Chapter 1

Signal Analysis

Lecture 4

Prepared by Prof

Mahmoud Ahmed Attia Ali

Department of Electronics and Communications
Faculty of Engineering
Tanta University
Egypt

Contents

Properties of Fourier Transform

10-Convolution

- **Convolution of two functions is the product of one function by a shifted reflected virgin of the other function.**
- **Its value is a function of shift.**

Convolution Mathematically

- Is given mathematically by:

$$f_1(t) * f_2(t) = \int_{-\infty}^{\infty} f_1(\tau)\, f_2(t-\tau)\, d\tau$$

- Convolution in time corresponds to multiplication in frequency and v.v:

$$f_1(t) * f_2(t) \leftrightarrow F_1(\omega) . F_1(\omega)$$

$$2\pi f_1(t) . f_2(t) \leftrightarrow F_1(\omega) * F_1(\omega)$$

Proof in Time Domain

$$\Im[f_1(t) * f_2(t)] = \int_{-\infty}^{\infty} [f_1(t) * f_2(t)]\, e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f_1(\tau)\, f_2(t-\tau)\, d\tau\right] e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} f_1(\tau) \left[\int_{-\infty}^{\infty} f_2(t-\tau)\, e^{-j\omega t} dt\right] d\tau$$

$$= \int_{-\infty}^{\infty} f_1(\tau)\left[F_2(\omega)\, e^{-j\omega\tau}\right] d\tau$$

$$= F_2(\omega) \int_{-\infty}^{\infty} f_1(\tau)\left[e^{-j\omega\tau}\right] d\tau = F_2(\omega)\, F_1(\omega)$$

Proof in Frequency Domain

$$\Im^{-1}[F_1(\omega) * F_2(\omega)]$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} [F_1(\omega) * F_2(\omega)]\, e^{j\omega t} d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} F_1(u)\, F_2(\omega - u)\, du\right] e^{j\omega t} d\omega$$

$$= \int_{-\infty}^{\infty} F_1(u) \left[\frac{1}{2\pi}\int_{-\infty}^{\infty} F_2(\omega - u)\, e^{j\omega t}\, d\omega\right] du$$

$$= \int_{-\infty}^{\infty} F_1(u)[f_2(t)\, e^{jut}]\, du =$$

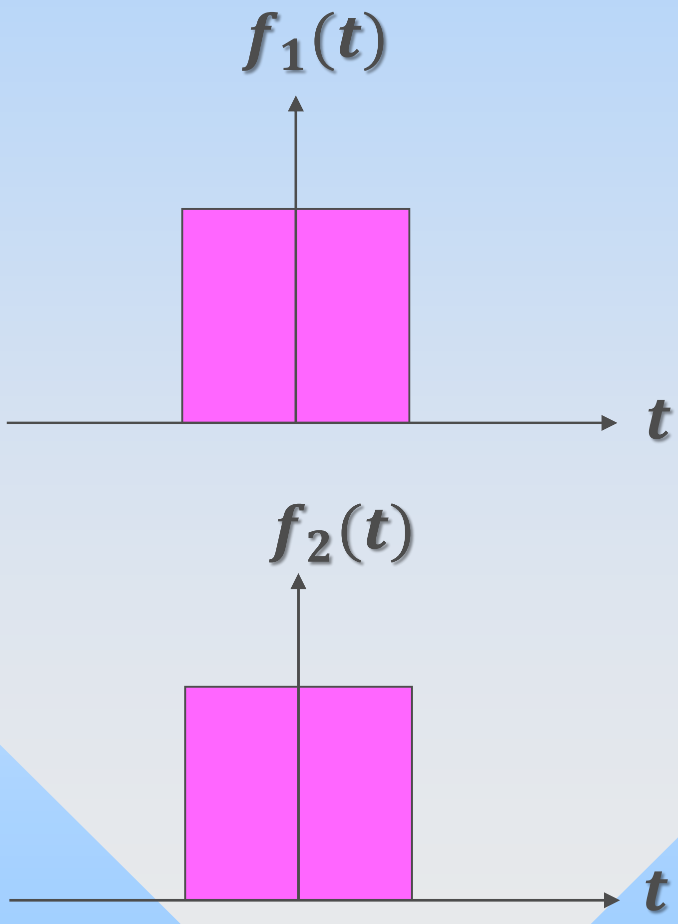
$$= 2\pi f_2(t) \frac{1}{2\pi}\int_{-\infty}^{\infty} F_1(u)[\, e^{jut}]du = 2\pi f_2(t)\, f_1(t)$$

Convolution Graphically

- ❑ Change t in $f_1(t)$ & $f_2(t)$ into τ to obtain $f_1(\tau)$ & $f_2(\tau)$
- ❑ Reflect $f_2(\tau)$ to get $f_2(-\tau)$
- ❑ Shift a distance t to obtain $f_2(t-\tau)$
- ❑ Calculate the results of product when second $f_2(t-\tau)$ passes first $f_1(\tau)$ (at all t)
- ❑ Plot these areas as a function of shifts t.

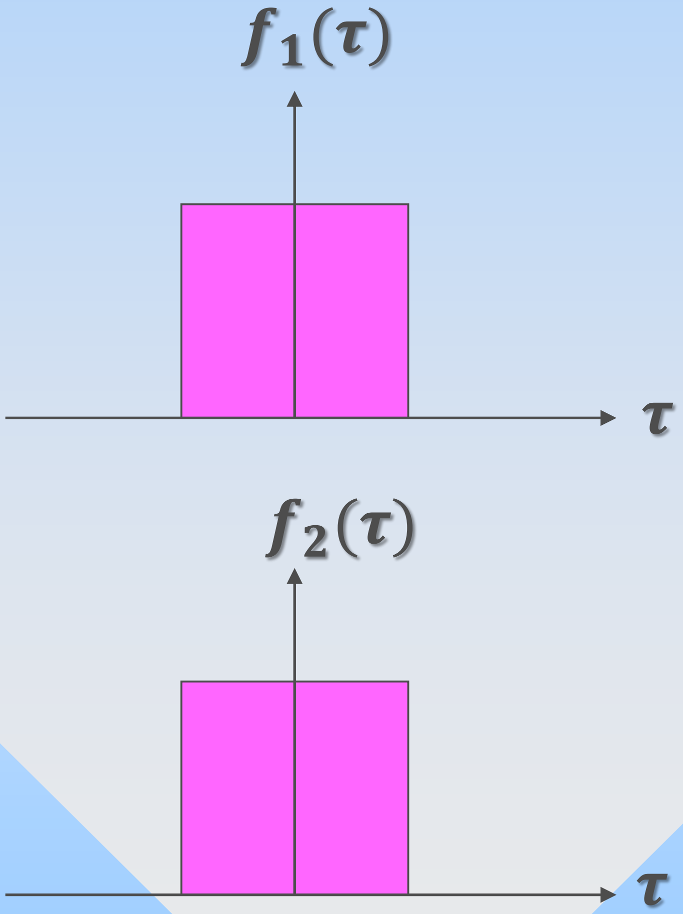
Example 1.21 on Convolution

Obtain the convolution of the two signal shown ?



