Numerical Analysis
Lec 4

Continued Chapter (2)
Numerical Integration

Numerical Integration:

- trapeziod
- simpson's
- Gaussian quadrature

1. Numerical Integration:

الغرض من هذه الطريقة إيجاد طريقة تقريبية للتكامل

$$I = \int_a^b f(x)\,dx$$

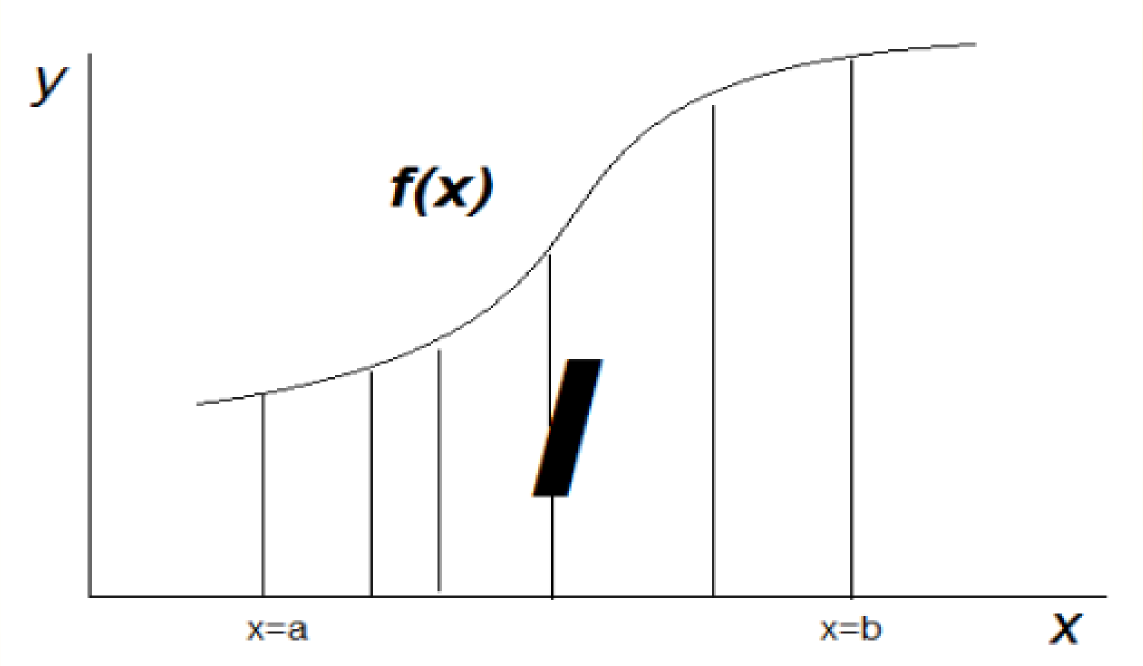


Figure (1) area under curve

خطوات الحل :

1- نقسم فترة التكامل عدد (n) من التقسيمات المتساوية طولها h

$$h = \frac{b-a}{n}$$

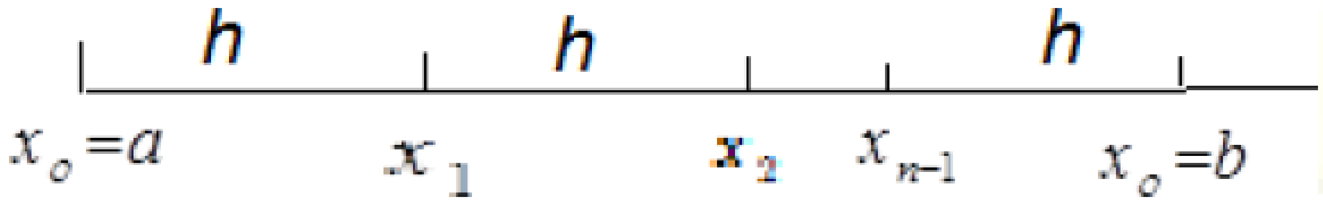


Figure (2) period of integration

2- نحسب قيمة $f(x)$ عند كل نقطة في الجدول

x	x_o	x_1	....	x_{n-1}	x_n
$y = f(x)$	y_o	y_1	....	y_{n-1}	y_n

3- نحدد الطريقة التي سوف نقوم بحساب التكامل بيها من الطرق الأتية

Trapezoidal Rule:

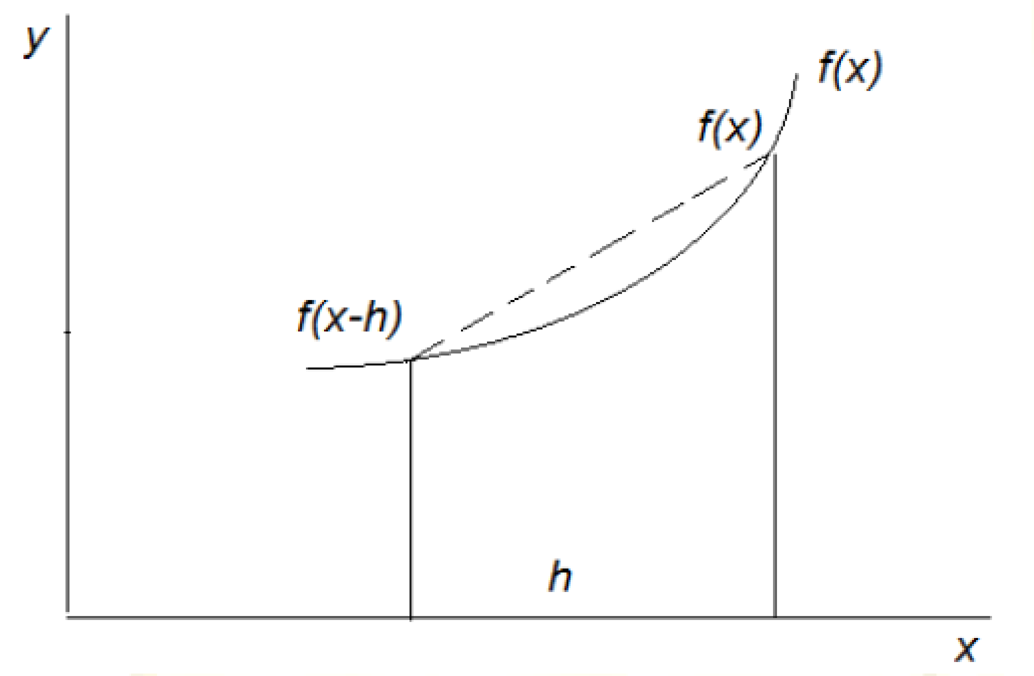


Figure (3) Trapezoidal Rule

$$I = \int_{x-h}^{x} f(x)dx = F(x)\Big|_{x-h}^{x}$$

$$I = F(x) - F(x-h) \rightarrow 1$$

$$\because F(x-h) = F(x) - hF'(x) + \frac{h^2}{2}F''(x) - \frac{h^3}{3!}F'''(x) + \ldots\ldots\ldots$$

$$\because f(x) = F'(x)$$

$$\therefore F(x-h) = F(x) - hf(x) + \frac{h^2}{2} f'(x) - \frac{h^3}{3!} f''(x) + \dots\dots\dots$$

$$\therefore F(x) - F(x-h) = hf(x) - \frac{h^2}{2} f'(x) + \frac{h^3}{3!} f''(x) + \dots\dots\dots$$

