Chapter 1

Signal Analysis

Lecture 2

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Introduction

4-Signup Function

Standard Signup:

$$f(t) = \text{sgn}(t) = \begin{cases} +1 & t > 0 \\ -1 & t < 0 \end{cases}$$

- ❑ Of unity strength on both sides.
- ❑ Centered at origin.

In General:

$$f(t) = k\, sgn(t - t_o) = \begin{cases} +k & t > t_o \\ -k & t < t_o \end{cases}$$

- ❑ Strength = k
- ❑ Shifted t_o to the right.

Exercise on Signup Function

Exersize.1.2: Show that:

i) $sgn(t) = 2u(t) - 1$

ii) $u(t) = 0.5[1 + sgn(t)]$

Then draw:

$\theta(\omega) = -0.5\pi \, sgn(\omega)$

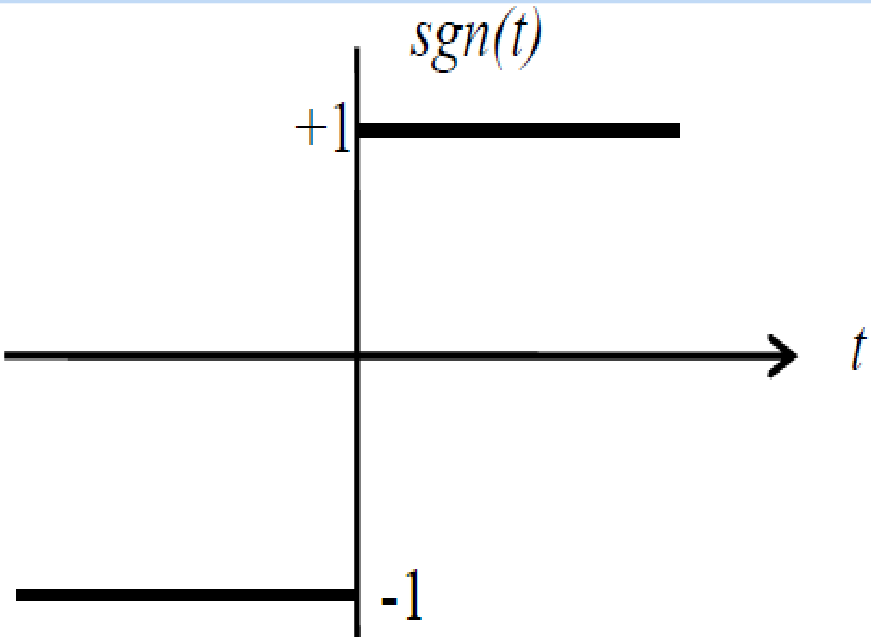


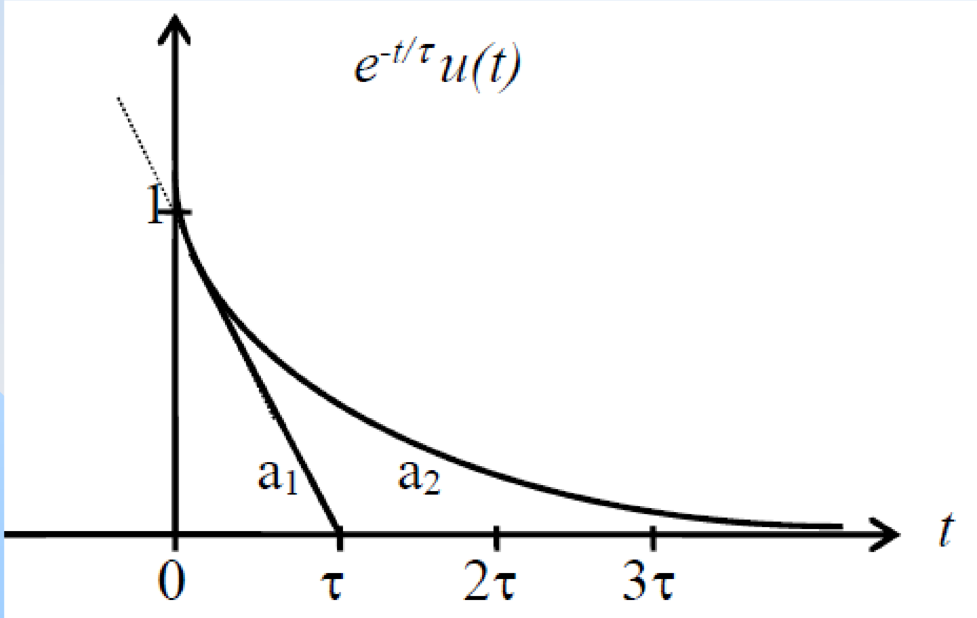
Fig.1.7: Signum Function

5-Single-sided Exponential

Standard:

$$f(t) = e^{-t/\tau}\, u(t)$$

- Of unity strength.
- Begins at origin.
- Area = τ
- Tangent splits area into 2 equal sections