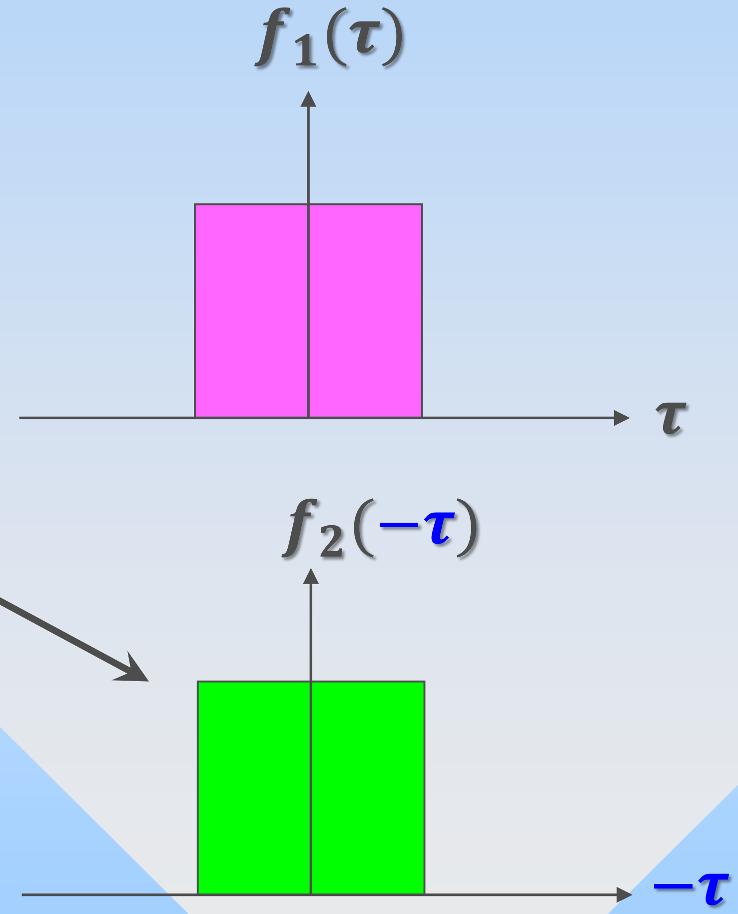
First: Get $f_1(\tau)$ and $f_2(\tau)$

Change t into τ for both signals



Second: Reflect $f_2(\tau)$ to get $f_2(-\tau)$

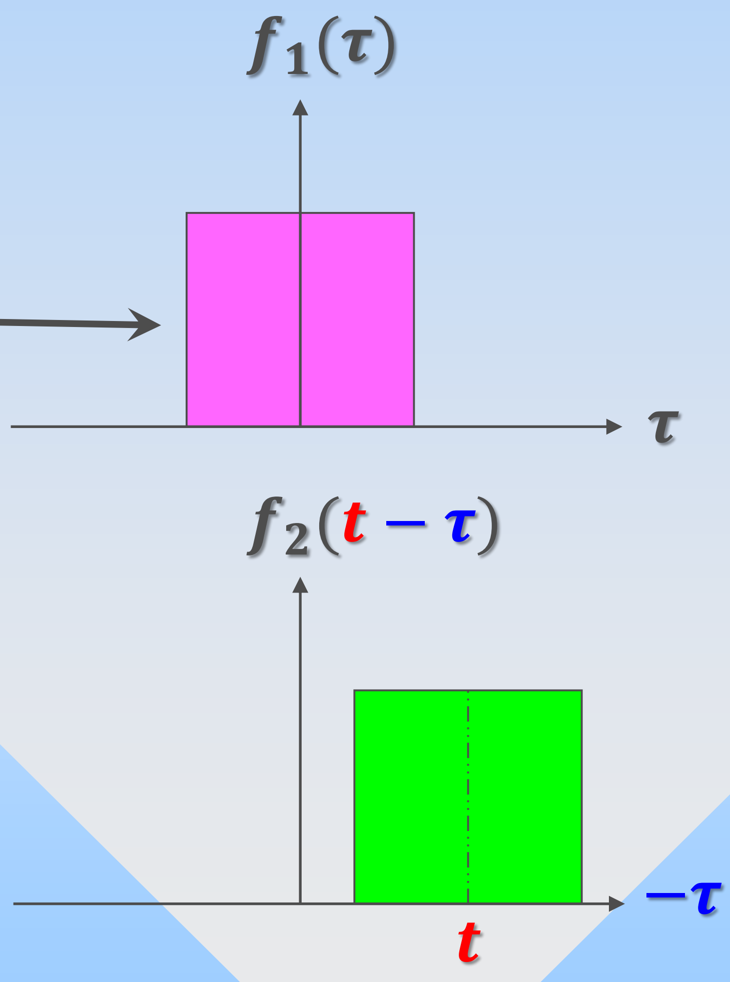
Here, these is no change with reflection because $f_2(\tau)$ is an even signal



