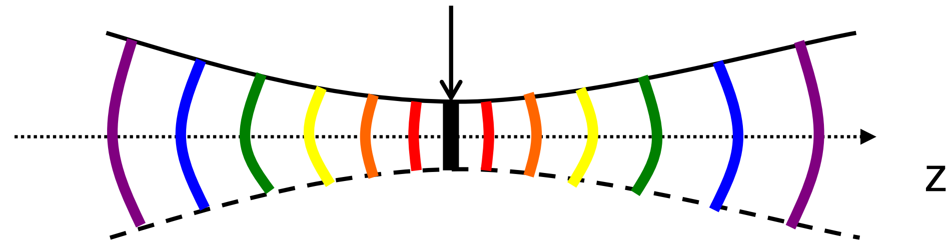
QUANTUM ELECTRONICS

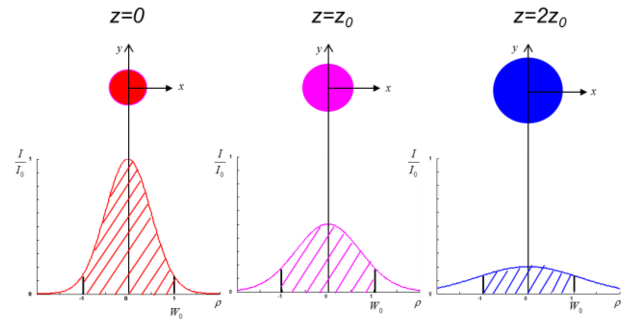
For atomic physics

Gaussian Beam

$$\boxed{\nabla_t^2\psi - 2ik\frac{\partial\psi}{\partial z} = 0,}$$



$$\psi(x,y,z) = \exp\left\{-i\left(P(z) + \frac{k}{2q(z)}r^2\right)\right\}$$



complex beam parameter q

$$\frac{1}{q(z)} = \frac{1}{R(z)} - i\frac{\lambda}{n\pi\omega(z)^2}$$

$$\omega(z) = \omega_0\left[1+\left(\frac{z}{z_R}\right)^2\right]^{1/2} \qquad z_R = \frac{n\pi\omega_0^2}{\lambda}$$

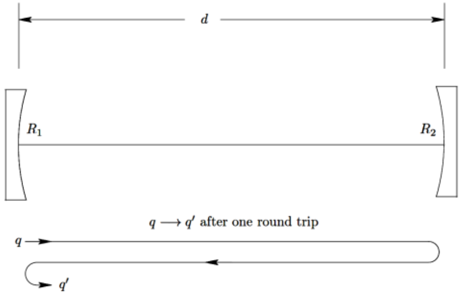
$$R(z) = z + \frac{z_R^2}{z}.$$

ABCD rule

$$q_1 = \frac{Aq_0 + B}{Cq_0 + D} \Longleftrightarrow \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

SUMMARY – GEOMETRIC PROPERTIES OF OPTICAL CAVITY

Stability condition



$$q = \frac{Aq + B}{Cq + D}$$

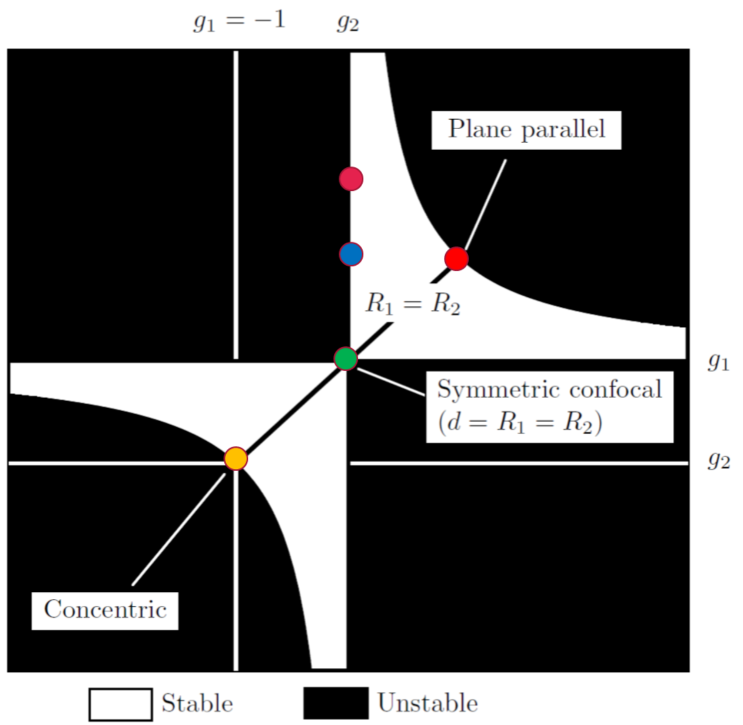
$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac{2}{R_1} & 1 \end{pmatrix} \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{2}{R_2} & 1 \end{pmatrix} \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}$$

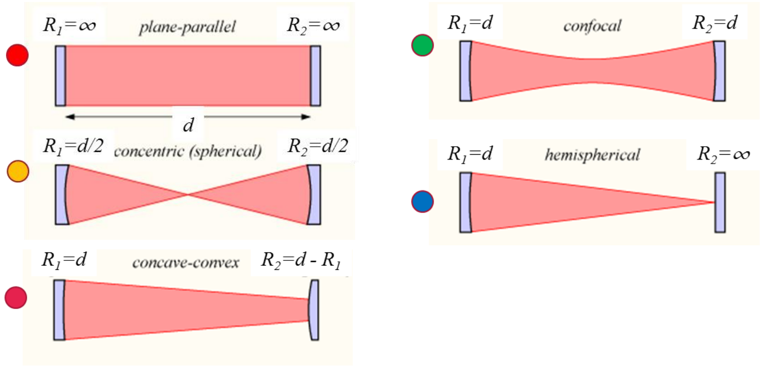
$$g_1 \equiv 1 - \frac{d}{R_1} \qquad g_2 \equiv 1 - \frac{d}{R_2}$$

$$\frac{1}{q} = \frac{g_1 - 1}{d} \pm \frac{i}{g_2 d}\sqrt{g_1 g_2 (1 - g_1 g_2)}$$

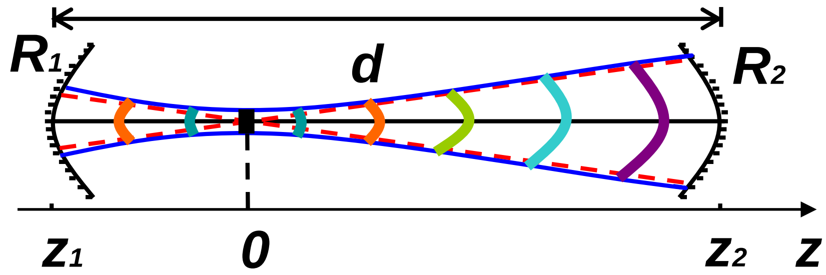
$$\frac{1}{q} = \frac{1}{R} - i\frac{\lambda}{n\pi\omega^2}$$

Stability criterion: $\quad 0 \le g_1 g_2 \le 1$





GEOMETRIC PROPERTIES OF OPTICAL CAVITY



$$\frac{1}{q} = \frac{g_1 - 1}{d} \pm \frac{i}{g_2 d}\sqrt{g_1 g_2 (1 - g_1 g_2)} \qquad \frac{1}{q} = \frac{1}{R} - i\frac{\lambda}{n\pi\omega^2}$$

$$q(z) = q_0 + z = i\frac{n\pi\omega_0^2}{\lambda} + z \quad = i\, z_R + z, \qquad \omega^2 = \left(\frac{\lambda d}{n\pi}\right)\sqrt{\frac{g_2}{g_1(1 - g_1 g_2)}}\ , R = \frac{d}{g_1 - 1} = -R_1,$$

