QUANTUM ELECTRONICS

For atomic physics

REVIEW

Gaussian Beam

$$\nabla_t^2 \psi - 2ik\frac{\partial \psi}{\partial z} = 0, \qquad \psi(x,y,z) = \exp\left\{-i\left(P(z) + \frac{k}{2q(z)}r^2\right)\right\}$$
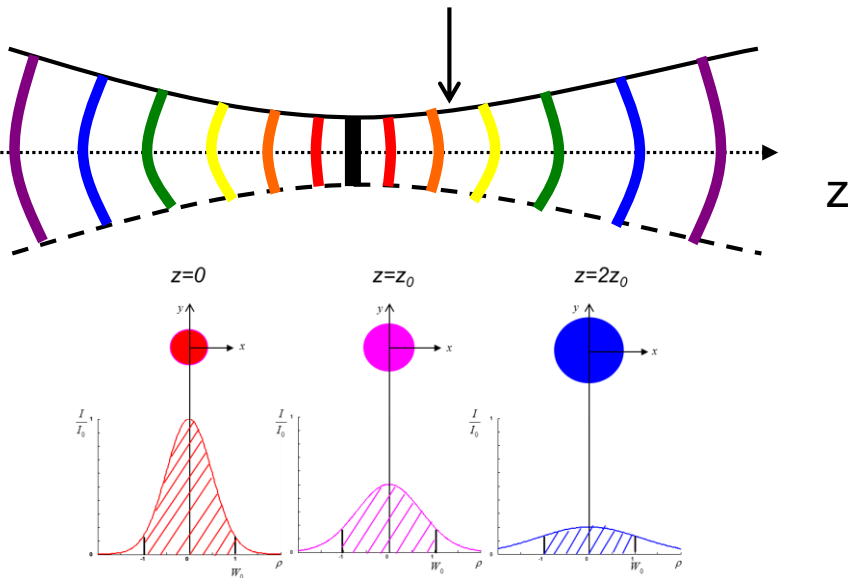
complex beam parameter q

$$\frac{1}{q} = \frac{1}{R} - i\frac{\lambda}{n\pi\omega^2}$$

z

$$\omega(z) = \omega_0\left[1 + \left(\frac{z}{z_R}\right)^2\right]^{1/2}$$

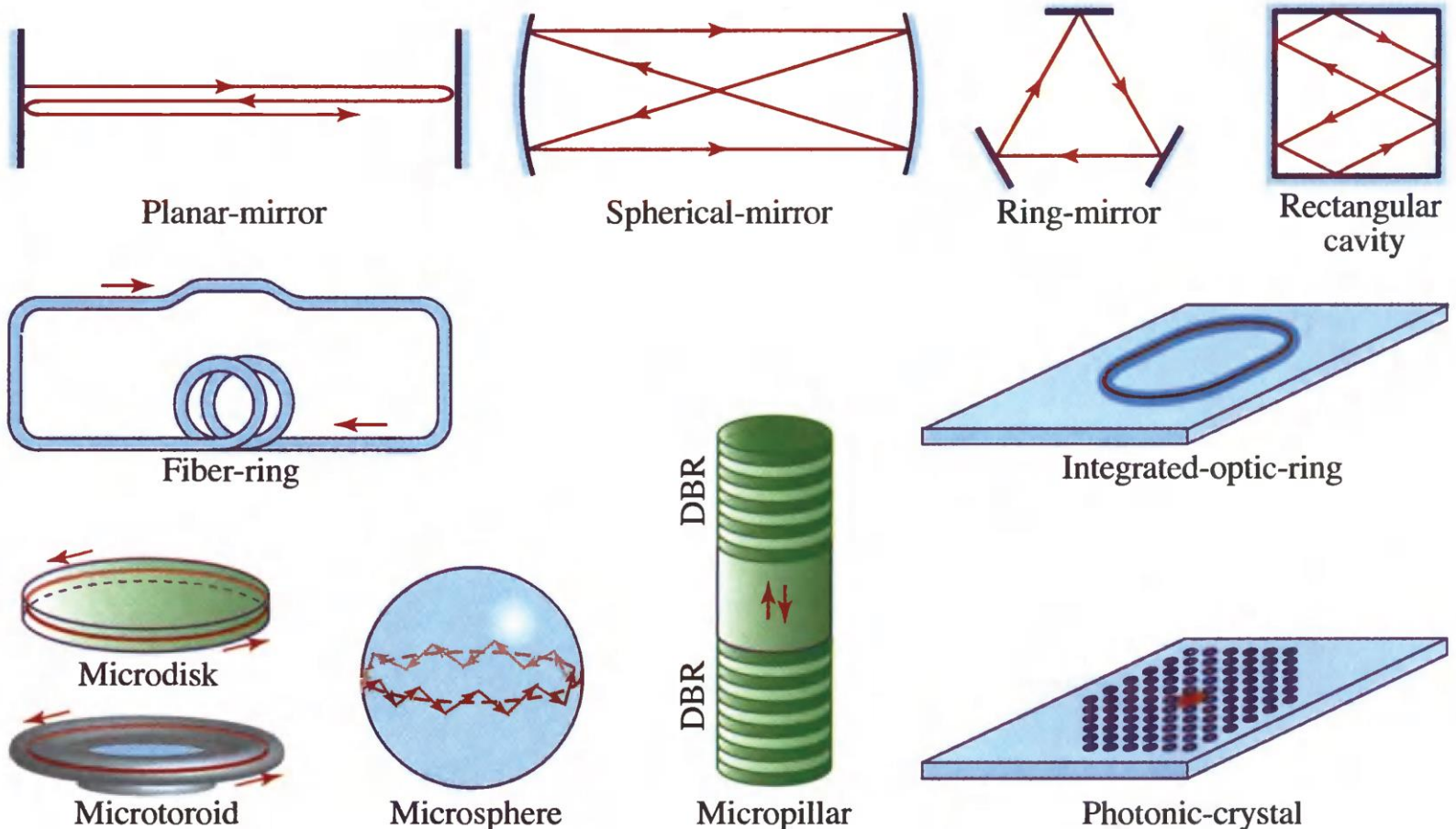$$R(z) = z + \frac{z_R^2}{z}.$$

$$z_R = \frac{n\pi\omega_0^2}{\lambda}$$

Case	q-transformation	Ray matrix
Free-space	$q_1 = q_0 + d = \dfrac{(1)q_0 + (d)}{(0)q_0 + (1)}$	$\begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}$
Thin lens	$q_1 = \dfrac{q_0}{1 - q_0/f} = \dfrac{(1)q_0 + (0)}{(-1/f)q_0 + (1)}$	$\begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix}$
Mirror	$q_1 = \dfrac{q_0}{1 - 2q_0/R} = \dfrac{(1)q_0 + (0)}{(-2/R)q_0 + (1)}$	$\begin{pmatrix} 1 & 0 \\ -2/R & 1 \end{pmatrix}$
Slab	$q_1 = q_0 + d/n = \dfrac{(1)q_0 + (d/n)}{(0)q_1 + (1)}$	$\begin{pmatrix} 1 & d/n \\ 0 & 1 \end{pmatrix}$