- $a_1 = a_2$

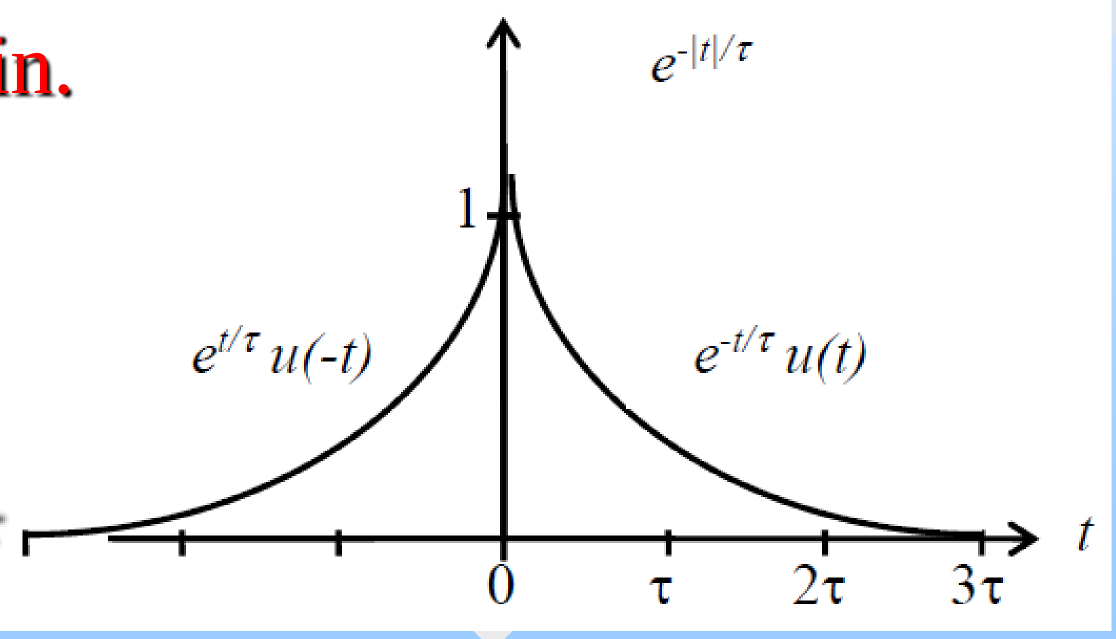
6-Dupple-sided Exponential

Standard: $f(t) = e^{-|t|/\tau}$

- Obtained by adding two single-sided functions:

$$f(t) = e^{-t/\tau}\,u(t) + e^{t/\tau}\,u(-t)$$

- Of unity strength.
- Centered at origin.
- Area = 2τ



Normalized:

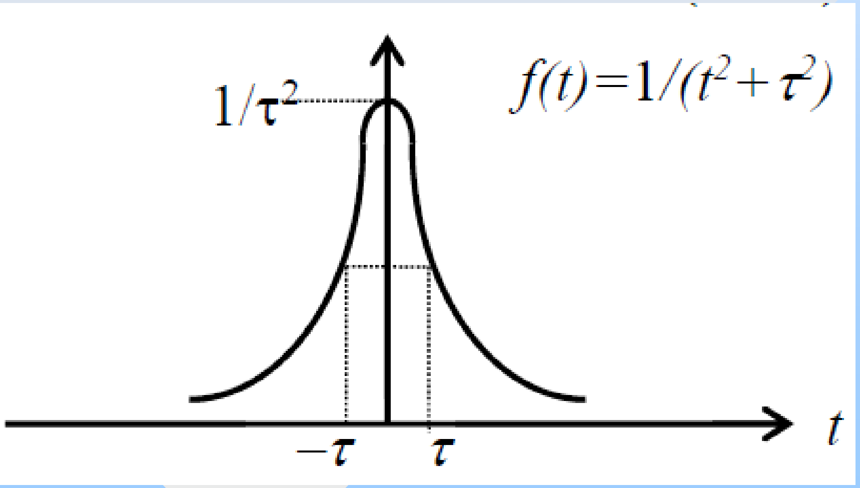
$$f(t) = \frac{1}{2\tau} e^{-|t|/\tau}$$

7-Lorenz Function

Standard: $f(t) = \frac{1}{t^2+\tau^2}$

- τ is a measure for time width or duration.
- Its strength = $1/\tau^2$
- Its area is given by the integration as follows:

$$Area = \int_{-\infty}^{\infty} \frac{1}{t^2+\tau^2} = \frac{\pi}{\tau}$$



Normalized Lorenz:

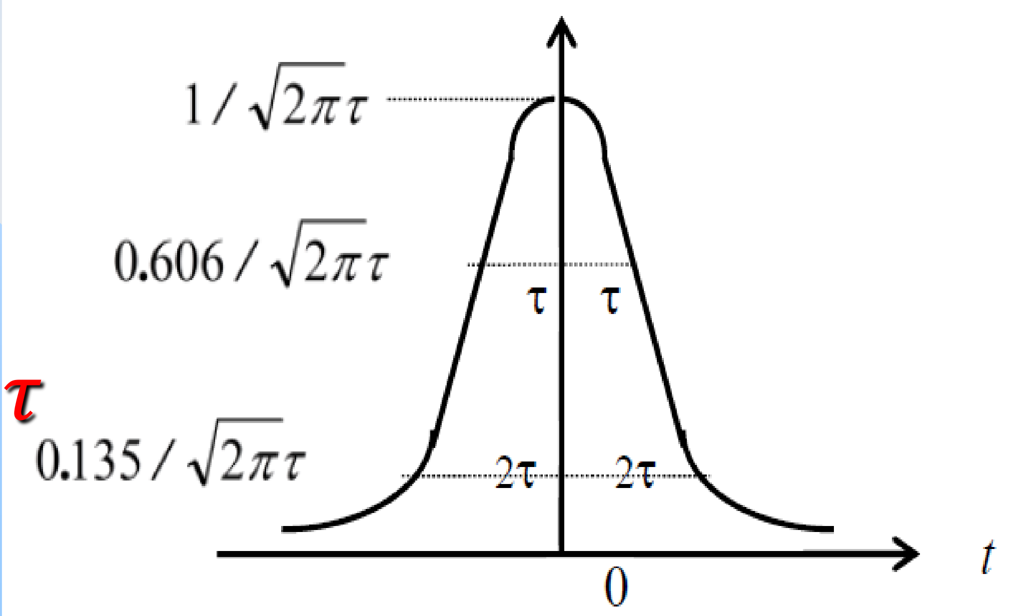
$$f(t) = \frac{\tau/\pi}{t^2+\tau^2}$$

8- Gaussian Function

Very important because so many natural experiments are characterized by random variables with Gaussian density.

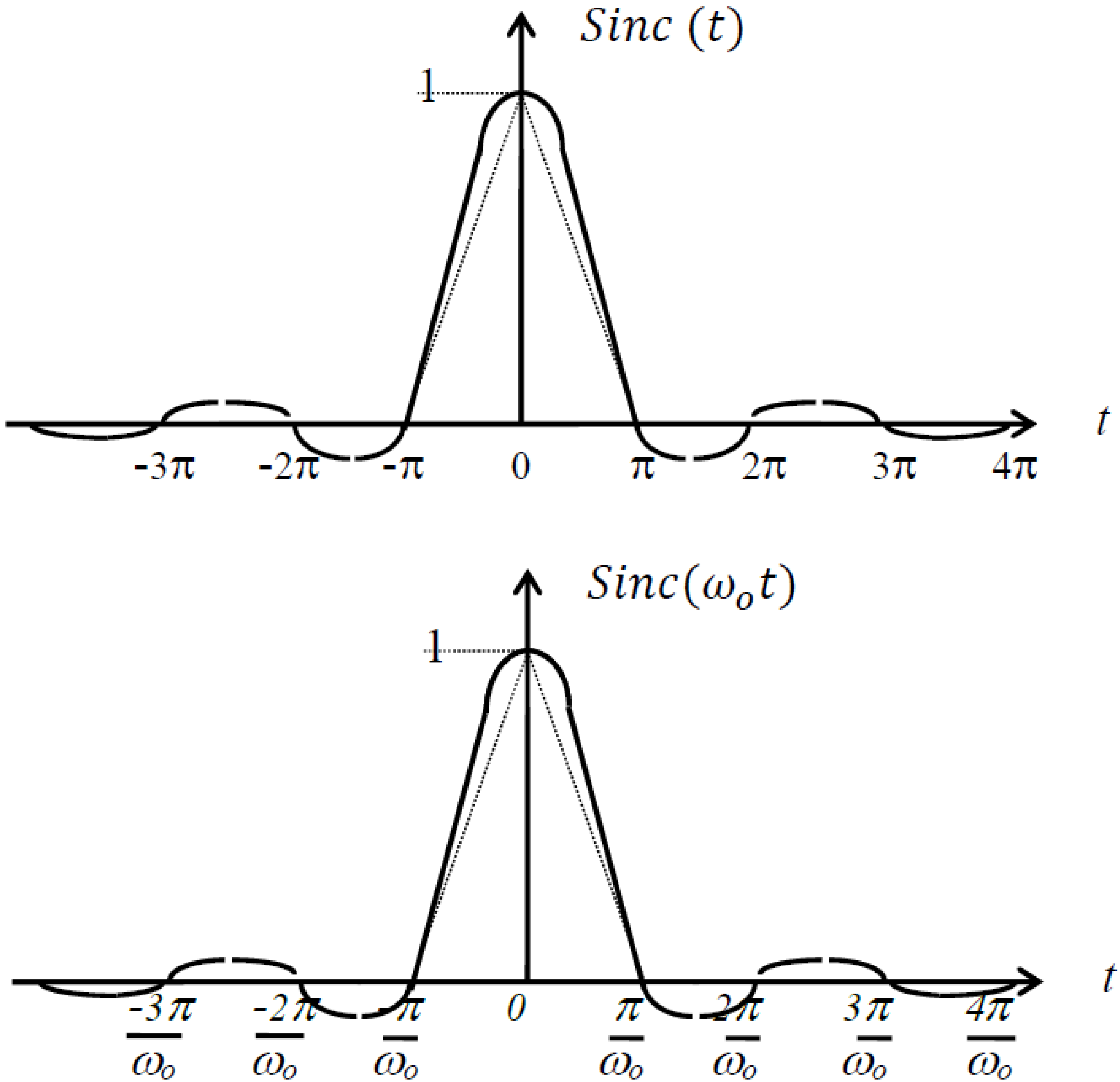
Normalized: $$f(t) = \frac{1}{\sqrt{2\pi}\,\tau}\, e^{-t^2/2\tau^2}$$