Third: Shift $f_2(-t)$ a distance t to obtain $f_2(t-\tau)$

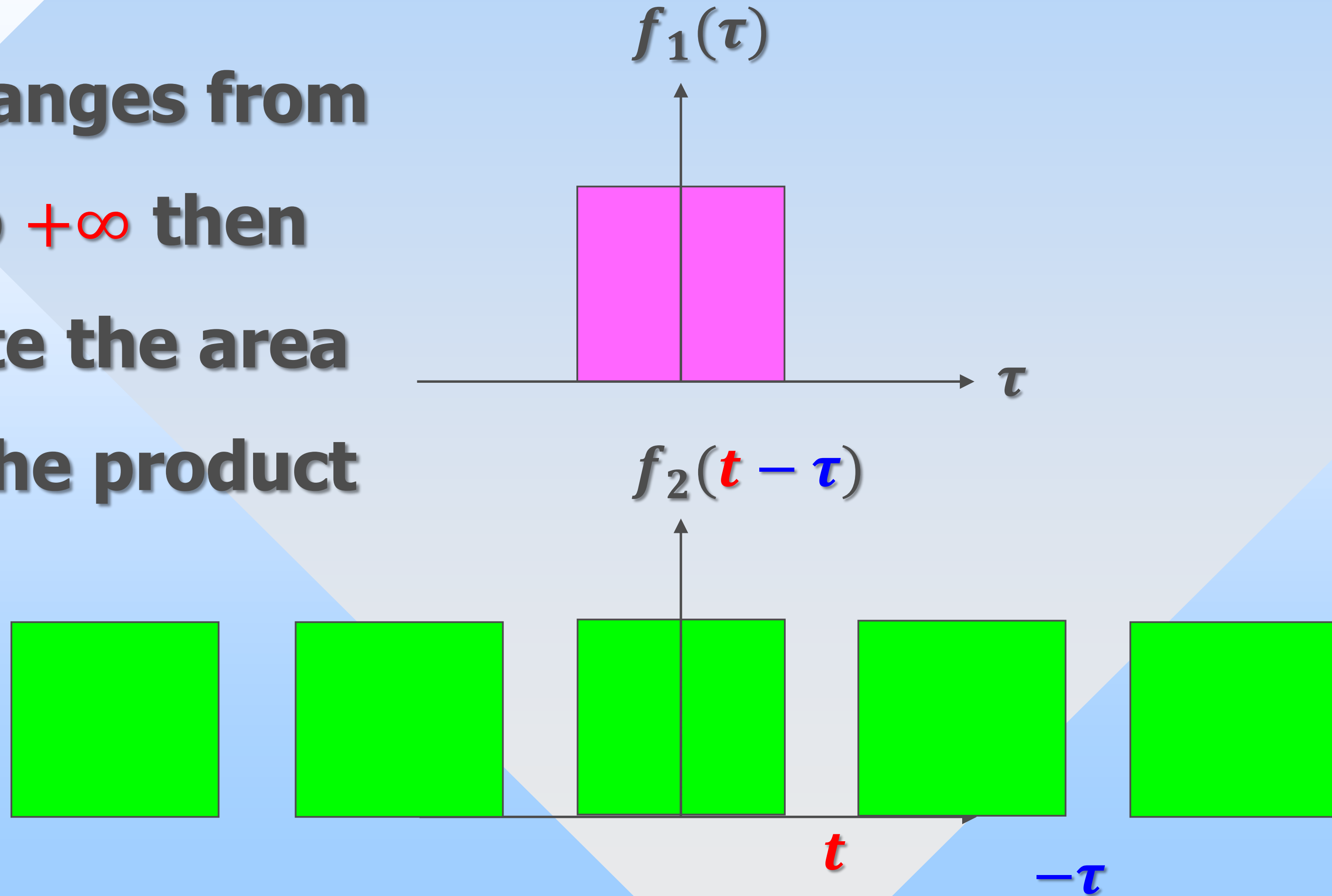
$f_1(\tau)$ without

Any change

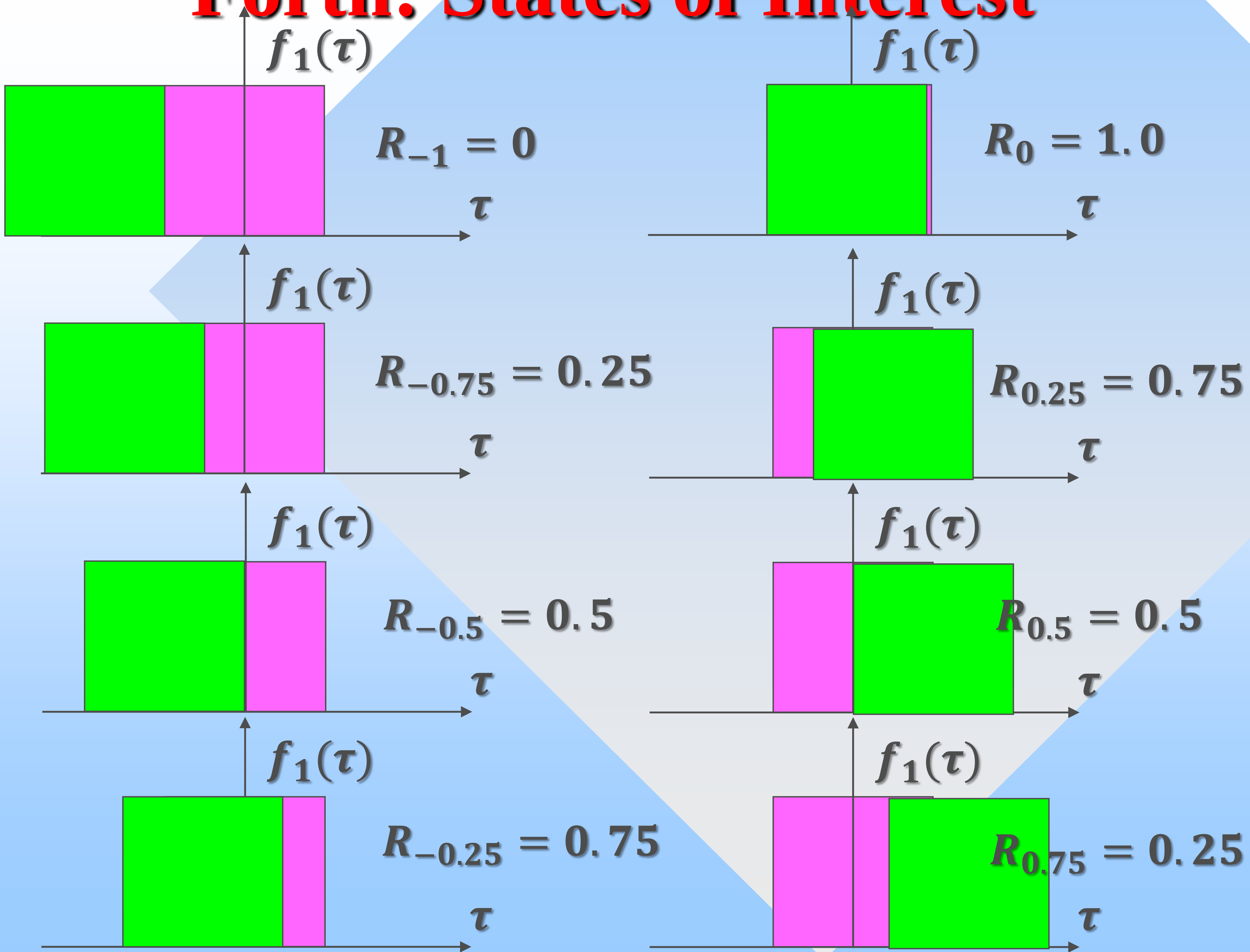


Forth: Calculate the Product when $f_2(t-\tau)$ Passes $f_1(\tau)$ (at all t)

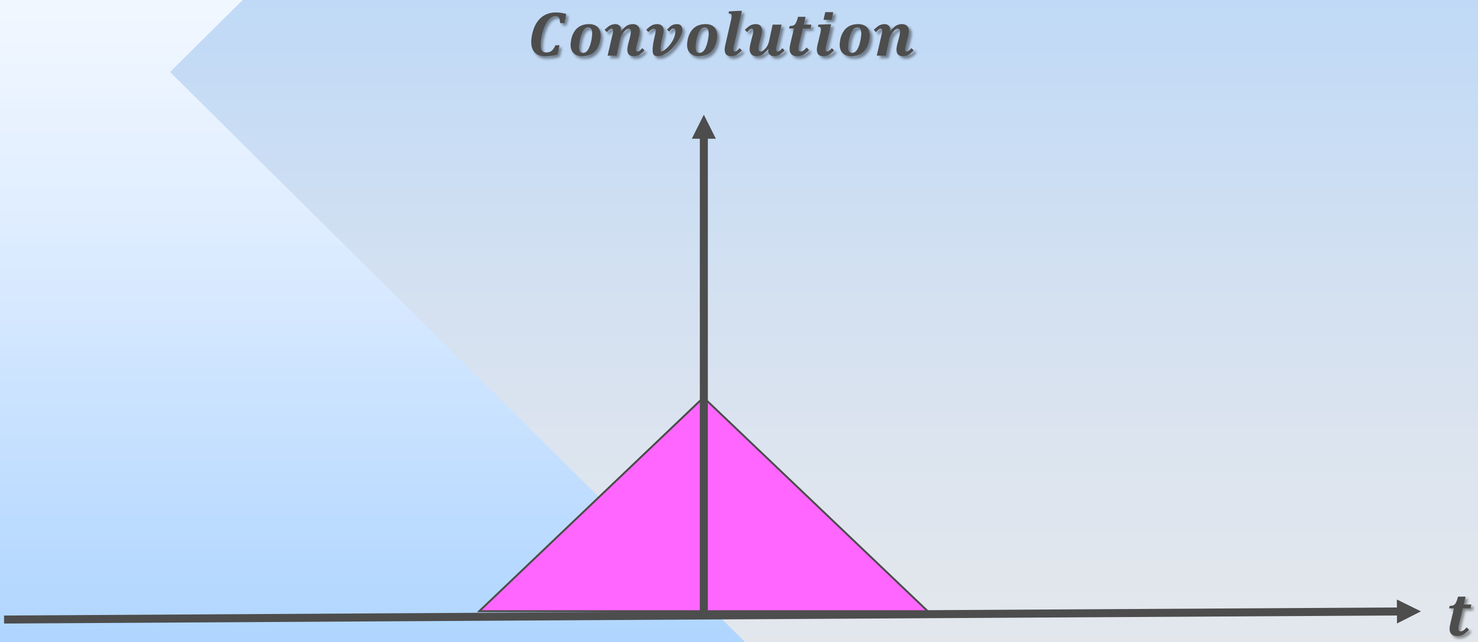
Let t changes from $-\infty$ to $+\infty$ then estimate the area under the product



