Chapter 1

Signal Analysis

Lecture 1

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Introduction

Overview and Objectives

- Analogy between Vectors and Signals
- Signal in Time and Frequency Domains
- Some Important Functions
- Modulation Theorem
- Convolution and Correlation
- Energy and Power Spectral Densities

Vector Space

and

Base Functions

Vector Space

Vector Space

- A vector in space is represented by 3 primary vectors:

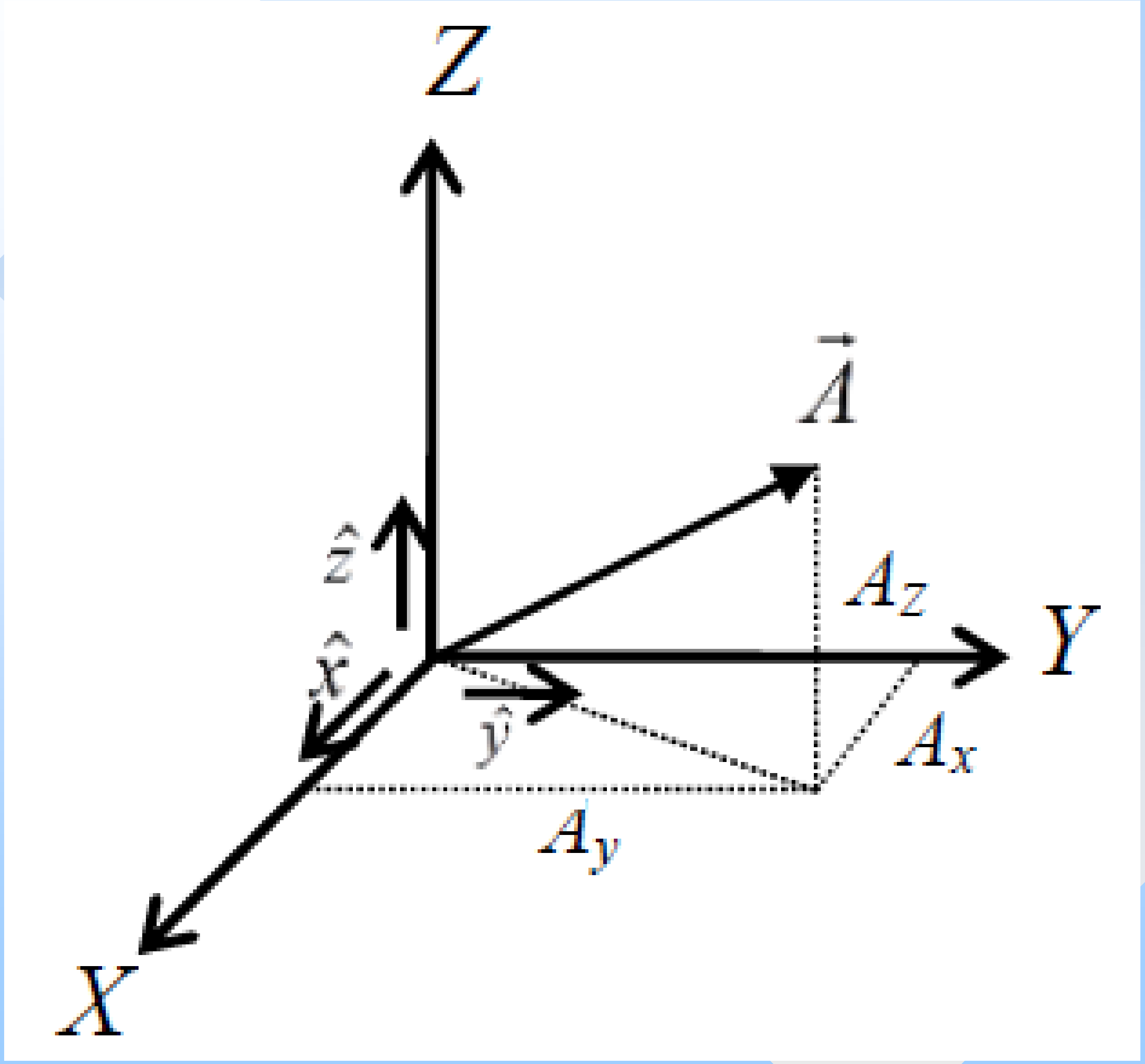
$$\vec{A} = A_x\,\hat{x} + A_y\,\hat{y} + A_z\,\hat{z}$$

- Ax, Ay, Az are the similarity between it and unit vectors
- Primary vectors are called orthogonal since there is no similarity between them:
- Orthogonality condition is: $\hat{x}.\hat{y} = \hat{y}.\hat{z} = \hat{x}.\hat{z} = 0$
- If primary vectors are all-possible directions, the representation is complete.
- In general, vector is represented in *n* dimensions:

$$\vec{A} = \sum_{i=1}^{n} A_i\,\hat{x}_i$$

- Condition of Orthogonality: $\hat{x}_i.\,\hat{x}_j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$

Vector Space



Base Functions

Base Functions

- To represent a time signal f(t) in terms of a set of base functions $\{g_i(t)\}$, which are assumed orthogonal:

$$f(t) = \sum_{i=1}^{n} C_i\, g_i(t) = C_1\, g_1(t) + C_2\, g_2(t) + \cdots$$

- By analogy to vectors, condition of orthogonality:

$$\int_{t_1}^{t_2} g_i(t)\, g_j(t)dt = \begin{cases} k & i = j \\ 0 & i \neq j \end{cases}$$

- Set $\{g_i(t)\}$ may be real or complex, condition is:

$$\int_{t_1}^{t_2} g_i(t) g_j^*(t)dt = \begin{cases} k & i = j \\ 0 & i \neq j \end{cases}$$

- Representation is limited to interval $t_1 \leq t \leq t_2$
- Is complete if no other signal having a weight of $f(t)$

Minimum Error

- We should choose coefficients C_i to give minimum errors, which at the complete set of functions must be zero.

