CONTROLLABILITY & OBSERVABILITY OF LINEAR SYSTEMS

Consider the linear system (A, B, C)

$$(*)\begin{cases} \dot{x} = Ax + Bu & \text{where } x \in \mathbb{R}^n \quad u \in \mathbb{R}^r \quad y \in \mathbb{R}^p \\ y = Cx & A \in \mathbb{R}^{n\times n} \quad B \in \mathbb{R}^{n\times r} \quad C \in \mathbb{R}^{n\times n} \end{cases}$$

Controllability of linear systems

A system is completely controllable if every state variable of the process can be affected or controlled to reach a certain objective in finite time by some unconstrained control $u(t)$.

Definition:

Consider $(*)$. The state $x(t)$ is said to be controllable at $t=t_0$ if $\exists$ a piecewise continuous $u(t)$ that will drive the state $x(t)$ to any final state $x(t_f)$ for a finite time $(t_f - t_0) \geq 0$.

If every state of the system is controllable in a finite time interval, the system is said to be completely state controllable or simply controllable.

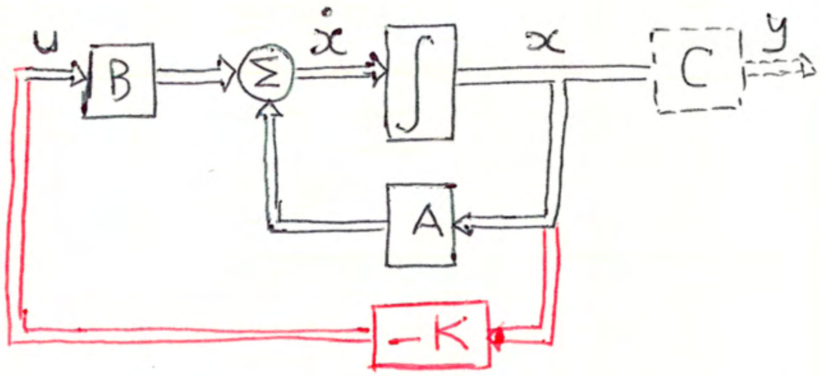
Theorem:

Consider $(*)$. It is completely state controllable iff the controllability matrix $\mathcal{C} = [B \; AB \; A^2B \; \dots \; A^{n-1}B] \in \mathbb{R}^{n\times(n\cdot r)}$ has rank n.

Note:

Pole Placement & Controllability:

$$(*)\begin{cases}\dot{x} = Ax + Bu \\ y = Cx\end{cases}$$



$$\left.\begin{array}{l}\dot{x} = Ax + Bu \\ \text{Let } u = -Kx\end{array}\right\} \Rightarrow \dot{x} = (A - BK)x \text{ : closed loop system}$$

If $[A, B]$ is controllable, then $\exists$ a constant feedback matrix K s.t. the eigenvalues of $(A - BK)$ "closed-loop poles" can be arbitrarily assigned.

Observability of Linear Systems

A system is completely observable if every state variable of the system affects some of the outputs.

Definition:

Given the linear time invariant system (*).
The state $x(t_0)$ is said to be observable if given any input $u(t)$, $\exists$ a finite time $t_f \geq t_0$ s.t. the knowledge of

- $u(t)$ for $t_0 \leq t \leq t_f$
- A, B, C
- the output $y(t)$ for $t_0 \leq t \leq t_f$

are sufficient to determine $x(t_0)$.
If every state of the system is observable for a finite time t_f, we say that the system is completely observable, or simply observable.

Theorem:

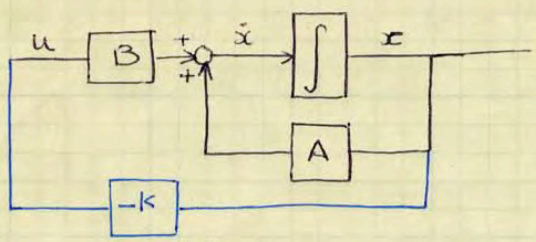
Consider (*). It is completely observable, or equivalently the pair $[A, C]$ is completely observable, iff the Observability matrix

$$\mathcal{O} = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix} \in \mathbb{R}^{(p \cdot n) \times n}$$

has rank n.

