Assignment #5

Name:Cao Mingming ID:2018311770
cmm18@mails.tsinghua.edu.cn

Tsinghua University — November 9, 2018

1 Problem 1

Derive Eq. 2.64 and Eq. 2.65 in the main textbook.

Solution 1

Assumed that the q parameter at the left of the lens is q_1 and the beam wasit at the right side is q_2, as is known in the figre 1. The ABCD matrix of lens is,

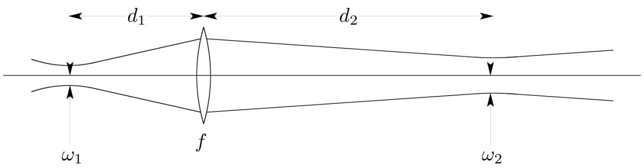


Figure 1: Mode mathcing

$$M = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} \tag{1}$$

Therefore we could get,

$$\begin{aligned} q_1 &= d_1 + i\frac{n\pi\omega_1^2}{\lambda} \\ q_2 &= \frac{Aq_1 + B}{Cq_1 + D} + d_2 \\ &= \frac{fq_1}{f - q_1} + d_2 \\ &= \frac{f(d_1 + iZ_{R1})(f - d_1 + iZ_{R1})}{(f - d_1)^2 + Z_{R1}^2} + d_2 \end{aligned} \tag{2}$$

q_2 is the q paramter at the right beam wasit therefore we know that,

$$\begin{aligned} q_2 &= iZ_{R2} \\ &= \frac{n\pi\omega_2^2}{\lambda} \end{aligned} \tag{3}$$

which is a pure imaginary number. Compare equaiton 2 with euqaiton 3 we can get that,

$$i\frac{f^2 Z_{R1}}{(f-d_1)^2+Z_{R1}^2}+\frac{f[d_1(f-d_1)^2-Z_{R1}^2+d_2(f-d_1)^2]}{(f-d_1)^2+Z_{R1}^2}+d_2=iZ_{R2} \tag{4}$$

which indicate that,

$$\begin{aligned}\tfrac{f^2 Z_{R1}}{(f-d_1)^2+Z_{R1}^2}&=Z_{R2}\\ \tfrac{f[d_1(f-d_1)^2-Z_{R1}^2+d_2(f-d_1)^2]}{(f-d_1)^2+Z_{R1}^2}+d_2&=0\end{aligned} \tag{5}$$

Solving above equation we could get,

$$\begin{aligned}d_1&=f\pm\frac{\omega_1}{\omega_2}\sqrt{f^2-f_0^2}\\ d_2&=f\pm\frac{\omega_2}{\omega_1}\sqrt{f^2-f_0^2}\end{aligned} \tag{6}$$

2 Problem 2

The mode-matching techniques for standing wave cavities described in the text did not take account of the refractive power of the input mirror which acts like a weak negative lens. Assume that one has an approximately confocal cavity with mirror separation d. Approximately how far is the apparent waist position shifted by refraction from the input mirror? (Treat the input mirror as a thin plano-concave lens with refractive index n.) Assume $n = 1.5$ and determine the actual shift in terms of d.

Solution

Suppose the radius of the cavity are $R1$ and R_2, due to that it is a confocal cavity we know that,

$$R_1 = R_2 = d \tag{7}$$

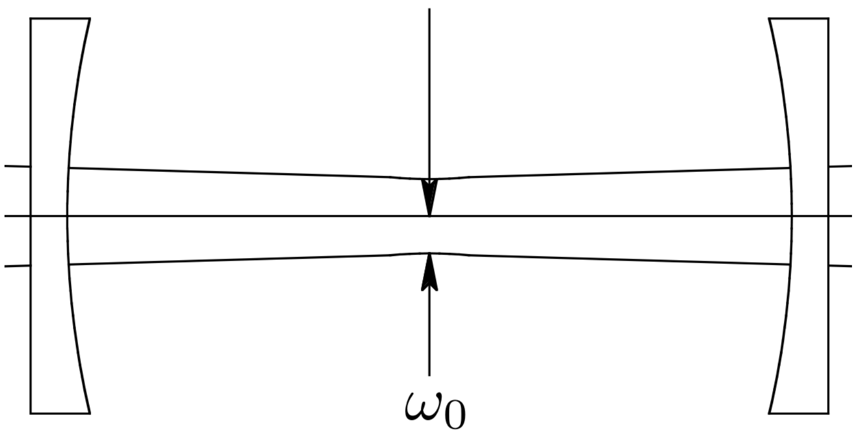
Using the equation of lens maker, the focal length of the plano-concave lens is f,

$$f = (\frac{n2}{n_1}-1)(\frac{1}{R_1}+\frac{1}{R_2}) \tag{8}$$

Substitute $n_1 = 1, \quad n_2 = 1.5, \quad R_1 = d, \quad R_2 = \infty$ into the above equation we can get,