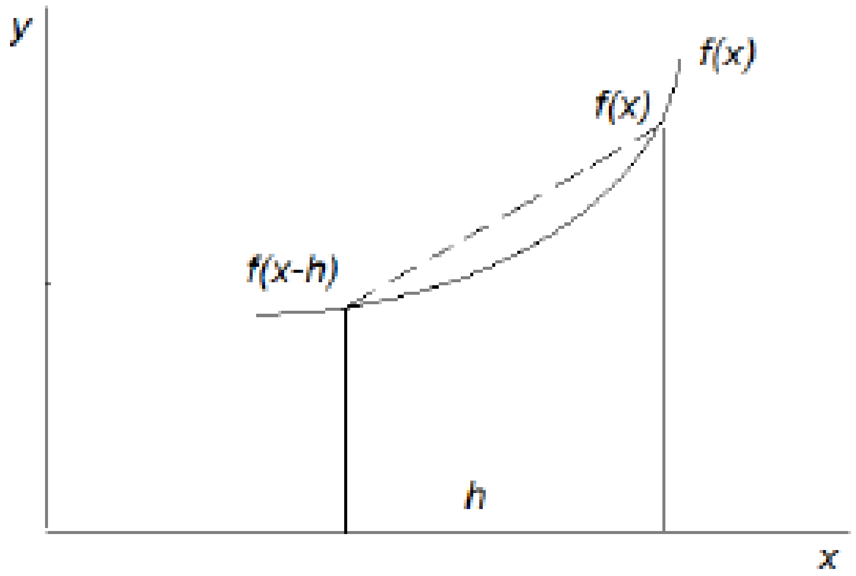
$$\therefore I = hf(x) - \frac{h^2}{2} f'(x) + \frac{h^3}{3!} f''(x) + \dots\dots\dots$$

$$\because f'(x) = \frac{f(x) - f(x-h)}{h} + \frac{h^2}{2} f''(x) + \dots\dots\dots \qquad (B.D)$$

$$\therefore I = hf(x) - \frac{h^2}{2}\left(\frac{f(x) - f(x-h)}{h} + \frac{h^2}{2} f''(x) + \dots\dots\dots\right) + \frac{h^3}{3!} f''(x) + \dots\dots\dots$$

$$\therefore I = hf(x) - \frac{h}{2} f(x) + \frac{h}{2} f(x-h) - \frac{h^3}{4} f''(x) + \frac{h^3}{6} f''(x) + \dots\dots\dots$$

$$\therefore I = \frac{h}{2} f(x) + \frac{h}{2} f(x-h) - \frac{h^3}{12} f''(x) + \ldots\ldots\ldots$$



T.E

$$I \cong \frac{h}{2}\left[f(x) + f(x-h)\right]$$ مساحة شبه المنحرف

$$T.E \le \frac{h^3}{12}\left|_{\max} f''(c)\right|, \qquad x-h \le c \le x$$

Composite Trapezoidal Rule:

$$I \cong \int_{x_o}^{x_1} f(x)dx + \int_{x_1}^{x_2} f(x)dx + \ldots\ldots \int_{x_{n-1}}^{x_n} f(x)dx$$

$$I = \frac{h}{2}\left[y_o + y_n + 2\left[y_2 + \ldots + y_{n-1}\right]\right]$$

باقي الحدود

$$T.E \le n\frac{h^3}{12}\left|_{\max} f''(c)\right|, \quad a \le c \le b$$

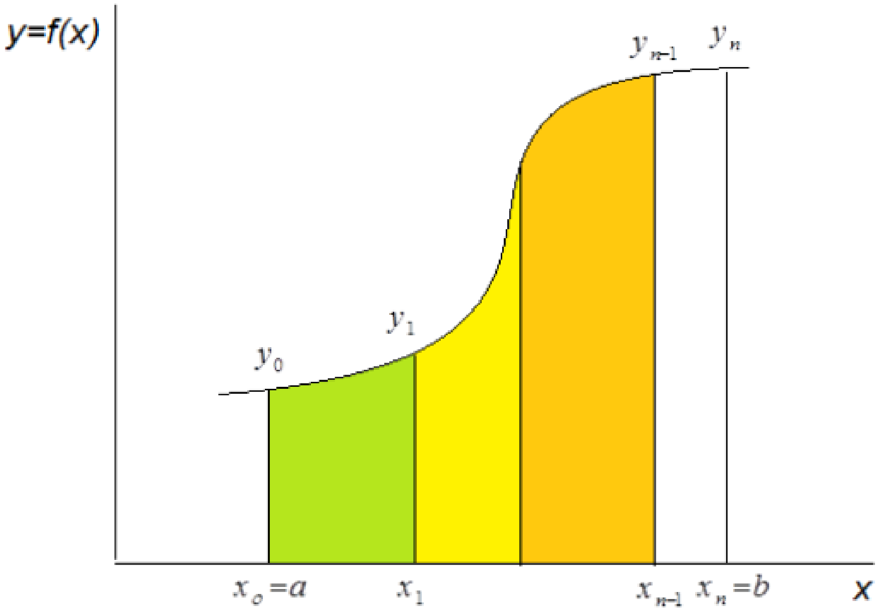


Figure (4) composite Trapezoidal Rule

$$T.E. \le \frac{(b-a)^3}{12\,n^2} M_2$$

Composite Trapezoidal Rule:

$$\boxed{I = \frac{h}{2}\left[y_o + y_n + 2(y_1 + y_2 + \cdots + y_{n-1})\right]}$$

$$T.E. \le \frac{(b-a)^3}{12\,n^2} M_2$$ $n=1$: Trapezoidal rule

M_2 هي أعلى قيمة لـ $f''(x)$ في الفترة $[a, b]$.

$I = \frac{h}{2}$ [الأول + الأخير + 2 (باقي الحدود y)]

Example (1):

Use the Composite Trapezoidal Rule with (n=2) to approximate $\int_0^3 x^2 e^x \, dx$ and find the true Truncation error

Solution:

$$f(x) = x^2 e^x, h = \frac{b-a}{n} = \frac{3-0}{2} = 1.5$$

x	0	1.5	3
$y = f(x)$	0	10.0838004	180.7698323
	y_0	y_1	y_2

$$I=\int_0^3 x^2e^x\,dx=\frac{h}{2}\left[y_o+y_2+2\left[y_1\right]\right]$$

$$I=\int_0^3 x^2e^x\,dx=\frac{1.5}{2}\left[0+180.7698323+2\left[10.0838004\right]\right]$$

$$I=150.7030748$$

$$f(x)=x^2e^x$$

$$f'(x)=2xe^x+x^2e^x$$

$$f''(x)=2e^x+2xe^x+2xe^x+x^2e^x$$

$$T.E\le n\frac{h^3}{12}\left|{}_{\max}f''(c)\right|,a\le c\le b$$

$$\max f''(x)\,at\,(x=3)$$

$$\therefore f''(3)=2e^3+2\times3e^3+2\times3e^3+3^2e^3$$

$$\therefore f''(3)=461.9673492$$

$$\therefore T.E \le n \frac{h^3}{12} \left| {}_{\max} f''(c) \right|, a \le c \le b =$$

$$T.E \le \frac{2(1.5)^3}{12} \times 461.9673492$$

$$T.E \le 259.856633925$$

Example (2):

Solve example (1) at (n=4)

Simpson's Rule:

طريقة الحل :

نأخذ ثلاث نقاط ونكون منهم كثيرة حدود من الدرجة الثانية ثم نحسب المساحة أسفل المنحني

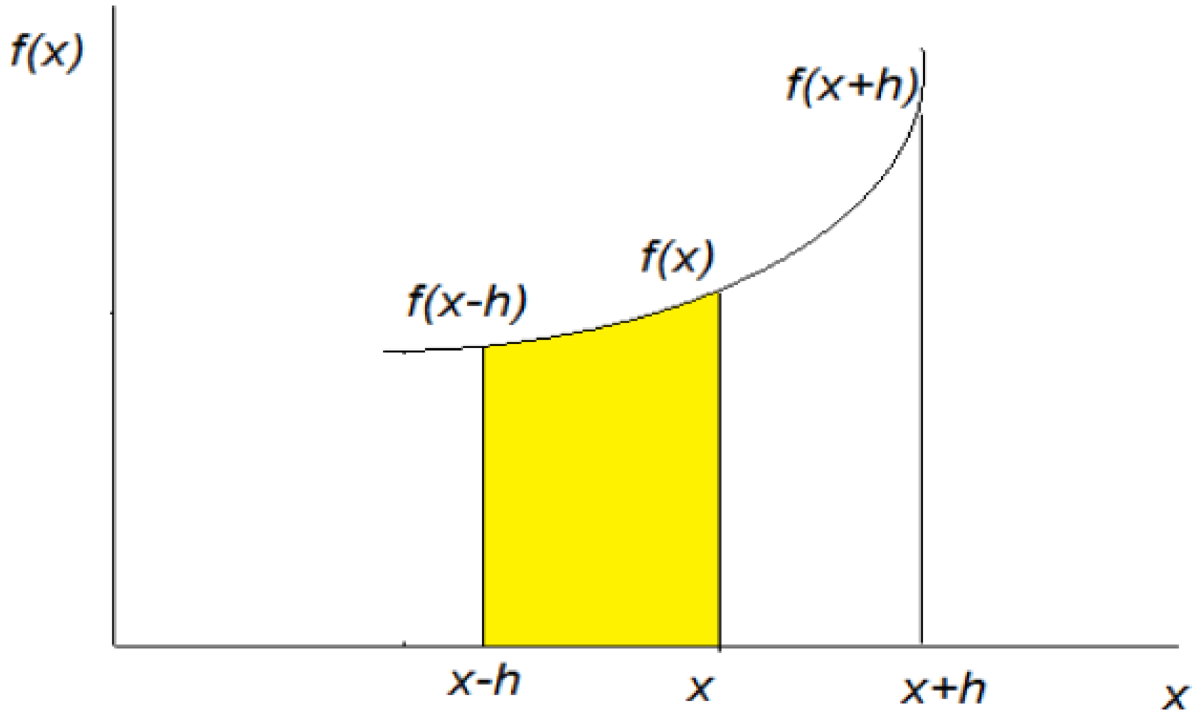


Figure (5) Simpson's Rule

$$I = \int_{x-h}^{x+h} f(x)dx = F(x)\Big|_{x-h}^{x+h}$$

$$I = F(x+h) - F(x-h) \rightarrow 1$$

$$\because F(x+h) = F(x) + hF'(x) + \frac{h^2}{2}F''(x) + \frac{h^3}{3!}F'''(x) + \frac{h^4}{4!}F^{(4)}(x)$$

$$\because F(x-h) = F(x) - hF'(x) + \frac{h^2}{2}F''(x) - \frac{h^3}{3!}F'''(x) + \frac{h^4}{4!}F^{(4)}(x)$$

$$\therefore F(x+h) - F(x-h) = I = 2hF'(x) + 2\frac{h^3}{3!}F'''(x) + 2\frac{h^5}{5!}F^{(5)}(x)$$

$$\because F'(x)=f(x),\ F''(x)=f'(x),\ F'''(x)=f(x)''$$

$$\therefore I=2hf(x)+2\frac{h^3}{3!}f''(x)+2\frac{h^5}{5!}f^{(4)}(x)$$

$$\because f''(x)=\frac{f(x+h)-2f(x)+f(x-h)}{h^2}-\frac{h^2}{12}f^{(4)}(x)+\ldots\ldots\ldots$$

$$\therefore I=2hf(x)+2\frac{h^3}{3!}\left(\frac{f(x+h)-2f(x)+f(x-h)}{h^2}-\frac{h^2}{12}f^{(4)}(x)+\ldots\ldots\ldots\right)+2\frac{h^5}{5!}f^{(4)}(x)$$

$$\therefore I=2hf(x)+\frac{h}{3}f(x+h)-2\frac{h}{3}f(x)+\frac{h}{3}f(x-h)-\frac{h^5}{36}f^{(4)}(x)+2\frac{h^5}{5!}f^{(4)}(x)+\ldots\ldots$$

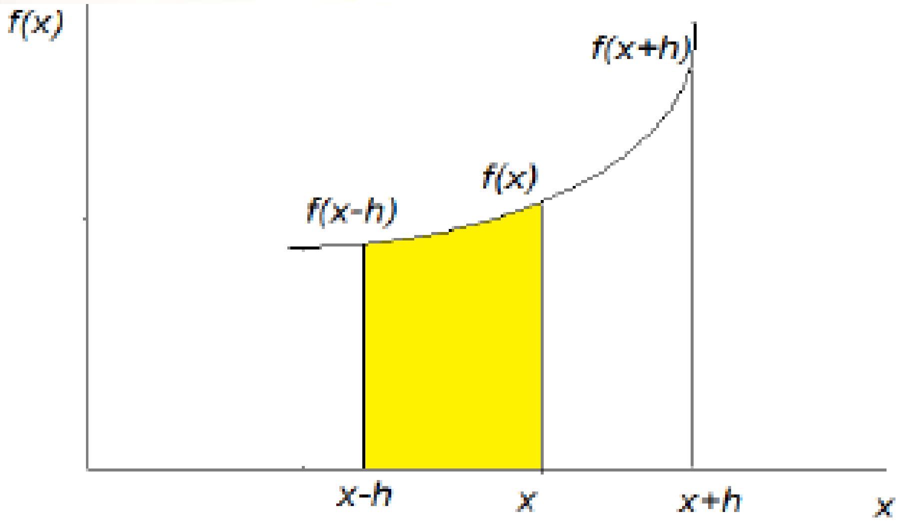
$$\therefore I=\frac{h}{3}[f(x+h)+4f(x)+f(x-h)]-\frac{h^5}{90}f^{(4)}(x)+\ldots$$

$$\therefore I = \frac{h}{3}\left[f(x+h) + 4f(x) + f(x-h)\right]$$

$[f(x+h)] \rightarrow$ الأخير

$[4f(x)] \rightarrow$ الأوسط

$[f(x-h)] \rightarrow$ الأول



$$T.E \leq \frac{h^5}{90}\left|_{\max} f^{(4)}(c)\right|, \qquad x-h \leq c \leq x+h$$

Composite Simpson's Rule:

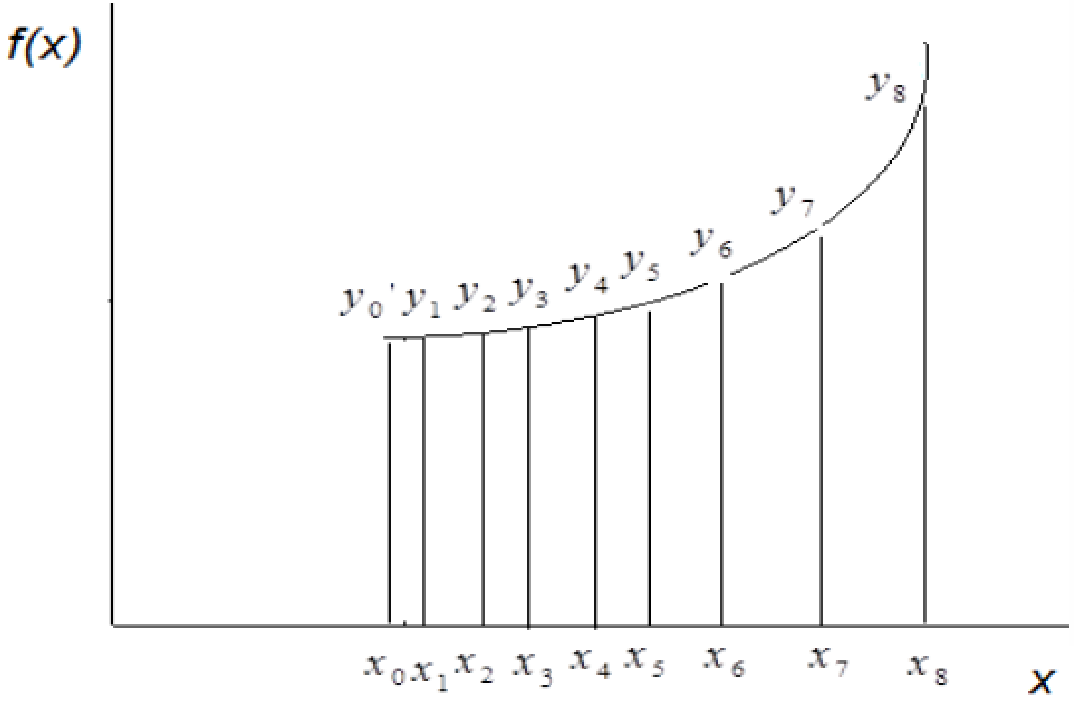


Figure (6) composite Simpson's Rule

$$I \cong \int_{x_0}^{x_2} f(x)dx + \int_{x_2}^{x_4} f(x)dx + \int_{x_4}^{x_6} f(x)dx \int_{x_6}^{x_8} f(x)dx$$

Composite Simpson's Rule:

$$\boxed{I = \frac{h}{3}\left[y_0 + y_n + 4(y_1 + y_3 + \dots) + 2(y_2 + y_4 + \dots)\right]}$$

$$T.E. \leq \frac{(b-a)^5}{180 n^4} M_4$$

M_4 هي أكبر قيمة لـ $f^{(4)}(x)$ في الفترة $[a, b]$.

$I = \frac{h}{3}$ [الأول + الأخير + 4(المفردي) + 2(الزوجي)]

n لابد أن تكون عدد صحيح زوجي.

$n = 2$: Simpson's rule

$$I=\frac{h}{3}\left[y_o+y_n+4\left[y_1+y_3+y_5+y_7\right]+2\left[y_2+y_4+y_6\right]\right]$$

$\left[y_1+y_3+y_5+y_7\right]\rightarrow$ الحدود الفردية , $\left[y_2+y_4+y_6\right]\rightarrow$ الحدود الزوجية

$$T.E\leq\left(\frac{n}{2}\right)\frac{h^5}{90}\left|_{max}f^{(4)}(c)\right|,\qquad x-h\leq c\leq x+h\approx 0\left(h^5\right)$$

$$\frac{n}{2}\rightarrow\frac{\text{عدد التقسيمات}}{2}$$

Example (3):

Use Composite Simpson's Rule with (n=6) to approximate the integration $I = \int_0^1 (7+14x^6)dx = 9$

then determine the absolute error and the true Truncation error

Solution:

$$f(x) = (7+14x^6), h = \frac{b-a}{n} = \frac{1-0}{6} = \frac{1}{6}$$

x	0	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{3}{6}$	$\frac{4}{6}$	$\frac{5}{6}$	1
$y = f(x)$	7	7.0003	7.0192	7.21875	8.2294	11.6874	21
	y_o	y_1	y_2	y_3	y_4	y_5	y_6

$$I = \frac{h}{3}\left[y_o + y_6 + 4[y_1 + y_3 + y_5] + 2[y_2 + y_4]\right]$$

$$I = \frac{1}{6\times 3}\left[7 + 21 + 4[7.003 + 7.21875 + 11.6874] + 2[7.0192 + 8.2294]\right]$$

$$I = 9.007059$$

$$\textbf{\textit{Absolute error}} = |9 - 9.007059| = 7.059\times 10^{-3}$$

Truncation error

$$T.E \le (\frac{n}{2})\frac{h^5}{90}\left|f^{(4)}(c)\right|, \qquad x - h \le c \le x + h \approx 0\left(h^5\right)$$

$$T.E \le \left(\frac{6}{2}\right)\frac{\left(\frac{1}{6}\right)^5}{90}\left|f^{(4)}(c)\right|$$

$$f(x) = \left(7 + 14x^6\right)$$

$$f'(x) = \left(14 \times 6x^5\right)$$

$$f''(x) = \left(14 \times 6 \times 5x^4\right)$$

$$f'''(x) = \left(14 \times 6 \times 5 \times 4x^3\right)$$

$$f^4(x) = \left(14 \times 6 \times 5 \times 4 \times 3x^2\right) \to \max(at\ (x = 1))$$

$$\therefore f^4(1) = \left(14 \times 6 \times 5 \times 4 \times 3 \times 1^2\right) = 14 \times 6 \times 5 \times 4 \times 3$$

$$\therefore T.E \le \left(\frac{6}{2}\right)\frac{\left(\frac{1}{6}\right)^5}{90}\left|(14 \times 6 \times 5 \times 4 \times 3)\right| = 0.021605$$

- **Gaussian quadrature**

Gaussian quadrature:

في الطرق السابقة كان إيجاد طرق التكامل بيعتمد علي تقسيم فترة التكامل إلي عدد من التقسيمات المتساوية h أما في هذه الطريقة فإن جاوس يفرض تكامل الدالة علي الصورة :

$$I = \int_a^b f(x)dx = \int_{-1}^{1} g(t) = \sum_i^n W_i g(t_i)$$

وذلك عن طريق تحويل التكامل من a إلي b $\int_a^b f(x)$ إلي تكامل $\int_{-1}^{1} g(t)$ عن طريق تغيير الدالة

$$I = \int_a^b f(x)dx \Rightarrow \int_{-1}^{1} g(t)dt = \sum_{i=1}^{n} W_i g(t_i)$$

Where W_i Constance

$$\int_a^b f(x)dx = \int_{-1}^{1} g(t)$$

نعوض في العلاقة

$$\frac{x-a}{b-a} = \frac{t+1}{2}$$

$$\therefore x = a + \frac{b-a}{2}(t+1)$$

$$\therefore dx = \frac{b-a}{2}dt$$

$$I = \int_a^b f(x)dx = \int_{-1}^{1} f\left(a + \frac{b-a}{2}(t+1)\right)\frac{b-a}{2}dt = \int_{-1}^{1} g(t)dt = \sum_{i=1}^{n} W_i g(t_i)$$

ثم نستخدم إحدي الطرق التالية لحساب $\sum_{i=1}^{n} W_i g(t_i)$:

Mid-point quadrature:

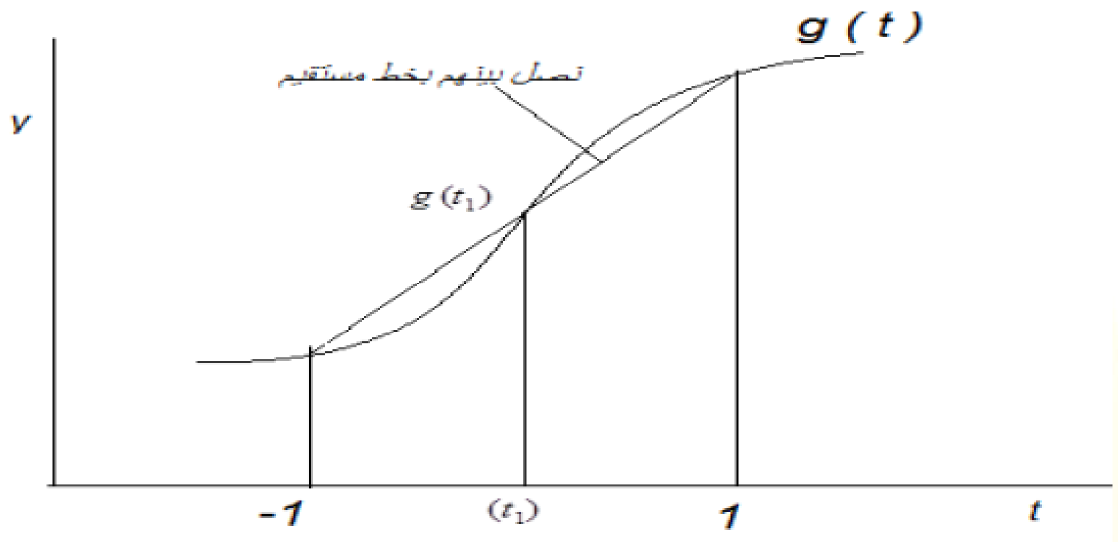


Figure (7) Mid-point quadrature

$$\therefore I = \int_{-1}^{1} g(t)dt = \sum_{i=1}^{n} W_i g(t_i)$$

$g(t)$

$$I = \int_{-1}^{1} g(t)dt = W_1 g(t_1) \rightarrow n = 1$$

عند (n=1) ويكون المطلوب حساب W_1 و t_1 وبالتالي نحتاج معادلتين في W_1 و t_1

ملحوظة هامة :

نفرض أن الدالة $g(t)$ دالة خطية linear polynomial

$g(t)=1,t$

At $g(t)=1$

$$\therefore \int_{-1}^{1} 1dt = t\Big|_{-1}^{1} = 1-(-1) = 2$$

$$\because \int_{-1}^{1} g(t)dt = W_1 g(t_1) \Rightarrow \therefore W_1 = 2$$

At $g(t)=t$

$$\therefore \int_{-1}^{1} tdt = \frac{t^2}{2}\Big|_{-1}^{1} = W_1 g(t_1)$$

$$\int_{-1}^{1} tdt = \frac{t^2}{2}\Big|_{-1}^{1} = W_1 t_1$$

$$\frac{t^2}{2}\Big|_{-1}^{1} = \frac{1^2-(-1)^2}{2} = \frac{1-1}{2} = 0 = 2t_1$$

$$\therefore t_1 = 0$$

$$\therefore \int_{-1}^{1} g(t)dt = 2g(0)$$

$$I = \int_{-1}^{1} g(t)dt = W_1 g(t_1) \qquad \therefore \int_{-1}^{1} g(t)dt = 2g(0)$$

Note that:

1- $\int_{-1}^{1} g(t)dt = 0, g(t)$ is odd function

2- $\int_{-1}^{1} g(t)dt = 2\int_{0}^{1} g(t)dt$, $g(t)$ is even function

2-Point quadrature: (n=2)