STATE SPACE DESIGN

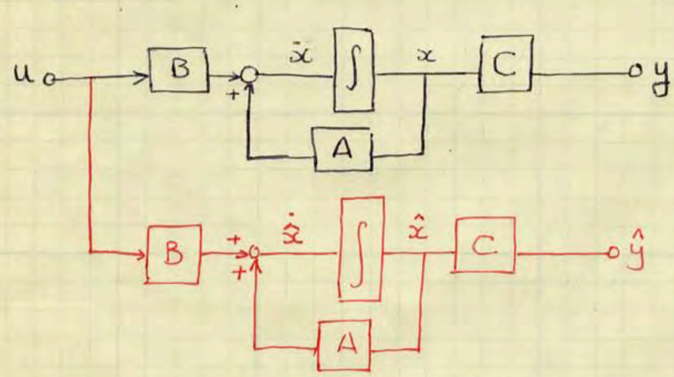
• Recall State Feedback and Pole Placement design



$$\left.\begin{aligned} \dot{x} &= Ax + Bu \\ u &= -kx \end{aligned}\right\} \overset{x(0)=x_0}{\Rightarrow} \quad \begin{aligned} \dot{x} &= Ax + B(-kx) \\ &= (A - BK)x. \end{aligned}$$

if $[A, B]$ is completely controllable $\Rightarrow$ arbitrary pole placement

• Open Loop Estimator



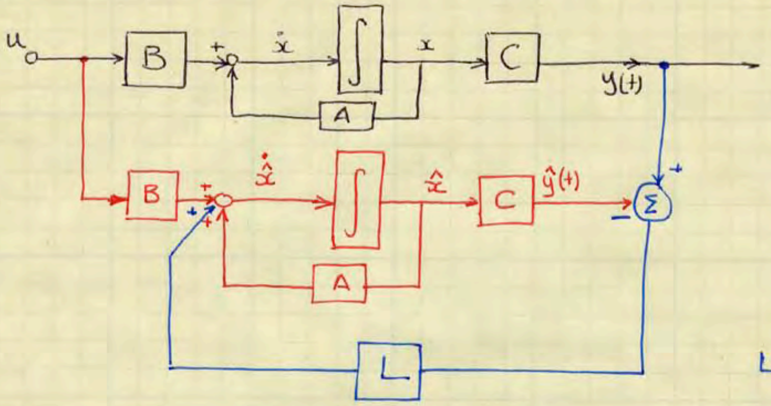
• If plant (A, B, C) are perfectly known & if $\hat{x}(0) = x(0)$ is known or close } Open loop estimator will be satisfactory.

Note: Let $\tilde{x} = x - \hat{x} \Rightarrow \dot{\tilde{x}} = \dot{x} - \dot{\hat{x}} = Ax + Bu - A\hat{x} - Bu$
$= A\tilde{x} \qquad \tilde{x}(0) = x(0) - \hat{x}(0)$

Issues ① if plant is stable (i.e. A Hurwitz) $\Rightarrow \tilde{x}$ (error in state) $\to 0$ as $t \to \infty$
② Can not influence the rate of convergence to zero.
③ $\tilde{x} \to 0$ at the same rate as the system natural dynamics

Invoke golden rule : $\boxed{\text{When in trouble, use Feedback}}$

- Closed-loop Estimator



L: gain matrix

$$\dot{x} = Ax + Bu$$

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - \hat{y})$$

$$= A\hat{x} + Bu + L(Cx - C\hat{x})$$

$$= A\hat{x} + Bu + LC(x - \hat{x})$$

$$\Rightarrow \quad \dot{\tilde{x}} = \dot{x} - \dot{\hat{x}} = Ax + Bu - A\hat{x} - Bu - LC\tilde{x}$$

$$= A\tilde{x} - LC\tilde{x}$$

$$= (A - LC)\tilde{x} \qquad \tilde{x}(0) = x(0) - \hat{x}(0)$$

- if $[A, C]$ is completely observable $\Rightarrow$ arbitrary observer pole placement

