Department of Mechanical Engineering
MECH 411/501 Fall 19
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Dynamics & Control of Mechanical Systems

Newtonian Particle Dynamics

Dynamics of a Particle

- A Particle is an idealization of a material body whose dimensions are very small when compared with the distance to other bodies.

  $\rightarrow$ can be regarded as a point mass.

- In our analysis, we assume existence of inertial systems of reference; i.e, systems of reference that are either at rest or moving with uniform velocity relative to a fixed reference frame.

- Absolute motion: motion measured relative to an inertial frame.

Newton's Laws

- <u>First Law:</u> "If there are no forces acting on a particle, then the particle will move in a straight line with constant velocity".

$$\text{If } \underbrace{\mathbf{F}}_{\text{Resultant force vector}} = 0\,,\quad \text{then } \underbrace{\mathbf{v}}_{\text{Absolute velocity vector}} = \text{constant}$$

- <u>Second Law:</u> "A particle acted on by a force moves so that the force vector is equal to the time rate of change of the linear momentum vector".

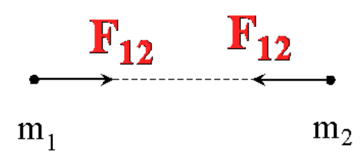
$$\text{linear momentum vecotr}: \quad p = \underbrace{m}_{\substack{\text{mass in}\\ \text{kilogram (kg)}}} \mathbf{v}$$

$$2^{\text{nd}} \text{ law}: \quad \underbrace{\mathbf{F}}_{\substack{\text{Newton}\\ 1\text{N}=1\text{kg}\cdot\text{m/s}^2}} = \frac{\mathrm{d}p}{\mathrm{dt}} = \frac{\mathrm{d}}{\mathrm{dt}}(m\mathbf{v}) = m \underbrace{\mathbf{a}}_{\substack{\text{absolute}\\ \text{acceleration}}}$$

$$\text{Note}: 2^{\text{nd}} \text{ law} \Rightarrow 1^{\text{st}} \text{ law}$$

- <u>Third Law:</u> (Law of Action & Reaction)

  "When two particles exert forces on one another, the forces act along the line joining the particles and they are equal in magnitude but opposite in directions"



- <u>Free Body Diagram:</u> a diagram containing all the forces acting on the particle.

Ex:

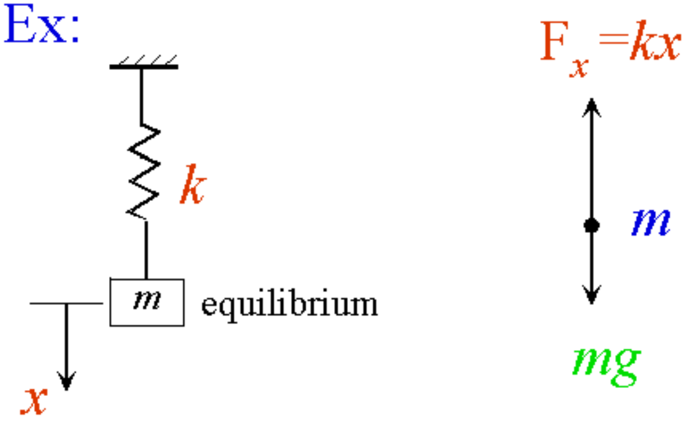
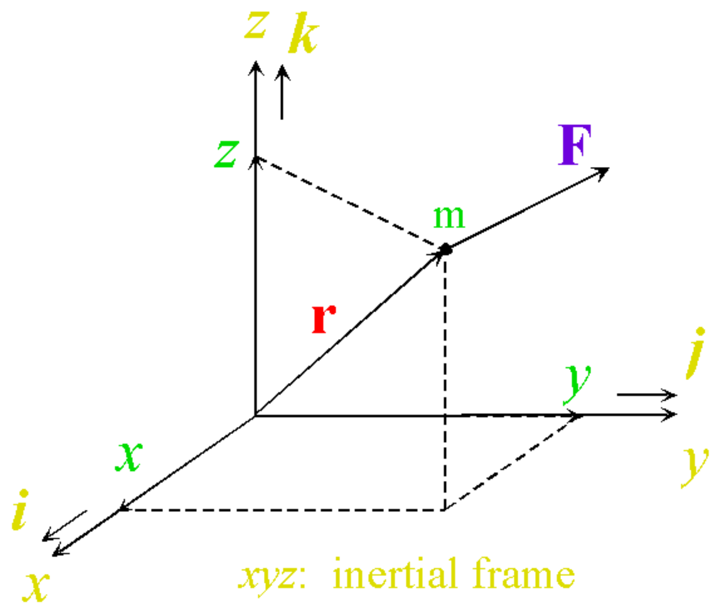


Figure28-29

The equations of Motion of a Particle



- $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$
- $\mathbf{v} = \dot{\mathbf{r}} = \dot{x}\mathbf{i} + \dot{y}\mathbf{j} + \dot{z}\mathbf{k}$
- $\mathbf{a} = \ddot{\mathbf{r}} = \ddot{x}\mathbf{i} + \ddot{y}\mathbf{j} + \ddot{z}\mathbf{k}$

- In general :

  $\mathbf{F} = \mathbf{F}(\mathbf{r}, \mathbf{v}, \mathrm{t})$

  $\quad = \mathbf{F}_x\mathbf{i} + \mathbf{F}_y\mathbf{j} + \mathbf{F}_z\mathbf{k}$
- $\mathbf{F}_x = \mathbf{F}_x(x, y, z, \dot{x}, \dot{y}, \dot{z}, t)$, etc...

