CHAPTER 2
Kinematics

2.1 INTRODUCTION

The motion of a mechanical system is governed by Newton's second law, which relates the forces acting on the system to its acceleration. A description of the acceleration requires a reference frame. In applying Newton's second law, it is necessary to use an inertial reference frame, that is, a reference frame fixed in an inertial space (Section 3.1). Quite often, however, it is more convenient to describe the motion in terms of moving reference frames. This is true in the case of rotating bodies, in which case rotating reference frames are more suitable than inertial frames.

It becomes clear from the above brief discussion that the description of the motion requires more effort than the description of the forces. In fact, the description of the motion represents a study in itself. The study of the motion of a body without regard to the forces causing the motion is known as *kinematics*. It is perhaps appropriate to think of kinematics as the geometry of motion. Because of its importance in the derivation of the differential equations governing the motion of mechanical systems by Newtonian mechanics, this chapter is devoted entirely to kinematics. The material is fundamental to the study of the dynamics of systems of particles and rigid bodies.

2.2 MOTION RELATIVE TO A FIXED REFERENCE FRAME. CARTESIAN COMPONENTS

Let us consider the motion of a particle P along curve C in a three-dimensional space (Fig. 2.1). To describe the motion, it is convenient to introduce a reference frame in the form of a rectangular set of axes xyz with the origin at the fixed point 0.

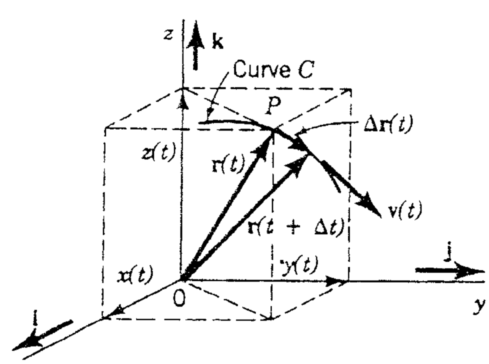

FIGURE 2.1

The axes xyz are assumed to maintain a fixed orientation in space. The position of the particle P in space is defined at any time t by the three *cartesian coordinates* $x(t)$, $y(t)$, and $z(t)$. These coordinates can be regarded as the components of the *position vector*, or *radius vector* $\mathbf{r}(t)$. An expression relating $\mathbf{r}(t)$ and its components can be provided by introducing the vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$, where the vectors have unit magnitude and directions along the axes x, y, and z, respectively. Such vectors are referred to as *unit vectors*. Because both the magnitude and direction of each of the vectors are constant, these are constant unit vectors. The position vector $\mathbf{r}(t)$ can then be expressed in terms of the components $x(t)$, $y(t)$, $z(t)$ and the unit vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ as follows:

$$\mathbf{r}(t)=x(t)\mathbf{i}+y(t)\mathbf{j}+z(t)\mathbf{k} \tag{2.1}$$

and we note that Eq. (2.1) is merely a convenient way of writing the three components of the position vector simultaneously. The position has units of length, which in SI units* is the meter (m).

The *velocity* of the particle P in space is defined as the time rate of change of the position. The velocity has in general three components also, so that it too can be regarded as a vector. Denoting the *velocity vector* of P by $\mathbf{v}(t)$ and using rules of differential calculus, we can refer to Fig. 2.1 and write

$$\mathbf{v}(t)=\lim_{\Delta t\to 0}\frac{\mathbf{r}(t+\Delta t)-\mathbf{r}(t)}{\Delta t}=\lim_{\Delta t\to 0}\frac{\Delta\mathbf{r}(t)}{\Delta t}=\frac{d\mathbf{r}(t)}{dt}=\dot{\mathbf{r}}(t) \tag{2.2}$$

where the overdot denotes differentiation with respect to time. The velocity has units meters per second (m/s). Introducing Eq. (2.1) into Eq. (2.2) and recalling that the unit vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are constant, we obtain

$$\begin{aligned}\mathbf{v}(t)&=\frac{dx(t)}{dt}\mathbf{i}+\frac{dy(t)}{dt}\mathbf{j}+\frac{dz(t)}{dt}\mathbf{k}=\dot{x}(t)\mathbf{i}+\dot{y}(t)\mathbf{j}+\dot{z}(t)\mathbf{k}\\&=v_x(t)\mathbf{i}+v_y(t)\mathbf{j}+v_z(t)\mathbf{k}\end{aligned} \tag{2.3}$$

where

$$v_x(t)=\dot{x}(t),\qquad v_y(t)=\dot{y}(t),\qquad v_z(t)=\dot{z}(t) \tag{2.4}$$

are the cartesian components of the velocity vector. From Fig. 2.1, we observe that as $\Delta t\to 0$ the increment $\Delta\mathbf{r}(t)$ in the position vector corresponding to the time increment Δt aligns itself with the curve C and becomes the differential $d\mathbf{r}(t)$. Hence, from Eq. (2.2), we conclude that *the velocity vector* $\mathbf{v}(t)$ *is tangent to the curve* C *at all times.*

The *acceleration* of the particle P is defined as the time rate of change of the velocity. As with the velocity, the acceleration can be regarded as a vector. Denoting the *acceleration vector* of P by $\mathbf{a}(t)$ and following the pattern established for the

*The acronym derives from the French name for the system of units, namely, *Système International d'Unités*.

velocity vector, we can write the acceleration vector in terms of cartesian components in the form

$$\mathbf{a}(t)=\dot{\mathbf{v}}(t)=\ddot{\mathbf{r}}(t)=a_x(t)\mathbf{i}+a_y(t)\mathbf{j}+a_z(t)\mathbf{k} \tag{2.5}$$

where

$$a_x(t)=\dot{v}_x(t)=\ddot{x}(t), \qquad a_y(t)=\dot{v}_y(t)=\ddot{y}(t), \qquad a_z(t)=\dot{v}_z(t)=\ddot{z}(t) \tag{2.6}$$

in which double overdots denote second derivatives with respect to time. The acceleration has units meters per second squared (m/s^2).

There are two special cases of motion of particular interest, namely, rectilinear motion and planar motion.

i. Rectilinear Motion

Rectilinear motion implies motion along a straight line. Because in this case there is only one component of motion, we can dispense with the vector notation and describe the motion in terms of scalar quantities. Denoting the line along which the motion takes place by s and the distance of the particle P from the fixed origin 0 by $s(t)$ (Fig. 2.2), the velocity of P can be written simply

$$v(t)=\frac{ds(t)}{dt}=\dot{s}(t) \tag{2.7}$$

and the acceleration of P

$$a(t)=\frac{dv(t)}{dt}=\frac{d^2s(t)}{dt^2}=\ddot{s}(t) \tag{2.8}$$

In Eqs. (2.7) and (2.8), the distance $s(t)$, the velocity $v(t)$, and the acceleration $a(t)$ are regarded as explicit functions of the time t. Using Eqs. (2.7) and (2.8), we can derive a relation among s, v, and a in which the time t is only implicit. To this end, let us use the chain rule for differentiation and write

$$a=\frac{dv}{dt}=\frac{dv}{ds}\frac{ds}{dt}=\frac{dv}{ds}v \tag{2.9}$$

Equation (2.9) can be rewritten in the form

$$a\,ds=v\,dv \tag{2.10}$$

which is the desired result.

Integrating Eq. (2.10) between the points $s=s_1, v=v_1$ and $s=s_2, v=v_2$, we obtain

$$\int_{s_1}^{s_2} a\,ds=\int_{v_1}^{v_2} v\,dv=\frac{1}{2}v^2\Big|_{v_1}^{v_2}=\frac{1}{2}(v_2^2-v_1^2) \tag{2.11}$$

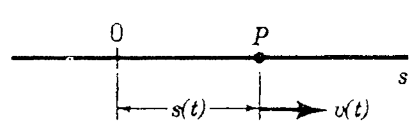

FIGURE 2.2

Example 2.1

An automobile starting from rest travels with the constant acceleration a_0 for 6 s (Fig. 2.3). Determine the value a_0 given that the automobile has reached the velocity $v_f = 108$ kilometers per hour (km/h) at the end of the 6 s. What is the distance traveled by the automobile?

Integrating Eq. (2.8), we obtain

$$\int_0^v dv = \int_0^t a\, dt = a_0 \int_0^t dt \tag{a}$$

or

$$v(t) = a_0 t \tag{b}$$

Letting $t = t_f = 6$ s, $v(t) = v(t_f) = v_f = 108$ km/h in Eq. (b), we obtain

$$a_0 = \frac{v_f}{t_f} = \frac{108 \times 1000}{3600} \frac{1}{6} = 5 \quad \text{m/s}^2 \tag{c}$$

To determine the distance traveled, we integrate Eq. (2.7), with the result

$$\int_0^s ds = \int_0^t v\, dt = \int_0^t a_0 t\, dt \tag{d}$$

so that

$$s = \tfrac{1}{2} a_0 t^2 \tag{e}$$

Letting $t = t_f = 6$ s in Eq. (e) and using Eq. (c), we obtain

$$s_f = \tfrac{1}{2} 5 \times 6^2 = 90 \quad \text{m} \tag{f}$$

Example 2.2

Determine the distance traveled by the automobile of Example 2.1 by using Eq. (2.11).

Letting $a = a_0 = \text{const}$, $s_1 = 0$, $s_2 = s_f$, $v_1 = 0$, $v_2 = v_f$ in Eq. (2.11), we obtain

$$a_0 s_f = \tfrac{1}{2} v_f^2 \tag{a}$$

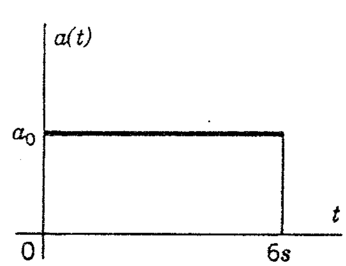

FIGURE 2.3

or

$$s_f = \frac{1}{2}\frac{v_f^2}{a_0} = \frac{1}{2}\left(\frac{108 \times 1000}{3600}\right)^2 \frac{1}{5} = 90 \quad \text{m} \tag{b}$$

which is the same result as that obtained in Example 2.1.

ii. Planar Motion. Trajectories

Let us consider a particle traveling in the plane xz with the constant acceleration

$$\mathbf{a} = \ddot{z}\mathbf{k} = -g\mathbf{k} \tag{2.12}$$

after being propelled with the initial velocity $\mathbf{v}_0$ from the origin 0 (Fig. 2.4), where g is the acceleration due to gravity. We propose to calculate the trajectory described by the particle in the plane xz. To this end, we first determine the motion as a function of time and then eliminate the explicit dependence of the motion on the time t.

Multiplying Eq. (2.5) by dt, integrating with respect to time, and considering Eq. (2.12), we obtain

$$\int_{\mathbf{v}_0}^{\mathbf{v}} d\mathbf{v} = \int_0^t \mathbf{a}\, dt = -gt\mathbf{k} \tag{2.13}$$

or

$$\mathbf{v} = \mathbf{v}_0 - gt\mathbf{k} \tag{2.14}$$

Multiplying Eq. (2.2) by dt, integrating with respect to time, and considering Eq. (2.14), we can write

$$\int_0^{\mathbf{r}} d\mathbf{r} = \int_0^t \mathbf{v}\, dt = \int_0^t (\mathbf{v}_0 - gt\mathbf{k})\, dt = \mathbf{v}_0 t - \tfrac{1}{2}gt^2\mathbf{k} \tag{2.15}$$

or

$$\mathbf{r} = \mathbf{v}_0 t - \tfrac{1}{2}gt^2\mathbf{k} \tag{2.16}$$

Letting v_0 be the magnitude of the initial velocity vector $\mathbf{v}_0$ and α the angle between $\mathbf{v}_0$ and the x axis, we can write the vector in terms of the cartesian components

$$\mathbf{v}_0 = v_0 \cos\alpha\, \mathbf{i} + v_0 \sin\alpha\, \mathbf{k} \tag{2.17}$$

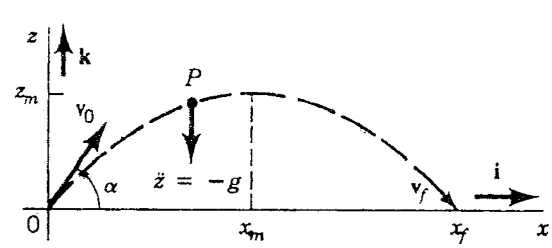

FIGURE 2.4

which permits us to express the position vector **r** in terms of Cartesian components as follows:

$$x = v_0 t \cos\alpha, \qquad z = v_0 t \sin\alpha - \tfrac{1}{2} g t^2 \tag{2.18}$$

Equations (2.18) give the position of the particle as an explicit function of time.