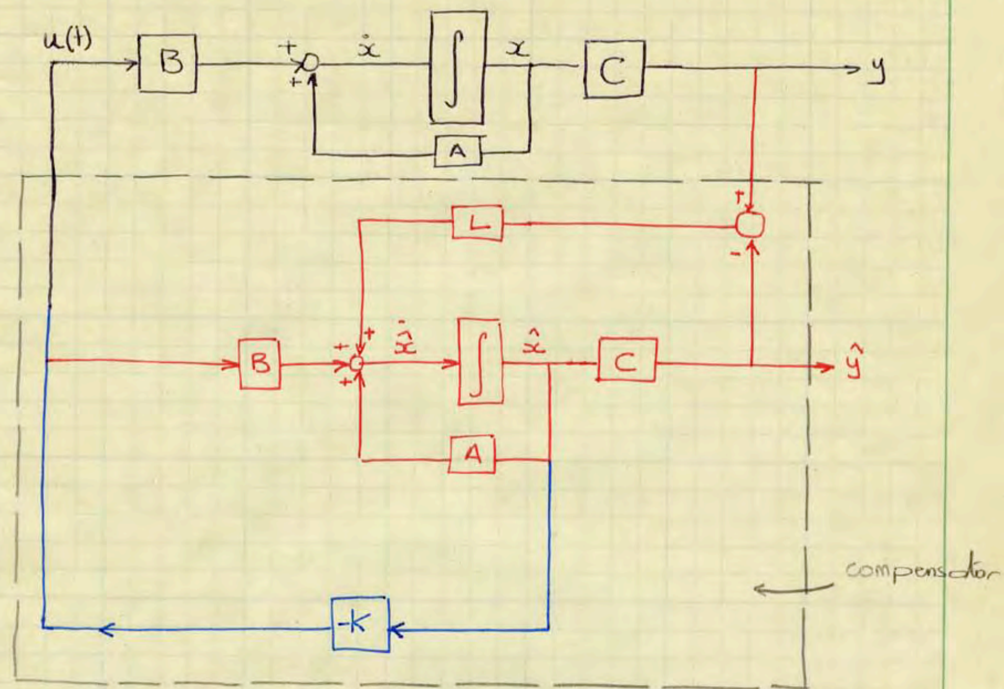
Notes: • we can choose L so that $\tilde{x}$ decays to zero at any desired speed independent of $u(t)$, $\tilde{x}(0)$!

- We are assuming that (A, B, C) of plant (physical) are identical to (A, B, C) of model (software).

In many cases, small discrepancy are still acceptable.

- Estimator pole selection : (Rule of thumb): Estimator poles can be chosen to be faster than the controller poles by a factor of 2 to 6

Combined State Feedback & Estimator.



- $\dot{x} = Ax - BK\hat{x}$
  $= Ax - BK(x - \tilde{x})$
  $= (A - BK)x + BK\tilde{x}$
- $\dot{\tilde{x}} = (A - LC)\tilde{x}$

$$\Rightarrow \begin{bmatrix} \dot{x} \\ \dot{\tilde{x}} \end{bmatrix} = \underbrace{\begin{bmatrix} A-BK & BK \\ 0 & A-LC \end{bmatrix}}_{\bar{A}} \begin{bmatrix} x \\ \tilde{x} \end{bmatrix}$$

$$\Rightarrow \text{C.E: } \det(sI - \bar{A}) \xrightarrow[\text{block triangular}]{\text{because it is}} \det\underbrace{(sI - A + BK)}_{\alpha_c(s)} \cdot \underbrace{\det(sI - A + LC)}_{\alpha_o(s)} = 0$$

!!! ⇒ Design of pole placement for controller & estimator can be carried out independently (**Separation principle**)

Example: State Feedback & Estimator Design of A second order system.

Plant: $G(s) = 1/s^2$ $\quad \dot{x} = \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}x + \begin{bmatrix}0\\ 1\end{bmatrix}u$ $\quad , y = [1, 0]\,x$.