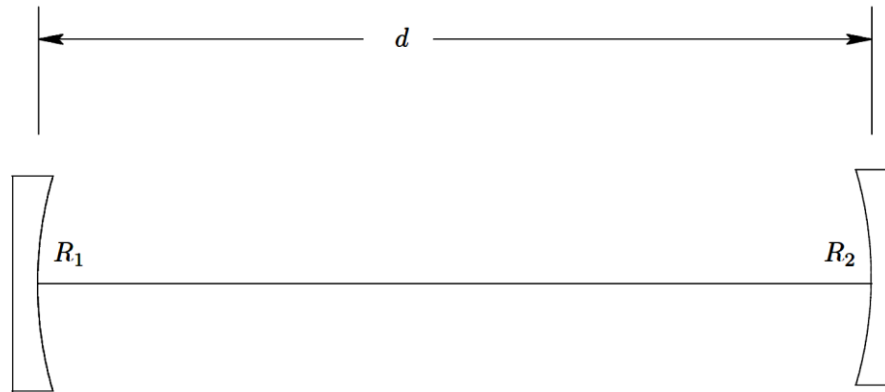$$q_1 = \frac{Aq_0 + B}{Cq_0 + D} \iff \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

$$q_0 \xrightarrow{ABCD} q_1 \xrightarrow{A'B'C'D'} q_2$$

2. OPTICAL RESONATORS – GEOMETRICAL PROPERTIES

Planar-mirror

Spherical-mirror

Ring-mirror

Rectangular cavity

Fiber-ring

Integrated-optic-ring

Microdisk

Microtoroid

Microsphere

DBR

DBR

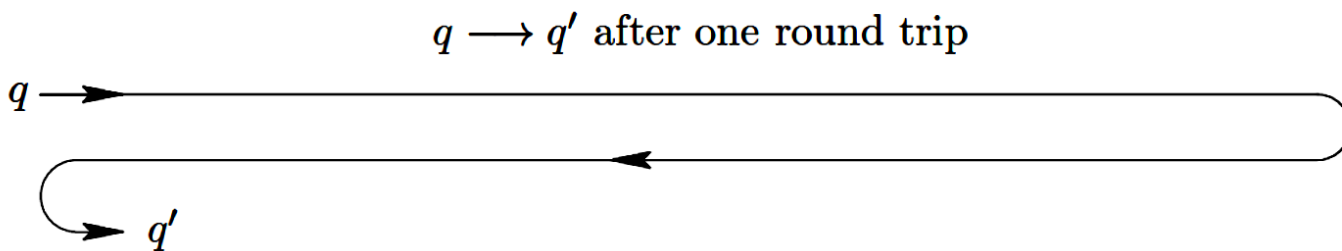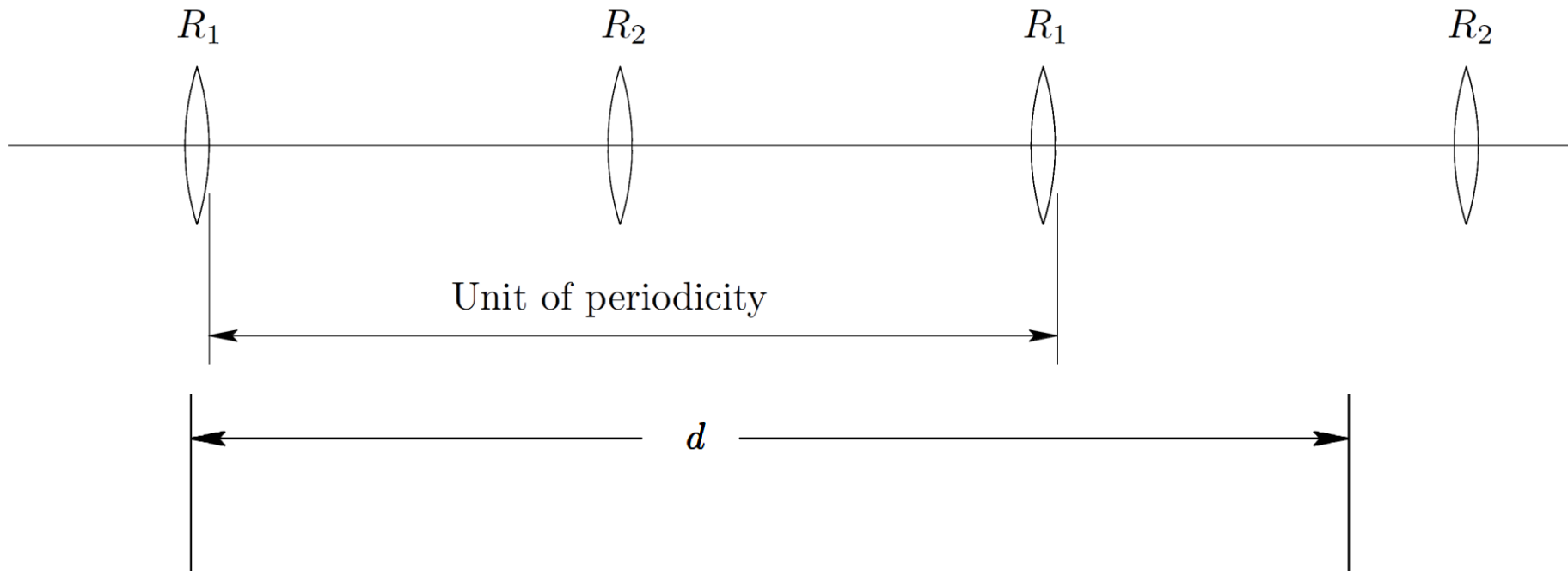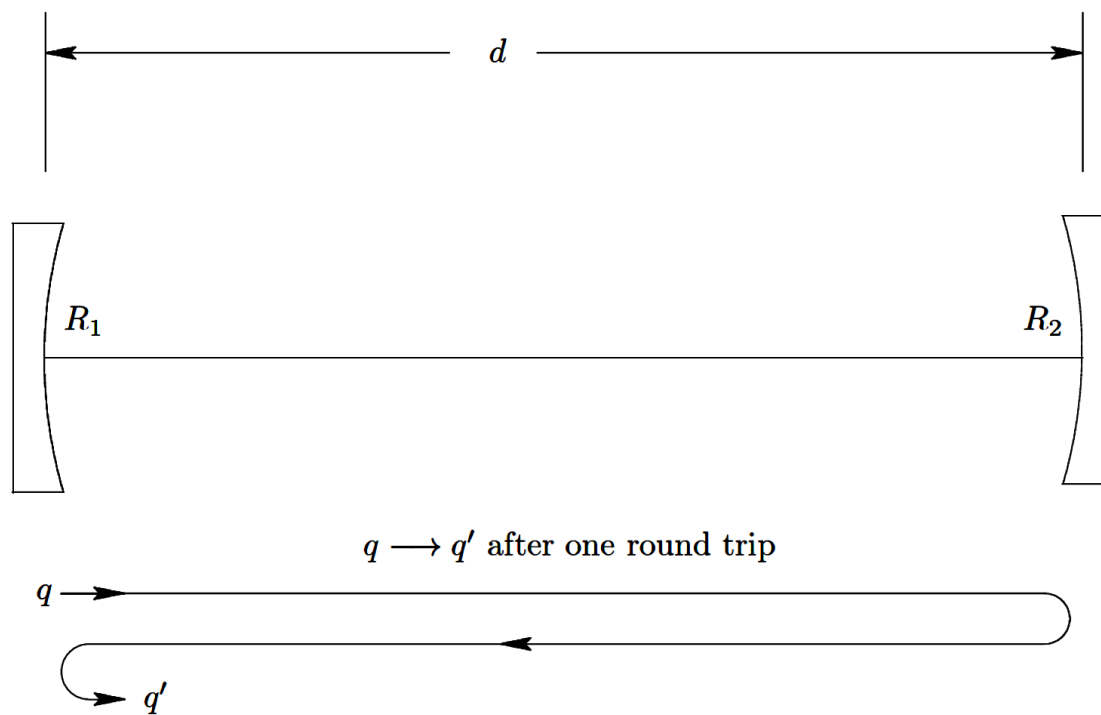Micropillar

