Laboratoire de Mécanique de Paris Saclay. Seminar LMPS

Non-smooth dynamical systems. Solving the contact problem with friction using methods derived from large-scale optimisation

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TRIPOP INRIA project team (LJK)

Research object:

Modeling, Simulation and Control of Nonsmooth Dynamics.

Main application:

Natural environmental risks in mountains.

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Nonsmooth

- nonsmooth = lack of differentiability ($\notin \mathcal{C}^1$),
- graphs with peaks, kinks, jumps.

Dynamics

- systems that evolves with time,
- branch of mechanics concerned with the motion of objects.

Where is nonsmoothness?

- nonsmooth solutions in time and space:
  - continuous, functions of bounded variations, measures and distributions.
- nonsmooth modeling of constitutive laws:
  - set–valued mapping, inequality constraints, complementarity, impact laws,
  - ODE with discontinuous r.h.s, differential inclusion, measure equation.

Scientific pairs: M. Jean, J.J. Moreau, M. Schatzman & C. Lemaréchal.

Our reference (bedside) books

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Variational inequalities
Differential inclusions
Switched systems
Nonsmooth Mechanical systems
Sliding mode control
Optimal control with state constraints
Hybrid systems and cyberphysical systems
Complementarity systems

Schneider Electric. C60 Circuit breaker

Watch escapment mechanisms

ESA Exomars Rover

Humanoid Robots

Robotics

Figure 6.1: Example of carrying hauling rope heaved by its top station at angular velocity $\dot{\omega}$ - The carriers are located via ($\bullet$)

Ropeways.

EDF Dam made for rocks and concrete
(ANR Saladyn project INRIA/EDF/LMGC/Schneider Electric)

Mines engineeging.
open pit of Chuquicamata (Chile)

Granular materials

Chair of Alpine Mass Movements. ETH Zurich. J. Gaume

Blatten debris flows

MPM simulation of Landslides (L. Guillet INRIA)

Natural gravity-driven hazards.

Contents

Solving the contact problem with friction using methods derived from large-scale optimisation.

1. Introduce a sufficiently generic and representative discrete 3D frictional contact problem.
2. Interpret this problem in the context of numerical optimisation and mathematical programming.
3. [SKIPPED] Provide an existence result, whose assumption can be verified numerically .
4. Compare the main existing numerical methods based on a large collection of problems (FCLIB) and a common implementation (SICONOS/Numerics).
5. Propose a new solution method based on the interior point method (IPM) and Asymptotic Numerical Method (ANM) .

The discrete frictional contact problem

An existence result via convex optimization

Existing numerical solvers

Benchmarking: Siconos/numerics and FCLIB

Interior Point Methods (IPM)

Conclusions & Perspectives

Signorini's condition and Coulomb's friction

- gap function $g_N = (C_B - C_A)\mathrm{N}$.
- local reaction forces r and velocities u

$$r = r_N \mathrm{N} + r_T, \quad \text{with } r_N \in \mathbb{R}, \quad r_T \in \mathbb{R}^2.$$

$$u = u_N \mathrm{N} + u_T, \quad \text{with } u_N \in \mathbb{R} \quad u_T \in \mathbb{R}^2.$$

- Signorini conditions

$$\text{position level : } 0 \leqslant g_N \perp r_N \geqslant 0.$$

$$\text{velocity level : } \begin{cases} 0 \leqslant u_N \perp r_N \geqslant 0 & \text{if } g_N \leqslant 0 \\ r_N = 0 & \text{otherwise.} \end{cases}$$

Complementarity relation : $0 \leqslant x \perp y \geqslant 0$ equivalent to

$$x \geqslant 0, y \geqslant 0 \text{ and } x^\top y = 0 \iff -x \in \mathbb{N}_{\boldsymbol{R}_+}(y) \iff -x \in \partial\Psi_{\boldsymbol{R}_+}(y)$$

Signorini's condition and Coulomb's friction

Coulomb's cone

Let μ be the coefficient of friction. The Coulomb friction cone K is

$$K = \{r \in \mathbb{R}^3 \mid \|r_\mathsf{T}\| \leqslant \mu r_\mathsf{N}\}. \tag{1}$$

A second order cone, a.k.a Lorentz cone, or ice-cream cone

Coulomb postulates two cases

- for the sticking case that

$$u_\mathsf{T} = 0, \quad r \in K, \tag{2}$$

- and for the sliding case that

$$u_\mathsf{T} \neq 0, \quad \|r_\mathsf{T}\| = \mu r_\mathsf{N}, \quad r_\mathsf{T} = -\frac{u_\mathsf{T}}{\|u_\mathsf{T}\|}\|r_\mathsf{T}\|. \tag{3}$$

Disjunctive formulation of the frictional contact behavior

$$\begin{cases} r = 0 & \text{if } g_\mathsf{N} > 0 \quad \text{(no contact)} \\ r = 0, u_\mathsf{N} \geqslant 0 & \text{if } g_\mathsf{N} \leqslant 0 \quad \text{(take–off)} \\ r \in K, u = 0 & \text{if } g_\mathsf{N} \leqslant 0 \quad \text{(sticking)} \\ r \in \partial K, u_\mathsf{N} = 0, r_\mathsf{T} = -\dfrac{u_\mathsf{T}}{\|u_\mathsf{T}\|}\|r_\mathsf{T}\| & \text{if } g_\mathsf{N} \leqslant 0 \quad \text{(sliding)} \end{cases} \tag{4}$$

Signorini's condition and Coulomb's friction

Second Order Cone Complementarity (SOCCP) formulation

- Bipotential approach (De Saxcé, 1992; Saxce and Feng, 1991)
  An extension of the framework of Moreau (Moreau, 1970, 1974)
- Modified relative velocity
$$\tilde{u} = u + \mu \|u_{\mathrm{T}}\| \mathrm{N}. \tag{5}$$
- Dual convex cone to K defined by
$$K^\star = \{u \in \mathbb{R}^3 \mid r^\top u \geqslant 0, \quad \text{for all } r \in K\} = \{u \in \mathbb{R}^3 \mid \mu \|u_{\mathrm{T}}\| \leqslant u_{\mathrm{N}}\}. \tag{6}$$
- Second-Order Cone Complementarity Problem (SOCCP)
  (Acary and Brogliato, 2008; Acary et al., 2011)
$$K^\star \ni \tilde{u} \perp r \in K \tag{7}$$
- Variational Inequality (VI) formulation
$$-\tilde{u} = -(u + \mu \|u_{\mathrm{T}}\| \mathrm{N}) \in \mathbb{N}_{\mathrm{K}}(\mathrm{r}) \tag{8}$$

Signorini's condition and Coulomb's friction

Figure: Coulomb's friction and the modified velocity $\tilde{u}$. The sliding case.

Discrete frictional contact problems

Problem 1 (General discrete frictional contact problem)

Given

- a symmetric positive de nite matrix $M \in \mathbb{R}^{n\,n}$,
- a vector $f \in \mathbb{R}^{n}$,
- a matrix $H \in \mathbb{R}^{n\,m}$,
- a vector $w \in \mathbb{R}^{m}$,
- a vector of coe cients of friction $\in \mathbb{R}^{n_c}$,

nd three vectors $v \in \mathbb{R}^{n}$, $u \in \mathbb{R}^{m}$ and $r \in \mathbb{R}^{m}$, denoted by FC=I(M; H; f; w;) such that

$$\begin{cases} Mv = Hr + f \\ u = H^{>} v + w \\ \tilde{u} = u + g(u) \\ K^{?} \ni \tilde{u} \perp r \in K \end{cases} \tag{9}$$

with $g(u) = [[\quad ku_T kN \quad]^{>}; \quad = 1 : : : n_c]^{>}$. □

Parameters of the simulation campaign

- More than 2500 problems
- Around 30 solvers with their variants
- More than 27000 runs between few seconds up to 400s.

Full error criteria

$$\text{error} = \frac{\|F^{\text{nat}}_{\text{vi-2}}(r)\|}{\|q\|}. \tag{23}$$

Performance profiles Dolan and Moré, 2002

- Given a set of problems $\mathcal{P}$
- Given a set of solvers $\mathcal{S}$
- A performance measure for each problem with a solver $t_{p,s}$ (cpu time, flops, ...)
- Compute the performance ratio

$$\tau_{p,s} = \frac{t_{p,s}}{\min\limits_{s \in \mathcal{S}} t_{p,s}} \geqslant 1 \tag{24}$$

- Compute the performance profile $\rho_s(\tau) : [1, +\infty] \to [0, 1]$ for each solver $s \in \mathcal{S}$

$$\rho_s(\tau) = \frac{1}{|\mathcal{P}|} \left| \{ p \in \mathcal{P} \mid \tau_{p,s} \leqslant \tau \} \right| \tag{25}$$

The value of $\rho_s(1)$ is the probability that the solver s will win over the rest of the solvers.

Comparisons by families of solvers

Benchmarking : conclusions

Conclusions

1. A bunch of articles in the literature
   47000 articles (2000–2020) on "Coulomb friction numerical method" in Google Scholar.
2. No 'Swiss army knife' solution: choose efficiency OR robustness.
3. Newton-based solvers (semi–smooth Newton)
   - best solvers with starting algorithm for well-posed FEM applications (robust and quadratic convergence)
   - robustness issues with rank of H
4. First-order iterative methods (VI, NSGS, PSORx)
   - solve all the problems but with a loose accuracy
   - very slow at tight accuracy
5. The rank of the H matrix ($\approx$ ratio number of contacts unknows/number of d.o.f) plays an important role on the robustness.
6. Solvers based on optimisation and the proximal-point algorithm are interesting, but it is difficult to forecast their efficiency.
7. Need for a second order method when H is rank-deficient
   Interior point method (IPM?).

Mode details in (Acary et al., 2018)

The discrete frictional contact problem

An existence result via convex optimization

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Interior Point Methods (IPM)

Conclusions & Perspectives

A Primer on IPM for convex QP

Convex QP

Consider the convex quadratic program in standard form:

$$\min_{x \in \mathbb{R}^n} \; f(x) := \tfrac{1}{2} x^T Q x + c^T x$$
$$\text{subject to} \;\; Ax = b, \quad x \geqslant 0,$$

where $Q \succeq 0$, $A \in \mathbb{R}^{m \times n}$.

First-order (KKT) conditions

First-order (KKT) conditions (existence of multipliers y, z):

$$\begin{aligned} Qx + c + A^T y - z &= 0 \quad \text{(stationarity)} \\ Ax - b &= 0 \quad \text{(primal feasibility)} \\ XZe = 0, \quad x \geqslant 0, \; z \geqslant 0 & \quad \text{(complementarity)} \end{aligned}$$

where $X = \mathrm{diag}(x)$, $Z = \mathrm{diag}(z)$, e is the vector of ones.

A Primer on IPM for convex QP

Interior-point idea

Enforce strict positivity $x > 0, z > 0$ and drive $XZe = \mu e$ to zero along the *central path* as $\mu \downarrow 0$.

Barrier formulation and Newton steps

Use a logarithmic barrier to keep $x > 0$:

$$\phi_\mu(x) = \tfrac{1}{2} x^T Q x + c^T x - \mu \sum_{i=1}^{n} \log x_i$$

with equality constraints $Ax = b$. For fixed $\mu > 0$ solve

$$\min_{x>0} \ \phi_\mu(x) \quad \text{s.t. } Ax = b.$$

The perturbed KKT conditions for $\mu > 0$ are:

$$\begin{aligned} Qx + c + A^T y - z &= 0, \\ Ax - b &= 0, \\ XZe &= \mu e. \end{aligned}$$

A Primer on IPM for convex QP

We linearize these equations at (x, y, z) to get the Newton system for $(\Delta x, \Delta y, \Delta z)$:

$$\begin{bmatrix} Q & A^T & -I \\ A & 0 & 0 \\ Z & 0 & X \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{bmatrix} = - \begin{bmatrix} r_d \\ r_p \\ r_c \end{bmatrix},$$

where

$$\begin{aligned} r_d &= Qx + c + A^T y - z, \\ r_p &= Ax - b, \\ r_c &= XZe - \mu e. \end{aligned}$$

This 3-block structure is the foundation of primal–dual interior-point methods. Take a damped Newton step and project back to positive orthant with step-length chosen to maintain $x > 0$.

Illustration of the central path

Definition: The *central path* is the trajectory of strictly feasible points $(x(\mu), y(\mu), z(\mu))$ satisfying

$$Qx(\mu) + c + A^T y(\mu) - z(\mu) = 0, \quad Ax(\mu) = b, \quad X(\mu)Z(\mu)e = \mu e, \ \mu > 0.$$

As $\mu \to 0$, the path converges to the optimal solution (x^*, y^*, z^*).

Geometric intuition:

- Each μ defines a barrier problem that smooths the feasible region.
- The central path traces the minimizers of these smoothed problems.
- Interior-point methods "follow/approximate" this path using Newton directions.

Interpretation: As μ decreases, the minimizer moves smoothly along the red curve toward the boundary optimum.

Central path in the positive orthant (boundary optimum)

The central path for a problem with $x \geqslant 0$ approaches the boundary smoothly as $\mu \to 0$. The iterates remain strictly positive until convergence.

Interpretation: As $\mu \downarrow 0$, $x_1(\mu) \to 0$ and the iterates approach the boundary while maintaining strict positivity, ensuring well-defined Newton steps.

Primal–dual interior-point method & practical notes

Primal–dual methods work with both primal and dual variables and maintain perturbed complementarity:

$$XZe = \sigma\mu e, \quad \mu = \frac{x^T z}{n}, \quad 0 < \sigma \leqslant 1.$$

A common predictor–corrector algorithm (Mehrotra) computes a predictor (affine-scaling) step, estimates a σ, then corrects to stay close to the central path. Key practical points:

- Use sparse symmetric linear algebra to solve the Newton/KKT system efficiently.
- Choose step-lengths to keep iterates positive; use Mehrotra predictor-corrector for fast practical convergence.
- Convergence: polynomial worst-case complexity (typically 30–60 iterations for large QPs in practice).

Interior-point methods follow the central path using Newton steps on a barrier-regularized KKT system; primal–dual variants maintain both primal and dual feasibility and usually reach high accuracy in few iterations.

Interior Point Methods for frictional contact

PhD thesis of Hoang Minh Nguyen(2025), with Paul Armand

Perturbation of the complementarity condition with a barrier parameter τ

Original problem	Perturbed problem
$Mv + f = H^\top r$	$Mv + f = H^\top r$
$Hv + w + se = \tilde{u}$	$Hv + w + se = \tilde{u}$
$s = \|\tilde{u}_\mathrm{T}\|$	$s = \|\tilde{u}_\mathrm{T}\|$
$\tilde{u} \circ r = 0$	$\tilde{u} \circ r = 2\tau e$
$(\tilde{u}, r) \in K^2$	$(\tilde{u}, r) \in \mathrm{int}(K^2)$

(26)

Convex case (s given fixed)(see Acary et al., 2023b)

- IPM solves very accurately ($\mathrm{tol} < 10^{-13}$) and efficiently the problem.
- Even when H is rank-deficient.
- Special care of sparse linear systems and conditioning with Nesterov-Todd scaling.

➜ Extension to general frictional contact problems: nonsmooth interior point method

Nonsmooth Interior-Point Method (NIPM)

Slater's assumption (SA)

$$\exists v \in \mathbb{R}^m \quad \text{such that} \quad Hv + w \in \text{int}(K)$$

Theorem 4

1. *Under SA, for each $\tau > 0$, the perturbed problem (26) has a solution $(v_\tau, \tilde{u}_\tau, r_\tau, s_\tau)$*
2. *Under SA, there exists an analytic central path $\{(v_\tau, \tilde{u}_\tau, r_\tau, s_\tau) : \tau > 0\}$, which converges to a solution of the original problem*

Proof ideas

- Convex optimization and analysis sith a given s
- Fixed point Brouwer's theorem,
- Curve selection lemma (semi-algebraic analysis)

Main theoretical outcome

- Alternative proof of solution existence
- The central path is not necessarily unique !

Nonsmooth Interior-Point Method (NIPM) - Linearization

Iterations and Jacobian matrix

$$G := \begin{bmatrix} Mv + f - H^\top r \\ Hv + w - \tilde{u} + se \\ s - \|\tilde{u}_\mathrm{T}\| \\ \tilde{u} \circ r \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2\tau e \end{bmatrix} \qquad J := \begin{bmatrix} M & -H^\top & 0 & 0 \\ H & 0 & -I & e \\ 0 & 0 & -L & 1 \\ 0 & \tilde{U} & R & 0 \end{bmatrix}$$

where $L = \begin{pmatrix} 0 & \partial\|\tilde{u}_\mathrm{T}\|^\top \end{pmatrix}$, with $\partial\|\tilde{u}_\mathrm{T}\| = \begin{cases} \dfrac{\tilde{u}_\mathrm{T}}{\|\tilde{u}_\mathrm{T}\|} & \text{if } \tilde{u}_\mathrm{T} \neq 0 \\ d \in \mathbb{B} & \text{if } \tilde{u}_\mathrm{T} = 0 \end{cases}$ (unit ball $\mathbb{B}$)

Linear system and Mehrotra's-Like algorithm

$$J\,d = -G + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2\sigma\tau e \end{bmatrix}$$

Predictor-corrector scheme $\sigma \in (0, 0.5)$: centralization parameter

Stopping test

$$\max\left\{ \|Hv + w - \tilde{u}\|_\infty,\ \|Mv + f - H^\top r\|_\infty, |s - \|\tilde{u}_\mathrm{T}\||,\ \|\tilde{u} \circ r\| \right\} \leqslant \text{tol}$$

Nonsmooth Interior-Point Method (NIPM)

Algorithm Mehrotra type Nonsmooth Interior Point Method (NIPM)

Parameters: Choose a starting point (v, r, u, s) such that $(r, u) \in \mathrm{int}(\mathcal{K}^2)$. Choose $\eta_1 \in (0, 1)$, $\eta_2 \geqslant 1$, $\eta_3 \geqslant 1$, $\gamma_1 \in (0, 1)$, $\gamma_2 \in (0, 1 - \gamma_1)$, $c_1 \geqslant 1$, $c_2 \in (0, 1)$, $\mathrm{tol} > 0$ and set $\gamma = 0.99$;

1: **if** the stopping criterion is satisfied **then** return (v, r, u, s) as solution of (**??**);
2: Set $\tau \leftarrow u^\top r/n$;
3: **if** $\|s - \ell(u)\|_2 \geqslant c_1 \|u \circ r\|_2$ **then** set $\sigma \leftarrow 0.998$, $d_a^u \circ d_a^r \leftarrow 0$ and go to 10,
4: **else** compute $d_a = (d_a^v, d_a^r, d_a^u, d_a^s)$ solution of

$$J(v, r, u, s)d_a = -G(v, r, u, s);$$

5: Find the greatest $\alpha_a \in (0, 1]$ such that $(r, u) + \alpha_a(d_a^r, d_a^u) \in \mathcal{K}^2$;
6: Set $\tau_a \leftarrow (u + \alpha_a d_a^u)^\top (r + \alpha_a d_a^r)/n$;
7: **if** $\tau > \eta_1$ **then** set $\beta \leftarrow \max\{1, \eta_2 \alpha_a^2\}$ **else** set $\beta \leftarrow \eta_3$;
8: Set $\sigma \leftarrow 0.998 \min\{1, (\tau_a/\tau)^\beta\}$;
9: **if** $\sigma \geqslant c_2$ **then** set $d_a^u \circ d_a^r \leftarrow 0$;
10: Compute $d = (d^v, d^r, d^u, d^s)$ solution of

$$J(v, r, u, s)d = -G(v, r, u, s) + \begin{bmatrix} 0 \\ 0 \\ -d_a^u \circ d_a^r + \sigma\tau e \\ 0 \end{bmatrix} \tag{27}$$

11: Find the greatest $\alpha \in (0, 1]$ such that $(r, u) + \alpha(d^r, d^u) \in (1 - \gamma)(r, u) + \mathcal{K}^2$;
12: Set $\gamma \leftarrow \gamma_1 + \alpha\gamma_2$;
13: Set $(v, r, u, s) \leftarrow (v, r, u, s) + \alpha(d^v, d^r, d^u, d^s)$ and goto 1.

Nonsmooth Interior-Point Method (NIPM)

Problem's size $11 \leqslant n \leqslant 100$

IPM (GFC3D) outperforms NSGS and ADMM in terms of efficiency and robustness

Nonsmooth Interior-Point Method (NIPM)

Problem's size $1000 \leqslant n$

IPM (GFC3D) suffers from robustness

Nonsmooth Interior-Point Method (NIPM) - failures

Failure #1: A special shape of the central path

Solution: $r^{\times} \in \text{int}(K)$, $\tilde{u}^* = 0$ (sticking)

The shape of the central path can cause iterations to become trapped at the boundary, at a point that is not the solution
A kind of "fake solution" at accuracy ($\approx 10^{-6}, 10^{-8}$)).

Nonsmooth Interior-Point Method (NIPM) - failures

Failure #2: Non-monotone parameterization of the central path

- Red-blue curve: Central path $\tau \to r(\tau)$ calculated by Asymptotic Numerical Method (ANM). Red: τ decreases. Blue: τ increases

- **Black** curve: the path of NIPM iterates

➜ Asymptotic Numerical Method (ANM): algorithm based on the computation of series to perform accurate numerical continuations of parameterized non-linear problems

Nonsmooth Interior-Point Method (NIPM) and ANM

Reformulation as bilinear problem

Let us write the perturbed problem (26) under the form

$$R(x,\tau) = L(x) + b + Q(x,x) - \tau \begin{pmatrix} 0 \\ 0 \\ 2e \\ 0 \end{pmatrix} = 0, \tag{28}$$

where we have a linear term L, a constant term b and bilinear function Q defined by

$$x = \begin{pmatrix} v \\ u \\ r \\ s \end{pmatrix}, \quad L = \begin{pmatrix} M & -H^\top & & 0 \\ -H & 0 & I & -E \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad Q(x,\tilde{x}) = \begin{pmatrix} 0 \\ 0 \\ u \circ \tilde{r} \\ s \bullet \tilde{s} - l(u) \bullet l(\tilde{u}) \end{pmatrix} \tag{29}$$

and

$$F = \begin{pmatrix} 0 & 0 & 2e & 0 \end{pmatrix}^\top, \quad b = \begin{pmatrix} -f & -w & 0 & 0 \end{pmatrix}^\top. \tag{30}$$

Nonsmooth Interior-Point Method (NIPM) and ANM

Principles of Asymptotic Numerical method (ANM)
Pottier Ferry, Cochelin, Damil, *et al.*

The principle of ANM is to start from an initial known vector (x_0, τ_0) such that $R(x_0, \tau_0) = 0$, then to calculate a solution of (28) under the form of truncated series

$$x(t) = \sum_{k=0}^{N} x_k a^k \quad \tau(t) = \sum_{k=0}^{N} \tau_k a^k,$$

➜ In our particular case, bilinear function $\Longrightarrow$ constant Jacobian for a computation of all the terms of the series.

$$J(x) = \begin{pmatrix} M & -H^\top & & 0 \\ -H & 0 & I & -E \\ 0 & U & R & 0 \\ 0 & 0 & -A & 2\,\mathrm{diag}(s) \end{pmatrix} \tag{31}$$

with

$$A = 2\,\mathrm{diag}(\begin{bmatrix} 0 & u_\alpha^\top \end{bmatrix}, \alpha \in \{1 \dots m\}]). \tag{32}$$

Nonsmooth Interior-Point Method (NIPM) and ANM

Principles of Asymptotic Numerical method (ANM)

Zeroth order ($x_0 a^0$). For $k = 0$, the substitution gives

$$R(x_0, \tau_0) = L(x_0) + b + Q(x_0, x_0) - \tau_0 F = 0 \tag{33}$$

First order ($x_1 a^1$).

$$\begin{cases} J(\bar{x}_1) = F \\ \tau_1^2 = \dfrac{s^2}{1 + \|\bar{x}_1\|^2} \\ x_1 = \tau_1 \bar{x}_1 \end{cases} \tag{34}$$

Higher orders ($x_k a^k$, $k \geqslant 2$).

$$\begin{cases} J\bar{x}_k = -\displaystyle\sum_{i=1}^{k-1} Q(x_i, x_{k-i}) \\ \tau_k = -\dfrac{\tau_1 x_1^\top \bar{x}_k}{s^2} \\ x_k = \dfrac{\tau_k}{\tau_1} x_1 + \bar{x}_k. \end{cases} \tag{35}$$

Nonsmooth Interior-Point Method (NIPM) and ANM

Principles of Asymptotic Numerical method (ANM)

Radius of convergence The radius of convergence for a user tolerance tol is approximated as follows:

$$a_{\max} = \left(\mathrm{tol} \frac{\|x_1\|}{\|x_N\|} \right)^{1/(N-1)} \tag{36}$$

ANM is able to calculate the central path with very tight tolerance ($\leqslant 10^{-14}$)

Nonsmooth Interior-Point Method (NIPM) and ANM

Figure: Values of τ_k w.r.t a for IPM-ANM for solving a problem with eleven contact points with accuracy 10^{-14}. Each marker indicates a new computation of an asymptotic sum. Bottom graph: a zoom on the last 4 ANM-IPM iterations.

Nonsmooth Interior-Point Method (NIPM) and ANM

Figure: Behavior of Algorithm 1 when solving PrimitiveSoup-ndof-6000-nc-1087-183. The projection of the central path in the cone 6 is represented by the red and blue curve with small circles. The dotted blue part of the curve corresponds to ANM iterations for which τ increases. The black curve represents the path of the interior point iterates. We can see that as τ decreases, the iterates of Algorithm 1 follow the central trajectory, but that as τ increases, the algorithm is no longer able to generate correct directions to continue following this trajectory.

Nonsmooth Interior-Point Method (NIPM) and ANM

Takeway

The central path is

- not monotonically parameterized,
- and sometimes slides on the boundary of the feasible domain.

➜ Need for a precise and fully tractable continuation method: ANM does the job

Nonsmooth Interior-Point Method (NIPM)

Moderate size problems

IPM with ANM is robust (but slow for large problems)

Conclusions & Perspectives

Conclusions

- Further research is needed to develop a robust AND efficient solver.
- The IPM and ANM numerical methods provide robust solvers.
- Coupling with other physical phenomena to obtain a monolithic variational inequality :
  - (non-associated) plasticity (Acary et al., 2023a; Guillet et al., 2025)
  - fracture with cohesive zone model (Collins-Craft et al., 2022)
  - damage mechanics.
  - and other funny mechanics...

Open software and data collections.

- Siconos/Numerics. A open source collection of solvers. https://github.com/siconos/siconos
- FCLIB: a open collection of discrete 3D Frictional Contact (FC) problems https://github.com/FrictionalContactLibrary contribute ...

Use and contribute ...

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