- Newton's 2nd law: **F** = ma

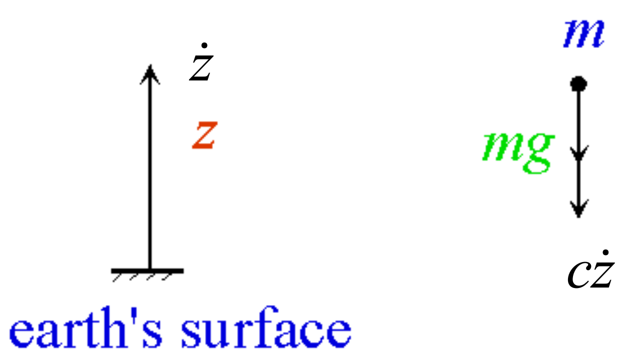
$$\begin{cases} m\ddot{x}(t) = \mathbf{F}_x(x,y,z,\dot{x},\dot{y},\dot{z},t) \\ m\ddot{y}(t) = \mathbf{F}_y(x,y,z,\dot{x},\dot{y},\dot{z},t) \\ m\ddot{z}(t) = \mathbf{F}_z(x,y,z,\dot{x},\dot{y},\dot{z},t) \end{cases}$$

- To solve this set of 3 equations in *x(t),y(t),z(t)* need $\begin{cases} x(0), \dot{x}(0) \\ y(0), \dot{y}(0) \\ z(0), \dot{z}(0) \end{cases}$

Figure30

Example:

- Mass m thrown vertically upward from the earth's surface with initial velocity $\mathbf{v_0}$. There is a force proportional to the velocity resisting the motion.



Newton's 2^{nd} law : $\quad \Sigma \mathbf{F}_z = -mg - c\dot{z} = m\ddot{z}$

$$\boxed{\text{Equation of motion : } \quad m\ddot{z}(t) + c\dot{z}(t) = -mg}$$

Initial Conditions : $\quad \begin{cases} z(0) = 0 \\ \dot{z}(0) = v_0 \end{cases}$

Figure31

Impulse and Momentum

- Linear Impulse Vector

$$\boxed{\widehat{\mathbf{F}} \equiv \int_{t_1}^{t_2} \mathbf{F}\mathrm{d}t \quad [\mathrm{N.s}]}$$

$$\widehat{\mathbf{F}} \equiv \int_{t_1}^{t_2} \mathbf{F}\mathrm{d}t = \int_{t_1}^{t_2} \frac{\mathrm{d}p}{\mathrm{d}t}\mathrm{d}t = \int_{t_1}^{t_2} \mathrm{d}p = p_2 - p_1 = m\mathbf{v}_2 - m\mathbf{v}_1$$

where $p_i = m\mathbf{v}_i = m\mathbf{v}(t_i)$

let $\underbrace{\Delta p = p_2 - p_1}$

change in linear momentum vector between t_1 & t_2

$$\Rightarrow \widehat{\mathbf{F}} = \Delta p$$

"Linear impulse vector corresponding to the times t_1 & t_2 is equal to the change in the linear momentum vector between the same two instants."

Principle of conservation of linear momentum

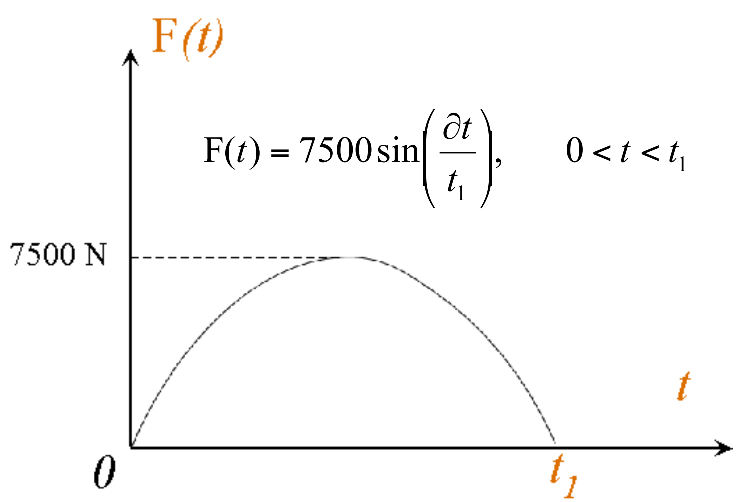
"In the absence of forces, the linear momentum doesn't change".

$$\boxed{\mathbf{F} = 0 \Rightarrow p_2 = p_1 = p = \text{constant}}$$

Note: Re-statement of Newton's 1^{st} law.

Problem 3.1

A bullet of mass 10^{-2} Kg leaves the gun barrel with a velocity of 0.8 km/s. If the firing is known to create a force on the bullet having a half-sine form of amplitude 7500 N, determine the duration of the force.



From definition of linear impulse

$$\boxed{\widehat{\mathbf{F}} \equiv \int_{t_1}^{t_2} \mathbf{F}\,\mathrm{d}t = m\mathbf{v}_2 - m\mathbf{v}_1}$$

So:

$$\int_0^{t_1} \mathbf{F}(t)\mathrm{d}t = 7500\int_0^{t_1} \sin\left(\frac{\pi t}{t_1}\right)\mathrm{d}t = \frac{15{,}000\,t_1}{\pi}$$