Photonic-crystal

2.1 INTRODUCTION
APPLICATIONS OF OPTICAL CAVITIES

- Feedback mechanism in laser oscillators

- Optical spectrum analyzer

- Stable frequency reference for laser stabilization

- Enhancement for the second harmonic generation

R_1 R_2 R_1 R_2

Unit of periodicity

d

R_1 R_2

$q \longrightarrow q'$ after one round trip

q

q'

$q \longrightarrow q'$ after one round trip

- Self-consistency requires

$$q = \frac{Aq + B}{Cq + D} \quad \text{or} \quad \frac{1}{q} = \frac{C + D\left(\frac{1}{q}\right)}{A + B\left(\frac{1}{q}\right)}$$

$$B\left(\frac{1}{q}\right)^2 + (A - D)\left(\frac{1}{q}\right) - C = 0,$$

$$\frac{1}{q} = \frac{D - A}{2B} \pm \frac{1}{2B}\sqrt{(A - D)^2 + 4BC}$$

$$\frac{1}{q} = \frac{D-A}{2B} \pm \frac{1}{2B}\sqrt{(A-D)^2 + 4BC}$$

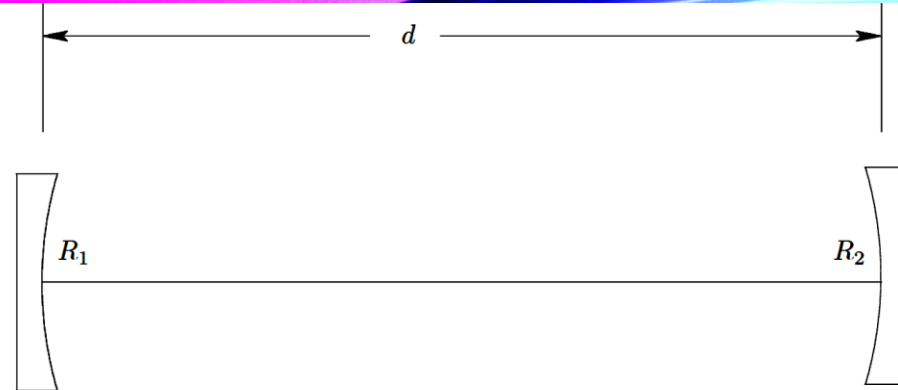$$\frac{1}{q} = \frac{1}{R} - i\frac{\lambda}{n\pi\omega^2}$$

$$R = \frac{2B}{D-A}$$

Here we use the fact, the determinant of the ABCD matrix is unity,

$$AD - BC = 1$$

$$\omega = \sqrt{\frac{2\lambda|B|}{n\pi\sqrt{4-(A+D)^2}}}$$

Stability criterion: $|A+D| \leq 2$

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac{2}{R_1} & 1 \end{pmatrix} \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{2}{R_2} & 1 \end{pmatrix} \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}$$

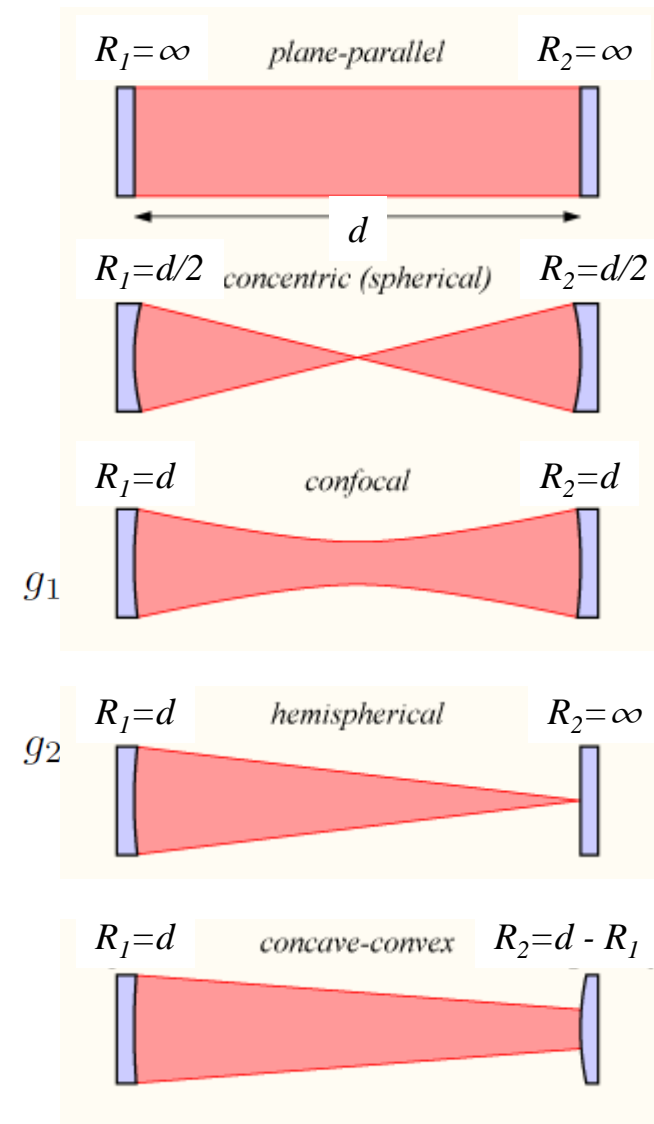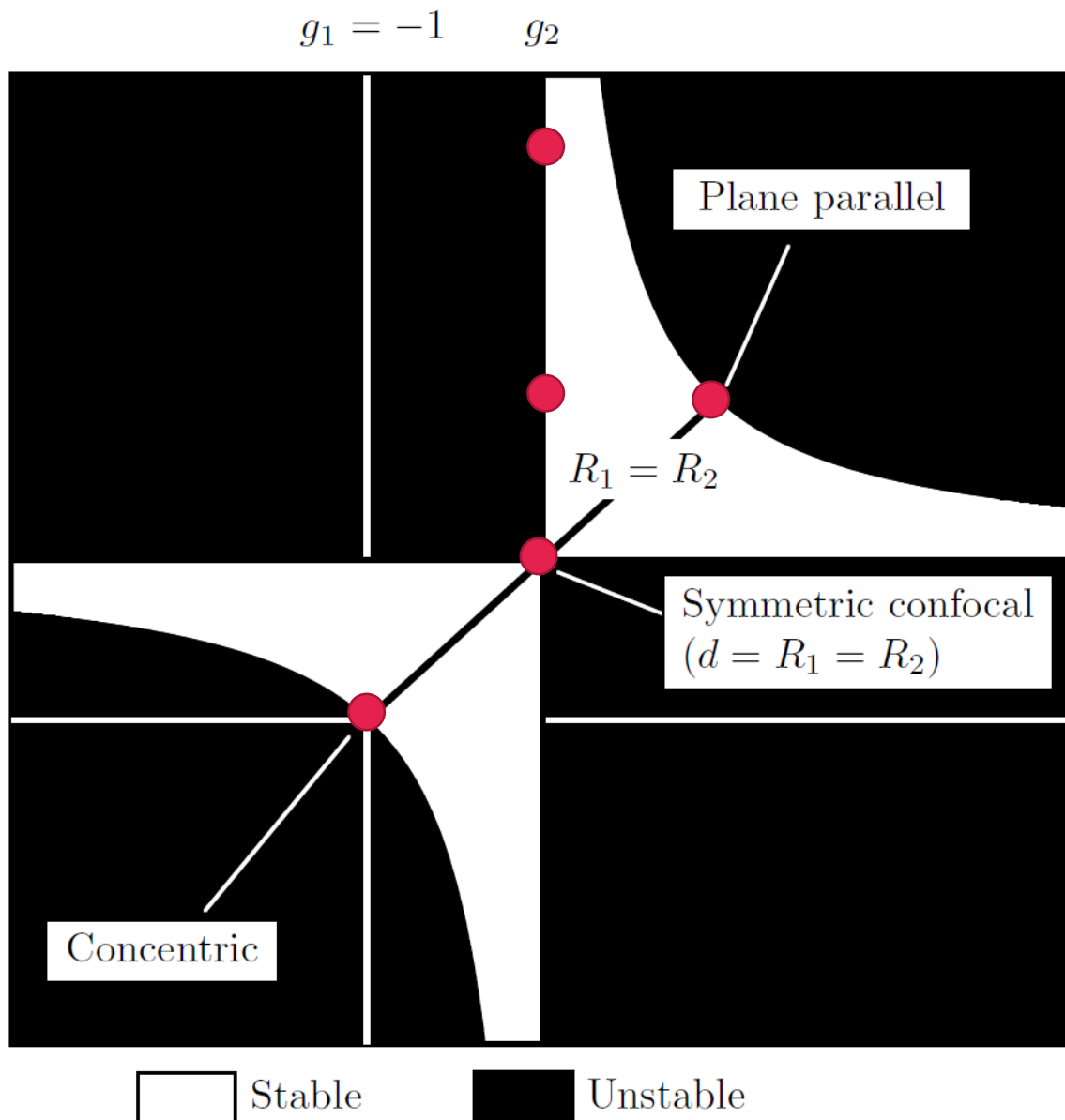$$g_1 \equiv 1 - \frac{d}{R_1} \qquad g_2 \equiv 1 - \frac{d}{R_2}$$

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{2}{d}(g_1 - 1) & 1 \end{pmatrix} \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{2}{d}(g_2 - 1) & 1 \end{pmatrix} \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2g_2 - 1 & 2g_2 d \\ \frac{2}{d}(2g_1 g_2 - g_1 - g_2) & 4g_1 g_2 - 2g_2 - 1 \end{pmatrix}.$$
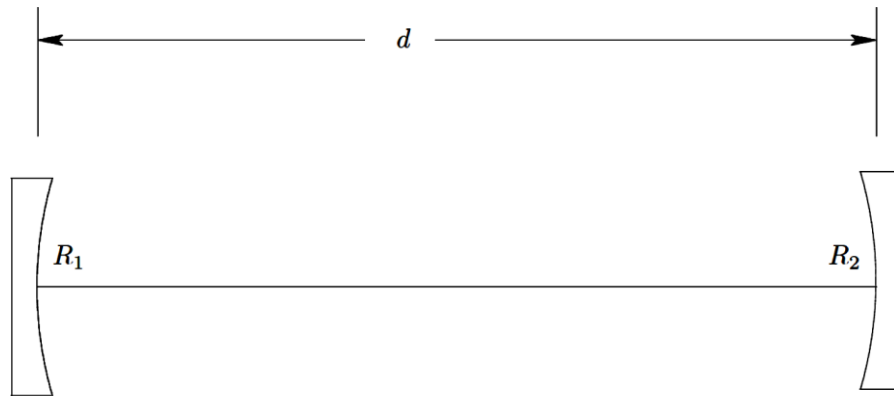
$$|A + D| \leq 2 \implies |4g_1 g_2 - 2| \leq 2$$