Forth: States of Interest



Fifth: Plot areas as a function of shifts t



Convolution Properties

- **Distributive:**

$$f_1(t) * [f_2(t) + f_3(t)] = [f_1(t) * f_2(t)] + [f_1(t) * f_3(t)]$$

- **Commutative:**

$$f_1(t) * f_2(t) = f_2(t) * f_1(t)$$

- **Associative:**

$$f_1(t) * f_2(t) * f_3(t) = [f_1(t) * f_2(t)] * f_3(t)$$
$$= [f_1(t) * f_3(t)] * f_2(t) = [f_2(t) * f_3(t)] * f_1(t)$$

11-Parseval's Theorem

If two functions are given as:

$$f_1(t) \leftrightarrow F_1(\omega) \ \ and \ \ f_2(t) \leftrightarrow F_2(\omega)$$

$$\int_{-\infty}^{\infty} f_1^*(t)\, f_2(t)\, dt \leftrightarrow \frac{1}{2\pi}\int_{-\infty}^{\infty} F_1^*(\omega) F_2(\omega)\, d\omega$$

For Same function:

$$\int_{-\infty}^{\infty} f^*(t)\, f(t)\, dt \leftrightarrow \frac{1}{2\pi}\int_{-\infty}^{\infty} F^*(\omega) F(\omega) d\omega$$

$$\int_{-\infty}^{\infty} |f(t)|^2\, dt \leftrightarrow \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(\omega)|^2\, d\omega$$

The

Correlation

Correlation

- Cross correlation is to measure the relation (similarity) between two functions one of them is shifted.
- Or Auto correlation between the function and itself after shift.
- Difference between convolution and correlation is that one of the functions is reflected.

Cross Correlation

- Generally, Average cross correlation:

$$\mathcal{R}_{f_1 f_2}(\tau) = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} f_1^*(t) f_2(t+\tau)\, dt$$

- If signals are periodic of period T_o:

$$\mathcal{R}_{f_1 f_2}(t) = \frac{1}{T_o}\int_{-T_o/2}^{T_o/2} f_1^*(t) f_2(t+\tau)\, dt$$

- If they are finite energy (non-periodic or pulse-type):

$$\mathcal{R}_{f_1 f_2}(t) = \int_{-\infty}^{\infty} f_1^*(t) f_2(t+\tau)\, dt$$

Auto Correlation

- Generally, Average auto correlation:

$$\mathcal{R}_f(t) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} f^*(t) f(t+\tau)\, dt$$

- If signal is periodic of period T_o:

$$\mathcal{R}_f(t) = \frac{1}{T_o} \int_{-T_o/2}^{T_o/2} f^*(t) f(t+\tau)\, dt$$

- If it is finite energy (non-periodic or pulse-type):

$$\mathcal{R}_f(t) = \int_{-\infty}^{\infty} f^*(t) f(t+\tau)\, dt$$

Exercise 1.23

- **Obtain the auto correlation of the rectangular pulse:**

$$\mathcal{R}_{rect}(\tau) = \int_{-\infty}^{\infty} rect\left(\frac{t}{x}\right) rect\left(\frac{t+\tau}{x}\right) dt$$

Answer

$$\mathcal{R}_{rect}(\tau) = x\, tri\left(\frac{\tau}{x}\right)$$

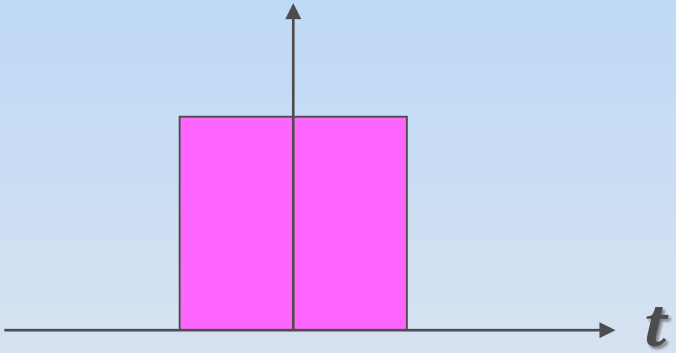
Correlation Graphically

- Draw first function $f_1(t)$ as it is.
- Shift second function $f_2(t)$ a distance τ to obtain $f_2(t+\tau)$
- Calculate the results of product when second $f_2(t+\tau)$ passes first $f_1(\tau)$, at all τ.
- Plot these areas as a function of shifts τ.

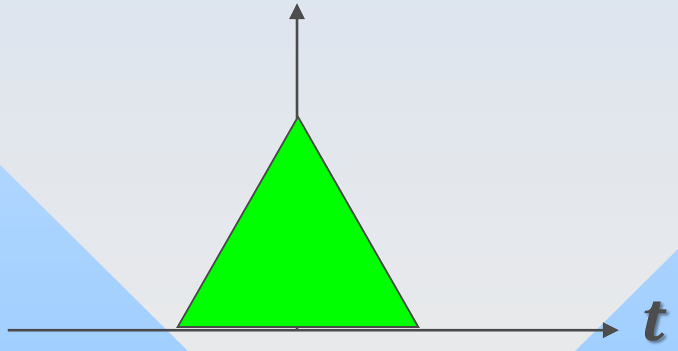
Example 1.24 on Correlation

Obtain the Correlation of the two signal shown ?