Place controller poles @ $s = -0.707 \pm 0.707j$ $\quad (\omega_n = 1\ , \zeta = 0.707)$

(hence desired $\alpha_c = s^2 + s\sqrt{2} + 1$)

• $\alpha_c(s) = |sI - A + BK|$

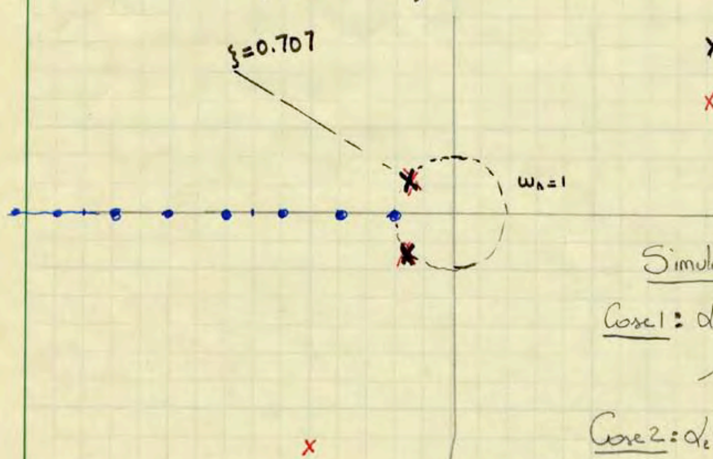
$$\overset{\det}{=} \left|\begin{bmatrix}s & -1\\ 0 & s\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix}[k_1 \;\; k_2]\right| = \left|\begin{bmatrix}s & -1\\ k_1 & s + k_2\end{bmatrix}\right| = s^2 + k_2 s + k_1$$

$$\Rightarrow \quad k_1 = 1, \quad k_2 = \sqrt{2}$$

• $\alpha_e(s) = \det(sI - A + LC)$

$$= \det\left\{\begin{bmatrix}s & -1\\ 0 & s\end{bmatrix} + \begin{bmatrix}l_1\\ l_2\end{bmatrix}[1 \;\; 0]\right\}$$

$$= \det\begin{Bmatrix}s + l_1 & -1\\ l_2 & s\end{Bmatrix} = s^2 + l_1 s + l_2$$



x : controller gain

x : estimator gains

Simulation cases 1

Case 1: $\alpha_e(s) = s^2 + s\sqrt{2} + 1 \Rightarrow \begin{Bmatrix}l_1 = \sqrt{2}\\ l_2 = 1\end{Bmatrix}$

$\omega_n = 1$, $\zeta = 0.707$

Case 2: $\alpha_e(s) = s^2 + 5s + 25 = (s + 2.5 \pm 4.3j)$

$\omega_n = 5$, $\zeta = 0.5$ $\quad \Rightarrow \begin{Bmatrix}l_1 = 5\\ l_2 = 25\end{Bmatrix}$

$$\begin{bmatrix} \dot{x} \\ \dot{\hat{x}} \end{bmatrix} = \begin{bmatrix} A-BK & BK \\ 0 & A-LC \end{bmatrix} \begin{bmatrix} x \\ \tilde{x} \end{bmatrix}$$

- $BK = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & \sqrt{2} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & \sqrt{2} \end{bmatrix}$

- $A - BK = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 1 & \sqrt{2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -\sqrt{2} \end{bmatrix}$

- $LC = \begin{bmatrix} l_1 \\ l_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} l_1 & 0 \\ l_2 & 0 \end{bmatrix}$

- $A - LC = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} l_1 & 0 \\ l_2 & 0 \end{bmatrix} = \begin{bmatrix} -l_1 & 1 \\ -l_2 & 0 \end{bmatrix}$