Stability criterion: $\qquad 0 \leq g_1 g_2 \leq 1$

Stability criterion: $0 \leq g_1 g_2 \leq 1$ $\qquad g_1 \equiv 1 - \dfrac{d}{R_1}$ $\quad g_2 \equiv 1 - \dfrac{d}{R_2}$

$g_1 = -1$ $\qquad g_2$

Plane parallel

$R_1 = R_2$

Symmetric confocal $(d = R_1 = R_2)$

Concentric

☐ Stable   ■ Unstable

g_1

g_2

$R_1 = \infty$ *plane-parallel* $R_2 = \infty$

d

$R_1 = d/2$ *concentric (spherical)* $R_2 = d/2$

$R_1 = d$ *confocal* $R_2 = d$

$R_1 = d$ *hemispherical* $R_2 = \infty$

$R_1 = d$ *concave-convex* $R_2 = d - R_1$

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

$$= \begin{pmatrix} 2g_2 - 1 & 2g_2 d \\ \frac{2}{d}(2g_1 g_2 - g_1 - g_2) & 4g_1 g_2 - 2g_2 - 1 \end{pmatrix}.$$
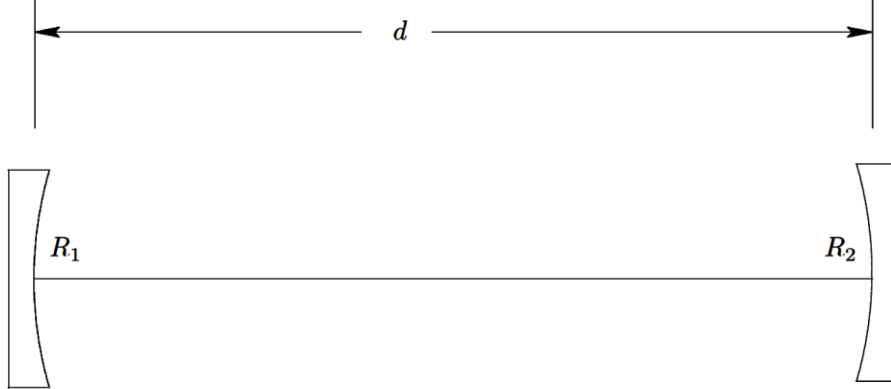
$$\frac{1}{q} = \frac{D - A}{2B} \pm \frac{1}{2B}\sqrt{(A-D)^2 + 4BC} \qquad \sqrt{(A+D)^2 - 4}$$

$$\frac{1}{q} = \frac{g_1 - 1}{d} \pm \frac{1}{g_2 d}\sqrt{g_1 g_2 (g_1 g_2 - 1)}$$

$$\omega^2 = \left(\frac{\lambda d}{n\pi}\right)\sqrt{\frac{g_2}{g_1(1 - g_1 g_2)}}$$

R=?

$$g_1 \equiv 1 - \frac{d}{R_1} \qquad g_2 \equiv 1 - \frac{d}{R_2}$$

$$\frac{1}{q} = \frac{g_1 - 1}{d} \pm \frac{1}{g_2 d}\sqrt{g_1 g_2(g_1 g_2 - 1)}$$

$$= \frac{g_1 - 1}{d} \pm i\frac{1}{g_2 d}\sqrt{g_1 g_2(1 - g_1 g_2)}$$
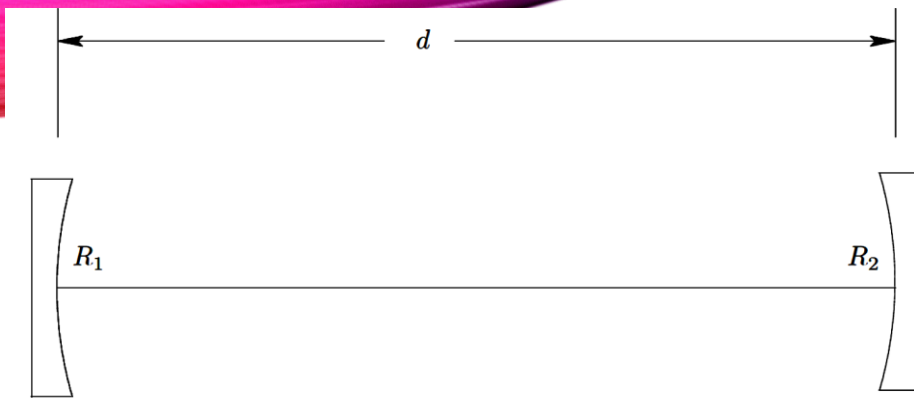
$$q(z) = q_0 + z = i \; z_R \; + z \qquad \frac{1}{q} = \frac{1}{R} - i\frac{\lambda}{n\pi\omega^2}$$

Rayleigh length: $\quad z_R = -\dfrac{Im\{q_1\}}{|q_1|^2} \qquad \left(q_1 \equiv \dfrac{1}{q}\right)$

Waist size: $\quad \omega_0^2 = \left(\dfrac{\lambda}{n\pi}\right) z_R = -\left(\dfrac{\lambda}{n\pi}\right)\dfrac{Im\{q_1\}}{|q_1|^2}$

$$z_R = d\frac{\sqrt{g_1 g_2(1 - g_1 g_2)}}{g_1 + g_2 - 2g_1 g_2} \qquad \omega_0^2 = \left(\frac{\lambda d}{n\pi}\right)\frac{\sqrt{g_1 g_2(1 - g_1 g_2)}}{g_1 + g_2 - 2g_1 g_2}$$

$$\frac{1}{q} = \frac{g_1 - 1}{d} \pm \frac{1}{g_2 d}\sqrt{g_1 g_2(g_1 g_2 - 1)}$$

$$= \frac{g_1 - 1}{d} \pm i\frac{1}{g_2 d}\sqrt{g_1 g_2(1 - g_1 g_2)}$$

$$q(z) = q_0 + z = i\ z_R\ + z \qquad \frac{1}{q} = \frac{1}{R} - i\frac{\lambda}{n\pi\omega^2}$$

