Numerical Differentiation and Integration

[1] Numerical Differentiation:

$f^{n}(x)$	Method	Numerical approximation	Truncation Error (T.E.)
1st derivative	[1] Forward	$f'(x)=\frac{1}{h}[f(x+h)-f(x)]+O(h)$	$T.E \leq \frac{h}{2}M_2$, $x \leq c \leq x+h$
	[2] backward	$f'(x)=\frac{1}{h}[f(x)-f(x-h)]+O(h)$	$T.E \leq \frac{h}{2}M_2$, $x-h \leq c \leq x$
	[3] Centeral	$f'(x)=\frac{1}{2h}[f(x+h)-f(x-h)]+O(h^2)$	$T.E \leq \frac{h^2}{6}M_3$, $x-h \leq c \leq x+h$
	[4] Richardson's Extrapolation	$f'(x)=[\frac{4}{3}\phi(\frac{h}{2})-\frac{1}{3}\phi(h)]+O(h^4)$ $\phi(h)=\frac{1}{2h}[f(x+h)-f(x-h)]$	$T.E \leq \frac{h^4}{480}M_5$, $x-h \leq c \leq x+h$
2nd derivative	[1] Forward	$f''(x)=\frac{1}{h^2}[f(x)-2f(x+h)+f(x+2h)]+O(h)$	$T.E \leq h M_3$, $x \leq c \leq x+2h$
	[2] backward	$f''(x)=\frac{1}{h^2}[f(x)-2f(x-h)+f(x-2h)]+O(h)$	$T.E \leq h M_3$, $x-2h \leq c \leq x$
	[3] Centeral	$f''(x)=\frac{1}{h^2}[f(x+h)-2f(x)+f(x-h)]+O(h^2)$	$T.E \leq \frac{h^2}{12}M_4$, $x-h \leq c \leq x+h$

[2] Numerical Integration:

$$M_n = \max_{\cdots \leq c \leq \cdots} |f^{(n)}(c)|$$

Method	Numerical approximation	Truncation Error (T.E)
[1] Trapezoidal rule	$\int_{a=x_0}^{b=x_n} f(x)\,dx = \frac{h}{2}[f_0+f_n+2(f_1+f_2+\cdots+f_{n-1})]$	$T.E \leq \frac{nh^3}{12}M_2$ $a \leq c \leq b$
[2] Simpson's rule	$\int_{a=x_0}^{b=x_n} f(x)\,dx = \frac{h}{3}[f_0+f_n+4(f_1+f_3+\cdots+f_{n-2})+2(f_2+f_4+\cdots+f_{n-1})]$ n is even زوجي	$T.E \leq \frac{nh^5}{180}M_4$ $a \leq c \leq b$
[3] Gaussian Quadrature	$\int_a^b f(x)\,dx = \int_{-1}^{1} g(t)\,dt = \sum_{i=1}^{n} w_i \cdot g(t_i)$, $\frac{x-a}{b-a}=\frac{t+1}{2}$, $dx=\frac{b-a}{2}dt$ [1] one-point (mid point) $n=1$ $[w_1=2,\ t_1=0]$ $I=\int_{-1}^{1} g(t)\,dt = 2g(0)$ [2] Two-point $n=2$ $[w_1=w_2=1,\ t_1=\frac{-1}{\sqrt{3}},\ t_2=\frac{1}{\sqrt{3}}]$ $I=\int_{-1}^{1} g(t)\,dt = g(\frac{-1}{\sqrt{3}})+g(\frac{1}{\sqrt{3}})$ [3] Three-point $n=3$ $[w_1=w_3=\frac{5}{9},\ w_2=\frac{8}{9},\ t_1=-\sqrt{\frac{3}{5}},\ t_2=0,\ t_3=\sqrt{\frac{3}{5}}]$ $I=\int_{-1}^{1} g(t)\,dt = \frac{5}{9}\cdot g(-\sqrt{\frac{3}{5}})+\frac{8}{9}\cdot g(0)+\frac{5}{9}\cdot g(\sqrt{\frac{3}{5}})$	

[1] For the following data, find $f'(6)$ with error $O(h)$ and $f'(6.3)$ with error $O(h^2)$

x	6	6.1	6.2	6.3	6.4
$f(x)$	-0.175	-0.1998	-0.2223	-0.2422	-0.2596

Sol.

