

Assignment #5

Name: Cao Mingming ID: 2018311770

cmm18@mails.tsinghua.edu.cn

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1 Problem 1

Derive Eq. 2.64 and Eq. 2.65 in the main textbook.

Solution 1

Assumed that the q parameter at the left of the lens is q_1 and the beam waist at the right side is q_2 , as is known in the figure 1. The ABCD matrix of lens is,

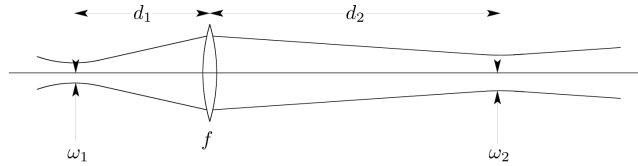


Figure 1: Mode matching

$$M = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} \quad (1)$$

Therefore we could get,

$$\begin{aligned} q_1 &= d_1 + i \frac{n\pi\omega_1^2}{\lambda} \\ q_2 &= \frac{Aq_1 + B}{Cq_1 + D} + d_2 \\ &= \frac{fq_1}{f - q_1} + d_2 \\ &= \frac{f(d_1 + iZ_{R1})(f - d_1 + iZ_{R1})}{(f - d_1)^2 + Z_{R1}^2} + d_2 \end{aligned} \quad (2)$$

q_2 is the q parameter at the right beam waist therefore we know that,

$$\begin{aligned} q_2 &= iZ_{R2} \\ &= \frac{n\pi\omega_2^2}{\lambda} \end{aligned} \quad (3)$$

which is a pure imaginary number. Compare equation 2 with equation 3 we can get that,

$$i \frac{f^2 Z_{R1}}{(f-d_1)^2 + Z_{R1}^2} + \frac{f[d_1(f-d_1)^2 - Z_{R1}^2 + d_2(f-d_1)^2]}{(f-d_1)^2 + Z_{R1}^2} + d_2 = iZ_{R2} \quad (4)$$

which indicate that,

$$\frac{f^2 Z_{R1}}{(f-d_1)^2 + Z_{R1}^2} = Z_{R2} \quad (5)$$

$$\frac{f[d_1(f-d_1)^2 - Z_{R1}^2 + d_2(f-d_1)^2]}{(f-d_1)^2 + Z_{R1}^2} + d_2 = 0$$

Solving above equation we could get,

$$d_1 = f \pm \frac{\omega_1}{\omega_2} \sqrt{f^2 - f_0^2} \quad (6)$$

$$d_2 = f \pm \frac{\omega_2}{\omega_1} \sqrt{f^2 - f_0^2}$$

2 Problem 2

The mode-matching techniques for standing wave cavities described in the text did not take account of the refractive power of the input mirror which acts like a weak negative lens. Assume that one has an approximately confocal cavity with mirror separation d . Approximately how far is the apparent waist position shifted by refraction from the input mirror? (Treat the input mirror as a thin plano-concave lens with refractive index n .) Assume $n = 1.5$ and determine the actual shift in terms of d .

Solution

Suppose the radius of the cavity are R_1 and R_2 , due to that it is a confocal cavity we know that,

$$R_1 = R_2 = d \quad (7)$$

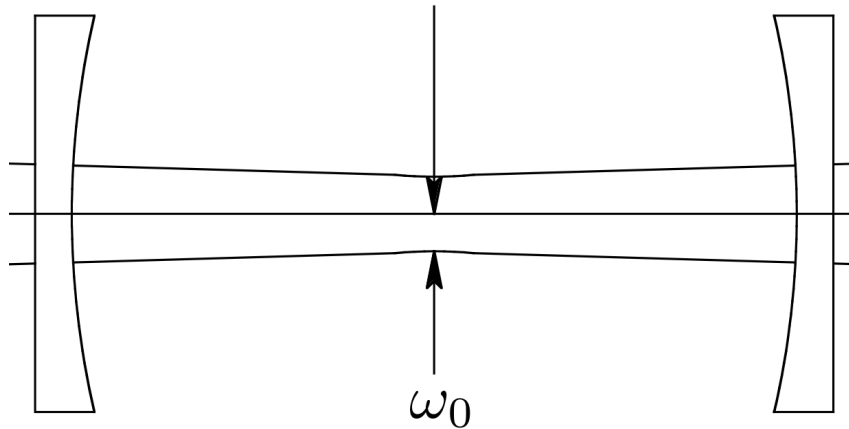
Using the equation of lens maker, the focal length of the plano-concave lens is f ,

$$f = \left(\frac{n^2}{n_1} - 1\right) \left(\frac{1}{R_1} + \frac{1}{R_2}\right) \quad (8)$$