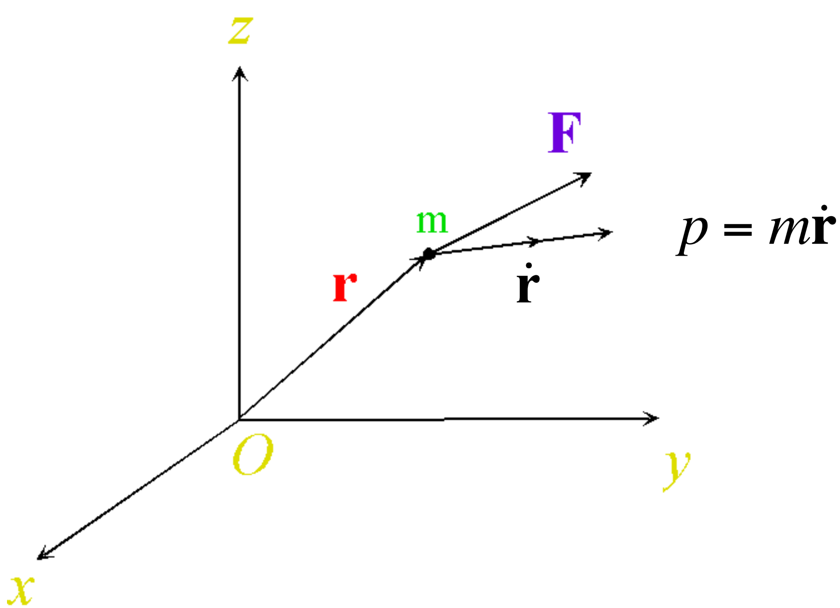
$$= m(\mathbf{v}_1 - \mathbf{v}_0) = m\mathbf{v}_1 = (0.8\times 10^3)\times 10^{-2} = 8$$

$$\Rightarrow \frac{15{,}000\,t_1}{\pi} = 8$$

$$\therefore t_1 = \frac{8\times\pi}{15{,}000} = \ldots\ldots$$

Figure32

Moment of a Force & Angular Momentum about a Fixed Point



Moment of **F** about O (torque)

$$\boxed{\mathbf{r} \times \mathbf{F} = \mathbf{M}_O}$$

- Moment of momentum $\mathbf{H}_0$, or angular momentum of m w.r.t. O is defined as the moment of **p** about O.

$$\boxed{\mathbf{H}_O = \mathbf{r} \times p = \mathbf{r} \times m\dot{\mathbf{r}}}$$

- Consider $\dot{\mathbf{H}}_O = \dot{\mathbf{r}} \times m\dot{\mathbf{r}} + \mathbf{r} \times m\ddot{\mathbf{r}} = \mathbf{r} \times m\ddot{\mathbf{r}}$

  (because $\dot{\mathbf{r}} \times m\dot{\mathbf{r}} = m\dot{\mathbf{r}} \times \dot{\mathbf{r}} = 0$)

  since $\mathbf{F} = m\ddot{\mathbf{r}}$ and $\boxed{\mathbf{r} \times \mathbf{F} = \mathbf{M}_O}$ $\leftarrow$ Moment of **F** about O (torque).

  $\Rightarrow \boxed{\mathbf{M}_O = \dot{\mathbf{H}}_O}$ $\leftarrow$ The moment of a force about a fixed point O is equal to the time rate of change of the moment of momentum about O.

- $$\left.\begin{array}{l} \left.\begin{array}{l} \mathbf{M}_O = \mathbf{r} \times \mathbf{F} \\ \mathbf{H}_O = m\mathbf{r} \times \dot{\mathbf{r}} \end{array}\right\} \text{In general not in same plane} \\ \mathbf{M}_O = \dot{\mathbf{H}}_O \leftarrow \mathbf{M}_O \,\&\, \dot{\mathbf{H}}_O \text{ in same plane} \end{array}\right\} \rightarrow \boxed{\begin{array}{r} \text{in the planar case} \\ \mathbf{M}_O, \mathbf{H}_O \,\&\, \dot{\mathbf{H}}_O \text{ are all} \\ \text{in the same plane} \end{array}}$$

- Angular impulse about O between the times t_1 & t_2

$$\boxed{\widehat{\mathbf{M}}_O \equiv \int_{t_1}^{t_2} \mathbf{M}_O \mathrm{d}t \quad [\text{N.m.s.}]}$$

Where M_O is the moment of **F** about O (torque).

$$\widehat{\mathbf{M}}_O \equiv \int_{t_1}^{t_2} \mathbf{M}_O \mathrm{d}t = \int_{t_1}^{t_2} \frac{\mathrm{d}\mathbf{H}_O}{\mathrm{d}t} \mathrm{d}t = \int_{t_1}^{t_2} \mathrm{d}\mathbf{H}_O = \mathbf{H}_{O2} - \mathbf{H}_{O1}$$

where $\underbrace{\mathbf{H}_{Oi}}_{\text{angular momentum at } t = t_1} = \mathbf{H}_O(t_i) = \mathbf{r}(t_i) \times m\dot{\mathbf{r}}(t_i) = \mathbf{r}(t_i) \times m\mathbf{v}(t_i)$

let $\Delta\mathbf{H} = \mathbf{H}_{O2} - \mathbf{H}_{O1}$ $\quad\Rightarrow\quad$ $\boxed{\widehat{\mathbf{M}}_O = \Delta\mathbf{H}}$

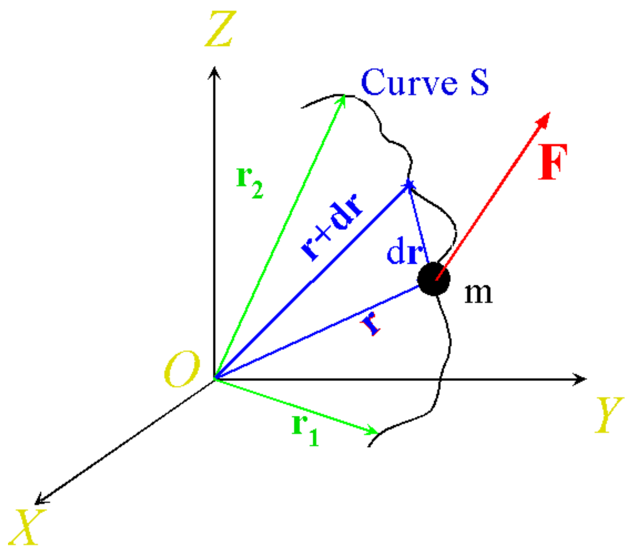
"The angular impulse vector about O between the times t_1 & t_2 is equal to the change in the angular momentum vector about O between the same two instants".