To derive the trajectory equation, let us solve the first of Eqs. (2.18) for the time t. The result is

$$t = \frac{x}{v_0 \cos\alpha} \tag{2.19}$$

Inserting Eq. (2.19) into the second of Eqs. (2.18), we obtain

$$z = x \tan\alpha - \frac{1}{2} g \frac{x^2}{v_0^2 \cos^2\alpha} \tag{2.20}$$

which is the *trajectory equation.* It represents a parabola. To determine its shape in more detail, we calculate the point at which the particle reaches its maximum altitude. From calculus, this point is characterized by zero slope, or

$$\frac{dz}{dx} = 0 \tag{2.21}$$

Inserting Eq. (2.20) into Eq. (2.21), we can write

$$\frac{dz}{dx} = \tan\alpha - g \frac{x}{v_0^2 \cos^2\alpha} = 0 \tag{2.22}$$

Denoting by $x = x_m$ the distance along the x axis corresponding to the maximum altitude $z = z_m$, we obtain from Eq. (2.22)

$$x_m = \frac{v_0^2}{g} \sin\alpha \cos\alpha \tag{2.23}$$

Letting $x = x_m$ and $z = z_m$ in Eq. (2.20) and using Eq. (2.23), we obtain the maximum altitude

$$z_m = \frac{1}{2} \frac{v_0^2}{g} \sin^2\alpha \tag{2.24}$$

The trajectory is symmetric with respect to the vertical through $x = x_m$, as shown in Fig. 2.4. We conclude from Fig. 2.4 that the particle hits the ground again at $x_f = 2x_m$. On the basis of geometric considerations, we also conclude from Fig. 2.4 that the impact velocity is

$$\mathbf{v}_f = v_0 \cos\alpha\, \mathbf{i} - v_0 \sin\alpha\, \mathbf{k} \tag{2.25}$$

2.3 PLANAR MOTION IN TERMS OF CURVILINEAR COORDINATES

There are instances in which the use of coordinates other than cartesian is advisable. This is often the case when the motion of the particle is not rectilinear, but follows a curved path. In such cases the use of curvilinear coordinates is likely to be more advantageous. There are two important sets of curvilinear coordinates in common use for planar motion: (i) radial and transverse coordinates and (ii) tangential and normal coordinates. In this section, we derive the velocity and acceleration expressions for both sets of coordinates.

i. Radial and Transverse Coordinates

The radial and transverse coordinates are commonly known as polar coordinates. Let us consider a particle P traveling along curve C, as shown in Fig. 2.5, and define the radial axis r as the one coinciding at all times with the direction of the radius vector $\mathbf{r}(t)$ from the origin 0 to the point P. The transverse axis θ is normal to the radial axis, as shown in Fig. 2.5. To describe the motion, it is convenient to introduce the unit vectors $\mathbf{u}_r(t)$ and $\mathbf{u}_\theta(t)$ in the radial and transverse directions, respectively. However, unlike the unit vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ of Section 2.2, the unit vectors $\mathbf{u}_r(t)$ and $\mathbf{u}_\theta(t)$ are not constant, but depend on time. This can be explained easily by observing that the radius vector $\mathbf{r}(t)$ changes directions continuously as the particle P moves along the curve C. Because the unit vector $\mathbf{u}_r(t)$ is aligned with $\mathbf{r}(t)$, $\mathbf{u}_r(t)$ also changes directions continuously. Hence, although the unit vector $\mathbf{u}_r(t)$ has constant magnitude (equal to unity), the mere fact that it changes directions makes it a time-dependent unit vector. Because $\mathbf{u}_\theta(t)$ is normal to $\mathbf{u}_r(t)$, $\mathbf{u}_\theta(t)$ is also a time-dependent unit vector.

Denoting the magnitude of the radius vector $\mathbf{r}(t)$ by $r(t)$, we can express the vector $\mathbf{r}(t)$ in the form

$$\mathbf{r}(t) = r(t)\mathbf{u}_r(t) \tag{2.26}$$

Following the pattern used to derive Eq. (2.2), the velocity $\mathbf{v}(t)$ of P is simply

$$\mathbf{v}(t) = \dot{\mathbf{r}}(t) = \dot{r}(t)\mathbf{u}_r(t) + r(t)\dot{\mathbf{u}}_r(t) \tag{2.27}$$

Equation (2.27) involves the time derivative $\dot{\mathbf{u}}_r(t)$ of the unit vector $\mathbf{u}_r(t)$. Before we can complete the expression of $\mathbf{v}(t)$ in terms of radial and transverse components, it is necessary to derive an expression for $\dot{\mathbf{u}}_r(t)$. We use this opportunity to evaluate $\dot{\mathbf{u}}_\theta(t)$ as well. From Fig. 2.6, we note that at the end of the time increment Δt the vector $\mathbf{u}_r(t)$ becomes the vector $\mathbf{u}_r(t+\Delta t)$, the difference between the two vectors

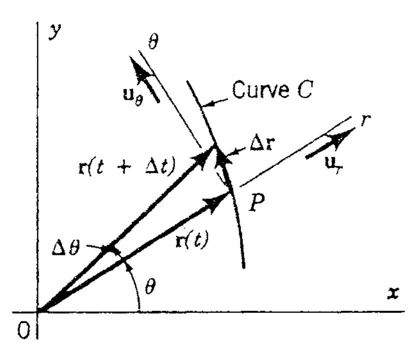

FIGURE 2.5

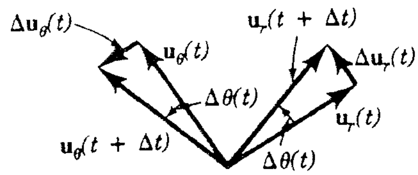

FIGURE 2.6

being $\Delta\mathbf{u}_r(t)$. For small increments Δt, the magnitude of $\Delta\mathbf{u}_r(t)$ is equal to the magnitude of $\mathbf{u}_r(t)$ multiplied by the correspondingly small angle $\Delta\theta(t)$, and the direction of $\Delta\mathbf{u}_r(t)$ is parallel to that of $\mathbf{u}_\theta(t)$. By analogy, the difference between the vectors $\mathbf{u}_\theta(t+\Delta t)$ and $\mathbf{u}_\theta(t)$ is the increment $\Delta\mathbf{u}_\theta(t)$, whose magnitude is equal to the magnitude of $\mathbf{u}_\theta(t)$ multiplied by $\Delta\theta(t)$, and its direction is opposite to that of $\mathbf{u}_r(t)$. Because $\mathbf{u}_r(t)$ and $\mathbf{u}_\theta(r)$ are unit vectors, we can write

$$\begin{aligned}\Delta\mathbf{u}_r(t)&=\mathbf{u}_r(t+\Delta t)-\mathbf{u}_r(t)=1\times\Delta\theta(t)\;\mathbf{u}_\theta(t)\\ \Delta\mathbf{u}_\theta(t)&=\mathbf{u}_\theta(t+\Delta t)-\mathbf{u}_\theta(t)=1\times\Delta\theta(t)[-\mathbf{u}_r(t)]\end{aligned}\tag{2.28}$$

Hence, dividing Eqs. (2.28) by Δt and taking the limit, we obtain the time derivatives of $\mathbf{u}_r(t)$ and $\mathbf{u}_\theta(t)$ in the form

$$\begin{aligned}\dot{\mathbf{u}}_r(t)&=\lim_{\Delta t\to 0}\frac{\Delta\mathbf{u}_r(t)}{\Delta t}=\lim_{\Delta t\to 0}\frac{\Delta\theta(t)}{\Delta t}\;\mathbf{u}_\theta(t)=\dot{\theta}(t)\mathbf{u}_\theta(t)\\ \dot{\mathbf{u}}_\theta(t)&=\lim_{\Delta t\to 0}\frac{\Delta\mathbf{u}_\theta(t)}{\Delta t}=\lim_{\Delta t\to 0}\frac{\Delta\theta(t)}{\Delta t}[-\mathbf{u}_r(t)]=-\dot{\theta}(t)\mathbf{u}_r(t)\end{aligned}\tag{2.29}$$

where $\dot{\theta}(t)$ is the angular rate of change of the radius vector $\mathbf{r}(t)$ as its tip moves along the curve C. Introducing the first of Eqs. (2.29) into Eq. (2.27), we obtain

$$\mathbf{v}(t)=\dot{r}\mathbf{u}_r+r\dot{\theta}\mathbf{u}_\theta=v_r\mathbf{u}_r+v_\theta\mathbf{u}_\theta\tag{2.30}$$

where

$$v_r=\dot{r},\qquad v_\theta=r\dot{\theta}\tag{2.31}$$

are the *radial and transverse components of the velocity vector*, respectively.

The acceleration of point P is obtained by simply taking the derivative of the velocity vector. Hence, using Eq. (2.30) and considering Eqs. (2.29), we can write

$$\begin{aligned}\mathbf{a}(t)=\dot{\mathbf{v}}(t)&=\ddot{r}\mathbf{u}_r+\dot{r}\dot{\mathbf{u}}_r+\dot{r}\dot{\theta}\mathbf{u}_\theta+r\ddot{\theta}\mathbf{u}_\theta+r\dot{\theta}\dot{\mathbf{u}}_\theta\\ &=(\ddot{r}-r\dot{\theta}^2)\mathbf{u}_r+(r\ddot{\theta}+2\dot{r}(\dot{\theta})\mathbf{u}_\theta=a_r\mathbf{u}_r+a_\theta\mathbf{u}_\theta\end{aligned}\tag{2.32}$$

where

$$a_r=\ddot{r}-r\dot{\theta}^2,\qquad a_\theta=r\ddot{\theta}+2\dot{r}\dot{\theta}\tag{2.33}$$

are the *radial and transverse components of the acceleration vector*, respectively.

It should be noted that by adding the coordinate z to the polar coordinates r and θ, we obtain the *cylindrical coordinates* r, θ, z. The velocity and acceleration vectors can be expressed in terms of cylindrical coordinates by simply adding $v_z\mathbf{k}=\dot{z}\mathbf{k}$ and $a_z\mathbf{k}=\ddot{z}\mathbf{k}$ to Eqs. (2.30) and (2.32), respectively.

Example 2.3

A cyclist enters the semicircular track of radius $R=50$ m shown in Fig. 2.7 with the velocity $v_A=15$ m/s, decelerates circumferentially at a uniform rate, and exits with the velocity $v_C=10$ m/s. Calculate the circumferential deceleration, the time it takes to complete the semicircle, and the velocity at point B.

Let us denote the magnitude of the circumferential deceleration by c, so that

$$a_\theta = -c = R\ddot{\theta} \tag{a}$$

Integrating Eq. (a) with respect to time, we obtain

$$v_\theta = R\dot{\theta} = v_A + \int_0^t a_\theta \, dt = v_A - ct \tag{b}$$

Letting $t=t_f$, $v_\theta = v_C$ in Eq. (b), we can write

$$ct_f = v_A - v_C = 15 - 10 = 5 \quad \text{m/s} \tag{c}$$

Integration of Eq. (b) with respect to time yields

$$R\theta = v_A t - \tfrac{1}{2}ct^2 \tag{d}$$

Introducing $\theta_f = \pi$, $t=t_f$ and $ct_f=5$ in Eq. (d) and solving for t_f, we obtain

$$t_f = \frac{R\theta_f}{v_A - \frac{1}{2}ct_f} = \frac{50\pi}{15-\frac{1}{2}\times 5} = 4\pi \quad \text{s} \tag{e}$$

so that, from Eqs. (c) and (e),

$$c = \frac{5}{4\pi} \quad \text{m/s}^2 \tag{f}$$

To obtain v_B, we make use of Eq. (2.11), recall that $a_\theta = \text{const}$, and write

$$a_\theta s_{AB} = \tfrac{1}{2}(v_B^2 - v_A^2) \tag{g}$$

so that

$$\begin{aligned} v_B &= \sqrt{v_A^2 + 2a_\theta s_{AB}} = \sqrt{v_A^2 - 2cR\pi/2} = \sqrt{15^2 - 2(5/4\pi)(50\pi/2)} \\ &= \sqrt{225 - 62.5} = 12.75 \quad \text{m/s} \end{aligned} \tag{h}$$

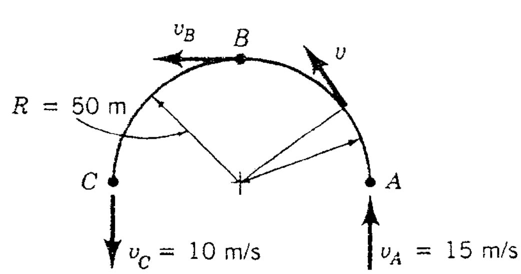

FIGURE 2.7

ii. Tangential and Normal Coordinates

Let us define the tangential and normal coordinates as a set of axes tangent and normal to curve C at the point P and time t, as shown in Fig. 2.8. The point $0'$ denotes the center of curvature of the curve C and ρ is the radius of curvature corresponding to the instantaneous position of the point P on the curve. As P moves along C, both the center of curvature and the radius of curvature change, except when the curve C is a circle. If Δs is the distance along C traveled by the particle P during the time increment Δt, we can write

$$\Delta \mathbf{r} \cong \Delta s\, \mathbf{u}_t \tag{2.34}$$

Hence, the velocity is simply

$$\mathbf{v} = \lim_{\Delta t \to 0} \frac{\Delta \mathbf{r}}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t}\, \mathbf{u}_t = \dot{s}\mathbf{u}_t \tag{2.35}$$

where $\dot{s}$ is the magnitude of the velocity vector. As expected, the velocity vector has only a tangential component, or the *velocity vector is tangent to the curve C at all times.* The acceleration vector is obtained by differentiating Eq. (2.35) with respect to time. The result is

$$\mathbf{a} = \ddot{s}\mathbf{u}_t + \dot{s}\dot{\mathbf{u}}_t \tag{2.36}$$

But, from Fig. 2.9, we can write

$$\dot{\mathbf{u}}_t = \lim_{\Delta t \to 0} \frac{\mathbf{u}_t(t+\Delta t) - \mathbf{u}_t(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta \mathbf{u}_t(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{1 \times \Delta\phi\, \mathbf{u}_n(t)}{\Delta t} = \dot{\phi}\mathbf{u}_n \tag{2.37}$$

where $\dot{\phi}$ is the angular rate of change of the vector $\boldsymbol{\rho}$ from O′ to P as its tip moves

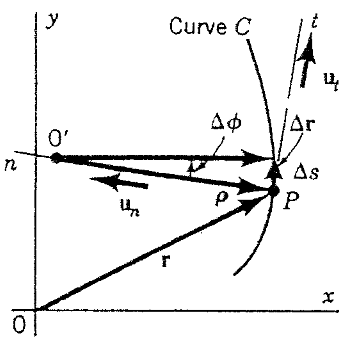

FIGURE 2.8

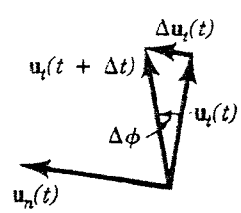

FIGURE 2.9

along C. Observing that

$$\Delta s = \rho \, \Delta \phi \tag{2.38}$$

where ρ is the magnitude of $\boldsymbol{\rho}$, we obtain

$$\dot{\phi} = \lim_{\Delta t \to 0} \frac{\Delta \phi}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta \phi}{\Delta s} \frac{\Delta s}{\Delta t} = \lim_{\Delta t \to 0} \frac{1}{\rho} \frac{\Delta s}{\Delta t} = \frac{\dot{s}}{\rho} \tag{2.39}$$

Hence, introducing Eqs. (2.37) and (2.39) into Eq. (2.36), we have

$$\mathbf{a} = \ddot{s}\mathbf{u}_t + \frac{\dot{s}^2}{\rho}\mathbf{u}_n = a_t \mathbf{u}_t + a_n \mathbf{u}_n \tag{2.40}$$

where

$$a_t = \ddot{s}, \qquad a_n = \frac{\dot{s}^2}{\rho} \tag{2.41}$$

are the *tangential and normal components of the acceleration vector*, and we observe that the normal component is directed toward the center of curvature.

Example 2.4
Consider the parabolic trajectory of Fig. 2.4 and calculate the radius of curvature at the top of the parabola.

From the second of Eqs. (2.41), the radius of curvature is

$$\rho = \frac{\dot{s}^2}{a_n} = \frac{v_t^2}{a_n} \tag{a}$$

At the top of the parabola, we have

$$a_n = g, \qquad v_t = v_0 \cos \alpha \tag{b}$$

so that

$$\rho = \frac{v_0^2 \cos^2 \alpha}{g} \tag{c}$$

2.4 MOVING REFERENCE FRAMES

In Sections 2.2 and 2.3, we described the motion of a particle in terms of a fixed reference frame. Yet, on many occasions it is more convenient to use moving reference frames. Hence, let us consider a reference frame xyz moving relative to the fixed reference frame XYZ (Fig. 2.10). We distinguish several cases: (i) the frame xyz translates relative to XYZ, (ii) the frame xyz rotates relative to XYZ, and (iii) the frame xyz both translates and rotates relative to XYZ. In all three cases, we can express the position of a point P relative to XYZ system in the form

$$\mathbf{R} = \mathbf{r}_A + \mathbf{r}_{AP} \tag{2.42}$$

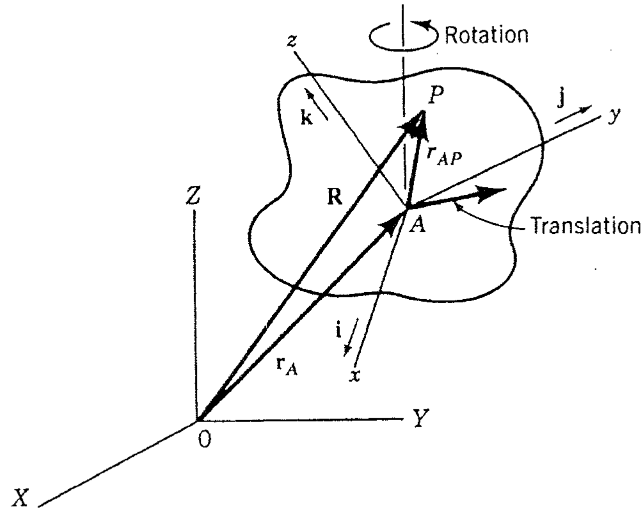

FIGURE 2.10

where $\mathbf{r}_A$ is the radius vector from the origin 0 of the fixed axes XYZ to the origin A of the moving axes xyz and $\mathbf{r}_{AP}$ is the radius vector from A to P, where the latter represents the position of P relative to A. The velocity of P relative to an inertial space is the time derivative of $\mathbf{R}$, or

$$\mathbf{v} = \dot{\mathbf{R}} = \mathbf{v}_A + \mathbf{v}_{AP} \tag{2.43}$$

where $\mathbf{v}_A = \dot{\mathbf{r}}_A$ is the velocity of point A and $\mathbf{v}_{AP} = \dot{\mathbf{r}}_{AP}$ is the velocity of P relative to A. Similarly, the acceleration of P relative to an inertial space is the time derivative of $\mathbf{v}$, or

$$\mathbf{a} = \dot{\mathbf{v}} = \ddot{\mathbf{R}} = \mathbf{a}_A + \mathbf{a}_{AP} \tag{2.44}$$

where $\mathbf{a}_A = \dot{\mathbf{v}}_A = \ddot{\mathbf{r}}_A$ is the acceleration of point A and $\mathbf{a}_{AP} = \dot{\mathbf{v}}_{AP} = \ddot{\mathbf{r}}_{AP}$ is the acceleration of P relative to A.

The evaluation of the velocity $\mathbf{v}_A$ and acceleration $\mathbf{a}_A$ of point A follows the same pattern as that established in Sections 2.2 and 2.3 for the motion of a particle relative to a fixed reference frame. On the other hand, the evaluation of the velocity $\mathbf{v}_{AP}$ and acceleration $\mathbf{a}_{AP}$ of point P relative to A depends on the case considered. If the frame xyz merely translates relative to the frame XYZ, then the calculation of $\mathbf{v}_{AP}$ and $\mathbf{a}_{AP}$ is similar to the calculation of $\mathbf{v}_A$ and $\mathbf{a}_A$, respectively. However, if the frame xyz rotates relative to the frame XYZ, then the evaluation of $\mathbf{v}_{AP}$ and $\mathbf{a}_{AP}$ becomes more involved and, in fact, requires the introduction of new concepts. In the following, the three cases indicated above are discussed separately.

i. The Frame xyz Translates Relative to XYZ

In this case it is convenient to let axes x, y, and z be parallel to axes X, Y, and Z, respectively. Because axes x, y, and z are in pure translation, they do not change directions in space, so that the unit vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ (Fig. 2.10) remain constant. The evaluation of the velocity and acceleration by means of Eqs. (2.43) and (2.44), respectively, represent mere vector additions, which implies simple addition of the corresponding components.

ii. The Frame *xyz* Rotates Relative to *XYZ*

In this section, we confine ourselves to the case in which *point P is fixed in the frame xyz*, so that *the motion of P relative to A is due entirely to the rotation of the frame xyz*. The case in which P moves relative to the frame xyz is discussed later in this chapter. The motion of P relative to A due to the rotation of the frame xyz can be best visualized by imagining that the system xyz is embedded in a rigid body, where a rigid body is defined as a body such that the distance between any two of its points is constant. Because the triad xyz is embedded in the rigid body, the rotation of the frame is identical to that of the rigid body.

Before we discuss the rotational motion of the triad xyz, it is necessary to introduce the concept of *angular velocity*. Let us assume that the rigid body is rotating instantaneously about a given axis AB, and consider a point P at a distance $\rho = |\boldsymbol{\rho}|$ from point C on axis AB, where C is the intersection between axis AB and a plane normal to AB and containing the point P (Fig. 2.11). In the time increment Δt the vector $\boldsymbol{\rho}$ from C to P sweeps an angle $\Delta\theta$ in a plane normal to AB. Hence, taking the limit, we can write

$$\lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \dot{\theta} \tag{2.45}$$

or the vector $\boldsymbol{\rho}$ rotates in a plane normal to the axis AB with the angular rate $\dot{\theta}$. Because the vector $\boldsymbol{\rho}$ is embedded in the rigid body, we conclude that the rigid body, and hence the triad xyz, rotates about axis AB at the same rate $\dot{\theta}$. Next, we propose to represent this angular rate as a vector. As Fig. 2.11 indicates, such a vector, denoted $\boldsymbol{\omega}$, is directed along the axis AB. Clearly, its magnitude must

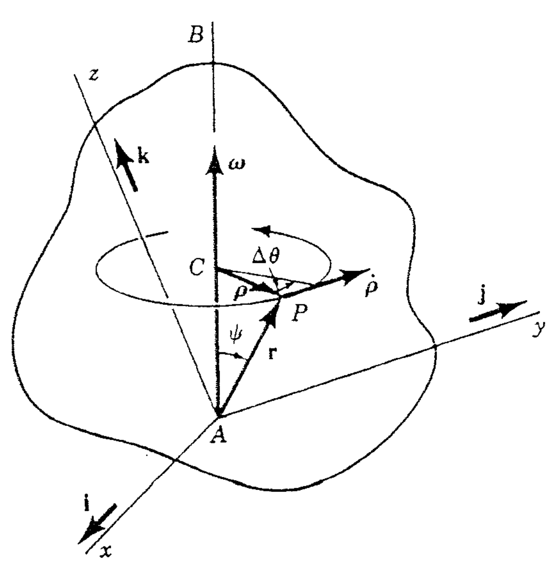

FIGURE 2.11

be $\dot{\theta}$, or

$$|\omega| = \dot{\theta} \tag{2.46}$$

If ω satisfies the rules of operations of vectors, then it can be used to represent the *angular velocity vector* of the rigid body, or of the frame xyz. The angular velocity has units radians per second (rad/s).