- The coefficients that insure minimum error is given by:

$$C_i = \frac{\int_{t_1}^{t_2} f(t) g_i^*(t)\, dt}{\int_{t_1}^{t_2} \left| g_i^2(t) \right|\, dt}$$

- The proof:

Proof of Minimum Error

- When $f(t)$ is represented in a complete set:

$$f(t) = \sum_{i=0}^{n} C_i \, g_i(t)$$

- i is a dummy index, it can be changed to j:

$$f(t) = \sum_{j=0}^{n} C_j \, g_j(t)$$

- Multiplying both sides by $g_i^*(t)$ and integrating:

$$\int_{t_1}^{t_2} f(t)\, g_i^*(t)\, dt = \int_{t_1}^{t_2} \sum_{j=0}^{n} C_j \, g_j(t)\, g_i^*(t)\, dt$$

- From orthogonality, all terms is 0 except for the term where j equals i:

$$\int_{t_1}^{t_2} f(t)\, g_i^*(t)\, dt = \int_{t_1}^{t_2} C_i \, g_i(t) g_i^*(t)\, dt = C_i \int_{t_1}^{t_2} |g_i(t)|^2 dt$$

Examples of Orthogonal Set

- Cosine Functions, $\{\cos(n\omega_o t)\}$:

$$\int_{t_o}^{t_o+\frac{2\pi}{\omega_0}} \cos n\omega_o t \cos m\omega_o t \, dt =$$

$$\frac{1}{2}\left(\int_{t_o}^{t_o+\frac{2\pi}{\omega_0}} \cos(n+m)\omega_o t \, dt + \int_{t_o}^{t_o+\frac{2\pi}{\omega_0}} \cos(n-m)\omega_o t \, dt \right) = \begin{cases} \dfrac{\pi}{\omega_o} & n = m \\ 0 & n \neq m \end{cases}$$

- Sine Functions, $\{\sin(n\omega_o t)\}$:
- Complete Set $\{\cos(n\omega_o t), \ \sin(n\omega_o t)\}$:

Generalized Fourier Series Expansion

Fourier Series Expansion

To represent a time function $f(t)$ in terms of complete orthogonal set: $\{\cos(n\omega_o t) + \sin(n\omega_o t)\}$: in interval $0 \le t \le T_o$, where $T_o = 2\pi/\omega_o$:

$$\boldsymbol{f(t) = a_o + \sum_{i=1}^{\infty} a_n \cos n\omega_o t + b_n \sin n\omega_o t}$$

- $a_o = \frac{1}{T_o}\int_0^{T_o} f(t)\, dt.$
- $a_n = \frac{2}{T_o}\int_0^{T_o} f(t)\, \cos n\omega_o t\, dt$
- $b_n = \frac{2}{T_o}\int_0^{T_o} f(t)\, \sin n\omega_o t\, dt$

Exercise

In representing $f(t)$ with complete orthogonal set $\{\cos(n\omega_o t) + \sin(n\omega_o t)\}$ in the interval $0 \le t \le T_o$, $T_o = 2\pi/\omega_o$ by the trigonometric expansion:

$$f(t) = a_o + \sum_{i=1}^{\infty} a_n \cos n\omega_o t + b_n \sin n\omega_o t$$

Prove that:

- $a_o = \frac{1}{T_o}\int_0^{T_o} f(t)\, dt.$
- $a_n = \frac{2}{T_o}\int_0^{T_o} f(t)\, \cos n\omega_o t\, dt$
- $b_n = \frac{2}{T_o}\int_0^{T_o} f(t)\, \sin n\omega_o t\, dt$

Periodic and Aperiodic

Generally, Fourier representing **periodic** function in the interval $0 \leq t \leq T_o$ as a linear combinations with different weights of periodic function in T_o.

When $f(t)$ is periodic in T_o, then the expansion is true for all values of t.

In this case $f(t)$ is denoted as $f_{T_o}(t)$.

Exponential

Fourier Series

Exponential Orthogonality

- Prove that exponential set $\{e^{jn\omega_o t}\}$ are orthogonal.