- Area =1
- Its mean is zero
- Its variance τ^2
- Standard deviation τ



9- Sampling or Sinc Function

- **Standard:** $Sa(t) = Sinc(t) = \frac{\sin t}{t}$
- Area under sampling function is equal to the area of principle triangle that equals π.
- **Normalized:** $\frac{1}{\pi}Sa(t) = \frac{1}{\pi}Sinc(t)$
- **Compressed:** $Sinc(t)$
- Difference of $Sinc(t)$ and $Sinc(\omega_o t)$ is that the second compressed with respect to first.
- The area of $Sinc(\omega_o t)$ is then π/ω_o.



Sinc (t)
1
t
-3π
-2π
-π
0
π
2π
3π
4π
Sinc(ωₒt)
1
t
-3π/ωₒ
-2π/ωₒ
-π/ωₒ
0
π/ωₒ
2π/ωₒ
3π/ωₒ
4π/ωₒ

Example 1.4: on Sinc

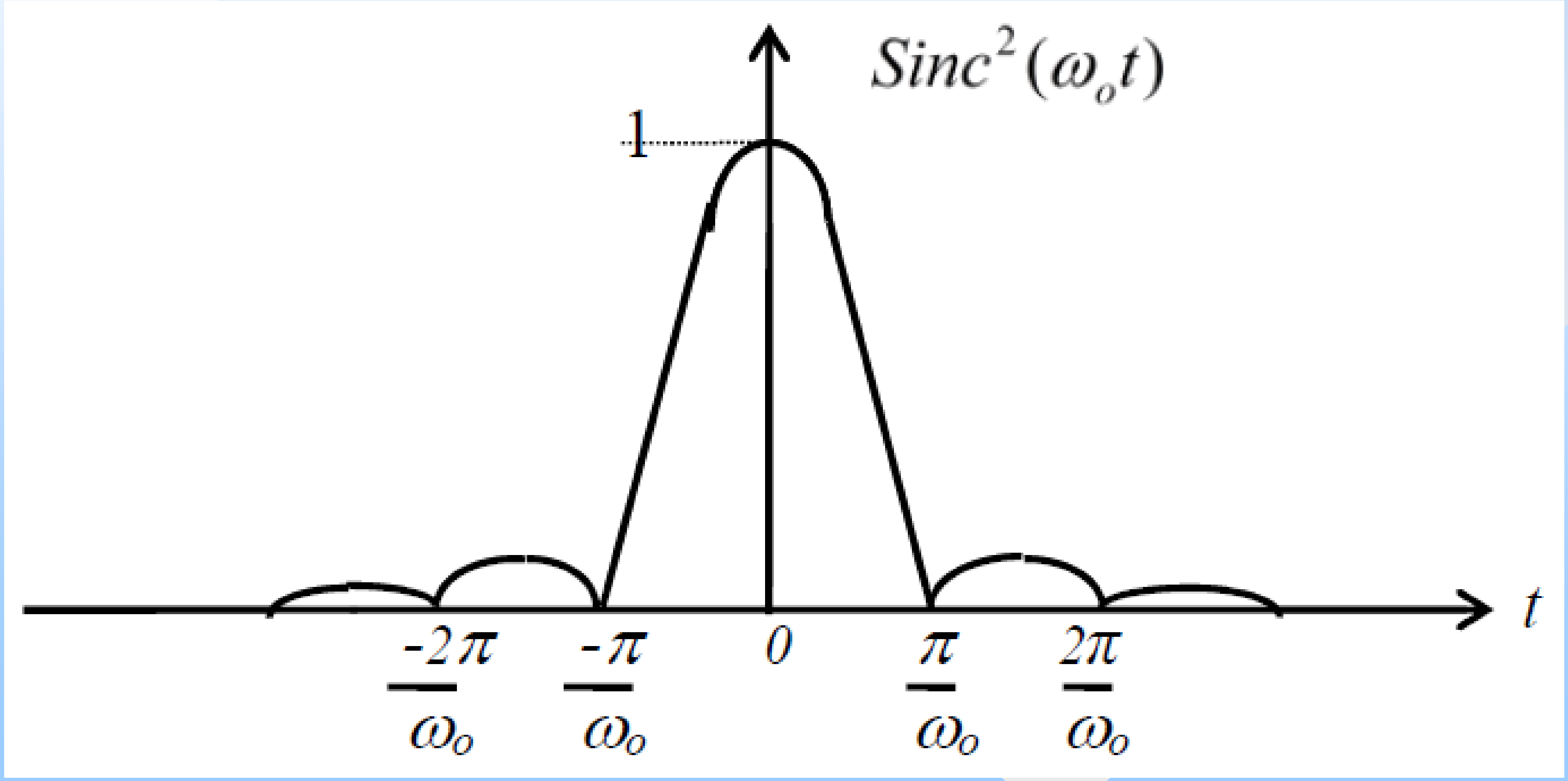
$$\text{Sinc}(\omega_o t) = \frac{\text{Sin}(\omega_o t)}{\omega_o t}$$

- The difference between $\text{Sinc}(t)$ and $\text{Sinc}(\omega_o t)$ is that the second will be compressed with respect to the first as shown.