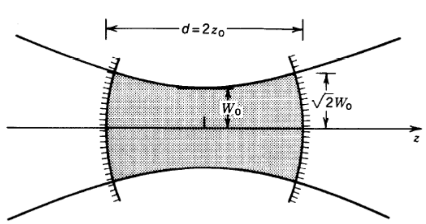
$$z_R = Im(q) = Im\left(\frac{1}{q^{-1}}\right) = Im\left(\frac{(q^{-1})^*}{q^{-1}(q^{-1})^*}\right) = \frac{-Im(q^{-1})}{|q^{-1}|^2} \qquad \omega_0^2 = \left(\frac{\lambda d}{n\pi}\right) z_R = \left(\frac{\lambda d}{n\pi}\right)\frac{\sqrt{g_1 g_2 (1 - g_1 g_2)}}{g_1 + g_2 - 2g_1 g_2}$$

Distance from waist $= Re\{q(z)\} = Re\{q^{-1}\}/\left|q^{-1}\right|^2$

$$z_1 = d\frac{g_2(g_1 - 1)}{g_1 + g_2 - 2g_1 g_2} \qquad z_2 = d\frac{g_1(1 - g_2)}{g_1 + g_2 - 2g_1 g_2} \qquad \because z_2 - z_1 = d$$

Confocal cavity

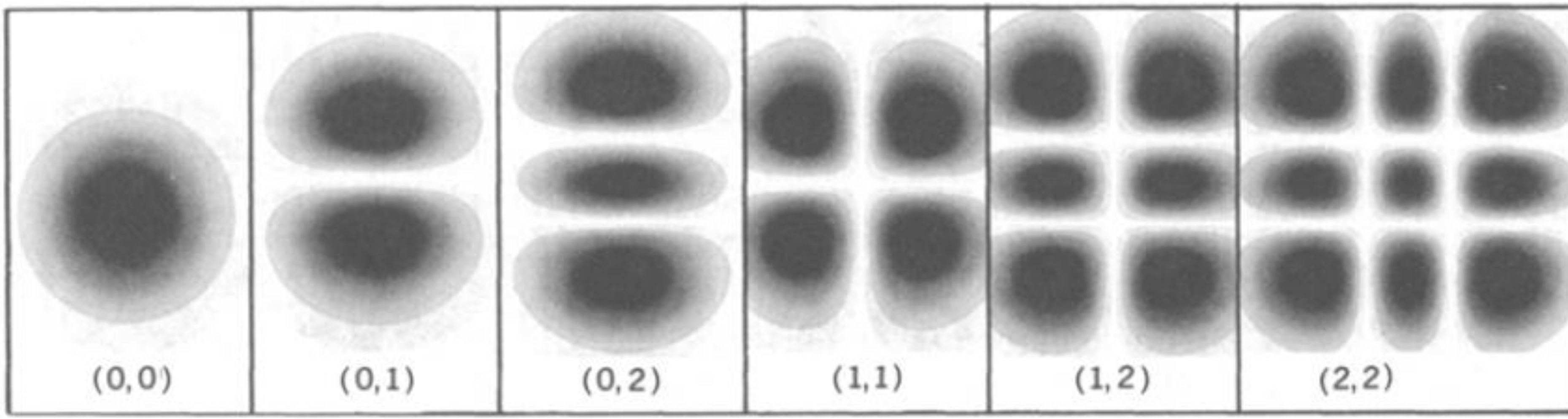
$$d = R$$



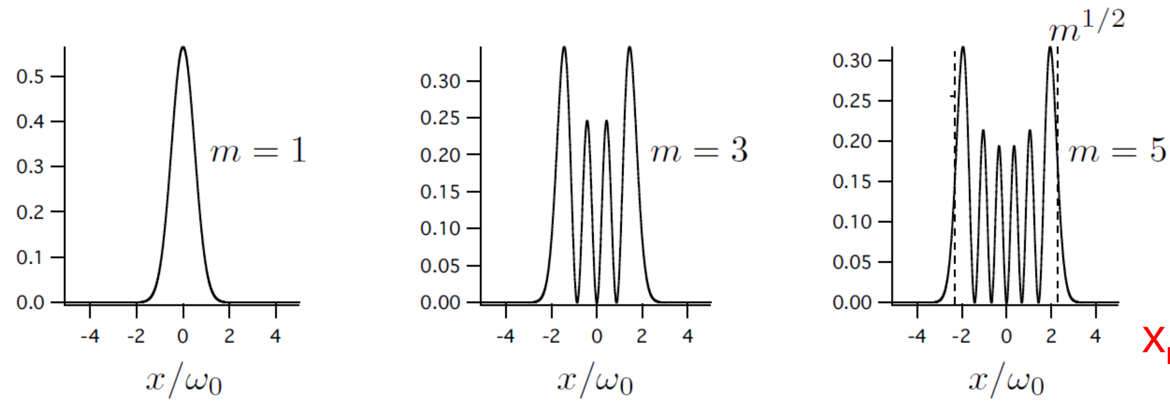
$$\omega_0 = \left(\frac{\lambda d}{2n\pi}\right)^{1/2}$$

$$\omega_{mirror} = \left(\frac{\lambda d}{n\pi}\right)^{1/2} = \sqrt{2}\omega_0$$

2.5 HIGHER-ORDER MODES



$$u(x,y,z)_{nm} =$$
$$\frac{\omega_0}{\omega(z)} H_n(\sqrt{2}\frac{x}{\omega}) H_m(\sqrt{2}\frac{y}{\omega})$$
$$\times \exp\left\{-i(kz - \Phi(m,n;z)) - i\frac{k}{2q}(x^2+y^2)\right\}$$



$x_m \sim \sqrt{m}\,\omega_0$

$$u(x,y,z)_{nm} =$$

$$\frac{\omega_0}{\omega(z)} H_n(\sqrt{2}\frac{x}{\omega}) H_m(\sqrt{2}\frac{y}{\omega})$$

$$\times \exp\left\{-i(kz - \Phi(m,n;z)) - i\frac{k}{2q}(x^2+y^2)\right\}$$

same q independent of m,n

Same ABCD rule

$$\frac{\omega_0}{\omega(z)} = \frac{1}{\sqrt{1+(z/z_R)^2}}$$

$$\Phi(n,m;z) = (n+m+1)\tan^{-1}(z/z_R)$$

HERMIT GAUSSIAN MODES

- They are all characterized by the *same* complex beam parameter, q, defined by

$$\frac{1}{q(z)} = \frac{1}{R(z)} - i\frac{\lambda}{n\pi\omega^2(z)}. \tag{2.28}$$

- They all satisfy the same ABCD rule as (lowest order) Gaussian beams. The *mode numbers*, m and n are preserved under all of the transformations discussed in this book.
- The mode *shape* is independent of z and scales with $\omega(z)$.
- The mode of index m has a half-width, x_m, (in one transverse coordinate), where