Substitute $n_1 = 1$, $n_2 = 1.5$, $R_1 = d$, $R_2 = \infty$ into the above equation we can get,

$$f = 2d$$

Therefore the ABCD matrix of the plano-concave lens is M_1 ,



$$M_1 = \begin{pmatrix} 1 & 0 \\ \frac{1}{2d} & 1 \end{pmatrix}$$

Assume that the q parameter at the left mirror is $q_1 = -\frac{d}{2} + iZ_R$ where,

$$Z_R = \frac{n\pi\omega_0^2}{\lambda} = \frac{d}{2}$$

using ABCD matrix of the plano-concave lens we can get,

$$q_2 = \frac{Aq_1 + B}{Cq_1 + D} \quad (9)$$

Substitute M_1 and q_1 into equation 9 we can get,

$$q_2 = \frac{-6d^3 + 8dZ_R^2 + (4d^2Z_R + 12d^2Z_R)i}{9d^2 + 4Z_R^2} \quad (10)$$

Therefore the distance of beam waist from the left mirror is,

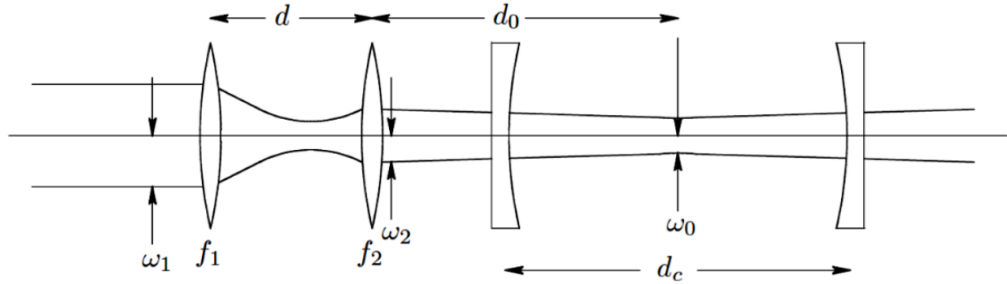
$$d_1 = \frac{-6d^3 + 8dZ_R^2}{9d^2 + 4Z_R^2} = -\frac{2}{5}d \quad (11)$$

Therefore the beam waist has a shift of $\frac{d}{10}$ to the left.

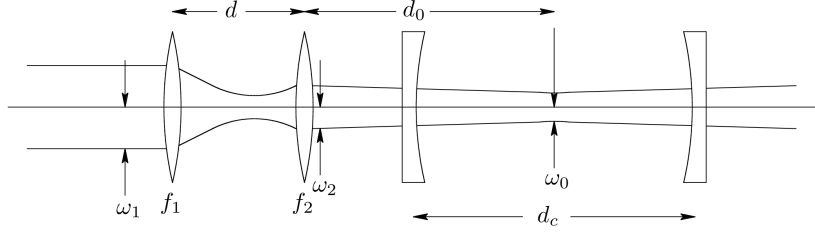
3 Problem 3

3. Can you explain the procedure #4 at page 32 of the main text book. How much do you need to separate the distance between two lens from f_1+f_2 ? Estimate the distance for the following two cases.

1) The mode matching to the following confocal cavity of $R = 10 \text{ cm}$ with the laser beam of $\lambda = 1 \mu\text{m}$. The waist of the input beam is $\omega_1 = 1 \text{ mm}$. Here you assume the beam waist is located at the position of lens f_1 , wherever it is. What are your choices of d_0, f_1, f_2 , and d for the proper mode matching? How much different the distance d from f_1+f_2 ?



2) The mode matching to the fiber, which requires the beam waist $\omega_0 = 20 \mu\text{m}$ at the surface of the fiber for $\lambda = 0.532 \mu\text{m}$. The waist of the input beam is $\omega_1 = 1 \text{ mm}$. Here you assume the beam waist is located at the position of lens f_1 , wherever it is. What are your choices of d_0, f_1, f_2 , and d for the proper mode matching? How much different the distance d from f_1+f_2 ?



Solution

3.1 PART I

The ABCD matrix of the lens is,

$$M_2 = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f_2} & 1 \end{pmatrix} \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{f_1} & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 - \frac{d}{f_1} & d \\ -\frac{1}{f_1} - \frac{1}{f_2} + \frac{d}{f_1 f_2} & 1 - \frac{d}{f_2} \end{pmatrix} \quad (12)$$

