CHAPTER 5
Dynamics of Systems of Particles

5.1 INTRODUCTION

Newton's second law was postulated for a single particle. Many physical phenomena can be explained by means of a model consisting of a single particle, but many more phenomena require a more elaborate model. We recall that, as a first approximation, planets and satellites can be treated as single particles moving around a fixed center of force. A more accurate theory reveals that the motion is really that of two particles moving around their common center of mass. In fact, an even more accurate theory would take into account the perturbing effect of more distant bodies, such as the moon, the other planets, or the sun, as the case may be. Many other dynamical systems can be regarded as systems of particles, including such seemingly disparate ones as rigid bodies and variable-mass systems.

In this chapter, we propose to extend many of the concepts and principles derived in Chapter 3 for a single particle to systems of particles. As applications, we consider the two-body problem and variable-mass systems. The first is a very important problem in dynamics, since two-body systems are encountered frequently. Examples of two-body systems are the earth and the sun, the earth and the moon, and the earth and a man-made satellite. Variable-mass systems are also very important. Indeed, the rocket engine is a prime example of a variable-mass system. The motion of rigid bodies is treated separately in the following chapter.

5.2 THE EQUATION OF MOTION FOR A SYSTEM OF PARTICLES

Let us consider the system of particles shown in Fig. 5.1 and denote the mass of a typical particle by $m_i\,(i=1,2,\ldots,n)$. For convenience, we distinguish between forces external and internal to the system and denote them by $\mathbf{F}_i$ and $\mathbf{f}_i$, respectively. The internal force is the resultant of the interaction forces $\mathbf{f}_{ij}$ exerted by the particles $m_j\,(j=1,2,\ldots,n, j\neq i)$ on the particle m_i (Fig. 5.2), or

$$\mathbf{f}_i=\sum_{j=1}^{n}\mathbf{f}_{ij} \tag{5.1}$$

Hence, according to Newton's second law, the equation of motion for particle m_i is

$$\mathbf{F}_i+\sum_{j=1}^{n}\mathbf{f}_{ij}=m_i\ddot{\mathbf{r}}_i=m_i\mathbf{a}_i \tag{5.2}$$

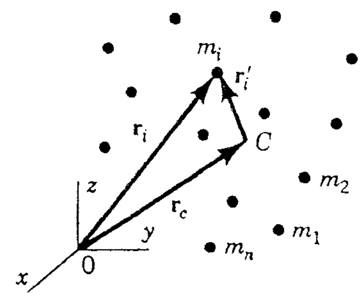


FIGURE 5.1

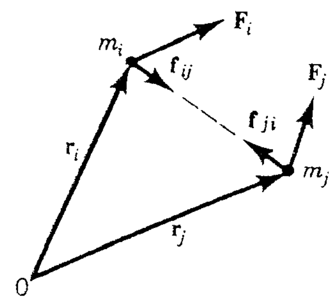


FIGURE 5.2

where $\ddot{\mathbf{r}}_i = \mathbf{a}_i$ is the acceleration of particle m_i relative to the inertial space xyz. Moreover, it is understood that the summation in Eq. (5.2) excludes the term for which $j = i$, as there is no corresponding force.

The equation of motion for the system of particles is obtained by extending Eq. (5.2) over the entire system of particles and then summing the corresponding equations. The result is

$$\sum_{i=1}^{n} \mathbf{F}_i + \sum_{i=1}^{n} \sum_{j=1}^{n} \mathbf{f}_{ij} = \sum_{i=1}^{n} m_i \ddot{\mathbf{r}}_i = \sum_{i=1}^{n} m_i \mathbf{a}_i \tag{5.3}$$

But, by Newton's third law,

$$\mathbf{f}_{ij} = -\mathbf{f}_{ji} \tag{5.4}$$

so that

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \mathbf{f}_{ij} = \mathbf{0} \tag{5.5}$$

because the internal forces cancel out in pairs. Note that Eq. (5.4) carries the implication that the forces $\mathbf{f}_{ij}$ pass through the particles m_i and m_j, which implies further that there are no internal torques. Introducing the resultant of the external force

$$\mathbf{F} = \sum_{i=1}^{n} \mathbf{F}_i \tag{5.6}$$

and considering Eq. (5.5), we obtain the equation of motion for the system of particles in the form

$$\mathbf{F} = \sum_{i=1}^{n} m_i \ddot{\mathbf{r}}_i = \sum_{i=1}^{n} m_i \mathbf{a}_i \tag{5.7}$$

Equation (5.7) is not very convenient for studying the motion of the system of particles, and in the next section we seek a more suitable form.

5.3 EQUATION OF MOTION IN TERMS OF THE MASS CENTER

Equation (5.7) represents a relation between the resultant force acting on the system of particles and the motion of the individual particles in the system. In many instances, however, the interest lies not in the motion of the individual particles but in an average motion of the system. In this section, we propose to derive an equation capable of describing such an average motion.

Let us consider the system of particles shown in Fig. 5.1 and denote by m the total mass of the system, so that

$$m = \sum_{i=1}^{n} m_i \tag{5.8}$$

The absolute position of the particle m_i is given by the radius vector $\mathbf{r}_i$ from the origin 0 of the inertial frame xyz to the particle m_i. The *center of mass* C of the system under consideration is defined as a point in space representing a *weighted average position of the system*, where the weighting factor for each particle is the mass of the particle. Denoting the radius vector from 0 to C by $\mathbf{r}_C$, the mathematical definition of the center of mass is given by

$$\mathbf{r}_C = \frac{1}{m} \sum_{i=1}^{n} m_i \mathbf{r}_i \tag{5.9}$$

Note that the point C is not necessarily a material point. Quite often, it is convenient to measure the motion not from the fixed point 0 but from the mass center C. Denoting the position of m_i relative to C by $\mathbf{r}'_i$ (Fig. 5.1), we can write

$$\mathbf{r}_i = \mathbf{r}_C + \mathbf{r}'_i \tag{5.10}$$

Introducing Eq. (5.10) into Eq. (5.9), we obtain

$$\mathbf{r}_C = \frac{1}{m} \sum_{i=1}^{n} m_i (\mathbf{r}_C + \mathbf{r}'_i) = \mathbf{r}_C + \frac{1}{m} \sum_{i=1}^{n} m_i \mathbf{r}'_i \tag{5.11}$$

which yields

$$\sum_{i=1}^{n} m_i \mathbf{r}'_i = \mathbf{0} \tag{5.12}$$

Hence, an equivalent definition of the mass center C is a point in space such that if the position of each particle is measured relative to that point, then the weighted average position is zero.

From Eq. (5.9), we obtain immediately the absolute velocity of the mass center

$$\mathbf{v}_C = \dot{\mathbf{r}}_C = \frac{1}{m} \sum_{i=1}^{n} m_i \dot{\mathbf{r}}_i = \frac{1}{m} \sum_{i=1}^{n} m_i \mathbf{v}_i \tag{5.13}$$

and the absolute acceleration of the mass center

$$\mathbf{a}_C = \ddot{\mathbf{r}}_C = \frac{1}{m} \sum_{i=1}^{n} m_i \ddot{\mathbf{r}}_i = \frac{1}{m} \sum_{i=1}^{n} m_i \mathbf{a}_i \tag{5.14}$$

where $\mathbf{v}_i$ and $\mathbf{a}_i$ are the absolute velocity and acceleration of particle m_i, respectively. As a matter of interest, we note from Eq. (5.12) that

$$\sum_{i=1}^{n} m_i \dot{\mathbf{r}}'_i = \sum_{i=1}^{n} m_i \mathbf{v}'_i = \mathbf{0} \tag{5.15}$$

and

$$\sum_{i=1}^{n} m_i \ddot{\mathbf{r}}'_i = \sum_{i=1}^{n} m_i \mathbf{a}'_i = \mathbf{0} \tag{5.16}$$

where $\mathbf{v}'_i$ and $\mathbf{a}'_i$ are the velocity and acceleration of m_i relative to C, respectively.

Equation (5.14) permits us to derive an equation for the average motion of the system, which is taken as the motion of the mass center. Indeed, introducing Eq. (5.14) into Eq. (5.7), we obtain the desired equation in the form

$$\mathbf{F} = m\mathbf{a}_C \tag{5.17}$$

Equation (5.17) implies that the motion of the mass center of the system of particles is the same as the motion of a fictitious body equal in mass to the total mass m of the system, concentrated at the mass center, and being acted on by the resultant of the external forces.

5.4 LINEAR MOMENTUM

Equation (5.17) can be written in a different form. To this end, let us introduce the linear momentum of m_i

$$\mathbf{p}_i = m_i \mathbf{v}_i \tag{5.18}$$

Then, if we consider Eq. (5.13), we can write the linear momentum for the system of particles in the form

$$\mathbf{p} = \sum_{i=1}^{n} \mathbf{p}_i = \sum_{i=1}^{n} m_i \mathbf{v}_i = m\mathbf{v}_C \tag{5.19}$$

The rate of change of the linear momentum is

$$\dot{\mathbf{p}} = m\dot{\mathbf{v}}_C = m\mathbf{a}_C \tag{5.20}$$

so that, comparing Eqs. (5.17) and (5.20), we conclude that

$$\mathbf{F} = \dot{\mathbf{p}} \tag{5.21}$$

or the resultant of the external forces acting on the system of particles is equal to the time rate of change of the system linear momentum.

Multiplying Eq. (5.21) by dt, integrating, and using a procedure analogous to that for a single particle (Section 3.4), we can write

$$\hat{\mathbf{F}} = \Delta \mathbf{p} \tag{5.22}$$

where $\hat{\mathbf{F}} = \int_{t_1}^{t_2} \mathbf{F}\, dt$ is the resultant of the external impulses on all particles and $\Delta \mathbf{p} = \mathbf{p}(t_2) - \mathbf{p}(t_1)$ is the change in the system linear momentum between the instants t_1 and t_2.

If $\mathbf{F} = \mathbf{0}$, Eq. (5.21) yields

$$\mathbf{p} = \text{const} \tag{5.23}$$

which states that in the absence of external forces the linear momentum of the system of particles remains constant. This statement represents the principle of *conservation of linear momentum* for a system of particles.

Example 5.1

A man is at rest at the rear end of a barge approaching the shore with the velocity v_0 relative to the shore, when he decides to walk toward the front end. If his velocity relative to the barge is uniform and equal to v_1, determine the velocity of the barge and plot it as a function of time. The mass of the man is m, that of the barge is M, and the length of the barge is L.

Ignoring the water friction, we conclude that the linear momentum is conserved. Denoting by v_2 the velocity of the barge while the man is walking (Fig. 5.3), we have

$$(m+M)v_0 = m(v_1 + v_2) + Mv_2 \tag{a}$$

which yields

$$v_2 = \frac{(m+M)v_0 - mv_1}{m+M} \tag{b}$$

Of course, when the man reaches the front of the barge, the velocity of the barge becomes once again v_0.

The velocity of the barge as a function of time is shown in Fig. 5.4, where t_0 denotes the time when the man beings his walk. Note that the shaded area in Fig. 5.4 is simply equal to the distance the barge is set back by the man's change in position relative to the barge. (Prove this statement.)

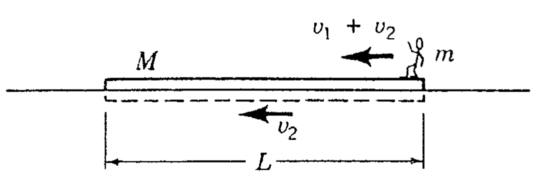


FIGURE 5.3

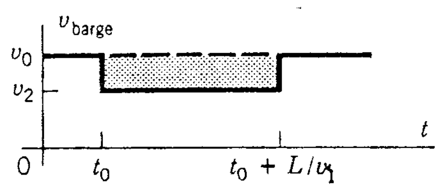


FIGURE 5.4

5.5 ANGULAR MOMENTUM

Next, let us define the angular momentum of the particle m_i about the fixed point 0 as

$$\mathbf{H}_{0i} = \mathbf{r}_i \times \mathbf{p}_i = \mathbf{r}_i \times m_i \mathbf{v}_i \tag{5.24}$$

so that the angular momentum about 0 of the system of particles is

$$\mathbf{H}_0 = \sum_{i=1}^{n} \mathbf{H}_{0i} = \sum_{i=1}^{n} \mathbf{r}_i \times m_i \mathbf{v}_i \tag{5.25}$$

Considering Eq. (5.2), we can write

$$\begin{aligned}\dot{\mathbf{H}}_0 &= \sum_{i=1}^{n} \dot{\mathbf{r}}_i \times m_i \mathbf{v}_i + \sum_{i=1}^{n} \mathbf{r}_i \times m_i \dot{\mathbf{v}}_i = \sum_{i=1}^{n} \mathbf{r}_i \times m_i \mathbf{a}_i \\ &= \sum_{i=1}^{n} \mathbf{r}_i \times \left(\mathbf{F}_i + \sum_{j=1}^{n} \mathbf{f}_{ij}\right) = \sum_{i=1}^{n} \mathbf{r}_i \times \mathbf{F}_i\end{aligned} \tag{5.26}$$

where we took into account that $\sum \dot{\mathbf{r}}_i \times m_i \mathbf{v}_i = \sum m_i \mathbf{v}_i \times \mathbf{v}_i = \mathbf{0}$ by the definition of the vector product and that

