before proceeding with the investigation of the motion, we recall from Section 2.3 that the absolute velocity vector has the expression

$$\mathbf{v} = v_r \mathbf{u}_r + v_\theta \mathbf{u}_\theta \tag{3.57}$$

where

$$v_r = \dot{r}, \qquad v_\theta = r\dot{\theta} \tag{3.58}$$

are the radial and transverse components of the velocity vector **v**, in which $\mathbf{r} = r\mathbf{u}_r$ is the radius vector. Moreover, the absolute acceleration has the form

$$\mathbf{a} = a_r \mathbf{u}_r + a_\theta \mathbf{u}_\theta \tag{3.59}$$

where

$$a_r = \ddot{r} - r\dot{\theta}^2, \qquad a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} \tag{3.60}$$

are the radial and transverse components of the acceleration vector $\mathbf{a}$.

The constancy of the angular momentum, Eq. (3.56), can be given an interesting geometric interpretation. Indeed, inserting Eqs. (3.57) and (3.58) into Eq. (3.56), we obtain

$$\mathbf{H}_0 = \mathbf{r} \times m\mathbf{v} = r\mathbf{u}_r \times m(\dot{r}\mathbf{u}_r + r\dot{\theta}\mathbf{u}_\theta) = mr^2\dot{\theta}\mathbf{u}_z = \text{const} \tag{3.61}$$

where we recognized that $\mathbf{u}_r \times \mathbf{u}_r = \mathbf{0}$ and that $\mathbf{u}_r \times \mathbf{u}_\theta = \mathbf{u}_z$. We have already established that the direction of $\mathbf{H}_0$, which coincides with the direction of the unit vector $\mathbf{u}_z$, is fixed in space. On the other hand, we observe that the triangular differential element of area dA shown in Fig. 3.12 can be written as

$$dA = \tfrac{1}{2}(r)(r\,d\theta) = \tfrac{1}{2}r^2\,d\theta \tag{3.62}$$

Dividing both sides of Eq. (3.62) by dt and considering Eq. (3.61), we obtain

$$\dot{A} = \tfrac{1}{2}r^2\dot{\theta} = H_0/2m = \text{const} \tag{3.63}$$

where

$$H_0 = mr^2\dot{\theta} = \text{const} \tag{3.64}$$

is the magnitude of the angular momentum vector $\mathbf{H}_0$. Equation (3.63) represents the mathematical statement of *Kepler's second law* for planetary motion: *Every planet moves in such a way that its radius vector sweeps over equal areas in equal times.*

Next, let us turn our attention to the equations of motion. Letting the central force have the general form

$$\mathbf{F} = F_r\mathbf{u}_r + F_\theta\mathbf{u}_\theta = F_r\mathbf{u}_r \tag{3.65}$$

the radial and transverse components of Newton's second law can be written as

$$ma_r = F_r, \qquad ma_\theta = F_\theta = 0 \tag{3.66}$$

Introducing Eqs. (3.60) into Eqs. (3.66), we obtain

$$m(\ddot{r} - r\dot{\theta}^2) = F_r \tag{3.67a}$$

$$m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = 0 \tag{3.67b}$$

Equation (3.67b) can be rewritten as

$$\frac{m}{r}\frac{d}{dt}(r^2\dot{\theta}) = 0 \tag{3.68}$$

which, upon integration, leads to a reaffirmation of Kepler's second law.

The above results, including Kepler's second law, were reached solely on the assumption of a force passing through the fixed center 0 and without specifying the explicit functional dependence of the central force.

3.8 THE INVERSE SQUARE LAW. ORBITS OF PLANETS AND SATELLITES

In Section 3.7, we discussed the motion of a particle m under a central force. A force of particular importance is that given by Newton's inverse square law. Strictly speaking, the force is along the line joining two moving particles. If the mass of one of the particles is considerably larger than the mass of the other, then the motion can be regarded as that of the small particle moving around the massive particle. This is certainly the case with many planets and satellites. Moreover, if we further assume that the massive particle is at rest in an inertial space, then the motion of the small particle can be regarded as taking place about a fixed point.