To verify the above proposition, let us consider the rate of change $\dot{\boldsymbol{\rho}}$ of the vector $\boldsymbol{\rho}$ due to the rotation of the body. We observe from Fig. 2.11 that the tip P of the vector $\boldsymbol{\rho}$ describes a circle of radius ρ, so that $\dot{\boldsymbol{\rho}}$ is tangent to the circle at P, and hence it is normal to the plane defined by the vectors $\boldsymbol{\omega}$ and $\boldsymbol{\rho}$. In the time increment Δt, the vector $\boldsymbol{\rho}$ undergoes the change in magnitude

$$\Delta \rho = \rho\, \Delta \theta \tag{2.47}$$

so that the rate of change of the magnitude of the vector $\boldsymbol{\rho}$ is

$$\dot{\rho} = \lim_{\Delta t \to 0} \frac{\Delta \rho}{\Delta t} = \lim_{\Delta t \to 0} \rho \frac{\Delta \theta}{\Delta t} = \rho \dot{\theta} \tag{2.48}$$

From vector analysis, however, it can be easily verified that the vector $\dot{\boldsymbol{\rho}}$ can be expressed as the cross product, or vector product,

$$\dot{\boldsymbol{\rho}} = \boldsymbol{\omega} \times \boldsymbol{\rho} \tag{2.49}$$

as both the direction and magnitude of $\dot{\boldsymbol{\rho}}$ are according to the definition of a vector product. Hence, the *angular velocity of the body, and of axes xyz, can be represented by a vector* $\boldsymbol{\omega}$ whose direction is along the axis of rotation AB.

Equation (2.49) can be generalized by observing that

$$\rho = r \sin \psi \tag{2.50}$$

where r is the magnitude of the vector $\mathbf{r}$ and ψ is the angle between AB and $\mathbf{r}$. Hence the rate of change $\dot{\mathbf{r}}$ of the vector $\mathbf{r}$ can be written as the vector product

$$\dot{\mathbf{r}} = \boldsymbol{\omega} \times \mathbf{r} \tag{2.51}$$

Equation (2.51) gives the time derivative of a vector constant in magnitude and embedded in a set of axes rotating relative to an inertial space with the angular velocity $\boldsymbol{\omega}$. The equation is valid for any vector, regardless of the physical meaning.

As pointed out earlier, the velocity of point P relative to point A is due entirely to the rotation of the frame xyz. Hence, replacing the vector $\mathbf{r}$ in Eq. (2.51) by the radius vector $\mathbf{r}_{AP}$, we obtain

$$\mathbf{v}_{AP} = \dot{\mathbf{r}}_{AP} = \boldsymbol{\omega} \times \mathbf{r}_{AP} \tag{2.52}$$

and, because point A is at rest, $\mathbf{v}_{AP}$ is also the absolute velocity of P, $\mathbf{v} = \mathbf{v}_{AP}$.

We demonstrated above that the angular velocity of a rotating set of axes can be represented by a vector coinciding with the instantaneous axis of rotation. The implication is that at a later time the rotation can take place about a different axis. Moreover, the rotation rate can be different, so that the angular velocity vector $\boldsymbol{\omega}$ can undergo changes both in direction and magnitude. We refer to the time rate of change of ω as the *angular acceleration*

$$\alpha = \dot{\omega} \tag{2.53}$$

The angular acceleration vector α is shown in Fig. 2.12. The angular acceleration has units radians per second squared (rad/s^2).

The acceleration of point P relative to A can be obtained from Eq. (2.52) by simply taking the time derivative, or

$$\mathbf{a}_{AP} = \dot{\mathbf{v}}_{AP} = \dot{\omega} \times \mathbf{r}_{AP} + \omega \times \dot{\mathbf{r}}_{AP} \tag{2.54}$$

so that, using Eqs. (2.52) and (2.53), we obtain

$$\mathbf{a}_{AP} = \alpha \times \mathbf{r}_{AP} + \omega \times (\omega \times \mathbf{r}_{AP}) \tag{2.55}$$

and, because point A is unaccelerated, $\mathbf{a}_{AP}$ is also the absolute acceleration of P, $\mathbf{a} = \mathbf{a}_{AP}$.

iii. The Frame xyz Translates and Rotates Relative to XYZ

When the origin A of the rotating frame xyz is not fixed, but moves relative to the inertial frame XYZ with the velocity $\mathbf{v}_A$ and acceleration $\mathbf{a}_A$, then the absolute velocity and acceleration of P are given by Eqs. (2.43) and (2.44), respectively, in which $\mathbf{v}_{AP}$ is given by Eq. (2.52) and $\mathbf{a}_{AP}$ by (2.55). Hence, the absolute velocity of P is

$$\mathbf{v} = \mathbf{v}_A + \mathbf{v}_{AP} = \mathbf{v}_A + \omega \times \mathbf{r}_{AP} \tag{2.56}$$

and the absolute acceleration of P is

$$\mathbf{a} = \mathbf{a}_A + \mathbf{a}_{AP} = \mathbf{a}_A + \mathbf{a} \times \mathbf{r}_{AP} + \omega \times (\omega \times \mathbf{r}_{AP}) \tag{2.57}$$

At times, it is advisable to use more than one rotating reference frame. Let us consider the case in which the reference frame $x_1 y_1 z_1$ rotates with the angular

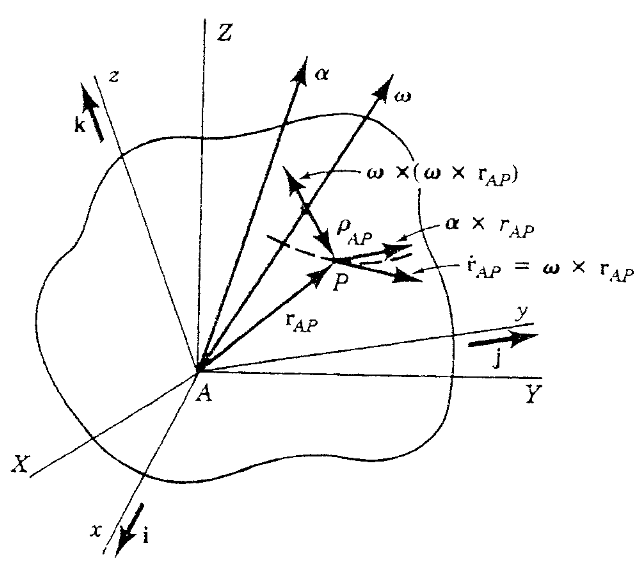

FIGURE 2.12

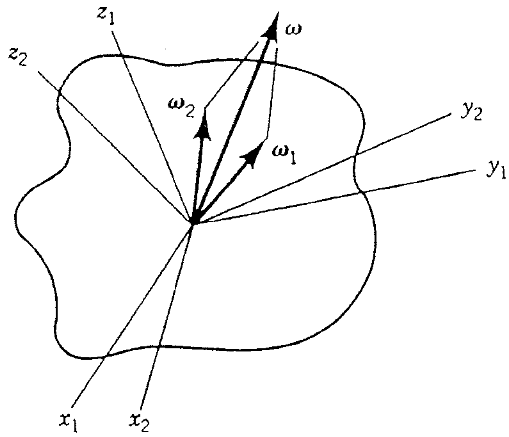

FIGURE 2.13

velocity ω_1 and the reference frame $x_2y_2z_2$ rotates relative to the frame $x_1y_1z_1$ with the angular velocity ω_2 (Fig. 2.13). Then, the angular velocity of the frame $x_2y_2z_2$ is simply

$$\omega=\omega_1+\omega_2 \tag{2.58}$$

The question arises as to the angular acceleration of the frame $x_2y_2z_2$. If we assume that the vector ω is expressed in terms of components along the frame $x_1y_1z_1$, the angular acceleration consists of two parts: the first due to the change in the components of ω relative to the frame $x_1y_1z_1$ and the second due to the fact that ω is expressed in terms of components along a rotating frame. Denoting the first part by $\boldsymbol{\alpha}'$ and recognizing that the second part can be obtained from Eq. (2.51) by replacing $\mathbf{r}$ by ω and ω by ω_1, we can write

$$\alpha=\dot{\omega}=\alpha'+\omega_1\times\omega=\alpha'+\omega_1\times\omega_2 \tag{2.59}$$

The interesting aspect of Eq. (2.59) is that the frame $x_2y_2z_2$ has an angular acceleration even when both ω_1 and ω_2 are constant. The explanation lies in the fact that the constancy of ω_2 implies only that the frame $x_2y_2z_2$ rotates uniformly relative to the frame $x_1y_1z_1$. But, as the frame $x_1y_1z_1$ rotates with the angular velocity ω_1, it causes an acceleration of the frame $x_2y_2z_2$ equal to $\omega_1\times\omega_2$, which is due entirely to a rate of change in the direction of ω_2.

Example 2.5

An airplane flies directly north at 1000 km/h, and it almost collides with another airplane. To the pilot of the first airplane it appears that the second airplane is flying at 800 km/h in a southwest path, at a 210° angle relative to the x-axis (Fig. 2.14). Determine the velocity of the second airplane.

Denoting the velocity of the first airplane by $\mathbf{v}_A$, the velocity of the second airplane by $\mathbf{v}_B$, and the velocity of the second airplane relative to the first by $\mathbf{v}_{AB}$, we can write

$$\mathbf{v}_B=\mathbf{v}_A+\mathbf{v}_{AB} \tag{a}$$

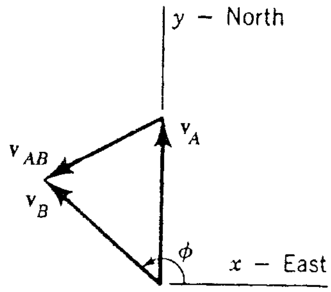

FIGURE 2.14

where, from Fig. 2.14,

$$\mathbf{v}_A = 1000\mathbf{j} \quad \text{km/h} \tag{b}$$

and

$$\mathbf{v}_{AB} = 800(\sin 210^\circ \mathbf{i} + \cos 210^\circ \mathbf{j}) = -693\mathbf{i} - 400\mathbf{j} \quad \text{km/h} \tag{c}$$

Hence,

$$\mathbf{v} = 1000\mathbf{j} - 693\mathbf{i} - 400\mathbf{j} = -693\mathbf{i} + 600\mathbf{j} \tag{d}$$

so that the second airplane is flying with the speed

$$v = [(-693)^2 + 600^2]^{1/2} \cong 917 \quad \text{km/h} \tag{e}$$

at an angle

$$\phi = \tan^{-1} \frac{600}{-693} = 139.11^\circ \tag{f}$$

as shown in Fig. 2.14.

