Sheet 4

Gauss's Law

&

Electric flux Density

$$\Psi = Q_{enclosed}$$

E : ~~at~~ electric flux intensity (V/m)

D : flux density (C/m²)

$$\boxed{D = \varepsilon_0 E}$$

$$\oint D \cdot ds = Q_{enc} \longrightarrow \text{Gauss Law}$$

$$\nabla \cdot D = \rho_v \longrightarrow \text{Divergence theorem}$$

$$Q_{enc} = \int \rho_v \, dv = \int (\nabla \cdot D) \, dv$$

- Point Charge $\quad \bar{D} = \frac{Q}{4\pi r^2} a_r$
- line Charge $\quad \bar{D} = \frac{\rho_L}{2\pi \rho} \bar{a}_\rho$
- Surface charge $\quad \bar{D} = \frac{\rho_s}{2} \bar{a}_n$

خصائص سطح جاوس

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[1] point charges 30nc, -20nC and 10 nc are Located at (-1,0,2), (0,0,0) and (3,4,0)
The total flux leaving a sphere of a diameter 6m centered at at the origin is

$r_1 = 1 \quad < 3$

$r_2 = 0 \quad < 3$

$r_3 = \sqrt{3^2+4^2} = 5 \quad > 3$

$Q_{enclosed} = 30 - 20 = \boxed{10\,nc}$

[2] An infinite sheet has a charge density of $150\,\mu c/m^2$

The flux density is

$D = \frac{\rho_s}{2} = \boxed{75\,\mu c/m^2}$

3

$Q = 100\,\mu C \rightarrow (0,0,0)$

ψ at

1 hemi sphere of radius = 2m

$0 < \theta < \frac{\pi}{2}$, $0 < \phi < 2\pi$

2 A spherical shell defined by $\theta_1 < \theta < \theta_2$

(a)

$$\psi = Q_{enclosed} = \int D \cdot dS$$

$$= \int_0^{2\pi}\int_0^{\frac{\pi}{2}} \frac{Q}{4\pi r^2} \cdot r^2 \sin\theta \, d\phi \, d\theta$$

$$= \frac{Q}{4\pi} \int_0^{2\pi} d\phi \int_0^{\frac{\pi}{2}} \sin\theta \, d\theta = \frac{Q}{4\pi} * 2\pi * (\cos 0 - \cancel{\cos\frac{\pi}{2}})$$

(1)

$$= \frac{Q}{2} = \frac{100}{2} = \boxed{50 * 10^{-6} \text{ line}} (\cancel{wb})$$

(b)

$$\psi = \frac{Q}{4\pi} \int_0^{2\pi} d\phi \int_{\theta_1}^{\theta_2} \sin\theta \, d\theta$$

$$\frac{Q}{2} [\cos\theta_1 - \cos\theta_2]$$

$$= \boxed{50 * 10^{-6} [\cos\theta_1 - \cos\theta_2]}$$

4 cylindrical surfaces

$\rho_{s_1} = 20 \ nc/m^2$ $\rho_1 = 2 \ m$

$\rho_{s_2} = -4 \ nc/m^2$ $\rho_2 = 4 \ m$

$\rho_{s_3} = \rho_{s0} \ nc/m^2$ $\rho_3 = 6 \ m$

[a] D at 1, 3. & 5m

solution

* $\rho = 1$

for $\rho < 2 \longrightarrow Q_{enclosed} = 0$

$\boxed{D = 0}$

* $\rho = 3$

$\oint D \cdot ds = Q_{enc} = \int \rho_s \cdot ds$

$D \int_0^{2\pi}\int_0^{l} \rho \, d\phi \, dz = \int_0^{2\pi}\int_0^{l} \rho_{s_1} \cdot \rho_1 \, d\phi \, dz$

$D * 3 * \cancel{2\pi} * \cancel{l} = \rho_{s_1} * 2 * \cancel{2\pi} * \cancel{l}$

$D = \frac{\rho_{s_1} * 2}{3} = \frac{2 * 20 * 1}{3} = \boxed{13.13 \ nC/m^2 \ \bar{a}_\rho}$

* $\rho = 5$

$D \int_0^{2\pi}\int_0^{l} \rho \, d\phi \, dz = \int_0^{2\pi}\int_0^{l} \rho_{s_1} \rho_1 \, d\phi \, dz + \int_0^{2\pi}\int_0^{l} \rho_{s_2} \rho_2 \, d\phi \, dz$

$D = \frac{\rho_{s_1} * \rho_1 + \rho_{s_2} \rho_2}{\rho}$

$$D = \frac{(20 * 2) + (-4 * 4)}{5} = \boxed{4.8 \ nC/m^2 \ \bar{a}_\rho}$$

(b) ρ_{s0} such that $D = 0$ at $\rho = 7m$

$$\oint D \cdot ds = Q_{enclosed} = 0$$

$$\rho_{s1} * 2\pi \rho_1 * l + \rho_{s2} * 2\pi \rho_2 * l + \rho_{s0} * 2\pi \rho_3 * l = 0$$

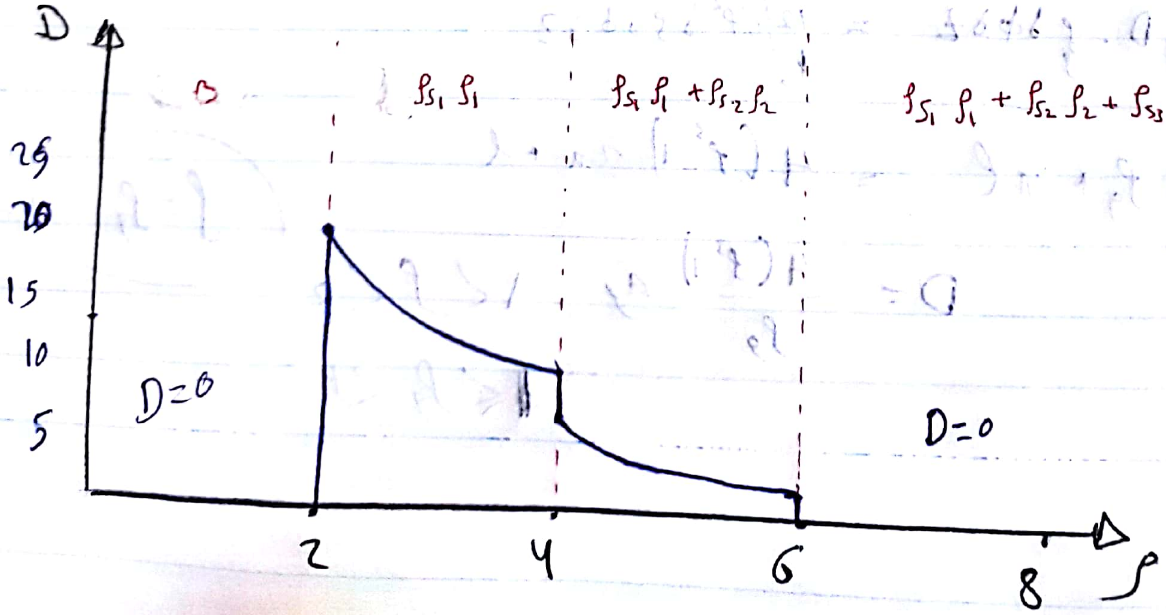
$$\rho_{s0} = -\frac{\rho_{s1}\rho_1 + \rho_{s2}\rho_2}{\rho_3} = -\frac{(2 \times 20) - (4 \times 4)}{6} \boxed{= -4 \ nc/m^2}$$

(c) Ψ at $\rho = 8$

$$\Psi = Q_{enclosed}$$

$$\rho_1, \rho_2, \rho_3 < 8$$

$$\Psi = Q_1 + Q_2 + Q_3 = \text{zero}$$



$\boxed{5}$

$$\rho_v = \begin{cases} 12\rho & nC/m^3 \quad 1<\rho<2 \\ 0 & \text{otherwise} \end{cases}$$

Dean where

$d\rho$
$\rho\, d\phi$
dz

$Q_{total} = \int \rho_v \, dv$

$= 12\rho\,(\rho\, d\rho\, d\phi\, dz)$

$= \int_1^2 \int_0^{2\pi} \int_0^{\ell} 12\rho^2\, d\rho\, d\phi\, dz$

$= 4\rho^3 \Big|_1^2 * 2\pi * \ell = 8\pi\ell * (2^3 - 1) = 56\pi\ell \quad nC$

$*\quad 0 < \rho < 1$

$Q_{enc} = 0 \qquad D = 0$

$*\quad 1 < \rho < 2$

$\int D \cdot ds = \int \rho_v \, dV$

$\int D \cdot \rho_g\, d\phi\, dz = 12 \int_1^{\rho} \int_0^{2\pi} \int_0^{\ell} \rho^2\, d\rho\, d\phi\, dz$

$D\, \rho_g * \pi\ell = 4(\rho^3 - 1)\, \cancel{2\pi * \ell}$

$D = \dfrac{4(\rho^3 - 1)}{\rho_g}\, a_\rho \qquad 1 < \rho < 2$

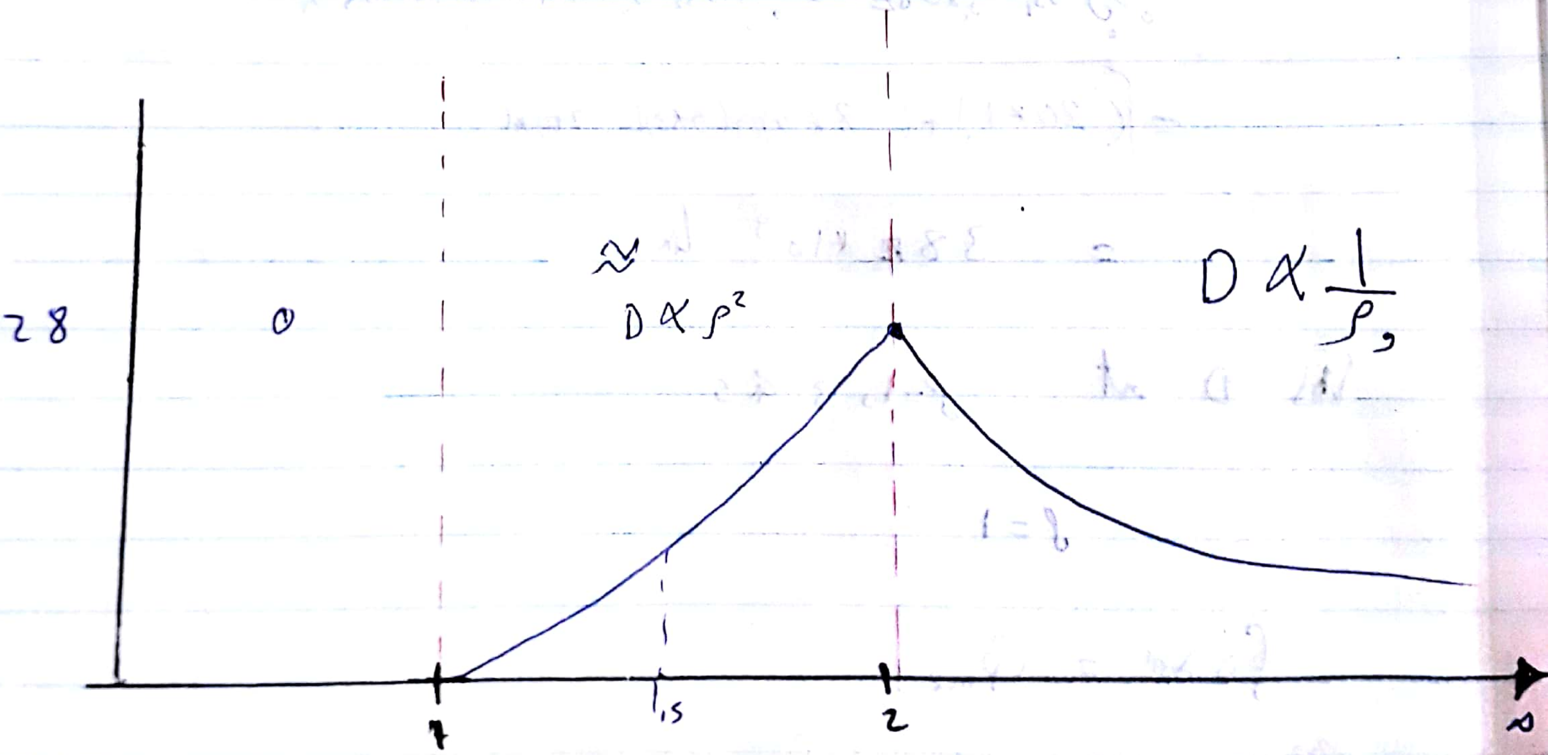
$\rho = \rho_g$

$1 < \rho_g < \rho$

* $\rho_g > 1$, $\underline{\rho_g > 2}$

$\rho = 2$

$$D = \frac{4(2^3 - 1)}{\rho_g} = \frac{28}{\rho_g} \text{ af}$$



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6 Cylindrical

$\rho_{s_1} = 20\ nC/m^2 \quad \rho_1 = 1$

$\rho_{s_2} = -8\ nC/m^2 \quad \rho_2 = 2$

$\rho_{s_3} = 5\ nC/m^2 \quad \rho_3 = 3$

a) Ψ at $\rho = 5 \qquad 0 < z < 1$

$$\Psi = Q_{enclosed}$$

$$= \int \rho_{s_1} ds + \int \rho_{s_2} ds + \int \rho_{s_3} ds$$

$$= \int_0^{2\pi}\int_0^1 \rho_{s_1}\ \rho_1 d\phi dz + \int_0^{2\pi}\int_0^1 \rho_{s_2}\ \rho_2 d\phi dz + \int_0^{2\pi}\int_0^1 \rho_{s_3}\ \rho_3 d\phi dz$$

$$= [(20*1) + (-8*2) + (5*3)]\ 2\pi*1$$

$$= 38\pi * 10^{-9} \text{ lines}$$

b) D at $\rho = 1, 3\ \&\ 5$

$\rho = 1$

$$\oint D \cdot ds' = Q_{enc}$$

$$\int_0^{2\pi}\int_0^1 D \cdot \rho\, d\phi dz = \int \rho_{s_1} \rho_1 d\phi dz$$

$$D * \cancel{1 * 2\pi * 1} = \rho_{s_1} * \cancel{1 * 2\pi * 1}$$

$$\boxed{D = \rho_{s_1} = 20\ nC/m^2}$$

$\rho = 3$

$$\oint D \cdot ds = \int \rho_{s_1} \, ds_1 + \int \rho_{s_2} \, ds_2 + \int \rho_{s_3} \, ds_3$$

$$D \int_0^{2\pi} \int_0^{1} \rho \, d\phi \, dz = \rho_{s_1} \int_0^{2\pi} \int_0^{1} \rho_1 \, d\phi \, dz + \rho_{s_2} \int \rho_2 \, d\phi \, dz + \rho_{s_3} \int_0^{2\pi} \int_0^{1} \rho_3 \, d\phi \, dz$$

$$D * \cancel{\rho} * \cancel{2\pi * 1} = \rho_{s_1} \rho * \cancel{2\pi * 1} + \rho_{s_2} \rho_2 * \cancel{2\pi * 1} + \rho_{s_3} \rho_3 * \cancel{2\pi * 1}$$

$$D = \frac{\rho_{s_1} \rho + \rho_{s_2} \rho_2 + \rho_{s_3} \rho_3}{\rho}$$

at $\rho = 3$

$$D = \frac{38 \cancel{\pi} \times \cancel{10^{-9}}}{3} \ \cancel{4\pi} \ nc/m^2$$

at $\rho = 5$

$$D = 7.6 \ nc/m^2$$

$\boxed{10}$

$\boxed{7}$

$$\rho_v = \frac{10}{r^2} \text{ mC/m}^3 \qquad 1 \leq r \leq 4$$

1) Ψ at $r=2$ & $r=6$

2) D at $r=1$ & $r=5$

$\Psi = Q_{enclosed} = \int \rho_v \, dv$

at $r = 2$

$$\Psi = \int_1^2 \int_0^\pi \int_0^{2\pi} \frac{10}{\cancel{r^2}} \cancel{r^2} \sin\theta \, dr \, d\theta \, d\phi$$

$$= 10r \Big|_1^2 \quad \cos\theta \Big|_\pi^0 \quad \phi \Big|_0^{2\pi}$$

$$= 10(2-1) \times 2 \times 2\pi = 40\pi \times 10^{-3}$$

$r = 6 > 4$

$$\Psi = \int_1^4 \int_0^\pi \int \frac{10}{r^2} r^2 \sin\theta \, dr \, d\theta \, d\phi$$

$$10r \Big|_1^4 \times 2 \times 2\pi$$

$$10(4-1) \times 4\pi = 120\pi \times 10^{-3}$$

$\boxed{11}$

$$\oint D \cdot ds = \oint \rho_v \cdot dv$$

dr

$r\sin\theta\, d\phi$

$r\, d\theta$

$$\int_0^{\pi}\int_0^{2\pi} D \cdot r_g^2 \sin\theta\, d\theta\, d\phi = \int_1^{r}\int_0^{\pi}\int_0^{2\pi} \frac{10}{\cancel{r^2}} \cancel{r^2} \sin\theta\, d\theta\, d\phi$$

$$D r_g^2 * \cancel{4\pi} = 10(r-1)\cancel{4\pi}$$

$$D = \frac{10(r-1)}{r_g^2} \qquad r = 1 \qquad r_g = 1$$

$$\boxed{D = 0 \quad mc/m^2}$$

at $r = 5$

$$D r_g^2 * \cancel{4\pi} = \cancel{120}^{30} \cancel{\pi} \times 10^{-3}$$

$$D = \frac{30 \times 10^{-3}}{r_g^2} = \frac{30 \times 10^{-3}}{(5)^2} = \boxed{1.2 \quad mc/m^2}$$

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[8] $\rho_v = 80 \mu c/m^3 \quad (8 < r < 10)$ mm

[1] Q total
[2] D_r at $r = 10$ mm
[3] D_r at $r = 20$ mm

$$Q_t = \int_8^{10}\int_0^{\pi}\int_0^{2\pi} \rho_v \, r^2 \sin\theta \, dr \, d\theta \, d\phi$$

$$= 80 \times 10^{-6} \times \frac{r^3}{3}\Big|_8^{10} \times -\cos\theta\Big|_0^{\pi} \times \phi\Big|_0^{2\pi}$$

$$= 80 \times 10^{-3} \times \frac{(10\times10^{-3})^3 - (8\times10^{-3})^3}{3} \times 2 \times 2\pi = 163.53 \text{ PC}$$

$r = 10$

$$\oint D \cdot ds = Q_{enc}$$

$$D \times 4\pi r^2 = 163.53$$

$$D = \frac{163.53}{4\pi \times (10\times10^{-3})^2} = 130.13 \text{ nC}/m^2$$

at $r = 20$ mm

$$D = \frac{163.53}{4\pi(20\times10^{-3})^2} = 32.533 \text{ nC}/m^2$$

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$Q = 10\mu C$ at $(0,0,0)$

$\rho_s = -1\mu C/m^2$ $\quad r_1 = 2$

(i) D at r=1 & r=3

(ii) ρ_v at $4<r<6$ for D to Vanish at r=7

<u>Solution</u>

$r = 1$

$Q_{enc} = 10\mu C$

$$D = \frac{Q}{4\pi r^2} a_r = \frac{10*10^{-6}}{4\pi*(1)^2} = 795.775 \ nC/m^2$$

$r = 3$

$$\int D \cdot ds = Q_{enc}$$

$$D \cdot 4\pi r^2 = Q_1 + \rho_s \iint r_1^2 \sin\theta \, d\theta \, d\phi$$

$$D = \frac{Q_1 + 4\pi r_1^2 \rho_s}{4\pi r^2} = \frac{10\times10^{-6} + (4\pi * 2^2 * -1*10^{-6})}{4\pi * (3)^2} = \frac{-4.0265*10^{-5}}{36\pi}$$

$$= -356.025 \ nC/m^2$$

$$0 = Q_1 + Q_2 + Q_3$$

$$Q_3 = -Q_1 - Q_2 = 4.0265*10^{-5} = \rho_v \int_4^6 \int_0^{\pi} \int_0^{2\pi} r^2 \sin\theta \, dr \, d\theta \, d\phi$$

$$\rho_v = \frac{-Q_1 - Q_2}{\frac{4\pi}{3}(6^3 - 4^3)} = 63.24 \ nC/m^3$$

[14]

$\boxed{10}$

$$D = \frac{2y}{z}\hat{a}_x + \frac{2x}{z}\hat{a}_y - \frac{4x}{z}a_z$$

$2 < x, y, z < 3$

(a) Volume integral

(b) Surface integral

$$\bar{\nabla}\cdot\bar{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}$$

$$= 0 + 0 - 4x\frac{\partial}{\partial z}z^{-1}$$

$$= \frac{4x}{z^2}$$

$$V = \int(\nabla\cdot D)\,dv$$

$$= \int_2^3\int_2^3\int_2^3 4x\,dx\,dy\,z^{-2}\,dz$$

$$= 2x^2\Big|_2^3\; y\Big|_2^3\; z^{-1}\Big|_3^2$$

$$2(3^2-2^2)(3-2)\left(\frac{1}{2}-\frac{1}{3}\right) = \boxed{\frac{5}{3} = 1.667}$$

$$\int D\cdot ds' = \int_{x=3} D_x\,dy\,dz\,\hat{a}_x + \int_{x=2} D\cdot dy\,dz\,(-\hat{a}_x)$$

$$+ \int_{y=3} D\cdot dx\,dz\,\hat{a}_y + \int_{y=2} D\cdot dx\,dz\,(-\hat{a}_y)$$

$$+ \int_{z=3} D\cdot dx\,dy\,\hat{a}_z + \int_{z=2} D\cdot dx\,dy\,(-\hat{a}_z)$$

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TSL

$$= \iint \cancel{\frac{2y}{z}\, dy\, dz} - \iint \cancel{\frac{2y}{z}}\, dy\, dz + \iint \cancel{\frac{2x}{z}\, dy\, dz} - \iint \cancel{\frac{2x}{z}\, dy\, dz}$$

$$\int_2^3\int_2^3 -\frac{4x}{3}\, dx\, dy + \int_2^3\int_2^3 \frac{4x}{2}\, dx\, dy$$

$$-\frac{2x^2 y}{3}\Big|_2^3\Big|_2^3 + \frac{2x^2 y}{3}\Big|_2^3\Big|_2^3$$

$$\frac{2(3^2-2^2)(3-2)}{2} - \frac{2(3^2-2^2)(3-2)}{3} = \frac{10}{2} - \frac{10}{3} = \boxed{\frac{5}{3}}$$

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$\boxed{11}$ $D = 15\rho^2 \sin 2\phi \, \bar{a}_\rho + 20\rho^2 \cos 2\phi \, \bar{a}_\phi$ C/m²

