Numerical Analysis
Lec 5

Chapter 3

Ordinary Differential Equations

(Initial Value Problems)

• Ordinary Differential Equations (Initial Value Problems) (I.V.P)

Topics

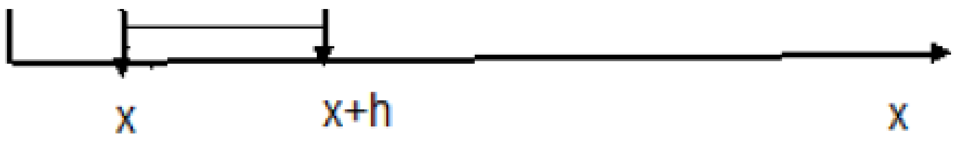
One – step method's:

- Euler's Method.
- Taylors Method.
- Runge-Kutta Methods.

Solving ordinary differential equation

$$\frac{dy}{dx} = f(x, y) \text{ or } \rightarrow y' = f(x, y)$$

Initial condition $y(x_o) = y_o$



$$f(x+h) = f(x) + f'(x)h + f''(x)\frac{h^2}{2!} + 0(h^3)$$

من مفكوك تايلور

1. ***Euler's Method:***

$$y_{i+1} = y_i + hy_i' + \frac{h^2}{2!}y_i'' + \underbrace{\ldots\ldots\ldots}_{\text{يهمل}}$$

Note that:

$$\because y_i' = f(x, y) = f(x_i, y_i)$$

$$\therefore y_{i+1} = y_i + h(y_i' \Rightarrow f(x_i, y_i))$$

$$\therefore y_{i+1} = y_i + hf(x_i, y_i)$$

$$h = \left(\frac{b-a}{n}\right)$$

local truncation error $T.E \le \frac{h^2}{2}|y''(\rho)|$

Where $y''(\rho)$ is

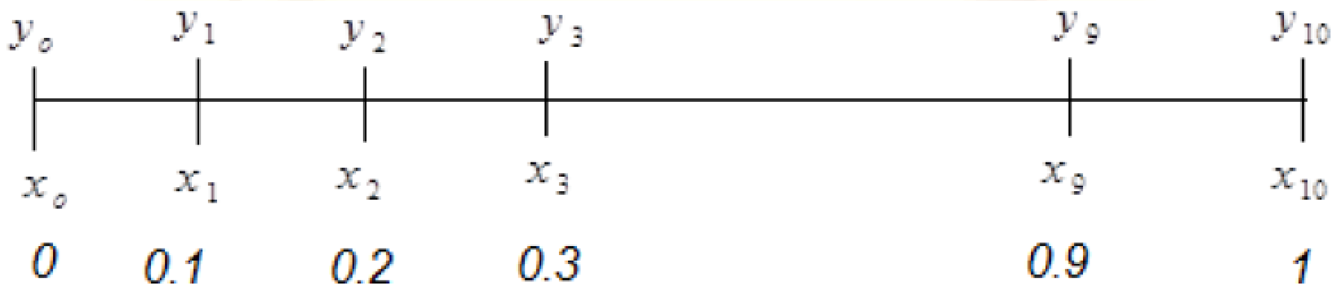
أكبر قيمة للمشتقة الثانية علي الفترة $[a,b]$

Example (1):

Solve the initial value problem (IVP) by using Euler method $dy = (2x - y)dx \rightarrow or \rightarrow \dfrac{dy}{dx} = (2x - y), x_o = 0, y_o = -1$

To get the value of (y) at (x=1) with (n=10) compare the values of the exact solution $y(x) = e^{-x} + 2x - 2$