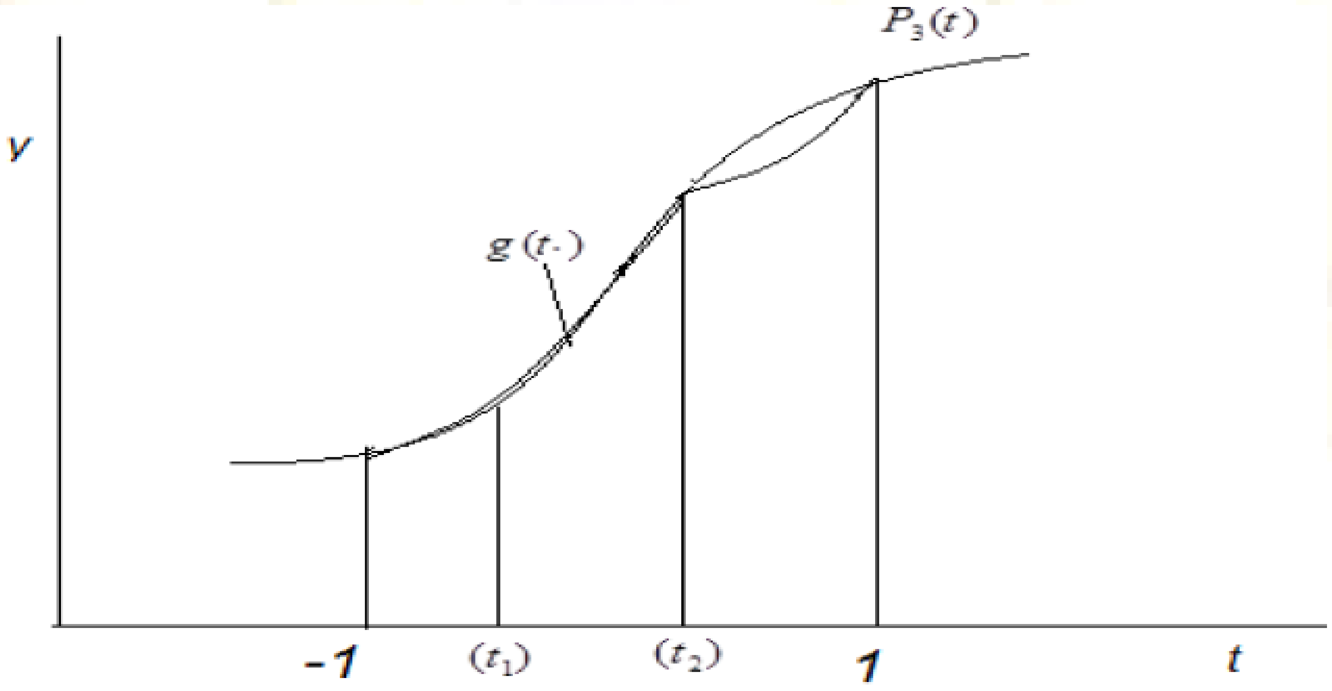


Figure (8) 2-point quadrature

عدد القراءات هنا 4

So that the polynomial of degree (3)

$$\therefore I = \int_{-1}^{1} g(t)dt = \sum_{i=1}^{2} W_i g(t_i) = W_1 g(t_1) + W_2 g(t_2)$$

بكدة لدينا 4 مجاهيل يلزمنا 4 معادلات للحل :

<u>طريقة الحل :</u>

نفرض كثيرة حدود من الدرجة الثالثة

$g(t) = 1, t, t^2, t^3$

At $g(t) = 1$

$$I = \int_{-1}^{1} g(t)dt = \int_{-1}^{1} dt = W_1 g(t_1) + W_2 g(t_2)$$

$$\because \int_{-1}^{1} g(t)dt = 2, g(t_1) = 1, g(t_2) = 1$$

$$\therefore \int_{-1}^{1} g(t)dt = \int_{-1}^{1} dt = W_1 g(t_1) + W_2 g(t_2) \Rightarrow W_1 + W_2 = 2 \rightarrow 1$$

At $g(t)=t$

$$\therefore I=\int_{-1}^{1} g(t)dt=\int_{-1}^{1} t\,dt=W_1 g(t_1)+W_2 g(t_2)$$

$$\because \int_{-1}^{1} t\,dt=0,\ g(t_1)=t_1,\ g(t_2)=t_2$$

$$\therefore \int_{-1}^{1} g(t)dt=\int_{-1}^{1} t\,dt=W_1 g(t_1)+W_2 g(t_2)\Rightarrow W_1 t_1+W_2 t_2=0\rightarrow 2$$

At $g(t)=t^2$

$$\therefore I=\int_{-1}^{1} g(t)dt=\int_{-1}^{1} t^2 dt=W_1 g(t^2{}_1)+W_2 g(t^2{}_2)$$

$$\because \int_{-1}^{1} t^2 dt=\left.\frac{t^3}{3}\right|_{-1}^{1}=\frac{1^3-(-1)^3}{3}=\frac{1-(-1)}{3}=\frac{2}{3},$$

$$g(t_1)=t^2{}_1,\ g(t_2)=t^2{}_2$$

$$\therefore \int_{-1}^{1} g(t)dt=\int_{-1}^{1} t^2 dt=W_1 g(t_1)+W_2 g(t_2)\Rightarrow W_1 t^2{}_1+W_2 t^2{}_2=\frac{2}{3}\rightarrow 3$$

At $g(t)=t^3$

$$\therefore I=\int_{-1}^{1} g(t)dt=\int_{-1}^{1} t^3 dt=W_1 g(t^3{}_1)+W_2 g(t^3{}_2)$$

$$\because \int_{-1}^{1} t^3 dt=\left.\frac{t^4}{4}\right|_{-1}^{1}=\frac{1^4-(-1)^4}{4}=\frac{1-1}{4}=0,$$

$$g(t_1)=t^3{}_1, g(t_2)=t^3{}_2$$

$$\therefore \int_{-1}^{1} g(t)dt=\int_{-1}^{1} t^3 dt=W_1 g(t_1)+W_2 g(t_2)\Rightarrow W_1 t^3{}_1+W_2 t^3{}_2=0\rightarrow 4$$

بحل المعادلات معا نجد أن :

$$W_1=W_2=1$$

$$t_1=-\frac{1}{\sqrt{3}}, t_2=\frac{1}{\sqrt{3}}$$

2-Point quadrature: (n=2)

$$\because I = \int_{-1}^{1} g(t)\,dt = g\left(-\frac{1}{\sqrt{3}}\right) + g\left(\frac{1}{\sqrt{3}}\right)$$

3- Point quadrature: (n=3)