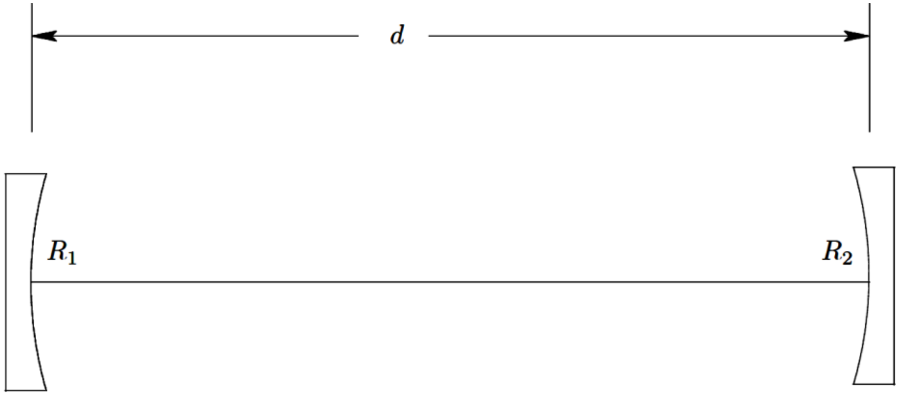
$$x_m \approx \sqrt{m} \times \omega. \tag{2.29}$$

  Thus, ω is no longer the beam size in higher-order modes. A focused spot has its size *degraded* by $\sqrt{m}$.
- The maximum mode number (m_{max}) at a waist which will "fit" into an aperture of radius a is:

$$m_{max} \approx (a/\omega_0)^2. \tag{2.30}$$

  This *spatial filtering* behavior of small apertures allows one to filter out modes whose mode number (in either coordinate) is greater than m_{max}.

2.6 RESONANT FREQUENCIES



$$\exp\left\{-i(kz-\Phi(m,n;z))-i\frac{k}{2q}(x^2+y^2)\right\}$$

$$\Phi(n,m;z)=(n+m+1)\tan^{-1}(z/z_R)$$

$$\delta = 2kd - 2(n+m+1)(\tan^{-1}(z_2/z_R)-\tan^{-1}(z_1/z_R))$$

$$\delta = q(2\pi) \Longrightarrow \frac{\omega d}{c} - (n+m+1)\cos^{-1}\pm\sqrt{g_1 g_2} = q\pi$$

$$\nu_{nmq} = \left(q + (n+m+1)\frac{\cos^{-1}\pm\sqrt{g_1 g_2}}{\pi}\right)\frac{c}{2d}$$

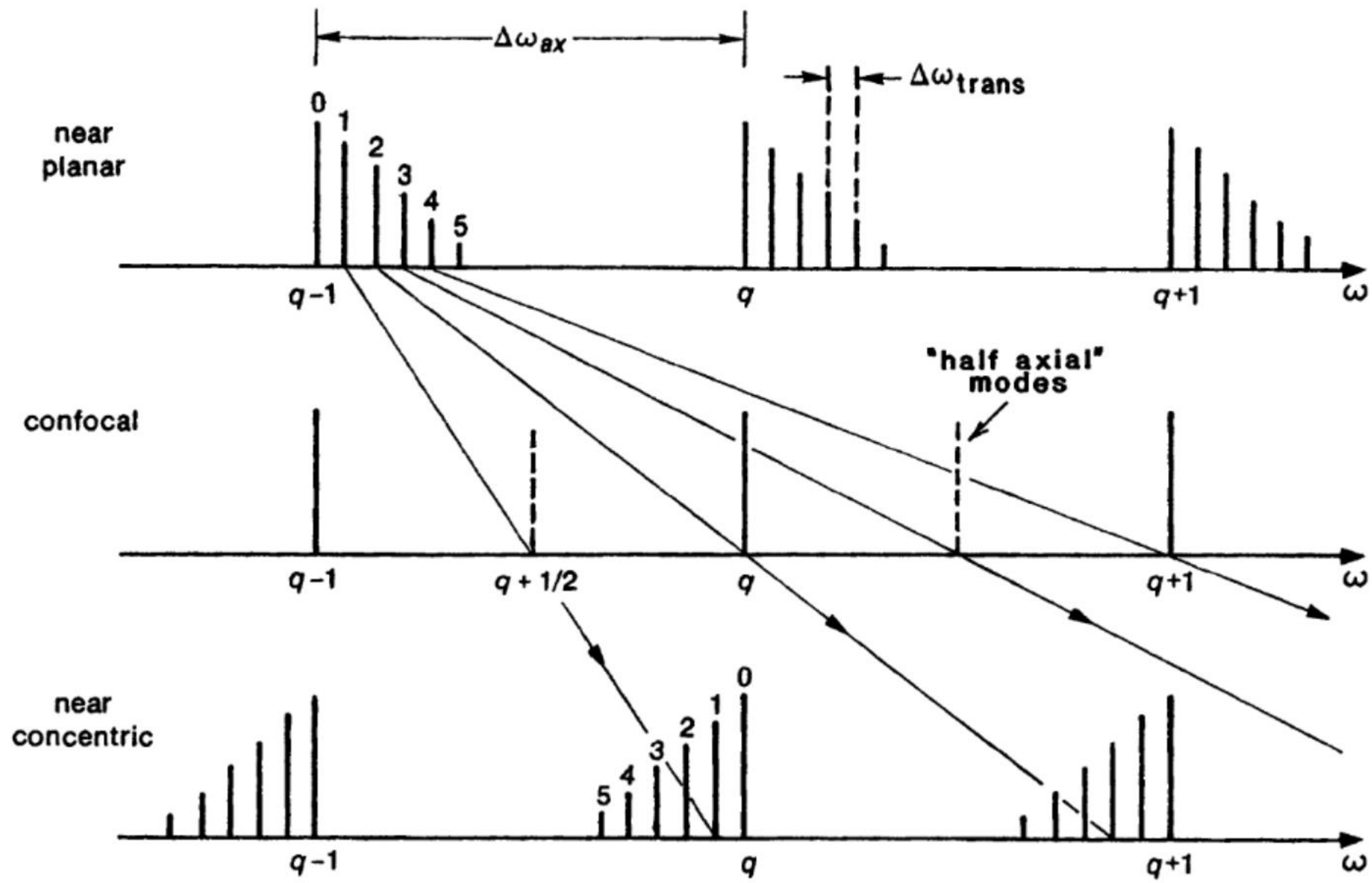
Free spectral range

$$\frac{\cos^{-1}\pm\sqrt{g_1 g_2}}{\pi} \approx \begin{cases} 0 & : \quad g_1, g_2 \to 1 \quad \text{(near-planar)} \\ 1/2 & : \quad g_1, g_2 \to 0 \quad \text{(near-confocal)} \\ 1 & : \quad g_1, g_2 \to -1 \quad \text{(near-spherical)} \end{cases}$$

