Fundamentals of Control Systems

MECH 420 / ELEC 436
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3. Laplace Transform Domain

MATHEMATICAL BACKGROUND

Mass-spring-damper equation

$$m\frac{d^2y(t)}{dt^2} + c\frac{dy(t)}{dt} + ky(t) = f(t) + \frac{df(t)}{dt}$$

Take ***<u>Laplace Transform</u>*** and assume zero initial conditions

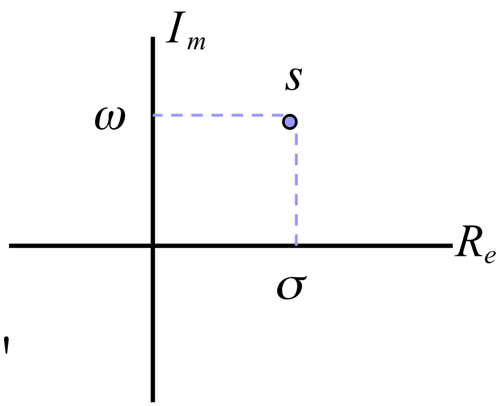
$$\frac{Y(s)}{F(s)} = \frac{s+1}{ms^2 + cs + k} = \frac{Output}{Input}$$

MATHEMATICAL BACKGROUND

1. Complex Variables

- $s = \sigma + j\omega$ $\quad\quad \sigma$: real

  ω : imaginary



- G(s) is a function of the complex variable "s"

$$\Rightarrow G(s) = R_e[G(s)] + jI_m[G(s)]$$

Analytic function: A function *G(s)* of *s* is called an analytic function in a region of the *s*-plane if *G(s)* and all its derivatives (Cauchy-Riemann condition) exist in the region.

Ex. $G(s) = \dfrac{1}{(s+1)(s+2)}$ $\quad\quad$ $G(s) \rightarrow \infty$ as $s \rightarrow -2$ and/or $s \rightarrow -1$

G(s) is analytic at every point in the s-plane except at s = -2 and s = -1

Poles of a function of complex variables

$\text{G(s)}: \text{ analytic and single-valued near } s_i$

$\text{G(s) has a } \underline{\text{pole}} \text{ of order "r" at } s = s_i \text{ if}$

$$\lim_{s \to s_i}\left[(s - s_i)^r G(s)\right] \text{ is } \underline{\text{finite}} \text{ and } \underline{\text{nonzero}}$$

$\text{If } r = 1,\ s = s_i \leftarrow \text{ simple pole}$

$$\text{Ex. } G(s) = \frac{s+1}{s(s+10)^2(s+2)}$$

has: - pole of order 2 @ s=-10
- simple poles @ s=0 and s=-2

Zeros of a function of complex variables

$\text{G(s) analytic @ } s = s_i$

$\text{G(s) has a } \underline{\text{zero}} \text{ of order "r" @ } s = s_i \text{ if}$

$$\lim_{s \to s_i}\left[(s - s_i)^{-r} G(s)\right] \text{ is } \underline{\text{finite}} \text{ and } \underline{\text{nonzero}}$$

$$\text{Ex. } G(s) = \frac{s+1}{s(s+10)^2(s+2)} \qquad \text{has: simple zero @ } s = -1$$

For a rational function of *s*, i.e. $\dfrac{Polynomial\ of\ s}{Polynomial\ of\ s} = \dfrac{Q(s)}{P(s)}$

$$\#\ \text{of poles} = \#\ \text{of zeros}$$

Example :

$$G(s) = \frac{s+1}{s(s+10)^2(s+2)}$$

$$\text{Poles} = 0,\ -10,\ -10,\ -2$$

$$\text{Zeros} = -1,\ \infty,\ \infty,\ \infty \qquad \left(\lim_{s\to\infty} G(s) = \lim_{s\to\infty} \frac{1}{s^3} = 0 \right)$$

2. Laplace Transform

f(t) = real function

$$F(s) = \mathcal{L}[f(t)] = \int_0^{\infty} f(t)e^{-st}dt$$ ← Laplace Transform of f(t)

$$f(t) = \mathcal{L}^{-1}[F(s)]$$ ← Inverse Laplace Transform

- Properties:

$\mathcal{L}[kf(t)] = kF(s)$ k = constant

$$\mathcal{L}[f_1(t) \pm f_2(t)] = F_1(s) \pm F_2(s)$$

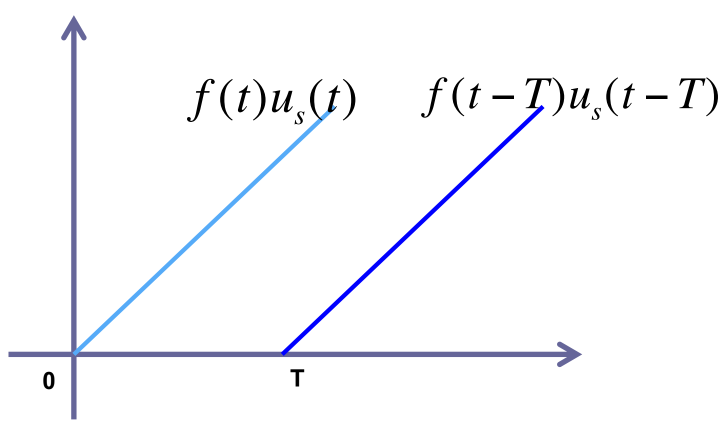
$$\mathcal{L}[\frac{d^n f(t)}{dt^n}] = s^n F(s) - s^{n-1}f(0) - s^{n-2}\frac{df}{dt}(0) - \ldots\ldots\ldots\ldots - \frac{d^{n-1}f(0)}{dt^{n-1}}$$

So $\mathcal{L}(\dot{f}) = sF(s) - f(0)$

$$\mathcal{L}[\int_0^{t_1}\int_0^{t_2}\ldots\ldots\int_0^{t_n} f(\tau)d\tau dt_1 dt_2 \ldots dt_{n-1}] = \frac{F(s)}{s^n}$$

So $\mathcal{L}[\int_0^{t_1} f(\tau)d\tau] = \frac{F(s)}{s}$

$\mathcal{L}[f(t-T)u_s(t-T)] = e^{-Ts}F(s)$ shift in Time by T



- Property: Initial Value Theorem:

$$\boxed{\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s) \qquad \text{If the time limit exists}}$$

$$e^{-\alpha t} \leftrightarrow \frac{1}{s+\alpha} \quad ; \quad sF(s) = \frac{s}{s+\alpha} = \frac{1}{1+\frac{\alpha}{s}}$$

- Property: Final Value Theorem:

Provided all of the poles of [sF(s)] lie strictly in the left half plane $\boxed{\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)}$

$$\text{Ex. } F(s) = \frac{5}{s(s+1)} \quad ; \quad sF(s) = \frac{5}{s+1} \quad \Rightarrow \quad \lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s) = 5$$

$$F(s) = \frac{\omega}{s^2+\omega^2} \quad ; \quad sF(s) = \frac{s\omega}{s^2+\omega^2} \quad \leftarrow \text{ has poles on the imaginary axis}$$

$\Rightarrow$ Final value Thm can not apply. In fact, $f(t) = \sin(\omega t)$

- Properties-continued

$$\mathcal{L}[e^{\pm\alpha t} f(t)] = F(s \pm \alpha) \qquad \leftarrow \text{Complex Shifting}$$

$$\mathcal{L}[\underbrace{f_1(t) * f_2(t)}] = F_1(s)F_2(s) \qquad \leftarrow \text{Convolution}$$

$$\int_0^t f_1(\tau) f_2(t-\tau)\,d\tau = \int_0^t f_2(\tau) f_1(t-\tau)\,d\tau$$

Partial-Fraction Expansion

$$X(s) = \frac{Q(s)}{P(s)}$$

(Useful in inverse Laplace transform)