Solution:



$$x_o = 0, x_n = 1$$

$$\therefore h = \frac{x_n - x_o}{n} = \frac{1-0}{10} = 0.1$$

$$f(x_i, y_i) = (2x_i - y_i)$$

$$\because y_{i+1} = y_i + hf(x_i, y_i)$$

$$i = 0$$

$$\therefore y_{0+1} = y_0 + hf(x_0, y_0) \rightarrow y_1 = y_0 + hf(x_0, y_0)$$

$$y_1 = -1 + 0.1(2(0.1) - (-0.9)) = -0.9$$

$$i = 1$$

$$\therefore y_{1+1} = y_1 + hf(x_1, y_1) \rightarrow y_2 = y_1 + hf(x_1, y_1)$$

$$y_2 = -0.9 + 0.1(2(0.1) - (-0.9)) = -0.79$$

ونكمل بنفس الطريقة حتي نحصل علي

$$i = 9$$

$$\therefore y_{10} = 0.348678$$

Exact Solution $y(1) = e^{-1} + 2\times 1 - 2 = 0.367879$

Error = $\left|Exact - Approximate\right| = \left|0.367879 - 0.348678\right| = 0.0192$

Example (2):

Solve the initial value problem (IVP) $(\sin x \cosh y)dx - (\cos x \sinh y)dy = 0, \;\; y(0) = 0.88137$

To get the value of (y) at (x=1) with (n=10) compare the values of the exact solution $\cos x \, \sinh y \; = 1$

Solution:

$$\frac{dy}{dx} = \frac{(\sin x \cosh y)}{(\cos x \sinh y)}$$

$$\frac{dy}{dx} = \tan x \coth y$$

$$f\;(x_i\,, y_i\,) = y\,'$$

$$\therefore f\;(x_i\,, y_i\,) = \tan x_i\; \coth y_i$$

$$x_o = 0, x_n = 1$$

$$\therefore h = \frac{x_n - x_o}{n} = \frac{1-0}{10} = 0.1$$

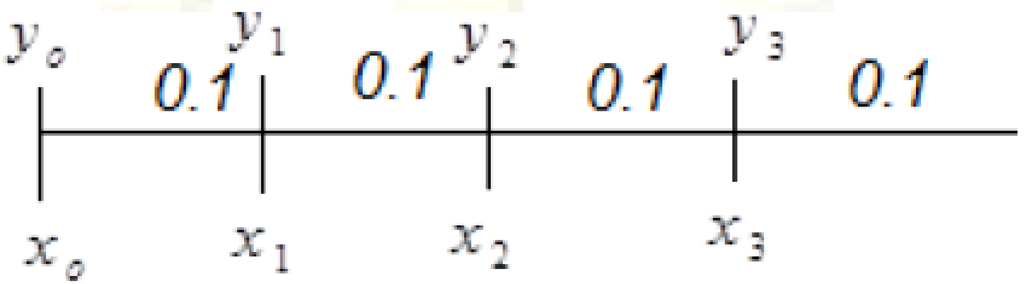
$$\because y_{i+1} = y_i + hf(x_i, y_i)$$

$$y_{i+1} = y_i + h(\tan x_i \coth y_i)$$

$$i = 0$$

$$\therefore y_{0+1} = y_0 + hf(x_0, y_0) \rightarrow y_1 = y_0 + h(\tan x_0 \coth y_0)$$

$$y_1 = 0.88137 + 0.1(\tan 0 \coth 0.88137) = 0.88137$$



$$i = 1$$

$$\therefore y_{1+1} = y_1 + hf(x_1, y_1) \rightarrow y_2 = y_1 + hf(x_1, y_1)$$

$$y_2 = 0.88137 + 0.1\left((\tan 0.1)(\coth 0.88137)\right) = 0.89555595$$

نكمل بنفس الطريقة

2- Taylor's Methods:

taylor's Method

2^{nd} order Taylor Method

3 rd order Taylor Method

من مفكوك تايلور

$$y_{i+1} = y_i + hy_i' + \frac{h^2}{2!}y_i'' + \frac{h^3}{3!}y_i''' + \dots\dots\dots$$

