Department of Physics and Engineering Mathematics
Numerical analysis (BAS127)
Mechatronics Engineering

Lecture (1)

Tanta University **Faculty of Engineering**

Chapter (1) Interpolation

الاستكمال

interpolation الإستكمال

Equal interval
الفروق المتساوية

Un equal interval
الفروق غير المتساوية

In the equal interval:

$$x_1 - x_o = x_2 - x_1 = \text{.............} = h$$

In the unequal interval:

$$x_1 - x_o \neq x_2 - x_1 \neq \text{............}$$

تعتمد الفكرة علي :

x	x_o	x_1	....	x_{n-1}	x_n
$y = f(x)$	y_o	y_o	....	y_{n-1}	y_n

ومطلوب إيجاد قيمة غير موجودة بالجدول :

عدد القراءات في الجدول ناقص 1

الحل :

1. نكون كثيرة حدود من الدرجة n

Polynomial of degree n $\qquad P_n(x) = \text{...........}$

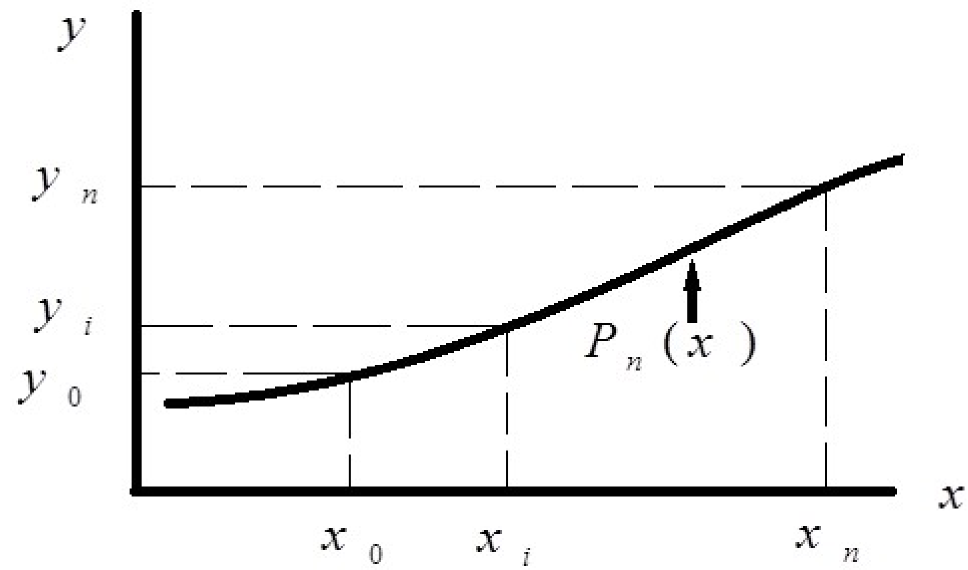


Figure (the equal interval

2. ثم بعد ذلك نوجد أي قيمة لل (x) مطلوبة

The unequal interval:

$$x_1 - x_o \neq x_2 - x_1 \neq ....$$

un equal interval الفروق غير المتساوية

Lagrange interpolation

Newton divied deferance

Lagrange interpolation:

$$P_n(x)=\frac{(x-x_1)(x-x_2).........(x-x_n)}{(x_o-x_1)(x_o-x_2).........(x_o-x_n)}y_o+\frac{(x-x_o)(x-x_2).........(x-x_n)}{(x_1-x_o)(x_1-x_2).........(x_1-x_n)}y_1+$$

$$...........+\frac{(x-x_o)(x-x_1).........(x-x_{n-1})}{(x_n-x_0)(x_n-x_1).........(x_n-x_{n-1})}y_n$$

Where:

$$L_o(x)=\frac{(x-x_1)(x_1-x_2).........(x-x_n)}{(x_o-x_1)(x_o-x_2).........(x_o-x_n)}$$

$$L_1(x)=\frac{(x-x_o)(x-x_2).........(x-x_n)}{(x_1-x_o)(x_1-x_2).........(x_1-x_n)}$$

$$L_n(x)=\frac{(x-x_o)(x-x_1).........(x-x_{n-1})}{(x_n-x_0)(x_n-x_1).........(x_n-x_{n-1})}$$

$$P_n(x)=\sum_{i=0}^{n}L_i(x)y_i=\sum_{i=0}^{n}L_i(x)f(x_i)$$

Where:

$L_i(x)$ Is agrange coefficient polynomial

Error terms and error bounds:

For unequal interval:

Error term:

$$E_n(x_i)=f(x_i)-P_n(x_i)$$

$$E_n(x)=\frac{(x-x_o)(x-x_1).........(x-x_n)}{(n+1)!}f^{(n+1)}(c)\rightarrow c\in[a,b]$$

Erro bounds:

$$E_n(x) \le \frac{\left|(x-x_o)(x-x_1)........(x-x_n)\right|}{(n+1)!} \quad {}_{n+1}$$

$$M_{n+1} = \max_{a\le c\le b}\left|f^{(n+1)}(c)\right|$$

عيوب هذه الطريقة :

- معقدة حسابيا
- لايمكن حذف أو إضافة أي قيمة داخل الجدول لأن ذلك يستلزم البدء من الاول لحساب $P_n(x)$ الجديدة .

For equal interval:

Error bounds:

$$E_1(x) \le \frac{h^2}{8} M_2, x \in [x_o, x_1]$$

$$E_2(x) \le \frac{h^3}{9\sqrt{3}} M_3, x \in [x_o, x_2]$$

$$E_1(x) \le \frac{h^4}{24} M_4, x \in [x_o, x_3]$$

Example (1):

Derive the LaGrange polynomial that interpolates

X	-1	0	3
Y	8	-2	4

Solution:

Therefore it follows that we need to compute

$$P_2(x) = f(-1)L_o(x) + f(0)L_1(x) + f(3)L_2(x) \leftrightarrow (1)$$

$$L_o(x) = \frac{(x-x_1)(x-x_2)}{(x_o-x_1)(x_o-x_2)} = \frac{(x-0)(x-3)}{(-1-0)(-1-3)} = \frac{1}{4}x(x-3)$$

$$L_1(x)=\frac{(x-x_o)(x-x_2)}{(x_1-x_o)(x_1-x_2)}=\frac{(x-(-1))(x-3)}{(0-(-1))(0-3)}=\frac{(x+1)(x-3)}{(0+1)(0-3)}=\frac{1}{3}(x+1)(x-3)$$

$$L_2(x)=\frac{(x-x_o)(x-x_1)}{(x_2-x_1)(x_2-x_1)}=\frac{(x-(-1))(x-0)}{(3-(-1))(3-0)}=\frac{(x+1)(x-0)}{(3-(-1))(3-0)}=\frac{1}{12}x\,(x+1)$$

Substitute in the equation (1)

The Lagrange interpolating polynomial is given by

$$P_2(x)=f(-1)L_o(x)+f(0)L_1(x)+f(3)L_2(x)$$
$$P_2(x)=2x(x-3)+\frac{2}{3}(x+1)(x-3)+\frac{1}{3}x(x+1)$$

Example (2):

Determine the coefficients of the Lagrange interpolating polynomial that interpolates the following data, and evaluate it at (x= 2.12)

X	2	2.1	2.2
Y	0.69315	0.74194	0.78846

Solution:

$$P_2(x)=f(2)L_o(x)+f(2.1)L_1(x)+f(2.2)L_2(x)$$

Therefore it follows that we need to compute

$$P_2(2.12)=f(2)L_o(2.12)+f(2.1)L_1(2.12)+f(2.2)L\ \ (2.12)\rightarrow(1)$$

$$L_o(2.12)=\frac{(x-x_1)(x-x_2)}{(x_o-x_1)(x_o-x_2)}=\frac{(2.12-2.1)(2.12-2.2)}{(2-2.1)(2-2.2)}=-0.08$$

$$L_1(2.12)=\frac{(x-x_o)(x-x_2)}{(x_1-x_o)(x_1-x_2)}=\frac{(2.12-2)(2.12-2.2)}{(2.1-2)(2.1-2.2)}=0.96$$

$$L_2(2.12)=\frac{(x-x_o)(x-x_1)}{(x_2-x_0)(x_2-x_1)}=\frac{(2.12-2)(2.12-2.1)}{(2.2-2)(2.2-2.1)}=0.12$$

Substitute in the equation (1)

The Lagrange interpolating polynomial is given by

$$P_2(2.12) = f\,(2)L_o(2.12) + f\,(2.1)L_1(2.12) + f\,(2.2)L\ \ (2.12)$$
$$P_2(2.12) = 0.619315(-0.08) + 0.74194(0.69) + 0.78846(0.12) = 0.75142$$

Example (3):

Consider $f\,(x) = \cos(x)$ over $[0,1.2]$ the following table Determine the error bounds for Lagrange interpolating polynomial $P_2(x)$

X	0	0.6	1.2
Y	1	0.825336	0.362358

Solution:

First, w   determine abound of Error

$$|E_2(x)| \le \frac{h^3}{9\sqrt{3}} M_3$$

Where:

$$M_3 = |f^{(3)}(x)| = |\sin(x)| \le |\sin(1.2)| = 0.932039$$

The spacing of the nodes $h = 0.6$ and its error bound is

$$|E_2(x)| \le \frac{(0.6)^3\,(0.932039)}{9\sqrt{3}} = 0.012915$$

Example (4):

Find the polynomial of degree $\le$ 3 that interpolates the following data using Lagrange interpolation.

x	1.2	2.1	3	3.6
y	0.7	8.1	27.7	45.1

Solution:

$$P_n(x) = \sum_{i=0}^{n} L_i(x) * y_i$$

$$L_0(x) = \frac{(x - 2.1)(x - 3)(x - 3.6)}{(1.2 - 2.1)(1.2 - 3)(1.2 - 3.6)} = -0.257 (x - 2.1)(x - 3)(x - 3.6)$$

$$L_1(x) = \frac{(x - 1.2)(x - 3)(x - 3.6)}{(2.1 - 1.2)(2.1 - 3)(2.1 - 3.6)} = 0.823 (x - 1.2)(x - 3)(x - 3.6)$$

$$L_2(x) = \frac{(x - 1.2)(x - 2.1)(x - 3.6)}{(3 - 1.2)(3 - 2.1)(3 - 3.6)} = -1.029 (x - 1.2)(x - 2.1)(x - 3.6)$$

$$L_3(x) = \frac{(x - 1.2)(x - 2.1)(x - 3)}{(3.6 - 1.2)(3.6 - 2.1)(3.6 - 3)} = 0.463 (x - 1.2)(x - 2.1)(x - 3)$$

Then

$$\begin{aligned} P_3(x) = & -0.18 (x - 2.1)(x - 3)(x - 3.6) + 6.67 (x - 1.2)(x - 3)(x - 3.6) \\ & - 28.49 (x - 1.2)(x - 2.1)(x - 3.6) \\ & + 20.8 \ (x - 1.2)(x - 2.1)(x - 3) \end{aligned}$$

Example (5):

Let $f(x) = 2x^2 e^x + 1$ construct a Lagrange polynomial of degree two using $x_0 = 0$, $x_1 = 0.5$ and $x_2 = 1$ hence find the value of $f(0.8)$ and its error.

Solution:

From the giving data we can form the following table

X	0	0.5	1
Y	1	1.824	6.437

Then

$P_n(x) = \sum_{i=0}^{n} L_i(x) * y_i$, n = 2

$P_2(x) = L_0(x)\, y_0 + L_1(x)\, y_1 + L_2(x)\, y_2$

$$\begin{aligned} P_2(x) = & \frac{(x - 0.5)(x - 1)}{(0 - 0.5)(0 - 1)}(1) + \frac{(x - 0)(x - 1)}{(0.5 - 0)(0.5 - 1)}(1.824) \\ & + \frac{(x - 0)(x - 0.5)}{(1 - 0)(1 - 0.5)}(6.437) \end{aligned}$$

$$P_2(x) = 2 (x - 0.5)(x - 1) - 7.296\, x (x - 1) + 12.8\, x (x - 0.5)$$

At $x = 0.8$

$$P_2(0.8) = 2*(0.8 - 0.5)*(0.8 - 1) - 7.296*0.8*(0.8 - 1) + 12.8*0.8*(0.8 - 0.5) \cong 4.1$$

Error calculation

$$|Error| \le \frac{M_{n+1}}{(n+1)!}|(x - x_0)(x - x_1)\dots(x - x_n)|, \text{n} = 2$$

$$f(x) = 2x^2 e^x + 1 \quad >>> \quad \frac{d^3 f}{dx^3}(x) = 2x^2 e^x + 12\,x\,e^x + 12\,e^x$$

$$\frac{d^3 f}{dx^3}(0) = 12 \qquad \text{And} \qquad \frac{d^3 f}{dx^3}(1) = 70.68 \quad >>> \quad M_{n+1} = 70.$$