Example 2.6

A bicycle travels on a circular track of radius R with the circumferential velocity v_A and acceleration a_A. Figure 2.15 shows one of the bicycle wheels. Determine the

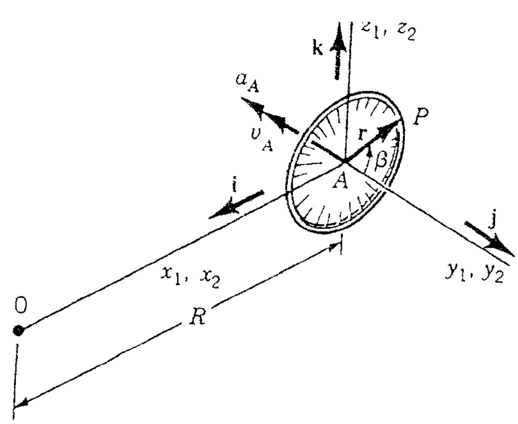

FIGURE 2.15

velocity $\mathbf{v}$ and acceleration $\mathbf{a}$ of a point P on the tire when the radius from the center of the wheel to the point P makes an angle β with respect to the horizontal plane. The radius of the wheel is r.

We consider two reference frames. The frame $x_1y_1z_1$ is attached to the bicycle, and the frame $x_2y_2z_2$ is attached to the wheel. The two frames are assumed to coincide instantaneously, although the frame $x_2y_2z_2$ rotates relative to the frame $x_1y_1z_1$. The velocity and acceleration of P are calculated according to Eqs. (2.56) and (2.57), where ω and α are given by Eqs. (2.58) and (2.59), respectively. In the first place, the velocity of the wheel center is simply

$$\mathbf{v}_A = -v_A\mathbf{j} \tag{a}$$

and the angular velocity of the frame $x_1y_1z_1$ is

$$\omega_1 = \frac{v_A}{R}\mathbf{k} \tag{b}$$

Moreover, the angular velocity of the frame $x_2y_2z_2$ relative to $x_1y_1z_1$ is

$$\omega_2 = \frac{v_A}{r}\mathbf{i} \tag{c}$$

so that the absolute angular velocity of $x_2y_2z_2$ is

$$\omega = \omega_1 + \omega_2 = \frac{v_A}{R}\mathbf{k} + \frac{v_A}{r}\mathbf{i} \tag{d}$$

The radius vector from A to P is

$$\mathbf{r}_{AP} = r(\cos\beta\ \mathbf{j} + \sin\beta\ \mathbf{k}) \tag{e}$$

Hence, the velocity of P is

$$\begin{aligned}\mathbf{v} &= \mathbf{v}_A + \omega\times\mathbf{r}_{AP} = -v_A\mathbf{j} + \left(\frac{v_A}{r}\mathbf{i} + \frac{v_A}{R}\mathbf{k}\right)\times r(\cos\beta\ \mathbf{j} + \sin\beta\ \mathbf{k}) \\ &= -\frac{v_A r}{R}\cos\beta\ \mathbf{i} - v_A(1+\sin\beta)\mathbf{j} + v_A\cos\beta\ \mathbf{k}\end{aligned} \tag{f}$$

The acceleration of A has two components: a tangential component due to the acceleration of the bicycle along the track and a normal component due to motion along a curvilinear track. Hence,

$$\begin{aligned}\mathbf{a}_A &= -a_A\mathbf{j} + \omega_1\times(\omega_1\times\mathbf{r}_{OA}) = -a_A\mathbf{j} + \frac{v_A}{R}\mathbf{k}\times\left[\frac{v_A}{R}\mathbf{k}\times(-R)\mathbf{i}\right] \\ &= \frac{v_A^2}{R}\mathbf{i} - a_A\mathbf{j}\end{aligned} \tag{g}$$

Using Eq. (2.59), the angular acceleration of the frame $x_2y_2z_2$ is

$$\alpha = \dot{\omega}'_1 + \dot{\omega}'_2 + \omega_1 \times \omega_2 = \frac{a_A}{R}\mathbf{k} + \frac{a_A}{r}\mathbf{i} + \frac{v_A}{R}\mathbf{k} \times \frac{v_A}{r}\mathbf{i}$$

$$= \frac{a_A}{r}\mathbf{i} + \frac{v_A^2}{Rr}\mathbf{j} + \frac{a_A}{R}\mathbf{k} \tag{h}$$

Hence, the acceleration of P is

$$\mathbf{a} = \mathbf{a}_A + \alpha \times \mathbf{r}_{AP} + \omega \times (\omega \times \mathbf{r}_{AP})$$

$$= \frac{v_A^2}{R}\mathbf{i} - a_A\mathbf{j} + \left(\frac{a_A}{r}\mathbf{i} + \frac{v_A^2}{Rr}\mathbf{j} + \frac{a_A}{R}\mathbf{k}\right) \times r(\cos\beta\,\mathbf{j} + \sin\beta\,\mathbf{k})$$

$$+ \left(\frac{v_A}{R}\mathbf{i} + \frac{v_A}{R}\mathbf{k}\right) \times \left[\left(\frac{v_A}{r}\mathbf{i} + \frac{v_A}{R}\mathbf{k}\right) \times r(\cos\beta\,\mathbf{j} + \sin\beta\,\mathbf{k})\right]$$

$$= \left[\frac{v_A^2}{R}(1 + 2\sin\beta) - \frac{a_A r}{R}\cos\beta\right]\mathbf{i} - \left[\left(\frac{1}{r} + \frac{r}{R^2}\right)v_A^2\cos\beta + a_A\sin\beta\right]\mathbf{j}$$

$$- \left(\frac{v_A^2}{r}\sin\beta - a_A\cos\beta\right)\mathbf{k} \tag{i}$$

2.5 PLANAR MOTION OF RIGID BODIES

In Section 2.4, we derived expressions for the general three-dimensional motion of rigid bodies. In a large number of applications, however, the motion is confined to two dimensions, so that it appears desirable to simplify the expressions for the velocity and acceleration, Eqs. (2.56) and (2.57), respectively. To this end we consider the case in which the body moves in the xy-plane, as shown in Fig. 2.16. We first express the motion in terms of cartesian components. Hence, we write

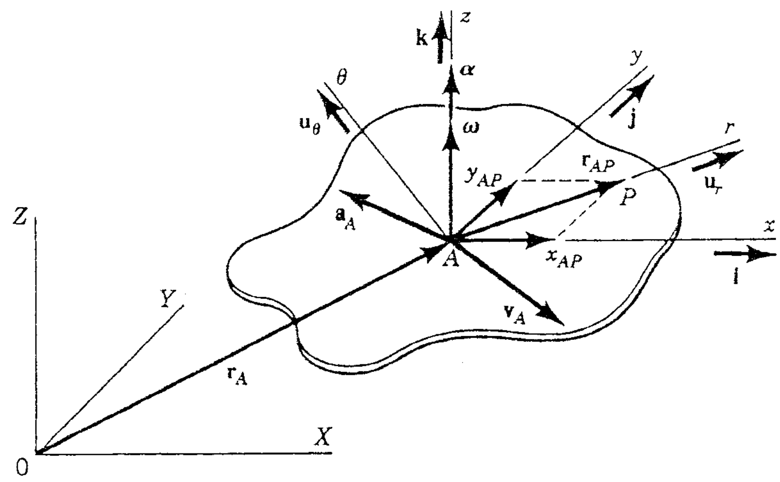

FIGURE 2.16

the radius vector $\mathbf{r}_{AP}$ in the simple form

$$\mathbf{r}_{AP} = x_{AP}\mathbf{i} + y_{AP}\mathbf{j} \tag{2.60}$$

where the various quantities are defined in Fig. 2.16. Planar motion implies that

$$\boldsymbol{\omega} = \omega\mathbf{k}, \qquad \boldsymbol{\alpha} = \alpha\mathbf{k} \tag{2.61}$$

where ω and α are the magnitude of the angular velocity and acceleration, respectively. Introducing Eqs. (2.60) and (2.61) into Eq. (2.56), we obtain the velocity

$$\mathbf{v} = \mathbf{v}_A + \boldsymbol{\omega} \times \mathbf{r}_{AP} = \mathbf{v}_A + \omega\mathbf{k} \times (x_{AP}\mathbf{i} + y_{AP}\mathbf{j}) = \mathbf{v}_A - \omega(y_{AP}\mathbf{i} - x_{AP}\mathbf{j}) \tag{2.62}$$

Moreover, inserting Eqs. (2.60) and (2.61) into Eqs. (2.57), we obtain the acceleration

$$\begin{aligned} \mathbf{a} &= \mathbf{a}_A + \boldsymbol{\alpha} \times \mathbf{r}_{AP} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_{AP}) = \mathbf{a}_A + \alpha\mathbf{k} \times (x_{AP}\mathbf{i} + y_{AP}\mathbf{j}) + \omega\mathbf{k} \times [\omega\mathbf{k} \times (x_{AP}\mathbf{i} + y_{AP}\mathbf{j})] \\ &= \mathbf{a}_A - \alpha(y_{AP}\mathbf{i} - x_{AP}\mathbf{j}) - \omega^2(x_{AP}\mathbf{i} + y_{AP}\mathbf{j}) = \mathbf{a}_A - \alpha(y_{AP}\mathbf{i} - x_{AP}\mathbf{j}) - \omega^2\mathbf{r}_{AP} \end{aligned} \tag{2.63}$$

At times, it is more convenient to express the motion in terms of radial and transverse components. To this end, we write the radius vector in the form

$$\mathbf{r}_{AP} = r_{AP}\mathbf{u}_r \tag{2.64}$$

where $\mathbf{u}_r$ is the unit vector in the radial direction. Then, it is easy to verify that the velocity vector is

$$\mathbf{v} = \mathbf{v}_A + \omega r_{AP}\mathbf{u}_\theta \tag{2.65}$$

and the acceleration vector is

$$\mathbf{a} = \mathbf{a}_A + \alpha r_{AP}\mathbf{u}_\theta - \omega^2\mathbf{r}_{AP} = \mathbf{a}_A + \alpha r_{AP}\mathbf{u}_\theta - \omega^2 r_{AP}\mathbf{u}_r \tag{2.66}$$

where $\mathbf{u}_\theta$ is the unit vector in the transverse direction.

Equation (2.62) regards the velocity of an arbitrary point P in a rigid body as consisting of two terms, the first representing the velocity of translation of a reference point A and the second representing the velocity due to rotation about A. However, there exists a point C such that the velocity of P can be regarded instantaneously as due entirely to rotation about C. It follows that the point C is instantaneously at rest. For this reason, the point C is called the *instantaneous center of rotation.* Point C does not necessarily lie inside the body. If the velocity vector is known in full (i.e., both the magnitude and direction are known) and if the angular velocity of the body is given, then the instantaneous center can be determined by a graphical procedure. Figure 2.17 shows the rigid body and the velocity vector $\mathbf{v}$ associated with point P. Using radial and transverse coordinates, the velocity vector can be written in the form

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r} = \omega\mathbf{k} \times r\mathbf{u}_r = \omega r\mathbf{u}_\theta = v\mathbf{u}_\theta \tag{2.67}$$

where

$$v = \omega r \tag{2.68}$$

is the magnitude of the velocity vector. Of course, the radius vector $\mathbf{r}$ must be normal to the velocity vector $\mathbf{v}$. From Eq. (2.68), we obtain the magnitude of the

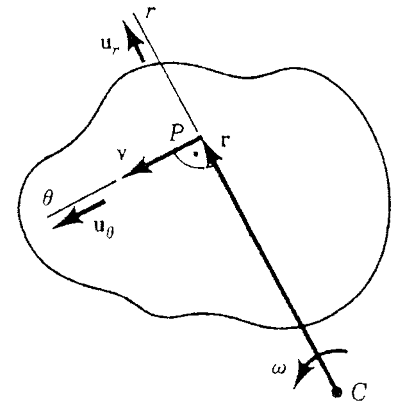

FIGURE 2.17

radius vector by writing simply

$$r = \frac{v}{\omega} \tag{2.69}$$

Figure 2.17 shows the instantaneous center C for the case in which the angular velocity ω is in the counterclockwise sense.

