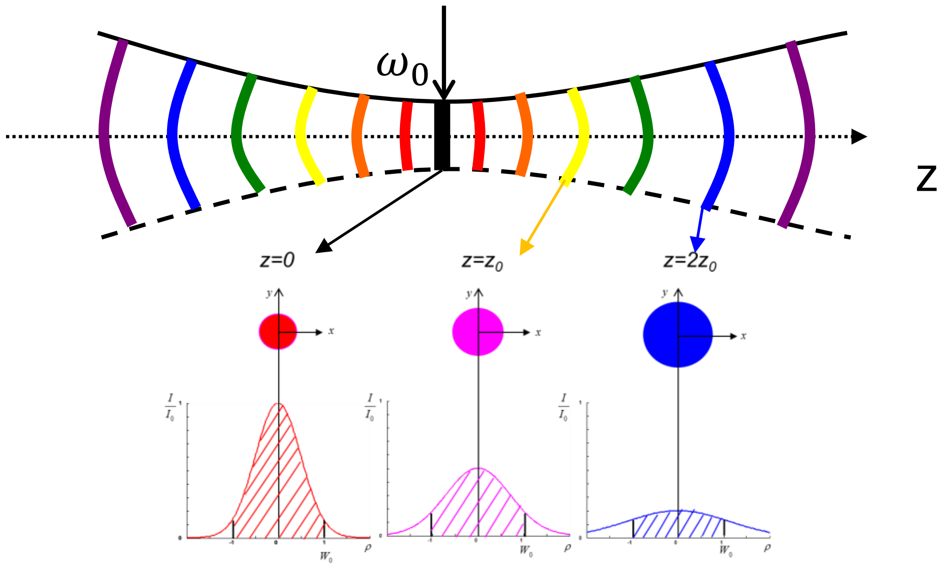
QUANTUM ELECTRONICS

For atomic physics

Gaussian Beam

$$\boxed{\nabla_t^2\psi - 2ik\frac{\partial\psi}{\partial z} = 0,}$$



$$\psi(x,y,z) = \exp\left\{-i\left(P(z) + \frac{k}{2q(z)}r^2\right)\right\}$$

complex beam parameter *q*

$$\frac{1}{q(z)} = \frac{1}{R(z)} - i\frac{\lambda}{n\pi\omega(z)^2}$$

$$\omega(z) = \omega_0\left[1 + \left(\frac{z}{z_R}\right)^2\right]^{1/2}$$

$$z_R = \frac{n\pi\omega_0^2}{\lambda}$$

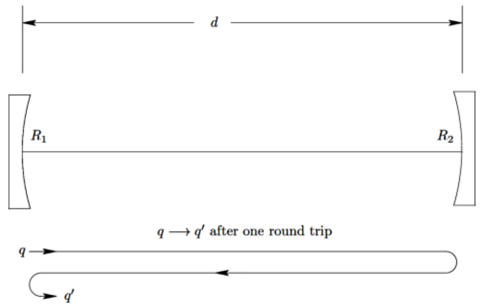
$$R(z) = z + \frac{z_R^2}{z}.$$

ABCD rule

$$q_1 = \frac{Aq_0 + B}{Cq_0 + D} \Longleftrightarrow \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

GEOMETRIC PROPERTIES OF OPTICAL CAVITY

Stability condition



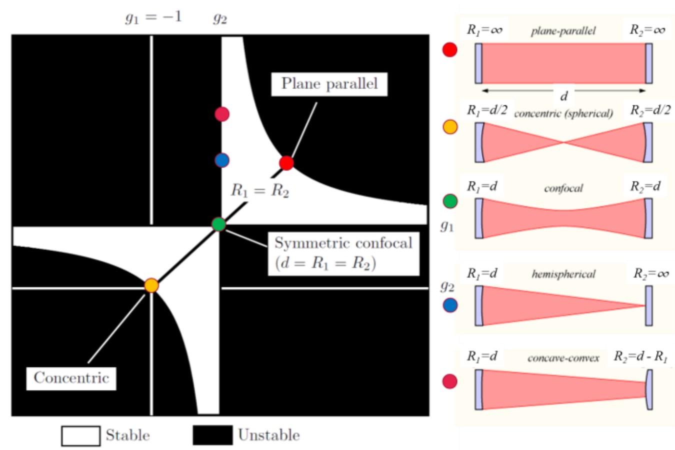
$$q = \frac{Aq+B}{Cq+D}$$

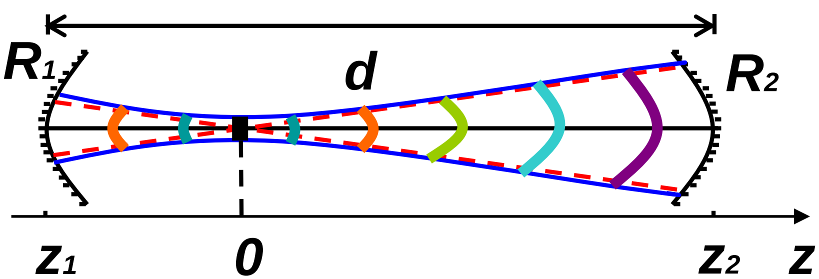
$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac{2}{R_1} & 1 \end{pmatrix} \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{2}{R_2} & 1 \end{pmatrix} \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}$$

$$g_1 \equiv 1 - \frac{d}{R_1} \qquad g_2 \equiv 1 - \frac{d}{R_2}$$

$$\frac{1}{q} = \frac{g_1 - 1}{d} \pm \frac{i}{g_2 d}\sqrt{g_1 g_2 (1 - g_1 g_2)}$$

Stability criterion: $\quad 0 \leq g_1 g_2 \leq 1$



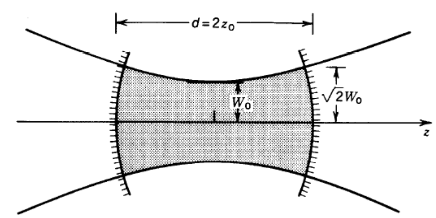


$$\omega^2 = \left(\frac{\lambda d}{n\pi}\right)\sqrt{\frac{g_2}{g_1(1-g_1 g_2)}}\ , R = \frac{d}{g_1 - 1} = -R_1,$$

$$\omega_0^2 = \left(\frac{\lambda d}{n\pi}\right)\frac{\sqrt{g_1 g_2 (1 - g_1 g_2)}}{g_1 + g_2 - 2 g_1 g_2}$$

Confocal cavity

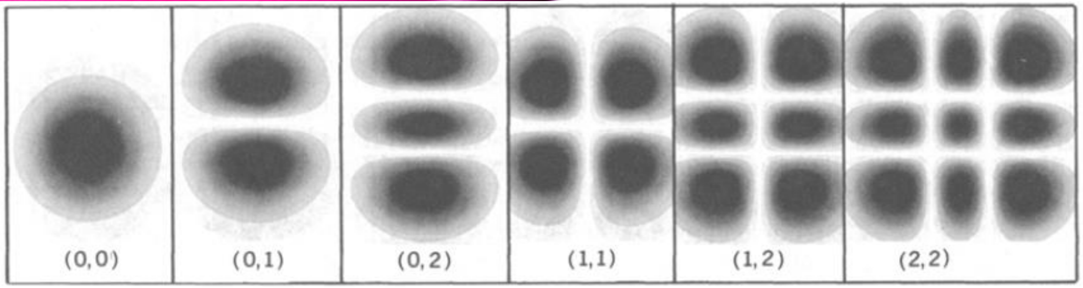
$$d = R$$



$$\omega_0 = \left(\frac{\lambda d}{2n\pi}\right)^{1/2}$$

$$\omega_{mirror} = \left(\frac{\lambda d}{n\pi}\right)^{1/2} = \sqrt{2}\omega_0$$