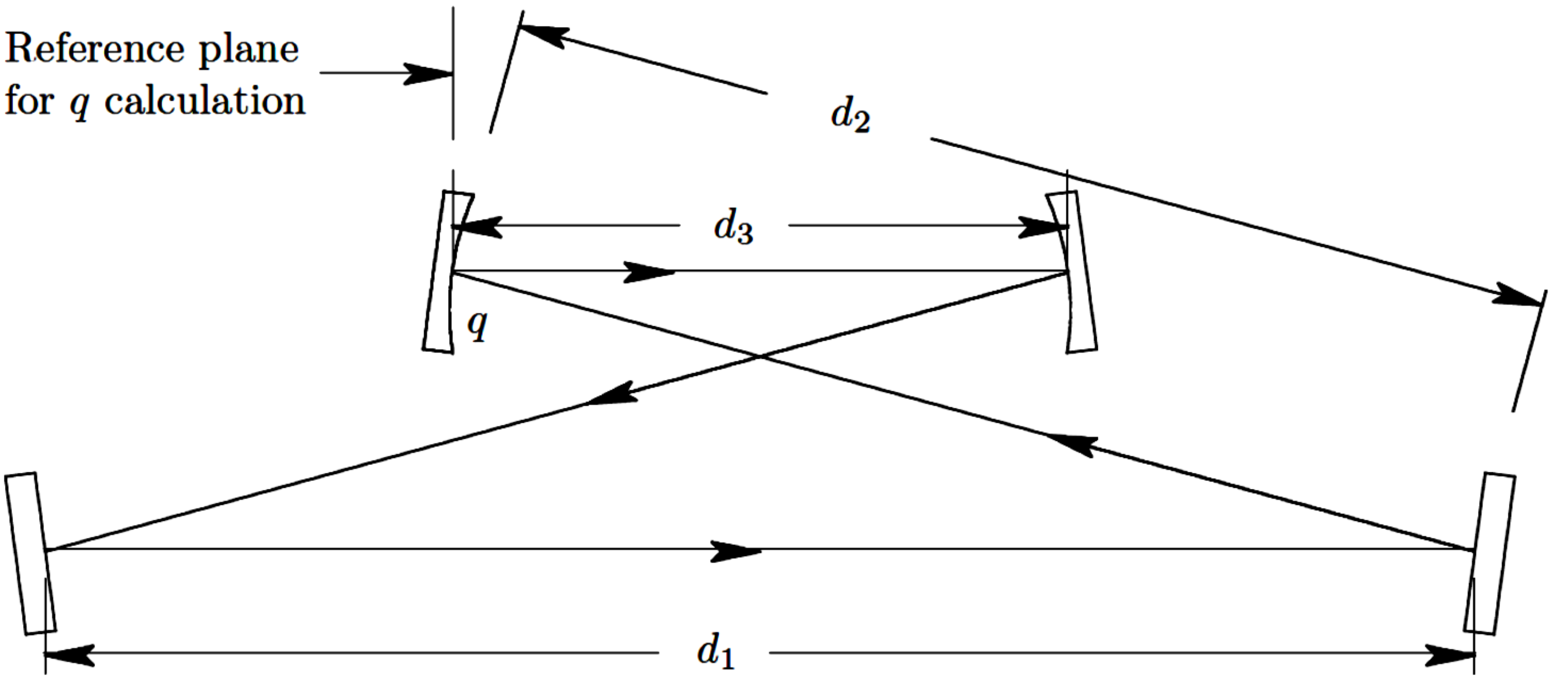
$$\nu_{nmq} = \left(q + (n+m+1)\frac{\cos^{-1} \pm\sqrt{g_1 g_2}}{\pi}\right)\frac{c}{2d}$$

Free spectral range

$$\frac{\cos^{-1} \pm\sqrt{g_1 g_2}}{\pi} \approx \begin{cases} 0 & : \quad g_1, g_2 \to 1 \quad \text{(near-planar)} \\ 1/2 & : \quad g_1, g_2 \to 0 \quad \text{(near-confocal)} \\ 1 & : \quad g_1, g_2 \to -1 \quad \text{(near-spherical)} \end{cases}$$



2.7 THE TRAVELING WAVE(RING) CAVITY



$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{-2}{R} & 1 \end{pmatrix} \begin{pmatrix} 1 & d_1 + 2d_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{-2}{R} & 1 \end{pmatrix} \begin{pmatrix} 1 & d_3 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{-2}{R} & 1 \end{pmatrix} \begin{pmatrix} 1 & d_1 + 2d_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{-2}{R} & 1 \end{pmatrix} \begin{pmatrix} 1 & d_3 \\ 0 & 1 \end{pmatrix}$$

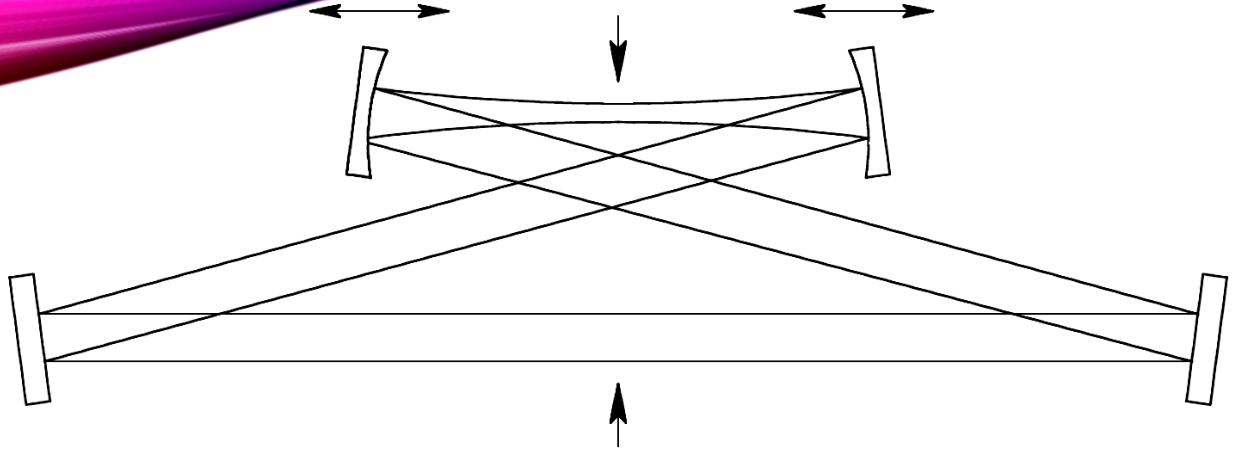
$$g_1 = 1 - \frac{d_1 + 2d_2}{R} \qquad g_2 = 1 - \frac{d_3}{R},$$

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{-2}{R} & 1 \end{pmatrix} \begin{pmatrix} 1 & R(1-g_1) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{-2}{R} & 1 \end{pmatrix} \begin{pmatrix} 1 & R(1-g_2) \\ 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 2g_1 - 1 & R(-2g_1g_2 + g_1 + g_2) \\ \frac{-4g_1}{R} & 4g_1g_2 - 2g_1 - 1 \end{pmatrix}.$$

$$|A + D| \leq 2 \Longrightarrow |4g_1g_1 - 2| \leq 2 \Longrightarrow 0 \leq g_1g_2 \leq 1$$