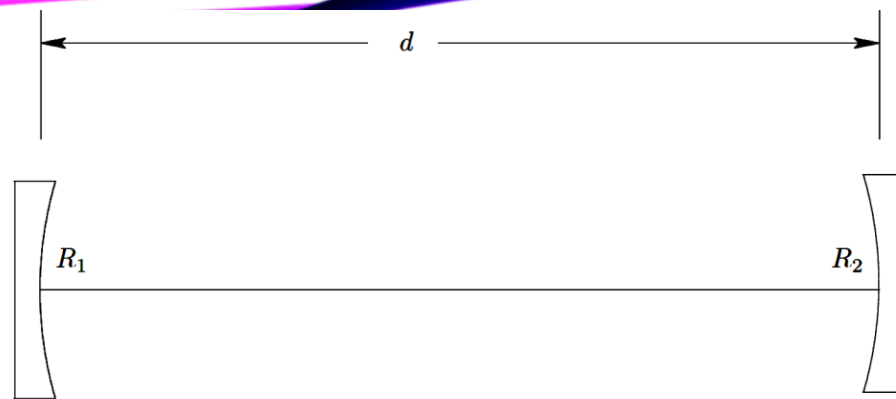Distance to waist $= -Re\{q(z)\}$, Distance from waist $= Re\{q(z)\}$

Distance to waist: $z = -\dfrac{Re\{q_1\}}{|q_1|^2}$ $\qquad \left(q_1 \equiv \dfrac{1}{q}\right)$

$$z_1 = d\frac{g_2(g_1 - 1)}{g_1 + g_2 - 2g_1 g_2}$$

$$g_1 \equiv 1 - \frac{d}{R_1} \quad g_2 \equiv 1 - \frac{d}{R_2}$$

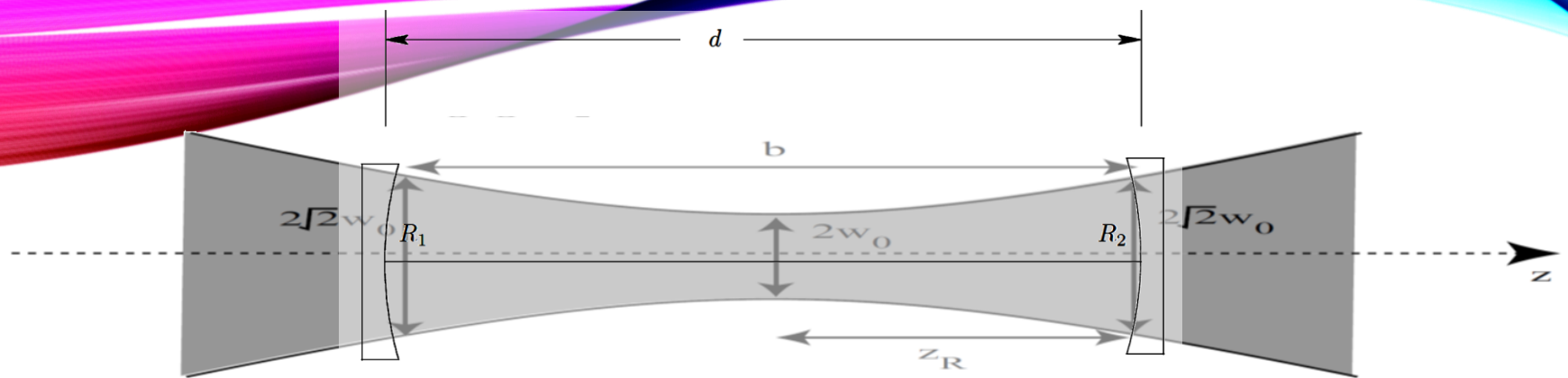$$z_2 = d\frac{g_1(1 - g_2)}{g_1 + g_2 - 2g_1 g_2}$$

$\because z_2 - z_1 = d$

$$\omega^2 = \left(\frac{\lambda d}{n\pi}\right) \sqrt{\frac{g_2}{g_1(1 - g_1 g_2)}} \qquad \omega_0^2 = \left(\frac{\lambda d}{n\pi}\right) \frac{\sqrt{g_1 g_2 (1 - g_1 g_2)}}{g_1 + g_2 - 2 g_1 g_2}$$

$$R_1 = R_2 = R$$
$$g_1 = g_2 = g = 1 - d/R$$

At mirror: $\quad \omega^2 = \left(\frac{\lambda d}{n\pi}\right) \sqrt{\frac{1}{1 - g^2}} = \left(\frac{\lambda R}{n\pi}\right) \sqrt{\frac{d}{2R - d}}$