Higher-order modes



$$u(x,y,z)_{nm} =$$

$$\frac{\omega_0}{\omega(z)} H_n(\sqrt{2}\frac{x}{\omega}) H_m(\sqrt{2}\frac{y}{\omega})$$

$$\times \exp\left\{-i(kz - \Phi(m,n;z)) - i\frac{k}{2q}(x^2+y^2)\right\}$$

- They are all characterized by the *same* complex beam parameter, q, defined by

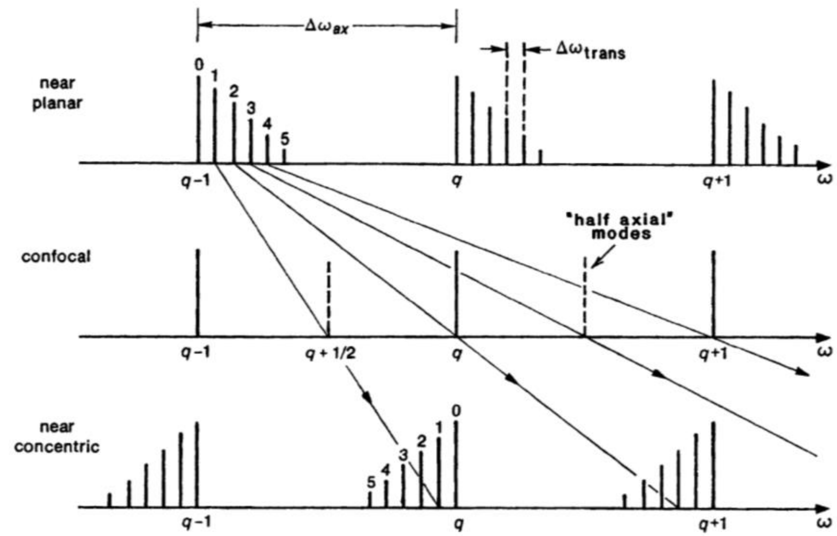
$$\frac{1}{q(z)} = \frac{1}{R(z)} - i\frac{\lambda}{n\pi\omega^2(z)}. \qquad (2.28)$$

- The mode of index m has a half-width, x_m, (in one transverse coordinate), where

$$x_m \approx \sqrt{m} \times \omega. \qquad (2.29)$$

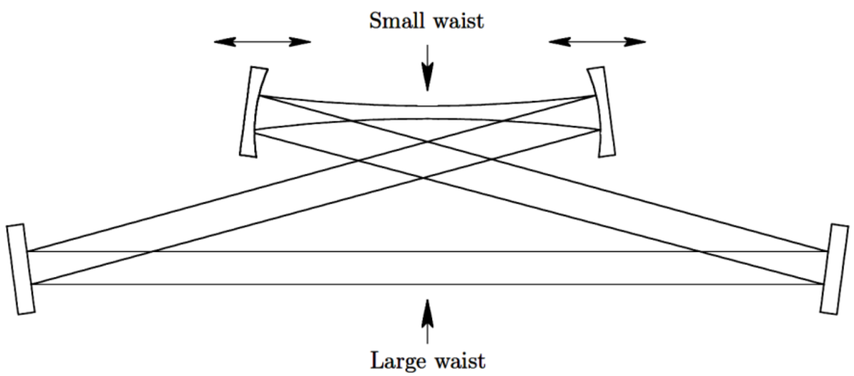
$$\nu_{nmq} = \left(q + (n+m+1)\frac{\cos^{-1} \pm\sqrt{g_1 g_2}}{\pi}\right)\frac{c}{2d}$$

$$\frac{\cos^{-1} \pm\sqrt{g_1 g_2}}{\pi} \approx \begin{cases} 0 & : \quad g_1, g_2 \to 1 \quad \text{(near-planar)} \\ 1/2 & : \quad g_1, g_2 \to 0 \quad \text{(near-confocal)} \\ 1 & : \quad g_1, g_2 \to -1 \quad \text{(near-spherical)} \end{cases}$$



SUMMARY – GEOMETRIC PROPERTIES OF OPTICAL CAVITY

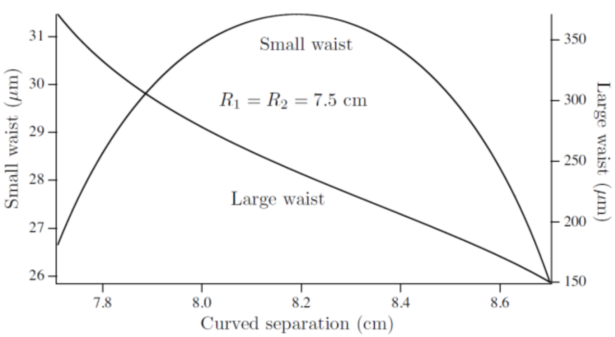
Ring Cavities



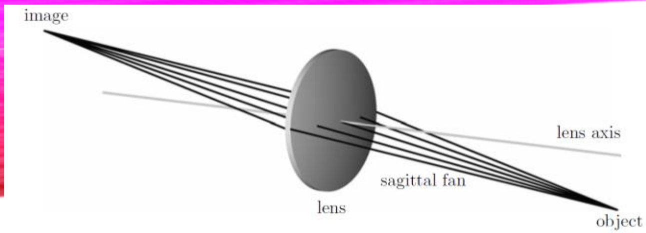
$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{-2}{R} & 1 \end{pmatrix} \begin{pmatrix} 1 & d_1 + 2d_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{-2}{R} & 1 \end{pmatrix} \begin{pmatrix} 1 & d_3 \\ 0 & 1 \end{pmatrix}$$

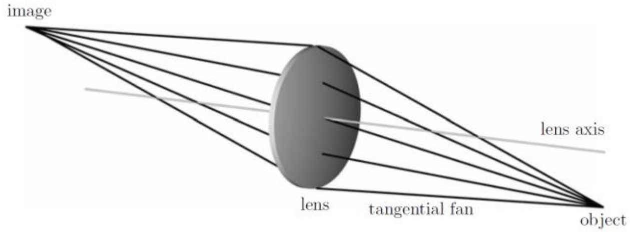
Stability: $R \leq d_3 \leq \frac{R(d_1+2d_2)}{d_1+2d_2-R}\left(\approx R\left(1+\frac{R}{d_1+2d_2}\right)\right)$