Assume that the q parameter at lens f_1 is $q_1 = iZ_1$, where $Z_1 = \frac{n\pi\omega_1^2}{\lambda}$, therefore we could get the q parameter at the lens f_2 as,

$$q_2 = \frac{Aq_1 + B}{Cq_1 + D}$$

$$= \frac{BD + ACZ_1^2}{B^2 + A^2Z_1^2} + i \frac{(BC - AD)Z_1}{B^2 + A^2Z_1^2} \quad (13)$$

q_2 is also a part of the beam of the cavity, therefore starting from the beam waist of the cavity we can write q_2 as,

$$q_2 = -d_0 + iZ_0 \quad (14)$$

where $Z_0 = \frac{n\pi\omega_0^2}{\lambda}$. Comparing equation 13 with equation 14 we could derive that,

$$Z_0 = \text{Im}(q_2) = \frac{(BC - AD)Z_1}{B^2 + A^2Z_1^2}$$

$$d_0 = -\text{Re}(q_2) = -\frac{BD + ACZ_1^2}{B^2 + A^2Z_1^2} \quad (15)$$

Substitute the element in M_2 into the above equation we can simplify equation as ,

$$Z_0[(f_1 f_2 - f_1 d)^2 + Z_1^2(f_1 + f_2 - d)^2] = f_1^2 f_2^2 Z_1 \quad (16)$$

Using mathematics to solve equation 17 we can get d as,

$$d = f_1 + f_2 + f_1^2 \frac{f_1 \pm \sqrt{-Z_1^2 + \frac{Z_1^3}{Z_0} \left(\frac{f_2^2}{f_1^2}\right) + \frac{Z_1}{Z_0} f_2^2}}{Z_1^2 + f_1^2} \quad (17)$$

Substitute $\frac{f_1}{f_2} = \frac{\omega_1}{\omega_2} = \frac{1}{\sqrt{1+(d_0/Z_0)^2}} \frac{\omega_1}{\omega_0}$ into equation 17 we can get,

$$d = (f_1 + f_2) + f_1^2 \frac{f_1 \pm \sqrt{\left(\frac{d_0}{Z_0}\right)^2 Z_1^2 + \frac{Z_1}{Z_0} f_2^2}}{Z_1^2 + f_1^2}$$

$$= (f_1 + f_2) + f_1^2 \frac{f_1 \pm \sqrt{d_0^2 Z_1^2 + f_2^2}}{Z_0(Z_1^2 + f_1^2)} \quad (18)$$

Therefore $\Delta d = f_1^2 \frac{f_1 \pm \sqrt{d_0^2 Z_1^2 + f_2^2}}{Z_0(Z_1^2 + f_1^2)}$ and we need to move slightly more than $f_1 + f_2$ to make the beam waist at the center of cavity.

3.2 Part II

However, we note that under the condition

$$\frac{f_1}{f_2} = \frac{\omega_1}{\omega_0} = \sqrt{\frac{Z_1}{Z_0}}$$

equation 17 can be simplified as,

$$q_2 = (f_1 + f_2) + \frac{f_1^2(f_1 \pm f_1)}{f_1^2 + Z_1^2} \quad (19)$$

which indicates that,

$$d = f_1 + f_2 \quad \text{or} \quad f_1 + f_2 + \frac{2f_1^3}{f_1^2 + Z_1^2} \quad (20)$$

therefore there exists a strict analytic solution $d = f_1 + f_2$ under such condition.

3.2.1 1)

Due to the radius of the confocal cavity is R therefore we could get $d = R = 10cm$, $Z_0 = \frac{d}{2} = 5cm$, $Z_1 = 3.14m$, $\omega_0 = \sqrt{\frac{\lambda d}{2n\pi}} \approx 0.178mm (n = 1)$. Choose $d_0 = R = 10cm$ and $\omega_2 = \sqrt{2}\omega_0 = 0.252mm$ and set

$$\frac{f_1}{f_2} = \frac{\omega_1}{\omega_0} \approx 4 \quad (21)$$

Hence we choose $f_1 = 40mm$, $f_2 = 10mm$ and $d = f_1 + f_2 = 50mm$ and $\Delta d = \pm 1mm$

3.2.2 2)

From the question we know that $\omega_0 = 20\mu m$, $\omega_1 = 1mm$ therefore we could get

$$\begin{aligned} Z_0 &= \frac{n\pi\omega_0^2}{\lambda} = 2.4mm \quad (n = 1) \\ Z_1 &= \frac{n\pi\omega_1^2}{\lambda} = 5.9m \\ \text{setting } d_0 &= 10 * Z_0 = 24mm \\ \omega_2 &= \omega_0 * \sqrt{1 + (d_0/z_0)^2} = 201\mu m \\ \frac{f_1}{f_2} &= \frac{\omega_1}{\omega_2} = 4.97 \end{aligned} \quad (22)$$

Set $d_0 = 10 * Z_0 = 24mm$ and $f_2 = 10mm$ we could get $f_1 \approx 50mm$, $d = f_1 + f_2 = 60mm$ and $\Delta d = -4.1mm \quad \text{or} \quad 4.3mm$