Principle of Conservation of Angular Momentum:

If $\mathbf{M}_O = 0 \Rightarrow \mathbf{H}_{O2} = \mathbf{H}_{O1} = \mathbf{H}_O = \text{constant}$

In the absence of torques about O, the angular momentum about O is constant.

6. Work & Energy



- A Particle m moving along a curve S under the action of a given force **F**.
- Definition: $\boxed{dW = \mathbf{F} \cdot d\mathbf{r}}$
  - dW: increment of work corresponding to the displacement of m from position **r** to **r**+d**r**;
  - Note: dW is a scalar.

- Note:

$$\mathbf{F} = m\ddot{\mathbf{r}} \quad \& \quad d\mathbf{r} = \dot{\mathbf{r}}dt$$

$$\Rightarrow dW = m\ddot{\mathbf{r}} \cdot \dot{\mathbf{r}}dt = m\frac{d\dot{\mathbf{r}}}{dt} \cdot \dot{\mathbf{r}}dt = m\dot{\mathbf{r}} \cdot d\dot{\mathbf{r}} = d\left(\frac{1}{2}m\dot{\mathbf{r}} \cdot \dot{\mathbf{r}}\right)$$

- Since Kinetic Energy $\boxed{T = \frac{1}{2}m\dot{\mathbf{r}} \cdot \dot{\mathbf{r}}}$ $\Rightarrow$ $\boxed{dW = dT}$

- Now consider work performed by **F** in moving m from $\mathbf{r}_1$ to $\mathbf{r}_2$:

$$W_{1-2} = \int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_{T_1}^{T_2} \mathrm{d}T = T_2 - T_1$$

$$\Rightarrow \qquad \boxed{W_{1-2} = T_2 - T_1}$$

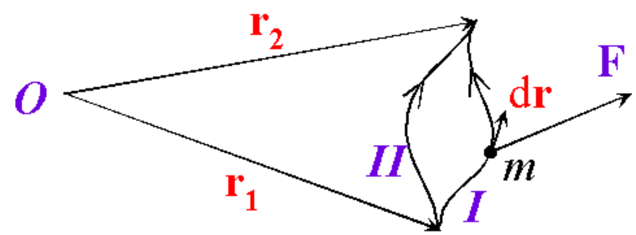
"Work performed by the force **F** in moving the particle m from position $\mathbf{r}_1$ to position $\mathbf{r}_2$ is equal to the change in the kinetic energy from T_1 to T_2."

- Question: $W_{1-2} = \int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{F} \cdot \mathrm{d}\mathbf{r}$

Does the work performed by F depend only on the terminal positions $\mathbf{r}_1$ to position $\mathbf{r}_2$?
Or does it depend on the path taken to travel from $\mathbf{r}_1$ to $\mathbf{r}_2$?

- Answer: ?

Figure33



- Let the force F belong to the class of forces s.t.:

$$\underset{\text{path I}}{\int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{F}\cdot d\mathbf{r}} = \underset{\text{path II}}{\int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{F}\cdot d\mathbf{r}}$$

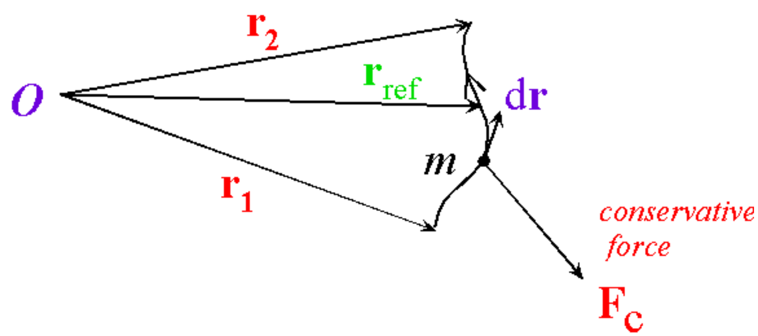
$$\Rightarrow \quad \boxed{\underset{\text{path I}}{\int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{F}\cdot d\mathbf{r}} - \underset{\text{path II}}{\int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{F}\cdot d\mathbf{r}} = \underset{\text{path I}}{\int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{F}\cdot d\mathbf{r}} + \underset{\text{-path II}}{\int_{\mathbf{r}_2}^{\mathbf{r}_1} \mathbf{F}\cdot d\mathbf{r}} = \oint \mathbf{F}\cdot d\mathbf{r}}$$

Note:

"Work doesn't depend on the path" $\Leftrightarrow$

"The work performed in traveling over a closed path(starting at a given point and returning to the same point) is zero"

$\rightarrow$ Such force are called <u>conservative forces</u> and denoted $\mathbf{F}_c$.



$$\boxed{\begin{aligned} W_{1-2c} &= \int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{F}_c \cdot d\mathbf{r} \\ &= \int_{\mathbf{r}_1}^{\mathbf{r}_{ref}} \mathbf{F}_c \cdot d\mathbf{r} + \int_{\mathbf{r}_{ref}}^{\mathbf{r}_2} \mathbf{F}_c \cdot d\mathbf{r} \end{aligned}}$$

- Define Potential Energy: work performed by a conservative force in moving a particle from position r to the reference position $\mathbf{r}_{ref}$.

$$\underbrace{V(\mathbf{r})}_{\text{scalar}} = \int_{\mathbf{r}}^{\mathbf{r}_{ref}} \mathbf{F}_c \cdot d\mathbf{r} \longrightarrow$$

$$\boxed{\begin{aligned} &\text{So:} \quad W_{1-2c} = V_1 - V_2 = -(V_2 - V_1) \\ &\text{where} \quad V_i = V(\mathbf{r}_i) \quad i = 1, 2. \end{aligned}}$$

"Work performed by a conservative force in moving a particle from $\mathbf{r}_1$ to $\mathbf{r}_2$ is equal to the negative of the change in the potential energy from V1 to V2."

