Equal interval:

equal interval الفروق المتساوية

Newton׳s forward

Newton׳s backward

Difference table:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
x_o	y_o	$(y_1 - y_o) = \Delta y_o$		*Newton' s forward*
x_1	y_1	$(y_2 - y_1) = \Delta y_1$	$\Delta y_1 - \Delta y_o = \Delta^2 y_o$	$\Delta^2 y_1 - \Delta^2 y_0 = \Delta^3 y_0$
x_2	y_2		$\Delta y_2 - \Delta y_1 = \Delta^2 y_1$	
x_3	y_3	$(y_3 - y_2) = \Delta y_2$	*Newton's backward*	

Newton' s forward:

Δ (Delta)

$x = x_o + Sh$, $S = \frac{x - x_o}{h}, \quad 0 < S < 1$

$$y = f(x) = y_o + S\Delta y_o + \frac{S(S-1)}{2!}\Delta^2 y_o + \frac{S(S-1)(S-2)}{3!}\Delta^3 y_o + \ldots\ldots$$

$$\frac{S(S-1)(S-2)\ldots\ldots\ldots\ldots(S-n+1)}{n!}\Delta^n y_o$$

Newton's backward:

(interval delta ∇) معكوس

$$S = \frac{x - x_n}{h}, \; S < 0$$

$$y = f(x) = y_n + S\nabla y_n + \frac{S(S+1)}{2!}\nabla^2 y_n + \frac{S(S+1)(S+2)}{3!}\nabla^3 y_o + \ldots\ldots$$

$$\frac{S(S+1)(S+2)\ldots\ldots\ldots\ldots(S+n-1)}{n!}\nabla^n y_o$$

Example (10):

From the following data given $f(x) = \sin(x)$ estimate the value of $\sin(30.2°)$, $\sin(31.3°)$

X	30.0	30.5	31.0	31.5
Y	0.50000	0.50754	0.51504	0.52250

Solution:

Difference is given by table

x	$F(x_i)$	$\Delta F(x_i)$	$\Delta^2 F(x_i)$	$\Delta^3 F(x_i)$
30.0	0.50000			
		0.00754		
30.5	0.50754		-0.00004	
		0.00750		0.00000
31.0	0.51504		-0.00004	
		0.00746		
31.5	0.52250			

$$h = 0.5 \text{ At } x = 30.2°$$

Newton' s forward

$$y = p_3(x) = y_o + S\,\Delta y_o + \frac{S\,(S-1)}{2!}\Delta^2 y_o + ...$$

$$S = \frac{x - x_o}{h} = \frac{30.2 - 30}{0.5} = 0.4$$

عدد موجب 0.4

$$p_3(x) = p_3(30.2) = 0.5 + 0.4\left(0.00754\right) + \frac{0.4\left(0.4-1\right)}{2!}\left(-0.00004\right) + 0$$

$$p_3(x) = p_3(30.2) = 0.50302$$

$h = 0.5$ At $x = 31.3°$

Newton's backward

$$S=\frac{-x_n+x}{h}=\frac{-31.5+31.3}{0.5}=-0.4$$

عدد سالب -0.4

$$P_3(x)=P_3(31.3)=y_n+S\nabla y_n+\frac{S\left(S+1\right)}{2!}\nabla^2 y_n+.......$$

$$P_3(31.3)=0.5225-0.4\left(0.00746\right)-0.4\frac{\left(-0.4+1\right)}{2!}\left(-0.00004\right)+0$$

$$P_3(31.3)=0.5195$$

Interpolation with Spline function:

هنا الفكرة تختلف عن الطرق السابقة في أنه نكون بين كل نقطتين في الجدول بدالة كثيرة حدود $S_i(x)$ أما في الطرق الأخرى هناك دالة كثيرة حدود واحدة تصل بين جميع النقاط $P_n(x)$

Linear Interpolation (First Degree Spline) :

نصل بين كل النقاط بخط مستقيم

$$S_i(x) = y_i + \frac{y_{i+1} - y_i}{x_{i+1} - x_i}\left(x - x_i\right)$$

$$S_i(x) = y_i + \delta_i\left(x - x_i\right) \to i = 0,1,2,3,....$$

Example (11):

Find a spline of degree one to interpolate the following data and use the resulting spline to approximate $f(2.2)$

X	1	1.5	2	2.5	3
$y = F(x)$	1	3	7	10	15

Solution:

$S_i(x) = y_i + \delta_i\left(x - x_i\right)$

At $i = 0$

$$S_0(x) = y_0 + \delta_0 (x - x_0)$$

$$S_0(x) = 1 + 4(x-1) \rightarrow S_0(x) = 4x - 3$$

At $i = 1$

$$S_1(x) = y_1 + \delta_1 (x - x_1)$$

$$S_1(x) = 3 + 8(x - 1.5) \rightarrow S_1(x) = 8x - 9$$

At $i = 2$

$$S_2(x) = y_2 + \delta_2 (x - x_2)$$

$$S_2(x) = 7 + 6(x - 2) \rightarrow S_2(x) = 6x - 5$$

At $i = 3$

$$S_3(x) = y_3 + \delta_3 (x - x_3)$$

$$S_3(x) = 10 + 10(x - 2.5) \rightarrow S_3(x) = 10x - 15$$

x	$y = F(x)$	δ_i
1	1	
		$\frac{3-1}{1.5-1} = 4 \rightarrow \delta_0$
1.5	3	
		$\frac{7-3}{2-1.5} = 8 \rightarrow \delta_1$
2	7	
		$\frac{10-7}{3-2.5} = 6 \rightarrow \delta_2$
2.5	10	
		$\frac{15-10}{3-2.5} = 10 \rightarrow \delta_3$
3	15	

$$\textbf{\textit{So that }} S_i(x) = \begin{cases} (4x - 3)\, if\ x \in [1, 1.5] \\ (8x - 9)\, if\ x \in [1.5, 2] \\ (6x - 5)\, if\ x \in [2, 2.5] \\ (10x - 15)\, if\ x \in [2.5, 3] \end{cases}$$

The value of $x = 2.2$ lies in $[2, 2.5]$

$$f(2.2) = 6(2.2) - 5 = 8.2$$

Nature Cubic Splines:

نصل بين كل نقطتين في الجدول بكثيرة حدود من الدرجة الثالثة ويكون علي الشكل:

$$S_i(x) = a_i + b_i(x - x_i) + c_i(x - x_i)^2 + d_i(x - x_i)^3 \rightarrow i = 0,1,2,3,....$$

Where:

a_i, b_i, c_i, d_i are constants

$$a_i = y_i \,,\; b_i = \frac{y_{i+1} - y_i}{h} - \frac{h}{3}(c_{i+1} + 2c_i) \,,\; d_i = \frac{(c_{i+1} - c_i)}{3h}$$

To calculate c_i

$$c_{i-1} + 4c_i + c_{i+1} = \frac{3}{h^2}[y_{i+1} - 2y_i + y_{i-1}] \rightarrow i = 1,2,3,......$$

$$c_0 = c_n = 0$$

ممكن كتابتها علي الصورة :

$$\begin{bmatrix} 4 & 1 & 0 & 0 & 0 \\ 1 & 4 & 1 & 0 & 0 \\ 0 & 1 & 4 & 1 & 0 \\ 0 & 0 & 1 & 4 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ . \end{bmatrix} = \frac{3}{h^2} \begin{bmatrix} y_2 - 2y_1 + y_0 \\ y_3 - 2y_2 + y_1 \\ . \\ . \end{bmatrix}$$

Example (12):

Use the values given by $f(x) = x^3 + 2$ at $x = 0, 0.2, 0.4, 0.6, 0.8$ and 1 to find approximation of $f(x)$ at $x = 0.1, 0.3, 0.5, 0.7, 0.9$ using the Natural Cubic Spline Interpolation

Solution:

X	0	0.2	0.4	0.6	0.8	1
$y = F(x)$	2	2.008	2.064	2.216	2.512	3

$$C_0 = C_5 = 0$$

$$\because c_{i-1}+4c_i+c_{i+1}=\frac{3}{h^2}\left[y_{i+1}-2y_i+y_{i-1}\right]$$

At $i=1$

$$\because c_0+4c_1+c_2=\frac{3}{h^2}\left[y_2-2y_1+y_0\right]$$

$$0+4c_1+c_2=\frac{3}{(0.2)^2}\left[2.064-2\times 2.008+2\right]$$

$$4c_1+c_2=3.6\rightarrow (1)$$

At $i=2$

$$\because c_1+4c_2+c_3=\frac{3}{h^2}\left[y_3-2y_2+y_1\right]$$

$$c_1+4c_2+c_3=\frac{3}{(0.2)^2}\left[2.216-2\times 2.064+2.008\right]$$

$$c_1+4c_2+c_3=7.2\rightarrow (2)$$

At $i=3$

$$\because c_2+4c_3+c_4=\frac{3}{h^2}\left[y_4-2y_3+y_2\right]$$

$$c_1+4c_2+c_3=\frac{3}{(0.2)^2}\left[2.512-2\times 2.216+2.0\quad\right]$$

$$c_2+4c_3+c_4=10.8\rightarrow (3)$$

At $i=4$

$$\because c_3+4c_4+c_5=\frac{3}{h^2}\left[y_5-2y_4+y_3\right]$$

$$c_3+4c_4+c_5=\frac{3}{(0.2)^2}\left[3-2\times 2.512+2.21\quad\right]$$

$$c_3 + 4c_4 + c_5 = c_3 + 4c_4 + 0 = c_3 + 4c_4 = 14.4 \rightarrow (4)$$

بحل المعادلات 1و2و3و4 معا نحصل علي الاتي :

$$\begin{cases} C_0 = 0 \\ C_1 = {}^{612}/_{1045} \\ C_2 = {}^{1314}/_{1045} \\ C_3 = {}^{1656}/_{1045} \\ C_4 = {}^{3348}/_{1045} \\ C_5 = 0 \end{cases}$$

بمعرفة c_i يمكن حساب باقي الثوابت:

$$a_i = y_i = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 2.008 \\ 2.064 \\ 2.216 \\ 2.512 \end{bmatrix}$$

$$\therefore b_i = \frac{y_{i+1} - y_i}{h} - \frac{h}{3}\left(c_{i+1} + 2c_i\right)$$

$$b_0 = \frac{y_1 - y_0}{0.2} - \frac{0.2}{3}[C_1 + 2C_0]$$

$$b_0 = \frac{2.0 \quad - 2}{0.2} - \frac{0.2}{3}\left[{}^{612}/_{1045} + 2(0)\right] = 9.56 \quad * 10^{-4}$$

$$b_1 = \frac{y_2 - y_1}{0.2} - \frac{0.2}{3}[C_2 + 2C_1]$$

$$b_1 = \frac{2.064 - 2.0}{0.2} - \frac{0.2}{3}\left[{}^{1314}/_{1045} + 2\left({}^{612}/_{1045}\right)\right] = {}^{617}/_{5225}$$

$$b_2 = \frac{y_3 - y_2}{0.2} - \frac{0.2}{3}[C_3 + 2C_2]$$

$$b_2 = \frac{2.216 - 2.064}{0.2} - \frac{0.2}{3}\left[{}^{1656}/_{1045} + 2\left({}^{1314}/_{1045}\right)\right] = {}^{2543}/_{5225}$$

$$b_3 = \frac{y_4 - y_3}{0.2} - \frac{0.2}{3}[C_4 + 2C_3]$$

$$b_3 = \frac{2.512 - 2.216}{0.2} - \frac{0.2}{3}\left[{}^{3348}/_{1045} + 2\left({}^{1656}/_{1045}\right)\right] = {}^{5513}/_{5225}$$

$$b_4 = \frac{y_5 - y_4}{0.2} - \frac{0.2}{3}[C_5 + 2C_4]$$

$$b_4 = \frac{3 - 2.512}{0.2} - \frac{0.2}{3}\left[0 + 2\left({}^{3348}/_{1045}\right)\right] = {}^{10517}/_{5225}$$

$$d_i = \frac{(c_{i+1} - c_i)}{3h}$$

$$d_0 = \frac{c_1 - c_0}{3(0.2)}$$

$$d_0 = \frac{{}^{612}/_{1045} - 0}{3(0.2)} = \frac{204}{209}$$

$$d_1 = \frac{c_2 - c_1}{3(0.2)}$$

$$d_1 = \frac{{}^{1314}/_{1045} - {}^{612}/_{1045}}{3(0.2)} = \frac{234}{209}$$

$$d_2 = \frac{c_3 - c_2}{3(0.2)}$$

$$d_2 = \frac{{}^{1656}/_{1045} - {}^{1314}/_{1045}}{3(0.2)} = \frac{6}{11}$$

$$d_3 = \frac{c_4 - c_3}{3(0.2)}$$

$$d_3 = \frac{{}^{3348}/_{1045} - {}^{1656}/_{1045}}{3(0.2)} = \frac{564}{209}$$

$$d_4 = \frac{c_5 - c_4}{3(0.2)}$$

$$d_4 = \frac{0 - {}^{3348}/_{1045}}{3(0.2)} = -\frac{1116}{209}$$

$$\because S_i(x) = a_i + b_i(x - x_i) + c_i(x - x_i)^2 + d_i(x - x_i)^3 \to i = 0,1,2,3,....$$