✓ ***Second order (2nd) Taylor's Method :(Three Terms)***

$$y_{i+1} = y_i + hy_i' + \frac{h^2}{2!} y_i'' = y_i + hf_i + \frac{h^2}{2!} f_i'$$

$$T.E \le \frac{h^3}{3!} \left| y'''(\rho) \right|$$

- ✓ ***Third order (3rd) Taylor's Method :(four Terms)***

$$y_{i+1} = y_i + hy_i' + \frac{h^2}{2!} y_i'' + \frac{h^3}{3!} y_i''' + \ldots\ldots\ldots\ldots$$

$$y_{i+1} = y_i + hy_i' + \frac{h^2}{2!} y_i'' + \frac{h^3}{3!} y_i''' = y_i + hf_i + \frac{h^2}{2!} f_i' + \frac{h3}{3!} f_i''$$

$$T.E \le \frac{h^4}{4!} \left| y^{(4)}(\rho) \right|$$

Example:

Drive an approximation solution to the initial value problem (IVP) $\frac{dy}{dx} = f(x, y), y(x_o) = y_o$

Using four terms in Taylor's series

Solution:

Taylors expansion:

$$y_{i+1} = y_i + hy_i' + \frac{h^2}{2!} y_i'' + \frac{h^3}{3!} y_i''' + \ldots\ldots\ldots\ldots$$

تهمل

$$\because \frac{dy}{dx} = f(x, y) \rightarrow y_i' = f(x_i, y_i) = f$$

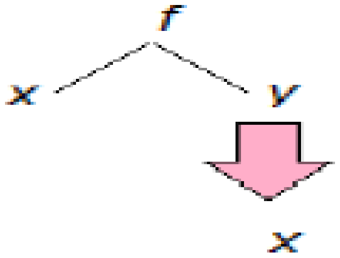
$$y' = f(x, y) \rightarrow y'' = f' = \frac{\partial}{\partial x} f \rightarrow y''' = f'' = \frac{\partial}{\partial x} f'$$

$$y_{i+1} = y_i + hf + \frac{h^2}{2!} f' + \frac{h^3}{3!} f'' + \ldots\ldots\ldots\ldots \qquad (1)$$

بتطبيق قاعدة السلسلة

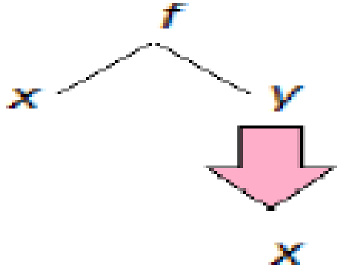
$$y' = f(x, y) \rightarrow y'' = f' = \frac{\partial}{\partial x} f = f_x + f_y f$$

$$\therefore y'' = f_x + f_y f$$



$$y'' = f'' = \frac{\partial}{\partial x} f' = \frac{\partial}{\partial x}\left(f_x + f_y f\right)$$

$$\therefore y''' = \left(f_{xx} + f_{yx} \cdot \frac{dy}{dx}\right) + f_y\left(f_x + f_y f\right) + \left(f_{yx} + f_{yy} \cdot \frac{dy}{dx}\right) f$$



$$f'' = f_{xx} + f_{yx} f + f_y\left(f_x + f_y f\right) + \left(f_{yx} + f_{yy} f\right) f$$

$$f'' = f_{xx} + 2f_{xy} f + f_{yy} f^2 + f_y f_x + f_y^2 f$$

بالتعويض في(1)

$$\therefore y_{i+1} = y_i + hf + \frac{h^2}{2}\left[f_x + f_y f\right] + \frac{h^3}{3!}\left[f_{xx} + 2f_{xy} f + f_{yy} f^2 + f_y f_x + f_y{}^2 f\right]$$

Example (4):

Use the third order Taylor's method for the initial value problem $\frac{dy}{dx} = (2x - y), x_o = 0, y_o = -1$ with (n=10) to approximate y (1). Where the exact solution is -0.36787944

Solution:

$$f(x, y) = (2x - y)$$

The first two derivatives of f (x , y) are :

$$f'(x,y) = \frac{\partial}{\partial x}(2x - y) = (2 - \frac{dy}{dx}) = (2 - y')$$

$$f'(x,y) = (2 - (2x - y))$$

$$f'(x,y) = 2(1 - x) + y \rightarrow 1$$

$$f''(x,y)=\frac{\partial}{\partial x}\left[2(1-x)+y\right]$$

$$f''(x,y)=-2+\frac{dy}{dx}=-2+y'=-2+(2x-y)$$

$$f''(x,y)=2(x-1)-y\rightarrow 2$$

$$\because y_{i+1}=y_i+hf+\frac{h^2}{2!}f'+\frac{h^3}{3!}f''$$

$$\therefore y_{i+1}=y_i+h(2x_i-y_i)+\frac{h^2}{2!}\left[2(1-x_i)+y_i\right]+\frac{h^3}{3!}\left[2(x_i-1)-y_i\right]$$