A more common case is that in which the velocity vector $\mathbf{v}_1$ of one point in the body is known in full and the direction of the velocity vector $\mathbf{v}_2$ of a different point is also known. Then, the instantaneous center C can be determined by the graphical procedure depicted in Fig. 2.18. The velocity vectors satisfy the equations

$$\mathbf{v}_1 = \boldsymbol{\omega} \times \mathbf{r}_1, \qquad \mathbf{v}_2 = \boldsymbol{\omega} \times r_2 \tag{2.70a, b}$$

so that the magnitudes of $\mathbf{v}_1$ and $\mathbf{v}_2$ satisfy

$$v_1 = \omega r_1, \qquad v_2 = \omega r_2 \tag{2.71a, b}$$

where r_1 and r_2 can be determined from Fig. 2.18. Then, from Eq. (2.71a), the angular velocity magnitude is simply

$$\omega = \frac{v_1}{r_1} \tag{2.72}$$

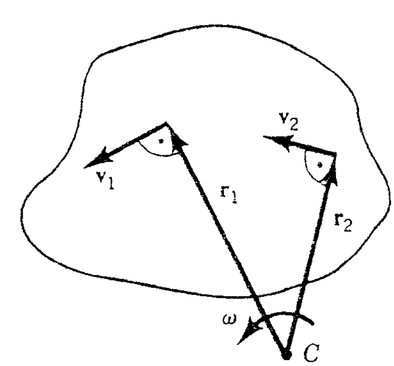

FIGURE 2.18

and, inserting Eq. (2.72) into Eq. (2.71b), we obtain the magnitude of $\mathbf{v}_2$ in the form

$$v_2 = v_1 \frac{r_1}{r_2} \tag{2.73}$$

Point C is an instantaneous center of rotation, which implies that the motion can be interpreted as pure rotation about C only for that particular instant. At a different time, the instantaneous center coincides with a different point. As the motion of the body unfolds, the point C traces a curve in the inertial space representing the locus of the instantaneous centers and known as the *space centrode.* At the same time, the point C traces a curve relative to the body and known as the *body centrode.* The two curves are tangent to one another at any given instant, as shown in Fig. 2.19, and the motion can be interpreted geometrically as the rolling of the body centrode on the space centrode.

The concept of instantaneous center can be used at times to solve kinematical problems.

Example 2.7

The planar framework shown in Fig. 2.20 consists of three rigid bars hinged at the points A, B, C and D. The rigid AB rotates with the angular velocity $\omega_{AB}=5$ rad/s and angular acceleration $\alpha_{AB}=2$ rad/s^2. Calculate the velocity $\mathbf{v}_C$ and acceleration $\mathbf{a}_C$ of point C.

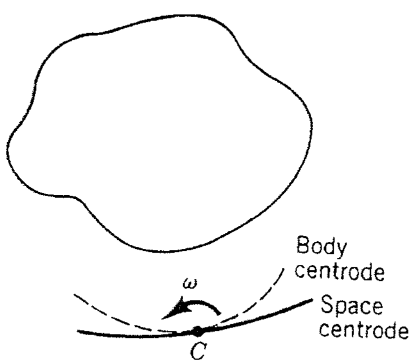

FIGURE 2.19

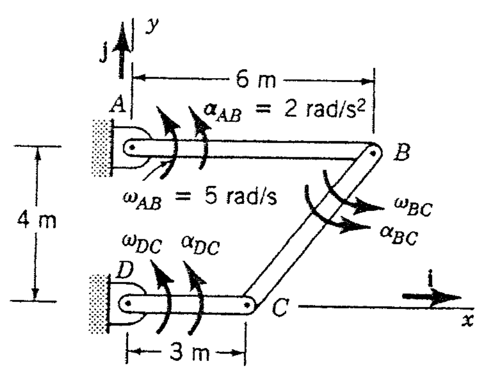

FIGURE 2.20

To solve the problem, we calculate the velocity and acceleration of point C by regarding point C in two ways: (1) as belonging to bar BC and (2) as belonging to bar DC. This procedure yields a sufficient number of equations to solve for all the unknown quantities, including $\mathbf{v}_C$ and $\mathbf{a}_C$.

The given quantities can be written in vector form in terms of cartesian components as follows:

$$\begin{gathered}\mathbf{r}_{AB}=6\mathbf{i}\quad \text{m},\qquad \omega_{AB}=\omega_{AB}\mathbf{k}=5\mathbf{k}\quad \text{rad/s},\\ \alpha_{AB}=\alpha_{AB}\mathbf{k}=2\mathbf{k}\quad \text{rad/s}^2,\qquad \mathbf{r}_{BC}=-3\mathbf{i}-4\mathbf{j}\quad \text{m},\qquad \mathbf{r}_{DC}=3\mathbf{i}\quad \text{m}\end{gathered}\tag{a}$$

where, from Fig. 2.20, the notation is obvious. First, we calculate $\mathbf{v}_C$ and $\mathbf{a}_C$ by regarding C as belonging to bar BC. Before we can do that, we must calculate $\mathbf{v}_B$ and $\mathbf{a}_B$. Considering Eq. (2.62) with $\mathbf{v}_A=\mathbf{0}$ and $P=B$, the velocity of point B is

$$\mathbf{v}_B=\boldsymbol{\omega}_{AB}\times\mathbf{r}_{AB}=-\omega_{AB}(y_{AB}\mathbf{i}-x_{AB}\mathbf{j})=\omega_{AB}x_{AB}\mathbf{j}=5\times 6\mathbf{j}=30\mathbf{j}\quad \text{m/s}\tag{b}$$

and considering Eq. (2.63) with $\mathbf{a}_A=\mathbf{0}$ the acceleration of point B is

$$\begin{aligned}\mathbf{a}_B&=\boldsymbol{\alpha}_{AB}\times\mathbf{r}_{AB}+\boldsymbol{\omega}_{AB}\times(\boldsymbol{\omega}_{AB}\times\mathbf{r}_{AB})=-\alpha_{AB}(y_{AB}\mathbf{i}-x_{AB}\mathbf{j})-\omega_{AB}^2\mathbf{r}_{AB}\\ &=-2\times 6\mathbf{j}-5^2\times 6\mathbf{i}=-150\mathbf{i}+12\mathbf{j}\quad \text{m/s}^2\end{aligned}\tag{c}$$

Similarly, using Eqs. (2.62) and (2.63) with B and C replacing A and P, respectively, we can write

$$\begin{aligned}\mathbf{v}_C&=\mathbf{v}_B+\boldsymbol{\omega}_{BC}\times\mathbf{r}_{BC}=30\mathbf{j}-\omega_{BC}(y_{BC}\mathbf{i}-x_{BC}\mathbf{j})\\ &=4\omega_{BC}\mathbf{i}+(30-3\omega_{BC})\mathbf{j}\quad \text{m/s}\end{aligned}\tag{d}$$

$$\begin{aligned}\mathbf{a}_C&=\mathbf{a}_B+\boldsymbol{\alpha}_{BC}\times\mathbf{r}_{BC}+\boldsymbol{\omega}_{BC}\times(\boldsymbol{\omega}_{BC}\times\mathbf{r}_{BC})\\ &=-150\mathbf{i}+12\mathbf{j}-\alpha_{BC}(y_{BC}\mathbf{i}-x_{BC}\mathbf{j})-\omega_{BC}^2\mathbf{r}_{BC}\\ &=(-150+4\alpha_{BC}+3\omega_{BC}^2)\mathbf{i}+(12-3\alpha_{BC}+4\omega_{BC}^2)\mathbf{j}\quad \text{m/s}^2\end{aligned}\tag{e}$$

Next, we regard point C as belonging to bar DC. The velocity of point C is simply

$$\mathbf{v}_C=\boldsymbol{\omega}_{DC}\times\mathbf{r}_{DC}=\omega_{DC}x_{DC}\mathbf{j}=3\omega_{DC}\mathbf{j}\quad \text{m/s}\tag{f}$$

and the acceleration of C is

$$\begin{aligned}\mathbf{a}_C&=\boldsymbol{\alpha}_{DC}\times\mathbf{r}_{DC}+\boldsymbol{\omega}_{DC}\times(\boldsymbol{\omega}_{DC}\times\mathbf{r}_{DC})\\ &=\alpha_{DC}x_{DC}\mathbf{j}-\omega_{DC}^2\mathbf{r}_{DC}=-3\omega_{DC}^2\mathbf{i}+3\alpha_{DC}\mathbf{j}\quad \text{m/s}^2\end{aligned}\tag{g}$$

Comparing the components of $\mathbf{v}_C$ in Eqs. (d) and (f), we conclude that

$$\omega_{BC}=0,\qquad \omega_{DC}=10\quad \text{rad/s}\tag{h}$$

so that

$$\mathbf{v}_C=30\mathbf{j}\quad \text{m/s}\tag{i}$$

Similarly, comparing the components of $\mathbf{a}_C$ in Eqs. (e) and (g), we obtain

$$\alpha_{BC}=-37.5\quad \text{rad/s}^2,\qquad \alpha_{DC}=-33.5\quad \text{rad/s}^2\tag{j}$$

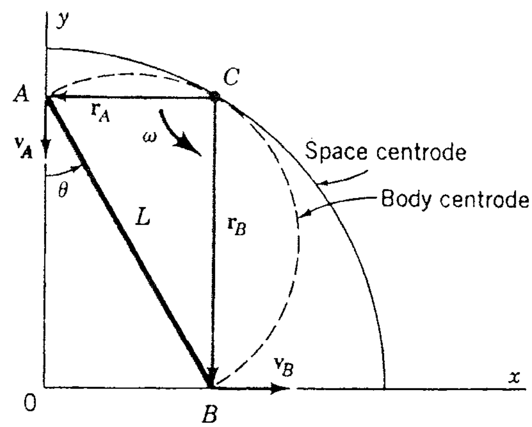

FIGURE 2.21

so that

$$\mathbf{a}_C = -300\mathbf{i} - 100.5\mathbf{j} \quad \text{m/s}^2 \tag{k}$$

Example 2.8
A rod of length $L = 10$ m slides with the end A touching the wall and the end B touching the floor (Fig. 2.21). If the velocity of A has the magnitude $v_A = 50$ m/s when the angle between the rod and the wall is $\theta = 30°$, determine the angular velocity of the rod and the velocity of B by using the concept of instantaneous center. Then, plot the body and space centrodes.

The instantaneous center lies at the intersection of the normal to the wall at A and the normal to the floor at B, as shown in Fig. 2.21. From Fig. 2.21, we obtain

$$r_A = L \sin\theta = 10 \times 0.5 = 5 \quad \text{m}, \qquad r_B = L\cos\theta = 10 \times 0.866 = 8.66 \quad \text{m} \tag{a}$$

Then, the angular velocity is simply

$$\omega = \frac{v_A}{r_A} = \frac{50}{5} = 10 \quad \text{rad/s} \tag{b}$$

in the counterclockwise sense and the magnitude of the velocity vector $\mathbf{v}_B$ is

$$v_B = \omega r_B = 10 \times 8.66 = 86.6 \quad \text{m/s} \tag{c}$$

Of course, the velocity is in the x-direction, as shown in Fig. 2.21.

From Fig. 2.21, we observe that the points A, 0, B, and C are the corners of a rectangle with the diagonals equal to L. Hence, C is always at a distance L from 0, so that the space centrode represents one quarter of a circle of radius L and with the center at 0. On the other hand, the vectors $\mathbf{r}_A$ and $\mathbf{r}_B$ make a 90° angle at C. From geometry, the locus of the points representing the corner of a right triangle corresponding to the right angle is a semicircle with the diameter coinciding with the side of the triangle opposing the right angle. This semicircle is the body centrode. The space and body centrodes are shown in Fig. 2.21 in solid and dashed lines, respectively. It is clear from Fig. 2.21 that, as the rod slides, the body centrode rolls on the space centrode.