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \mathbf{r}_i \times \mathbf{f}_{ij} = \mathbf{0} \tag{5.27}$$

Equation (5.27) is true because $\mathbf{f}_{ij} = -\mathbf{f}_{ji}$ and, in addition, the two vectors are collinear (see Fig. 5.2), so that $\mathbf{r}_i \times \mathbf{f}_{ij} = -\mathbf{r}_j \times \mathbf{f}_{ji}$. Recognizing that

$$\mathbf{M}_0 = \sum_{i=1}^{n} \mathbf{r}_i \times \mathbf{F}_i \tag{5.28}$$

is the moment about 0 of the external forces acting on the system of particles, we can rewrite Eq. (5.26) in the form

$$\mathbf{M}_0 = \dot{\mathbf{H}}_0 \tag{5.29}$$

or, the moment about the fixed point 0 of the external forces acting on the system of particles is equal to the time rate of change of the system angular momentum about the same fixed point.

By analogy with Eq. (5.22) for the linear momentum, we can use Eq. (5.29) to obtain

$$\hat{\mathbf{M}}_0 = \Delta \mathbf{H}_0 \tag{5.30}$$

where $\hat{\mathbf{M}}_0 = \sum \mathbf{r}_i \times \hat{\mathbf{F}}_i$ is the resultant of all the external angular impulses about the fixed point 0 and $\Delta \mathbf{H}_0$ is the change in the system angular momentum about 0.

If $\mathbf{M}_0 = \mathbf{0}$, Eq. (5.29) yields

$$\mathbf{H}_0 = \text{const} \tag{5.31}$$

which states that in the absence of external torques about the fixed point 0 the angular momentum of the system of particles about 0 remains constant. This statement is known as the *conservation of angular momentum about a fixed point.*

The above moment-angular momentum relations were derived for the fixed point 0. The question arises whether similar relations exist also for a moving point. The answer is affirmative, provided the moving point is the mass center of the system of particles. To show this, let us consider the angular momentum of the particle m_i about C. Because the radius vector from C to m_i is $\mathbf{r}'_i$, we have

$$\mathbf{H}_{Ci} = \mathbf{r}'_i \times \mathbf{p}_i = \mathbf{r}'_i \times m_i \mathbf{v}_i \tag{5.32}$$

so that the angular momentum about C of the system of particles is

$$\mathbf{H}_C = \sum_{i=1}^{n} \mathbf{H}_{Ci} = \sum_{i=1}^{n} \mathbf{r}'_i \times m_i \mathbf{v}_i \tag{5.33}$$

But, taking the time derivative of Eq. (5.10), we can write

$$\mathbf{v}_i = \mathbf{v}_C + \mathbf{v}'_i \tag{5.34}$$

Inserting Eq. (5.34) into Eq. (5.33) and considering Eq. (5.12), we obtain

$$\begin{aligned} \mathbf{H}_C = \sum_{i=1}^{n} \mathbf{r}'_i \times m_i(\mathbf{v}_C + \mathbf{v}'_i) &= \left(\sum_{i=1}^{n} m_i \mathbf{r}'_i \right) \times \mathbf{v}_C + \sum_{i=1}^{n} \mathbf{r}'_i \times m_i \mathbf{v}'_i \\ &= \sum_{i=1}^{n} \mathbf{r}'_i \times m_i \mathbf{v}'_i \end{aligned} \tag{5.35}$$

Introducing the definition

$$\mathbf{H}'_C = \sum_{i=1}^{n} \mathbf{r}'_i \times m_i \mathbf{v}'_i \tag{5.36}$$

where $\mathbf{H}'_C$ is known as an *apparent angular momentum* and represents the angular momentum of the system of particles as seen by an observer whose position coincides with that of C at all times, we have

$$\mathbf{H}_C = \mathbf{H}'_C \tag{5.37}$$

or the apparent angular momentum about C of a system of particles is equal to the actual angular momentum about C. This property is characteristic of the mass center and does not extend to any other moving point.

Next, let us consider Eqs. (5.33) and (5.34) and write

$$\begin{aligned}\dot{\mathbf{H}}_C &= \sum_{i=1}^{n} \dot{\mathbf{r}}'_i \times m_i \mathbf{v}_i + \sum_{i=1}^{n} \mathbf{r}'_i \times m_i \dot{\mathbf{v}}_i \\ &= \sum_{i=1}^{n} \mathbf{v}'_i \times m_i(\mathbf{v}_C + \mathbf{v}'_i) + \sum_{i=1}^{n} \mathbf{r}'_i \times m_i \mathbf{a}_i \\ &= \left(\sum_{i=1}^{n} m_i \mathbf{v}'_i\right) \times \mathbf{v}_C + \sum_{i=1}^{n} m_i \mathbf{v}'_i \times \mathbf{v}'_i + \sum_{i=1}^{n} \mathbf{r}'_i \times m_i \mathbf{a}_i = \sum_{i=1}^{n} \mathbf{r}'_i \times m_i \mathbf{a}_i \end{aligned} \tag{5.38}$$

where we recognize that $\sum m_i \mathbf{v}'_i = \mathbf{0}$ by virtue of Eq. (5.15). Inserting Eq. (5.2) into Eq. (5.38), we obtain

$$\dot{\mathbf{H}}_C = \sum_{i=1}^{n} \mathbf{r}'_i \times \left(\mathbf{F}_i + \sum_{j=1}^{n} \mathbf{f}_{ij}\right) = \sum_{i=1}^{n} \mathbf{r}'_i \times \mathbf{F}_i \tag{5.39}$$

where, consistent with Eq. (5.27), $\sum\sum \mathbf{r}'_i \times \mathbf{f}_{ij} = \mathbf{0}$. Recognizing that the right side of Eq. (5.39) is the moment of the external forces about C, or

$$\mathbf{M}_C = \sum_{i=1}^{n} \mathbf{r}'_i \times \mathbf{F}_i \tag{5.40}$$

we can rewrite Eq. (5.39) in the form

$$\mathbf{M}_C = \dot{\mathbf{H}}_C \tag{5.41}$$

or the moment of the external forces about C is equal to the time rate of change of the system angular momentum about C. Note that the same result is obtained if one uses the apparent angular momentum $\mathbf{H}'_C$ instead of the actual angular momentum $\mathbf{H}_C$. Moreover, using the analogy with Eq. (5.30), we have

$$\hat{\mathbf{M}}_C = \Delta \mathbf{H}_C \tag{5.42}$$

where the notation is obvious.