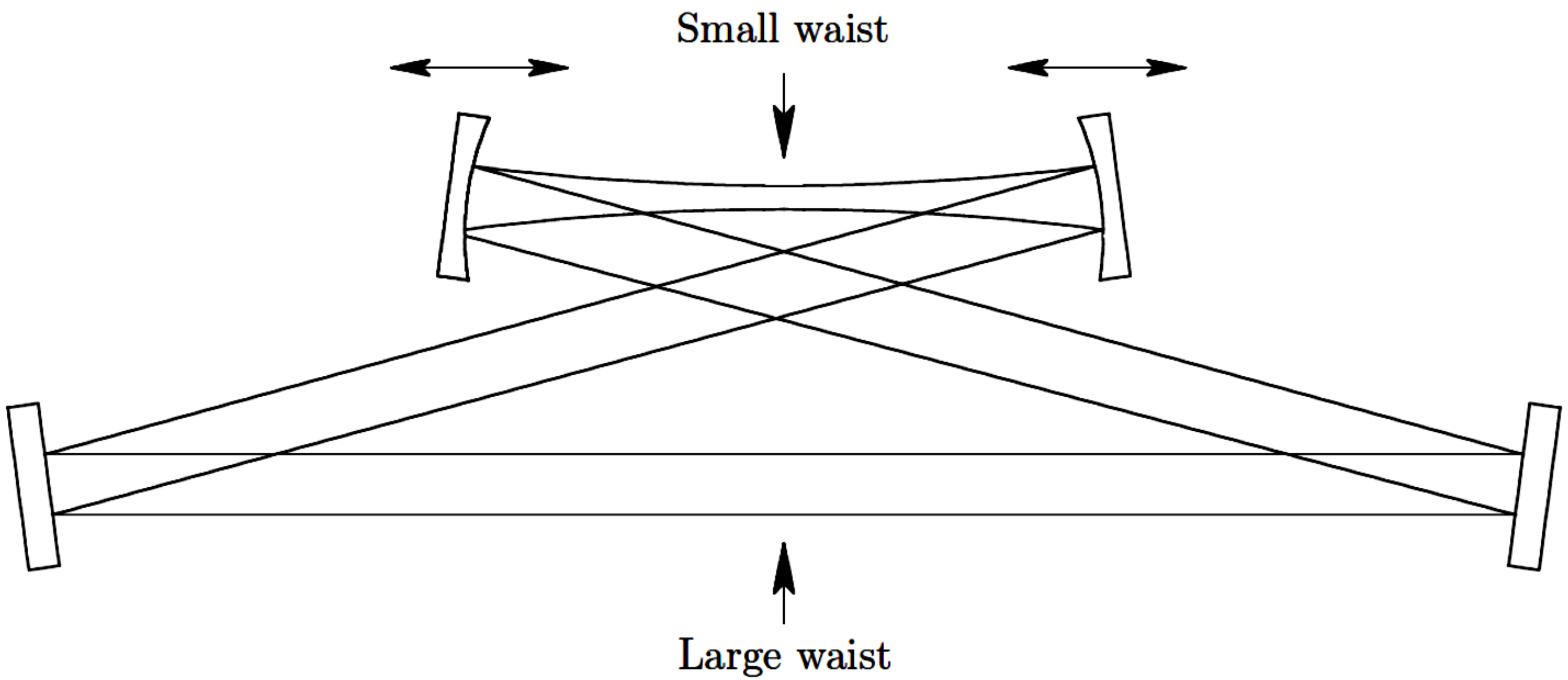
Stability: $\quad R \leq d_3 \leq \dfrac{R(d_1 + 2d_2)}{d_1 + 2d_2 - R}$

Range: $\quad \Delta d_3 \approx \dfrac{R^2}{d_1 + 2d_2} \qquad$ when $\; d_1 + 2d_2 \gg R.$

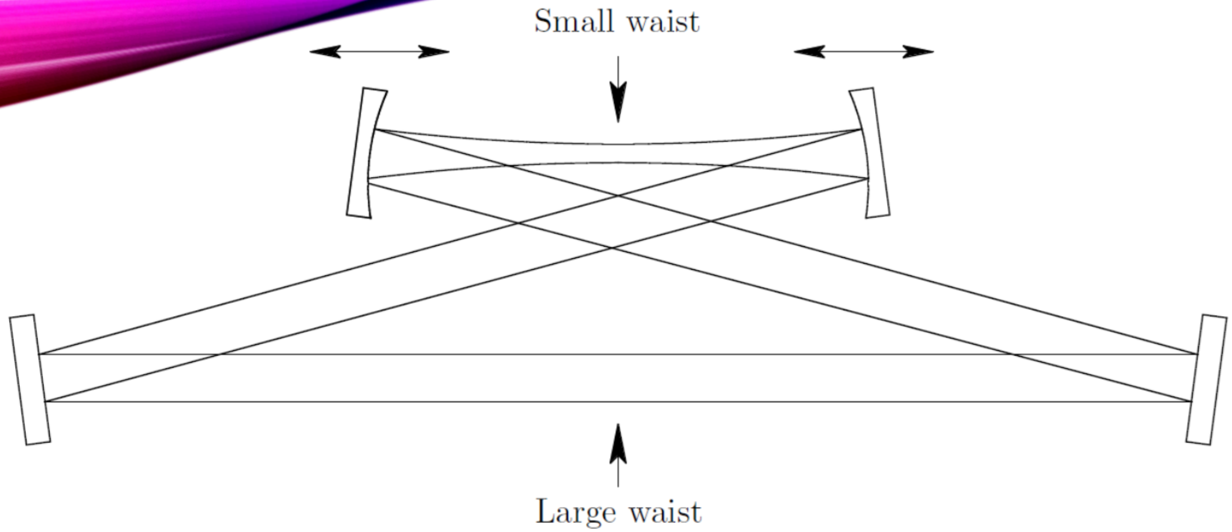


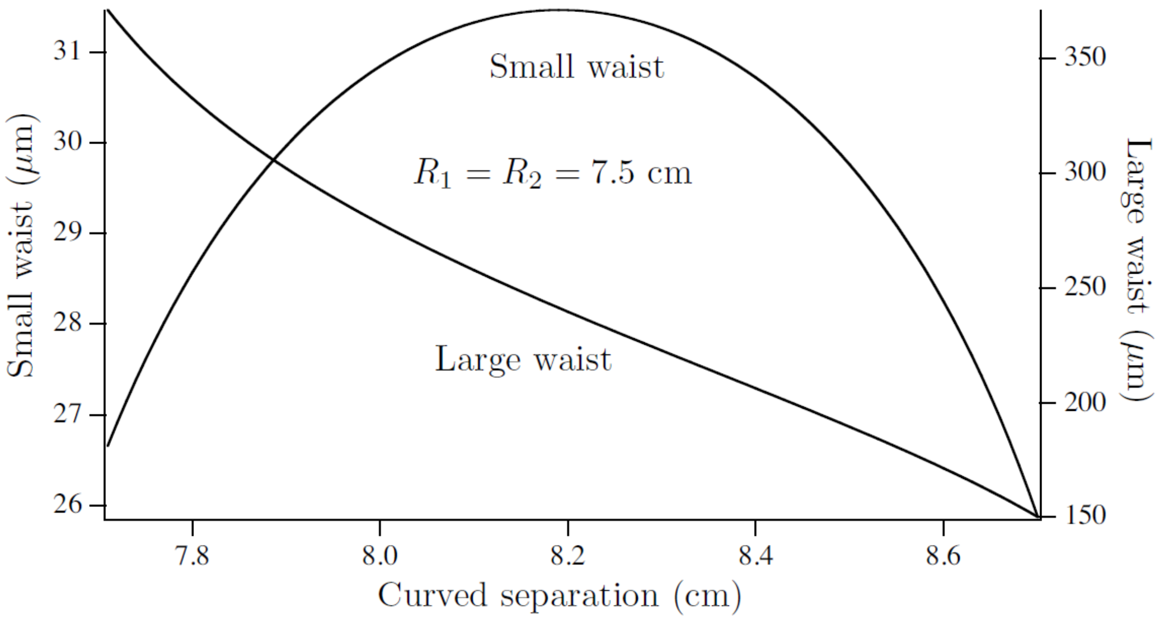
Property	Standing wave	Symmetric ring
g_1	$1-\frac{d}{R_1}$	$1-\frac{d_1+2d_2}{R}$
g_2	$1-\frac{d}{R_2}$	$1-\frac{d_3}{R}$
Path length	$2d$	$L=d_1+2d_2+d_3$
Waist(s)	$\omega^2=\left(\frac{\lambda d}{n\pi}\right)\frac{\sqrt{g_1g_2(1-g_1g_2)}}{g_1+g_2-2g_1g_2}$	$\boxed{\omega_0^2=\left(\frac{\lambda R}{2n\pi}\right)\frac{\sqrt{g_1g_2(1-g_1g_2)}}{g_1} \quad {\omega_0'}^2=\left(\frac{\lambda R}{2n\pi}\right)\frac{\sqrt{g_1g_2(1-g_1g_2)}}{g_2}}$
ν_{nmq}	$\left(q+(n+m+1)\frac{\cos^{-1}\pm\sqrt{g_1g_2}}{\pi}\right)\frac{c}{2d}$	$\left(q+(n+m+1)\frac{\cos^{-1}\pm\sqrt{g_1g_2}}{\pi}\right)\frac{c}{L}$
FSR	$\frac{c}{2d}$	$\frac{c}{L}$