Let us consider a particle m moving around a massive particle M under the action of a force according to Newton's gravitational law, Eq. (3.7). Letting $m_1 = m$ and $m_2 = M$ in Eq. (3.7), introducing the notation

$$GM = \mu \tag{3.69}$$

and dividing through by m, we can write Eq. (3.67a) in the form

$$\ddot{r} - r\dot{\theta}^2 = \frac{F_r}{m} = -\frac{\mu}{r^2} \tag{3.70}$$

where the minus sign on the right side of the equation can be explained by the fact that the force is in the opposite direction to that shown in Fig. 3.12. Moreover, Kepler's second law can be rewritten in the form

$$r^2\dot{\theta} = h = \text{const} \tag{3.71}$$

where the constant h can be identified as the angular momentum per unit mass, $h = H_0/m$, and is sometimes referred to as *specific angular momentum.*

The object now is to eliminate the time dependence from Eq. (3.70) and obtain an equation in terms of the polar coordinates r and θ alone. This equation, in which the time does not appear explicitly, is known as the *orbit equation.* As it turns out, to derive the orbit equation, we do not work with the radial distance r but with its reciprocal. Hence, let us consider Eq. (3.71) and write

$$\dot{r} = \frac{dr}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt} = \frac{h}{r^2}\frac{dr}{d\theta} = -h\frac{d}{d\theta}\left(\frac{1}{r}\right) = -h\frac{du}{d\theta} \tag{3.72}$$

where u is the reciprocal of r, or

$$u = 1/r \tag{3.73}$$

Moreover, we have

$$\ddot{r} = \frac{d\dot{r}}{dt} = \frac{d\dot{r}}{d\theta}\frac{d\theta}{dt} = \frac{h}{r^2}\frac{d\dot{r}}{d\theta} = -h^2u^2\frac{d^2u}{d\theta^2} \tag{3.74}$$

Introducing Eqs. (3.71), (3.73), and (3.74) into Eq. (3.70), and dividing through by $-h^2u^2$, we obtain the second-order differential equation

$$\frac{d^2u}{d\theta^2} + u = \frac{\mu}{h^2} \tag{3.75}$$

which does not contain the time explicitly. The complete solution of Eq. (3.75) is

$$u = \frac{\mu}{h^2} + C\cos(\theta - \theta_0) \tag{3.76}$$

and it represents the orbit equation. The first term on the right side of Eq. (3.76) is constant and is recognized as the particular solution; the second term is harmonic and can be identified as the homogeneous solution, in which C and θ_0 are the two constants of integration. There are several types of orbits possible, with the type of orbit depending on the constants of integration, and in particular on C. The constant θ_0 turns out to be of minor importance. In the following, we shall determine the relation between the type of orbit and the constant C.

Physically, the orbit is determined by the initial conditions, and hence by the energy of the orbiting body. Because for bodies in the solar system there is virtually no energy dissipation, the orbits of planets will remain the same. Actually, this statement is only approximately true, as there are always perturbing forces present. In the case of artificial earth satellites, the energy dissipation is negligible if the orbits are sufficiently high to avoid atmospheric drag. Note that Eqs. (3.70) and (3.71) imply no energy dissipation. Because there is no force component in the transverse direction, the *potential energy per unit mass* can be written

$$V(r) = \int_r^{r_0} \left(-\frac{\mu}{\xi^2}\right) d\xi = \left.\frac{\mu}{\xi}\right|_r^{r_0} = \frac{\mu}{r_0} - \frac{\mu}{r} \tag{3.77}$$

where we replaced r on the right side of Eq. (3.70) by ξ, a dummy variable of integration. We observe that the term μ/r_0 is constant. As pointed out earlier, in comparing the potential energy associated with two different positions, the constant is immaterial. Hence, in such cases it will prove convenient to choose as reference position $r_0 = \infty$, which has the advantage that it renders this constant zero. We shall return to this subject later in this section. Using Eqs. (3.58), we can write the *kinetic energy per unit mass*

$$T = \tfrac{1}{2}(v_r^2 + v_\theta^2) = \tfrac{1}{2}(\dot{r}^2 + r^2\dot{\theta}^2) \tag{3.78}$$