Case 1: All poles are simple and real

$$X(s) = \frac{5s+3}{(s+1)(s+2)(s+3)}$$

$$= \frac{K_1}{s+1} + \frac{K_2}{s+2} + \frac{K_3}{s+3}$$

$$K_1 = (s+1)X(s)\big|_{s=-1}$$

$$= \frac{5(-1)+3}{(-1+2)(-1+3)} = -1$$

$$K_2 = (s+2)X(s)\big|_{s=-2}$$

$$= \frac{5(-2)+3}{(-2+2)(-2+3)} = 7$$

$$K_3 = (s+3)X(s)\big|_{s=-3}$$

$$= \frac{5(-3)+3}{(-3+2)(-3+3)} = -6$$

Case 2: Some poles are of multiple order

$$X(s) = \frac{1}{s(s+1)^3(s+2)}$$

$$= \frac{K_0}{s} + \frac{K_1}{s+2} + \frac{A_1}{(s+1)} + \frac{A_2}{(s+1)^2} + \frac{A_3}{(s+1)^3}$$

$$K_0 = sX(s)\big|_{s=0} = \frac{1}{2}$$

$$K_1 = (s+2)X(s)\big|_{s=-2} = \frac{1}{2}$$

$$A_3 = (s+1)^3 X(s)\big|_{s=-1} = -1$$

$$A_2 = \frac{d}{ds}[(s+1)^3 X(s)]\big|_{s=-1} = \frac{d}{ds}\left[\frac{1}{s(s+2)}\right]\bigg|_{s=-1} = 0$$

$$A_1 = \frac{1}{2}\frac{d^2}{ds^2}[(s+1)^3 X(s)]\big|_{s=-1} = \frac{1}{2}\frac{d^2}{ds^2}\left[\frac{1}{s(s+2)}\right]\bigg|_{s=-1} = -1$$

Case 3: Simple complex-conjugate poles

$$X(s) = \frac{\omega n^2}{s^2 + 2\xi\omega_n s + \omega_n{}^2}$$

$(\xi,\ \omega_{\mathrm{n}}, s.t$ poles are complex)

$$X(s) = \frac{K_1}{s+\alpha-j\omega} + \frac{K_2}{s+\alpha+j\omega} \quad ; \quad \alpha = \xi\omega_n \ ;\ \omega = \omega_n\sqrt{1-\xi^2}$$

$$K_1 = (s+\alpha-j\omega)X(s)\Big|_{s=-\alpha+j\omega} = \frac{\omega_n^2}{2j\omega}$$

$$K_2 = (s+\alpha+j\omega)X(s)\Big|_{s=-\alpha-j\omega} = -\frac{\omega_n^2}{2j\omega}$$

Case 4: General case

Use intuition

Example: *Find the solution x(t) of* $\ddot{x} + 2\dot{x} + 5x = 3 \quad ; \quad x(0) = 0, \quad \dot{x}(0) = 0$

$$s^2 X(s) + 2sX(s) + 5X(s) = \frac{3}{s}$$

$$X(s)\left[s^2 + 2s + 5\right] = \frac{3}{s} \quad \Rightarrow \quad X(s) = \frac{3}{s(s^2 + 2s + 5)} = \frac{3}{s\left[(s+1)^2 + 2^2\right]}$$

$$\text{So } \ X(s) = \frac{3}{s\left[(s+1)^2 + 2^2\right]} = \frac{A}{s} + \frac{B}{(s+1)^2 + 2^2} = \frac{A\left[(s+1)^2 + 2^2\right] + Bs}{s\left[(s+1)^2 + 2^2\right]} = \frac{A(s^2 + 2s + 5) + Bs}{s\left[(s+1)^2 + 2^2\right]}$$

$$= \frac{5A + As^2 + (2A + B)s}{s\left[(s+1)^2 + 2^2\right]}$$

$$3 = 5A \Rightarrow A = \frac{3}{5}$$

$$\frac{3}{5}s^2 + \frac{6}{5}s + Bs = 0 \quad \Rightarrow \quad B = -\frac{3}{5}s - \frac{6}{5} = -\frac{3}{5}(s+2)$$

$$\Rightarrow X(s) = \frac{3}{5}\frac{1}{s} - \frac{3}{5}\frac{s+2}{(s+1)^2 + 2^2} = \frac{3}{5}\frac{1}{s} - \frac{3}{5}\left(\frac{1 + s + 1}{(s+1)^2 + 2^2}\right) = \frac{3}{5}\frac{1}{s} - \frac{3}{20}\frac{2^2}{(s+1)^2 + 2^2} - \frac{3}{5}\frac{s+1}{(s+1)^2 + 2^2}$$

$$\boxed{\Rightarrow x(t) = \frac{3}{5}u(t) - \frac{3}{20}e^{-t}\sin 2t - \frac{3}{5}e^{-t}\cos 2t}$$