① $f'(6) \rightarrow$ using forward difference formula

$$f'(6) = \frac{f(6.1)-f(6)}{0.1} = \frac{-0.1998+0.175}{0.1} = -0.248$$

② $f'(6.3) \rightarrow$ using centeral difference formula.

$$f'(6.3) = \frac{f(6.4)-f(6.2)}{2*0.1} = \frac{-0.2596+0.2223}{0.2} = -0.183$$

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[2] Find $f'(1.4)$ using backward difference formula, centeral difference formula and Richardson's extrapolation where $F(x) = x\cos x - 3x$ with $h=0.2$. Then find the absolute and Trunction erro

Sol

[1] backward difference formula:

$$f'(x) = \frac{1}{h}[f(x)-f(x-h)] \Rightarrow f'(1.4) = \frac{1}{0.2}[f(1.4)-f(1.2)] = -3.984376527$$

$$\because F'(x) = -x\sin x + \cos x - 3 \Rightarrow F'(1.4) = -4.209662479$$

$$Er(1.4) = |F'(1.4) - f'(1.4)| = 2.252859 * 10^{-1}$$

$$T.E \leq \frac{h}{2}|F''(c)| \quad , \quad 1.2 \leq c \leq 1.4$$

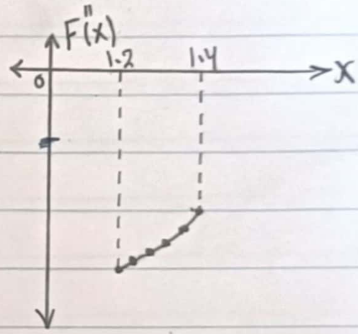
$$\leq \frac{0.2}{2}|F''(1.2)|$$

$$\leq 2.298989 * 10^{-1}$$

$$, F''(x) = -x\cos x - 2\sin x$$

$$F''(1.2) = -2.2989074$$

$$F''(1.4) = -2.208853$$



[2] Centeral difference formula:

$$f'(x) = \frac{1}{2h}[f(x+h) - f(x-h)] \Rightarrow f'(1.4) = \frac{1}{0.4}[f(1.6)-f(1.2)] = -4.203871353$$

$$Er(1.4) = |F'(1.4) - f'(1.4)| = 5.7911265 * 10^{-3}$$

$$T.E \leq \frac{h^2}{6}|F'''(c)| \quad , \quad 1.2 \leq c \leq 1.6$$

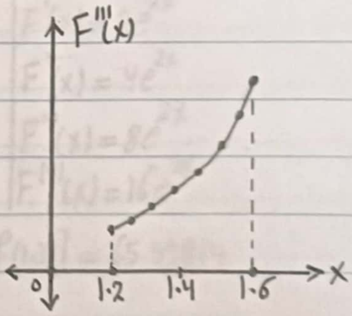
$$\leq \frac{(0.2)^2}{6}|F'''(1.6)|$$

$$\leq 1.12461 * 10^{-2}$$

$$, F'''(x) = x\sin x - 3\cos x$$

$$F'''(1.2) = 0.0313736$$

$$F'''(1.6) = 1.6869163$$



[3] Richardson's extrapolation:

$$f'(x) = \frac{4}{3}\phi\left(\frac{h}{2}\right) - \frac{1}{3}\phi(h) \quad , \quad \phi(h) = \frac{1}{2h}[f(x+h) - f(x-h)]$$

$$f'(1.4) = \frac{4}{3}\phi(0.1) - \frac{1}{3}\phi(0.2)$$

$$= -4.209660709$$

$$\phi(0.2) = \frac{1}{2*0.2}[f(1.4+0.2) - f(1.4-0.2)] = -4.203871353$$

$$\phi(0.1) = \frac{1}{2*0.1}[f(1.4+0.1) - f(1.4-0.1)] = -4.2082133$$

$$Er(1.4) = |F'(1.4) - f'(1.4)| = 1.77 * 10^{-6}$$

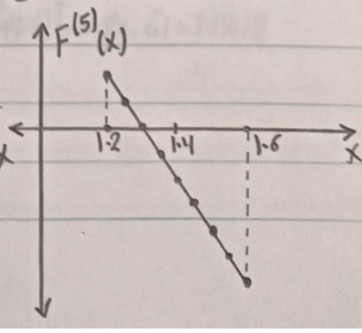
$$T.E \leq \frac{h^4}{480}|F^{(5)}(c)| \quad , \quad 1.2 \leq c \leq 1.6$$

$$\leq \frac{(0.2)^4}{480}|F^{(5)}(1.6)|$$

$$\leq 5.8177 * 10^{-6}$$

$$, F''''(x) = x\cos x + 4\sin x$$

$$F^{(5)}(x) = -x\sin x + 5\cos x$$



3 Find $f'(2.3)$ with $h=0.2$ where $f(x)=xe^x-x^2$ using Richardson's extrapolation, find T.E

Sol

$f'(2.3)=\frac{4}{3}\phi(\frac{0.2}{2})-\frac{1}{3}\phi(0.2)$, $\phi(0.2)=\frac{1}{2*0.2}[f(2.3+0.2)-f(2.3-0.2)]=28.66819521$

$=\frac{4}{3}\phi(0.1)-\frac{1}{3}\phi(0.2)$, $\phi(0.1)=\frac{1}{2*0.1}[f(2.3+0.1)-f(2.3-0.1)]=28.40296807$

$f'(2.3)=\frac{4}{3}*28.40296807-\frac{1}{3}*28.66819521=28.31455903$

$F'(x)=xe^x+e^x-2x=(x+1)e^x-2x \Rightarrow F'(2.3)=28.3148021$

$Er(2.3)=|F'(2.3)-f'(2.3)|=2.430708*10^{-4}$

$T.E\leq\frac{h^4}{180}|F^{(5)}(c)|$, $2.1\leq c\leq 2.5$

$\leq\frac{(0.2)^4}{180}|F^{(5)}(2.5)|$

$\leq 8.12166*10^{-4}$

$F(x)=xe^x-x^2$
$F'(x)=(x+1)e^x-2x$
$F''(x)=(x+2)e^x-2$
$F'''(x)=(x+3)e^x$
$F^{(4)}(x)=(x+4)e^x$
$F^{(5)}(x)=(x+5)e^x$

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4 Given $f(x)=e^{2x}$ and $h=0.1$. Estimate $f''(1.5)$ using
1) Forward difference formula.
2) Backward difference formula.
3) Centeral difference formula.

Then find the absolute error and trunction error.

Sol

1 Forward difference formula:

$f''(x)=\frac{1}{h^2}[f(x)-2f(x+h)+f(x+2h)] \Rightarrow f''(1.5)=\frac{1}{0.1^2}[f(1.5)-2f(1.6)+f(1.7)]=98.45765764$

$Er(1.5)=|F''(1.5)-f''(1.5)|=18.1155099$

$F''(1.5)=80.34214769$

$T.E\leq h|F'''(c)|$, $1.5\leq c\leq 1.7$

$\leq 0.1|F'''(1.7)|\leq 23.97128$

$F(x)=e^{2x}$
$F'(x)=2e^{2x}$
$F''(x)=4e^{2x}$
$F'''(x)=8e^{2x}$
$F^{(4)}(x)=16e^{2x}$

2 Backward difference formula:

$f''(x)=\frac{1}{h^2}[f(x)-2f(x-h)+f(x-2h)] \Rightarrow f''(1.5)=\frac{1}{0.1^2}[f(1.5)-2f(1.4)+f(1.3)]=65.99814$

$Er(1.5)=|F''(1.5)-f''(1.5)|=14.344$

$T.E\leq h|F'''(c)|$, $1.3\leq c\leq 1.5$

$\leq 0.1|F'''(1.5)|\leq 16.068429$

3 Centeral difference formula:

$f''(x)=\frac{1}{h^2}[f(x-h)-2f(x)+f(x+h)] \Rightarrow f''(1.5)=\frac{1}{0.1^2}[f(1.4)-2f(1.5)+f(1.6)]=80.61031218$

$Er(1.5)=|F''(1.5)-f''(1.5)|=2.6816*10^{-1}$

$T.E\leq\frac{h^2}{12}|F^{(4)}(c)|$, $1.4\leq c\leq 1.6$

$\leq\frac{0.1^2}{12}|F^{(4)}(1.6)|\leq 3.271*10^{-1}$

5 Using Taylor's series, deduce a forward difference formula for the first derivative such that the truncation error is of order $O(h^2)$

Sol

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \frac{h^4}{4!}f^{(4)}(x) + \cdots$$

$$f(x+2h) = f(x) + 2hf'(x) + \frac{4h^2}{2!}f''(x) + \frac{8h^3}{3!}f'''(x) + \frac{16h^4}{4!}f^{(4)}(x) + \cdots$$

$$4f(x+h) = 4f(x) + 4hf'(x) + \frac{4h^2}{2!}f''(x) + \frac{4h^3}{3!}f'''(x) + \frac{4h^4}{4!}f^{(4)}(x) + \cdots$$

$$4f(x+h) - f(x+2h) = 3f(x) + 2hf'(x) - \frac{4h^3}{3!}f'''(x) - \frac{12h^4}{4!}f^{(4)}(x) - \cdots$$

$$f'(x) = \frac{1}{2h}\left[-3f(x) + 4f(x+h) - f(x+2h)\right] + \frac{h^2}{3}f'''(x) + \frac{h^3}{4}f^{(4)}(x) + \cdots$$

$$\therefore f'(x) = \frac{1}{2h}\left[-3f(x) + 4f(x+h) - f(x+2h)\right] + O(h^2) \quad , \quad T.E \leq \frac{h^2}{3}|f'''(c)| \quad , \quad x \leq c \leq x+2h$$

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6 Using Taylor's series, deduce a central difference formula for the 3rd derivative such that the truncation error is of order $O(h^2)$

Sol

$$a \quad f(x+2h) = f(x) + 2hf'(x) + \frac{4h^2}{2!}f''(x) + \frac{8h^3}{3!}f'''(x) + \frac{16h^4}{4!}f^{(4)}(x) + \frac{32h^5}{5!}f^{(5)}(x) + \frac{64h^6}{6!}f^{(6)}(x) + \cdots \quad *a$$

$$b \quad f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \frac{h^4}{4!}f^{(4)}(x) + \frac{h^5}{5!}f^{(5)}(x) + \frac{h^6}{6!}f^{(6)}(x) + \cdots \quad *b$$

$$f(x-h) = f(x) - hf'(x) + \frac{h^2}{2!}f''(x) - \frac{h^3}{3!}f'''(x) + \frac{h^4}{4!}f^{(4)}(x) - \frac{h^5}{5!}f^{(5)}(x) + \frac{h^6}{6!}f^{(6)}(x) + \cdots$$

$$c \quad f(x+2h) = f(x) - 2hf'(x) + \frac{4h^2}{2!}f''(x) - \frac{8h^3}{3!}f'''(x) + \frac{16h^4}{4!}f^{(4)}(x) - \frac{32h^5}{5!}f^{(5)}(x) + \frac{64h^6}{6!}f^{(6)}(x) + \cdots \quad *c$$

$$\therefore a f(x+2h) + b f(x+h) + f(x-h) + c f(x-2h) = (a+b+1+c)f(x) + h(2a+b-1-2c)f'(x) + \frac{h^2}{2}(4a+b+1+4c)f''(x)$$
$$+ \frac{h^3}{3!}(8a+b-1-8c)f'''(x) + \frac{h^4}{4!}(16a+b+1+16c)f^{(4)}(x) + \frac{h^5}{5!}(32a+b-1-32c)f^{(5)}(x) + \cdots$$

$$\therefore \left.\begin{array}{ll} 2a+b-1-2c=0 & \rightarrow 2a+b-2c=1 \\ 4a+b+1+4c=0 & \rightarrow 4a+b+4c=-1 \\ 16a+b+1+16c=0 & \rightarrow 16a+b+16c=-1 \end{array}\right\} a=\frac{1}{2} \;,\; b=-1 \;,\; c=\frac{-1}{2}$$

$$\frac{1}{2}f(x+2h) - f(x+h) + f(x-h) - \frac{1}{2}f(x-2h) = h^3 f'''(x) + \frac{h^5}{4}f^{(5)}(x) + \cdots$$

$$f'''(x) = \frac{1}{2h^3}\left[f(x+2h) - 2f(x+h) + 2f(x-h) - f(x-2h)\right] - \frac{h^2}{4}f^{(5)}(x) - \cdots$$

$$T.E \leq \frac{h^2}{4}M_5$$

[8] Use composite trapezoidal rule and Simpson's rule to get $\int_1^2 \frac{2}{\sqrt{x}}\,dx$ with $n=8$ then find the absolute and trunction error.

Sol.

$h=\frac{b-a}{n}=\frac{2-1}{8}=0.125$, $f(x)=\frac{2}{\sqrt{x}}$

[1] Trapezoidal rule:

$I=\frac{h}{2}[f_0+f_8+2[f_1+f_2+\cdots+f_7]]$

$=\frac{0.125}{2}\left[\frac{2}{\sqrt{1}}+\frac{2}{\sqrt{2}}+2\left[\frac{2}{\sqrt{1.125}}+\frac{2}{\sqrt{1.25}}+\frac{2}{\sqrt{1.375}}+\frac{2}{\sqrt{1.5}}+\frac{2}{\sqrt{1.625}}+\frac{2}{\sqrt{1.75}}+\frac{2}{\sqrt{1.875}}\right]\right]=1.657694825$

$I_{exact}=\int_1^2\frac{2}{\sqrt{x}}dx=4\sqrt{x}\Big|_1^2=4[\sqrt{2}-1]=1.656854249$

$f=\frac{2}{\sqrt{x}}=2x^{-\frac{1}{2}}$

$f'=-x^{-\frac{3}{2}}$

$f''=\frac{3}{2}x^{-\frac{5}{2}}=\frac{1.5}{x^{2.5}}$

$Er=|I_{exact}-I_{app}|=8.40575*10^{-4}$

$T.E\leqslant\frac{nh^3}{12}M_2$, $M_2=\max_{a\leqslant c\leqslant b}|f''(c)|$

$\leqslant\frac{8*(0.125)^3}{12}\cdot\max_{1\leqslant c\leqslant 2}\left|\frac{1.5}{c^{2.5}}\right|$

$\leqslant\frac{8*(0.125)^3}{12}*\frac{1.5}{1^{2.5}}=1.953125*10^{-3}$

[2] Simpson's rule $(O(h^5))$

$I=\frac{h}{3}[f_0+f_8+4(f_1+f_3+f_5+f_7)+2(f_2+f_4+f_6)]$

$=\frac{0.125}{3}\left[\frac{2}{\sqrt{1}}+\frac{2}{\sqrt{2}}+4\left(\frac{2}{\sqrt{1.125}}+\frac{2}{\sqrt{1.375}}+\frac{2}{\sqrt{1.625}}+\frac{2}{\sqrt{1.875}}\right)+2\left(\frac{2}{\sqrt{1.25}}+\frac{2}{\sqrt{1.5}}+\frac{2}{\sqrt{1.75}}\right)\right]=1.656858748$

$Er=4.499389*10^{-6}$

$f'''=\frac{-15}{4}x^{-\frac{7}{2}}$

$f^{(4)}=\frac{105}{8}x^{-\frac{9}{2}}=\frac{105}{8x^{4.5}}$

$T.E\leqslant\frac{nh^5}{180}M_4$, $M_4=\max_{a\leqslant c\leqslant b}|f^{(4)}(c)|$

$\leqslant\frac{8*(0.125)^5}{180}\cdot\max_{1\leqslant c\leqslant 2}\left|\frac{105}{8c^{4.5}}\right|$

$\leqslant\frac{8*(0.125)^5}{180}*\frac{105}{8\cdot(1)^{4.5}}=1.780192*10^{-5}$

using Simpson's rule $(O(h^7))$

$I=\frac{2h}{45}[7(f_0+f_8)+32(f_1+f_3+f_5+f_7)+12(f_2+f_6)+14(f_4)]$

$=\frac{2*0.125}{45}\left[7\left(\frac{2}{\sqrt{1}}+\frac{2}{\sqrt{2}}\right)+32\left(\frac{2}{\sqrt{1.125}}+\frac{2}{\sqrt{1.375}}+\frac{2}{\sqrt{1.625}}+\frac{2}{\sqrt{1.875}}\right)+12\left(\frac{2}{\sqrt{1.25}}+\frac{2}{\sqrt{1.75}}\right)+14\left(\frac{2}{\sqrt{1.5}}\right)\right]=1.65685461$

$Er=3.6604*10^{-7}$

$f^{(5)}=\frac{-945}{16}x^{-\frac{11}{2}}$

$T.E\leqslant\frac{2nh^7}{945}M_6=\frac{2*8*(0.125)^7}{945}*\frac{10395}{32*(1)^{6.5}}=2.6226*10^{-6}$

$f^{(6)}=\frac{10395}{32}x^{-\frac{13}{2}}=\frac{1039}{32x}$

[9] Use Simpson's rule to get $\int_1^2 (2x+1)\,dx$ with $n=5$

Sol

$h = \frac{b-a}{n} = \frac{2-1}{5} = 0.2$, $f(x) = 2x+1$

	f_0	f_1	f_2	f_3	f_4	f_5
x	1	1.2	1.4	1.6	1.8	2
f(x)	3	3.4	3.8	4.2	4.6	5

Trapezoidal   Simpson

$$I = I_T + I_S$$
$$= \frac{h}{2}[f_0+f_1] + \frac{h}{3}[f_1+f_5+2f_3+4(f_2+f_4)]$$
$$= \frac{0.2}{2}[3+3.4] + \frac{0.2}{3}[3.4+5+4[3.8+4.6]+2[4.2]]$$
$$= 4$$
$$I_{exact} = \int_1^2 2x+1\,dx = x^2+x\Big|_1^2 = 4$$
$$Er = 0$$

[10] Find the number of subintervals to approximate $\int_0^1 \frac{1}{x^2+6}\,dx$ with error less than 10^{-4} by trapezoidal rule.

Sol

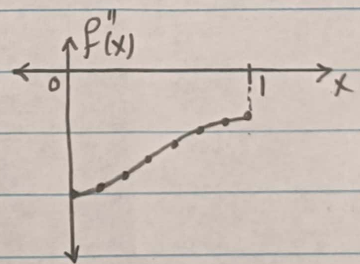
$T.E \leq \frac{nh^3}{12} M_2$, $h = \frac{b-a}{n} = \frac{1}{n}$, $M_2 = \max_{0\leq c\leq 1} |f''(c)|$

$f = \frac{1}{(x^2+6)} = (x^2+6)^{-1}$

$f' = -2x(x^2+6)^{-2}$

$f'' = \frac{6x^2-12}{(x^2+6)^3}$

$$\leq \frac{1}{12n^2} \cdot \max_{0\leq c\leq 1} \left|\frac{6c^2-12}{(c^2+6)^3}\right|$$



$$\leq \frac{1}{12n^2}\left|\frac{0-12}{(0+6)^3}\right|$$
$$\leq \frac{1}{216n^2} = 10^{-4}$$
$$\therefore n^2 = \frac{10^4}{216} \rightarrow n \simeq 6.8 \simeq 7$$

[11] Use ... Point and 3-Point Gaussian quadrature method to get $\int^{2.2} x e^x\,dx$