$i=0$

$$y_{0+1}=y_0+h(2x_0-y_0)+\frac{h^2}{2!}\left[2(1-x_0)+y_0\right]+\frac{h^3}{3!}\left[2(x_0-1)-y_0\right]=-0.8951666$$

$i=1$

$$y_{1+1}=y_1+h(2x_1-y_1)+\frac{h^2}{2!}\left[2(1-x_1)+y_1\right]+\frac{h^3}{3!}\left[2(x_1-1)-y_1\right]=-0.78127652$$

ونكمل

$$i = 9$$

$$y_{9+1} = y_9 + h\left(2x_9 - y_9\right) + \frac{h^2}{2!}\left[2\left(1 - x_9\right) + y_9\right] + \frac{h^3}{3!}\left[2\left(x_9 - 1\right) - y_9\right] = -0.36786301$$

$$\text{Error} = \left|Exact - Approximate\right|$$

$$= \left|-0.36787944 - (-0.36786301)\right|$$

$$= 1.6431 \times 10^{-5}$$

3- Runge – Kutta Method

Runge – Kutta Method

- **Mid-point Runge-Kutta**
- **second order (2^{nd})order Runge-Kutta**
- **fourth order (4^{th})order Runge-Kutta**

Deduce the Runge-Kutta Method ?

$$\text{Given } y' = f(x_i, y_i), y(x_o) = y_o$$

تعتمد فكرة الحل علي حساب قيمة $f(x, y)$ عند النقطة بين x_i و x_{i+1} و إستخدامها لحساب قيمة y_{i+1}

<u>الصورة العامة second order (2nd)Runge-Kutta هي</u>

$k_1 = f(x_i, y_i) \longrightarrow$ قيمة y' عند x_i

$k_2 = f(x_i + ah, y_i + bhk_1) \longrightarrow$ قيمة y' عند نقطة مابين x_i و x_{i+1}

$$y_{i+1} = y_i + h(w_1 k_1 + w_2 k_2) \rightarrow (1)$$

حيث (a, b, w_1, w_2) ثوابت نبدأ في تعينها

$$y_{i+1} = y_i + h y_i' + \frac{h^2}{2!} y_i'' + \frac{h^3}{3!} y_i''' + \dots\dots\dots$$

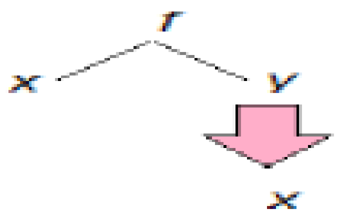
$$y'(x_i) = f(x_i, y_i)$$

$$y''(x_i) = f_x + f_y f$$

Chain rule f

$$y''(x_i) = f_x + f_y \frac{dy}{dx}$$

$$y''(x_i) = f_x + f_y f$$



بالتعويض في مفكوك تايلور

$$y_{i+1} = y_i + h f(x_i, y_i) + \frac{h^2}{2!}\left[f_x + f_y f\right] + Oh^3 \rightarrow (2)$$

مفكوك تايلور ل

$$f(x_i + ah, y_i + bhk_1) = f(x_i, y_i) + \left[ahf_x + bhk_1 f_y\right] \rightarrow \because k_1 = f(x_i, y_i)$$

$$f(x_i + ah, y_i + bhk_1) = f(x_i, y_i) + \left[ahf_x + bhff_y\right] \rightarrow \qquad (3)$$

من معادلة (1)

$$y_{i+1} = y_i + h\left(w_1 k_1 + w_2 k_2\right)$$

$$y_{i+1} = y_i + h\left(w_1 f + w_2\left(f\left(x_i, y_i\right) + \left[ahf_x + bhff_y\right]\right)\right)$$

$$y_{i+1} = y_i + hf\left(w_1 + w_2\right) + h^2\left(aw_2 f_x + bw_2 ff_y\right) \rightarrow \qquad (4)$$

بمقارنة معادلة رقم (4) بمعادلة (2) نجد أن

$$w_1 + w_2 = 1, aw_2 = \frac{1}{2}, bw_2 = \frac{1}{2}$$

ثلاث معادلات في أربع مجاهيل نفرض واحد ونحسب الباقي

$$a = 1 \rightarrow \therefore b = 1, w_1 = w_2 = \frac{1}{2}$$

So that 2nd order Runge Kutta :

$$
\begin{cases}
y_{i+1} = y_i + \dfrac{h}{2}\left[k_1 + k_2\right] \qquad i = 0,1,\ldots\ldots,n-1 \\
k_1 = f\left(x_i, y_i\right) \\
k_2 = f\left(x_i + h, y_i + hk_1\right)
\end{cases}
$$

Mid-point Method: modified

$$a = \frac{1}{2}, b = \frac{1}{2}, w_1 = 0, w_2 = 1$$

$$\begin{cases} y_{i+1} = y_i + hk_2 \\ k_1 = f\left(x_i, y_i\right) \\ k_2 = f\left(x_i + \frac{h}{2}, y_i + \frac{h}{2}k_1\right) \end{cases}$$

4^th order Runge Kutta method "classic Runge-Kutta":

$$a=\frac{1}{2}, b=\frac{1}{2}, c=\frac{1}{2}, d=\frac{1}{2}, e=1, f=1,$$

$$w_1=w_4=\frac{1}{6}, w_2=w_3=\frac{1}{3}$$

$$\begin{cases} y_{i+1}=y_i+\frac{h}{6}\left[k_1+2k_2+2k_3+k_4\right] \quad i=0,1,\ldots\ldots,n-1 \\ k_1=f\left(x_i,y_i\right) \\ k_2=f\left(x_i+\frac{h}{2},y_i+\frac{h}{2}k_1\right) \\ k_3=f\left(x_i+\frac{h}{2},y_i+\frac{h}{2}k_2\right) \\ k_4=f\left(x_i+h,y_i+hk_3\right) \end{cases}$$

Example (5):

Use the Mid-point Runge-Kutta method to obtain an approximation to the solution of the initial value problem (IVP) $\frac{dy}{dx} = (2x - y), x_o = 0, y_o = -1$ *with (n=10) to approximate y at x=1*

Solution:

$$f(x_i, y_i) = (2x_i - y_i)$$

$$k_1 = f(x_i, y_i) \rightarrow 1$$

$$k_2 = f\left(x_i + \frac{h}{2}, y_i + \frac{h}{2} k_1\right) \rightarrow 2$$

$$y_{i+1} = y_i + hk_2$$

$$i = 0$$

$$y_{0+1} = y_1 = y_0 + hk_2$$

$$\therefore k_1 = f\left(x_0, y_0\right) = 2\left(0\right) - \left(-1\right) = 1$$

$$\therefore k_2 = 2\left(\left(x_0 + \frac{h}{2}\right) - \left(y_0 + \frac{h}{2} \times 1\right)\right) = 2\left(\left(x_0 + \frac{0.1}{2}\right) - \left(-1 + \frac{0.1}{2} \times 1\right)\right) = 1.05$$

$$\therefore y_1 = -1 + 0.1\left(1.05\right) = -0.895$$

بالمثل :

$$y_{i+1} = y_i + hk_2$$

$$i = 1$$

$$y_{1+1} = y_2 = y_1 + hk_2$$

$$\therefore k_1 = f(x_1, y_1) = 2(0.1) - (-0.895) = 1.095$$

$$\therefore k_2 = 2\left(\left(x_1 + \frac{h}{2}\right) - \left(y_1 + \frac{h}{2} \times 1\right)\right) = 2\left(\left(0.1 + \frac{0.1}{2}\right) - \left(-0.895 + \frac{0.1}{2} \times 1\right)\right) =$$

$$k_2 = 2(0.15) - (-0.84025) = 1.14025$$

$$\therefore y_2 = -0.895 + 0.1(1.14025) = -0.780975$$

Example (6):