If $\mathbf{M}_C = \mathbf{0}$, then Eq. (5.41) yields

$$\mathbf{H}_C = \text{const} \tag{5.43}$$

or in the absence of external torques about C the system angular momentum about C remains constant, which is the statement of *conservation of angular momentum about the mass center.*

It should be stressed again that the simple relations between the moment and angular momentum, Eqs. (5.29) and (5.41), hold true only if the reference point is a fixed point or the center of mass. If the reference point for the moment and angular momentum is an arbitrary moving point, then an extra term generally appears in the relation.

Example 5.2

The system shown in Fig. 5.5 consists of three mass particles, $m_1 = 2m$, $m_2 = m_3 = 3m$, connected by rigid bars welded at point 0. Particle m_1 is struck by a force generating

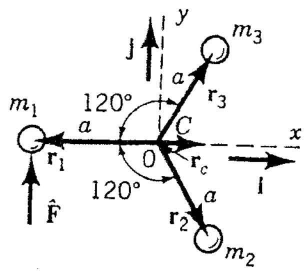


FIGURE 5.5

the impulse $\hat{\mathbf{F}}$, as shown. Calculate the velocity of each particle at the termination of the impulse. The system is initially at rest.

We solve the problem by working with the center of mass. First, we calculate the velocity of the mass center and then the velocity of each particle relative to the mass center. Because the initial linear momentum is zero, we can write from Eq. (5.22)

$$\hat{\mathbf{F}} = \hat{F}\mathbf{j} = \Delta\mathbf{p} = \mathbf{p} \tag{a}$$

where $\mathbf{p}$ is the linear momentum at the termination of the impulse, or

$$\mathbf{p} = \left(\sum_{i=1}^{3} m_i\right)\mathbf{v}_C = 8m\mathbf{v}_C \tag{b}$$

Comparing Eqs. (a) and (b), we obtain the velocity of the mass center

$$\mathbf{v}_C = \frac{\hat{F}}{8m}\mathbf{j} \tag{c}$$

Next, we wish to calculate the velocities of the particles relative to C. From Eq. (5.42), we can write

$$\hat{\mathbf{M}}_C = (\mathbf{r}_1 - \mathbf{r}_C)\times\hat{\mathbf{F}} = \Delta\mathbf{H}_C = \mathbf{H}_C \tag{d}$$

where $\mathbf{H}_C$ is the angular momentum about C. To calculate $\hat{\mathbf{M}}_C$ and $\mathbf{H}_C$, we need the position of the mass center. Assuming that point 0 coincides initially with the origin of the inertial system xy, we have

$$\mathbf{r}_C = \frac{\sum_{i=1}^{3} m_i\mathbf{r}_i}{\sum_{i=1}^{3} m_i} = \frac{1}{8m}\left[2m(-a\mathbf{i}) + 3m\left(\frac{a}{2}\mathbf{i} - \frac{\sqrt{3}}{2}a\mathbf{j}\right) + 3m\left(\frac{a}{2}\mathbf{i} + \frac{\sqrt{3}}{2}a\mathbf{j}\right)\right] = \frac{a}{8}\mathbf{i} \tag{e}$$

Denoting by $\boldsymbol{\omega} = \omega\mathbf{k}$ the angular velocity of the system following the impulse, the angular momentum about C is

$$\mathbf{H}_C = \mathbf{H}'_C = \sum_{i=1}^{3}\mathbf{r}'_i\times m_i\mathbf{v}'_i = \sum_{i=1}^{3}(\mathbf{r}_i - \mathbf{r}_C)\times m_i\mathbf{v}'_i = \sum_{i=1}^{3}(\mathbf{r}_i - \mathbf{r}_C)\times[m_i\boldsymbol{\omega}\times(\mathbf{r}_i - \mathbf{r}_C)]$$

$$= 2m\left(-a\mathbf{i} - \frac{a}{8}\mathbf{i}\right)\times\left[\omega\mathbf{k}\times\left(-a\mathbf{i} - \frac{a}{8}\mathbf{i}\right)\right]$$

$$+ 3m\left(\frac{a}{2}\mathbf{i} - \frac{\sqrt{3}a}{2}\mathbf{j} - \frac{a}{8}\mathbf{i}\right)\times\left[\omega\mathbf{k}\times\left(\frac{a}{2}\mathbf{i} - \frac{\sqrt{3}a}{2}\mathbf{j} - \frac{a}{8}\mathbf{i}\right)\right]$$

$$+3m\left(\frac{a}{2}\mathbf{i}+\frac{\sqrt{3}a}{2}\mathbf{j}-\frac{a}{8}\mathbf{i}\right)\times\left[\omega\mathbf{k}\times\left(\frac{a}{2}\mathbf{i}+\frac{\sqrt{3}a}{2}\mathbf{j}-\frac{a}{8}\mathbf{i}\right)\right]$$

$$=ma^2\omega\left\{2\left(\frac{9}{8}\right)^2+3\left[\left(\frac{3}{8}\right)^2+\left(-\frac{\sqrt{3}}{2}\right)^2\right]+3\left[\left(\frac{3}{8}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2\right]\right\}\mathbf{k}$$

$$=\frac{63}{8}ma^2\omega\mathbf{k} \tag{f}$$

so that Eq. (d) yields

$$\left(-a\mathbf{i}-\frac{a}{8}\mathbf{i}\right)\times\hat{F}\mathbf{j}=-\frac{9}{8}a\hat{F}\mathbf{k}=\frac{63}{8}ma^2\omega\mathbf{k} \tag{g}$$

from which we obtain

$$\omega=-\frac{1}{7}\frac{\hat{F}}{ma} \tag{h}$$

The velocities of the particles relative to C are

$$\mathbf{v}_1'=\omega\times(\mathbf{r}_1-\mathbf{r}_C)=-\frac{1}{7}\frac{\hat{F}}{ma}\mathbf{k}\times\left(-a\mathbf{i}-\frac{a}{8}\mathbf{i}\right)=\frac{9}{56}\frac{\hat{F}}{m}\mathbf{j}$$

$$\mathbf{v}_2'=\omega\times(\mathbf{r}_2-\mathbf{r}_C)=-\frac{1}{7}\frac{\hat{F}}{ma}\mathbf{k}\times\left(\frac{a}{2}\mathbf{i}-\frac{\sqrt{3}a}{2}\mathbf{j}-\frac{a}{8}\mathbf{i}\right)=-\frac{1}{56}(4\sqrt{3}\mathbf{i}+3\mathbf{j})\frac{\hat{F}}{m} \tag{i}$$

$$\mathbf{v}_3'=\omega\times(\mathbf{r}_3-\mathbf{r}_C)=-\frac{1}{7}\frac{\hat{F}}{ma}\mathbf{k}\times\left(\frac{a}{2}\mathbf{i}+\frac{\sqrt{3}a}{2}\mathbf{j}-\frac{a}{8}\mathbf{i}\right)=\frac{1}{56}(4\sqrt{3}\mathbf{i}-3\mathbf{j})\frac{\hat{F}}{m}$$