- The area of $\text{Sinc}(\omega_o t)$ is then π/ω_o.

Sinc Square Function

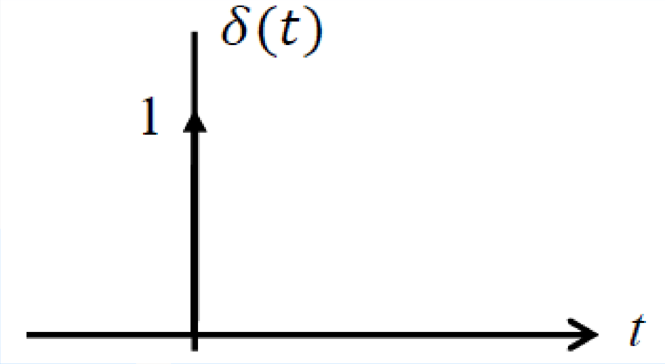
$$\mathrm{Sinc}^2(\omega_o t) = \left[\frac{\sin(\omega_o t)}{\omega_o t}\right]^2$$



10- Delta Dirac Function

- Delta function is only a mathematical function not found in nature.
- Introduced by a physical scientist "Dirac"
- He used it in quantum mechanics.
- Then has been used by mathematicians.
- It represents a phenomena occurs at certain point whereas it vanish after or before this point.