Then $|Error| \le \frac{70.68}{(3)!}|(0.8 - 0)(0.8 - 0.5)(0.8 - 1)| = 0.56544$

The newton divided difference form:

تعتمد الفكرة علي عمل جدول الفروق :

Divided difference table

x	y	First divided difference 1st DD δ	Second divided difference 2nd DD δ^2	Third divided difference 3rd DD δ^3
x_o	y_o	$\frac{(y_1 - y_o)}{(x_1 - x_o)} = \delta_o$		¦
x_1	y_1	$\frac{(y_2 - y_1)}{(x_2 - x_1)} = \delta_1$	$\frac{(\delta_1 - \delta_o)}{(x_2 - x_o)} = \delta_o^2$	¦
x_2	y_2	¦	¦	¦
x_n	y_n			¦

$$P_n(x) = y_o + (x - x_o)\delta_o + (x - x_o)(x - x_1)\delta_o^2 + ...(x - x_o)...(x - x_n)\delta_o^n$$

Example (6):

From a divided –difference table for the following data and obtain Newton's interpolating polynomial

X	0	4	6	8
Y	4	8	14	16

Solution:

Divided difference table

x	$F[x_i]$	$F[x_i, x_{i+1}]$	$F[x_i, x_{i+1}, x_{i+2}]$	$F[x_i, x_{i+1}, x_{i+2}, x_{i+3}]$
0	4			
		$\frac{(8-4)}{(4-0)} = 1$		
4	8		$\frac{(3-1)}{(6-0)} = \frac{1}{3}$	
		$\frac{(14-8)}{(6-4)} = 3$		$\frac{\left(-\frac{1}{2} - \frac{1}{3}\right)}{(8-0)} = \frac{-5}{48}$
6	14		$\frac{(1-3)}{(8-4)} = \frac{-1}{2}$	
		$\frac{(16-4)}{(8-6)} = 1$		
8	16			

From the diagonal Newton's interpolating polynomial is then

$$P_3(x) = 4 + x + \frac{1}{3}x\,(x-4) - \frac{5}{48}x\,(x-4)(x-6)$$

Example (7):

From a divided –difference table for the following data and obtain Newton's interpolating polynomial

X	1	2	3	5	7
Y	3	5	9	11	15

Solution:

Divided difference table

x	y	δ	δ^2	δ^3	δ^4
1	3				
		$\frac{(5-3)}{(2-1)} = 2$			
2	5		$\frac{(4-2)}{(3-1)} = 1$		
		$\frac{(9-5)}{(3-2)} = 4$		$\frac{(-1-1)}{(5-1)} = \frac{-1}{2}$	
3	9		$\frac{(1-4)}{(5-2)} = -1$		$\frac{\left(\frac{1}{4}+\frac{1}{2}\right)}{(7-1)} = \frac{1}{8}$
		$\frac{(11-9)}{(5-3)} = 1$		$\frac{\left(\frac{1}{4}+1\right)}{(7-2)} = \frac{1}{4}$	
5	11		$\frac{(2-1)}{(7-3)} = \frac{1}{4}$		
		$\frac{(15-11)}{(7-5)} = 2$			
7	15				

From the diagonal Newton's interpolating polynomial is then

$$P_4(x) = 3 + 2(x-1) + (x-1)(x-2) - \frac{1}{2}x\,(x-1)(x-2)(x-3) + \frac{1}{8}x\,(x-1)(x-2)(x-3)(x-5)$$

Example (8):

Determine the polynomial of degree ≤ 5 using Newton's divided difference that interpolates the following data, hence use the result to estimate the value of y at $x = 4.5$ and compare it with the exact value71.375.

X	1	2	3	4	5	6
Y	14.5	19.5	30.5	53.5	94.5	159.5

Solution:

Form the difference table as below:

x_i	y_i	δ	δ^2	δ^3	δ^4	δ^5
1	14.5	$\frac{19.5-14.5}{2-1} = 5$	$\frac{11-5}{3-1} = 3$	$\frac{6-3}{4-1} = 1$	$\frac{1-1}{5-1} = 0$	$\frac{0-0}{6-1} = 0$
2	19.	$\frac{30.5-19.5}{3-2} = 11$	$\frac{23-11}{4-2} = 6$	$\frac{9-6}{5-2} = 1$	$\frac{1-1}{6-2} = 0$	
3	30.5	$\frac{53.5-30.5}{4-3} = 2$	$\frac{41-23}{5-3} = 9$	$\frac{12-9}{6-3} = 1$		
4	53.	$\frac{94.\ \ -53.5}{5-4} = 41$	$\frac{65-41}{6-3} = 1$			
5	94.5	$\frac{159.5-30.5}{4-3} = 6$				
6	159.5					