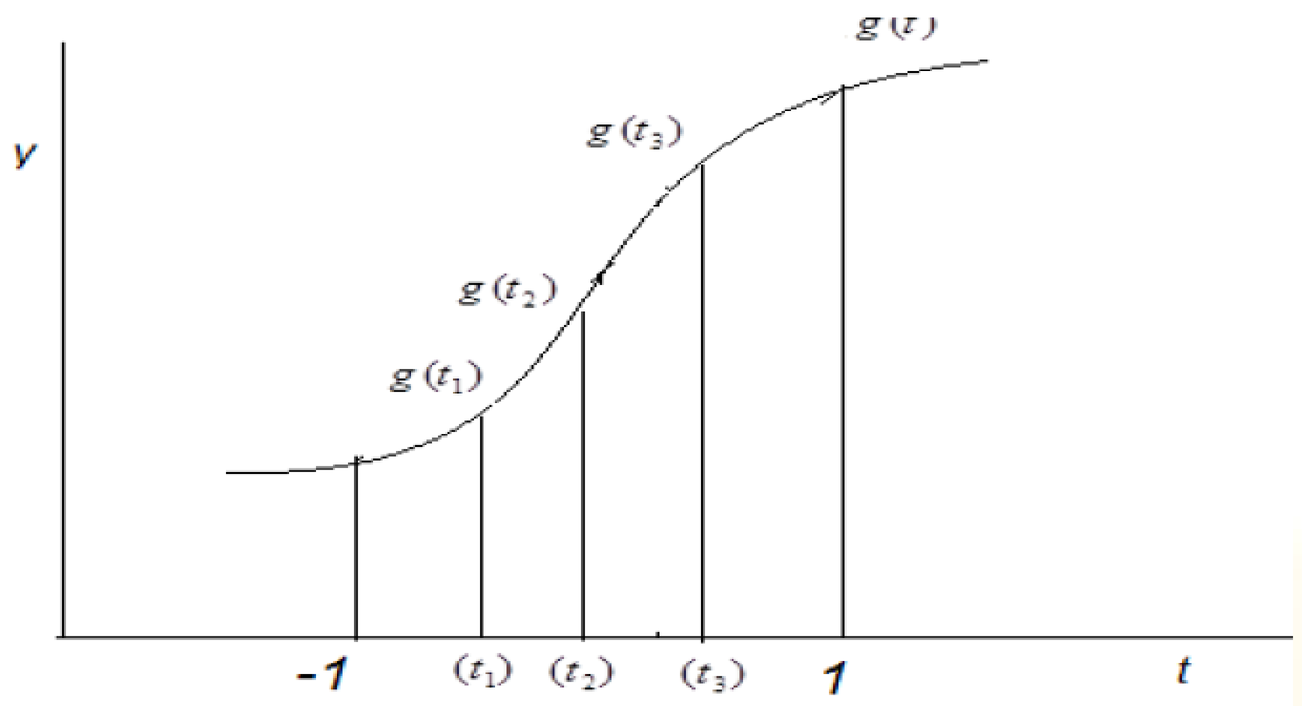


Figure (9) 3-point quadrature

$$I=\int_{-1}^{1} g(t)dt=\sum_{i=1}^{3} W_i g(t_i)=W_1 g(t_1)+W_2 g(t_2)+W_3 g(t_3)$$

لدينا 6 مجاهيل يلزمنا 6 معادلات للحل :

So that we assume the polynomial of degree (5)

طريقة الحل :

نفرض كثيرة حدود من الدرجة الخامسة

$g(t)=1,t,t^2,t^3,t^4,t^5$

At $g(t)=1$

$$I=\int_{-1}^{1} g(t)dt=\int_{-1}^{1} dt=W_1 g(t_1)+W_2 g(t_2)+W_3 g(t_3)$$

$$\because \int_{-1}^{1} g(t)dt=2, g(t_1)=1, g(t_2)=1, g(t_3)=1$$

$$\therefore \int_{-1}^{1} g(t)dt=\int_{-1}^{1} dt=W_1 g(t_1)+W_2 g(t_2)+W_3 g(t_3)$$

$$\therefore \int_{-1}^{1} dt=W_1+W_2+W_3=2 \rightarrow 1$$

At $g(t) = t$

$$\therefore I = \int_{-1}^{1} g(t)dt = \int_{-1}^{1} t dt = W_1 g(t_1) + W_2 g(t_2) + W_3 g(t_3)$$

$$\because \int_{-1}^{1} t dt = 0, g(t_1) = t_1, g(t_2) = t_2, g(t_3) = t_3$$

$$\therefore \int_{-1}^{1} g(t)dt = \int_{-1}^{1} t dt = W_1 g(t_1) + W_2 g(t_2) + W_3 g(t_3)$$

$$\int_{-1}^{1} t dt = W_1 t_1 + W_2 t_2 + W_3 t_3 = 0 \rightarrow 2$$

At $g(t)=t^2$

$$\therefore I=\int_{-1}^{1} g(t)dt=\int_{-1}^{1} t^2 dt=W_1 g(t^2{}_1)+W_2 g(t^2{}_2)+W_3 g(t^2{}_3)$$

$$\because \int_{-1}^{1} t^2 dt=\left.\frac{t^3}{3}\right|_{-1}^{1}=\frac{1^3-(-1)^3}{3}=\frac{1-(-1)}{3}=\frac{2}{3},$$

$$g(t_1)=t^2{}_1, g(t_2)=t^2{}_2, g(t_3)=t^2{}_3$$

$$\therefore \int_{-1}^{1} g(t)dt=\int_{-1}^{1} t^2 dt=W_1 g(t_1)+W_2 g(t_2)+W_3 g(t_3)$$

$$\int_{-1}^{1} t^2 dt=W_1 t^2{}_1+W_2 t^2{}_2+W_3 t^2{}_3=\frac{2}{3}\rightarrow 3$$

At $g(t)=t^3$

$$\therefore I=\int_{-1}^{1} g(t)dt=\int_{-1}^{1} t^3 dt=W_1 g(t^3{}_1)+W_2 g(t^3{}_2)+W_3 g(t^3{}_3)$$

$$\because \int_{-1}^{1} t^3 dt=\left.\frac{t^4}{4}\right|_{-1}^{1}=\frac{1^4-(-1)^4}{4}=\frac{1-1}{4}=0,$$

$$g(t_1)=t^3{}_1,\ g(t_2)=t^3{}_2,\ g(t_3)=t^3{}_3$$

$$\therefore \int_{-1}^{1} g(t)dt=\int_{-1}^{1} t^3 dt=W_1 g(t_1)+W_2 g(t_2)+W_3 g(t_3)$$

$$\int_{-1}^{1} t^3 dt=W_1 t^3{}_1+W_2 t^3{}_2+W_3 t^3{}_3=0\rightarrow 4$$

At $g(t)=t^4$

$$\therefore I=\int_{-1}^{1} g(t)dt=\int_{-1}^{1} t^4 dt=W_1 g(t^4{}_1)+W_2 g(t^4{}_2)+W_3 g(t^4{}_3)$$

$$\because \int_{-1}^{1} t^4 dt=\left.\frac{t^5}{5}\right|_{-1}^{1}=\frac{1^5-(-1)^5}{5}=\frac{1-(-1)}{5}=\frac{2}{5},$$

$$g(t_1)=t^4{}_1, g(t_2)=t^4{}_2, g(t_3)=t^4{}_3$$

$$\therefore \int_{-1}^{1} g(t)dt=\int_{-1}^{1} t^4 dt=W_1 g(t_1)+W_2 g(t_2)+W_3 g(t_3)$$

$$\int_{-1}^{1} t^4 dt=W_1 t^4{}_1+W_2 t^4{}_2+W_3 t^4{}_3=\frac{2}{5}\rightarrow 5$$