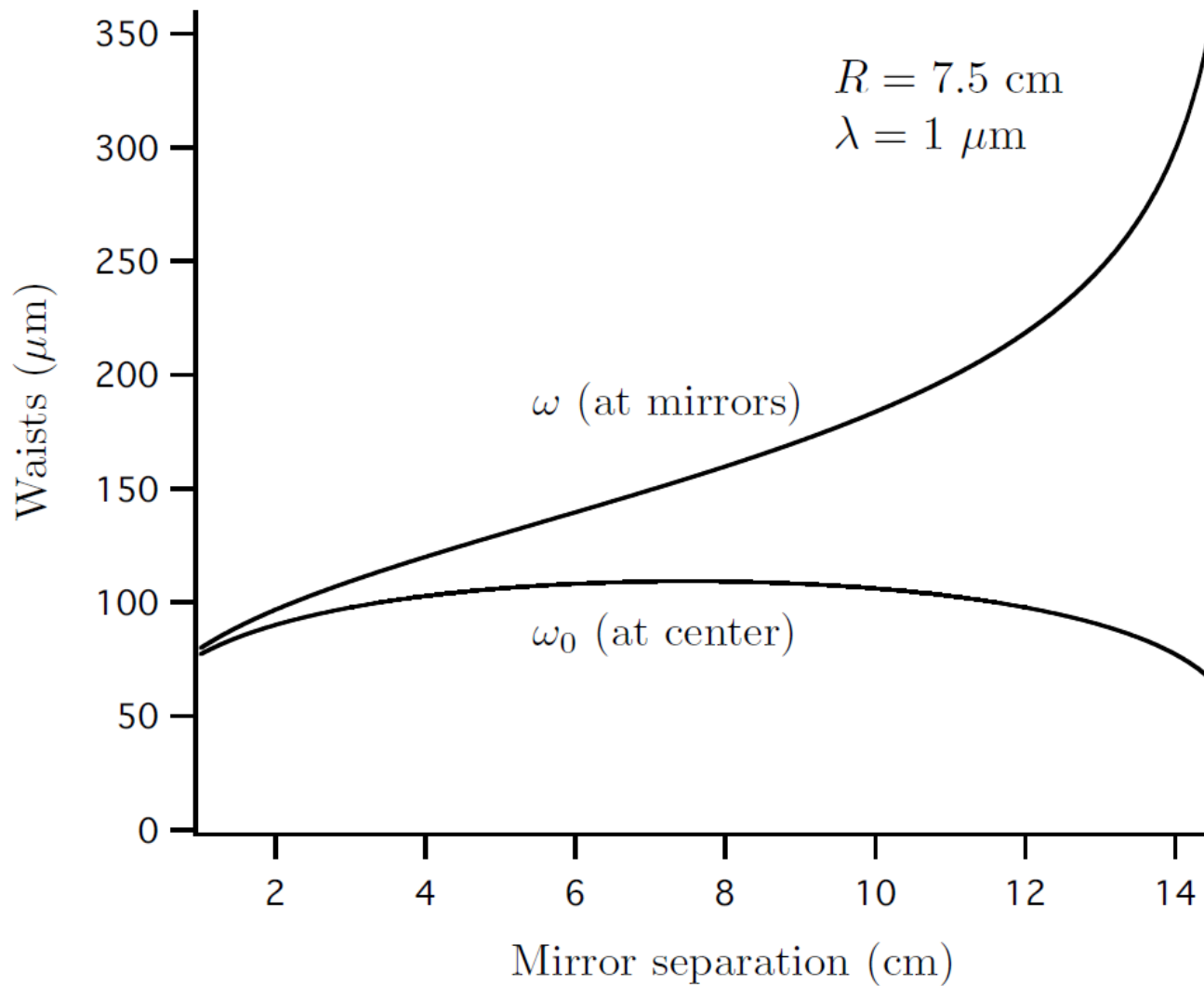Waist: $\quad \omega_0^2 = \left(\frac{\lambda d}{2n\pi}\right) \sqrt{\frac{1 + g}{1 - g}} = \left(\frac{\lambda}{n\pi}\right) \sqrt{\frac{dR}{2} - \frac{d^2}{4}}$

Confocal Cavity $\quad d = R$

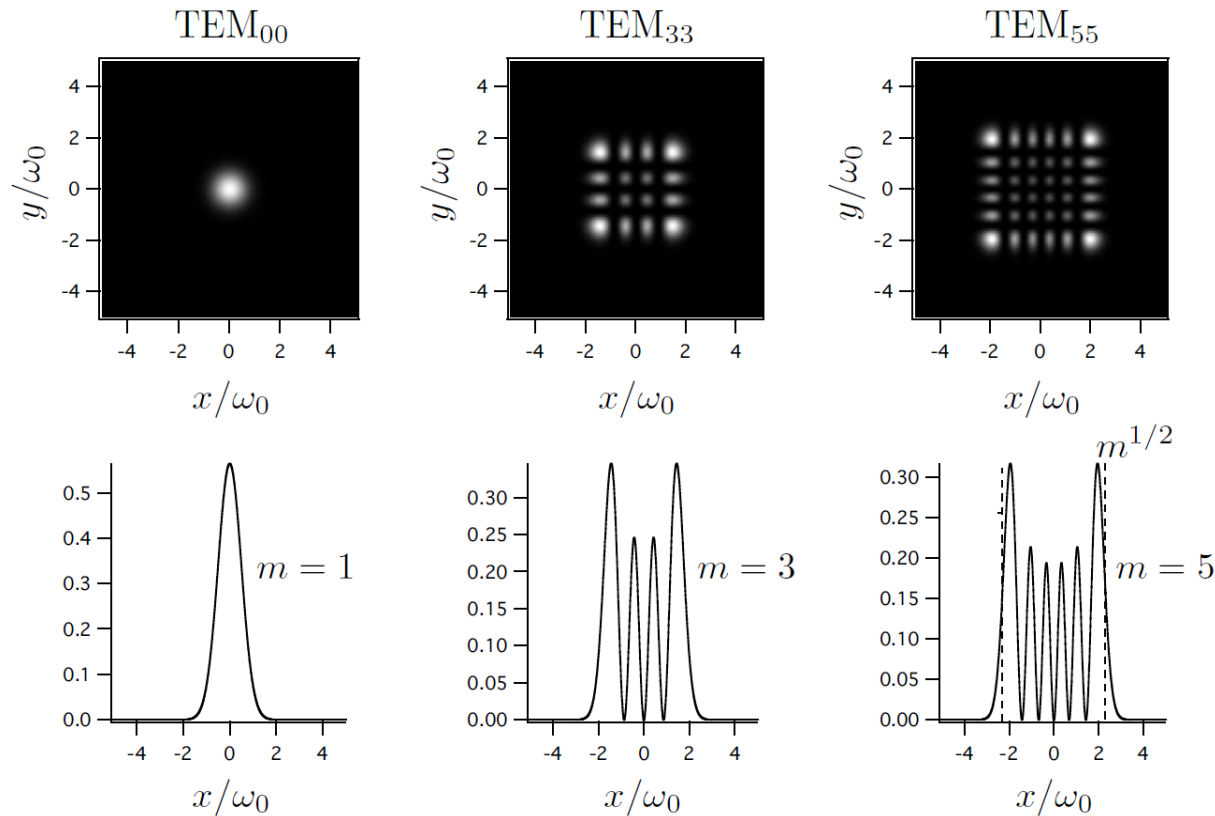$$\omega_0 = \left(\frac{\lambda d}{2n\pi} \right)^{1/2}$$

$$\omega_{mirror} = \left(\frac{\lambda d}{n\pi} \right)^{1/2} = \sqrt{2}\omega_0$$