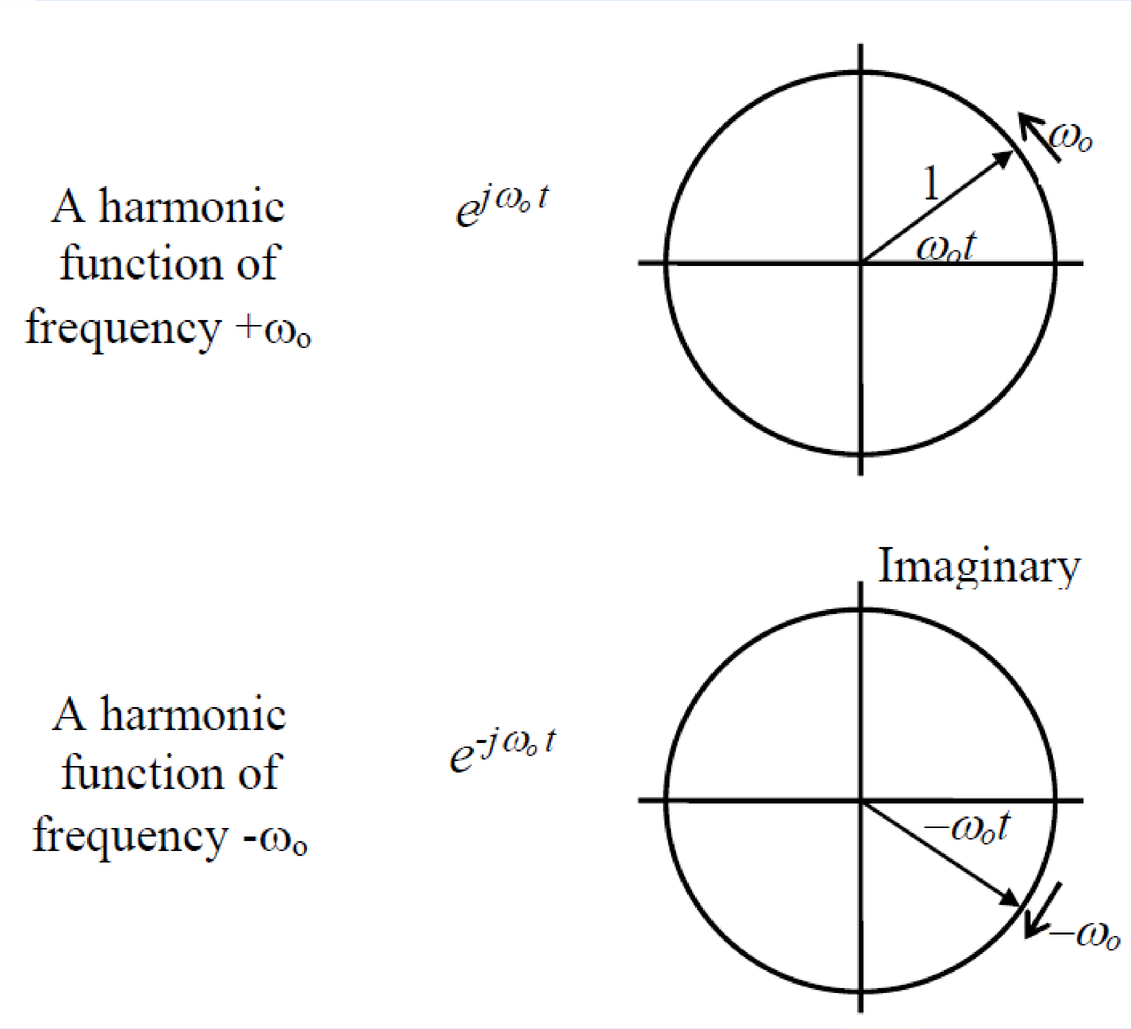
- By applying the orthogonality condition:

$$\int_0^{T_o} e^{jn\omega_o t}(e^{jm\omega_o t})^* dt = \int_0^{T_o} e^{j(n-m)\omega_o t} dt =$$

$$\int_0^{T_o} [\cos(n-m)\omega_o t + j\sin(n-m)\omega_o t] dt = \begin{cases} T_o & if\ n = m \\ 0 & if\ n \neq m \end{cases}$$

Meaning of Exponential Function

- **Exponential function means a phasor diagram**
- **Its value is represented as: $e^{j\omega_o t} = 1\langle\omega_o t$**



Exponential Fourier Series

- Advantages: exponential representation is a complex and complete orthogonal set. ,
- This set is also complete for all values of n: positive, zero, and negative.
- Any function (real or complex) can be expanded in interval $0 \leq t \leq T_o$ as:

$$f(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_o t}$$

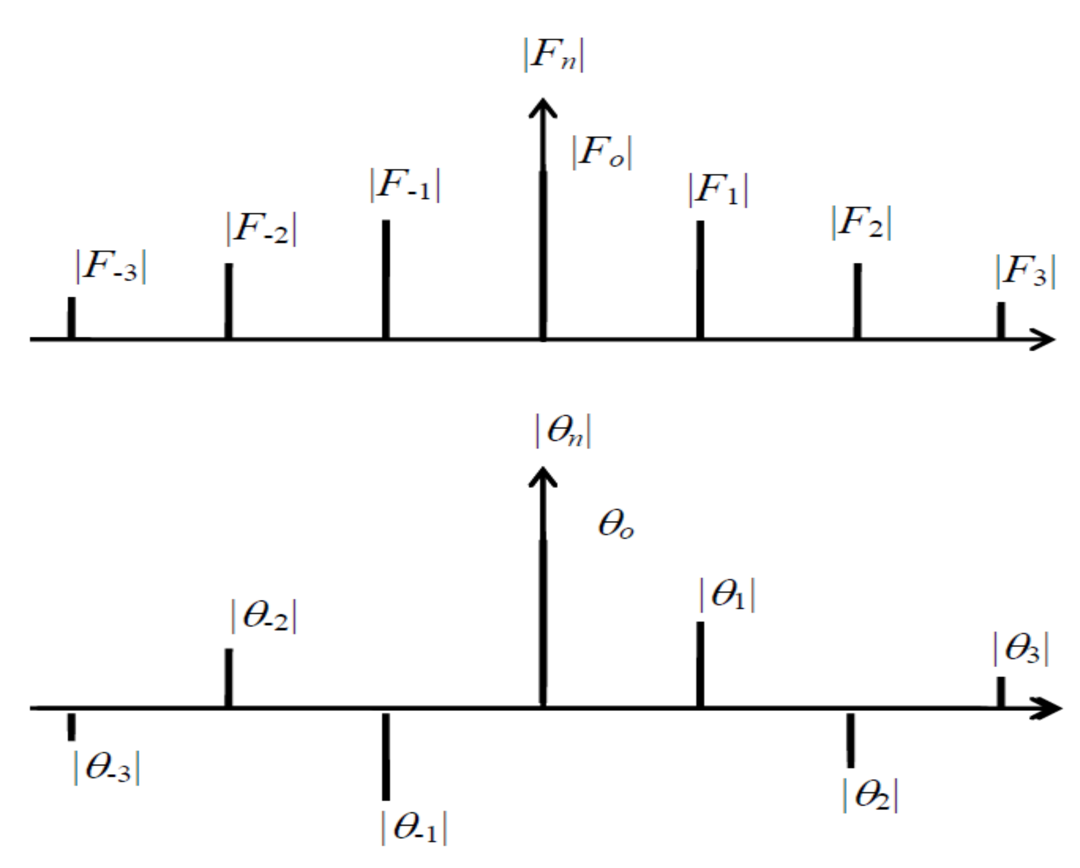
$$F_n = \frac{1}{T_o} \int_0^T f(t) e^{-jn\omega_o t}\, dt$$