At $g(t)=t^5$

$$\therefore I=\int_{-1}^{1} g(t)dt=\int_{-1}^{1} t^5 dt=W_1 g(t^5{}_1)+W_2 g(t^5{}_2)+W_3 g(t^5{}_3)$$

$$\because \int_{-1}^{1} t^5 dt=\left.\frac{t^6}{6}\right|_{-1}^{1}=\frac{1^6-(-1)^6}{6}=\frac{1-1}{6}=0$$

$$g(t_1)=t^5{}_1,\ g(t_2)=t^5{}_2,\ g(t_3)=t^5{}_3$$

$$\therefore \int_{-1}^{1} g(t)dt=\int_{-1}^{1} t^5 dt=W_1 g(t_1)+W_2 g(t_2)+W_3 g(t_3)$$

$$\int_{-1}^{1} t^5 dt=W_1 t^5{}_1+W_2 t^5{}_2+W_3 t^5{}_3=0 \rightarrow 6$$

بحل المعادلات معا نجد أن :

$$W_1 = W_3 = \frac{5}{9}, W_2 = \frac{8}{9}$$

$$t_1 = -\sqrt{\frac{3}{5}}, t_2 = 0, t_3 = \sqrt{\frac{3}{5}}$$

3- Point quadrature: (n=3)

$$\because I = \int_{-1}^{1} g(t)dt = \frac{5}{9}g(-\sqrt{\frac{3}{5}}) + \frac{8}{9}g(0) + \frac{5}{9}g(\sqrt{\frac{3}{5}})$$

Example (4):

Use Gaussian quadrature (1- midpoint, 2- points, and 3- points) formula to evaluate the integral

$I = \int_0^1 \frac{dx}{1+x^2}$ then determine the absolute error.

Solution:

Mid-point:

$$\int_{-1}^{1} g(t)dt = 2g(0)$$

$$a = 0, b = 1$$

$$\because \frac{x-a}{b-a} = \frac{t+1}{2}$$

$$\therefore x = a + \frac{b-a}{2}(t+1) = 0 + \frac{1-0}{2}(t+1) = \frac{1}{2}(t+1)$$

$$\therefore dx = \frac{b-a}{2}dt = \frac{1-0}{2}dt = \frac{1}{2}dt$$

$$\therefore I = \int_0^1 \frac{dx}{1+x^2} = \int_{-1}^{1} \frac{\frac{1}{2}dt}{1+\left(\frac{1}{2}(t+1)\right)^2} = \frac{1}{2}\int_{-1}^{1} \frac{dt}{1+\frac{(t+1)^2}{4}} = 2\int_{-1}^{1} \frac{dt}{t^2+2t+5}$$

$$\therefore I = \int_{-1}^{1} \frac{2dt}{t^2+2t+5}$$

$$\therefore g(t) = \frac{2}{t^2+2t+5}$$

$$\therefore g(0) = \frac{2}{0^2+2\times 0+5} = \frac{2}{5}$$

$$\therefore I = \int_0^1 \frac{dx}{1+x^2} = 2g(0) = 2\times(\frac{2}{5}) = \frac{4}{5} = 0.8$$

So the absolute error $= \left|\frac{\pi}{4} - 0.8\right| = 0.014602$

Two point:

$$\therefore I=\int_0^1 \frac{dx}{1+x^2}=\int_{-1}^{1} g(t)=g\left(-\frac{1}{\sqrt{3}}\right)+g\left(\frac{1}{\sqrt{3}}\right)$$

$$\therefore g\left(-\frac{1}{\sqrt{3}}\right)=g\left(-\frac{1}{\sqrt{3}}\right)=\frac{2}{\left(-\frac{1}{\sqrt{3}}\right)^2+2\times-\frac{1}{\sqrt{3}}+5}=0.47863$$

$$\therefore g\left(\frac{1}{\sqrt{3}}\right)=g\left(\frac{1}{\sqrt{3}}\right)=\frac{2}{\left(\frac{1}{\sqrt{3}}\right)^2+2\times\frac{1}{\sqrt{3}}+5}=0.30826$$

$$\therefore I=\int_0^1 \frac{dx}{1+x^2}=0.47863+0.30826=0.78689$$

So the absolute error $=\left|\frac{\pi}{4}-0.78689\right|=0.001492$

3-points:

$$\therefore I = \int_{-1}^{1} g(t)dt = \frac{5}{9} g(-\sqrt{\frac{3}{5}}) + \frac{8}{9} g(0) + \frac{5}{9} g(\sqrt{\frac{3}{5}})$$

$$\because g(t) = \frac{2}{t^2 + 2t + 5}$$

$$\therefore g(0) = g(0) = \frac{2}{0^2 + 2\times 0 + 5} = \frac{2}{5}$$

$$\therefore g(-\sqrt{\frac{3}{5}}) = g(-\sqrt{\frac{3}{5}}) = \frac{2}{\left(-\sqrt{\frac{3}{5}}\right)^2 + 2\times -\sqrt{\frac{3}{5}} + 5} = 0.493728821$$

$$\therefore g(\sqrt{\frac{3}{5}}) = g(\sqrt{\frac{3}{5}}) = \frac{2}{\left(\sqrt{\frac{3}{5}}\right)^2 + 2\times \sqrt{\frac{3}{5}} + 5} = 0.279751841$$

$$I = \int_0^1 \frac{dx}{1+x^2} = \int_{-1}^1 g(t) = \frac{5}{9} g(-\sqrt{\frac{3}{5}}) + \frac{8}{9} g(0) + \frac{5}{9} g(\sqrt{\frac{3}{5}})$$

$$\therefore I = \frac{5}{9} \times 0.49372881 + \frac{8}{9} \times \frac{2}{5} + \frac{5}{9} \times 0.279751841$$

$$\therefore I = 0.78526668$$

So the absolute error $= \left| \frac{\pi}{4} - 0.7852668 \right| = 0.000131$

The relative error $= \left| \frac{\text{absolute error}}{\text{Exact solution}} \right| * 100 = \left| \frac{0.000131}{\frac{\pi}{4}} \right| * 100 = 1.6640\%$

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Thank You