Figure36

- Note 1:

$$
\begin{aligned}
\text{Recall}\quad W_{1-2c} &= V_1 - V_2 \\
&= \int_{\mathbf{r}_1}^{\mathbf{r}_{\text{ref}}} \mathbf{F}_c \cdot d\mathbf{r} - \int_{\mathbf{r}_2}^{\mathbf{r}_{\text{ref}}} \mathbf{F}_c \cdot d\mathbf{r} \\
&= \int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{F}_c \cdot d\mathbf{r}
\end{aligned}
$$

  - Reference position is arbitrary!
  - We are interested in changes in potential energy rather than the potential energy itself !

- Note 2:

  In contrast to potential energy, kinetic energy represents an absolute quantity since expressed in terms of velocities relative to an inertial frame.

- In general, we have both conservative and non-conservative forces

$$\boxed{W_{1-2} = W_{1-2c} + W_{1-2nc}\ , \quad \text{where } W_{1-2nc} = \int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{F}_{\mathrm{nc}} \cdot d\mathbf{r}}$$

Non-conservative work   Non-conservative force

- Recall $\boxed{\begin{aligned} W_{1-2} &= T_2 - T_1 \\ W_{1-2c} &= -(V_2 - V_1) \end{aligned}}$ $\Rightarrow$ $\boxed{T_2 - T_1 = -(V_2 - V_1) + W_{1-2nc}}$

- Define: $\boxed{E = T + V}$

Total Energy   Kinetic Energy   Potential Energy

$$\Rightarrow \boxed{W_{1-2nc} = E_2 - E_1}$$

"The work performed by the non-conservative force $\mathbf{F}_{\mathbf{nc}}$ in moving a particle from $\mathbf{r}_1$ to $\mathbf{r}_2$ is equal to the change in the total energy from E_1 to E_2."

- Definition: $$\boxed{\mathbf{P} = \mathbf{F} \cdot \dot{\mathbf{r}}}$$
  Power

- Recall
  - $W_{1-2c} = -(V_2 - V_1) \quad \Rightarrow \quad dW_c = -dV$
  - $W_{1-2} = W_{1-2c} + W_{1-2nc} \quad \Rightarrow \quad dW = dW_c + dW_{nc}$
  - $dW = dT$
  - $E = T + V$ $\quad \Rightarrow \quad \boxed{dW_{nc} = dE}$

- but $\quad W_{1-2nc} = \int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{F}_{\mathrm{nc}} \cdot d\mathbf{r} \Rightarrow \quad dW_{nc} = \mathbf{F}_{\mathrm{nc}} \cdot d\mathbf{r}$

  $\Rightarrow \mathbf{F}_{\mathrm{nc}} \cdot d\mathbf{r} = dE \quad \Rightarrow \quad \boxed{\mathbf{F}_{\mathrm{nc}} \cdot \dot{\mathbf{r}} = \dot{E}}$

"The power associated with the non-conservative force $\mathbf{F}_{\mathbf{nc}}$ is equal to the rate of change of the total energy E."

- A non-conservative force can add energy to a system: Ex: applied force

  It can also dissipate energy: Ex: damping force

- If there are no non-conservative forces

$$\mathbf{F}_{nc} \cdot \dot{\mathbf{r}} = \dot{E} \quad \text{becomes} \quad \boxed{E = \text{const}}$$

"In the absence of non-conservative forces the total energy remains constant."

Principle of Conservation of Energy

- So

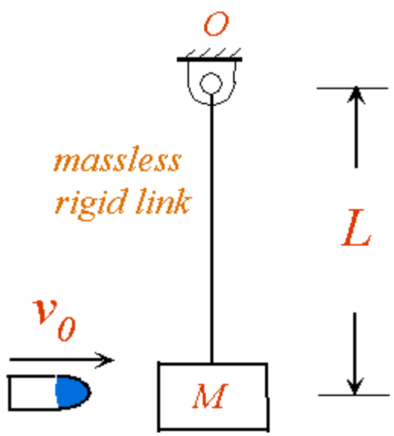
$$T_2 - T_1 = -(V_2 - V_1) + \underbrace{\cancel{W_{1-2nc}}}_{0}$$

$$\Rightarrow \quad \boxed{T_2 - T_1 = V_1 - V_2}$$

$$\xrightarrow{\text{or}} \quad \boxed{E = T_2 + V_2 = T_1 + V_1}$$

Problem 5.3

A bullet of mass m is fired with a velocity v_0 into a block of wood of mass M. If the bullet becomes embedded in the block, calculate the maximum height reached by the block and the bullet?