Combining Eqs. (c) and (i), we obtain the absolute velocities

$$\mathbf{v}_1=\mathbf{v}_C+\mathbf{v}_1'=\frac{\hat{F}}{8m}\mathbf{j}+\frac{9}{56}\frac{\hat{F}}{m}\mathbf{j}=\frac{2}{7}\frac{\hat{F}}{m}\mathbf{j}$$

$$\mathbf{v}_2=\mathbf{v}_C+\mathbf{v}_2'=\frac{\hat{F}}{8m}\mathbf{j}-\frac{1}{56}(4\sqrt{3}\mathbf{i}+3\mathbf{j})\frac{\hat{F}}{m}=-\frac{1}{14}(\sqrt{3}\mathbf{i}-\mathbf{j})\frac{\hat{F}}{m} \tag{j}$$

$$\mathbf{v}_3=\mathbf{v}_C+\mathbf{v}_3'=\frac{\hat{F}}{8m}\mathbf{j}+\frac{1}{56}(4\sqrt{3}\mathbf{i}-3\mathbf{j})\frac{\hat{F}}{m}=\frac{1}{14}(\sqrt{3}\mathbf{i}+\mathbf{j})\frac{\hat{F}}{m}$$

5.6 KINETIC ENERGY

Let us now derive an expression for the kinetic energy of a system of particles. The kinetic energy for particle m_i is simply

$$T_i=\tfrac{1}{2}m_i\dot{\mathbf{r}}_i\cdot\dot{\mathbf{r}}_i \tag{5.44}$$

so that the kinetic energy of the system of particles is

$$T=\sum_{i=1}^{n}T_i=\frac{1}{2}\sum_{i=1}^{n}m_i\dot{\mathbf{r}}_i\cdot\dot{\mathbf{r}}_i \tag{5.45}$$

Using Eq. (5.10) and recalling Eq. (5.15), we can write

$$
\begin{aligned}
T &= \frac{1}{2}\sum_{i=1}^{n} m_i(\dot{\mathbf{r}}_C+\dot{\mathbf{r}}_i')\cdot(\dot{\mathbf{r}}_C+\dot{\mathbf{r}}_i') \\
&= \frac{1}{2}\dot{\mathbf{r}}_C\cdot\dot{\mathbf{r}}_C\sum_{i=1}^{n} m_i+\dot{\mathbf{r}}_C\cdot\sum_{i=1}^{n} m_i\dot{\mathbf{r}}_i'+\frac{1}{2}\sum_{i=1}^{n} m_i\dot{\mathbf{r}}_i'\cdot\dot{\mathbf{r}}_i' \\
&= \frac{1}{2}m\dot{\mathbf{r}}_C\cdot\dot{\mathbf{r}}_C+\frac{1}{2}\sum_{i=1}^{n} m_i\dot{\mathbf{r}}_i'\cdot\dot{\mathbf{r}}_i' \qquad (5.46)
\end{aligned}
$$

Introducing the notation

$$T_{\mathrm{tr}}=\frac{1}{2}m\dot{\mathbf{r}}_C\cdot\dot{\mathbf{r}}_C \qquad (5.47)$$

$$T_{\mathrm{rel}}=\frac{1}{2}\sum_{i=1}^{n} m_i\dot{\mathbf{r}}_i'\cdot\dot{\mathbf{r}}_i' \qquad (5.48)$$

we obtain

$$T=T_{\mathrm{tr}}+T_{\mathrm{rel}} \qquad (5.49)$$

so that the kinetic energy of a system of particles can be separated into two parts, the first representing the kinetic energy that would be obtained if all the particles were translating with the velocity of the mass center and the second representing the kinetic energy due to motion relative to the mass center. This separation is possible only if the moving reference point is the mass center.

Example 5.3

Calculate the kinetic energy for the system of Example 5.2.

The kinetic energy is given by Eq. (5.49), where from Eq. (5.47),

$$T_{\mathrm{tr}}=\frac{1}{2}\left(\sum_{i=1}^{3} m_i\right)v_C^2=\frac{1}{2}8m\left(\frac{\hat{F}}{8m}\right)^2=\frac{1}{16}\frac{\hat{F}^2}{m} \qquad \text{(a)}$$

and, from Eq. (5.48),

$$
\begin{aligned}
T_{\mathrm{rel}}=\frac{1}{2}\sum_{i=1}^{3} m_i(v_i')^2=\frac{1}{2}\Bigg\{&2m\left(\frac{9}{56}\frac{\hat{F}}{m}\right)^2+3m\left(-\frac{1}{56}\right)^2[(4\sqrt{3})^2+3^2]\left(\frac{\hat{F}}{m}\right)^2 \\
&+3m\left(\frac{1}{56}\right)^2[(4\sqrt{3})^2+(-3)^2]\left(\frac{\hat{F}}{m}\right)^2\Bigg\}=\frac{151}{784}\frac{\hat{F}^2}{m} \qquad \text{(b)}
\end{aligned}
$$

5.7 THE TWO-BODY PROBLEM

Let us consider a system consisting of two particles moving under mutual attraction forces, with no external forces present. We propose to show that the system can be treated as a single particle under a central force.