Homework : Drive the marked equations

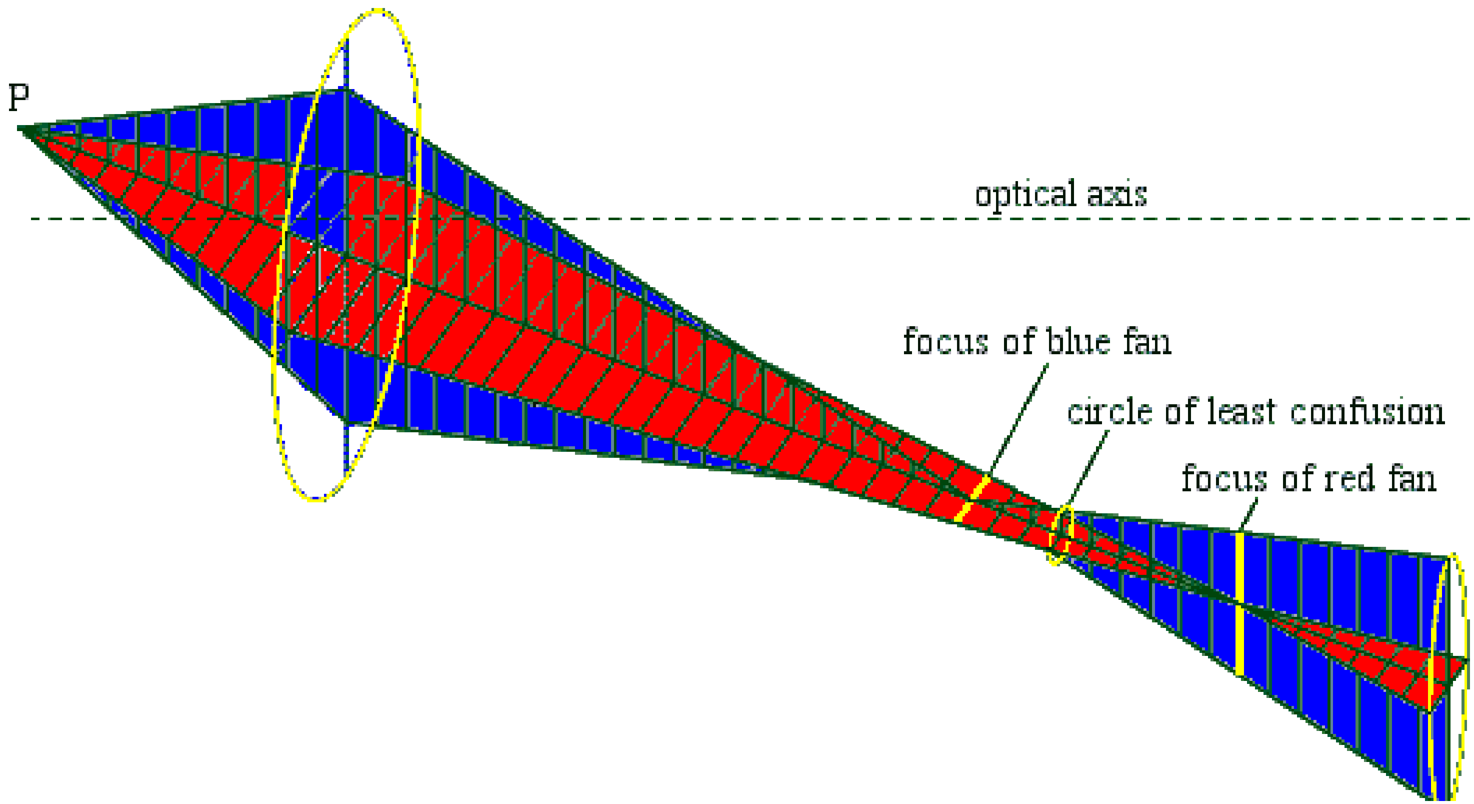


Homework: Find and draw the beam radius and Radius of curvature everywhere inside the cavities with R = 6 cm, d_3 = 7 cm, d_1+2d_2 = 18 cm, λ = 0.74 um.


Small waist
Large waist

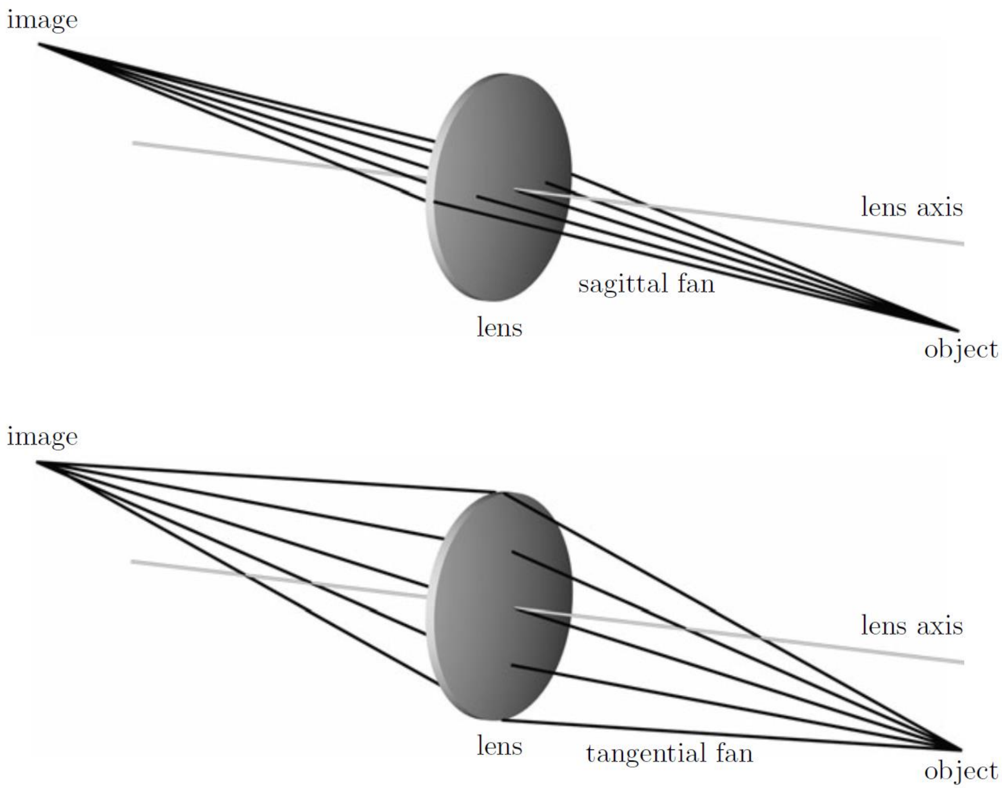

Small waist
$R_1 = R_2 = 7.5$ cm
Large waist
Small waist (μm)
Large waist (μm)
Curved separation (cm)
31
30
29
28
27
26
350
300
250
200
150
7.8
8.0
8.2
8.4
8.6