Homework: draw the curves with the parameters that you are interested in. (e.g. R = 15 cm, λ = 0.37 um or R = 20 cm, λ = 0.78 um

$$u(x, y, z)_{nm} =$$

$$\frac{\omega_0}{\omega(z)} H_n(\sqrt{2}\frac{x}{\omega}) H_m(\sqrt{2}\frac{y}{\omega})$$

$$\times \exp\left\{ -i(kz - \Phi(m, n; z)) - i\frac{k}{2q}(x^2 + y^2) \right\}$$

$$x_m \sim \sqrt{m}\, \omega_0$$
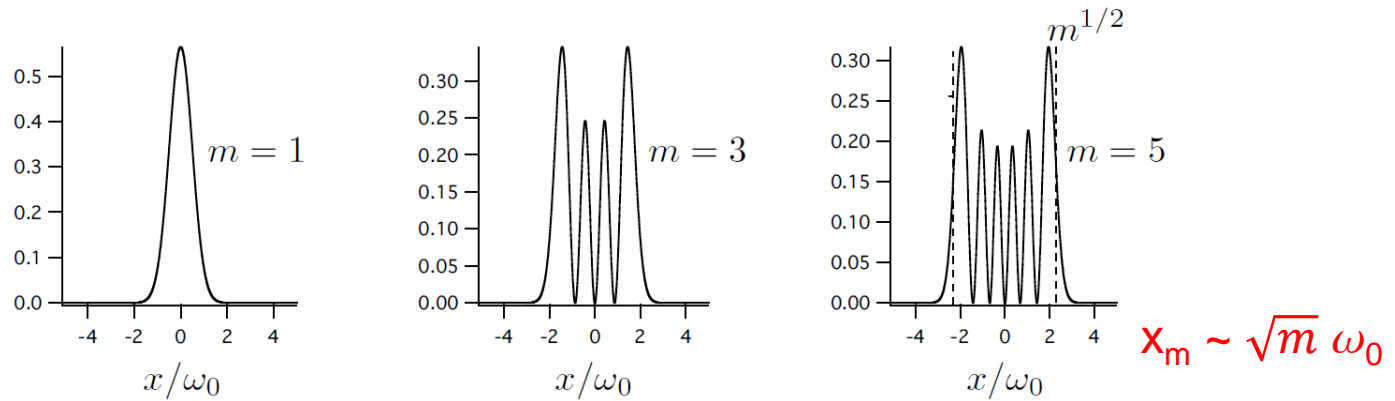
$$u(x, y, z)_{nm} =$$

$$\frac{\omega_0}{\omega(z)} H_n\left(\sqrt{2}\frac{x}{\omega}\right) H_m\left(\sqrt{2}\frac{y}{\omega}\right)$$

$$\times \exp\left\{-i(kz - \Phi(m, n; z)) - i\frac{k}{2q}(x^2 + y^2)\right\}$$

same q independent of m,n
Same ABCD rule

$$\frac{\omega_0}{\omega(z)} = \frac{1}{\sqrt{1 + (z/z_R)^2}}$$

$$\Phi(n, m; z) = (n + m + 1)\tan^{-1}(z/z_R)$$

HERMIT GAUSSIAN MODES

- They are all characterized by the *same* complex beam parameter, q, defined by

$$\frac{1}{q(z)} = \frac{1}{R(z)} - i\frac{\lambda}{n\pi\omega^2(z)}. \tag{2.28}$$

- They all satisfy the same ABCD rule as (lowest order) Gaussian beams. The *mode numbers*, m and n are preserved under all of the transformations discussed in this book.
- The mode *shape* is independent of z and scales with $\omega(z)$.
- The mode of index m has a half-width, x_m, (in one transverse coordinate), where
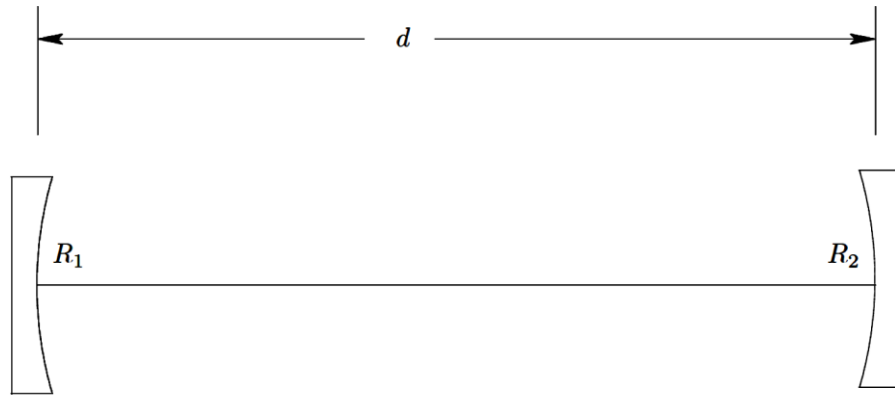
$$x_m \approx \sqrt{m} \times \omega. \tag{2.29}$$

Thus, ω is no longer the beam size in higher-order modes. A focused spot has its size *degraded* by $\sqrt{m}$.

- The maximum mode number (m_{max}) at a waist which will "fit" into an aperture of radius a is:

$$m_{max} \approx (a/\omega_0)^2. \tag{2.30}$$

This *spatial filtering* behavior of small apertures allows one to filter out modes whose mode number (in either coordinate) is greater than m_{max}.

$$\exp\left\{-i(kz - \Phi(m,n;z)) - i\frac{k}{2q}(x^2 + y^2)\right\}$$

$$\Phi(n,m;z) = (n+m+1)\tan^{-1}(z/z_R)$$

$$\delta = 2kd - 2(n+m+1)(\tan^{-1}(z_2/z_R) - \tan^{-1}(z_1/z_R))$$

$$\delta = q(2\pi) \implies \frac{\omega d}{c} - (n+m+1)\cos^{-1}\pm\sqrt{g_1 g_2} = q\pi$$

$$\nu_{nmq} = \left(q + (n+m+1)\frac{\cos^{-1}\pm\sqrt{g_1 g_2}}{\pi}\right)\frac{c}{2d}$$

Free spectral range

$$\frac{\cos^{-1}\pm\sqrt{g_1 g_2}}{\pi} \approx \begin{cases} 0 & : & g_1, g_2 \to 1 \quad \text{(near-planar)} \\ 1/2 & : & g_1, g_2 \to 0 \quad \text{(near-confocal)} \\ 1 & : & g_1, g_2 \to -1 \quad \text{(near-spherical)} \end{cases}$$