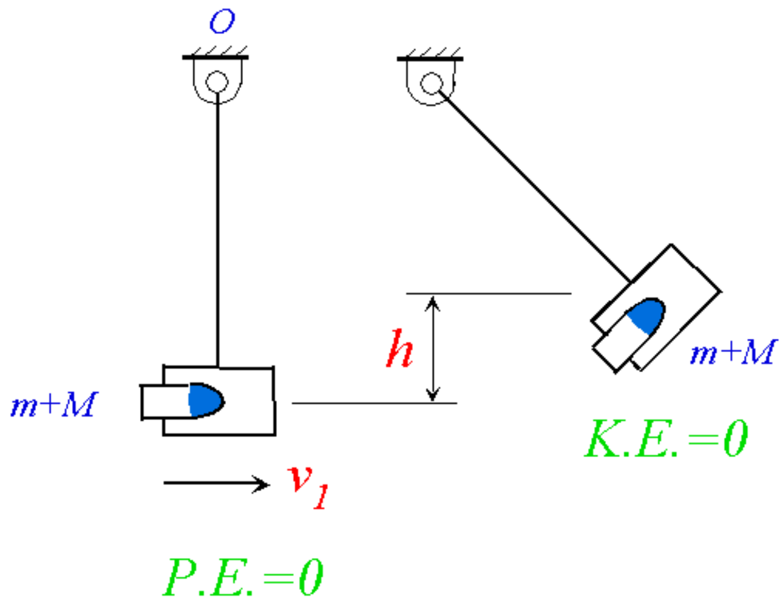
- Before the block starts moving, the angular momentum H_0 about O is conserved.
- Denote the velocity of the block at the termination of the impact by v_1

$$\mathbf{H}_0 = \mathbf{H}_1 \quad \Leftrightarrow \quad \mathbf{r}_O \times m\mathbf{v}_0 = \mathbf{r}_O \times (m+M)\mathbf{v}_1$$

$$mv_0 L = (m+M)v_1 L \quad \Rightarrow \quad \boxed{v_1 = \frac{mv_0}{m+M}}$$

Figure37

- After the termination of the impact, the energy is conserved:

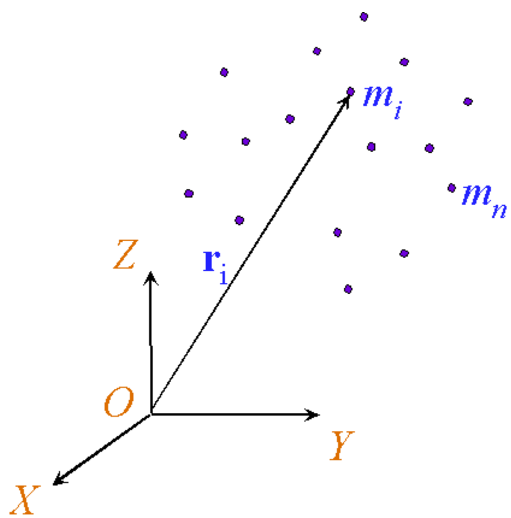


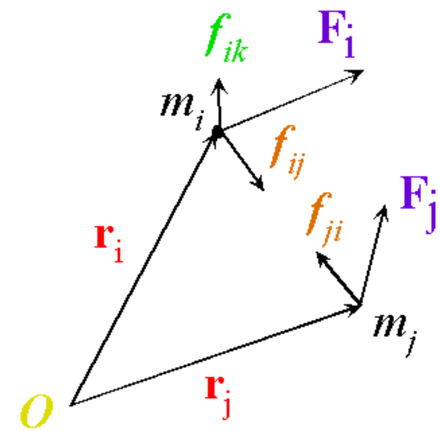
$$\frac{1}{2}(m+M)v_1^{\;2} = (m+M)gh$$

$$\Rightarrow \qquad \boxed{h = \frac{v_1^{\;2}}{2g} = \frac{(mv_0)^2}{2g(m+M)^2}}$$

Figure38

Dynamics of Systems of Particles





- Consider n particles m_i, $i=1,2,\ldots,n$;
- External forces are denoted $\mathbf{F}_i$;

  Internal forces are denoted by $\mathbf{f}_i$.

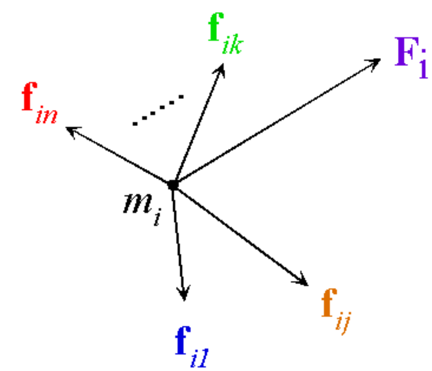
Resultant of the interaction forces $\mathbf{f}_{ij}$ exerted by the particle m_j $(j=1,2,\ldots,n,\ j\neq i)$ on the particle m_i

i.e. $$\boxed{\mathbf{f}_i = \sum_{\substack{j=1\\ j\neq i}}^{n} \mathbf{f}_{ij}}$$

Figure39-40

- Newton's 2nd law for particle m_i:

$$\underbrace{\mathbf{F}_i}_{\text{External forces}} + \underbrace{\sum_{\substack{j=1\\ j\neq i}}^{n} \mathbf{f}_{ij}}_{\text{Internal forces}} = m_i\ddot{\mathbf{r}}_i = m_i\mathbf{a}_i$$



- Newton's 2nd law for the entire system of particles

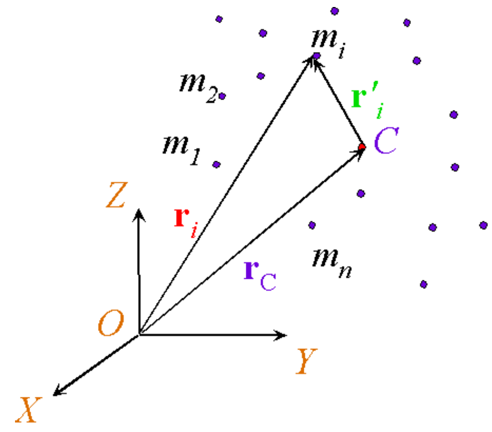
$$\sum_{i=1}^{n}\mathbf{F}_i + \underbrace{\sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n}\mathbf{f}_{ij}}_{\substack{=0!\ \text{ since by Newton's}\\ \text{3rd law,}\quad f_{ij}=-f_{ji}}} = \sum_{i=1}^{n} m_i\ddot{\mathbf{r}}_i = \sum_{i=1}^{n} m_i\mathbf{a}_i$$