$f_1(t) = rect(t)$

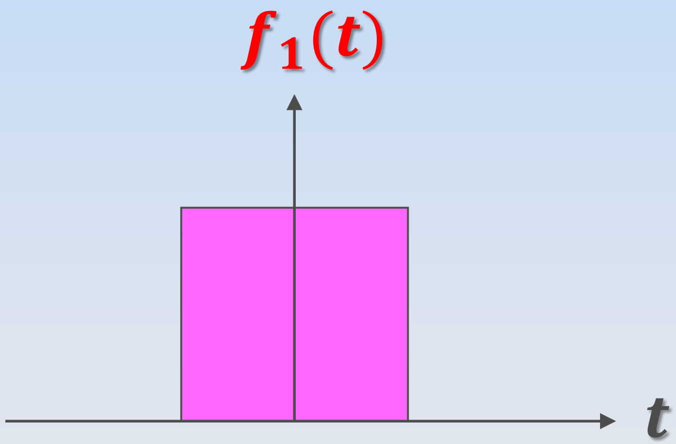


$f_2(t) = tri(t)$



First: Draw $f_1(t)$

Without any Change

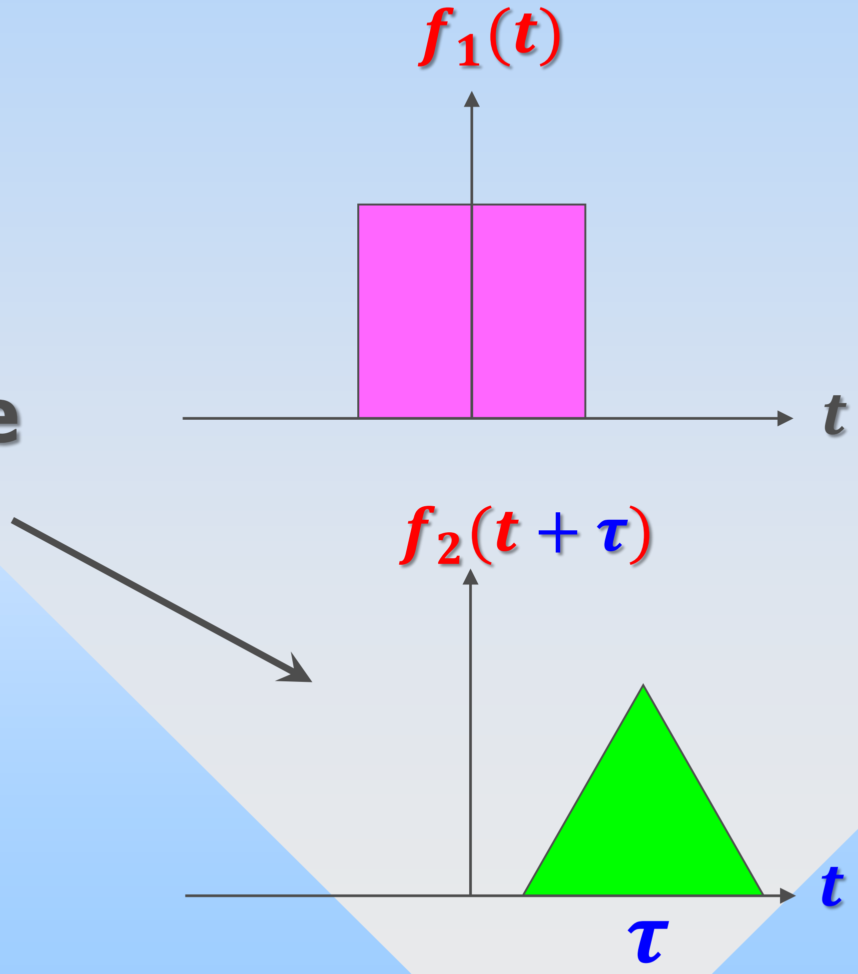


Second: Shift second function $f_2(t)$ a distance τ to obtain $f_2(t+\tau)$

The second

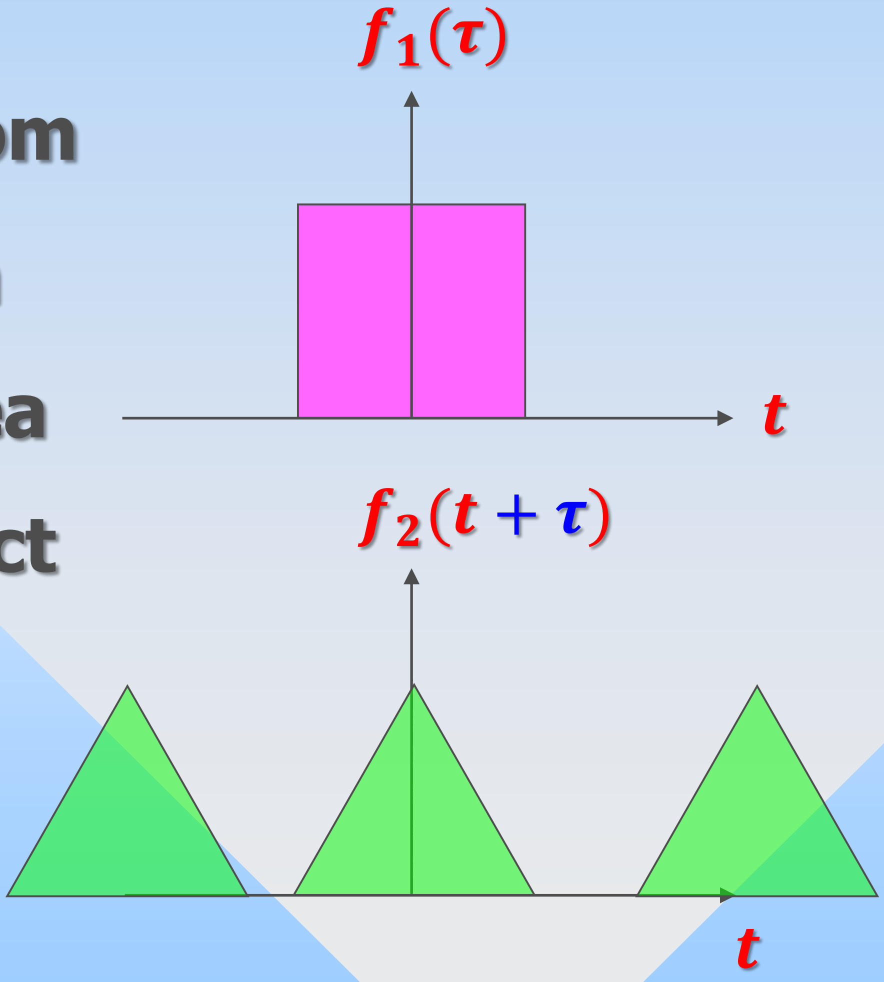