2.6 GENERAL CASE OF MOTION

Let us return to the system of Section 2.4 and consider the case in which the particle P is no longer at rest relative to the moving frame xyz, but can move relative to that frame. To treat this case, it is advisable to derive first an expression for the time derivative of a vector with time-dependent magnitude and embedded in a rotating reference frame. Considering an arbitrary vector $\mathbf{r}$ and referring to Fig. 2.11, we can write

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \tag{2.74}$$

where x, y, and z are the cartesian components of the vector and $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are unit vectors along these axes. In the case at hand, the components x, y, and z of the vector $\mathbf{r}$ are not constant, and, of course, the unit vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are not constant either, as they rotate with the same angular velocity ω as the moving frame. Hence, the time derivative of the vector $\mathbf{r}$ can be written in the form

$$\dot{\mathbf{r}} = \dot{x}\mathbf{i} + x\dot{\mathbf{i}} + \dot{y}\mathbf{j} + y\dot{\mathbf{j}} + \dot{z}\mathbf{k} + z\dot{\mathbf{k}} \tag{2.75}$$

where the time derivative of the unit vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ can be obtained from Eq. (2.51), as the derivative of any vector embedded in a rotating reference frame. Hence, replacing $\mathbf{r}$ by $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ in Eq. (2.51) in sequence, we obtain

$$\dot{\mathbf{i}} = \omega \times \mathbf{i}, \qquad \dot{\mathbf{j}} = \omega \times \mathbf{j}, \qquad \dot{\mathbf{k}} = \omega \times \mathbf{k} \tag{2.76}$$

Inserting Eqs. (2.76) into Eq. (2.75), we obtain the time derivative of $\mathbf{r}$

$$\dot{\mathbf{r}} = \dot{x}\mathbf{i} + \dot{y}\mathbf{j} + \dot{z}\mathbf{k} + \omega \times (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) = \dot{\mathbf{r}}' + \omega \times \mathbf{r} \tag{2.77}$$

where

$$\dot{\mathbf{r}}' = \dot{x}\mathbf{i} + \dot{y}\mathbf{j} + \dot{z}\mathbf{k} \tag{2.78}$$

can be identified as the time rate of change of $\mathbf{r}$ regarding the reference frame xyz as inertial.

Equation (2.77) can be used to derive the velocity and acceleration of point P. From Fig. 2.22, the position vector of point P has the form

$$\mathbf{R} = \mathbf{r}_A + \mathbf{r}_{AP} \tag{2.79}$$

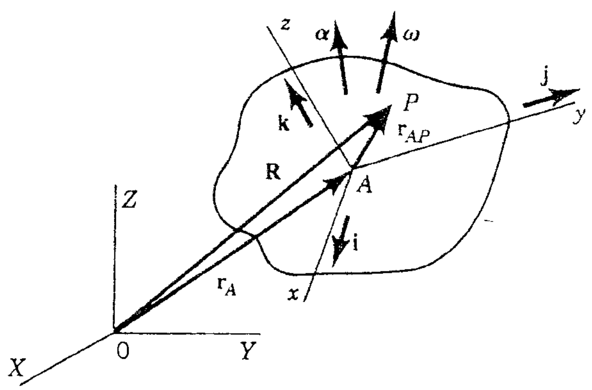

FIGURE 2.22

so that, using Eq. (2.77), we can write the absolute velocity of P as follows:

$$\mathbf{v} = \dot{\mathbf{R}} = \dot{\mathbf{r}}_A + \dot{\mathbf{r}}_{AP} = \mathbf{v}_A + \mathbf{v}'_{AP} + \boldsymbol{\omega} \times \mathbf{r}_{AP} \tag{2.80}$$

where $\mathbf{v}_A$ is the velocity of the origin A relative to the inertial space,

$$\mathbf{v}'_{AP} = \dot{x}_{AP}\mathbf{i} + \dot{y}_{AP}\mathbf{j} + \dot{z}_{AP}\mathbf{k} \tag{2.81}$$

is the velocity of P relative to the moving frame xyz, in which x_{AP}, y_{AP}, and z_{AP} are the cartesian components of $\mathbf{r}_{AP}$, and $\boldsymbol{\omega} \times \mathbf{r}_{AP}$ is the velocity of P due entirely to the rotation of the frame xyz. Similarly, from Eq. (2.80), we write the absolute acceleration of P in the form

$$\begin{aligned}
\mathbf{a} = \dot{\mathbf{v}} &= \dot{\mathbf{v}}_A + \frac{d}{dt}(\mathbf{v}'_{AP}) + \dot{\boldsymbol{\omega}} \times \mathbf{r}_{AP} + \boldsymbol{\omega} \times \dot{\mathbf{r}}_{AP} \\
&= \mathbf{a}_A + \mathbf{a}'_{AP} + \boldsymbol{\omega} \times \mathbf{v}'_{AP} + \boldsymbol{\alpha} \times \mathbf{r}_{AP} + \boldsymbol{\omega} \times (\mathbf{v}'_{AP} + \boldsymbol{\omega} \times \mathbf{r}_{AP}) \\
&= \mathbf{a}_A + \mathbf{a}'_{AP} + 2\boldsymbol{\omega} \times \mathbf{v}'_{AP} + \boldsymbol{\alpha} \times \mathbf{r}_{AP} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_{AP})
\end{aligned} \tag{2.82}$$

where $\mathbf{a}_A = \dot{\mathbf{v}}_A$ is the acceleration of A relative to the inertial space,

$$\mathbf{a}'_{AP} = \ddot{x}_{AP}\mathbf{i} + \ddot{y}_{AP}\mathbf{j} + \ddot{z}_{AP}\mathbf{k} \tag{2.83}$$

is the acceleration of P relative to the rotating frame xyz, $2\boldsymbol{\omega} \times \mathbf{v}'_{AP}$ is the so-called *Coriolis acceleration*, and $\boldsymbol{\alpha} \times \mathbf{r}_{AP} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_{AP})$ is the acceleration of P due entirely to the rotation of the frame xyz, in which $\boldsymbol{\alpha} = \dot{\boldsymbol{\omega}}$ is the angular acceleration of the frame.

Example 2.9

An automobile travels north at the constant speed v relative to the ground. The earth rotates with the angular velocity $\boldsymbol{\Omega}$ relative to the inertial space, where $\boldsymbol{\Omega}$ can be assumed to be constant. Ignore the motion of the earth's center 0 relative to the inertial space, and calculate the velocity and acceleration of the automobile when the automobile is at a latitude λ.

Let us consider a reference frame xyz, with x pointing south, y pointing east, and z pointing to the zenith (Fig. 2.23). Then, recognizing that point P coincides

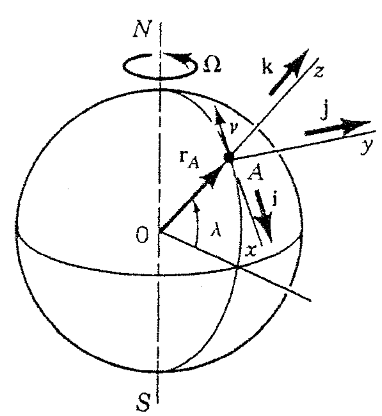

FIGURE 2.23

with point A, and that the automobile travels on a circular path of radius r_A, we can write

$$\mathbf{r}_A = r_A\mathbf{k}, \qquad \boldsymbol{\omega} = \Omega(-\cos\lambda\,\mathbf{i} + \sin\lambda\,\mathbf{k}),$$
$$\mathbf{r}_{AP} = \mathbf{0}, \qquad \mathbf{v}'_{AP} = -v\mathbf{i}, \qquad \mathbf{a}'_{AP} = -\frac{v^2}{r_A}\mathbf{k} \tag{a}$$

where r_A and Ω are the magnitudes of $\mathbf{r}_A$ and $\boldsymbol{\Omega}$, respectively. From Eq. (2.80), we can write the velocity of the automobile

$$\mathbf{v} = \mathbf{v}_A + \mathbf{v}'_{AP} = \mathbf{v}_A - v\mathbf{i} \tag{b}$$

where the velocity of the origin A is

$$\mathbf{v}_A = \boldsymbol{\omega}\times\mathbf{r}_A = \Omega(-\cos\lambda\,\mathbf{i} + \sin\lambda\,\mathbf{k})\times r_A\mathbf{k} = \Omega r_A\cos\lambda\,\mathbf{j} \tag{c}$$

Hence,

$$\mathbf{v} = -v\mathbf{i} + \Omega r_A\cos\lambda\,\mathbf{j} \tag{d}$$

Using Eq. (2.82), we can write the acceleration of the automobile in the form

$$\begin{aligned}\mathbf{a} &= \mathbf{a}_A + \mathbf{a}'_{AP} + 2\boldsymbol{\omega}\times\mathbf{v}'_{AP} = \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}_A) + \mathbf{a}'_{AP} + 2\boldsymbol{\omega}\times\mathbf{v}'_{AP}\\ &= \Omega(-\cos\lambda\,\mathbf{i} + \sin\lambda\,\mathbf{k})\times\Omega r_A\cos\lambda\,\mathbf{j} - \frac{v^2}{r_A}\mathbf{k} + 2\Omega(-\cos\lambda\,\mathbf{i} + \sin\lambda\,\mathbf{k})\times(-v\mathbf{i})\\ &= -\Omega^2 r_A\sin\lambda\cos\lambda\,\mathbf{i} - 2\Omega v\sin\lambda\,\mathbf{j} - \left(\frac{v^2}{r_A} + \Omega^2 r_A\cos^2\lambda\right)\mathbf{k}\end{aligned} \tag{e}$$

Example 2.10

A sprinkler rotates with the constant angular velocity ω, as shown in Fig. 2.24. The pipe is in the form of an arc of a circle of radius of curvature ρ. Determine the velocity and acceleration of a particle of water as it leaves the pipe with the constant velocity $\dot{s}$ relative to the pipe.

Let us introduce a reference frame xyz, with x and y as shown in Fig. 2.24 and with z perpendicular to x and y, and let $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ be unit vectors along these axes. In addition, let t and n be tangential and normal directions with associated unit vectors $\mathbf{u}_t$ and $\mathbf{u}_n$, where $\mathbf{u}_t$ makes a 30° angle relative to $\mathbf{i}$. Note that point A

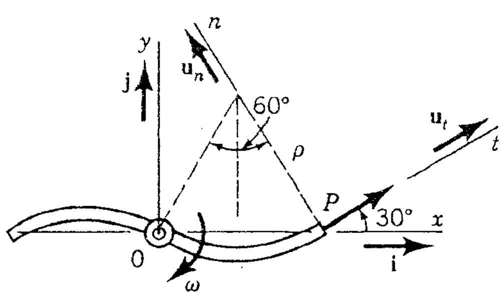

FIGURE 2.24

coincides with point 0. In terms of this notation, we have

$$\mathbf{r}_{AP}=\rho\mathbf{i}, \qquad \boldsymbol{\omega}=-\omega\mathbf{k}, \qquad \mathbf{v}'_{AP}=v_t\mathbf{u}_t=\dot{s}\mathbf{u}_t=\frac{\dot{s}}{2}(\sqrt{3}\mathbf{i}+\mathbf{j}) \tag{a}$$

Then, using Eq. (2.80) and recognizing that $\mathbf{v}_A=\mathbf{0}$, we can write

$$\begin{aligned}\mathbf{v}&=\mathbf{v}'_{AP}+\boldsymbol{\omega}\times\mathbf{r}_{AP}=\frac{\dot{s}}{2}(\sqrt{3}\mathbf{i}+\mathbf{j})+(-\omega\mathbf{k})\times\rho\mathbf{i}\\&=\frac{\sqrt{3}}{2}\dot{s}\mathbf{i}+\left(\frac{\dot{s}}{2}-\omega\rho\right)\mathbf{j}\end{aligned} \tag{b}$$

To calculate the acceleration, we recognize that, although $\dot{s}$ is constant in magnitude, we have the relative acceleration

$$\mathbf{a}'_{AP}=a_n\mathbf{u}_n=\frac{\dot{s}^2}{\rho}\mathbf{u}_n=\frac{\dot{s}^2}{2\rho}(-\mathbf{i}+\sqrt{3}\mathbf{j}) \tag{c}$$

Moreover, $\mathbf{a}_A=\dot{\boldsymbol{\omega}}=\mathbf{0}$, so that Eq. (2.82) yields

$$\begin{aligned}\mathbf{a}&=\mathbf{a}'_{AP}+2\boldsymbol{\omega}\times\mathbf{v}'_{AP}+\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}_{AP})\\&=\frac{\dot{s}^2}{2\rho}(-\mathbf{i}+\sqrt{3}\mathbf{j})+2(-\omega)\mathbf{k}\times\frac{\dot{s}}{2}(\sqrt{3}\mathbf{i}+\mathbf{j})+(-\omega\mathbf{k})\times(-\omega\rho\mathbf{j})\\&=\left(-\frac{\dot{s}^2}{2\rho}+\omega\dot{s}-\omega^2\rho\right)\mathbf{i}+\left(\frac{\sqrt{3}\dot{s}^2}{2\rho}-\sqrt{3}\omega\dot{s}\right)\mathbf{j}\end{aligned} \tag{d}$$

PROBLEMS

2.1 An automobile moves according to the formula $s(t)=-0.05t^3+2t^2+100$, where s is the distance in meters and t is the time in seconds. Determine the following: (1) the initial acceleration, (2) the velocity reached in 10 s, (3) the maximum velocity reached and the time when this occurs, and (4) the distance at $t=20$ s.