- Let $\sum_{i=1}^{n}\mathbf{F}_i \equiv \mathbf{F}$ $\Rightarrow$ $$\boxed{\mathbf{F} = \sum_{i=1}^{n} m_i\ddot{\mathbf{r}}_i = \sum_{i=1}^{n} m_i\mathbf{a}_i}$$

Equation of motion of system of particles

Figure41

Equations of motion in terms of the mass center



- Definition: Center of Mass C

  A point in space representing a weighted average position of the system, where the weighting factor for each particle is the mass of the particle.

$$\boxed{\mathbf{r}_C = \frac{1}{m}\sum_{i=1}^{n} m_i \mathbf{r}_i} \qquad \text{where } m = \sum_{i=1}^{n} m_i$$

- Let

$$\boxed{\mathbf{r}_i = \mathbf{r}_C + \mathbf{r}'_i}$$

Figure42

- $$\mathbf{r}_C = \frac{1}{m}\sum_{i=1}^{n} m_i \mathbf{r}_i = \frac{1}{m}\sum_{i=1}^{n} m_i[\mathbf{r}_C + \mathbf{r}_i'] = \underbrace{\frac{1}{m}\sum_{i=1}^{n} m_i \mathbf{r}_C}_{\mathbf{r}_C} + \underbrace{\frac{1}{m}\sum_{i=1}^{n} m_i \mathbf{r}_i'}_{\Rightarrow\ = 0}$$

Equivalent definition of mass center

- $$\mathbf{v}_C = \dot{\mathbf{r}}_C = \frac{1}{m}\sum_{i=1}^{n} m_i \dot{\mathbf{r}}_i = \frac{1}{m}\sum_{i=1}^{n} m_i \mathbf{v}_i$$

- $$\mathbf{a}_C = \ddot{\mathbf{r}}_C = \frac{1}{m}\sum_{i=1}^{n} m_i \ddot{\mathbf{r}}_i = \frac{1}{m}\sum_{i=1}^{n} m_i \mathbf{a}_i$$

- Recall $\mathbf{F} = \sum_{i=1}^{n} m_i \mathbf{a}_i \quad \Rightarrow \quad \boxed{\mathbf{F} = m\mathbf{a}_C}$

"The motion of the mass center of the system of particles is the same as the motion of a fictitious body equal in mass to the total mass *m* of the system, concentrated at the mass center, and being acted on by the resultant of the external forces."

Linear Momentum

- Linear momentum of particle m_i

$$\boxed{p_i = m_i \mathbf{v}_i}$$

- Linear momentum of system of particles

$$p = \sum_{i=1}^{n} p_i = \sum_{i=1}^{n} m_i \mathbf{v}_i = \sum_{i=1}^{n} m_i \dot{\mathbf{r}}_i = \sum_{i=1}^{n} m_i [\dot{\mathbf{r}}_C + \dot{\mathbf{r}}_i']$$

$$= \underbrace{\sum_{i=1}^{n} m_i \dot{\mathbf{r}}_C}_{m\mathbf{v}_C} + \underbrace{\sum_{i=1}^{n} m_i \dot{\mathbf{r}}_i'}_{=0} \Rightarrow \boxed{p = m\mathbf{v}_C} \Rightarrow \boxed{\dot{p} = m\dot{\mathbf{v}}_C = m\mathbf{a}_C}$$

- Recall $\mathbf{F} = m\mathbf{a}_C \Rightarrow \boxed{\mathbf{F} = \dot{p}}$
- Similar to single particle $\boxed{\hat{\mathbf{F}} = \Delta p}$ $\leftarrow$ $\Delta p = p(t_2) - p(t_1)$

Resultant of external impulses $\leftarrow$ $\hat{\mathbf{F}} = \int_{t_1}^{t_2} \mathbf{F} \mathrm{d}t$

Change in system linear momentum between t_1 & t_2.

- $\boxed{\text{If } \mathbf{F} = 0 \quad \Rightarrow p = \text{const}}$

Principle of conservation of linear momentum for system of particles:

"In the absence of external forces, linear momentum of system of particles remains constant."

Angular Momentum

- Angular momentum of particle m_i about the fixed point O:

$$\mathbf{H}_{Oi} = \mathbf{r}_i \times p_i = \mathbf{r}_i \times m_i \mathbf{v}_i$$

- For a system of particles

$$\boxed{\mathbf{H}_O = \sum_{i=1}^{n} \mathbf{H}_{Oi} = \sum_{i=1}^{n} \mathbf{r}_i \times m_i \mathbf{v}_i}$$

Recall $\quad \mathbf{F}_i + \sum_{\substack{j=1 \\ j\neq i}}^{n} \mathbf{f}_{ij} = m_i \ddot{\mathbf{r}}_i = m_i \mathbf{a}_i$

$$\Rightarrow \quad \dot{\mathbf{H}}_O = \underbrace{\sum_{i=1} \dot{\mathbf{r}}_i \times m\mathbf{v}_i}_{=0} + \sum_{i=1} \mathbf{r}_i \times m\dot{\mathbf{v}}_i = \sum_{i=1}^{n} \mathbf{r}_i \times \left[\mathbf{F}_i + \sum_{\substack{j=1 \\ j\neq i}}^{n} \mathbf{f}_{ij} \right]$$

$$= \sum_{i=1}^{n} \mathbf{r}_i \times \mathbf{F}_i + \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j\neq i}}^{n} \mathbf{r}_i \times \mathbf{f}_{ij} = \sum_{i=1}^{n} \mathbf{r}_i \times \mathbf{F}_i$$