Denoting by C the position of the mass center and by $\mathbf{r}_C$ the radius vector from

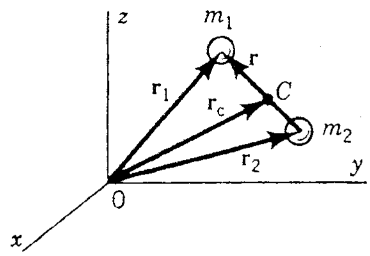


FIGURE 5.6

the fixed point 0 to C (Fig. 5.6), we can write

$$\mathbf{r}_C = \frac{m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2}{m_1 + m_2} \tag{5.50}$$

Then, denoting the radius vector from m_2 to m_1 by

$$\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2 \tag{5.51}$$

and eliminating $\mathbf{r}_2$ and $\mathbf{r}_1$ from Eq. (5.50) in sequence, we obtain

$$\mathbf{r}_1 = \mathbf{r}_C + \frac{m_2}{m_1 + m_2} \mathbf{r}, \qquad \mathbf{r}_2 = \mathbf{r}_C - \frac{m_1}{m_1 + m_2} \mathbf{r} \tag{5.52}$$

Letting $\mathbf{f}_{12}$ be the force exerted by m_2 on m_1 and $\mathbf{f}_{21}$ the force exerted by m_1 on m_2, we can write Newton's second law for the two particles in the form

$$\mathbf{f}_{12} = m_1 \ddot{\mathbf{r}}_1 = m_1 \ddot{\mathbf{r}}_C + \frac{m_1 m_2}{m_1 + m_2} \ddot{\mathbf{r}} \tag{5.53a}$$

$$\mathbf{f}_{21} = m_2 \ddot{\mathbf{r}}_2 = m_2 \ddot{\mathbf{r}}_C - \frac{m_1 m_2}{m_1 + m_2} \ddot{\mathbf{r}} \tag{5.53b}$$

Adding Eqs. (5.53) and recalling that $\mathbf{f}_{21} = -\mathbf{f}_{12}$, we conclude that

$$\ddot{\mathbf{r}}_C = \mathbf{0} \tag{5.54}$$

or, the mass center C moves with uniform velocity. This is consistent with the fact that in the absence of external forces the system linear momentum is conserved.

In view of Eq. (5.54), Eq. (5.53a) reduces to

$$\mathbf{f}_{12} = \frac{m_1 m_2}{m_1 + m_2} \ddot{\mathbf{r}} \tag{5.55}$$

which can be interpreted as describing the motion of a single body of equivalent mass $m_1 m_2/(m_1 + m_2)$. Because C is unaccelerated, it can be regarded as the origin of an inertial system. Moreover, because $\mathbf{f}_{12}$ passes through C at all times, it is a central force, or

$$\mathbf{f}_{12} = -f\mathbf{u}_r \tag{5.56}$$

where f is the magnitude of $\mathbf{f}_{12}$ and $\mathbf{u}_r$ is a unit vector along $\mathbf{r}$.

As a matter of interest, let us introduce the notation $m_1 = m$ and $m_2 = M$. Then, inserting Eq. (5.56) into Eq. (5.55), we obtain

$$\frac{mM}{m+M}\ddot{\mathbf{r}} = -f\mathbf{u}_r \tag{5.57}$$

If $M >> m$, Eq. (5.57) can be approximated by

$$m\ddot{\mathbf{r}} = -f\mathbf{u}_r \tag{5.58}$$

and, under the same assumption, it also follows from the second of Eqs. (5.52) that

$$\mathbf{r}_2 = \mathbf{r}_C \tag{5.59}$$

so that C coincides with the center of the massive particle M. This is essentially the justification for the assumptions made in the beginning of Section 3.8.

There are many two-body systems in nature. The sun and any one of the planets in the solar system form a two-body system. Of course, this implies that the effect of other planets is negligible, which is indeed the case. Another two-body system consists of the earth and the moon. More modern examples are the earth and any man-made satellite orbiting the earth.

Finally, we wish to reconsider Kepler's third law. To this end, we use Newton's inverse square law to write the radial component of Eq. (5.57) in the form

$$\frac{mM}{m+M}(\ddot{r} - r\dot{\theta}^2) = -\frac{GMm}{r^2} \tag{5.60}$$

Dividing Eq. (5.60) through by $mM/(m+M)$, we obtain

$$\ddot{r} - r\dot{\theta}^2 = -\frac{\mu}{r^2} \tag{5.61}$$

where

$$\mu = G(m+M) \tag{5.62}$$

Introducing Eq. (5.62) into Eq. (3.98), we conclude that the period for elliptic orbits is

$$T = 2\pi\frac{a^{3/2}}{[G(m+M)]^{1/2}} \tag{5.63}$$

so that the orbital period of a planet depends not only on the semimajor axis a, as stated by Kepler's third law, but also on the mass m, which varies from planet to planet. Because the mass of the sun is considerably larger than the mass of any planet in the solar system, the correction to the period given by Kepler's third law is relatively small.

Example 5.4

Let the earth and the sun constitute a two-body system and calculate the orbital period of the earth in two ways, first by using Eq. (3.98) and then by using Eq. (5.63).

Compare the results, and verify the statement made at the end of this section. Pertinent data are as follows:

Semimajor axis of the earth's orbit	$a = 1.49527 \times 10^{11}$ m
Universal gravitational constant	$G = 6.668462 \times 10^{-11}$ $\text{m}^3/(\text{kg}\cdot\text{s}^2)$
Mass of the sun	$M = 1.987323 \times 10^{30}$ kg
Mass of the earth	$m = 5.977414 \times 10^{24}$ kg

Using Eq. (3.98), we obtain

$$\begin{aligned} T &= 2\pi \frac{a^{3/2}}{(GM)^{1/2}} = 2\pi \frac{(1.49527 \times 10^{11})^{3/2}}{(6.668462 \times 1.987323 \times 10^{10})^{1/2}} \\ &= 3.15582 \times 10^{7} \quad \text{s} \\ &= 365.256949 \quad \text{days} \end{aligned} \tag{a}$$

On the other hand, using Eq. (5.63), we can write

$$\begin{aligned} T &= 2\pi \frac{a^{3/2}}{G^{1/2}(M+m)^{1/2}} = 2\pi \frac{a^{3/2}}{(GM)^{1/2}} \left(1 + \frac{m}{M}\right)^{-1/2} \\ &= 2\pi \frac{a^{3/2}}{(GM)^{1/2}} \left[1 - \frac{1}{2}\frac{m}{M} + \frac{3}{8}\left(\frac{m}{M}\right)^2 + \cdots\right] \\ &= 365.256949(1 - 1.50388249 \times 10^{-6}) = 365.2564 \quad \text{days} \end{aligned} \tag{b}$$