Property	Standing wave	Symmetric ring
g_1	$1-\frac{d}{R_1}$	$1-\frac{d_1+2d_2}{R}$
g_2	$1-\frac{d}{R_2}$	$1-\frac{d_3}{R}$
Path length	$2d$	$L = d_1 + 2d_2 + d_3$
Waist(s)	$\omega^2 = \left(\frac{\lambda d}{n\pi}\right)\frac{\sqrt{g_1 g_2(1-g_1 g_2)}}{g_1+g_2-2g_1 g_2}$	$\omega_0^2 = \left(\frac{\lambda R}{2n\pi}\right)\frac{\sqrt{g_1 g_2(1-g_1 g_2)}}{g_1}$
		${\omega'_0}^2 = \left(\frac{\lambda R}{2n\pi}\right)\frac{\sqrt{g_1 g_2(1-g_1 g_2)}}{g_2}$
ν_{nmq}	$\left(q+(n+m+1)\frac{\cos^{-1}\pm\sqrt{g_1 g_2}}{\pi}\right)\frac{c}{2d}$	$\left(q+(n+m+1)\frac{\cos^{-1}\pm\sqrt{g_1 g_2}}{\pi}\right)\frac{c}{L}$
FSR	$\frac{c}{2d}$	$\frac{c}{L}$



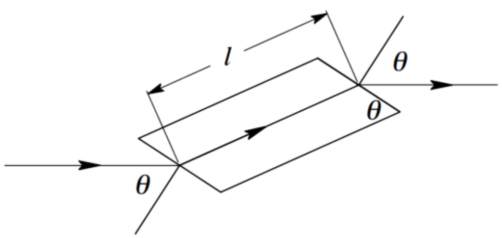
2.8 ASTIGMATISM IN A RING CAVITY





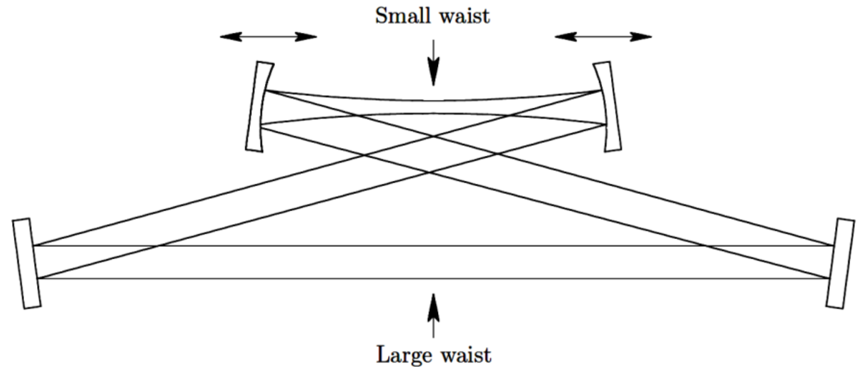
$$M_T = \begin{pmatrix} 1 & 0 \\ \frac{-2}{R\cos\theta} & 1 \end{pmatrix}$$

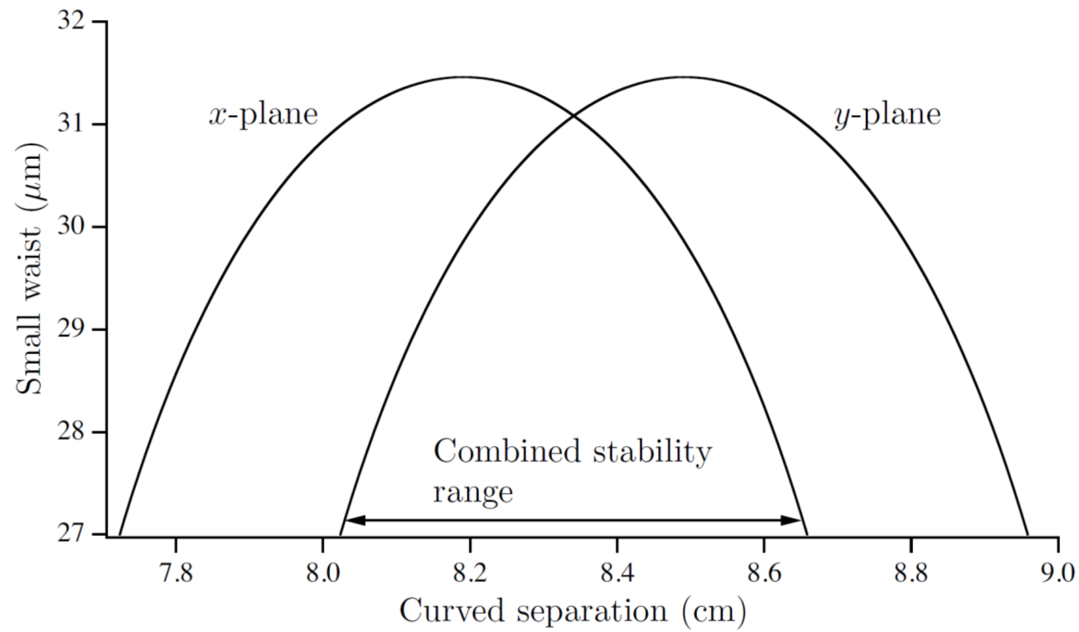
$$M_S = \begin{pmatrix} 1 & 0 \\ \frac{-2\cos\theta}{R} & 1 \end{pmatrix}$$



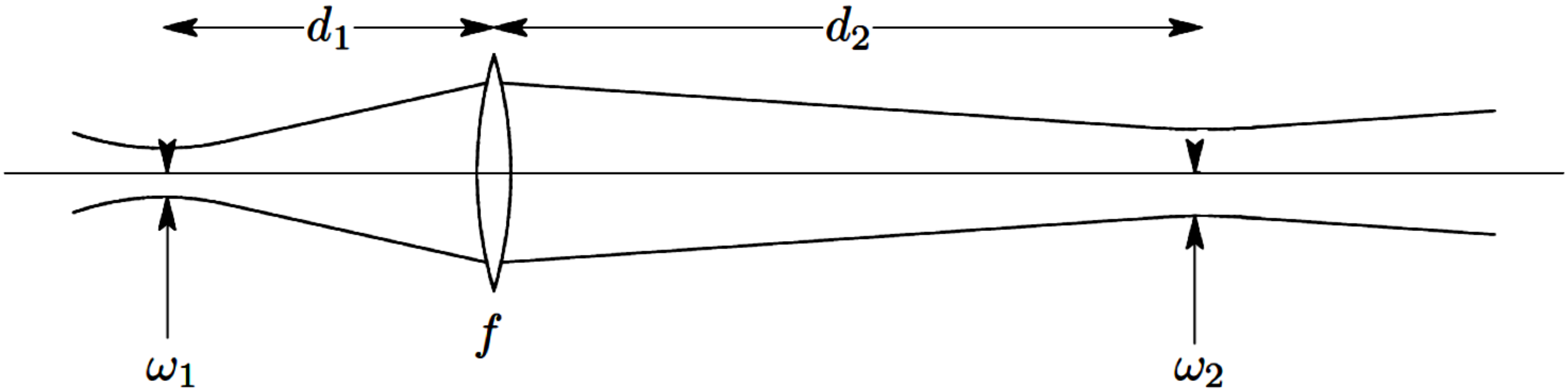
$$M_T = \begin{pmatrix} 1 & l/n^3 \\ 0 & 1 \end{pmatrix}$$

$$M_S = \begin{pmatrix} 1 & l/n \\ 0 & 1 \end{pmatrix}.$$





2.9 MODE MATCHING

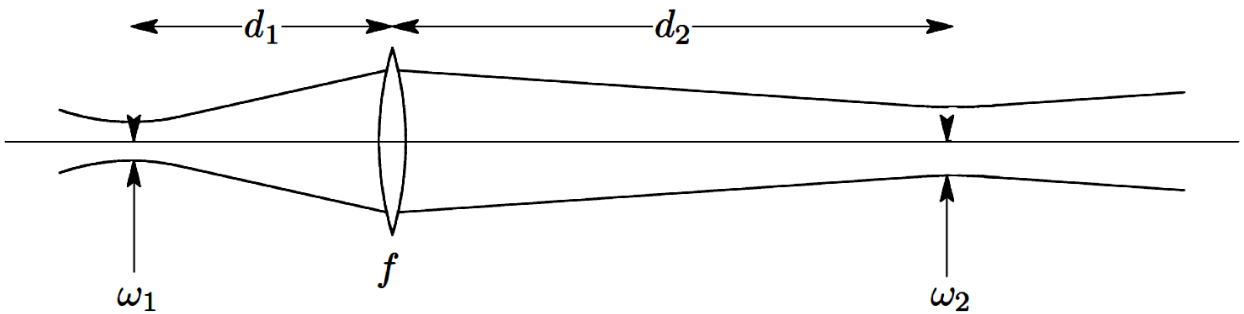


$$f_0 = n\pi\omega_1\omega_2/\lambda = \sqrt{z_{R1}z_{R2}}$$

$$d_1 = f \pm \frac{\omega_1}{\omega_2}\sqrt{f^2 - f_0^2}$$
$$d_2 = f \pm \frac{\omega_2}{\omega_1}\sqrt{f^2 - f_0^2}$$

Homework: drive the equation for d1 and d2

2.9 MODE MATCHING



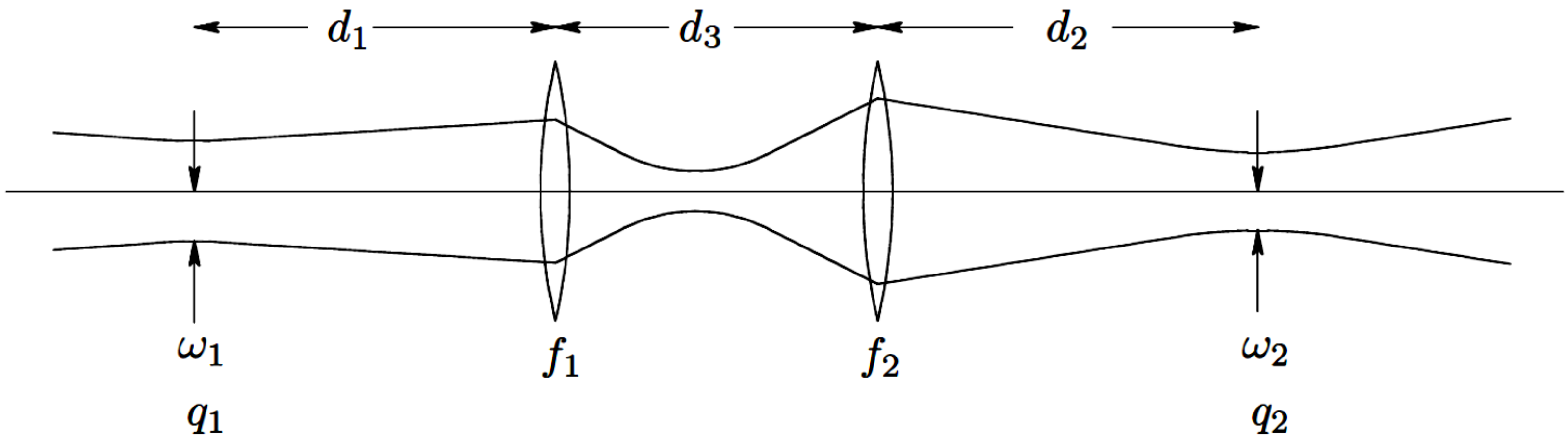
$$\frac{1}{q_1} = -i\frac{\lambda}{n\pi\omega_1^2} = \frac{-i}{z_{R1}} \qquad \frac{1}{q_2} = -i\frac{\lambda}{n\pi\omega_2^2} = \frac{-i}{z_{R2}}$$

$$q_1 = iz_{R1} \qquad q_2 = iz_{R2}$$

$$q_2 = \frac{Aq_1 + B}{Cq_1 + D} = iz_{R2} = \frac{Aiz_{R1} + B}{Ciz_{R1} + D} \qquad B + z_{R1}z_{R2}C = 0, z_{R1}A - z_{R2}D = 0$$

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & d_2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix}\begin{pmatrix} 1 & d_1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 - d_2/f & d_1 + d_2 - d_1 d_2/f \\ -1/f & 1 - d_1/f \end{pmatrix}$$

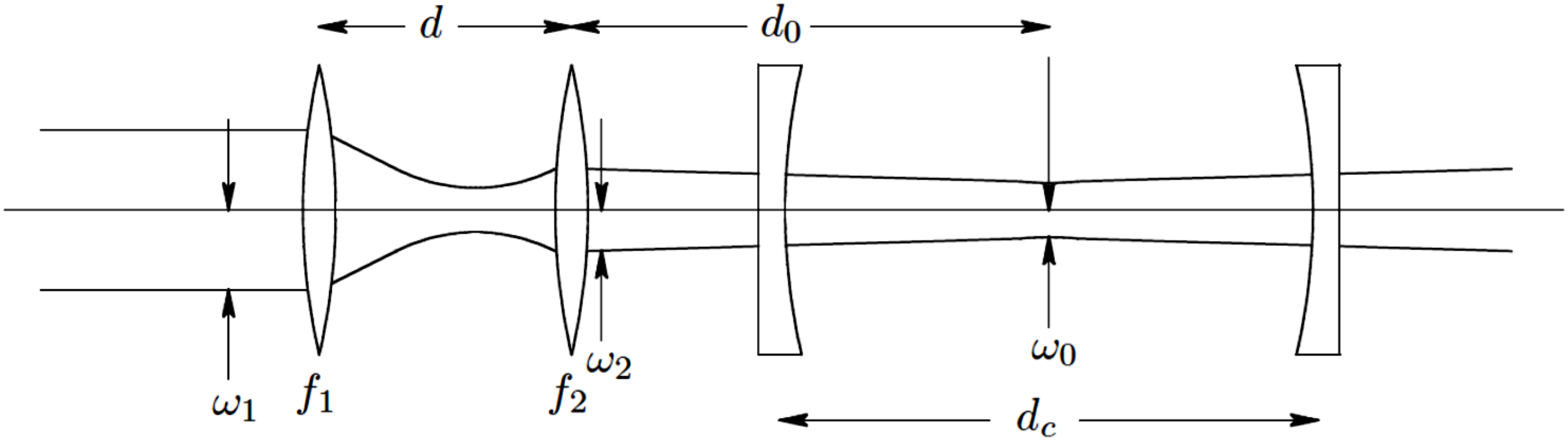
Then solve the equations.



