Assignment #1

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1 Solution 1

Take $\mu(x, y, z) = \psi(x, y, z)e^{-ikz}$ into the wave equation $\nabla^2\mu + k^2 u = 0$,

$$\nabla_t^2(\psi e^{-ikz}) + \frac{\partial^2}{\partial z^2}(\psi e^{-ikz}) + k^2\psi e^{-ikz} = 0 \tag{1}$$

where

$$\nabla_t^2 = (\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})$$

Expand $\frac{\partial^2}{\partial z^2}(\psi e^{-ikz})$ we can get,

$$\frac{\partial^2}{\partial z^2}(\psi e^{-ikz}) = (\frac{\partial^2\psi}{\partial z^2} - 2ik\frac{\partial\psi}{\partial z} - k^2\psi)e^{-ikz} \tag{2}$$

Take equation 2 into equation 1,

$$\nabla_t^2\psi + \frac{\partial^2\psi}{\partial z^2} - 2ik\frac{\partial\psi}{\partial z} = 0 \tag{3}$$

In order for paraxial condition to be satisfied, which means $\psi(x, y, z)$ must vary much more slowly along the z-axis than it does along the x- and y-axes.Thus, when substituting equation2 into the wave equation, $\partial^2\psi/\ \partial z^2$ can be ignored compared to the other second derivatives.So we can rewrite equation 1 as

$$\nabla_t^2\psi - 2ik\frac{\partial\psi}{\partial z} = 0 \tag{4}$$

Take the trial function as,

$$\psi(x, y, z) = exp\{-i(P(z) + \frac{k}{2q(z)}r^2)\}, r^2 = x^2 + y^2 \tag{5}$$

and in the cylindrical coordinate $\nabla_t^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}$,substitute equation 5 into equation 4 and organized by the order of r.

$$\frac{k^2}{q(z)^2}(\frac{dq(z)}{dz} - 1)r^2 - 2k(\frac{dP(z)}{dz} + \frac{i}{q(z)}) = 0 \tag{6}$$

Due to that equation 6 is always equal to zero no matter how r varies, which indicates the parameter equals to zero,

$$\frac{dq(z)}{dz} - 1 = 0$$

$$\frac{dP(z)}{dz} + \frac{i}{q(z)} = 0$$

Finally, we can get

$$\begin{aligned}\frac{dq(z)}{dz} &= 1\\ \frac{dP(z)}{dz} &= -\frac{i}{q(z)}\end{aligned} \tag{7}$$

2 Solution 2

By solving equation 7 we could get, $q(z) = q(z_1) + (z_2 - z_1)$.Write the complex beam parameter in two real parameters, R and ω,

$$\frac{1}{q} = \frac{1}{R} - i\frac{\lambda}{n\pi\omega^2}. \tag{8}$$

Substitute equation 8 into $\mu(x, y, z)$,

$$\mu(x, y, z) = exp\{-i(P(z) + kz + k\frac{r^2}{2R}) - \frac{r^2}{\omega^2}\}, \tag{9}$$

where $k = 2n\pi/\lambda$ is used. Now consider about the surface of constant phase which is r-dependence, which means the phase parameter in equation 9 is a constant,

$$z + \frac{r^2}{2R} = constant \tag{10}$$

and $P(z)$ has been ignore for that it varies much more slowly than other two terms. R can be interpreted as the radius of wavefront by the equation of calculating the curvature under the paraxial approximation $r \ll R$.

$$\rho = \frac{ds}{d\theta} = \frac{(1 + f^{'}(x))^{3/2}}{f^{''}(x)}. \tag{11}$$

Substitute

$$z = f(r) = constant - \frac{r^2}{2R},$$

into equation 11,

$$\rho = R(1 + \frac{r}{R})^{3/2} \tag{12}$$

So the curvature of wave front $\rho = R$ under the paraxial condition $r \ll R$. Now consider the minimum size of the spot when the wave front becomes a plan ($R = \infty$), the corresponding value of q is:

$$\text{At waist:} \quad q \equiv q_0 = i\frac{n\pi\omega^2}{\lambda} \tag{13}$$

Setting z as the distance from the waist,ω_0 as the radius of waist, the values of q is,

$$q(z) = q_0 + z = i\frac{n\pi\omega^2}{\lambda} + z, \tag{14}$$

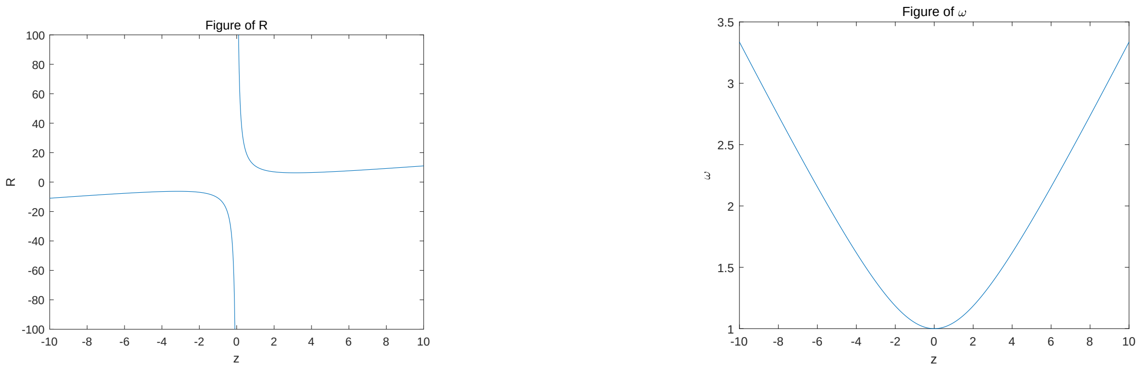
From equation 14 we can derive that,

$$\begin{aligned}&\text{Distanace to waist:} \quad d = -Re\{q(z)\}\\ &\text{Rdius of waist:} \quad \omega_0 = \sqrt{\frac{\lambda}{n\pi}Im\{q(z)\}}\end{aligned} \tag{15}$$

By solving equation 8 and 14,we can obtain ω and R as the function of propagation distance z from waist:

$$\begin{aligned}\omega(z) &= \omega_0[1 + (\frac{\lambda z}{n\pi\omega_0^2})^2]^{1/2}\\ R(z) &= z[1 + (\frac{n\pi\omega_0^2}{\lambda z})^2].\end{aligned} \tag{16}$$

The figure of $\omega(z)$ and $R(z)$ is drew as in figure 1,where we have assumed that $\omega_0 = 1, \lambda = 1, n = 1$ for simplification.



(a) Figure of $R(z)$

(b) Figure of $\omega(z)$

Figure 1: