Numerical Analysis
Chapter2

Numerical differentiation and Integration

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Numerical differentiation:

- **Approximation of First derivatives:**
- **Forward difference.**
- **Backward difference.**
- **Central difference.**
- **Approximation of second derivatives:**
- **Forward difference.**
- **Backward difference.**
- **Central difference.**

Approximation of First derivatives:

-Forward difference.

Given function by using Taylor's expansion: $f(x) \Rightarrow$

$$f(x+h) = f(x) + f'(x)h + f''(x)\frac{h^2}{2!} + 0\left(h^3\right)$$

$$f'(x)h = f(x+h) - f(x) - f''(x)\frac{h^2}{2!} + 0\left(h^3\right)$$

$$f'(x) = \frac{f(x+h) - f(x)}{h} - \frac{1}{2}hf''(x) + 0\left(h^2\right)$$

***Forward differentiation* formula for $f'(x)$**

$$f'(x) \approx \frac{f(x+h) - f(x)}{h}$$

Error = 0 (h)

$$T.E \le \frac{1}{2}h\left|f''(c)\right|, x \le c \le x+h$$

Backward difference:

$$f(x-h)=f(x)-f'(x)h+f''(x)\frac{h^2}{2!}+0\left(h^3\right)$$

$$f'(x)h=f(x)-f(x-h)+f''(x)\frac{h^2}{2!}+0\left(h^3\right)$$

$$f'(x)=\frac{f(x)-f(x-h)}{h}+\frac{1}{2}hf''(x)+0\left(h^2\right)$$

__Back ward differentiation__ **__formula__** **for** $f'(x)$

$$f'(x)\approx\frac{f(x)-f(x-h)}{h}$$

$$\text{Error}=0\left(h\right)$$

$$T.E\leq\frac{1}{2}h\left|f''(c)\right|,x-h\leq c\leq x$$

Central difference:

Central difference differentiation formula for $f'(x)$

$$f(x+h)=f(x)+hf'(x)+\frac{h^2}{2!}f''(x)+\frac{h^3}{3!}f'''(x)+0\left(h^4\right)$$

$$f(x-h)=f(x)-hf'(x)+\frac{h^2}{2!}f''(x)-\frac{h^3}{3!}f'''(x)+0\left(h^4\right)$$

$$\therefore f(x+h)-f(x-h)=2hf'(x)+2\frac{h^3}{3!}f''(x)+0\left(h^5\right)$$

$$2hf'(x)=f(x+h)-f(x-h)-\frac{1}{3}f''(x)h^3+0\left(h^5\right)$$

$$f'(x)\approx\frac{f(x+h)-f(x-h)}{2h}$$

$$\text{Error}=0\left(h^2\right)$$

$$T.E\le\frac{h^2}{3!}\left|f'''(c)\right|, x-h\le c\le x+h$$

Example (1):

Given $f(x) = e^x$ approximate $f'(1.5)$ using forward, back ward and central formulas with h= 0.1 compare results with the exact value $f'(x) = e^{1.5}$

Solution:

Forward differentiation

$$f'(x) \approx \frac{f(x+h) - f(x)}{h}$$

$$f'(1.5) \approx \frac{e^{1.6} - e^{1.5}}{0.1} \approx 4.713433$$

Back ward differentiation

$$f'(x) \approx \frac{f(x) - f(x-h)}{h}$$

$$f'(1.5) \approx \frac{e^{1.5} - e^{1.4}}{0.1} \approx 4.264891035$$

h=0.1

X – h=

1.5-0.1=1.4

X=1.5

Central difference differentiation

$$f'(x) \approx \frac{f(x+h) - f(x-h)}{2h}$$

$$f'(1.5) \approx \frac{e^{1.6} - e^{1.4}}{2 \times 0.1} \approx 4.4891623$$

Absolute errors are:

Forward differentiation $\left|e^{1.5} - 4.713433\right| = 0.231744$

Back ward differentiation $\left|e^{1.5} - 4.264891\right| = 0.216798$

Central difference differentiation $\left|e^{1.5} - 4.4891623\right| = 0.007473$

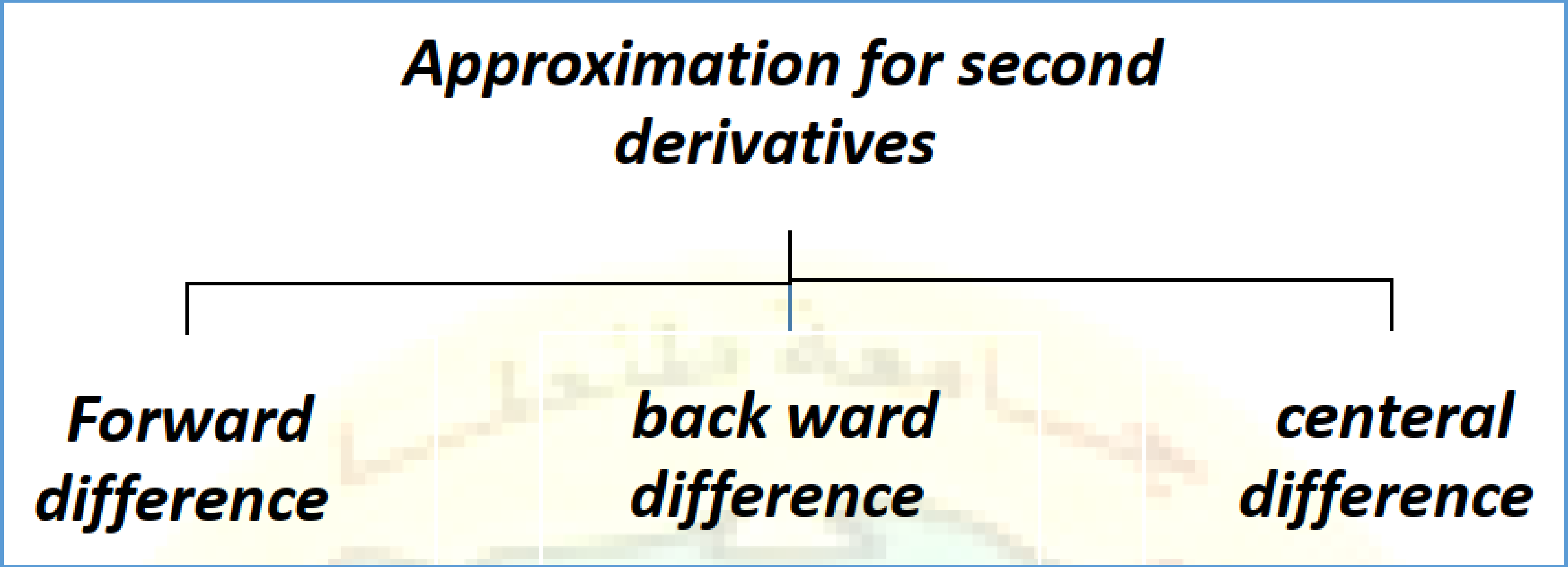
Truncation errors

$$T.E \le \frac{h}{2}\left|f''(c)\right| \le \frac{0.1}{2}e^{1.6} \approx 0.2477, 1.5 \le c \le 1.6$$

$$T.E \le \frac{h}{2}\left|f''(c)\right| \le \frac{0.1}{2}e^{1.5} \approx 0.22408, 1.4 \le c \le 1.5$$

$$T.E \le \frac{h^2}{3!}\left|f'''(c)\right| \le \frac{(0.1)^2}{3!}e^{1.6} \approx 0.0083, 1.4 \le c \le 1.6$$

Approximation of second derivatives:



Forward difference formula for $f\,''(x)$

$$f(x+2h)=f(x)+f'(x)2h+\frac{1}{2}f''(x)(2h)^2+\frac{1}{6}f'''(x)(2h)^3+0\left(h^4\right)$$

$$f(x+h)=f(x)+f'(x)h+\frac{1}{2}f''(x)(h)^2+\frac{1}{6}f'''(x)(h)^3+0\left(h^4\right)$$

$$\therefore f(x+2h)-2f(x+h)=-f(x)+f''(x)h^2\left[2-2\times\frac{1}{2}\right]+f'''(x)(h)^3\left[\frac{4}{3}-\frac{1}{3}\right]+0\left(h^4\right)$$

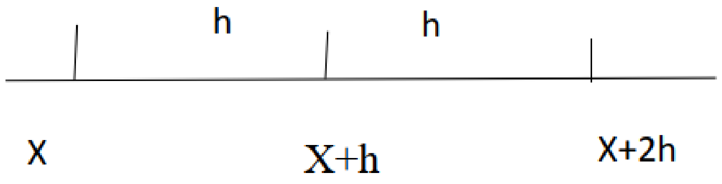
$$\therefore f''(x)h^2=f(x+2h)-f(x+h)+f(x)+f'''(x)h^3+0\left(h^4\right)$$

$$\therefore f''(x)=\frac{f(x+2h)-f(x+h)+f(x)+f'''(x)h^3+0\left(h^4\right)}{h^2}$$

$$\therefore f''(x)=\frac{f(x+2h)-f(x+h)+f(x)}{h^2}+hf'''(x)+0\left(h^2\right)$$

$$f\,''(x)\approx\frac{f(x+2h)-2f(x+h)+f(x)}{h^2}$$

Error $=0(h)$ $\qquad T.E\le h\left|f'''(c)\right|,\quad x\le c\le x+2h$



Backward difference formula for $f''(x)$

$$f(x-2h) = f(x) - f'(x)2h + \frac{1}{2}f''(x)(2h)^2 + \frac{1}{3!}f'''(x)(2h)^3 + 0\left(h^4\right)$$

$$f(x-h) = f(x) - f'(x)h + \frac{1}{2}f''(x)(h)^2 - \frac{1}{3!}f'''(x)(h)^3 + 0\left(h^4\right) \rightarrow \times -2$$

$$\therefore f(x-2h) - 2f(x-h) = -f(x) + f''(x)h^2\left[2 - 2\times\frac{1}{2}\right] - f'''(x)(h)^3\left[\frac{4}{3} - \frac{1}{3}\right] + 0\left(h^4\right)$$

$$\therefore f''(x)h^2 = f(x-2h) - 2f(x-h) + f(x) + f'''(x)h^3 + 0\left(h^4\right)$$

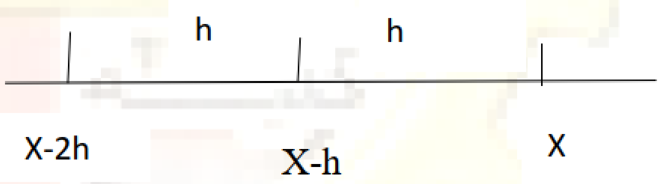
$$\therefore f''(x) = \frac{f(x-2h) - 2f(x-h) + f(x) + f'''(x)h^3 + 0\left(h^4\right)}{h^2}$$

$$\therefore f''(x) = \frac{f(x-2h) - 2f(x-h) + f(x)}{h^2} + hf'''(x) + 0\left(h^2\right)$$

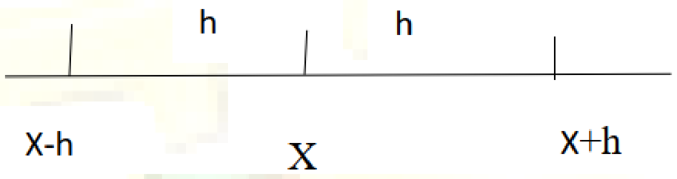
$$f''(x) \approx \frac{f(x-2h) - 2f(x-h) + f(x)}{h^2}$$

Error $= 0(h)$

$$T.E \le h\left|f'''(c)\right|, \quad x-2h \le c \le x$$



Central difference formula for $f''(x)$



$$f(x+h) = f(x) + f'(x)h + \frac{1}{2}f''(x)(h)^2 + \frac{1}{6}f'''(x)(h)^3 + \frac{1}{4!}f^{(4)}(x)(h)^4 + 0(h^5)$$

$$f(x-h) = f(x) - f'(x)h + \frac{1}{2}f''(x)(h)^2 - \frac{1}{6}f'''(x)(h)^3 - \frac{1}{4!}f^{(4)}(x)(h)^4 + 0(h^5) \rightarrow$$

$$\therefore f(x+h) + f(x-h) = 2f(x) + f''(x)h^2 + \frac{1}{12}f''''(x)h^4 + 0(h^6)$$

$$f''(x) \approx \frac{f(x+h) - 2f(x) + f(x-h)}{h^2}$$

$$\text{Error} = 0(h^2)$$

$$T.E \le \frac{h^2}{12}\left|f^{(4)}(c)\right|, \quad x-h \le c \le x+h$$

Example (2):

Let $f(x) = \sin x$, use forward, back ward and central difference formulas with h= 0.1 to approximate $f''(0.5)$. Compare with the true results $f''(0.5) = -0.47942554$, then find T.E

Solution:

Forward difference formula for $f''(x)$

h= 0.1 h=0.1

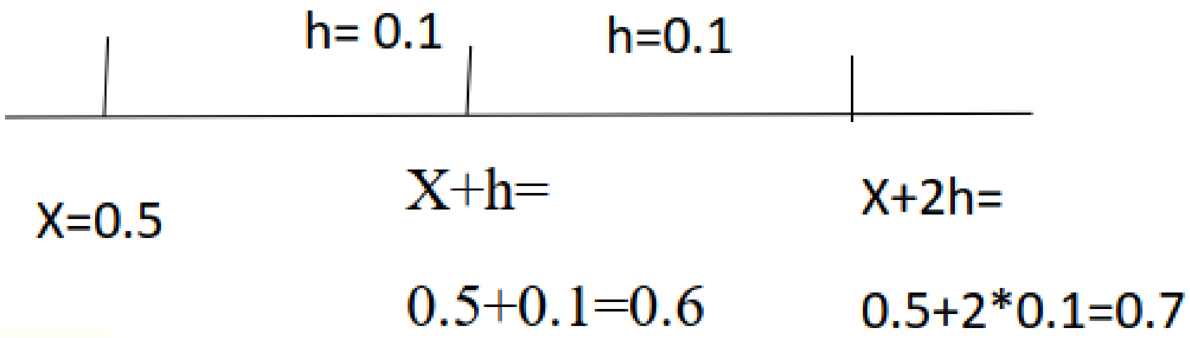
X=0.5 X+h= 0.5+0.1=0.6 X+2h= 0.5+2*0.1=0.7

$$f''(x) \approx \frac{f(x+2h) - 2f(x+h) + f(x)}{h^2}$$

$$f''(x) \approx \frac{\sin(0.7) - 2\sin(0.6) + \sin(0.5)}{(0.1)^2} = -0.564172$$

Solution:

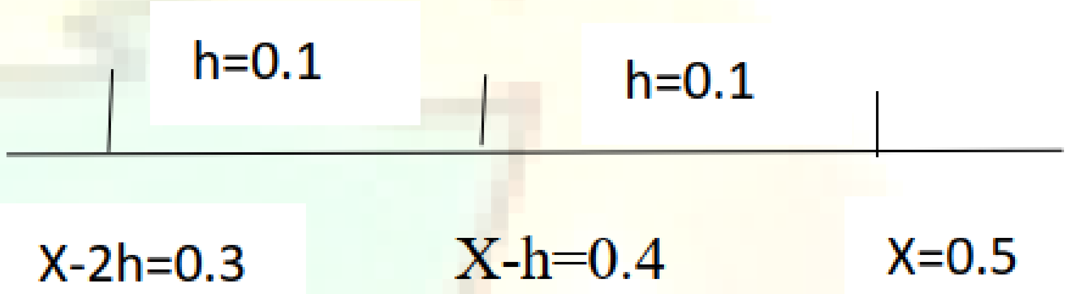
Forward difference formula for $f''(x)$



$$f''(x) \approx \frac{f(x+2h) - 2f(x+h) + f(x)}{h^2}$$

$$f''(x) \approx \frac{\sin(0.7) - 2\sin(0.6) + \sin(0.5)}{(0.1)^2} = -0.564172$$

Back ward difference formula for $f''(x)$



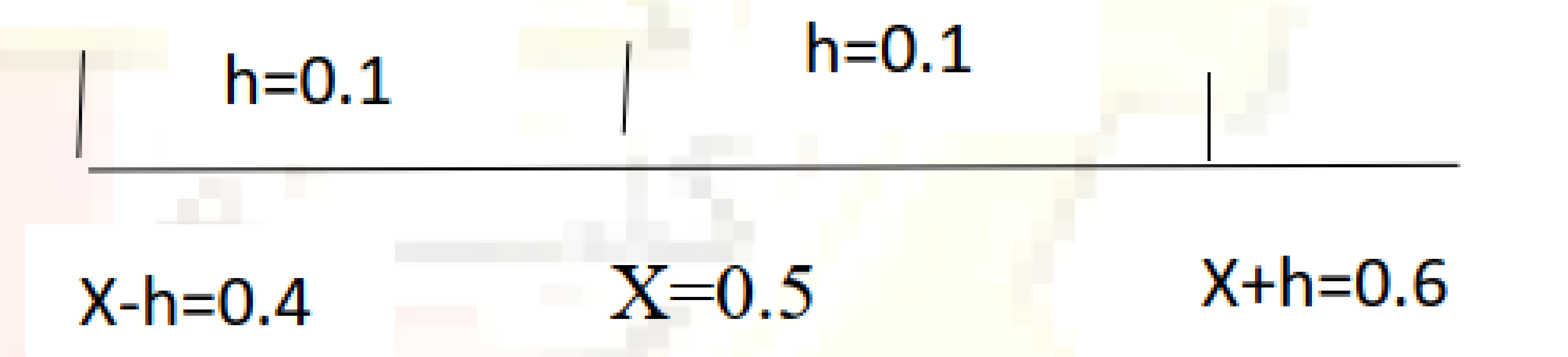
$$f''(x) \approx \frac{f(x-2h) - 2f(x-h) + f(x)}{h^2}$$

$$f''(x) \approx \frac{\sin(0.3) - 2\sin(0.4) + \sin(0.5)}{(0.1)^2} = -0.389094$$

Central difference formula for $f''(x)$

$$f''(x) \approx \frac{f(x+h) - 2f(x) + f(x-h)}{h^2}$$

$$f''(x) \approx \frac{\sin(0.6) - 2\sin(0.5) + \sin(0.4)}{(0.1)^2} \approx -0.479027$$



Absolute Error

$$f(x) = \sin x$$
$$f'(x) = \cos(x)$$
$$f''(x) = -\sin(x)$$
$$\therefore f''(x) = -\sin(0.5 \times \frac{180}{\pi}) = -\sin(28.647885)$$
$$\therefore f''(x) = -0.4794255$$

Forward difference

$$\text{Absolute Error} = \left| -\sin(0.5) + 0.564172 \right| = 0.0847465$$

Back ward difference

$$\text{Absolute Error} = \left| -\sin(0.5) + 0.389094 \right| = 0.0903315$$

Central difference

$$\text{Absolute Error} = \left| -\sin(0.5) - (-0.479027) \right| = 0.000398$$

Truncation Error

Forward difference

$$T.E \le h\left|f'''(c)\right|, x \le c \le x+2h$$

$$T.E \le 0.1\left|\cos(0.5+\frac{180}{\pi})\right|, \quad 0.5 \le c \le 0.7$$

$$T.E = 0.8775856$$

$$f(x) = \sin x$$

$$f'''(x) = -\cos(x)$$

$$f'''(0.7) = -0.76481$$

$$f'''(0.5) = -0.8775856$$

Back ward difference

$$T.E \le h\left|f'''(c)\right|, x-2h \le c \le x$$

$$T.E \le 0.1\left|(-\cos(0.3\times\frac{180}{\pi})\right|, 0.3 \le c \le 0.5$$

$$T.E = 0.0955336$$

$$f(x) = \sin x$$

$$f'''(x) = -\cos(x)$$

$$f'''(0.3) = -\cos(0.3\times\frac{180}{\pi})$$

$$f'''(0.3) = 0.955336$$

Central difference

$$T.E \le \frac{h^2}{12}\left|f^{(4)}(c)\right|, x-h \le c \le x+h$$

$$T.E \le \frac{1}{12}(0.1)^2\left|f^{(4)}(0.6)\right|, 0.4 \le c \le 0.6$$

$$T.E \le \frac{1}{12}(0.1)^2\left|(0.5646425)\right| \le 0.00047035$$

$$f(x) = \sin x$$

$$f'''(x) = -\cos(x)$$

$$f^{(4)}(x) = \sin(x)$$

$$\therefore f^{(4)}(c) = \sin(0.6), \; 0.4 \le c \le 0.6$$

Richardson Extrapolation:

Forward difference $\approx o(h)$

Backward difference $\approx o(h)$

Central difference $\approx o(h^2)$

Our target T.E $\approx o(h^4)$ or $o(h^6)$

$$f(x+h)=f(x)+hf'(x)+\frac{h^2}{2!}f''(x)+\frac{h^3}{3!}f'''(x)+\frac{h^4}{4!}f^4(x)+\frac{h^5}{5!}f^5(x)+\ldots\ldots\ldots$$

$$f(x-h)=f(x)-hf'(x)+\frac{h^2}{2!}f''(x)-\frac{h^3}{3!}f'''(x)+\frac{h^4}{4!}f^4(x)-\frac{h^5}{5!}f^5(x)+\ldots\ldots\ldots$$

$$\therefore f(x+h)-f(x-h)=$$

$$\left[f(x)+hf'(x)+\frac{h^2}{2!}f''(x)+\frac{h^3}{3!}f'''(x)+\frac{h^4}{4!}f^4(x)+\frac{h^5}{5!}f^5(x)\right]$$

$$-\left[f(x)-hf'(x)+\frac{h^2}{2!}f''(x)-\frac{h^3}{3!}f'''(x)+\frac{h^4}{4!}f^4(x)-\frac{h^5}{5!}f^5(x)\right]$$

$$\therefore f(x+h)-f(x-h)=2hf'(x)+2\frac{h^3}{3!}f'''(x)+2\frac{h^5}{5!}f^5(x)+.........$$

$$\therefore \frac{f(x+h)-f(x-h)}{2h}=f'(x)+\frac{h^2}{3!}f'''(x)+\frac{h^4}{5!}f^5(x)$$

$$f'(x)=\frac{f(x+h)-f(x-h)}{2h}-h^2\frac{f'''(x)}{3!}-h^4\frac{f^5(x)}{5!}$$

$$f'(x)=\Phi(h)+c_2h^2+c_4h^4.....\rightarrow 1$$

Replace $h=\frac{h}{2}$

$$f'(x) = \Phi(\frac{h}{2}) + c_2 \frac{h^2}{4} + c_4 \frac{h^4}{16} \ldots\ldots \to 2$$

لحل 1 مع 2 للتخلص من h^2

$$(4 \times 2 - 1)$$

$$3f'(x) = 4\Phi(\frac{h}{2}) - \Phi(h) + 4c_2 \frac{h^2}{4} - c_2 h^2 + 4c_4 \frac{h^4}{16} - c_4 h^4 + \ldots\ldots$$

$$\therefore f'(x) = \frac{4}{3}\Phi(\frac{h}{2}) - \frac{1}{3}\Phi(h) - \frac{1}{4} c_4 h^4 + \ldots\ldots\ldots\ldots$$

$$\therefore f'(x) \cong \frac{4}{3}\Phi(\frac{h}{2}) - \frac{1}{3}\Phi(h) \text{ error} \approx o\left(h^4\right)$$

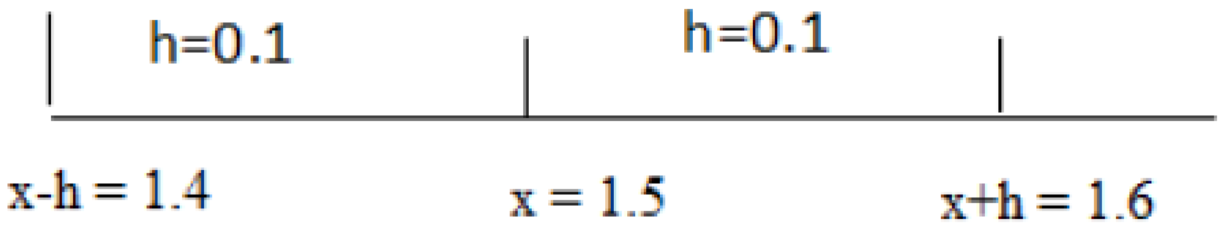
$$T.E \le \left| \frac{1}{4} c_4 h^4 \right|$$

$$T.E \le \left| \frac{h^4}{4} \frac{f^{(5)}(x)}{5!} \right| \le \left| \frac{h^4}{480} f^{(5)}_{\max}(\rho) \right| \to (x-h) \le \rho \le (x+h)$$

Example (8):

Given $f(x) = e^x$, approximate $f'(1.5)$ using Richard Extrapolation formula with h= 0.1. Compare with the value of $f'(1.5) = e^{1.5}$, then find Truncation Error.

Solution:



$$\Phi(h) = \frac{f(x+h) - f(x-h)}{2h}$$

$$\Phi(0.1) = \frac{e^{(1.5+0.1)} - e^{(1.5-0.1)}}{2\times 0.1} \cong \frac{e^{(1.6)} - e^{(1.4)}}{2\times 0.1} \cong 4.4891623$$

$$\Phi\left(\frac{h}{2}\right) = \frac{f\left(x+\frac{h}{2}\right) - f\left(x-\frac{h}{2}\right)}{2\frac{h}{2}}$$

$$\Phi(0.1) = \frac{e^{\left(1.5+\frac{0.1}{2}\right)} - e^{\left(1.5-\frac{0.1}{2}\right)}}{2\times\frac{0.1}{2}} \cong \frac{e^{(1.55)} - e^{(1.45)}}{2\times\frac{0.1}{2}} \cong 4.4835566$$

$$\because f'(x) \cong \frac{4}{3}\Phi\left(\frac{h}{2}\right) - \frac{1}{3}\Phi(h)$$

$$\therefore f'(x) \cong \frac{4}{3}(4.4835566) - \frac{1}{3}\Phi(4.4891623) = 4.556351813$$

$$\text{Absolute error} = \left|e^{1.5} - 4.481688136\right| = 9.3406\times10^{-7}$$

$$T.E \le \left|\frac{h^4}{4}\frac{f^{(5)}(x)}{5!}\right| \le \left|\frac{h^4}{480} f^{(5)}_{\max}(\rho)\right| \rightarrow 1.4 \le \rho \le 1.6$$

$$f^{(5)}(x) = e^x \rightarrow \therefore f^{(5)}(\rho) = e^{1.6}$$

$$T.E \le \left|\frac{(0.1)^4}{480} e^{1.6}\right| = 1.0319\times10^{-5}$$

Thank you