so that, letting $r_0 = \infty$ in Eq. (3.77), the *total energy per unit mass* becomes

$$E = T + V = \tfrac{1}{2}(\dot{r}^2 + r^2\dot{\theta}^2) - \mu/r \tag{3.79}$$

Inserting Eqs. (3.71)–(3.73) into Eq. (3.79) and considering solution (3.76), we obtain

$$\begin{aligned} E &= \frac{1}{2}\left[\left(-h\frac{du}{d\theta}\right)^2 + (hu)^2\right] - \mu u = \frac{h^2}{2}\left\{[-C\sin(\theta-\theta_0)]^2\right. \\ &\quad \left. + \left[\frac{\mu}{h^2} + C\cos(\theta-\theta_0)\right]^2\right\} - \frac{\mu^2}{h^2} - \mu C\cos(\theta-\theta_0) \\ &= \frac{h^2}{2}\left[C^2 - \left(\frac{\mu}{h^2}\right)^2\right] \end{aligned} \tag{3.80}$$

from which it follows that

$$C = \frac{\mu}{h^2}\sqrt{1+\frac{2Eh^2}{\mu^2}} = \frac{\mu}{h^2}\,e \tag{3.81}$$

where

$$e = \sqrt{1+\frac{2Eh^2}{\mu^2}} \tag{3.82}$$

Introducing Eq. (3.81) into Eq. (3.76) and recalling Eq. (3.73), we obtain the orbit equation in the form

$$r = \frac{h^2}{\mu}\,\frac{1}{1+e\cos(\theta-\theta_0)} \tag{3.83}$$

which represents geometrically the equation of a *conic section* in polar form with the *focus* at the center 0. The shape of the orbit is determined by the parameters h^2/μ and e, where the latter is known as the orbit *eccentricity*. The type of conic section depends mathematically on the eccentricity e and physically on the total energy E. In the following, we present a brief discussion of the various types of orbits possible.

There are two classes of orbits: closed and open. Among the closed orbits we find the *circle* and the *ellipse*, and among the open orbits the *parabola* and *hyperbola*. A point characterized by $dr/d\theta = 0$ is known as an *apsis*, and it represents either the point on the orbit closest to the focus or that farthest from the focus. A circle has an infinity of *apsides*, because at every point of a circle $dr/d\theta = 0$, which is consistent with the fact that all points on a circle are equidistant from the center. An ellipse has two apsides. On the other hand, open orbits such as the parabola and hyperbola have only one apsis. The shorter apsis is called *pericenter*, and the longer one is called *apocenter*. For orbits with the focus at the earth's center they are known as *perigee* and *apogee*, and for orbits with the focus at the sun's center they are known as *perihelion* and *aphelion*, respectively. The angle θ_0 can be identified as the angle from a given reference line to the pericenter. If θ is measured from the pericenter, then $\theta_0 = 0$. In future discussion, we shall assume that this is the case. Introducing the notation

$$h^2/\mu = p \tag{3.84}$$

where p is known as the *semi-latus rectum*, the orbit equation can be written in the simple form

$$r = \frac{p}{1+e\cos\theta} \tag{3.85}$$

Note that the semi-latus rectum is the radial distance corresponding to $\theta = \pi/2$. The various possible orbits, together with the eccentricity e and the total energy E, are listed below:

1. Circle: $e=0$, $E = -\mu^2/2h^2$.
2. Ellipse: $0<e<1$, $-\mu^2/2h^2 < E < 0$.

3. Parabola: $e=1$, $E=0$.
4. Hyperbola: $e>1$, $E>0$.

The fact that the total energy can be negative has no particular physical significance and is only a reflection of choosing the reference position r_0 for the potential energy as infinity. As pointed out earlier, this is an acceptable choice if the energy thus calculated is used only to compare energy requirements for various orbits and not as an absolute energy requirement. Indeed, if one is interested in the energy necessary to place an earth satellite in orbit, then the reference position for the potential energy must be the surface of the earth, at a distance R from the earth's center. This adds the constant μ/R to the total energy, Eq. (3.79), so that E is no longer negative.