- If $f(t)$ is periodic expansion is valid for all t

F_n Usually Complex

- Generally, F_n is complex having a **real** and **imaginary** parts (**magnitude** and **phase**) as:

$$F_n = |F_n| e^{j\theta_n}$$



Some Important Functions

1-Rectangular or Gate

Standard Gate:

$$f(t) = rect(t)$$

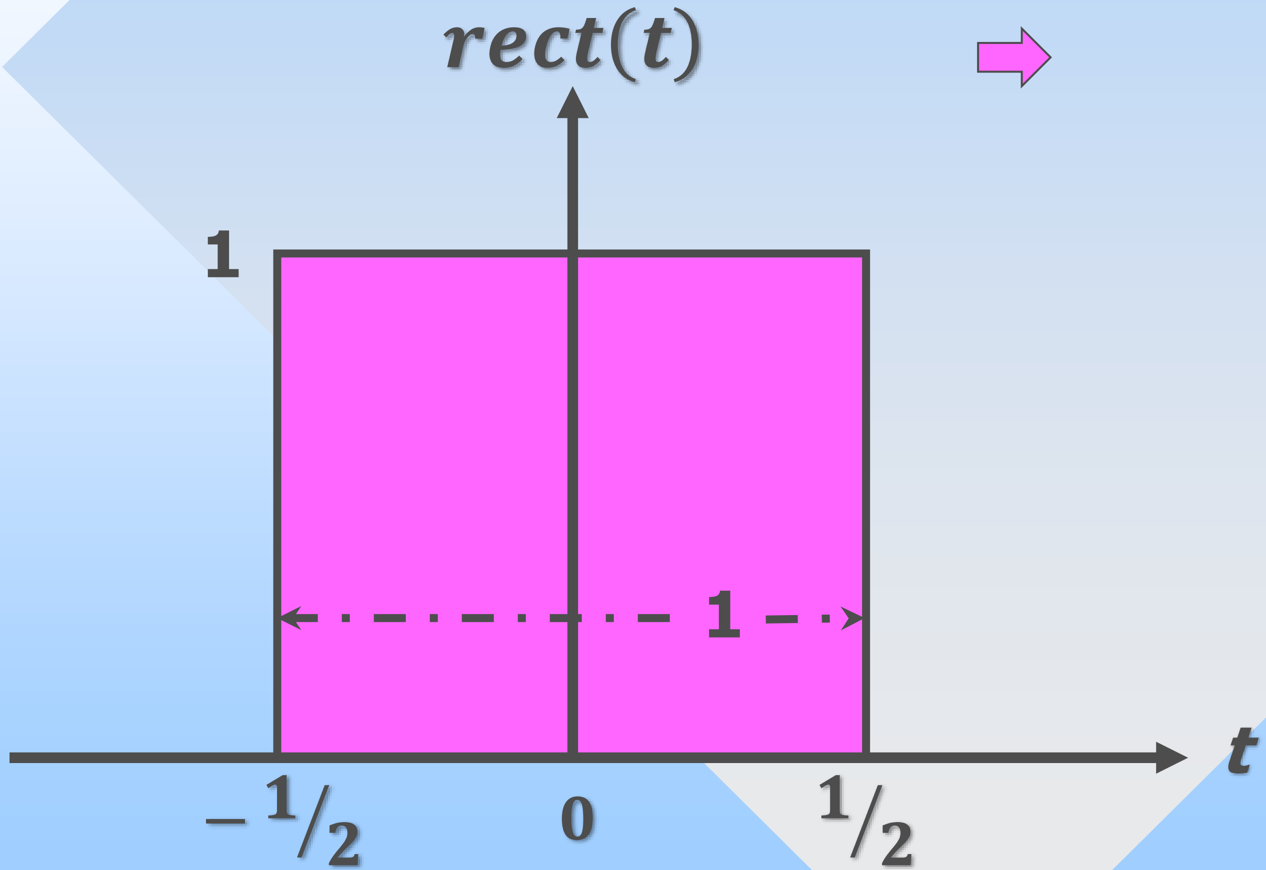
- ❑ Its height or strength = 1
- ❑ Its width = 1
- ❑ Area = 1
- ❑ Centered to the origin

In General:

$$f(t) = k\, rect\left(\frac{t - t_o}{\tau}\right)$$

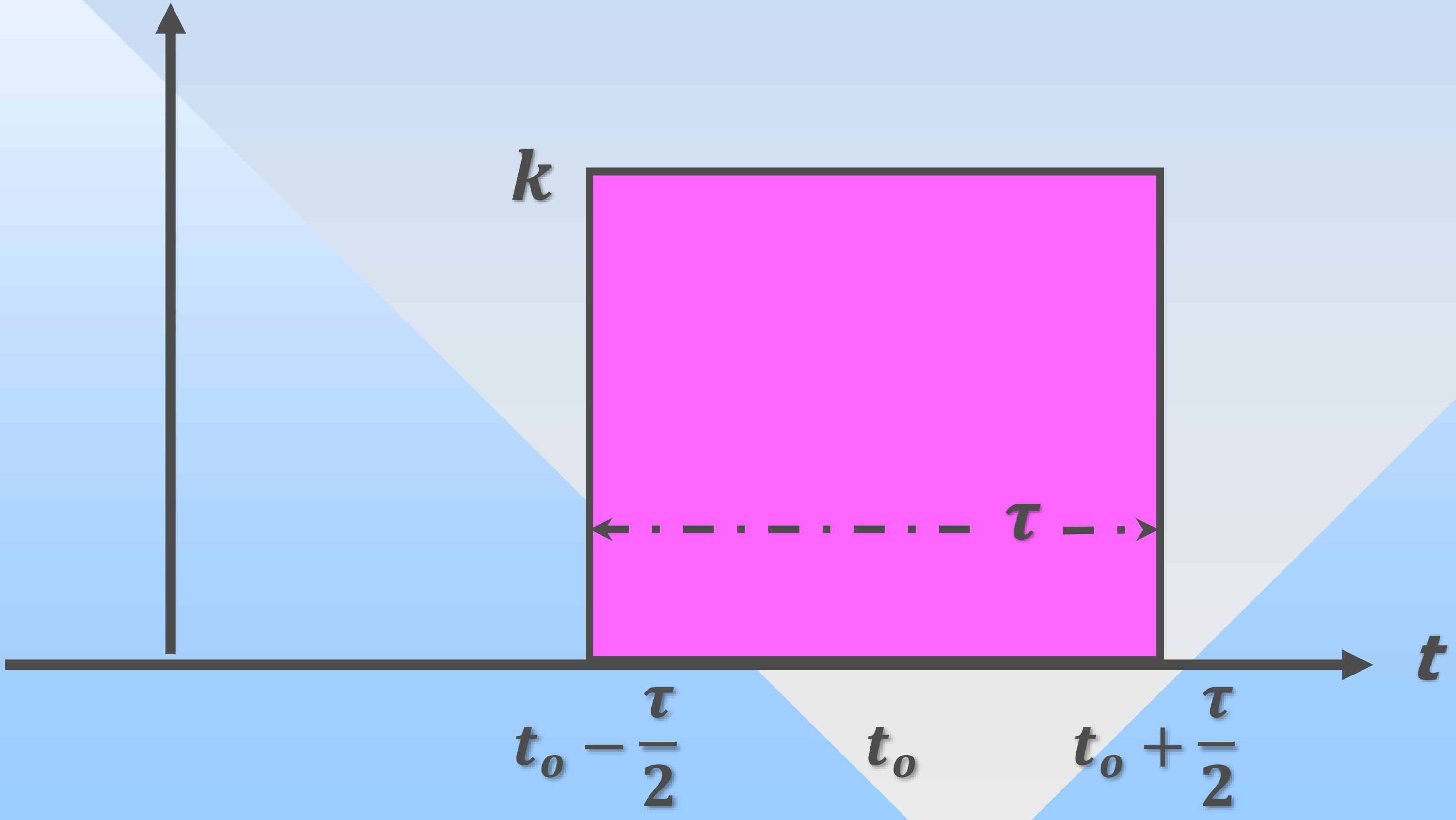
- ❑ Strength or Length = k
- ❑ Width = τ
- ❑ Area = $k\ \tau$
- ❑ Shifted t_o to the right

Standard Gate



General Gate

$$k\,rect\left(\frac{t-t_o}{\tau}\right)$$



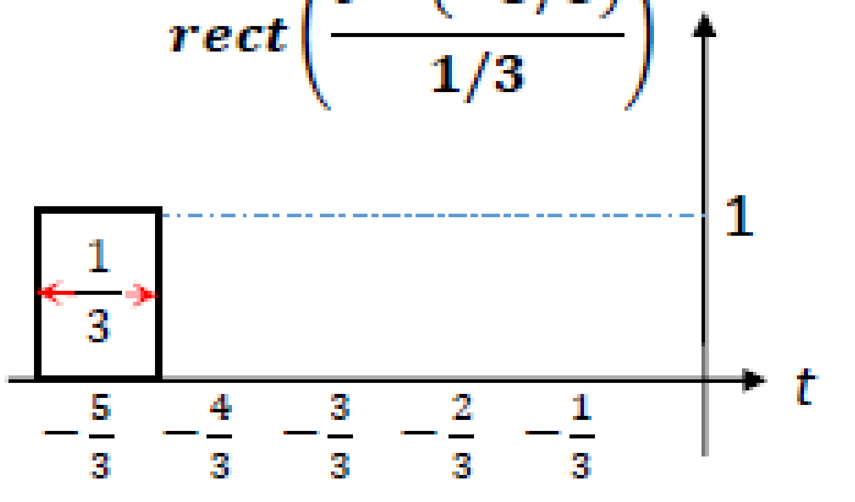
Examples of Gate Function

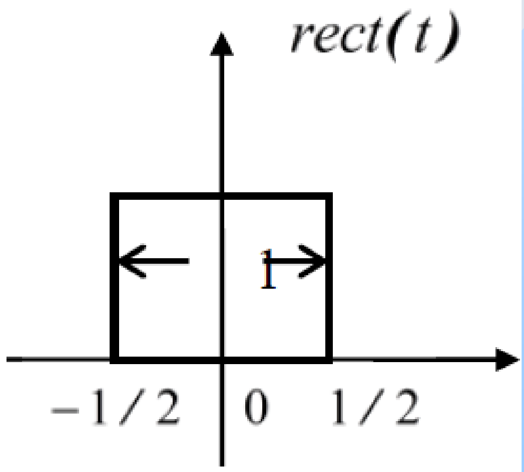
$$rect\left(\frac{t-t_o}{\tau}\right)=G\left(\frac{t-t_o}{\tau}\right)$$