$f_2(t)$ after the

shift τ

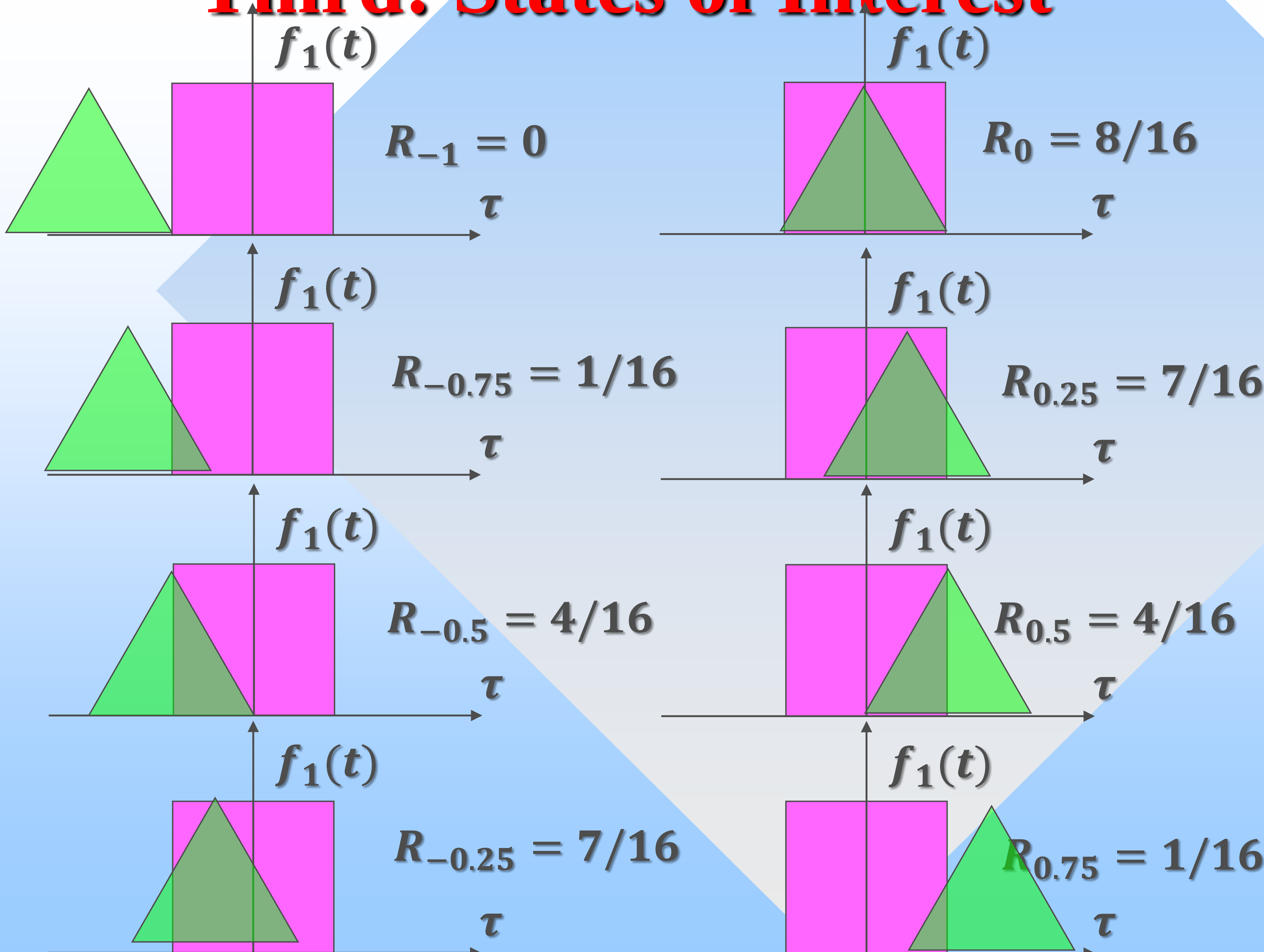


Third: Calculate the Product when $f_2(t+\tau)$ Passes $f_1(t)$ (at all τ)

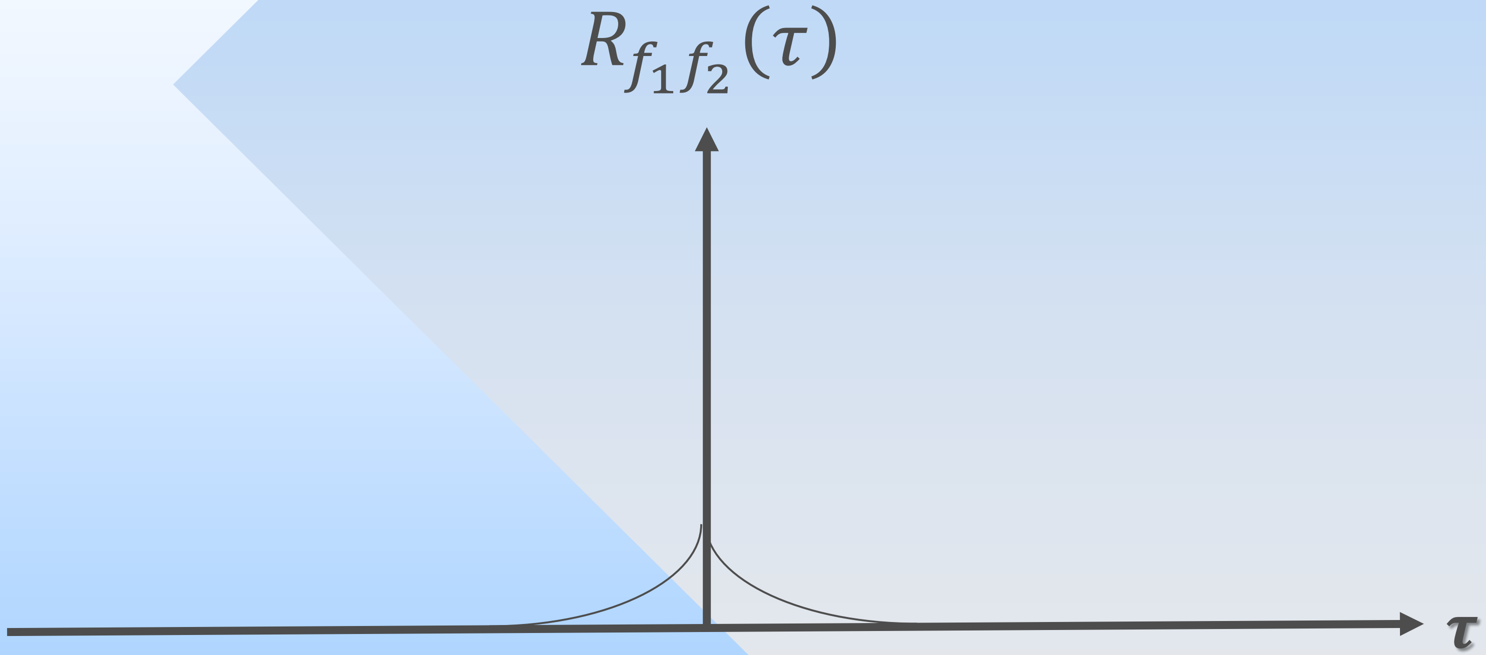
Let τ changes from $-\infty$ to $+\infty$ then estimate the area under the product