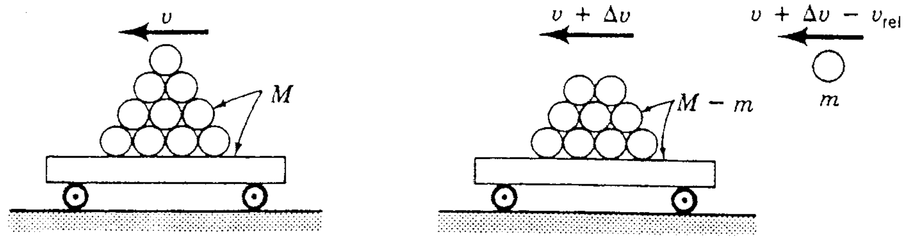
From Eqs. (a) and (b), we conclude that the difference in the two results is very small, thus corroborating the statement made at the end of this section. The difference is not insignificant, however. Indeed, the measured value of the orbital period of the earth is

$$T = 365.25636048 \quad \text{days} \tag{c}$$

so that the value given by Eq. (b) is appreciably closer to the measured value. The difference in the values given in (b) and (c) can be attributed to perturbation from various sources, such as the other planets in the solar system.

5.8 VARIABLE-MASS SYSTEMS. ROCKET MOTION

In Example 5.1, we presented an application of the conservation of linear momentum for a system consisting of two bodies subjected to internal forces alone. As an introduction to variable-mass systems, let us examine a similar system. Figure 5.7*a* shows a cart full of rocks traveling with velocity v at time t_1. Someone on the cart throws away one of the rocks in a direction opposite to that of the motion. As a result, at time t_2 a lighter cart is traveling with the velocity $v + \Delta v$, and the thrown rock is traveling with the velocity $v + \Delta v - v_{\text{rel}}$, as shown in Fig. 5.7*b*, where v_{rel} is the velocity of the rock relative to the cart. It is assumed that there are no forces between the ground and the cart. Because the force between the cart and the rock



(a) System at time t_1 (b) System at time t_2

FIGURE 5.7

is internal, the momentum is conserved, which permits us to write

$$Mv = (M - m)(v + \Delta v) + m(v + \Delta v - v_{\text{rel}}) \tag{5.64}$$

where M is the mass of the cart and the rocks at time t_1 and m is the mass of the thrown rock. Canceling terms in Eq. (5.64) and rearranging, we obtain

$$\Delta v = \frac{m v_{\text{rel}}}{M} \tag{5.65}$$

which is the velocity with which the cart is propelled forward by the force expelling the mass m. If rocks are being thrown off the cart repeatedly, then the cart will accelerate accordingly. This is essentially the principle of jet propulsion, the main difference being that in jet propulsion there is a continuous stream of particles being expelled at very high relative velocity.

Let us now consider a simple model simulating a rocket engine, as shown in Fig. 5.8. Assuming that there is an external force $F(t)$ acting on the system and letting Δt be a small time increment, the impulse–momentum principle yields

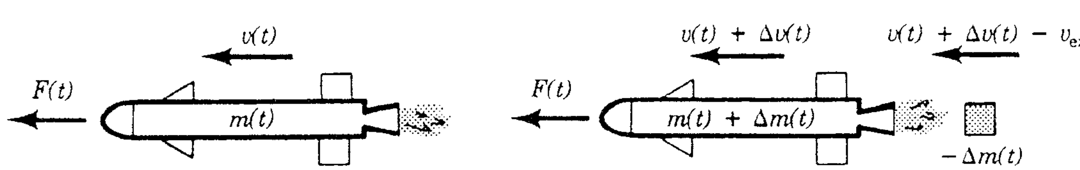
$$F(t)\,\Delta t = [m(t) + \Delta m(t)][v(t) + \Delta v(t)] - \Delta m[v(t) + \Delta v(t) - v_{\text{ex}}] - m(t)v(t) \tag{5.66}$$

where v_{ex} is the exhaust velocity, which is the velocity of the hot gases flowing relative to the engine. The exhaust velocity v_{ex} is a constant depending on the type of fuel. Canceling terms and rearranging, we conclude that Eq. (5.66) yields

$$F(t) = m(t)\frac{\Delta v(t)}{\Delta t} + \frac{\Delta m(t)}{\Delta t} v_{\text{ex}} \tag{5.67}$$

Letting $\Delta t \to 0$, we obtain the differential equation governing the rocket motion

$$F(t) = m(t)\frac{dv(t)}{dt} + \frac{dm(t)}{dt} v_{\text{ex}} \tag{5.68}$$



(a) System at time t (b) System at time $t + \Delta t$

FIGURE 5.8

Introducing the notation

$$-\frac{dm(t)}{dt} v_{ex} = T(t) \tag{5.69}$$

where $T(t)$ can be identified as the *engine thrust*, we can rewrite Eq. (5.68) in the form

$$T(t)+F(t)=m(t)\frac{dv(t)}{dt} \tag{5.70}$$

Note that in the above derivation we assumed that the rocket gains mass when in fact it loses mass. This fact is taken into account ordinarily by letting dm/dt be negative. Moreover, the force $F(t)$ is in general due to drag and gravity, so that $F(t)$ is usually negative and hence acting in a direction opposite to that shown in Fig. 5.8.

As an illustration, let us consider the vertical flight of a sounding rocket (Fig. 5.9) and calculate the altitude it will achieve at burnout (i.e., when the entire fuel is exhausted). Neglecting air resistance, we can assume that the force $F(t)$ is simply due to gravity, where g is assumed to be constant. Hence, the differential equation of motion is

$$-m(t)g=m(t)\frac{dv(t)}{dt}+\frac{dm(t)}{dt}v_{ex} \tag{5.71}$$

Rearranging Eq. (5.71) yields

$$dv=-g\,dt-v_{ex}\frac{dm}{m} \tag{5.72}$$

Letting $v(0)=v_0$, $m(0)=m_0$ and integrating Eq. (5.72), we obtain

$$v(t)=v_0-gt-v_{ex}\ln\frac{m}{m_0} \tag{5.73}$$

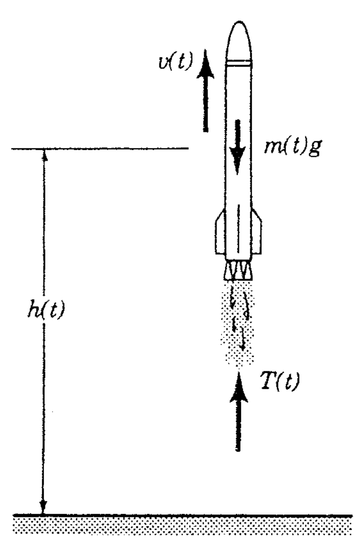


FIGURE 5.9

Denoting the burnout time by t_{bo} and the mass at burnout by m_{bo}, and assuming that mass is expelled at constant rate, we can write

$$m(t) = m_0 - \frac{m_0 - m_{bo}}{t_{bo}} t \tag{5.74}$$

so that Eq. (5.73) becomes

$$v(t) = v_0 - gt - v_{ex} \ln\left(1 - \frac{m_0 - m_{bo}}{m_0 t_{bo}} t\right) \tag{5.75}$$