Using Newton's difference form gives

$$P_n(x) = y_0 + \delta\,(x-x_0) + \delta^2(x-x_0)\,(x-x_1) + \cdots + \delta^n(x-x_0)\ldots(x-x_{n-1})\,, \mathrm{n} = 5$$

Then

$P(x) = 14.5 + 5\,(x-1) + 3\,(x-1)\,(x-2) + \,(x-1)\,(x-2)\,(x-3)$

At $x = \ 4.5$

$P(4.5) = 14.5 + 5\,(4.5-1) + 3\,(4.5-1)\,(4.5-2) + \,(4.5-1)\,(4.5-2)\,(4.5-3)$

$P(4.5) = 71.375$ Which equal to the exact value 71.375.

Example (9):

From the following table find the number of students who obtained **less than** 45 marks.

Marks	30 – 40	40 – 50	50 – 60	60 – 70	70 – 80
No. of Students	31	42	51	35	31

Solution

From the given data, we can form the less than marks table as bel

Marks	40	50	60	70	80
No. of Students	31	31 + 42 = 73	124	159	190

To obtain the students obtained less than 45 marks use Newton's forward method as follows:

x_i	y_i	Δ	Δ^2	Δ^3	Δ^4
40	31	$73 - 31 = 42$	$51 - 42 = 9$	$-16 - 9 = -25$	$12 + 25 = 37$
50	73	$124 - 73 =$	$35 - 51 \quad -16$	$-4 + 16 = 12$	
60	124	$159 - 124 = 35$	$31 - 35 \quad -4$		
70	15	$190 - 159 =$			
80	190				

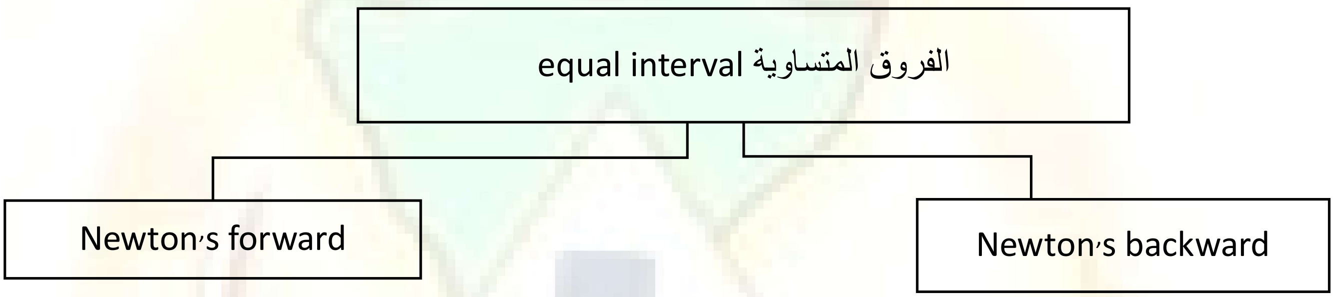
From the table >> $h = 1$ and $s = \frac{45-40}{10} = 0.5$

Using Newton's forward formula, we get:

$$P(45) = 31 + (0.5)(42) + \frac{(0.5)(0.5\ -1)}{2!}(9) + \frac{(0.5)(0.5\ -1)(0.5\ -2)}{3!}(-25)$$
$$+ \frac{(0.5)(0.5\ -1)(0.5\ -2)(0.5\ -3)}{4!}(37)$$

$P(45) = 47.867$ Then the number of students obtained less than 45 marks equal 48 students

Equal interval:



Difference table:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
x_o	y_o	$(y_1 - y_o) = \Delta y_o$		*Newton' s forward*
x_1	y_1	$(y_2 - y_1) = \Delta y_1$	$\Delta y_1 - \Delta y_o = \Delta^2 y_o$	$\Delta^2 y_1 - \Delta^2 y_0 = \Delta^3 y_0$
x_2	y_2	$(y_3 - y_2) = \Delta y_2$	$\Delta y_2 - \Delta y_1 = \Delta^2 y_1$	
x_3	y_3		*Newton's backward*	