$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{-1}{f_2} & 1 \end{pmatrix} \begin{pmatrix} 1 & d_3 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{-1}{f_1} & 1 \end{pmatrix} \begin{pmatrix} 1 & d_1 \\ 0 & 1 \end{pmatrix}$$

$$q_2 = \frac{Aq_1 + B}{Cq_1 + D} + d_2$$

$$R_2 = 1/Re\{1/q_2\} \quad \text{and} \quad \omega_2 = \sqrt{\frac{-\lambda}{n\pi Im\{1/q_2\}}}$$



Procedure

1. Determine ω_0 and z_R from the cavity equations.

2. Given d_0, the distance from waist to lens 2, determine ω_2 using $\omega_2 = \omega_0\sqrt{1+(d_0/z_R)^2}$.

3. Given input spot size (ω_1), find two lenses whose focal lengths are in ratio ω_1/ω_2.

4. Separate lenses by slightly more than $f_1 + f_2$ so waist is at center of cavity.

Example

(Confocal) $d_c = 10$ cm, $\lambda = 1\ \mu$m, $\omega_0 = 126\ \mu$m, $z_R = 5$ cm

$d_0 = 10$ cm, $\omega_2 = 282\ \mu$m

$\omega_1 = .05$ cm, ratio ≈ 1.8, $f_1 = 1.8$ cm, $f_2 =1$ cm

$f_1 + f_2 = 2.8$ cm

Homework: Problem 2.7

2.10 BEAM QUALITY CHARACTERIZATION: THE M2 PARAMETERS

Gaussian beam is an *ideal, diffraction-limited* laser beam

How to characterize the deviation of the actual beam from Gaussian beam?

Keep in mind: A beam is properly characterized by both intensity profile and its divergence

Rectangular slit with 2a: $E(x,z) \sim \frac{\sin(2\pi a x/z\lambda)}{2\pi a x/z\lambda}$

$$\Delta x_0 \times \Delta x = 0.5 \times \lambda z$$

Gaussian beam: $\omega^2(z) = \omega_0^2\left(1 + \left(\frac{z}{z_R}\right)^2\right)$

$$\omega_0 \times \omega(z) = 1/\pi \times \lambda z$$

2.10 BEAM QUALITY CHARACTERIZATION: THE M^2 PARAMETERS

M^2 parameters

Variance of width $\sigma_x^2 = \frac{\int_{-\infty}^{\infty}(x-x_0)^2 I(x,y)dxdy}{\int_{-\infty}^{\infty} I(x,y)dxdy}, \sigma_y^2 = \frac{\int_{-\infty}^{\infty}(y-y_0)^2 I(x,y)dxdy}{\int_{-\infty}^{\infty} I(x,y)dxdy}$

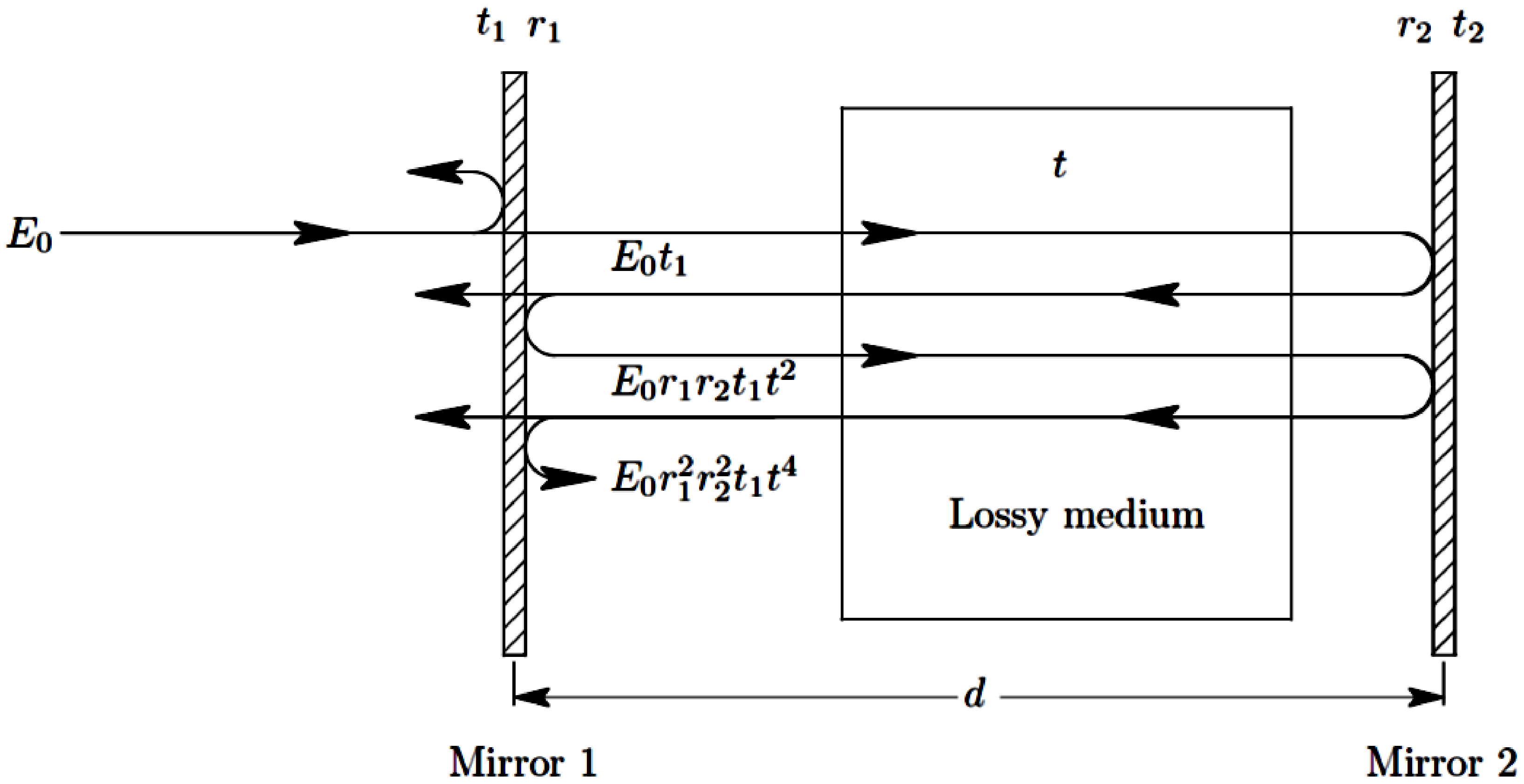
$$\sigma_x^2 = \sigma_{0x}^2 + \sigma_{\theta x}^2 \times z^2$$
$$\sigma_y^2 = \sigma_{0y}^2 + \sigma_{\theta y}^2 \times z^2$$