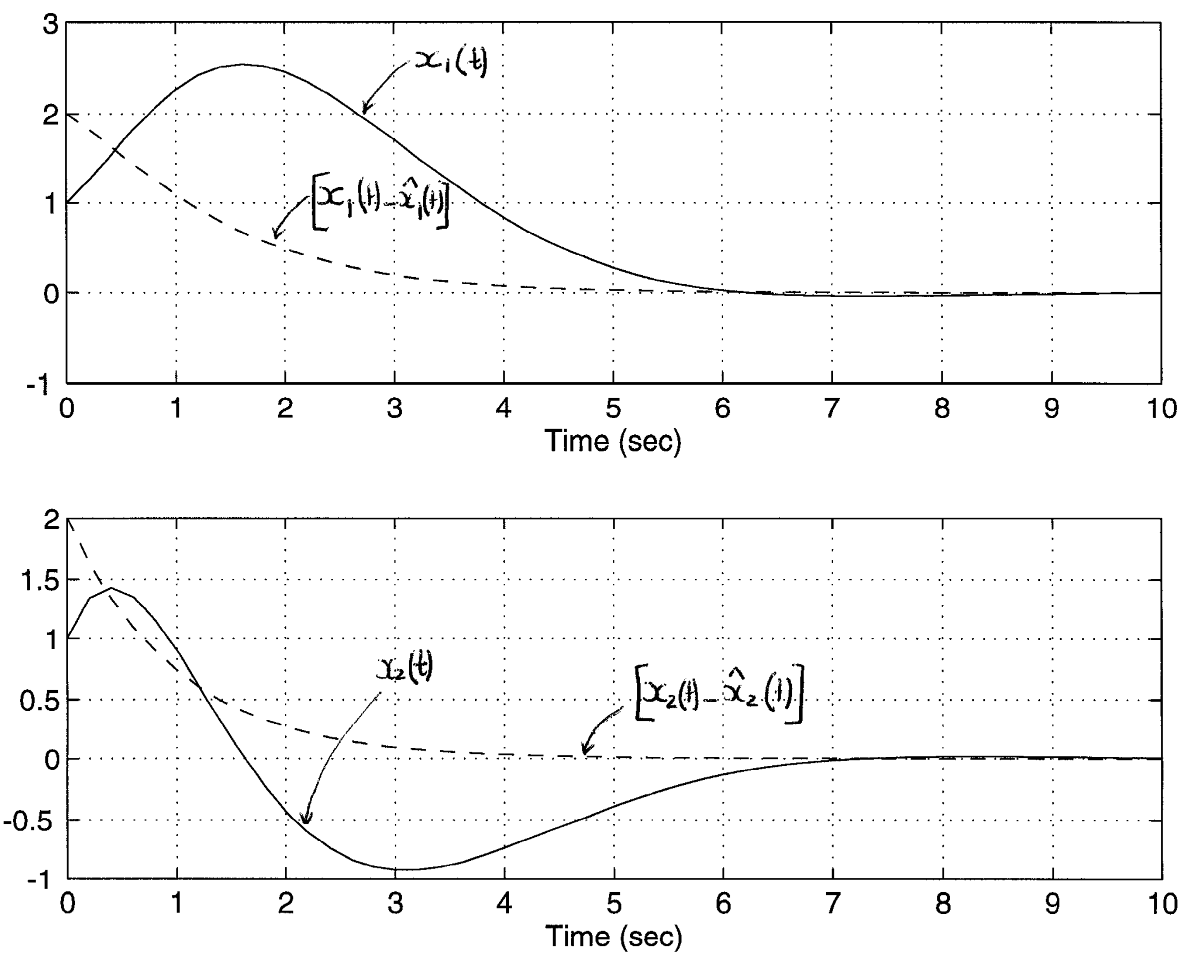
So

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{\hat{x}}_1 \\ \dot{\hat{x}}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & -\sqrt{2} & 1 & \sqrt{2} \\ 0 & 0 & -l_1 & 1 \\ 0 & 0 & -l_2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \hat{x}_1 \\ \hat{x}_2 \end{bmatrix}$$

$$\Rightarrow \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1 - \sqrt{2}\, x_2 + \hat{x}_1 + \sqrt{2}\, \hat{x}_2 \\ \dot{\hat{x}}_1 = -l_1 \hat{x}_1 + \hat{x}_2 \\ \dot{\hat{x}}_2 = -l_2 \hat{x}_1 \end{cases}$$

$$K = \begin{bmatrix} 1 & \sqrt{2} \end{bmatrix} \Longrightarrow \alpha_c = s^2 + s\sqrt{2} + 1 \Rightarrow \omega_n = 1,\ \xi = 0.707$$

$$L = \begin{bmatrix} \sqrt{2} \\ 1 \end{bmatrix} \Longrightarrow \alpha_o = s^2 + s\sqrt{2} + 1 \Rightarrow \omega_n = 1,\ \xi = 0.707$$



$$K = \begin{bmatrix} 1 & \sqrt{2} \end{bmatrix} \Rightarrow \alpha_c = s^2 + s\sqrt{2} + 1 \Rightarrow \omega_n = 1, \ \xi = 0.707$$

$$L = \begin{bmatrix} 5 \\ 25 \end{bmatrix} \Rightarrow \alpha_o = s^2 + 5s + 25 \Rightarrow \omega_n = 5, \ \xi = 0.5$$