Observing that $e=0$ and $e=1$ are merely two values out of an entire range of values of e, we conclude that circles and parabolas are really limiting cases of ellipses and hyperbolas, respectively. We shall discuss the various orbits separately.

1. *Circle:* $e=0$, $E=-\mu^2/2h^2$. For zero eccentricity, Eq. (3.85) yields the orbit equation $r=r_c=p=\text{const}$, where r_c is the radius of the circular orbit. In a circular orbit, the gravitational force and the centrifugal force are in balance, so that the radial velocity component is zero and the transverse component is simply equal to the circular velocity v_c. The circular velocity can be obtained from Eq. (3.79). Indeed, letting $\dot{r}=0$, $r\dot{\theta}=v_c$, and $E=-\mu^2/2h^2$ in Eq. (3.79), we obtain

$$\frac{1}{2}v_c^2-\frac{\mu}{r_c}=-\frac{\mu^2}{2h^2} \tag{3.86}$$

But, from Eqs. (3.84) and (3.85), we conclude that

$$h^2=\mu r_c \tag{3.87}$$

so that, inserting Eq. (3.87) into Eq. (3.86), we obtain the circular velocity

$$v_c=\sqrt{\mu/r_c} \tag{3.88}$$

Defining the period T as the time necessary for one complete revolution, $\theta=2\pi$, integrating $dt=(r_c/v_c)\,d\theta$ over one period and using Eq. (3.88), we obtain

$$T=\frac{2\pi r_c}{v_c}=2\pi\frac{r_c^{3/2}}{\mu^{1/2}} \tag{3.89}$$

2. *Ellipse:* $0<e<1$, $-\mu^2/2h^2<E<0$. Because this is a closed orbit, the motion is periodic. Denoting by r_p and r_a the pericenter and apocenter, respectively, and letting $\theta=0$ and $\theta=\pi$ in Eq. (3.75), we obtain

$$r_p=\frac{p}{1+e\cos 0}=\frac{p}{1+e} \tag{3.90a}$$

$$r_a=\frac{p}{1+e\cos\pi}=\frac{p}{1-e} \tag{3.90b}$$

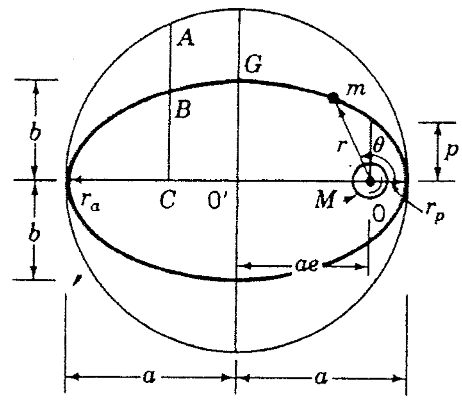


FIGURE 3.13

A typical ellipse is shown in Fig. 3.13, in which r_p, r_a, e, and p are identified. Another quantity of interest is the *semimajor axis* given by

$$a=\tfrac{1}{2}(r_p+r_a) \tag{3.91}$$

Using Eqs. (3.90), the semimajor axis is simply

$$a=\frac{1}{2}\left(\frac{p}{1+e}+\frac{p}{1-e}\right)=\frac{p}{1-e^2} \tag{3.92}$$

Moreover, recalling Eqs. (3.82) and (3.84), we obtain

$$a=\frac{h^2}{\mu}\left(-\frac{\mu^2}{2Eh^2}\right)=-\frac{\mu}{2E} \tag{3.93}$$

To construct the ellipse, we recall from analytic geometry that the ellipse is defined as the locus of points for which the sum of the distances from the two foci of the ellipse is constant and equal to $2a$. A simpler way of constructing the ellipse is by recognizing that the ellipse and the circle are related. Indeed, the relation between the ellipse and the circle is defined by the ratio (see Fig. 3.13)

$$\overline{BC}/\overline{AC}=b/a=\text{const} \tag{3.94}$$

where b is the ellipse *semiminor axis*. If we consider Pythagoras's theorem for the triangle $00'G$, we conclude that the semiminor axis is

$$b=\sqrt{a^2-(ae)^2}=a\sqrt{1-e^2} \tag{3.95}$$

so that

$$\overline{BC}/\overline{AC}=\sqrt{1-e^2} \tag{3.96}$$

The importance of the ellipse in planetary motion is underscored by *Kepler's first law* which states: *Each planet revolves in an elliptic orbit about the sun at one focus.*