$$\delta(t) = \begin{cases} \infty & t > 0 \\ 0 & t < 0 \end{cases}$$



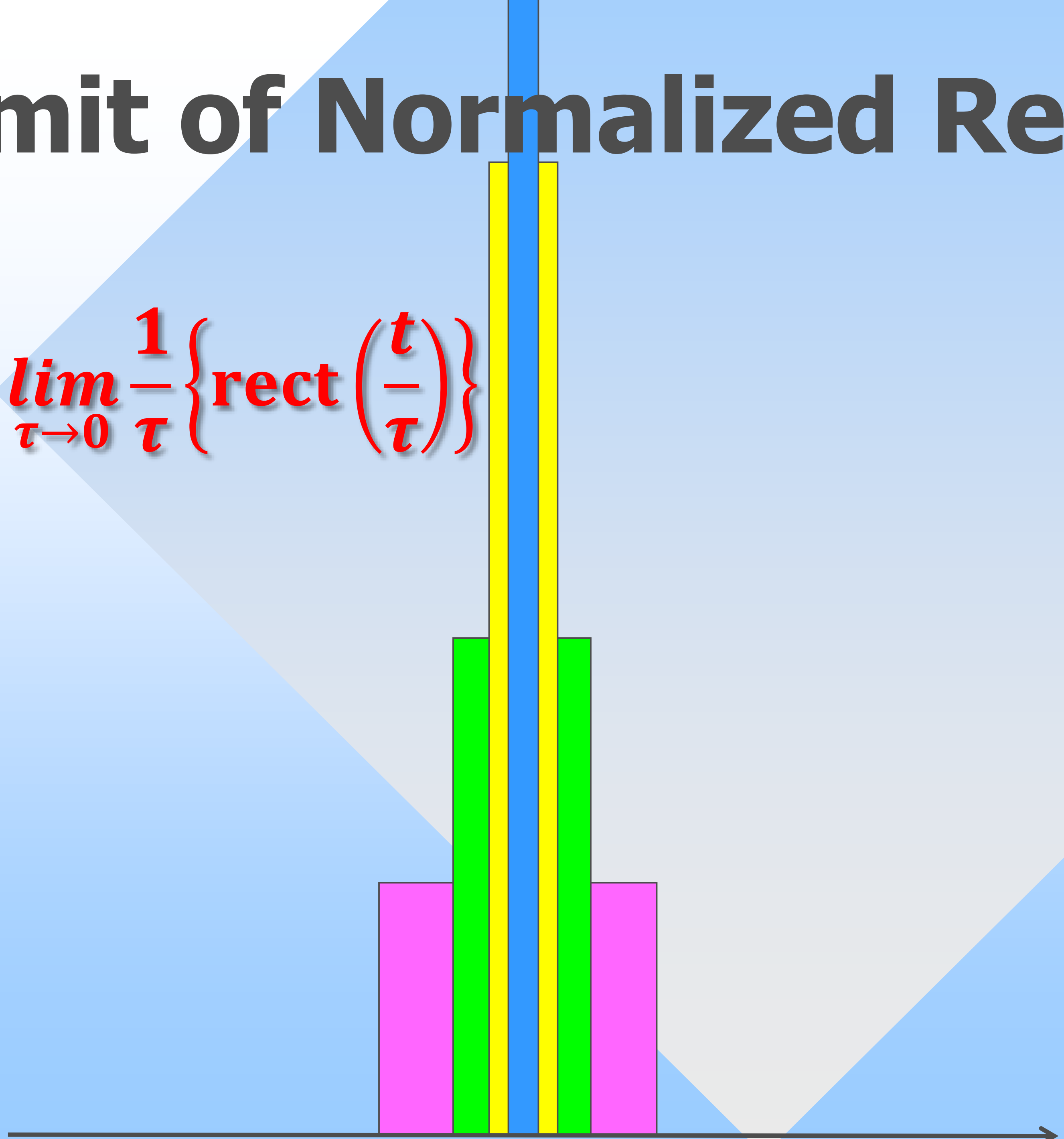
Delta as a Limit of Rect

- Delta function can be defined from the functions defined before when their widths tend to zero in the limit.
- For example:
  - Form the normalized rect function:

$$\delta(t) = \lim_{\tau \to 0} \frac{1}{\tau}\left\{\operatorname{rect}\left(\frac{t}{\tau}\right)\right\}$$

Limit of Normalized Rect

$$\delta(t) = \lim_{\tau \to 0} \frac{1}{\tau} \left\{ \mathrm{rect}\left(\frac{t}{\tau}\right) \right\}$$



Other Definitions for Delta

- Tri: $\delta(t) = \lim_{\tau \to 0} \frac{1}{\tau}\left\{\text{tri}\left(\frac{t}{\tau}\right)\right\}$
- Two Sided Exponential:

$$\delta(t) = \lim_{\tau \to 0} \frac{1}{2\tau}\left\{e^{-|t|/\tau}\right\} = \lim_{\tau \to \infty} \frac{1}{\tau}\left\{e^{-\tau|t|}\right\}$$

- Lorenz: $\delta(t) = \lim_{\tau \to 0} \frac{\tau}{\pi}\left\{\frac{1}{t^2+\tau^2}\right\}$
- Gaussian: $\delta(t) = \lim_{\tau \to 0} \frac{1}{\sqrt{2\pi}\tau}\left\{e^{-t^2/2\tau^2}\right\}$
- Sinc: $\delta(t) = \lim_{\omega \to \infty} \frac{\omega}{\pi}\left\{\frac{\sin \omega t}{\omega t}\right\}$

Properties

of

Delta

1- Unity Area

- Area under the delta function is unity:

$$\int_{-\infty}^{+\infty} \delta(t)\ dt = \int_{-0}^{+0} \delta(t)\ dt = 1$$

- Also for shifted delta:

$$\int_{t_o-0}^{t_o+0} \delta(t - t_o)\ dt = 1$$

2- Sampling Property

$$\int_{-\infty}^{+\infty} f(t)\,\delta(t-t_o)\ \mathrm{dt} = f(t_o)$$

- It means that $f(t)$ is multiplied by zero at all values of time except for t_o at which $\delta(t-t_o)$ has a value equal to infinity.
- This value when multiplied by $f(t)$ gives $f(t_o)$ only since $f(t)$ cannot varied rapidly.
- This is only valid when $f(t)$ is continuous at t_o.
- Example.1.6:

$$\int_{-\infty}^{+\infty}(t^2-3\,t+4)\ \delta(t-1)=2$$

3- Product or Multiplication

$$f(t)\,\delta(t-t_o) = f(t_o)\,\delta(t-t_o)$$

- **Example.1.7:**

$$(t^2-1)\,\delta(t) = (t^2-1)_{t=0}\,\delta(t) = -\,\delta(t)$$

- **Example.1.8:**

$$\therefore\; t\,\delta(t) = 0$$

$$\therefore\; \omega\,\delta(\omega) = 0$$

4- Unit Step Differentiation

$$\delta(t) = \acute{u}(t) = \frac{d\,u(t)}{dt}$$

Proof: Differentiation is given by:

$$\acute{u}(t) \overset{\Delta}{\rightarrow} \lim_{\tau\to 0} \frac{u\left(t+\frac{\tau}{2}\right) - u\left(t-\frac{\tau}{2}\right)}{\tau}$$

$$\therefore \acute{}u(t) = \lim_{\tau\to 0} \frac{1}{\tau}\left\{rect\left(\frac{t}{\tau}\right)\right\} = \delta(t)$$

Example 1.9

Draw following expression, then get and draw its differentiation:

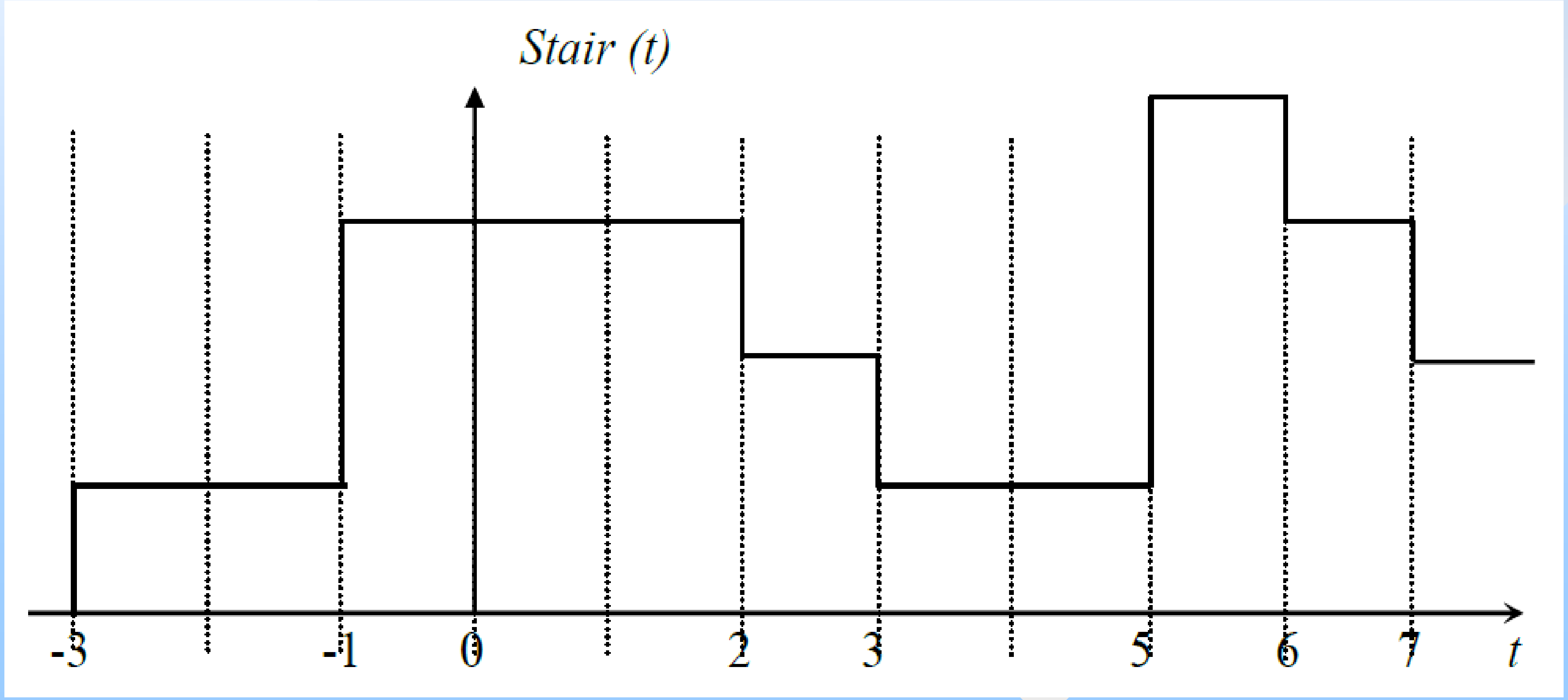
$$Stair(t) = u(t+3) + 2u(t+1) - u(t-2) - u(t-3) + 3(t-5) - u(t-6) - u(t-7)$$

Answer

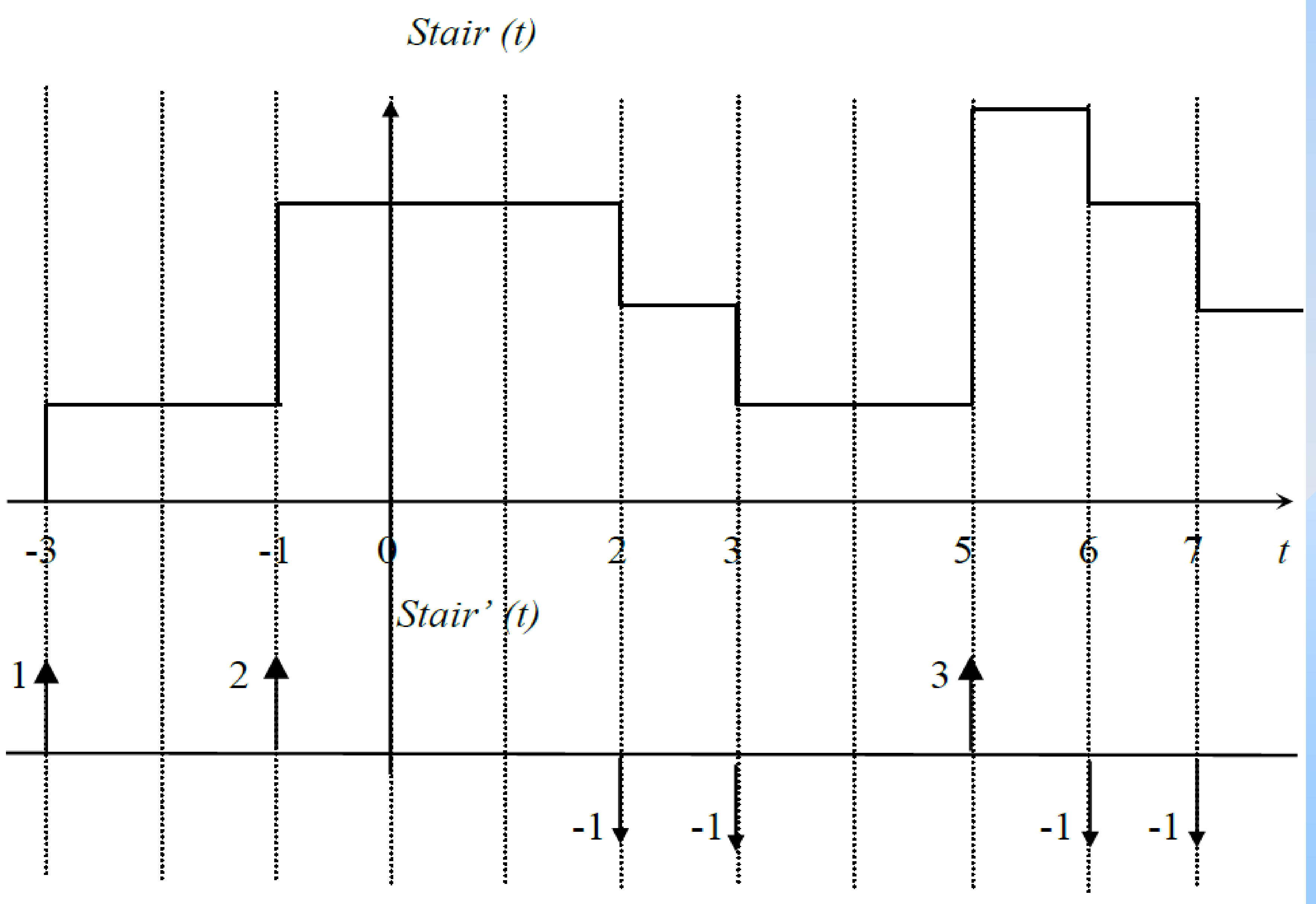
$$St\acute{a}ir(t) = \delta(t+3) + 2\delta(t+1) - \delta(t-2) - \delta(t-3) + 3\delta(t-5) - \delta(t-6) - \delta(t-7)$$