Use the Runge-Kutta of order 2 and 4 method to obtain an approximation to the solution of the initial value problem (IVP) $\frac{dy}{dx} = (2x - y), x_o = 0, y_o = -1$ with (n=10)

Solution:

Runge-Kutta of order 2:

$x_0 = 0$

$y_0 = -1$

$h = \frac{x_n - x_0}{n} = \frac{1 - 0}{10} = 0.1$

$f(x_i, y_i) = (2x_i - y_i)$

$k_1 = f(x_i, y_i)$

$k_2 = f(x_i + h, y_i + hk_1)$

$y_{i+1} = y_i + \frac{h}{2}(k_1 + k_2)$

At i = 0:

$$k_1 = 2x_0 - y_0 = 2 * 0 + 1 = 1$$

$$k_2 = 2(x_0 + h) - (y_0 + hk_1) = 2(0 + 0.1) - (-1 + 0.1 * 1) = 1.1$$

$$y_1 = y_0 + \frac{h}{2}(k_1 + k_2) = -1 + \frac{0.1}{2}(1 + 1.1) = -0.895$$

At i = 1:

$$k_1 = 2x_1 - y_1 = 2 * 0.1 + 0.895 = 1.095$$

$$k_2 = 2(x_1 + h) - (y_1 + hk_1) = 2(0.1 + 0.1) - (-0.895 + 0.1 * 1.095) = 1.1855$$

$$y_2 = y_1 + \frac{h}{2}(k_1 + k_2) = -0.895 + \frac{0.1}{2}(1.095 + 1.1855) = -0.78097$$

At i = 9:

$$k_1 = 2x_9 - y_9 = 2 * 0.9 + 0.2072 = 1.5928$$

$$k_2 = 2(x_9 + h) - (y_9 + hk_1) = 2(0.9 + 0.1) - (0.2072 + 0.1 * 1.5928) = 1.6335$$

$$y_{10} = y_9 + \frac{h}{2}(k_1 + k_2) = 0.2072 + \frac{0.1}{2}(1.5928 + 1.6335) = 0.368703$$

the exact solution

$$y(1) = e^{-1} + 2(1) - 2 = 0.367879$$

$$\text{Error} = |Exact - Approximate|$$

$$= |0.367879 - 0.368703|$$

$$= 8.24 \times 10^{-4}$$

4^{th} order Runge Kutta method "classic Runge-Kutta":

$$\begin{cases} y_{i+1} = y_i + \frac{h}{6}\left[k_1 + 2k_2 + 2k_3 + k_4\right] \quad i = 0,1,\ldots\ldots,n-1 \\ k_1 = f\left(x_i, y_i\right) \\ k_2 = f\left(x_i + \frac{h}{2}, y_i + \frac{h}{2}k_1\right) \\ k_3 = f\left(x_i + \frac{h}{2}, y_i + \frac{h}{2}k_2\right) \\ k_4 = f\left(x_i + h, y_i + hk_3\right) \end{cases}$$

At i = 0:

$$k_1 = 2x_0 - y_0 = 2(0) - (-1) = 1$$

$$k_2 = 2\left(x_0 + \frac{h}{2}\right) - \left(y_0 + \frac{h}{2}k_1\right) = 2\left(0 + \frac{0.1}{2}\right) - \left(-1 + \frac{0.1}{2} * 1\right) = 1.05$$

$$k_3 = 2\left(x_0 + \frac{h}{2}\right) - \left(y_0 + \frac{h}{2}k_2\right) = 2\left(0 + \frac{0.1}{2}\right) - \left(-1 + \frac{0.1}{2} * 1.05\right) = 1.0475$$

$$k_4 = 2\left(x_0 + \frac{h}{2}\right) - \left(y_0 + \frac{h}{2}k_3\right) = 2\left(0 + \frac{0.1}{2}\right) - \left(-1 + \frac{0.1}{2} * 1.0475\right) = 1.04762$$

$$y_1 = y_0 + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4) = -1 + \frac{0.1}{6}(1 + 2 * 1.05 + 2 * 1.0475 + 1.04762)$$
$$= -0.89596$$