But the velocity is the rate of change in altitude, $v(t) = dh(t)/dt$, so that integrating once again, we obtain the altitude at burnout

$$h(t_{bo}) = v_0 t_{bo} - \frac{1}{2} g t_{bo}^2 - v_{ex} \int_0^{t_{bo}} \ln\left(1 - \frac{m_0 - m_{bo}}{m_0 t_{bo}} t\right) dt \tag{5.76}$$

To evaluate the integral, let us introduce the change of variable

$$1 - \frac{m_0 - m_{bo}}{m_0 t_{bo}} t = \tau, \qquad dt = -\frac{m_0 t_{bo}}{m_0 - m_{bo}} d\tau \tag{5.77}$$

which calls for the following change in the limits of integration

$$t = 0 \rightarrow \tau = 1, \qquad t = t_{bo} \rightarrow \tau = m_{bo}/m_0 \tag{5.78}$$

Hence, the integral can be evaluated as follows:

$$\begin{aligned}
\int_0^{t_{bo}} \ln\left(1 - \frac{m_0 t_{bo}}{m_0 - m_{bo}} t\right) dt &= -\frac{m_0 t_{bo}}{m_0 - m_{bo}} \int_1^{m_{bo}/m_0} \ln \tau \, d\tau \\
&= -\frac{m_0 t_{bo}}{m_0 - m_{bo}} \tau(\ln \tau - 1) \Big|_1^{m_{bo}/m_0} \\
&= -\frac{m_0 t_{bo}}{m_0 - m_{bo}} \left[\frac{m_{bo}}{m_0}\left(\ln \frac{m_{bo}}{m_0} - 1\right) + 1\right] \\
&= -t_{bo} + \frac{m_{bo} t_{bo}}{m_0 - m_{bo}} \ln \frac{m_0}{m_{bo}}
\end{aligned} \tag{5.79}$$

so that the altitude at burnout is

$$h(t_{bo}) = v_0 t_{bo} - \frac{1}{2} g t_{bo}^2 + v_{ex} t_{bo} \left(1 - \frac{m_{bo}}{m_0 - m_{bo}} \ln \frac{m_0}{m_{bo}}\right) \tag{5.80}$$

PROBLEMS

5.1 A dumbell consisting of two particles of mass m_1 and m_2 and connected by a rigid link of length L is falling with the translational velocity v_0 and with zero rotation when m_2 strikes a ledge, as shown in Fig. 5.10. Assume that m_2 rebounds with the velocity v_2, and calculate the following: (1) the velocity v_1

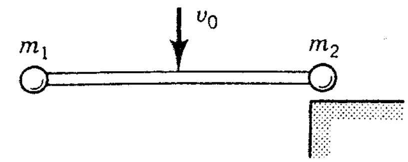


FIGURE 5.10

of m_1 immediately after impact and (2) the impulse $\hat{F}$ imparted to the dumbbell by the ledge.

5.2 A bullet of mass m is traveling with the velocity $\mathbf{v}_0 = 50\mathbf{i}$ m/s when it hits a rock of mass $2m$ lying at rest (Fig. 5.11*a*). As a result of the impact, the rock breaks into two fragments, one of mass $m/2$ traveling with a velocity of 30 m/s and in a direction making an angle of 30° with the vector $\mathbf{v}_0$ and the second of mass $3m/2$ traveling with a velocity of 20 m/s and in a direction making an angle of $-15°$ with $\mathbf{v}_0$, as shown in Fig. 5.11*b*. Find the velocity $\mathbf{v}_1$ of the bullet after impact by magnitude and direction, and calculate the energy lost during impact.

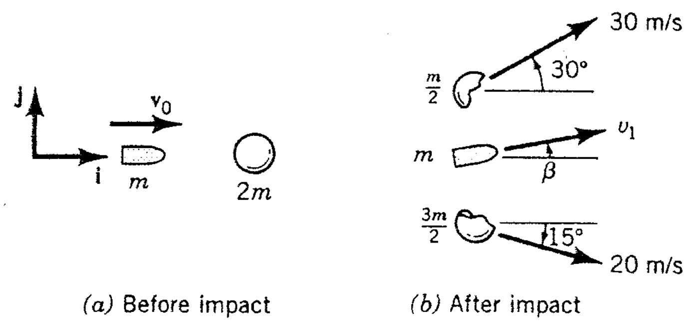


FIGURE 5.11

5.3 A bullet of mass m is fired with a velocity v_0 into a block of wood of mass M (Fig. 5.12). If the bullet becomes embedded in the block, calculate the maximum height reached by the block and the bullet.

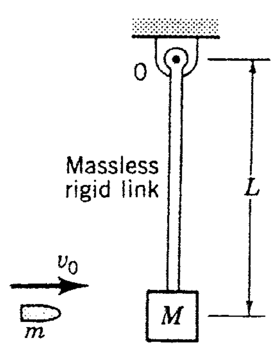


FIGURE 5.12

5.4 Two particles of mass m each are connected by an inextensible string of length L passing through a small hole at point 0, as shown in Fig. 5.13. Initially the string is held fixed at 0 while one of the particles moves on the frictionless

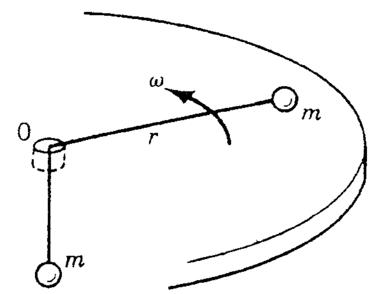


FIGURE 5.13

table in a circle of radius r_0 with the constant angular velocity ω_0. Upon release of the string, the hanging mass begins to fall. Find the minimum value of r and the maximum tension in the string.

5.5 A sounding rocket is launched vertically from the earth's surface. Assuming that the exhaust velocity is $v_{ex} = 2500$ m/s, the burnout time is $t_{bo} = 10$ s and the mass at burnout is 10% of the initial mass, calculate the maximum velocity and the maximum altitude reached by the rocket. The acceleration due to gravity can be assumed to be constant.