Satisfy Divergence theorem

$1 < \rho < 2$ m

$1 < \phi < 2$ rad

$1 < z < 2$ m

$$\rho_v = \nabla \cdot D = \frac{1}{\rho}\frac{\partial(\rho * 15\rho^2 \sin 2\phi)}{\partial\rho} + \frac{1}{\rho}\frac{\partial}{\partial\phi} 20\rho^2 \cos 2\phi$$

$$= \frac{45\rho^2 \sin 2\phi}{\rho} - \frac{40\rho^2 \sin 2\phi}{\rho} = 5\rho \sin(2\phi)$$

$$Q_{enc} = \int \rho_v \, dv = \int_1^2\int_1^2\int_1^2 5\rho \sin(2\phi) \, \rho \, d\rho \, d\phi \, dz$$

$$= \frac{5}{3}\rho^3\Big|_1^2 \quad \frac{\cos 2\phi}{2}\Big|_2^1 * z\Big|_1^2$$

$$= \frac{35}{3} * 0.1187 * 1 = \boxed{1.38539\,C}$$

$$Q_{enc} = \int D \cdot ds$$

$$= \iint_{\rho=2} 15\rho^2 \sin 2\phi \, \rho \, d\phi \, dz - \iint_{\rho=1} 15\rho^2 \sin 2\phi \, \rho \, d\phi \, dz$$

$$+ \iint_{\phi=2} 20\rho^2 \cos 2\phi \, d\rho \, dz - \iint_{\phi=1} 20\rho^2 \cos 2\phi \, d\rho \, dz$$

$\boxed{17}$

$$= \frac{15}{3}\left(2^3 - 1^3\right) \int_1^2\int_1^2 \sin 2\phi \, d\phi \, dz + 20\left[\cos(4) - \cos 2\right] \int_1^2\int_1^2 \rho^2 \, d\rho \, dz$$

$$\frac{15}{3} \times (2-1) \times \frac{\cos(2) - \cos(4)}{2} + 20\left(\cos(4) - \cos(2)\right) \times \frac{1}{3}\left(2^3 - 1^3\right)(2-1)$$

$$= 1.38539 \, C$$

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$$D = \frac{16}{r}\cos 2\theta \; a_\theta \quad C/m^2$$

$1 < r < 2$ m

$1 < \theta < 2$ rad

$1 < \phi < 2$ rad

Q total

$$\because \nabla \cdot D = \frac{1}{r^2}\frac{\partial r^2 D_r}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial \sin\theta D_\theta}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial D_\phi}{\partial \phi}$$

$$\rho_v = \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}\left[\frac{16}{r}\cos 2\theta \sin\theta\right]$$

$$= \frac{16 \quad \cos\theta\cos 2\theta - 2\sin\theta\sin 2\theta}{r^2 \sin\theta}$$

$$Q = \int \rho_v \, dv = \iiint \frac{16(\cos\theta\cos 2\theta - \sin\theta\sin 2\theta)}{\cancel{r^2\sin\theta}} \cancel{r^2\sin\theta}\, dr\, d\theta\, d\phi$$

$$= r\Big|_1^2 \; \phi\Big|_1^2 \left[\frac{\sin 3\theta}{3}\Big|_1^2 - \int 2\sin^2\theta\cos\theta\, d\theta\right]$$

$$= 16 \times (2-1)(2-1)\left[\left[\frac{\sin 6 - \sin 3}{3}\right] - \frac{2}{3}\sin^3\theta\Big|_1^2\right]$$

$$\boxed{= -3.9068 \; C}$$

$Q_{enc} = \oint D \cdot ds$

dr

$r\, d\theta$

$r \sin\theta\, d\phi$

$= \int \frac{16}{r} \cos 2\theta \times r \sin\theta\, dr\, d\phi \Big|_{\theta=2} - \int \frac{16}{r} \cos 2\theta \times r \sin\theta\, dr\, d\phi \Big|_{\theta=1}$

$\cancel{16 \left[\cos(2) - \cos(1)\right] \int}$

$16 \left[\cos 4 \sin 2 - \cos 2 \sin 1\right] \; r\Big|_1^2 \; \phi\Big|_1^2$

$$\boxed{= -3.90689 \quad C}$$

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