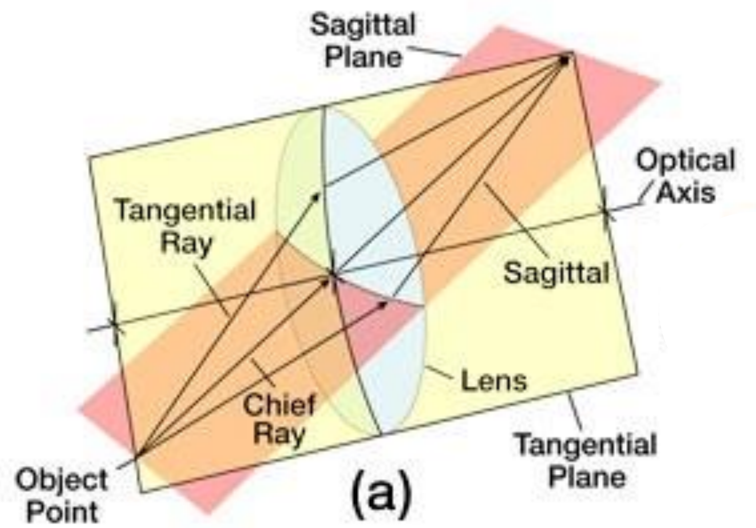
2.8 ASTIGMATISM IN A RING CAVITY



$$M_T = \begin{pmatrix} 1 & 0 \\ \frac{-2}{R\cos\theta} & 1 \end{pmatrix}, \qquad M_S = \begin{pmatrix} 1 & 0 \\ \frac{-2\cos\theta}{R} & 1 \end{pmatrix}$$

2.8 ASTIGMATISM IN A RING CAVITY





$$M_T = \begin{pmatrix} 1 & 0 \\ \frac{-2}{R\cos\theta} & 1 \end{pmatrix}, \qquad M_S = \begin{pmatrix} 1 & 0 \\ \frac{-2\cos\theta}{R} & 1 \end{pmatrix}$$

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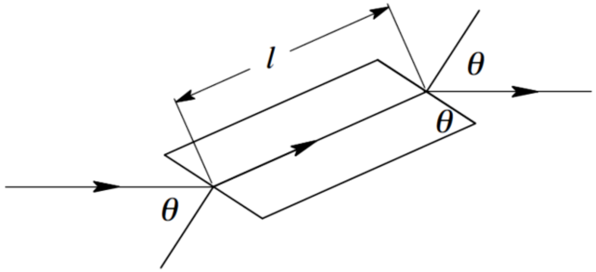
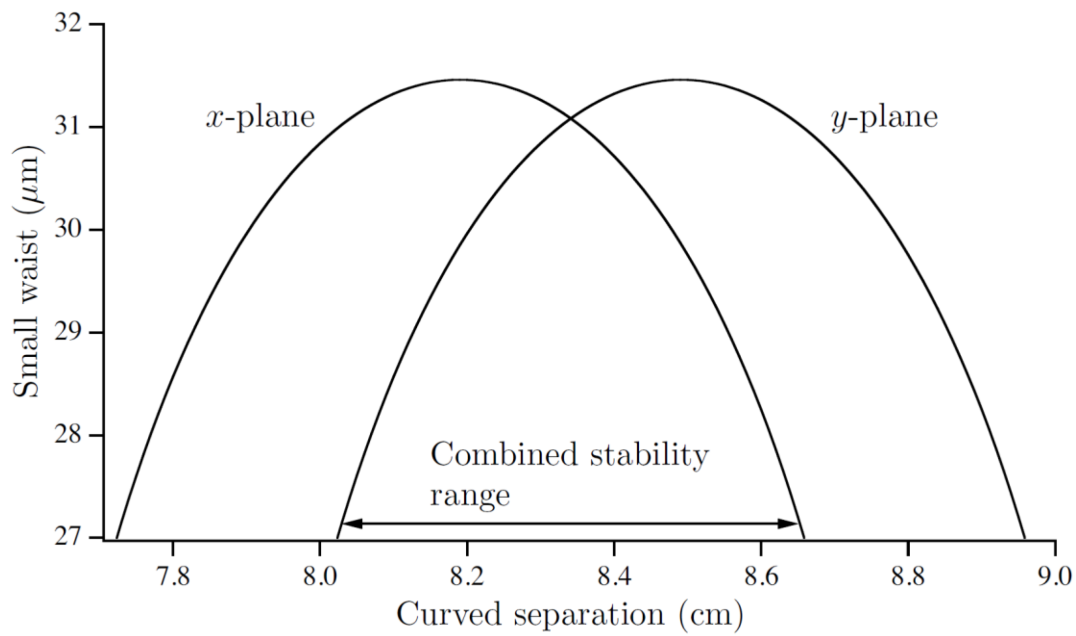
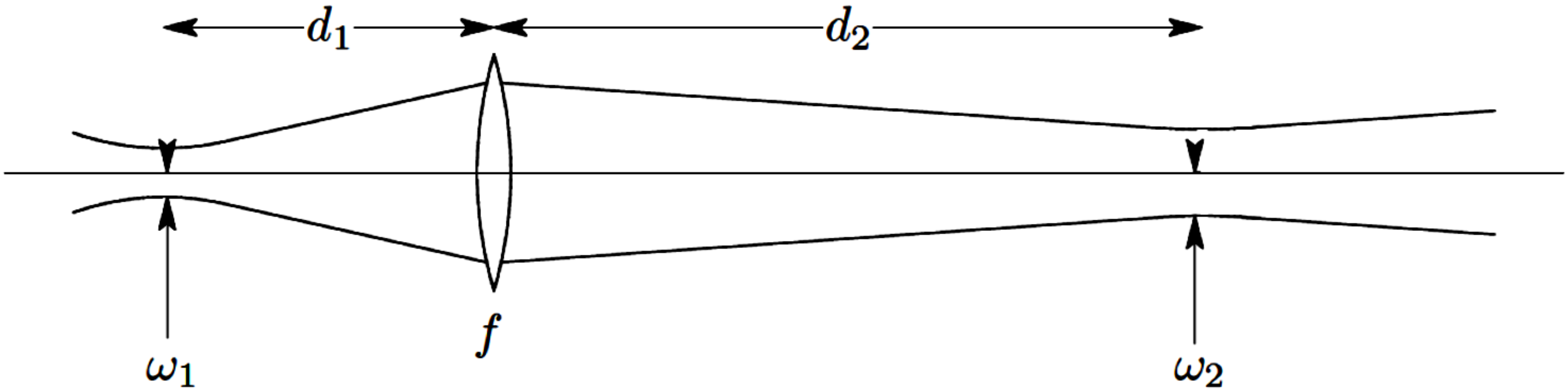


$$M_T = \begin{pmatrix} 1 & l/n^3 \\ 0 & 1 \end{pmatrix}$$

$$M_S = \begin{pmatrix} 1 & l/n \\ 0 & 1 \end{pmatrix}.$$



2.9 MODE MATCHING



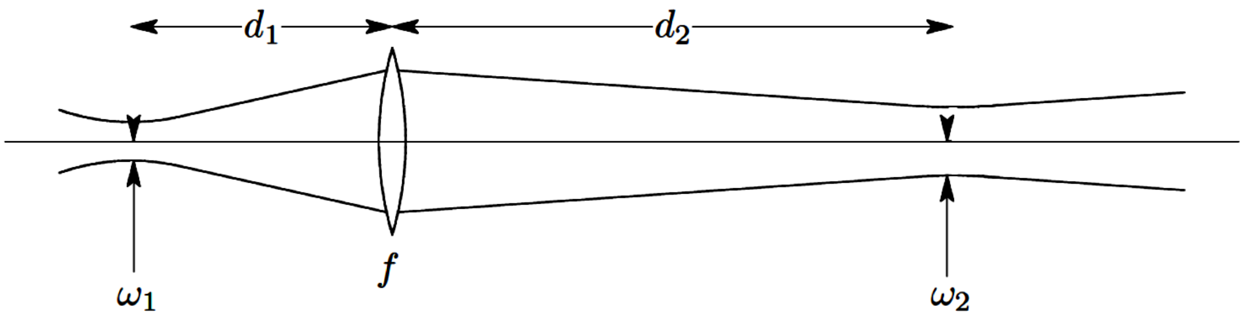
$$f_0 = n\pi\omega_1\omega_2/\lambda = \sqrt{z_{R1}z_{R2}}$$