Third: States of Interest



Forth: Plot areas as a function of shifts τ



Properties of Auto correlation

$\mathcal{R}_f(0) = P_f$

- Autocorrelation function at $\tau = 0$ equals the average power of the signal. **Proof:**

$$\mathcal{R}_f(0) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} f^*(t) f(t+\tau)\, dt \Big|_{\tau=0}$$

$$\mathcal{R}_f(0) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} f^*(t) f(t)\, dt$$

$$\mathcal{R}_f(0) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |f(t)|^2\, dt$$

$$\mathcal{R}_f(0) = P_f$$

$\mathcal{R}_f(0) \geq \mathcal{R}_f(\tau)$

- **Autocorrelation is a decreasing function since increasing shift decreases similarity. Proof: ???**

$$\mathcal{R}_f(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathcal{S}_f(\omega)\, e^{j\omega\tau} d\omega$$

$$\left|\mathcal{R}_f(\tau)\right| \leq \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathcal{S}_f(\omega)\, d\omega = \mathcal{R}_f(0)$$

$$\therefore \mathcal{R}_f(0) \geq \mathcal{R}_f(\tau)$$

For Real $f(t)$

$$\mathcal{R}_f(-\tau) = \mathcal{R}_f(\tau)$$

- **Autocorrelation is even function of τ for real $f(t)$**

$$\mathcal{R}_f(-\tau) = \mathcal{R}_f(\tau)$$

$$\mathcal{R}_f(\tau + T_o) = \mathcal{R}_f(\tau)$$

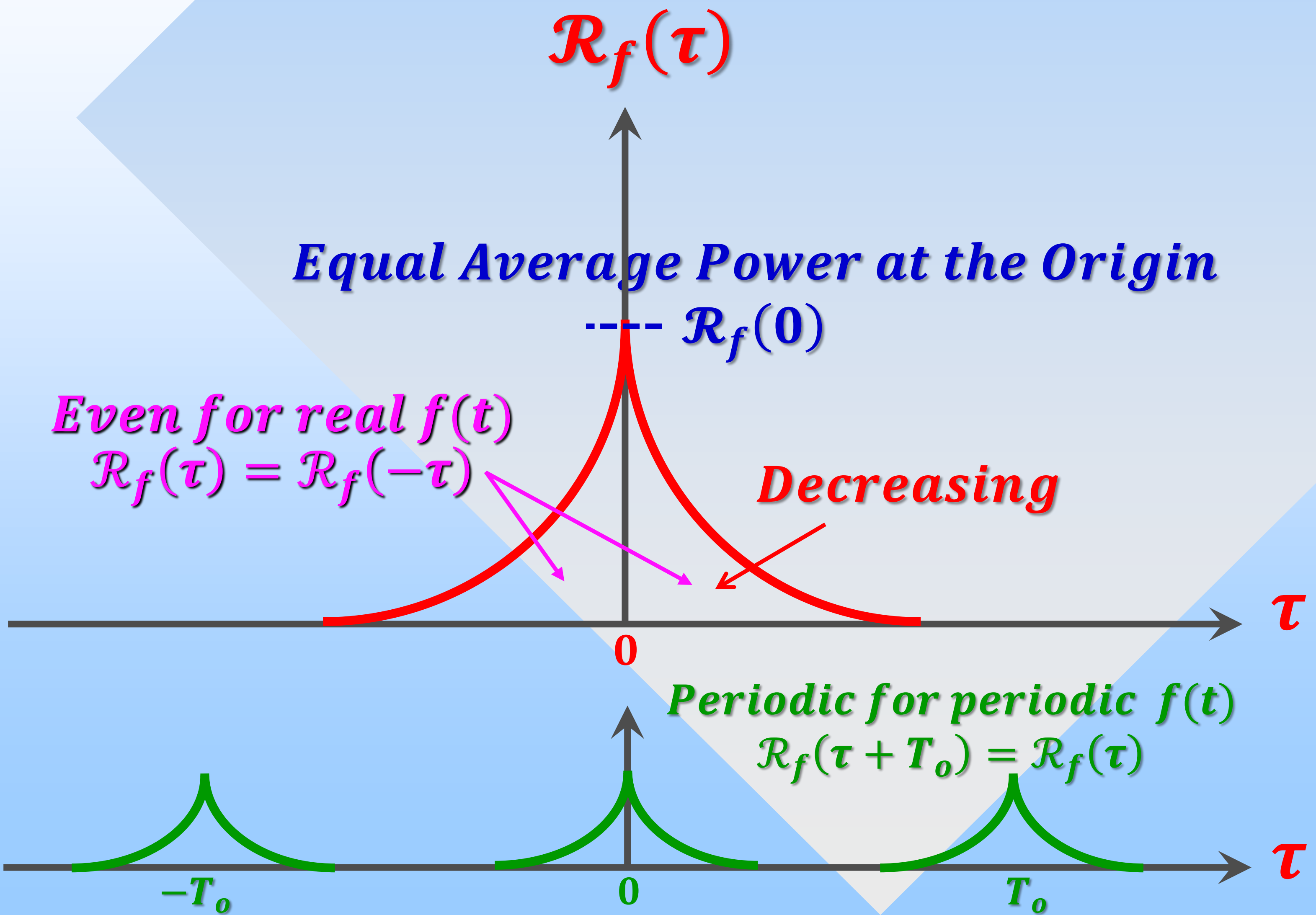
- **Autocorrelation of a periodic signal is also periodic:**

$When: f(t)$ is periodic:

$$\therefore f(t + T_o) = f(t)$$

$$\therefore \mathcal{R}_f(\tau + T_o) = \mathcal{R}_f(\tau)$$

Autocorrelation Properties



Correlation and Fourier Transform

- Cross correlation of 2 functions forms a Fourier transform pair with multiplication of their spectrums.

$$\mathcal{R}_{f_1 f_2}(\tau) = \int_{-\infty}^{\infty} f_1^*(t) f_2(t+\tau)\, dt \leftrightarrow F_1^*(\omega).\, F_2(\omega)$$

- Autocorrelation integral of a function $f(t)$ forms a Fourier transform pair with the magnitude square of its spectrum.

$$\mathcal{R}_f(\tau) = \int_{-\infty}^{\infty} f^*(t) f(t+\tau)\, dt \leftrightarrow F^*(\omega).\, F(\omega)$$

$$\mathcal{R}_f(\tau) \leftrightarrow |F(\omega)|^2$$

Uncorrelated Signals

- Functions $f_1(t)$ and $f_2(t)$ are uncorrelated if their cross correlation is independent on τ:

$$\mathcal{R}_{f_1 f_2}(\tau) = \overline{f_1^*(t).f_2(t+\tau)} = \overline{f_1^*(t)}.\overline{f_2(t+\tau)}$$

- Example.1.24:

$$f_1(t) = e^{j\omega_1 t} \quad \& \quad f_2(t) = e^{j\omega_2 t}$$

$$\mathcal{R}_{f_1 f_2}(\tau) = \overline{e^{-j\omega_1 t}}.\overline{e^{j\omega_2 t+\tau}} = 1.1 \; independent \; on \; \tau$$

- Example.1.25:

$$f_1(t) = e^{j\omega_1 t} \quad \& \quad f_2(t) = 1 + e^{j\omega_2 t}$$

Correlation Receiver

- **Example.26:**

Explain the correlation between information signal $f(t)$ and the random noise $n(t)$, then show how to use a correlation receiver to suppress it.

Answer

There is no correlation between the information signal $f(t)$ and the random noise $n(t)$ because the noise has a zero average.

To check how to use the correlation receiver to suppress such noise, see next slide.