$$W_x \equiv 2\sigma_x$$
$$W_y \equiv 2\sigma_y$$

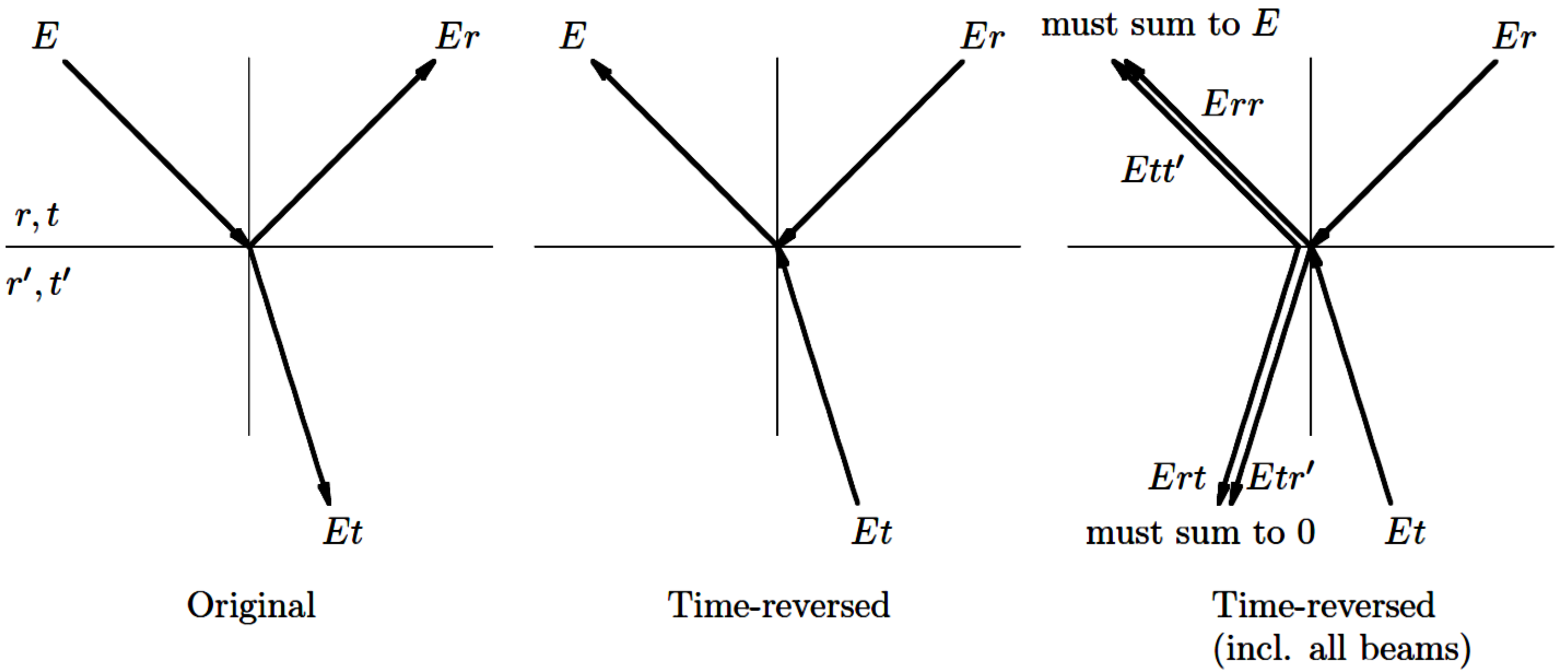
$$W_x^2 \equiv W_{0x}^2\ (1 + M_x^4 \times \left(\frac{z}{Z_{Rx}}\right)^2)$$
$$W_y^2 \equiv W_{0y}^2\ (1 + M_y^4 \times \left(\frac{z}{Z_{Ry}}\right)^2)$$

$$W_{0x} \times W_x\ (z) \sim M_x^2 \times \frac{z\ \lambda}{\pi}$$
$$W_{0x} \times W_y\ (z) \sim M_y^2 \times \frac{z\ \lambda}{\pi}$$

3. ENERGY RELATIONS IN OPTICAL CAVITIES



3.2 REFLECTION AND TRANSMISSION AT AN INTERFACE

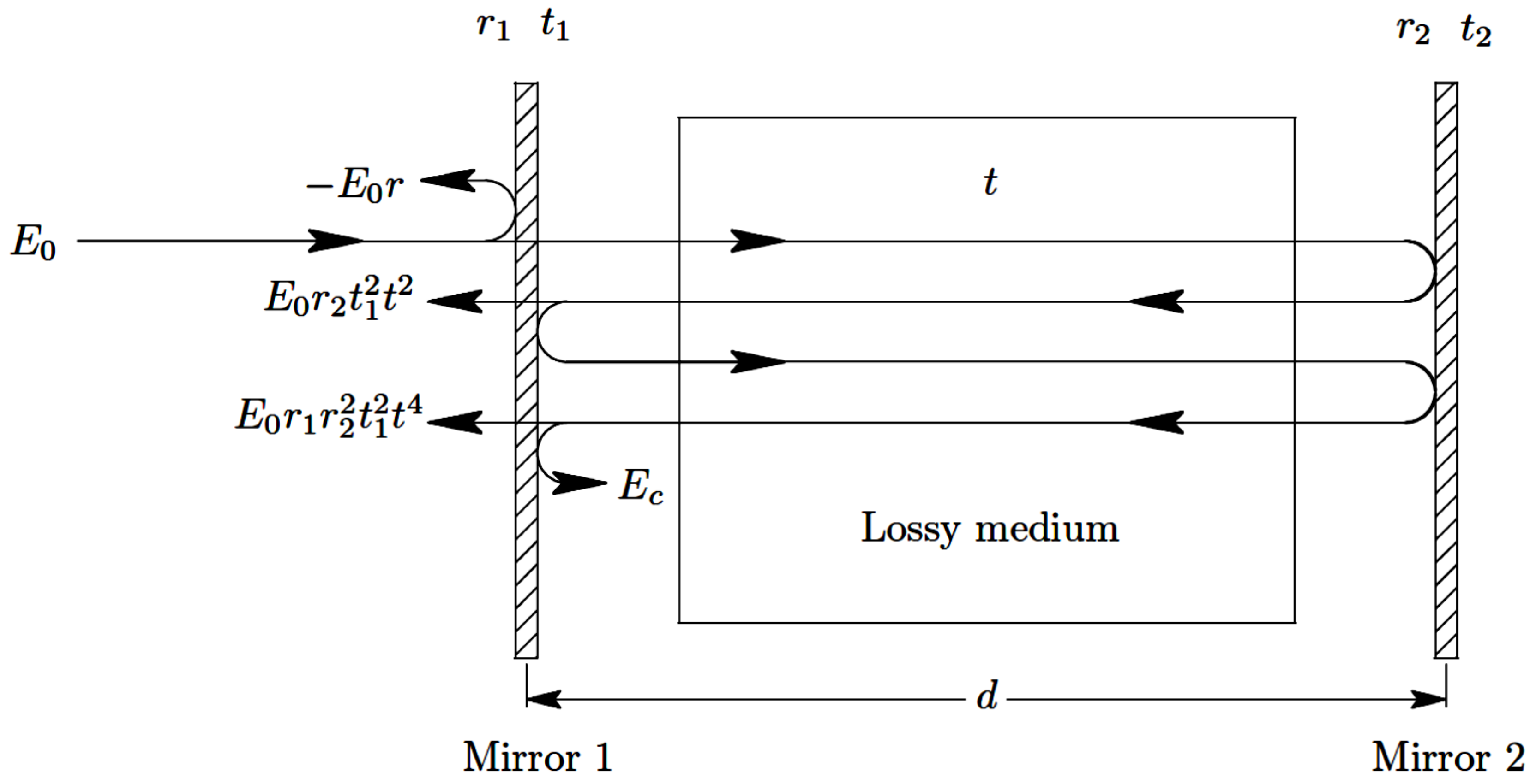


$$E = Ett' + Err \quad \text{and} \quad 0 = Ert + Er't,$$

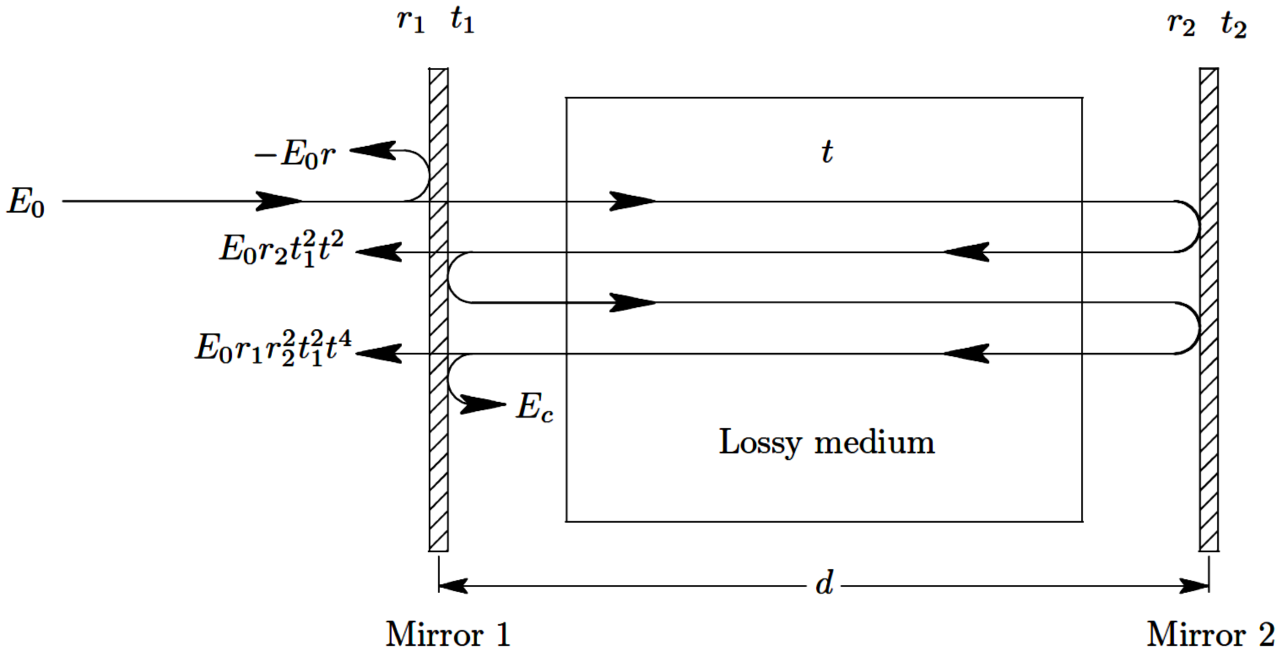
$$tt' + r^2 = 1 \quad \text{and} \quad r = -r.'$$

$$R + T = 1, \qquad \text{where} \quad R = r^2, T = t^2. \qquad \therefore t = t'$$

3.3 REFLECTED FIELDS FROM STANDING WAVE CAVITY



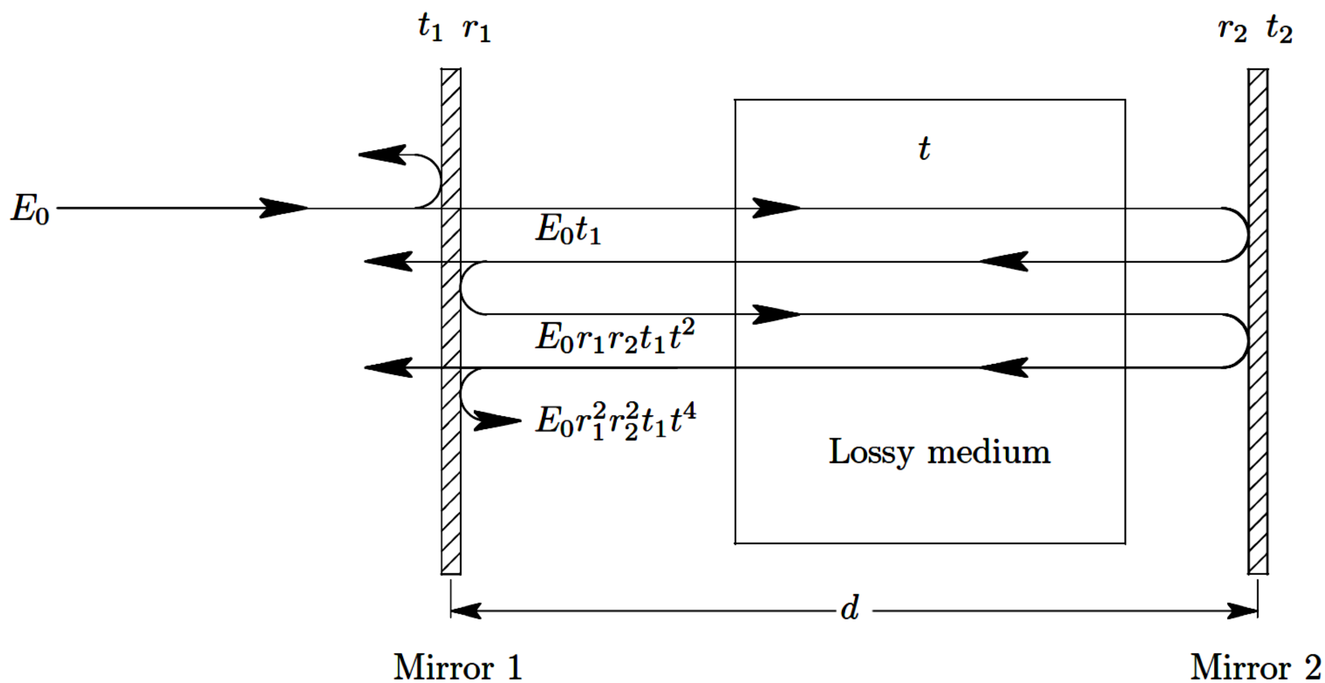
Find a big problem in this drawing!