At i = 1:

$$k_1 = 2x_1 - y_1 = 2(0.1) - (-0.89596) = 1.09596$$

$$k_2 = 2\left(x_1 + \frac{h}{2}\right) - \left(y_1 + \frac{h}{2}k_1\right) = 2\left(0.1 + \frac{0.1}{2}\right) - \left(-0.89596 + \frac{0.1}{2} * 1.09596\right) = 1.14116$$

$$k_3 = 2\left(x_1 + \frac{h}{2}\right) - \left(y_1 + \frac{h}{2}k_2\right) = 2\left(0.1 + \frac{0.1}{2}\right) - \left(-0.89596 + \frac{0.1}{2} * 1.14116\right) = 1.1389$$

$$k_4 = 2\left(x_1 + \frac{h}{2}\right) - \left(y_1 + \frac{h}{2}k_3\right) = 2\left(0.1 + \frac{0.1}{2}\right) - \left(-0.89596 + \frac{0.1}{2} * 1.1389\right) = 1.139$$

$$y_2 = y_1 + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= -0.89596 + \frac{0.1}{6}(1.09596 + 2 * 1.14116 + 2 * 1.1389 + 1.139) = -0.7827$$

At i = 2:

$$k_1 = 2x_2 - y_2 = 2(0.2) - (-0.7827) = 1.1828$$

$$k_2 = 2\left(x_2 + \frac{h}{2}\right) - \left(y_2 + \frac{h}{2}k_1\right) = 2\left(0.2 + \frac{0.1}{2}\right) - \left(-0.7827 + \frac{0.1}{2} * 1.1828\right) = 1.22356$$

$$k_3 = 2\left(x_2 + \frac{h}{2}\right) - \left(y_2 + \frac{h}{2}k_2\right) = 2\left(0.2 + \frac{0.1}{2}\right) - \left(-0.7827 + \frac{0.1}{2} * 1.22356\right) = 1.22152$$

$$k_4 = 2\left(x_2 + \frac{h}{2}\right) - \left(y_2 + \frac{h}{2}k_3\right) = 2\left(0.2 + \frac{0.1}{2}\right) - \left(-0.7827 + \frac{0.1}{2} * 1.22152\right) = 1.22162$$

$$y_3 = y_2 + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= -0.7827 + \frac{0.1}{6}(1.1828 + 2 * 1.22356 + 2 * 1.22152 + 1.22162)$$

$$= -0.66112$$

At i = 9:

$$k_1 = 2x_9 - y_9 = 2(0.9) - (0.20337) = 1.59663$$

$$k_2 = 2\left(x_9 + \frac{h}{2}\right) - \left(y_9 + \frac{h}{2}k_1\right) = 2\left(0.9 + \frac{0.1}{2}\right) - \left(0.20337 + \frac{0.1}{2} * 1.59663\right) = 1.61679$$

$$k_3 = 2\left(x_9 + \frac{h}{2}\right) - \left(y_9 + \frac{h}{2}k_2\right) = 2\left(0.9 + \frac{0.1}{2}\right) - \left(0.20337 + \frac{0.1}{2} * 1.61679\right) = 1.61579$$

$$k_4 = 2\left(x_9 + \frac{h}{2}\right) - \left(y_9 + \frac{h}{2}k_3\right) = 2\left(0.9 + \frac{0.1}{2}\right) - \left(0.20337 + \frac{0.1}{2} * 1.61579\right) = 1.61584$$

$$y_{10} = y_9 + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= 0.20337 + \frac{0.1}{6}(1.59663 + 2 * 1.61679 + 2 * 1.61579 + 1.61584)$$

$$= 0.364664$$

the exact solution

$$y(1) = e^{-1} + 2(1) - 2 = 0.367879$$

$$\text{Error} = |Exact - Approximate|$$

$$= |0.367879 - 0.364664|$$

$$= 3.245 \times 10^{-3}$$

Thank you