$$d_1 = f \pm \frac{\omega_1}{\omega_2}\sqrt{f^2 - f_0^2}$$

$$d_2 = f \pm \frac{\omega_2}{\omega_1}\sqrt{f^2 - f_0^2}$$

Homework: drive the equation for d1 and d2

2.9 MODE MATCHING



$$\frac{1}{q_1} = -i\frac{\lambda}{n\pi\omega_1^2} = \frac{-i}{z_{R1}}$$

$$\frac{1}{q_2} = -i\frac{\lambda}{n\pi\omega_2^2} = \frac{-i}{z_{R2}}$$

$$q_1 = iz_{R1}$$

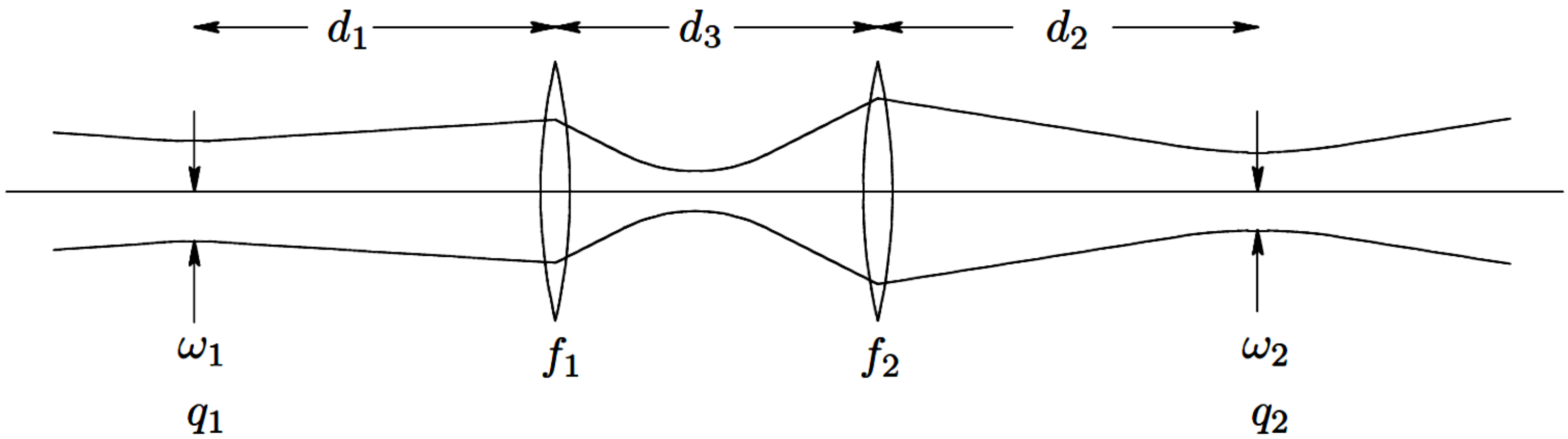
$$q_2 = iz_{R2}$$

$$q_2 = \frac{Aq_1 + B}{Cq_1 + D} = iz_{R2} = \frac{Aiz_{R1} + B}{Ciz_{R1} + D}$$

$$B + z_{R1}z_{R2}C = 0, z_{R1}A - z_{R2}D = 0$$

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & d_2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix}\begin{pmatrix} 1 & d_1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 - d_2/f & d_1 + d_2 - d_1 d_2/f \\ -1/f & 1 - d_1/f \end{pmatrix}$$

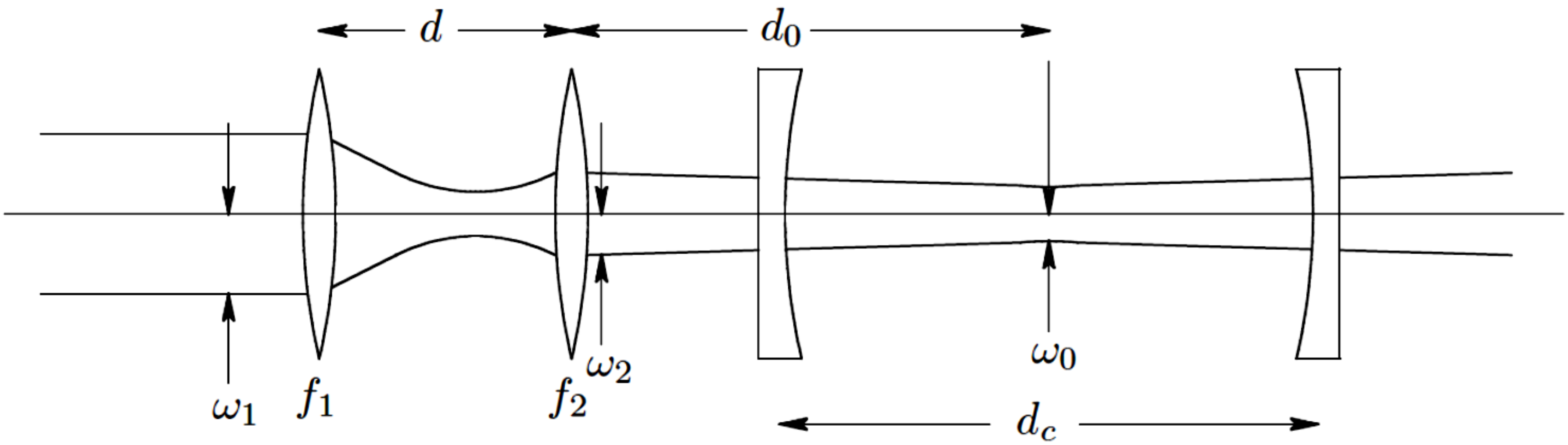
Then solve the equations.



$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{-1}{f_2} & 1 \end{pmatrix} \begin{pmatrix} 1 & d_3 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{-1}{f_1} & 1 \end{pmatrix} \begin{pmatrix} 1 & d_1 \\ 0 & 1 \end{pmatrix}$$

$$q_2 = \frac{Aq_1 + B}{Cq_1 + D} + d_2$$

$$R_2 = 1/Re\{1/q_2\} \quad \text{and} \quad \omega_2 = \sqrt{\frac{-\lambda}{n\pi Im\{1/q_2\}}}$$



Procedure	**Example**
1. Determine ω_0 and z_R from the cavity equations.	(Confocal) $d_c = 10$ cm, $\lambda = 1\ \mu$m, $\omega_0 = 126\ \mu$m, $z_R = 5$ cm
2. Given d_0, the distance from waist to lens 2, determine ω_2 using $\omega_2 = \omega_0\sqrt{1 + (d_0/z_R)^2}$.	$d_0 = 10$ cm, $\omega_2 = 282\ \mu$m
3. Given input spot size (ω_1), find two lenses whose focal lengths are in ratio ω_1/ω_2.	$\omega_1 = .05$ cm, ratio ≈ 1.8, $f_1 = 1.8$ cm, $f_2 = 1$ cm
4. Separate lenses by slightly more than $f_1 + f_2$ so waist is at center of cavity.	$f_1 + f_2 = 2.8$ cm

Homework: Problem 2.7