Assuming no other loss,

$$\begin{aligned} E_r &= -E_0r_1 + E_0r_2t_1^2t^2e^{-i\delta} + E_0r_1r_2^2t_1^2t^4e^{-2i\delta} + \cdots \\ &= -E_0r_1 + \frac{E_0t_1^2}{r_1}\left(r_1r_2t^2e^{-i\delta} + (r_1r_2t^2e^{-i\delta})^2 + \cdots\right) \\ &= E_0\frac{r_2t^2e^{-i\delta} - r_1}{1 - r_1r_2t^2e^{-i\delta}}, \end{aligned}$$

3.4 INTERNAL FIELD IN A STANDING WAVE CAVITY



$$
\begin{aligned}
E_c &= E_0 t_1 + E_0 r_1 r_2 t_1 t^2 e^{-i\delta} + E_0 r_1^2 r_2^2 t_1 t^4 e^{-2i\delta} + \cdots \\
&= E_0 t_1 \left(1 + r_1 r_2 t^2 e^{-i\delta} + (r_1 r_2 t^2 e^{-i\delta})^2 + \cdots\right) \\
&= \frac{E_0 t_1}{1 - r_1 r_2 t^2 e^{-i\delta}}.
\end{aligned}
$$

3.5 REFLECTED AND INTERNAL INTENSITIES

$$\frac{I_{r,c}}{I_0} = \left|\frac{E_{r,c}}{E_0}\right|^2 \qquad\qquad r_m \equiv r_2 t^2$$

$$\text{Field:} \quad E_r = E_0 \frac{r_m e^{-i\delta} - r_1}{1 - r_1 r_m e^{-i\delta}}$$

$$\text{Intensity:} \quad I_r = I_0 \frac{(r_1 - r_m)^2 + 4 r_1 r_m \sin^2 \delta/2}{(1 - r_1 r_m)^2 + 4 r_1 r_m \sin^2 \delta/2}$$

$$\text{Field:} \quad E_c = E_0 \frac{t_1}{1 - r_1 r_m e^{-i\delta}}$$

$$\text{Intensity:} \quad I_c = I_0 \frac{t_1^2}{(1 - r_1 r_m)^2 + 4 r_1 r_m \sin^2 \delta/2}$$

Homework: Calculate the transmission of the electric field and the intensity

Homework: Draw I_r and I_c in terms of δ with $r_1 = r_m = 0.995$ and 0.95.

3.6 THE RESONANT CHARACTER OF THE REFLECTED AND CIRCULATING INTENSITIES

$$I_c = I_0 \frac{t_1^2}{(1 - r_1 r_m)^2 + 4 r_1 r_m \sin^2 \delta/2}$$

a maximum when $\delta = 0$, $2\pi, 4\pi, .., 2q\pi, ..$ $\quad \delta = 2\pi \frac{2d}{\lambda} = 2\pi\nu \frac{2d}{c}$

Free spectral range in phase: $\delta = 2\pi$

Free spectral range in distance: $d = \lambda/2$

Free spectral range in frequency: $\nu = c/2d$

half maximum intensity occurs at a phase $(\delta_{1/2})$

$$(1 - r_1 r_m)^2 = 4 r_1 r_m \sin^2 \delta_{1/2}/2$$

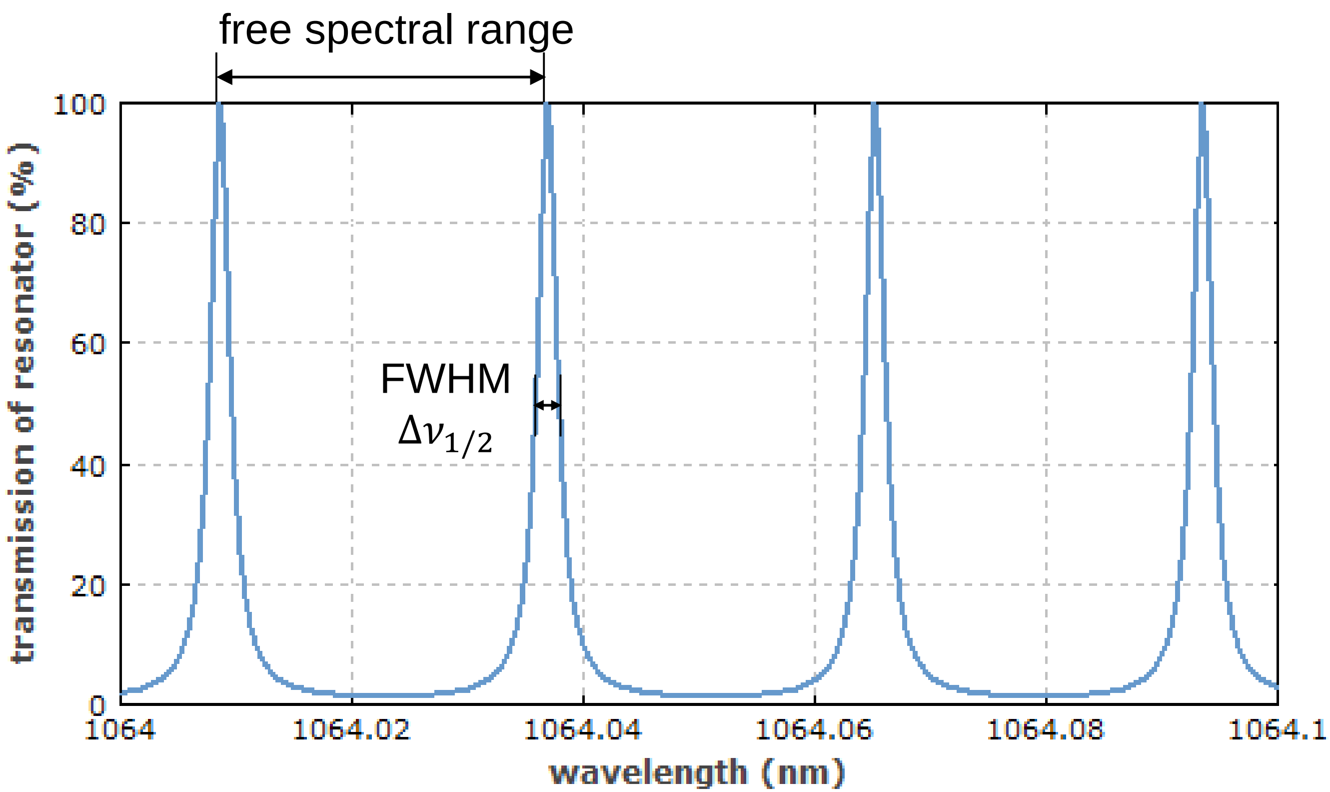
$$\delta_{1/2} = 2 \sin^{-1} \left(\frac{1 - r_1 r_m}{2\sqrt{r_1 r_m}} \right) \approx \frac{1 - r_1 r_m}{\sqrt{r_1 r_m}}.$$

$$\text{FWHM} \equiv \Delta\nu_{1/2} = 2\delta_{1/2} = \frac{2(1 - r_1 r_m)}{\sqrt{r_1 r_m}}$$

$$\text{FWHM} \equiv \Delta\nu_{1/2} = 2\delta_{1/2} = \frac{2(1 - r_1 r_m)}{\sqrt{r_1 r_m}}$$

$$\text{Finesse: } \mathcal{F} = \frac{\text{free spectral range}}{\Delta\nu_{1/2}} = \frac{\pi\sqrt{r_1 r_m}}{1 - r_1 r_m}$$

$$\mathcal{F} = \frac{\pi\sqrt{r_1^2}}{1 - r_1^2} = \frac{\pi\sqrt{R}}{1 - R} \qquad (t = 1, r_1 = r_2, R = r_1^2).$$



Finesse: 14