A quantity of particular interest is the orbital period T, which can be obtained from Eq. (3.71) in the form

$$T=\int_0^{2\pi}\frac{r^2}{h}\,d\theta \tag{3.97}$$

Introducing Eqs. (3.84), (3.85), and (3.92) into Eq. (3.97), we can write

$$\begin{aligned} T &= \int_0^{2\pi} \frac{1}{h} \frac{p^2\, d\theta}{(1+e\cos\theta)^2} = \frac{2a^{3/2}(1-e^2)^{3/2}}{\mu^{1/2}} \int_0^{\pi} \frac{d\theta}{(1+e\cos\theta)^2} \\ &= \frac{2a^{3/2}(1-e^2)^{3/2}}{\mu^{1/2}} \left[\frac{-\sin\theta}{1+e\cos\theta} + \frac{2}{(1-e)^{1/2}} \tan^{-1} \frac{(1-e)^{1/2}\tan\theta/2}{1+e} \right]_0^{\pi} \\ &= 2\pi \frac{a^{3/2}}{\mu^{1/2}} \end{aligned} \tag{3.98}$$

Equation (3.98) is the mathematical statement of *Kepler's third law: The squares of the periodic times of the planets are proportional to the cubes of the semimajor axes of the ellipses.* As we shall see later, this law is only approximately valid.

Finally, we wish to calculate the velocity at any point on the ellipse. From Eqs. (3.77), (3.79), and (3.93), we obtain simply

$$v^2 = (\dot{r}^2 + r^2\dot{\theta}^2) = 2(E - V) = \mu\left(\frac{2}{r} - \frac{1}{a}\right) \tag{3.99}$$

Clearly, the highest velocity is at the pericenter and the lowest at the apocenter.

Note that Eqs. (3.96) and (3.97) are also valid for the circle if one simply replaces both a and r by r_c.

An interesting question arises as to what happens if a satellite expected to be placed in a circular earth orbit fails to achieve the circular velocity v_c, as given by Eq. (3.88). In this case, the satellite goes into a subcircular elliptic orbit, with the launching point playing the role of apogee instead of the expected perigee. Of course, if $r = r_a$ and $v = v_a$ are not large enough, the orbit will intersect the earth's surface (which is another way of saying that the satellite will crash or will burn up because of atmospheric friction).

3. *Parabola:* $e = 1$, $E = 0$. The parabola is the open orbit (Fig. 3.14) requiring the least amount of energy. We shall not go into the geometry of the parabola, but concentrate our attention on the velocity required to achieve parabolic

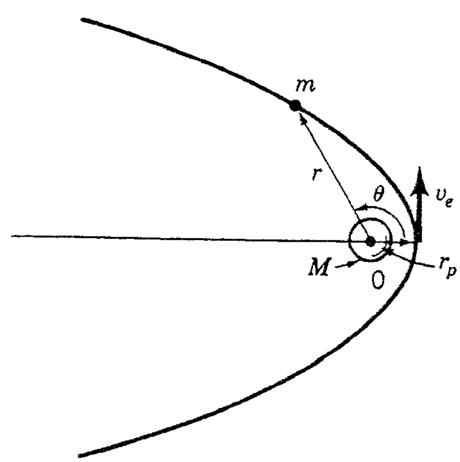


FIGURE 3.14

orbit. Recalling that for a parabolic orbit $E=0$, we can use Eq. (3.79) to write

$$\frac{1}{2}v^2-\frac{\mu}{r}=0 \tag{3.100}$$

where $v=\sqrt{\dot{r}^2+r^2\dot{\theta}^2}$ is the magnitude of the velocity vector. Hence, from Eq. (3.100), the velocity in a parabolic orbit is

$$v=\sqrt{2\mu/r} \tag{3.101}$$

It is obvious from Eq. (3.101) that the velocity tends to zero as the satellite approaches infinity. Denoting the velocity at the pericenter r_p by v_e, Eq. (3.101) yields

$$v_e=\sqrt{2\mu/r_p} \tag{3.102}$$

Because this is the velocity necessary to escape the gravitational pull of the massive particle M, v_e is referred to as the *escape velocity*, which explains the subscript e. Clearly, v_e is the minimum velocity for which an open orbit is achieved. In the case of an earth satellite, it is the velocity that must be imparted to the satellite at perigee to escape the gravitational pull of the earth.