Correlation of Received Signal

- The received signal $r(t)$ will be as follows:

$$r(t) = f(t) + n(t)$$

- Taking the autocorrelation to received signal:

$$\mathcal{R}_{r(t)} = \overline{[f(t) + n(t)]\cdot[f(t+\tau) + n(t+\tau)]}$$

$$\mathcal{R}_r(\tau) = \overline{f^*(t)\, f(t+\tau)} + \overline{f^*(t)\, n(t+\tau)}$$

$$+\overline{n^*(t)\, f(t+\tau)} + \overline{n^*(t)\, n(t+\tau)}$$

$$\mathcal{R}_r(\tau) = \mathcal{R}_f(\tau) + 0 + 0 + \mathcal{R}_n(\tau)$$

$$\mathcal{R}_r(\tau) = \mathcal{R}_f(\tau) + \mathcal{R}_n(\tau)$$

So, autocorrelation of received signal is equal to autocorrelation of the signal in addition to the autocorrelation of noise signal.

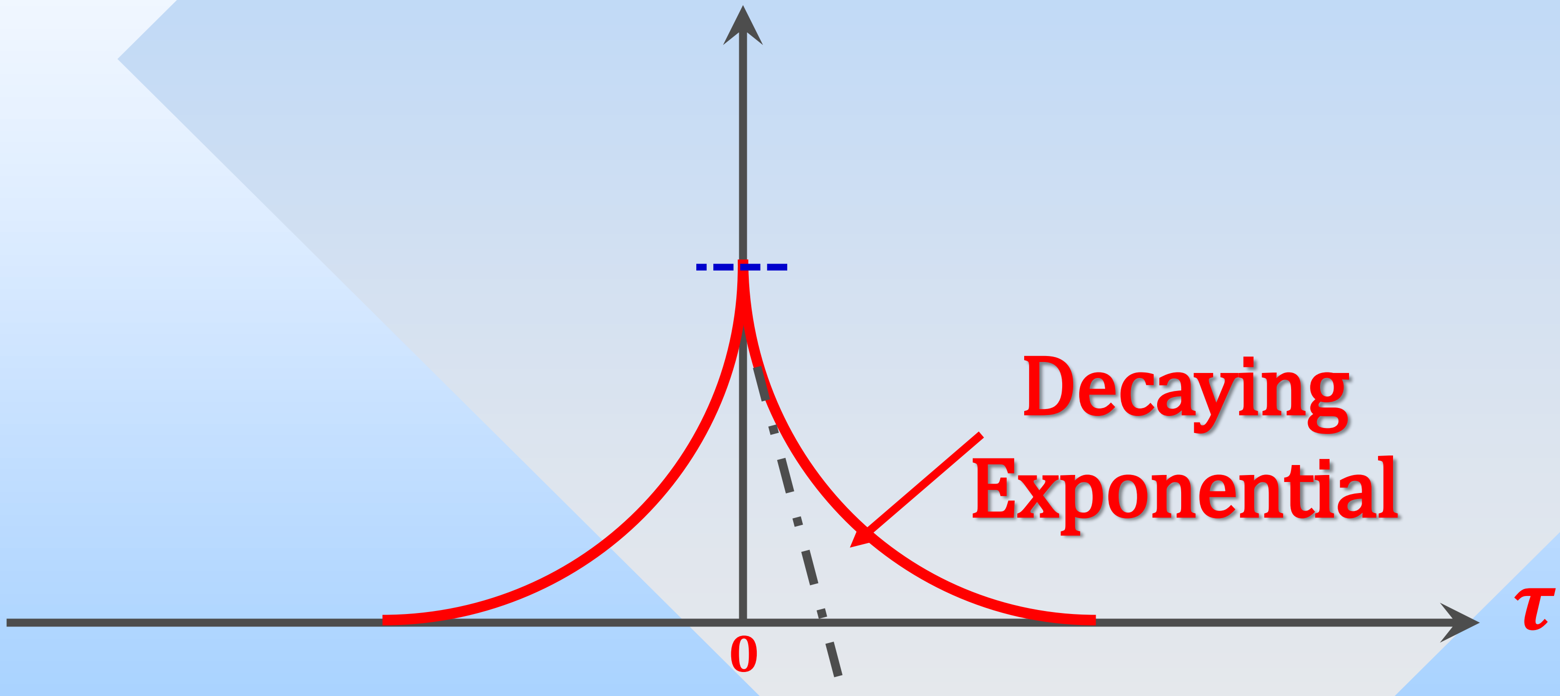
Usefulness of Correlation

$$\mathcal{R}_r(\tau) = \mathcal{R}_f(\tau) + \mathcal{R}_n(\tau)$$

- Noise autocorrelation $\mathcal{R}_n(\tau)$ is a decreasing exponential.
- So by multiplying the received signal by itself with enough time shift $\tau \gg \tau'$:
  - Noise autocorrelation $\mathcal{R}_n(\tau)$ dimenshies
  - Receiver correlation output $\mathcal{R}_r(\tau)$ is almost the autocorrelation of desired signal alone $\mathcal{R}_f(\tau)$.

Autocorrelation of Noise

$$\mathcal{R}_n(\tau) = e^{-\alpha|\tau|}$$



$$\acute{\tau} = \frac{1}{Noise\ Bandwidth} \approx \frac{1}{10^8} = 10^{-8}$$

Electromagnetic waves travels by the speed of light = 3 x 10^8

Energy and Power Signals

Energy Signal

- Energy signal is characterized by:
  - finite value and
  - finite duration

  so that it has a:
  - finite energy
  - but zero average power along all time.
- Examples: Non-periodic or pulsed signals.
- Total energy of a signal $f(t)$ in 1 ohm is by:

$$E = \int_{-\infty}^{\infty} |f(t)|^2 dt = \int_{-\infty}^{\infty} f(t) f^*(t) dt$$

$$E = \int_{-\infty}^{\infty} F(\omega) F^*(\omega) d\omega = \int_{-\infty}^{\infty} |F(\omega)|^2 d\omega$$

Power Signal

- Power signal has:
  - Infinite energy
  - But finite average power
- Examples: Periodic signals and noise.
- Average power in a 1 ohm resistance is:

$$P = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |f(t)|^2 dt$$

- For Periodic Signals:

$$P = \frac{1}{T_o} \int_{-T_o/2}^{T_o/2} |f(t)|^2 dt = \sum_{n=1}^{\infty} |F_n|^2$$

Energy Spectral Density

- Energy density spectrum shows how the energy of the function is distributed among the different frequency components:

$$\xi_f(\omega) = |F(\omega)|^2 = 2\pi \frac{dE}{d\omega}$$

- Total energy of a signal $f(t)$ is related to the spectral energy by:

$$E = \frac{1}{2\pi}\int_{-\infty}^{\infty} \xi_f(\omega)\, d\omega$$

Power Spectral Density

- Power density spectrum $\mathcal{S}_f(\omega)$, distribution of the average power of the signal among the frequency components of the spectrum:

$$P = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathcal{S}_f(\omega)\,d\omega$$

- The autocorrelation function $R_f(\tau)$ and the power density spectrum $\mathcal{S}_f(\omega)$ forms a Fourier transform pair:

$$R_f(\tau) \leftrightarrow \mathcal{S}_f(\omega)$$