At $i = 0$

$$\therefore S_0(x) = a_0 + b_0(x - x_0) + c_0(x - x_0)^2 + d_0(x - x_0)^3$$

$$\therefore S_0(x) = 2 + 0.001(x - 0) + 0(x - 0)^2 + 0.976(x - 0)^3$$

$$\therefore S_0(x) = 2 + 0.001x + 0.976x^3, x \in [0, 0.2]$$

At $i = 1$

$$\therefore S_1(x) = a_1 + b_1(x - x_1) + c_1(x - x_1)^2 + d_1(x - x_1)^3$$

$$\therefore S_1(x) = 2.008 + 0.118(x - 0.2) + 0.586(x - 0.2)^2 + 1.12(\quad - 0.2)^3$$

$$\therefore S_1(x) = \surd, x \in [0.2, 0.4]$$

$$S_2(x) = a_2 + b_2(x - x_2) + c_2(x - x_2)^2 + d_2(x - x_2)^3$$

$$= 2.064 + {}^{2543}/_{5225}(x - 0.4) + {}^{1314}/_{1045}(x - 0.4)^2 + \frac{6}{11}(x - 0.4)^3$$

$$S_3(x) = a_3 + b_3(x - x_3) + c_3(x - x_3)^2 + d_3(x - x_3)^3$$

$$= 2.216 + {}^{5513}/_{5225}(x - 0.6) + {}^{1656}/_{1045}(x - 0.6)^2 + \frac{564}{209}(x - 0.6)^3$$

$$S_4(x) = a_4 + b_4(x - x_4) + c_4(x - x_4)^2 + d_4(x - x_4)^3$$

$$= 2.512 + {}^{10517}/_{5225}(x - 0.8) + {}^{3348}/_{1045}(x - 0.8)^2 - \frac{1116}{209}(x - 0.8)^3$$

لحساب $S(0.1) \to \in [0, 0.2]$

$$\therefore S(0.1) = 2 + 0.001 \times 0.1 + 0.976(0.1)^3 = 2.001, x \in [0, 0.2]$$

لحساب $S(0.3) \to \in [0.2, 0.4]$

$$\therefore S(0.3) = 2.008 + 0.118(0.3 - 0.2) + 0.586(0.3 - 0.2)^2 + 1.12(0.3 - 0.2)^3 = 2.02678$$

At x=0.5

$\therefore S_2(0.5) = 2.125789$

At x=0.7

$\therefore S_3(0.7) = 2.340057$

At x=0.9

$\therefore S_4(0.9) = 2.739981$

Exercises

1- Find a polynomial of order (2) that interpolates the table

X	0.2	0.4	0.6
Y	-0.95	-0.82	-0.65

2- Determine a polynomial of degree ≤3 that interpolates the data

X	1.2	2.1	3	3.6
Y	0.7	8.1	27.7	45.1

3- Determine the Lagrange polynomial that interpolates the data in the following table

X	0	2	4	6
Y	1	-1	3	4

4- Let $f(x) = 2x^2e^x + 1$ contrast a Lagrange polynomial of degree two or less using $x_o = 0, x_1 = 0.5 and\ x_2 = 1$ approximate $f(0.8)$.

5- Determine a polynomial of degree ≤5 using Newton's divided differences that interpolate the data in the following table

X	1	2	3	4	5	6
Y	14.5	19.5	30.5	53.5	94.5	159.5

Use the resulting polynomial to estimate the value of $f(4.5)$. Compare of the exact value of $f(4.5) = 71.375$

6- To investigate the relationship between yield of Potatoes (y) and level of fertilizer application (x). An experimenter divided a field into (5) pots of equal size and applied differing amounts of fertilizers to each. the data recorded for each plot are given by the table (in Kgs)

X	1	2	3	4	5
Y	22	23	25	30	28

- Find the interpolation polynomial for this table.

According to the interpolating polynomial, approximately how many Kgs would you expect from a plot to which 2.5 Kgs of fertilizer had been applied?

Best wishes

Dr. Ashraf Almahallawy