$$f = 2d$$

Therefore the ABCD matrix of the plano-concave lens is M_1,



$$M_1 = \begin{pmatrix} 1 & 0 \\ \frac{1}{2d} & 1 \end{pmatrix}$$

Assume that the q parameter at the left mirror is $q_1 = -\frac{d}{2} + iZ_R$ where,

$$Z_R = \frac{n\pi\omega_0^2}{\lambda} = \frac{d}{2}$$

using ABCD matrix of the plano-concave lens we can get,

$$q_2 = \frac{Aq_1 + B}{Cq_1 + D} \tag{9}$$

Substitute M_1 and q_1 into equation 9 we can get,

$$q_2 = \frac{-6d^3 + 8dZ_R^2 + (4d^2Z_R + 12d^2Z_R)i}{9d^2 + 4Z_R^2} \tag{10}$$

Therefore the distance of beam waist from the left mirror is,

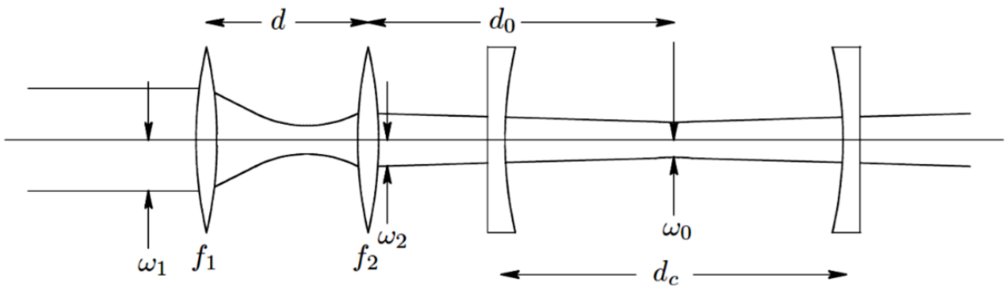
$$d_1 = \frac{-6d^3 + 8dZ_R^2}{9d^2 + 4Z_R^2} = -\frac{2}{5}d \tag{11}$$

Therefore the beam waist has a shift of $\frac{d}{10}$ to the left.

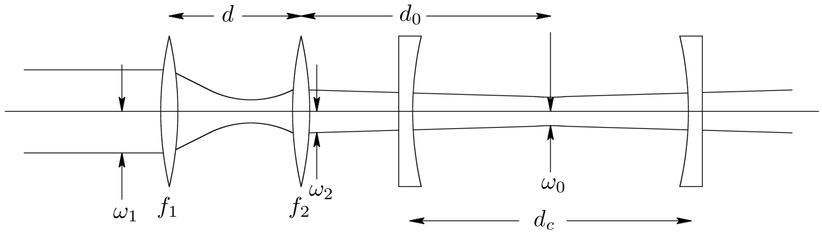
3 Problem 3

3. Can you explain the procedure #4 at page 32 of the main text book. How much do you need to separate the distance between two lens from f_1+f_2? Estimate the distance for the following two cases.

1) The mode matching to the following confocal cavity of R = 10 *cm* with the laser beam of $\lambda = 1\ \mu m$. The waist of the input beam is $\omega_1 = 1\ mm$. Here you assume the beam waist is located at the position of lens f_1, wherever it is. What are your choices of d_0, f_1, f_2, and d for the proper mode matching? How much different the distance *d* from f_1+f_2?



2) The mode matching to the fiber, which requires the beam waist $\omega_0 = 20\ \mu m$ at the surface of the fiber for $\lambda = 0.532\ \mu m$. The waist of the input beam is $\omega_1 = 1\ mm$. Here you assume the beam waist is located at the position of lens f_1, wherever it is. What are your choices of d_0, f_1, f_2, and d for the proper mode matching? How much different the distance *d* from f_1+f_2?



Solution

3.1 PART I

The ABCD mtrix of the lens is,

$$
\begin{aligned}
M_2 &= \begin{pmatrix} 1 & 0 \\ -\frac{1}{f_2} & 1 \end{pmatrix} \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{f_1} & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 - \frac{d}{f_1} & d \\ -\frac{1}{f_1} - -\frac{1}{f_2} + \frac{d}{f_1 f_2} & 1 - \frac{d}{f_2} \end{pmatrix}
\end{aligned} \tag{12}
$$

Assume that the q parameter at lens f_1 is $q_1 = iZ_1$, where $Z_1 = \frac{n\pi\omega_1^2}{\lambda}$, therefore we could get the q parameter at the lens f_2 as,

$$
\begin{aligned}
q_2 &= \frac{Aq_1 + B}{Cq_1 + D} \\
&= \frac{BD + ACZ_1^2}{B^2 + A^2 Z_1^2} + i\frac{(BC - AD)Z_1}{B^2 + A^2 Z_1^2}
\end{aligned} \tag{13}
$$

q_2 is also a part of the beam of the cavity, therefore starting from the beam wasit of the cavity we can write q_2 as,

$$
q_2 = -d_0 + iZ_0 \tag{14}
$$

where $Z_0 = \frac{n\pi\omega_0^2}{\lambda}$. Comparing equation 13 with equation 14 we could derive that,

$$
\begin{aligned}
Z_0 &= Im(q_2) = \tfrac{(BC-AD)Z_1}{B^2+A^2Z_1^2} \\
d_0 &= -Re(q_2) = -\tfrac{BD+ACZ_1^2}{B^2+A^2Z_1^2}
\end{aligned} \tag{15}
$$

Substitute the element in M_2 into the above equation we can simplify equation as ,

$$
Z_0[(f_1 f_2 - f_1 d)^2 + Z_1^2(f_1 + f_2 - d)^2] = f_1^2 f_2^2 Z_1 \tag{16}
$$

Using mathematics to solve equation 17 we can get d as,

$$
d = f_1 + f_2 + f_1^2 \frac{f_1 \pm \sqrt{-Z_1^2 + \frac{Z_1^3}{Z_0}(\frac{f_2^2}{f_1^2}) + \frac{Z_1}{Z_0} f_2^2}}{Z_1^2 + f_1^2} \tag{17}
$$

Substitue $\frac{f_1}{f_2} = \frac{\omega_1}{\omega_2} = \frac{1}{\sqrt{1+(d_0/Z_0)^2}} \frac{\omega_1}{\omega_0}$ into eqaution 17 we can get,

$$
\begin{aligned}
d &= (f_1 + f_2) + f_1^2 \frac{f_1 \pm \sqrt{(\frac{d_0}{Z_0})^2 Z_1^2 + \frac{Z_1}{Z_0} f_2^2}}{Z_1^2 + f_1^2} \\
&= (f_1 + f_2) + f_1^2 \frac{f_1 \pm \sqrt{d_0^2 Z_1^2 + f_2^2}}{Z_0(Z_1^2 + f_1^2)}
\end{aligned} \tag{18}
$$

Therefore $\Delta d = f_1^2 \frac{f_1 \pm \sqrt{d_0^2 Z_1^2 + f_2^2}}{Z_0(Z_1^2+f_1^2)}$ and we need to move slightly more than $f_1 + f_2$ to make the beam waist at the center of cavity.

3.2 Part II

However, we note that under the condition

$$\frac{f_1}{f_2} = \frac{\omega_1}{\omega_0} = \sqrt{\frac{Z_1}{Z_0}}$$

equation 17 can be simplified as,

$$q_2 = (f_1 + f_2) + \frac{f_1^2(f_1 \pm f_1)}{f_1^2 + Z_1^2} \tag{19}$$

which indicates that,

$$d = f_1 + f_2 \quad or \quad f_1 + f_2 + \frac{2f_1^3}{f_1^2 + Z_1^2} \tag{20}$$

therefore there exists a strict analytic solution $d = f_1 + f_2$ under such condition.

3.2.1 1)

Due to the radius of the confocal cavity is R therefore we could get $d = R = 10cm$, $Z_0 = \frac{d}{2} = 5cm$, $Z_1 = 3.14m$, $\omega_0 = \sqrt{\frac{\lambda d}{2n\pi}} \approx 0.178mm(n = 1)$. Choose $d_0 = R = 10cm$ and $\omega_2 = \sqrt{2}\omega_0 = 0.252mm$ and set

$$\frac{f_1}{f_2} = \frac{\omega_1}{\omega_0} \approx 4 \tag{21}$$

Hence we choose $f_1 = 40mm$, $f_2 = 10mm$ and $d = f_1 + f_2 = 50mm$ and $\Delta d = \pm 1mm$

3.2.2 2)

Frome the question we know that $\omega_0 = 20\mu m$, $\omega_1 = 1mm$ therefore we could get

$$\begin{aligned}
&Z_0 = \tfrac{n\pi\omega_0^2}{\lambda} = 2.4mm \quad (n = 1) \\
&Z_1 = \tfrac{n\pi\omega_0^2}{\lambda} = 5.9m \\
&setting d_0 = 10 * Z_0 = 24mm \\
&\omega_2 = \omega_0 * \sqrt{1 + (d_0/z_0)^2} = 201\mu m \\
&\tfrac{f_1}{f_2} = \tfrac{\omega_1}{\omega_2} = 4.97
\end{aligned} \tag{22}$$

Set $d_0 = 10 * Z_0 = 24mm$ and $f_2 = 10mm$ we could get $f_1 \approx 50mm$, $d = f_1 + f_2 = 60mm$ and $\Delta d = -4.1mm \quad or \quad 4.3mm$