2.2 Plot the distance, velocity, and acceleration as functions of time for the automobile of Problem 2.1 for $0<t<20$ s.

2.3 An automobile is traveling at 72 km/h when the driver observes that a traffic light 240 m ahead is turning red. The driver knows that the traffic light is timed to stay red for 20 s. Determine the following: (1) the uniform deceleration that would permit the driver to reach the traffic light just as it turns green and (2) the speed of the automobile as it reaches the traffic light. Derive an expression for the distance as a function of time.

2.4 Water is dripping from a leaking faucet at constant intervals of time. At the instant a drop is ready to fall, the preceding drop has already fallen 1.25 m. Determine the following: (1) the interval of time between the beginning of free fall of two succeeding water drops and (2) the distance between the two drops as a function of time.

2.5 Rocks leave the chute shown in Fig. 2.25 with the velocity $v = 5$ m/s. The rocks are to drop in a basket 10 m below. At what distance B should the basket be placed? Determine the velocity and the angle with respect to the horizontal at which the rocks enter the basket.

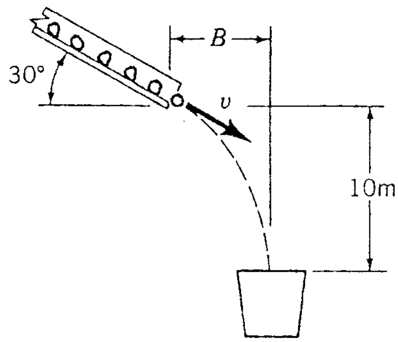

FIGURE 2.25

2.6 A particle travels in a resisting medium with the acceleration $a = -0.01v^2$. If the particle begins its motion with the velocity $v_0 = 15$ m/s, determine the distance traveled by the time the velocity falls to $v = 10$ m/s.

2.7 A particle moves with the acceleration $a = -cs$, where c is a constant and s is the distance. Determine an equation relating the velocity v and the distance s, and plot the function $v = v(s)$ for the initial values $s_0 = 0$, $v_0 = 2$, and the constant $c = 4$.

2.8 A satellite moves around the earth in a circular orbit of radius $R = 6600$ km. Assuming that the radial acceleration (due to the attraction of the earth) is equal to $0.94g$, where g is the acceleration due to gravity at sea level, determine the velocity of the satellite and the orbital period (the time necessary for one complete revolution).

2.9 The orbit of a satellite moving around the earth is given by the expression

$$r = \frac{p}{1 + e \cos \theta}$$

where r is the magnitude of the radius vector from the center of the earth to the satellite and θ is the angle between a given reference direction and the radius vector. Knowing that the satellite motion satisfies the equation

$$r^2 \dot{\theta} = h = \text{const}$$

determine the velocity and acceleration of the satellite as a function of θ.

2.10 A cam rotates with the constant angular velocity ω, as shown in Fig. 2.26. The follower is constrained so as to move vertically only. Assuming that the roller is always in contact with the cam, determine the velocity and acceleration of the follower if the shape of the cam is given by

$$r = e - b\cos\theta, \qquad e > b$$

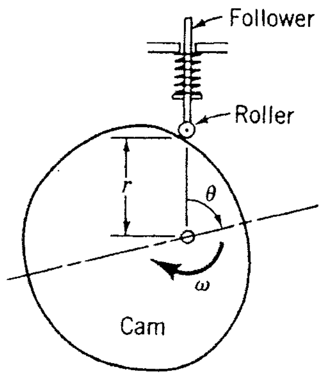

FIGURE 2.26

2.11 Consider the particle of Fig. 2.4, and determine the radius of curvature of the trajectory immediately after the particle leaves point 0.

2.12 Determine the radius of curvature of the trajectory described by the rocks of Problem 2.5 at the time they enter the basket.

2.13 During a rainstorm the raindrops have been observed from a side window of an automobile stopped at a red light to fall with the velocity of 12 m/s at an angle of 10° with respect to the vertical. After the light has changed, the automobile resumed its motion and reached a given cruising speed at which time the raindrops were observed from the same side window to fall at an angle of 40° with respect to the vertical. Determine the velocity of the automobile and of the raindrops relative to the automobile.

2.14 A boy on a bicycle traveling at 5 m/s wants to throw a ball into an open truck directly ahead of him and traveling at 10 m/s on a street making a 90° angle with the street on which he is traveling. If he can throw the ball with the velocity of 20 m/s relative to the bicycle, determine the direction in which he must throw the ball and the velocity of the ball relative to the ground.

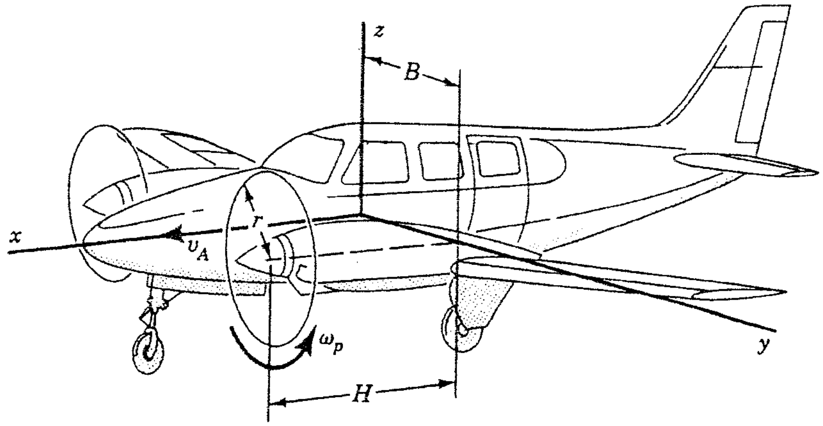

FIGURE 2.27

2.15 The propeller airplane shown in Fig. 2.27 is taxiing in a straight line with the uniform velocity v_A while the propeller rotates with the angular velocity ω_p. Determine the velocity and acceleration of a point at the tip of the propeller blade when the blade is horizontal.

2.16 Repeat Problem 2.15 for the case in which the airplane makes a turn of radius R to the right.

2.17 The gun turret shown in Fig. 2.28 rotates with the angular velocity ω_t and angular acceleration α_t when the gun barrel is being raised with the angular velocity ω_b. Determine the velocity and acceleration of a point at the end of the barrel when the barrel makes an arbitrary angle θ with respect to the horizontal.

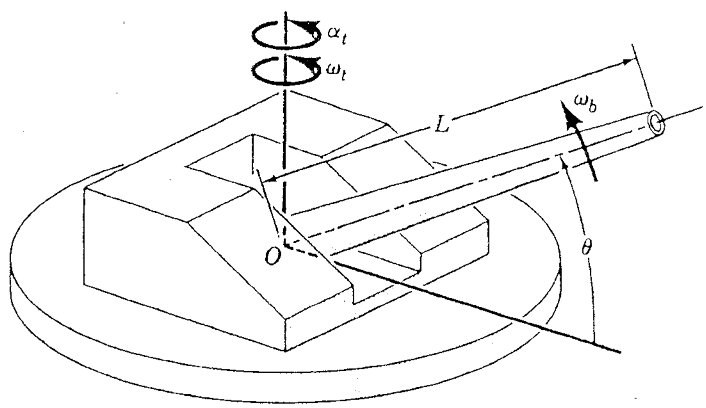

FIGURE 2.28

2.18 A disk of radius r rolls without slip inside a circular cylinder of radius R, as shown in Fig. 2.29. Determine the acceleration of point A and the velocity and acceleration of point B. Express the results in terms of $\dot{\theta}$ and $\ddot{\theta}$.

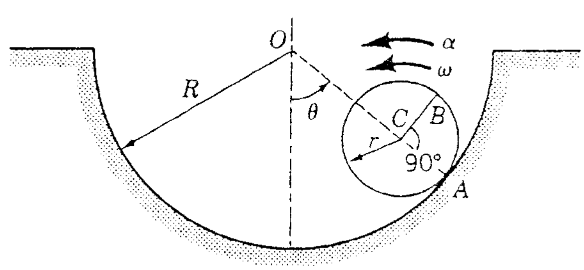

FIGURE 2.29

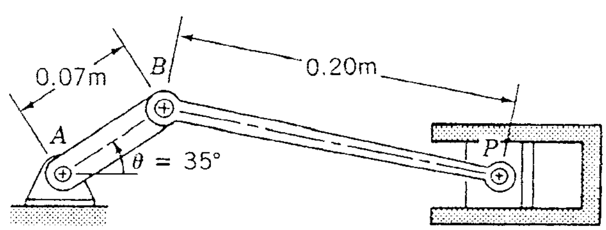

FIGURE 2.30

2.19 The crank AB of the automobile engine shown in Fig. 2.30 rotates counterclockwise with the constant angular velocity of 3000 rotations per minute (rpm). Determine the following: (1) the angular velocity of the connecting rod BP, (2) the velocity of the piston P, (3) the angular acceleration of the connecting rod, and (4) the acceleration of the piston.

2.20 In the position shown in Fig. 2.31, bar AB rotates counterclockwise with the constant angular velocity $\omega_{AB} = 10$ rad/s. Determine the following: (1) the angular velocity of bar BC, (2) the velocity of point C, (3) the angular velocity of bar CD, (4) the angular acceleration of bar BC, (5) the angular acceleration of bar CD, and (6) the acceleration of point C.

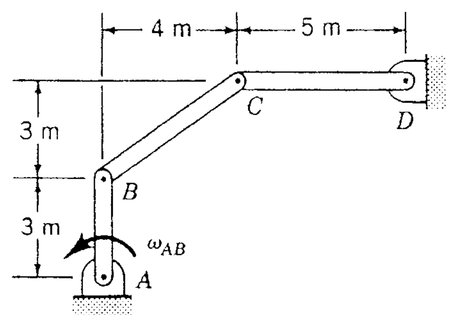

FIGURE 2.31

2.21 Solve the problem of Example 2.6 by using only the rotating reference frame xyz and by regarding the motion of P as motion relative to that frame.

2.22 Solve Problem 2.16 by using only one reference frame, namely, one attached to the airplane.

2.23 Solve Problem 2.17 by using only one reference frame, namely, one attached to the turret.

2.24 A river is flowing east with the constant speed of 3 km/h. Determine the acceleration of a drop of water. The latitude of the river is $\lambda = 30°$.