Example.1.1: $rect(3t+5)=rect\left(\frac{t-(-5/3)}{(1/3)}\right)$

Example.1.2: $rect(t)=rect\left(\frac{t-0}{1}\right)$

$$rect\left(\frac{t-(-5/3)}{1/3}\right)$$





2-Triangular Function

Standard Triangle:

$$f(t) = tri(t)$$

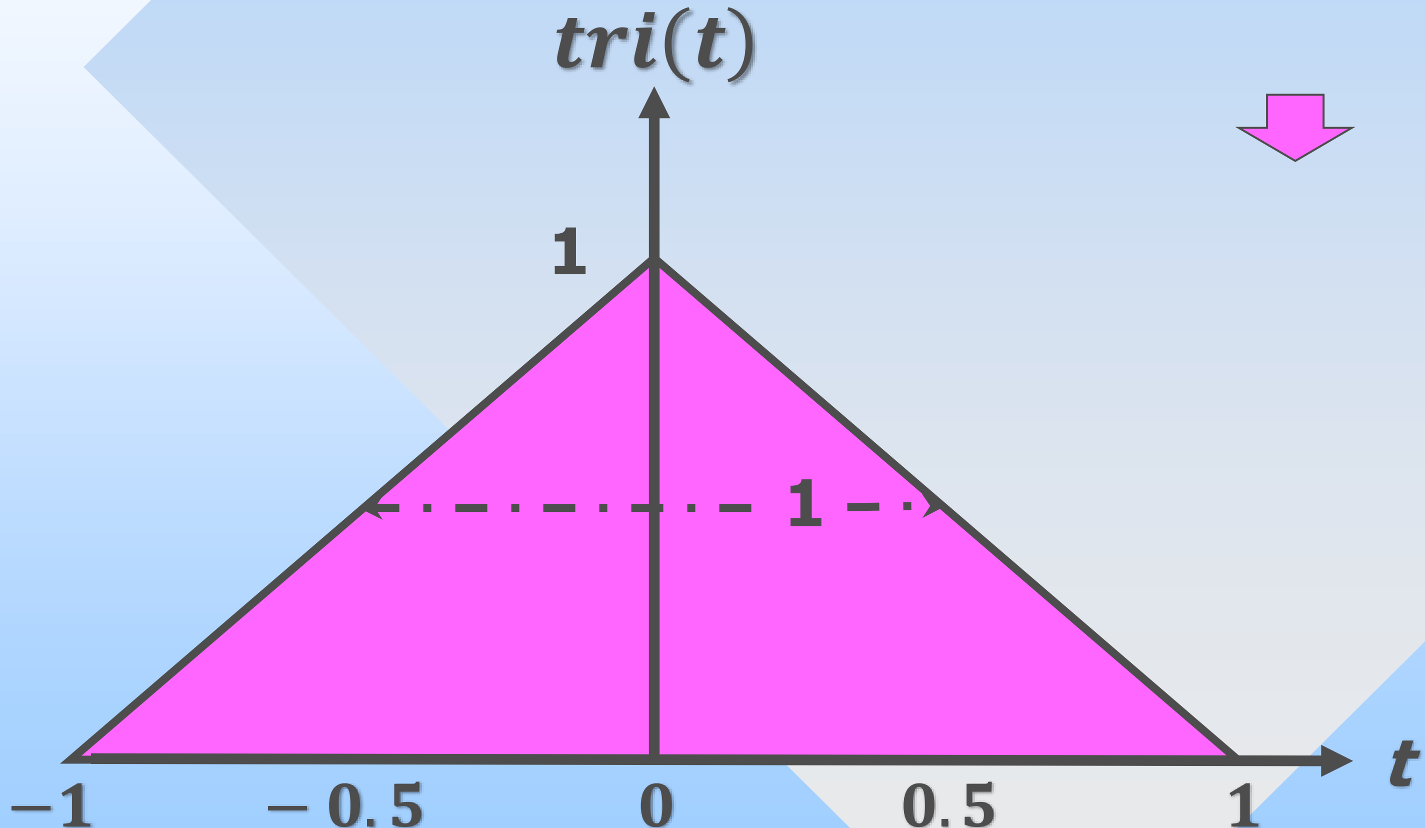
- Of unity strength, unity Area and width is two.
- Centered to the origin.

General Triangular:

$$f(t) = k\,tri\left(\frac{t - t_o}{\tau}\right)$$

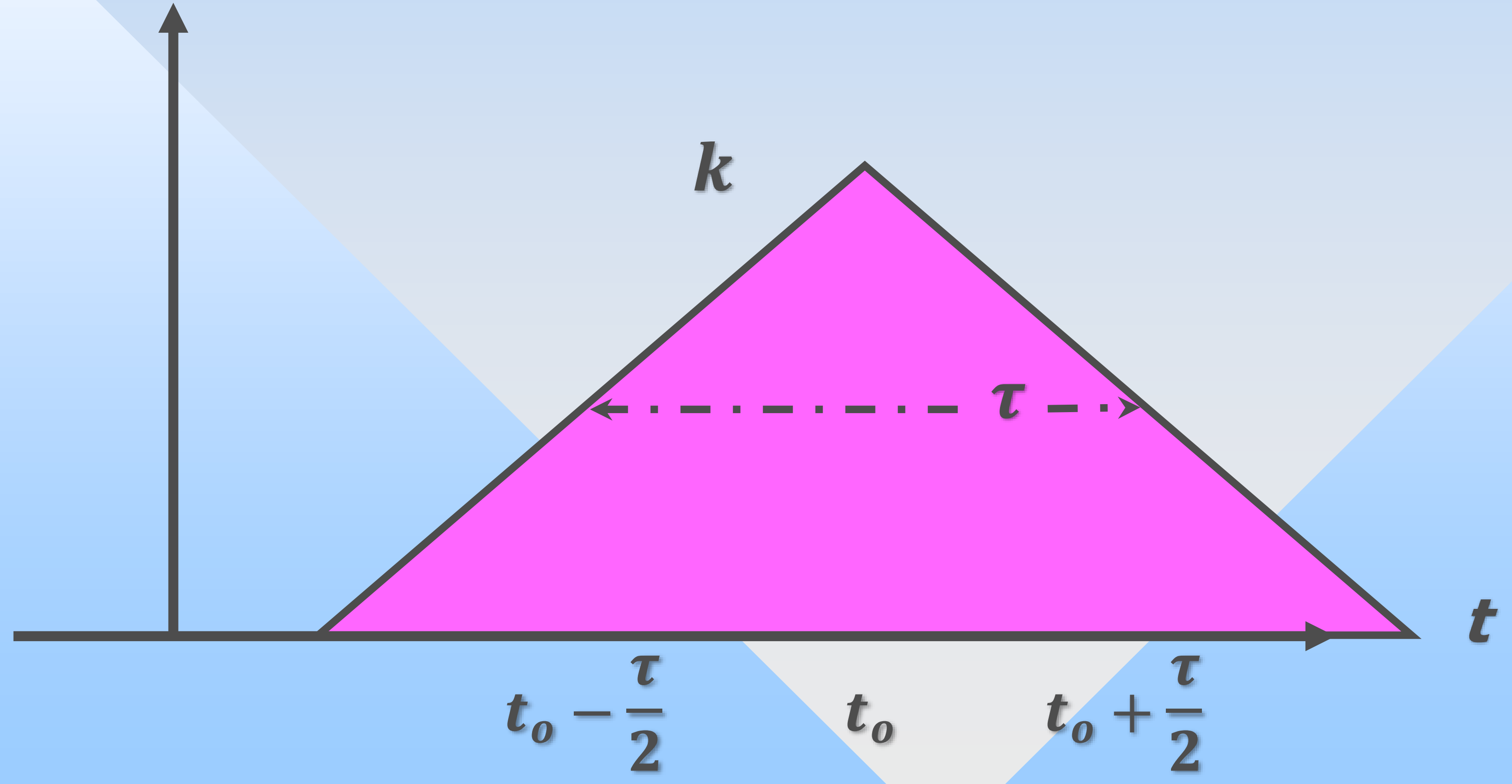
- Strength or length = k
- Width = 2 τ.
- Area = $k\,\tau$
- Shifted t_o to the right.

Standard Triangular

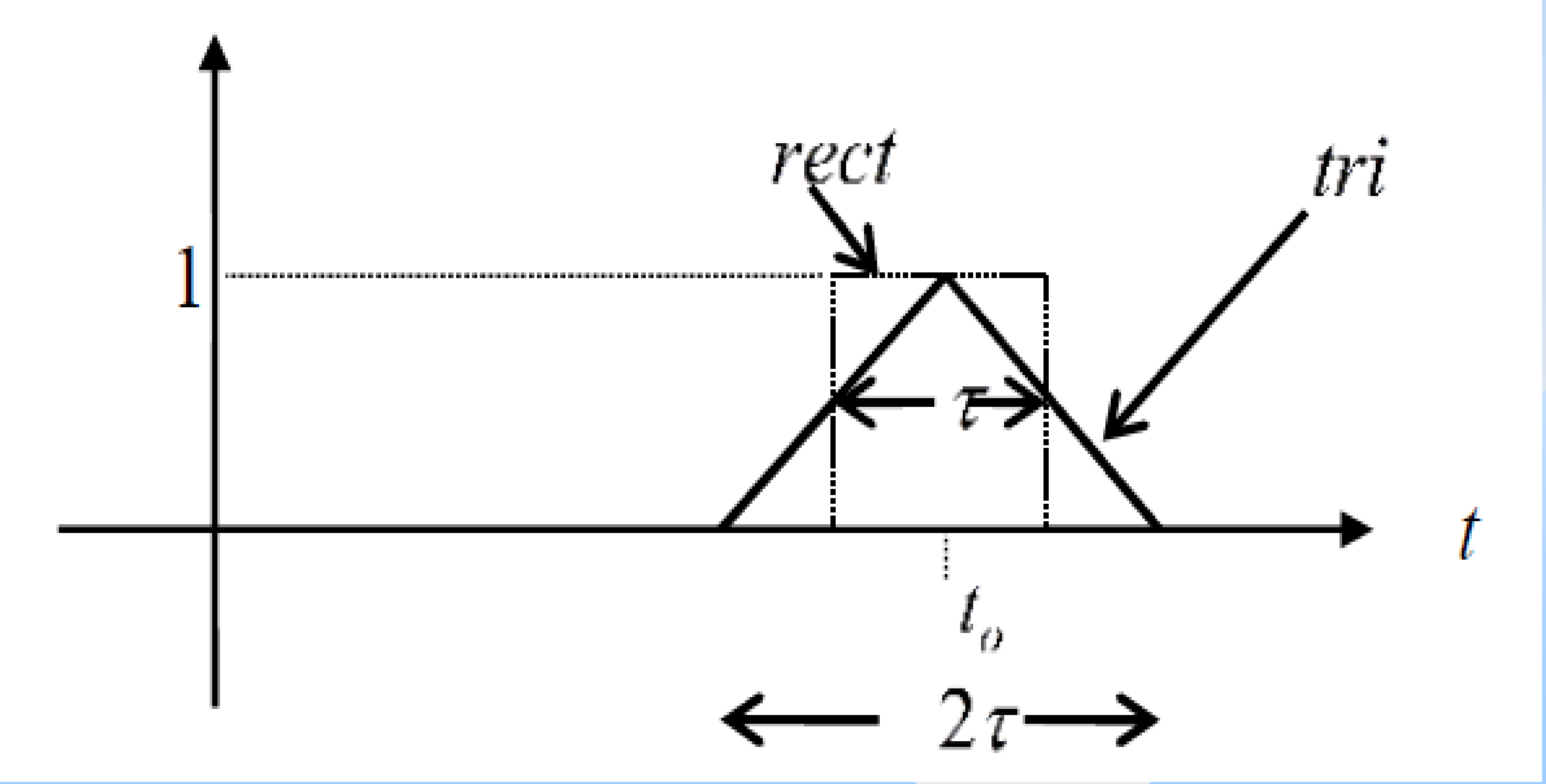


General Triangular

$$k\,tri\left(\frac{t-t_o}{\tau}\right)$$



Triangular Function



3-Unit Step Function

Standard Unit Step:

$$f(t) = u(t) = \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases}$$

- Of unity strength.
- Begins at origin.

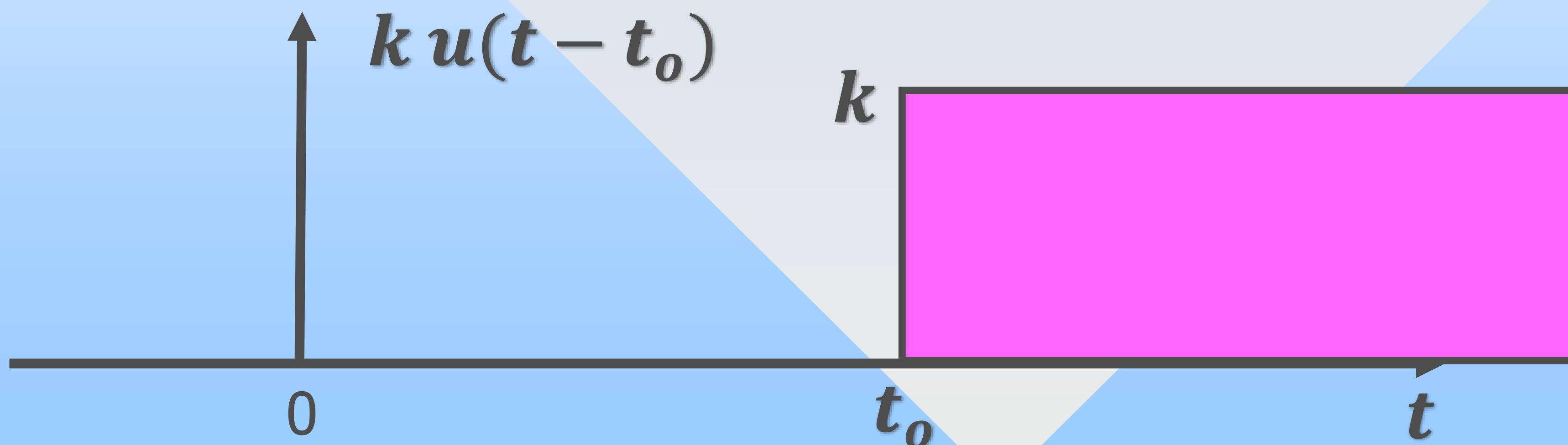
In General:

$$f(t) = k\,u(t - t_o) = \begin{cases} k & t > t_o \\ 0 & t < t_o \end{cases}$$

- Strength = k
- Shifted t_o to the right.

Standard Unit Step

General Unit Step



Unit Step Examples

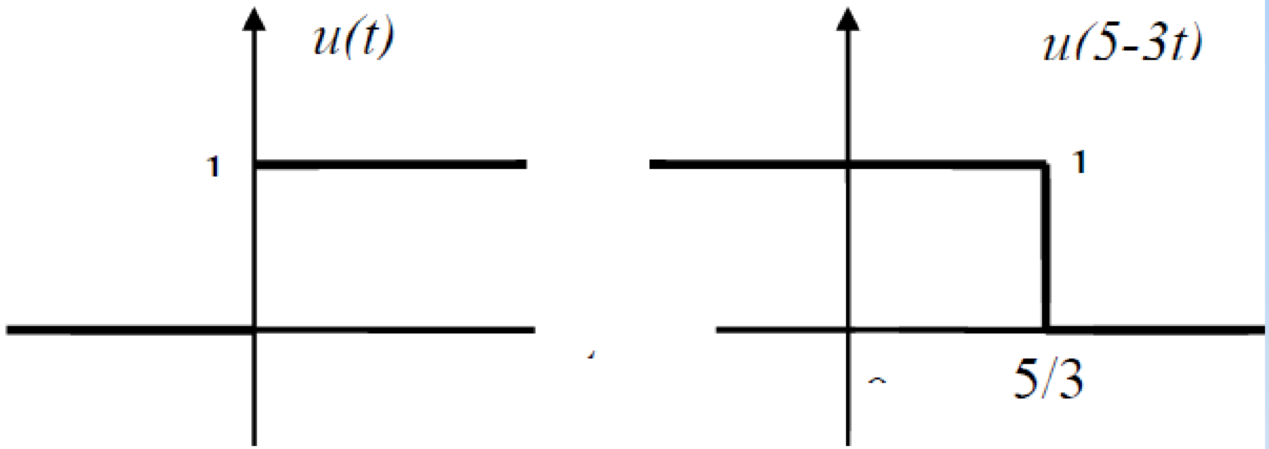


Fig.1.6: Unit Step Function

Example.1.3: $u(5-3t)=\begin{cases}1 & if\ 5-3t>0 \quad or\ t<5/3\\ 0 & if\ 5-3t<0 \quad or\ t>5/3\end{cases}$

Exersize.1.1:

Show that the rectangular function can be represented as two unit step functions as follows:

$$f(t)=u\left(\frac{t+\tau}{2}\right)-u\left(\frac{t-\tau}{2}\right)\equiv rect\left(\frac{t}{\tau}\right)$$