Chapter 1

Signal Analysis

Lecture 3

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Fourier

Transform

Meaning of Fourier Transform

- Any signal $f(t)$ in the interval $-\frac{T_o}{2} \le t \le \frac{T_o}{2}$ can be represented by a set of **periodic** exponentials $e^{jn\omega_o t}$ (discrete sum), $\omega_o = 2\pi/T_o$.
- If the signal is periodic $f_{T_o}(t)$, the representation is valid for all t.

$$f_{T_o}(t) = \sum_{-T_o/2}^{+T_o/2} F_n e^{jn\omega_o t}, F_n = \frac{1}{T_o} \int_{-T_o/2}^{+T_o/2} f_{T_o}(t)\, e^{-jn\omega_o t}\, dt$$

- So, any discrete sum in frequency domain can be represented by a periodic function in time domain.
- Is it possible to have a representation valid for all t for non-periodic functions?

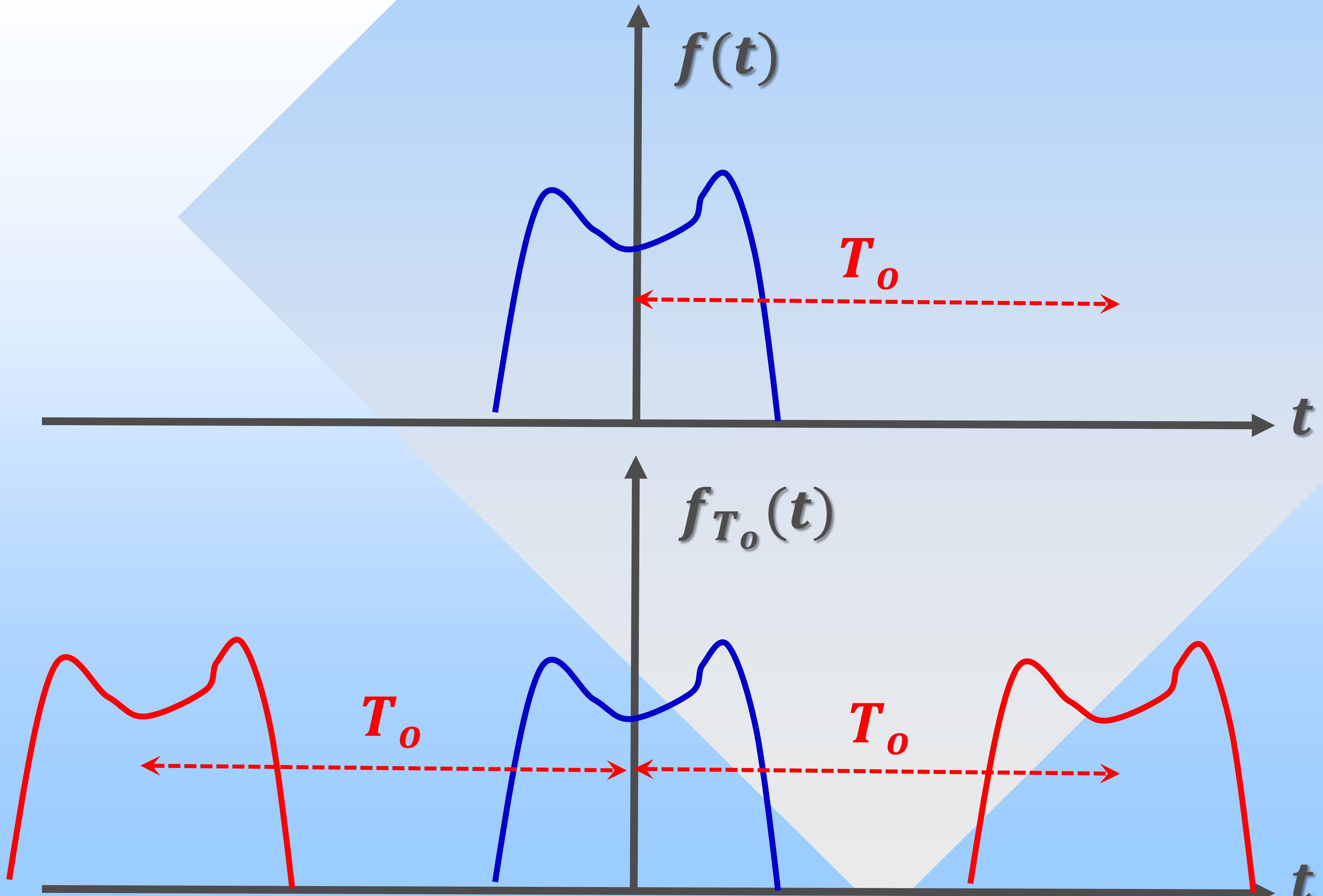
Achieving Fourier Transform

- Given a non periodic or aperiodic or finite duration signal $f(t)$ of duration T_o
- Repeat this aperiodic function $f(t)$ to be periodic with period T_o.
- So, represent it with the exponentials:

$$f_{T_o}(t) = \sum_{-\infty}^{\infty} F_n e^{jn\omega_o t}, F_n = \frac{1}{T_o}\int_{-T_o/2}^{+T_o/2} f_{T_o}(t)\, e^{-jn\omega_o t}\, dt$$

- Now, letting T_o goes to infinity.
  - Periodic signal returns back to be aperiodic.
  - So, aperiodic signal is represented well.

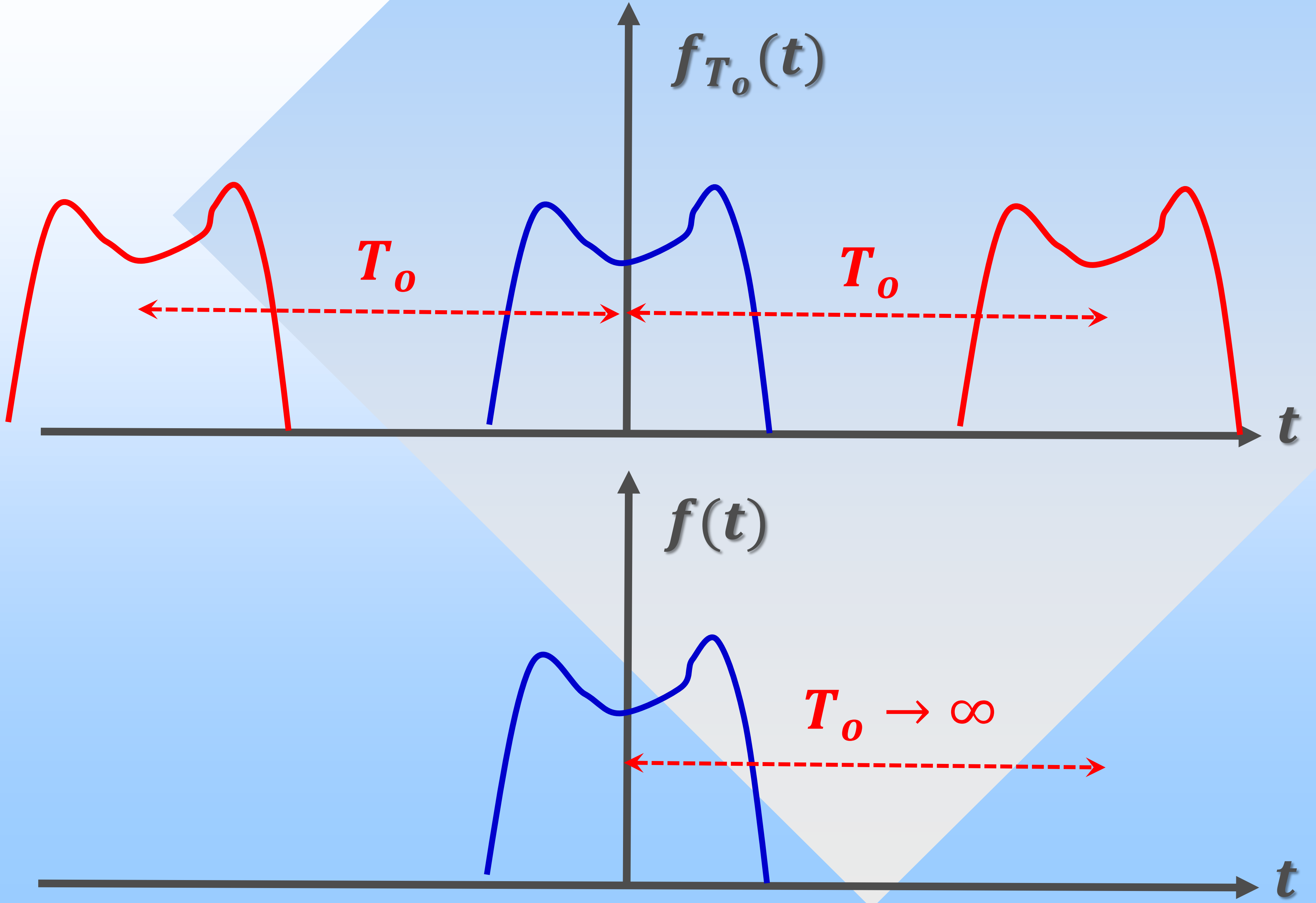
Aperiodic being Periodic



Results of Transform

- When letting T_o goes to infinity, then:
- F_n spectrums will be very condensed with very small amplitudes tends to zero.
- Instead, a relative continuous spectrum $F(\omega)$ can be obtained from the multiplication $F_n T_o$.
- Moreover, a continuous spectrum ω is obtained from the multiplication $n\omega_o$

Periodic being back Aperiodic



When: $T_o \to \infty$

$$f_{T_o}(t) \to f(t)\ Nonperiodic$$

$$n\omega_o \to \omega$$

$$F_n T_o \to F(\omega)$$

$$\omega_o = \frac{2\pi}{T_o} \to d\omega\ Contineous$$

$$\sum_{n=-\infty}^{\infty} \to \int_{-\infty}^{+\infty}$$

Fourier Transform Pair

- **Fourier Transform**

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t} dt$$

- **Inverse Fourier Transform:**

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega)\, e^{j\omega t} d\omega$$

Frequency and Time Domains

- $F(\omega)$ is the frequency domain of the signal whereas $f(t)$ is time domain.
- Any signal can be represented in both time and frequency domains.
- Properties of the signal along the time t-axis are reciprocal to that along the frequency ω -axis.
- $F(\omega)$ is the spectrum of $f(t)$.

Existence of Fourier Transform

Existence of Fourier Transform

- FT exist if its absolute $|F(\omega)|$ is finite
- But, absolute value of the summation is less than the sum of the individual values.

$$|F(\omega)| = \left|\int_{-\infty}^{\infty} f(t)\, e^{-j\omega t} dt\right| \leq \int_{-\infty}^{\infty} \left|f(t)\, e^{-j\omega t}\right| dt$$

$$\leq \int_{-\infty}^{\infty} |f(t)|\, dt \leq \infty$$

- This is a sufficient but not necessary for $F(\omega)$
- Some functions not absolute integral but could have in the limit a FT: $1 \leftrightarrow 2\pi\delta(\omega)$.

Examples on Fourier Transform

1-FT of Rectangular Pulse

$$f(t) = rect\left(\frac{t}{\tau}\right)$$

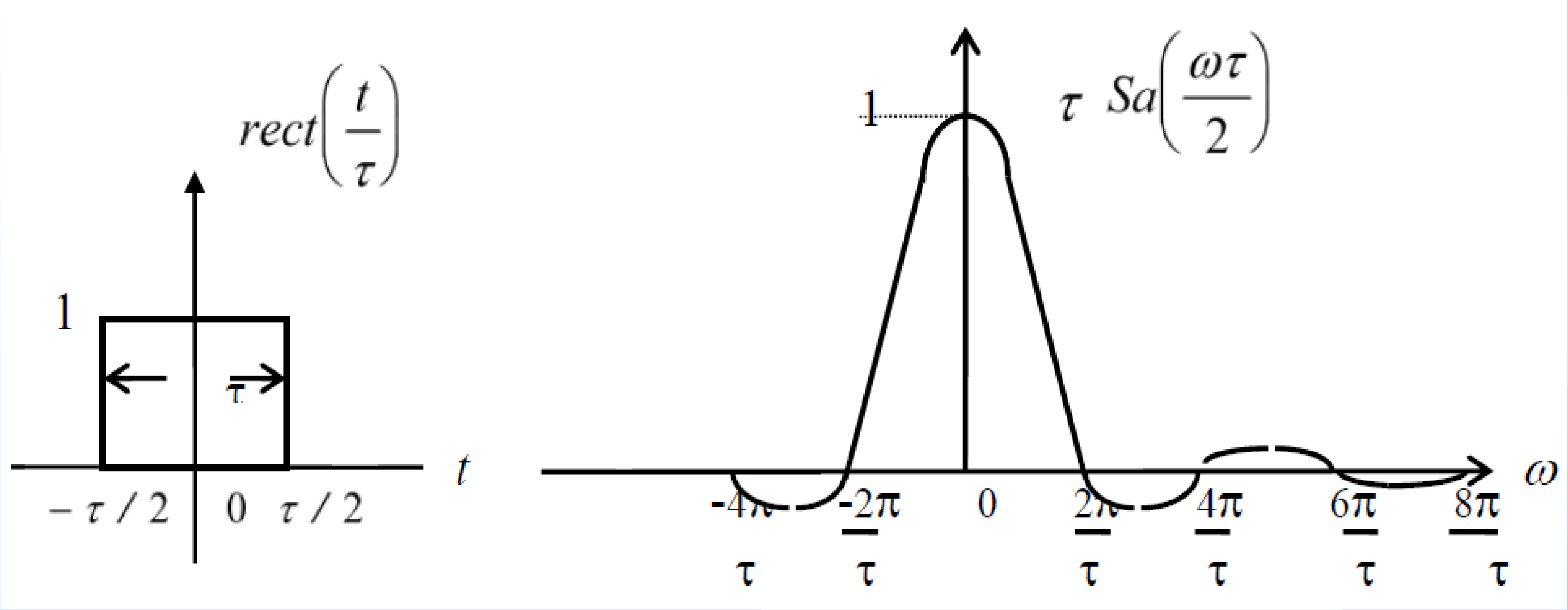
$$F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t} dt = \int_{-\infty}^{\infty} rect\left(\frac{t}{\tau}\right) e^{-j\omega t} dt$$

$$F(\omega) = \int_{-\tau/2}^{\tau/2} e^{-j\omega t} dt = \int_{-\tau/2}^{\tau/2} \cos\omega t - j\sin\omega t\, dt$$

$$F(\omega) = 2\int_{0}^{\tau/2} \cos\omega t\, dt = 2\left.\frac{\sin\omega t}{\omega}\right|_{0}^{\tau/2} = 2\frac{\sin\omega\tau/2}{\omega}$$

$$F(\omega) = \tau\frac{\sin\omega\tau/2}{\omega\tau/2} = \tau\, Sinc\left(\frac{\omega\tau}{2}\right)$$

Time and Frequency Domains of Rect



Time Domain **Frequency Domain**

2-FT of One-sided Exponential

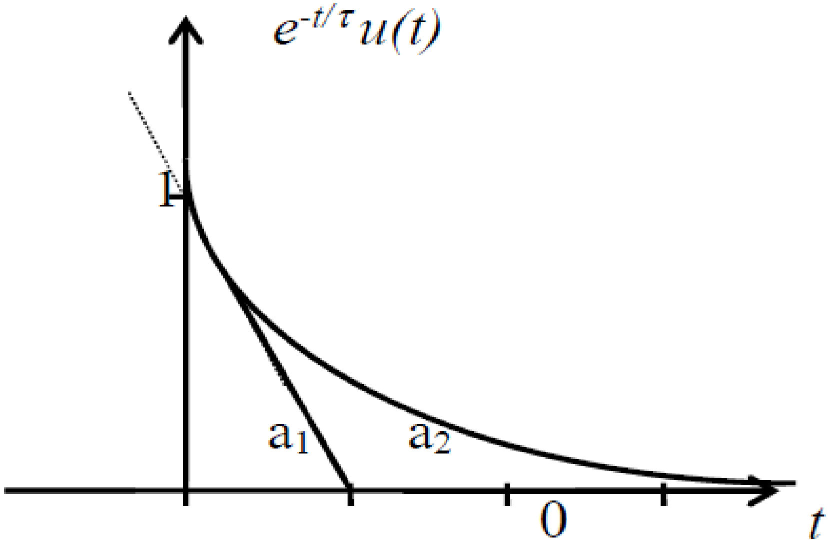
$$\Im\left[e^{-t/\tau}u(t)\right] = \int_{-\infty}^{\infty} e^{-t/\tau}u(t)\, e^{-j\omega t}dt$$

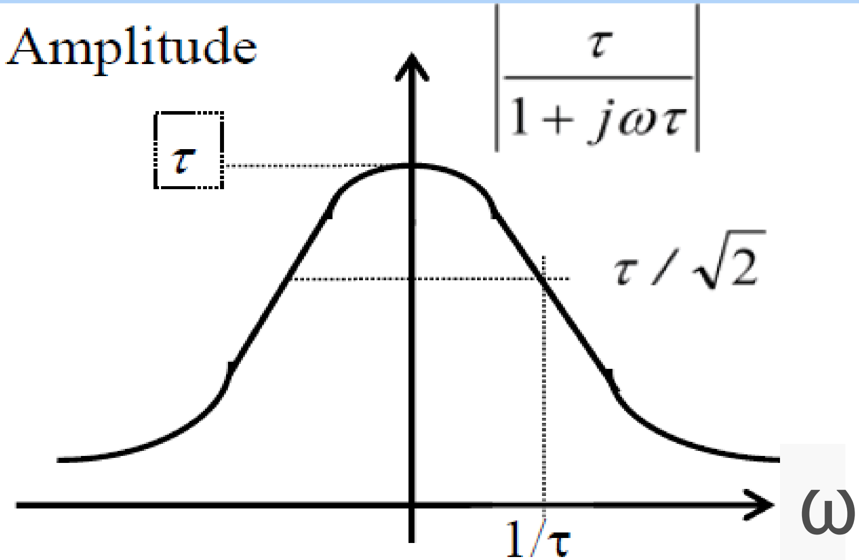
$$= \int_{0}^{\infty} e^{-t/\tau}\, e^{-j\omega t}dt = \int_{0}^{\infty} e^{-\left(\frac{1}{\tau}+j\omega\right)t}dt$$

$$= \left.\frac{e^{-\left(\frac{1}{\tau}+j\omega\right)t}}{-\left(\frac{1}{\tau}+j\omega\right)}\right|_{0}^{\infty} = \frac{0-1}{-\left(\frac{1}{\tau}+j\omega\right)}$$

$$\Im\left[e^{-t/\tau}u(t)\right] = \frac{1}{\frac{1}{\tau}+j\omega} = \frac{\tau}{1+j\omega\tau}$$

One-sided Exponential





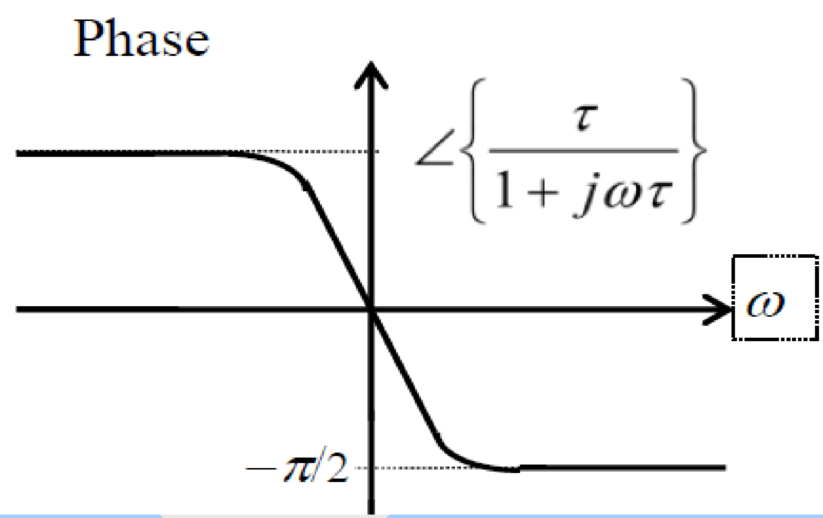
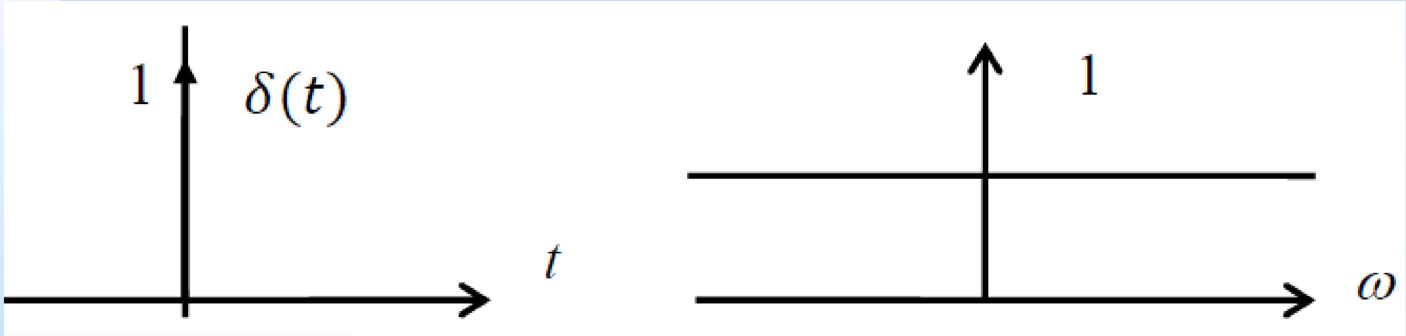


Fig.1.19: Fourier Transform of a single sided Exponential (It is a complex spectrum of an amplitude and phase as indicated in figures *b* and *c*.

Time Domain **Frequency Domain**

3-FT of Delta Function

$$\Im[\delta(t)] = \int_{-\infty}^{\infty} \delta(t)\, e^{-j\omega t} dt = e^{-j\omega t}\Big|_{t=0} = e^{0} = 1$$



- Impulse is very rich by frequencies since it contains all frequencies with the same weight.
- So, Delta function can then be defined as:

$$\delta(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{j\omega t}\, d\omega$$

Another Definition of Delta

$$\delta(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{j\omega t}\, d\omega$$

Put $t = -x$

$$\delta(-x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-j\omega x}\, d\omega$$

Put $\omega = t, d\omega = dt, -\infty \leq \omega \leq \infty, -\infty \leq t \leq \infty$

$$\delta(-x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-jtx}\, dt$$

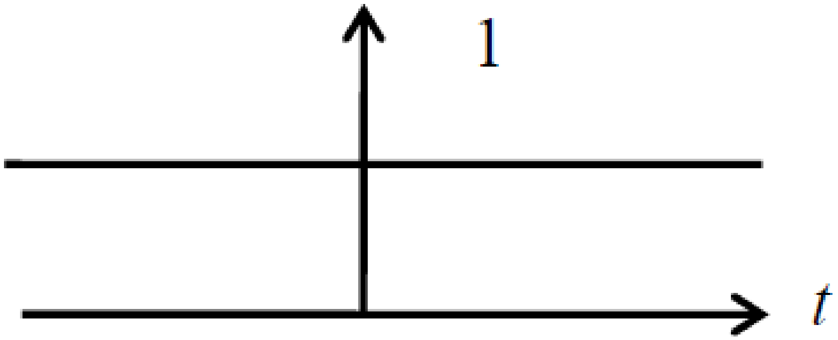
Since Delta is even $\delta(-x) = \delta(x)$, Put $x = \omega$,

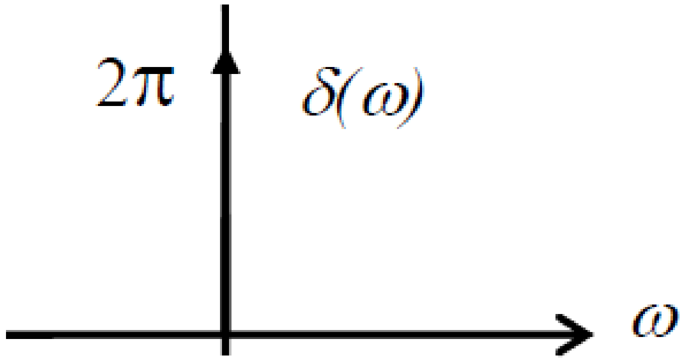
$$\delta(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-j\omega t}\, dt$$

4-FT of the Constant

$$\Im[1] = \int_{-\infty}^{\infty} 1\, e^{-j\omega t} dt$$

$$= 2\pi \left\{ \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-j\omega t} dt \right\} = 2\pi\, \delta(\omega)$$





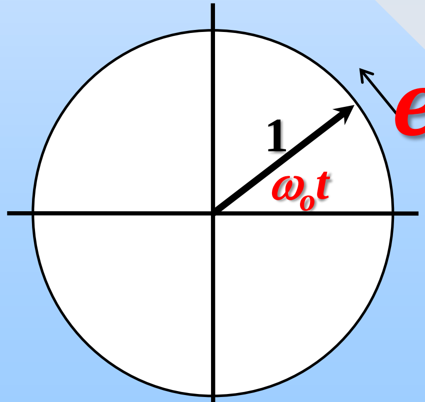
Time Domain **Frequency Domain**

5-FT of Exponential

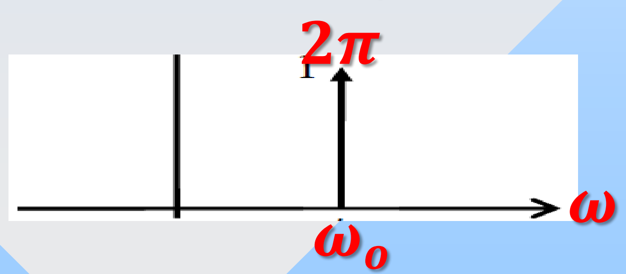
$$\Im[e^{j\omega_o t}] = \int_{-\infty}^{\infty} e^{j\omega_o t}\, e^{-j\omega t} dt$$

$$= 2\pi \left\{ \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-j(\omega-\omega_o)t} dt \right\}$$

$$= 2\pi\, \delta(\omega - \omega_o)$$



$$e^{j\omega_o t} \leftrightarrow 2\pi\, \delta(\omega - \omega_o)$$



Time Domain

Frequency Domain

Properties of Fourier Transform

1-Linearity

- FT is a linear operator.
  - Multiplying by constant in time corresponds to multiplying the frequency by the same constant.

$$f_1(t) \leftrightarrow F_1(\omega)$$

$$\therefore af_1(t) \leftrightarrow aF_1(\omega)$$

$$bf_2(t) \leftrightarrow bF_2(\omega)$$

$$\therefore af_1(t) + bf_2(t) \leftrightarrow aF_1(\omega) + bF_2(\omega)$$

- Addition of signals in time domain corresponding to addition of their spectrums in frequency domain.

2-Scaling

- Compression in time corresponds to an expansion in frequency domain and v.v:

$$f(at) \leftrightarrow \frac{1}{|a|}F\left(\frac{\omega}{a}\right) \quad \text{or} \quad f\left(\frac{t}{\alpha}\right) \leftrightarrow \alpha F(\alpha\omega)$$

$$\frac{1}{|c|}f\left(\frac{t}{c}\right) \leftrightarrow F(c\omega) \quad \text{or} \quad c\,f(ct) \leftrightarrow F\left(\frac{\omega}{c}\right)$$

- Also, a reflection in time corresponds to a reflection in frequency domain.

$$f(-t) \leftrightarrow F(-\omega)$$

Proof of Scaling

$$\Im[f(at)] = \int_{-\infty}^{\infty} f(at)\, e^{-j\omega t\, \frac{a}{a}}\, dt$$

$$= \int_{-\infty}^{\infty} f(at)\, e^{-j\left(\frac{\omega}{a}\right) at}\, dt$$

$$= \frac{1}{a} \int_{-\infty}^{\infty} f(at)\, e^{-j\left(\frac{\omega}{a}\right) at}\, d(at)$$

$$\therefore\ f(at) \leftrightarrow \frac{1}{a} F\left(\frac{\omega}{a}\right)$$

Example 1.10

If $\frac{1}{1+t^2} \leftrightarrow \pi e^{-|\omega|}$, Find $\Im\left[\frac{1}{\propto^2+t^2}\right]$

Answer

$$\Im\left[\frac{1}{\propto^2+t^2}\right] = \Im\left[\frac{1/\propto^2}{1+\left(\frac{t}{\propto}\right)^2}\right] = \frac{1}{\propto^2}\Im\left[\frac{1}{1+\left(\frac{t}{\propto}\right)^2}\right]$$

$$= \frac{1}{\propto^2} \propto \pi e^{-|\propto\omega|} = \frac{\pi}{\propto} e^{-|\propto\omega|}$$

3-Shift

- A time domain shift t_o corresponds to a phase shift $-\omega t_o$ in frequency domain:

$$f(t - t_o) \leftrightarrow F(\omega)e^{-j\omega t_o}$$

- While a frequency shift ω_o corresponds to multiplying time domain by $e^{j\omega_o t}$ that is to increase its speed.

$$f(t)e^{j\omega_o t} \leftrightarrow F(\omega - \omega_o)$$

Proof of Time Shift

$$\Im[f(t-t_o)] = \int_{-\infty}^{\infty} f(t-t_o)\, e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} f(t-t_o)\, e^{-j\omega(t-t_o)} e^{-j\omega t_o} dt$$

$$= e^{-j\omega t_o} \int_{-\infty}^{\infty} f(t-t_o)\, e^{-j\omega(t-t_o)} dt$$

$Put\ t - t_o = \tau, \therefore dt = d\tau$, if $-\infty \le t \le \infty,\ then -\infty \le \tau \le \infty$:

$$\Im[f(t-t_o)] = e^{-j\omega t_o} \int_{-\infty}^{\infty} f(\tau)\, e^{-j\omega\tau} d\tau$$

$$f(t-t_o) \leftrightarrow e^{-j\omega t_o} F(\omega)$$

4-Modulation

- Any low speed time signal $f(t)$ will has low frequencies around origin in frequency domain
- Multiplying $f(t)$ by high speed time signal $\cos \omega_o t$ increases its speed in time domain.
- This modulates the frequency domain so that the low frequencies around the origin become around a high frequency corresponds to the high speed time signal:

$$f(t)\cos \omega_o t \leftrightarrow \frac{1}{2}[F(\omega-\omega_o)+F(\omega+\omega_o)]$$

Proof of Modulation

$$\Im[f(t)\cos\omega_o t] = \int_{-\infty}^{\infty} f(t)\cos\omega_o t\, e^{-j\omega t}dt$$

$$= \int_{-\infty}^{\infty} f(t)\left(\frac{e^{j\omega_o t} + e^{-j\omega_o t}}{2}\right)e^{-j\omega t}dt$$

$$= \frac{1}{2}\left[\int_{-\infty}^{\infty}\left(f(t)e^{j\omega_o t}\right)e^{-j\omega t}dt + \int_{-\infty}^{\infty}\left(f(t)e^{-j\omega_o t}\right)e^{-j\omega t}dt\right]$$

$$= \frac{1}{2}\left[\int_{-\infty}^{\infty} f(t)\, e^{-j(\omega-\omega_o)t}dt + \int_{-\infty}^{\infty} f(t)\, e^{-j(\omega+\omega_o)t}dt\right]$$

$$= \frac{1}{2}[F(\omega-\omega_o) + F(\omega+\omega_o)]$$

Examples

Example 1.12: if $\sin\omega_o t = \frac{\pi}{j}[\delta(\omega-\omega_o)-\delta(\omega+\omega_o)]$

$$\Im[f(t)\sin\omega_o t] = \frac{1}{2j}[F(\omega-\omega_o)-F(\omega+\omega_o)]$$

Example 1.13: If $\frac{1}{1+t^2} \leftrightarrow \pi e^{-|\omega|}$

$$\Im\left[\frac{1}{1+t^2}\cos 5t\right] = \frac{\pi}{2}\left[e^{-|\omega-5|}+e^{-|\omega+5|}\right]$$

5-Differentiation

- Differentiating a time function corresponds to multiplying it by $j\omega$ in frequency domain.

$$\frac{d^n f(t)}{dt^n} \Leftrightarrow (j\omega)^n F(\omega)$$

- Whereas differentiation in frequency domain corresponds to multiplying it by $-jt$ in time domain.

$$(-jt)^n f(t) \Leftrightarrow \frac{d^n F(\omega)}{d\omega^n}$$

Proof

$$\frac{d}{dt}[f(t)] = \frac{d}{dt}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\, e^{j\omega t} d\omega\right]$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} j\omega\, F(\omega)\, e^{j\omega t} d\omega$$

$$\frac{d}{dt}[f(t)] \leftrightarrow j\omega\, F(\omega)$$

$$\frac{d^2}{dt^2}[f(t)] = \frac{d^2}{dt^2}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\, e^{j\omega t} d\omega\right]$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} (j\omega)^2\, F(\omega)\, e^{j\omega t} d\omega$$

$$\frac{d^2}{dt^2}[f(t)] \leftrightarrow (j\omega)^2\, F(\omega)$$

6-Integrentiation

- The function that has zero average does not has dc component.
- For a signal without dc, integration in time domain corresponds to a division by $j\omega$ in the frequency domain and vice versa.

$$\int_{-\infty}^{t} f(\tau)d\tau \Leftrightarrow \frac{F(\omega)}{j\omega} + \pi F(0)\delta(\omega)$$

$$\pi f(0)\delta(t) + \frac{f(t)}{-jt} \Leftrightarrow \int_{-\infty}^{\omega} F(u)du$$

7-Symmetry [Duality]

- If a shape f in time domain corresponds to a shape F in frequency, then the shape F in time corresponds to the shape f in frequency domain but reflected and multiplied by 2π.

$$F(t) = 2\pi\, f(-\omega) = 2\pi\, f(\omega)_{f(t)even\, real}$$

- For an even real time function there is no need for reflection.

Proof

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega)\, e^{j\omega t} d\omega$$

$2\pi\, f(t) = \int_{-\infty}^{\infty} F(\omega)\, e^{j\omega t} d\omega$ Put $t = -x$

$2\pi\, f(-x) = \int_{-\infty}^{\infty} F(\omega)\, e^{-j\omega x} d\omega$ Put $\omega = t$

So, $d\omega = dt$, if $-\infty \leq \omega \leq \infty, \therefore -\infty \leq t \leq \infty$:

$2\pi\, f(-x) = \int_{-\infty}^{\infty} F(t)\, e^{-jtx} dt$ Put $x = \omega$

$$2\pi\, f(-\omega) = \int_{-\infty}^{\infty} F(t)\, e^{-j\omega t} dt$$

Example 1.14

Get the FT of the Constant 1

$$\delta(t) \leftrightarrow 1$$

$$1 \leftrightarrow 2\pi\,\delta(-\omega)$$

But since Delta is even function:

$$1 \leftrightarrow 2\pi\,\delta(\omega)$$

Example 1.15

Deduce the time function for the even rectangular spectrum of width *w* given that the Fourier transform of gate function is as:

$$rect\left(\frac{t}{\tau}\right) \leftrightarrow \tau\, Sinc\left(\frac{\omega\tau}{2}\right)$$

Answer

$$rect\left(\frac{t}{\tau}\right) \leftrightarrow \tau\, Sinc\left(\frac{\omega\tau}{2}\right)$$

$$w\, Sinc\left(\frac{wt}{2}\right) \leftrightarrow 2\pi\, rect\left(-\frac{\omega}{w}\right)$$

$$\frac{w}{2\pi}\, Sinc\left(\frac{wt}{2}\right) \leftrightarrow rect\left(\frac{\omega}{w}\right)$$

8-Conjugation

- **Conjugation** in one domain is corresponding to **conjugation** and **reflection** of the other.

$$f^*(t) \leftrightarrow F^*(-\omega)$$

$$f^*(-t) \leftrightarrow F^*(\omega)$$

Proof in Time

$$f^*(t) = \left[\frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\, e^{j\omega t} d\omega\right]^*$$

$$f^*(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F^*(\omega)\, e^{-j\omega t} d\omega \quad \textbf{put } \omega = -\omega$$

$$\therefore f^*(t) = \frac{1}{2\pi}\int_{+\infty}^{-\infty} F^*(-\omega)\, e^{j\omega t} d(-\omega)$$

$$\therefore f^*(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} F^*(-\omega)\, e^{j\omega t} d(\omega)$$

$$f^*(t) \leftrightarrow F^*(-\omega)$$

Proof in Frequency Domain

$$F^*(\omega) = \left[\int_{-\infty}^{\infty} f(t)\, e^{-j\omega t} dt\right]^*$$

$$F^*(\omega) = \int_{-\infty}^{\infty} f^*(t)\, e^{j\omega t} dt \quad \text{put } t = -t$$

$$\therefore F^*(\omega) = \int_{+\infty}^{-\infty} f^*(-t)\, e^{-j\omega t} d(-t)$$

$$\therefore F^*(\omega) = \int_{-\infty}^{+\infty} f^*(-t)\, e^{-j\omega t} d(t)$$

$$f^*(-t) \leftrightarrow F^*(\omega)$$

9-Real and Complex

- For a complex time function spectrum is also complex:

$$f(t) = r(t) + jx(t) \leftrightarrow F(\omega) = R(\omega) + jX(\omega)$$

- This does not mean:

$$\cancel{r(t) \leftrightarrow R(\omega)}$$

$$\cancel{x(t) \leftrightarrow X(\omega)}$$

$f(t)$ Real

- Spectrum has even magnitude and odd phase:

$$f(t) = r(t) \leftrightarrow F(\omega) = R(\omega)_{Even} + jX(\omega)_{Odd}$$

  - $f(t)$ Real Even:

$$F(\omega) = R(\omega)_{Even}$$

$$X(\omega) = 0$$

  - $f(t)$ Real Odd:

$$R(\omega) = 0$$

$$F(\omega) = jX(\omega)_{Odd}$$

$f(t)$ Pure Real

FT of One-sided Exponential

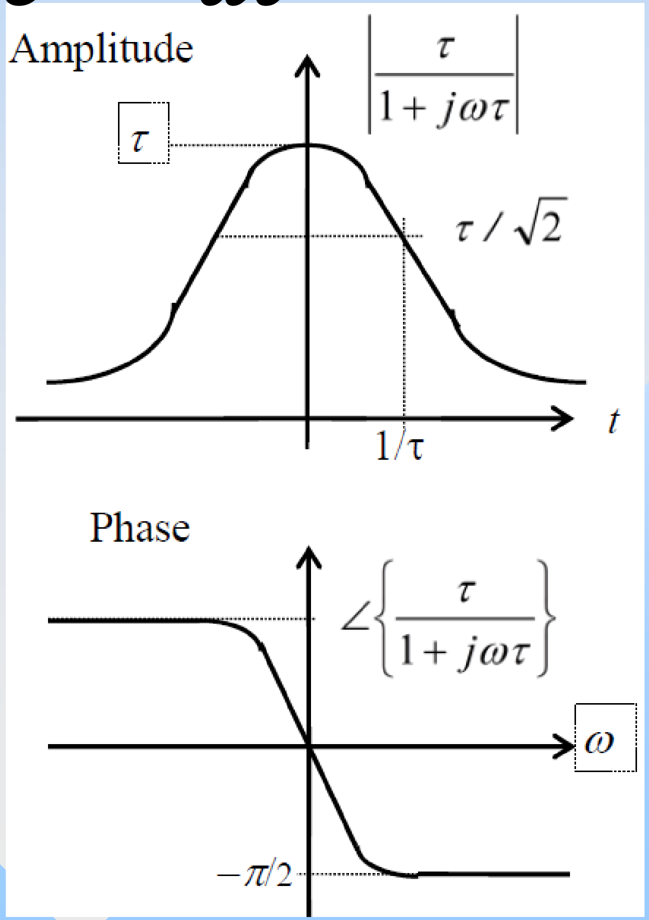
$$\Im\left[e^{-t/\tau}u(t)\right] = \int_{-\infty}^{\infty} e^{-t/\tau}u(t)e^{-j\omega t}dt$$

$$= \int_0^{\infty} e^{-t/\tau}e^{-j\omega t}dt$$

$$= \int_0^{\infty} e^{-\left(\frac{1}{\tau}+j\omega\right)t}dt$$

$$= \left(\frac{-1}{1/\tau + j\omega}\right)e^{-\left(\frac{1}{\tau}+j\omega\right)t}\Big|_0^{\infty}$$

$$= \frac{1}{1/\tau + j\omega} = \frac{\tau}{1 + j\omega\tau}$$



$f(t)$ Even Pure Real

- **Example 1.16:**

$$rect\left(\frac{t}{\tau}\right) \leftrightarrow \tau\, Sinc\left(\frac{\omega\tau}{2}\right)$$

- **Example 1.17:**

$$tri\left(\frac{t}{\tau}\right) \leftrightarrow \tau\, Sinc^2\left(\frac{\omega\tau}{2}\right)$$

- **Example 1.18:**

$$\cos\omega_o t \leftrightarrow \pi[\delta(\omega-\omega_o)+\delta(\omega+\omega_o)]$$

- **Example 1.19:**

$$\delta(t) \leftrightarrow 1$$

$f(t)$ Odd Pure Real

- **<u>Example 1.20:</u>**

$$\sin \omega_o t \leftrightarrow$$

$$\frac{\pi}{j}[\delta(\omega - \omega_o) - \delta(\omega + \omega_o)]$$

$f(t)$ Imaginary

- **Spectrum has odd magnitude and even phase:**

$$f(t) = j\,x(t) \leftrightarrow F(\omega) = R(\omega)_{Odd} + jX(\omega)_{Even}$$

  - **$f(t)$ Imaginary Odd:**

$$F(\omega) = R(\omega)_{Odd}$$

$$X(\omega) = 0$$

  - **$f(t)$ Imaginary Even:**

$$R(\omega) = 0$$

$$F(\omega) = jX(\omega)_{Even}$$