$= 0,$ because $\mathbf{r}_i \times \mathbf{f}_{ij} = -\mathbf{r}_j \times \mathbf{f}_{ji}$

Recognize that

$$\boxed{\mathbf{M}_O = \sum_{i=1}^{n} \mathbf{r}_i \times \mathbf{F}_i} \longleftarrow \text{Moment about O of external forces.} \quad \Rightarrow \quad \boxed{\dot{\mathbf{H}}_O = \mathbf{M}_O}$$

- $$\boxed{\dot{\mathbf{H}}_O = \mathbf{M}_O}$$

- $$\boxed{\hat{\mathbf{M}}_O = \Delta \mathbf{H}_O}$$ ← Change in the system angular momentum about O .

- $$\boxed{\hat{\mathbf{M}}_O = \sum_{i=1}^{n} \mathbf{r}_i \times \hat{\mathbf{F}}_i}$$ ← Resultant of all external angular impulses about O.

$$\boxed{\text{If} \quad \mathbf{M}_O = 0 \quad \Rightarrow \quad \mathbf{H}_O = \text{const}}$$ ← Conservation of angular momentum about a fixed point .

- Question:

  Above relations hold for the fixed point O.

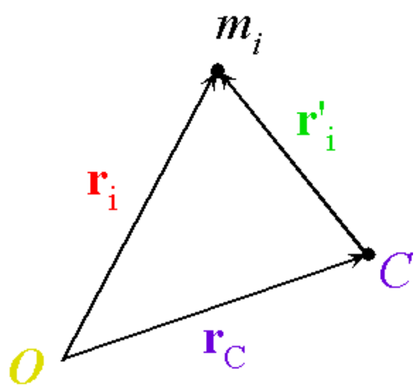
  Do similar relations exist also for a moving point X?

$$\boxed{\mathbf{M}_X = \dot{\mathbf{H}}_X \ ?} \qquad \boxed{\hat{\mathbf{M}}_X = \Delta \mathbf{H}_X \ ?}$$

- Answer:

  Yes, if the moving point is the mass center of the system of particles!

- Angular momentum about C of particle m_i:



$$\boxed{\mathbf{H}_{C\,i} = \mathbf{r}_i' \times p = \mathbf{r}_i' \times m\mathbf{v}_i}$$

- Angular momentum about C of the system of particles:

$$\mathbf{H}_C = \sum_{i=1}^{n} \mathbf{H}_{C\,i} = \sum_{i=1}^{n} \mathbf{r}_i' \times m_i \mathbf{v}_i$$

$$= \sum_{i=1}^{n} \mathbf{r}_i' \times m_i [\mathbf{v}_C + \mathbf{v}_i']$$

$$= \underbrace{\left(\sum_{i=1}^{n} m_i \mathbf{r}_i'\right)}_{=0} \times \mathbf{v}_C + \underbrace{\sum_{i=1}^{n} \mathbf{r}_i' \times m_i \mathbf{v}_i'}_{\equiv \mathbf{H}_C'}$$

$$\Rightarrow \quad \boxed{\mathbf{H}_C = \mathbf{H}_C' = \sum_{i=1}^{n} \mathbf{r}_i' \times m_i \mathbf{v}_i'}$$

Apparent Angular Momentum .

Figure43

- We can also derive the following :

$$\boxed{\dot{\mathbf{H}}_C = \mathbf{M}_C \equiv \sum_{i=1}^{n} \mathbf{r}_i' \times \mathbf{F}_i}$$

Time rate of change of system angular momentum about C.

Moment of external forces about C .

$$\boxed{\hat{\mathbf{M}}_C = \Delta \mathbf{H}_C}$$

Angular impulse about C.

Change in angular momentum .

$$\boxed{\text{If} \quad \mathbf{M}_C = 0 \Rightarrow \mathbf{H}_C = \text{const}}$$

Kinetic Energy:

- Kinetic energy for particle m_i : $$T_i = \frac{1}{2} m_i \, \dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}_i$$

- Kinetic energy of the system of particles:

$$T = \sum_{i=1}^{n} T_i = \frac{1}{2} \sum_{i=1}^{n} m_i \, \dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}_i = \frac{1}{2} \sum_{i=1}^{n} m_i \, (\dot{\mathbf{r}}_C + \dot{\mathbf{r}}_i') \cdot (\dot{\mathbf{r}}_C + \dot{\mathbf{r}}_i')$$

$$= \left(\frac{1}{2} \dot{\mathbf{r}}_C \cdot \dot{\mathbf{r}}_C \sum_{i=1}^{n} m_i \right) + \underbrace{\left(\dot{\mathbf{r}}_C \cdot \sum_{i=1}^{n} m_i \dot{\mathbf{r}}_i' \right)}_{=0} + \left(\frac{1}{2} \sum_{i=1}^{n} m_i \dot{\mathbf{r}}_i' \cdot \dot{\mathbf{r}}_i' \right)$$

- So

$$\boxed{T = \overbrace{\underbrace{\frac{1}{2} m \, \dot{\mathbf{r}}_C \cdot \dot{\mathbf{r}}_C}}^{T_{tr}} + \overbrace{\underbrace{\frac{1}{2} \sum_{i=1}^{n} m_i \dot{\mathbf{r}}_i' \cdot \dot{\mathbf{r}}_i'}}^{T_{rel}}}$$

kinetic energy with velocity of mass center.

kinetic energy due to motion relative to mass center.