Drawing of Stair Function

$$Stair(t) = u(t+3) + 2u(t+1) - u(t-2)$$
$$-u(t-3) + 3u(t-5) - u(t-6)$$
$$-u(t-7)$$



Drawing of Differentiation

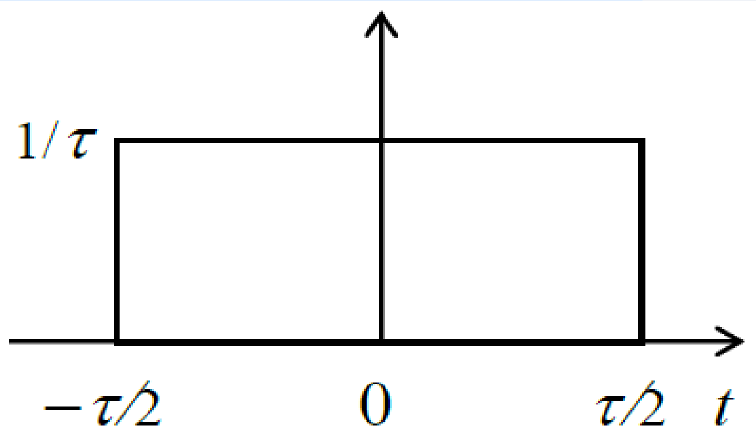


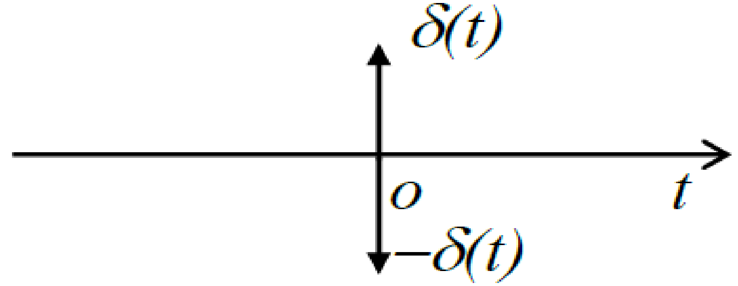
5- Differentiation (Double or Doublet Impulses

$$\frac{d}{dt}[\delta(t)] = \frac{d}{dt}\left[\lim_{\tau\to 0}\frac{1}{\tau}\left\{rect\left(\frac{t}{\tau}\right)\right\}\right]$$

$$= \frac{d}{dt}\left[\lim_{\tau\to 0}\frac{u\left(t+\frac{\tau}{2}\right)-u\left(t-\frac{\tau}{2}\right)}{\tau}\right]$$

$$= \lim_{\tau\to 0}\left\{\delta\left(t+\frac{\tau}{2}\right)-\delta\left(t-\frac{\tau}{2}\right)\right\} = \delta(t)-\delta(t)$$





6- Nth Differentiation

$$\int_{-\infty}^{+\infty} f(t)\,\delta^{n}(t)\;dt = (-1)^{n}\,f(0)$$

Examples:

$$\int_{-\infty}^{+\infty} f(t)\,\delta'(t)\;dt = -1\,f(0)$$

$$\int_{-\infty}^{+\infty} f(t)\,\delta''(t)\;dt = \;f(0)$$

7- Integration

$$\int_{-\infty}^{t} \delta(t)\; dt = \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases} = u(t)$$