Note that very elongated (high eccentricity) ellipses can be mistaken for parabolas, which had been the case with the orbit of the comet Halley. The orbit is an elongated ellipse with the sun at one focus and having a period of close to 76 years.

4. *Hyperbola:* $e>1$, $E>0$. If the velocity at the pericenter is larger than v_e, then the resulting orbit is a hyperbola. Because in this case the energy E is higher than the minimum required for an open orbit, the hyperbola is known as a high-energy orbit. Letting $r_0=\infty$ in Eq. (3.77), we observe that $V\to 0$ as $r\to\infty$. But, because $E>0$, which implies that $T>0$, there will be some residual velocity as $r\to\infty$. The velocity at infinity is called the *hyperbolic excess velocity* and characterizes the energy of a hyperbolic orbit.

Example 3.8

A satellite is launched in an earth orbit. At injection in orbit, the satellite is at an altitude of $H=2\times10^5$ m, and its velocity is parallel to the earth surface and has the magnitude $v=8\times10^3$ m/s. Determine the following: (1) the orbit eccentricity, (2) the semimajor axis, and (3) the orbital period. Use the following data: $M=5.9774\times10^{24}$ kg, $G=6.6685\times10^{-11}$ m^3/kg·s^2 and $R=6.3781\times10^6$ m, where M is the mass of the earth, G is the universal gravitational constant, and R is the radius of the earth.

Because the velocity at injection is parallel to the earth surface, the injection point corresponds to the perigee. (Note that the velocity is sufficiently high to achieve orbit.) Hence, we have

$$\begin{aligned} r_p&=R+H=(6.3781+0.2)10^6\text{ m}=6.5781\times10^6\text{ m}\\ v_p&=v=8\times10^3\text{ m/s}\end{aligned} \tag{a}$$

To calculate the eccentricity e, we observe from Eq. (3.82) that we need the total energy E, the specific angular momentum h, and $\mu = GM$. First, let us calculate

$$\mu = GM = 6.6685 \times 10^{-11} \times 5.9774 \times 10^{24} = 3.9860 \times 10^{14} \quad \text{m}^3/\text{s}^2 \tag{b}$$

Because $t = 0$ at perigee, from Eq. (3.79), we can write

$$\begin{aligned} E &= \frac{1}{2} v_\text{p}^2 - \frac{\mu}{r_\text{p}} = \frac{1}{2} \times 8^2 \times 10^6 - \frac{3.9860 \times 10^{14}}{6.5781 \times 10^6} \\ &= -2.8595 \times 10^7 \quad \text{m}^2/\text{s}^2 \end{aligned} \tag{c}$$

Moreover, the specific angular momentum is

$$h = r_\text{p} v_\text{p} = 6.5781 \times 10^6 \times 8 \times 10^3 = 5.2625 \times 10^{10} \quad \text{m}^2/\text{s} \tag{d}$$

Using Eq. (3.82), we obtain

$$\begin{aligned} e &= \sqrt{1 + \frac{2Eh^2}{\mu^2}} = \sqrt{1 - \frac{2 \times 2.8595 \times 10^7 \times (5.2625 \times 10^{10})^2}{(3.9860 \times 10^{14})^2}} \\ &= 0.0561 \end{aligned} \tag{e}$$

from which we conclude that the orbit is elliptical.

The semimajor axis can be calculated by means of Eq. (3.93). The result is

$$a = -\frac{\mu}{2E} = \frac{3.9860 \times 10^{14}}{2 \times 2.8595 \times 10^7} = 6.9697 \times 10^6 \quad \text{m} \tag{f}$$

Finally, from Eq. (3.98) the orbital period is

$$\begin{aligned} T &= 2\pi \sqrt{\frac{a^3}{\mu}} = 2\pi \sqrt{\frac{(6.9697 \times 10^6)^3}{3.9860 \times 10^{14}}} \\ &= 5.7907 \times 10^3 \quad \text{s} = 1.6085 \quad \text{h} \end{aligned} \tag{g}$$

PROBLEMS

3.1 A bullet of mass 10^{-2} kg leaves the gun barrel with a velocity of 0.8 km/s. If the firing is known to create a force on the bullet having a half-sine form of amplitude 7500 N, determine the duration of the force.

3.2 A pendulum consists of a rigid rod of length L and a bob of mass m (Fig. 3.15). The plane of the pendulum is made to rotate with the constant angular velocity Ω about a vertical axis through 0. Determine the forces on the bob and the torque on the rotating shaft. Note that the angle θ can vary with time, $\theta = \theta(t)$.

3.3 Determine the energy dissipation in the system of Example 3.4.

3.4 The bob of a pendulum of mass $m = 5$ kg and length $L = 2$ m is released from rest in a position defined by the angle $\theta_0 = 60°$ with respect to the vertical. Assuming that the string is inextensible, (1) determine the tension in the

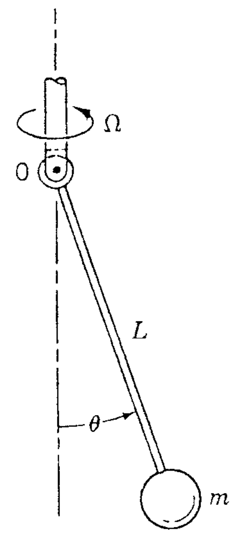


FIGURE 3.15

string when the bob is in the positions $\theta_1 = 30°$ and $\theta_2 = 0°$ and (2) calculate the angular impulse about the point of support between the times $t = t_0 = 0$ and $t = t_2$, where t_2 is the time corresponding to $\theta = \theta_2$.

3.5 An airplane goes into a dive. If at the bottom of the dive the airplane flies in a curve of radius $R = 2.5$ km with the velocity 900 km/h, calculate the force exerted by the seat on the pilot.

3.6 An airplane flying at 1000 km/h must make a circular turn of radius $R = 20$ km. At what angle relative to the plane of motion should the pilot bank the airplane? Assume that the airplane is flying in a horizontal plane.

3.7 A highway makes a bend with a radius of curvature $R = 100$ m. Determine the angle of the bank so as to permit a safe speed of 90 km/h if the coefficient of friction between the tires of a vehicle and the road is 0.2.

3.8 Determine the energy required to place a 1000-kg satellite in a circular orbit about the earth at an altitude of 300 km above the earth's surface. Calculate the period of the orbit.

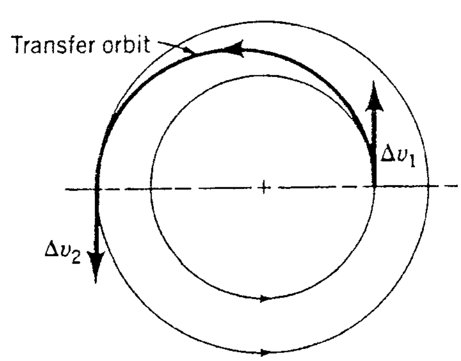


FIGURE 3.16

Pertinent data are as follows:

The universal gravitation constant $G = 6.6685 \times 10^{-11}$ m^3/kg$\cdot$s^2.

The mass of the earth $M = 5.9774 \times 10^{24}$ kg.

The radius of the earth $R = 6.3781 \times 10^6$ m.

3.9 A synchronous satellite is one that maintains a fixed position relative to the rotating earth. Find the altitude of an earth synchronous orbit.

3.10 Find the energy required to transfer the satellite of Problem 3.8 from the 300-km circular orbit to a 400-km circular orbit. The transfer is to be performed by means of two impulses, the first placing the satellite in an elliptic transfer orbit and the second placing the satellite in the modified orbit, as shown in Fig. 3.16.