Non-smooth dynamics and optimization:

solving the contact problem with friction using the interior point method and the numerical asymptotic method.

CAS, Mines de Paris, October 2025

Vincent Acary

Inria - Centre de l'Université Grenoble Alpes - Laboratoire Jean Kuntzmann

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Pierre Alart, Paul Armand, Florent Cadoux, Frédéric Dubois,
Claude Lémaréchal, Jérôme Malick and Mathieu Renouf

Nonsmooth

Dynamics

- nonsmooth = lack of differentiability ($\notin \mathcal{C}^1$),
- graphs with peaks, kinks, jumps.

- systems that evolves with time,
- branch of mechanics concerned with the motion of objects.

Where is nonsmoothness?

- nonsmooth solutions in time and space:
  - continuous, functions of bounded variations, measures and distributions.
- nonsmooth modeling of constitutive laws:
  - set–valued mapping, inequality constraints, complementarity, impact laws,
  - ODE with discontinuous r.h.s, differential inclusion, measure equation.

TRIPOP INRIA project team (LJK)

Research object:

Modeling, Simulation and Control of Nonsmooth Dynamics.

Main application:

Natural environmental risks in mountains.

- Research Scientists
  - **Vincent Acary** (INRIA, LJK)
  - Bernard Brogliato (INRIA, LJK)
  - **Franck Bourrier** (INRAE LJK & IGE)
  - Olivier Goury (INRIA, LJK)
  - Filippo Masi (INRIA, LJK)
  - Felix Miranda Villatoro (INRIA. LJK)
  - Arnaud Tonnelier (INRIA, LJK)
- Engineer
  - Franck Pérignon (CNRS, LJK)
  - Maurice Brémond (INRIA, SED)
  - Samuel Heidman (INRIA, SED)
- External collaborators
  - **Thierry Faug** (INRAE IGE)
  - Christophe Prieur (CNRS, Gipsa-Lab)
- PhD
  - Quang Hung Pham (UGA 2022-2025)
  - **Louis Guillet** (INRIA 2023–2025)
  - **Florian Vincent** (INRIA 2023–2026)
  - **Mattéo Oziol** (G-INP 2023–2026)
  - **Chloé Gergely** (UGA, 2024–2027)
- Post-doc
  - Ritesh Gupta (INRIA, 2023-2024)
  - Simon Le Berre (INRIA, 2024)

Scientific pairs: M. Jean, J.J. Moreau, M. Schatzman & C. Lemaréchal.

Our reference (bedside) books

- Moreau, J. J. (1966). *Fonctionnelles convexes.* Séminaire Jean Leray, Collège de France.
- Rockafellar, R.T. (1970). *Convex Analysis.* Princeton UP.
- Duvaut, G., & Lions, J.-L. (1972). *Les inégalités variationnelles en mécanique et en physique.* Dunod.
- Brézis, H. (1973). *Opérateurs maximaux monotones et semi–groupes de contractions dans les espaces de Hilbert.* Math. Studies.
- Germain, P. (1973) *Cours de mécanique des milieux continus.* Tome 1. Masson & Cie.
- Moreau, J. J. (1973) *On Unilateral Constraints, Friction and Plasticity.* Springer, 2011.
- Moreau, J. J., & Panagiotopoulos, P. D. (Eds.). (1988). *Nonsmooth mechanics and applications.* Springer.
- Cottle, R. W., Pang, J. S., & Stone, R. E. (1992). *The linear complementarity problem.* SIAM.
- Hiriart-Urruty, J.-B., & Lemaréchal, C. (1996). *Convex analysis and minimization algorithms.* Springer.
- Brogliato, B. (1999). *Nonsmooth mechanics.* Springer.
- Nguyen, Q. S. (2000). *Stability and nonlinear solid mechanics.* Wiley.
- Facchinei, F. & Pang J. S. (2003). *Finite-dimensional variational inequalities and complementarity problems.* Springer.
- Rockafellar, R. T., & Wets, R. J. B. (2009). *Variational analysis (Vol. 317).* Springer.

Variational inequalities

Differential inclusions

Switched systems

Nonsmooth Mechanical systems

Sliding mode control

Optimal control with state constraints

Hybrid systems and cyberphysical systems

Complementarity systems

Elasto-dynamics with plasticity, contact and impact.

A second order sweeping process

$$
\begin{cases}
v^+ = \dot{q}^+ & \text{(velocity of bounded variations)} \\[2ex]
M(q)dv + F(q, v^+)dt + B^\top \sigma dt = \iota & \text{(differential measure)} \\[2ex]
\dot{\sigma} = E(Bv - \dot{\varepsilon}^p) & \text{(elasticity)} \\[2ex]
\dot{\varepsilon}^p \in N_C(\sigma) & \text{(plasticity)} \\[2ex]
-\iota \in N_{T_M(q)}(v^+ + ev^-) & \text{(impact and contact)}
\end{cases}
$$

Motivations & contents

1. Introduce a sufficiently generic and representative discrete 3D frictional contact problem.
2. Interpret this problem in the context of numerical optimisation and mathematical programming.
3. Provide an existence result, whose assumption can be verified numerically.
4. Compare the main existing numerical methods based on a large collection of problems (FCLIB) and a common implementation (SICONOS/Numerics).
5. Propose a new solution method based on the interior point method.

The discrete frictional contact problem

An existence result via convex optimization

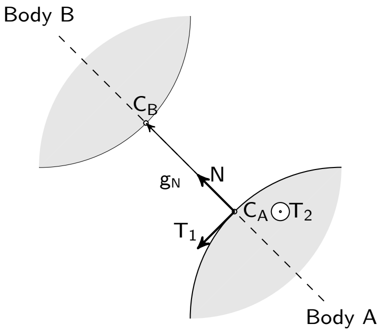
Existing numerical solvers

Benchmarking: Siconos/numerics and FCLIB

Interior Point Methods (IPM)

Conclusions & Perspectives

Signorini's condition and Coulomb's friction



- gap function $g_N = (C_B - C_A)N$.
- reaction forces and velocities

$$r = r_N N + r_T, \quad \text{with } r_N \in \mathbb{R}, \quad r_T \in \mathbb{R}^2.$$

$$u = u_N N + u_T, \quad \text{with } u_N \in \mathbb{R} \quad u_T \in \mathbb{R}^2.$$

- Signorini conditions

$$\text{position level : } 0 \leqslant g_N \perp r_N \geqslant 0.$$

$$\text{velocity level : } \begin{cases} 0 \leqslant u_N \perp r_N \geqslant 0 & \text{if } g_N \leqslant 0 \\ r_N = 0 & \text{otherwise.} \end{cases}$$

Signorini's condition and Coulomb's friction

Coulomb friction modeling assumption

Let μ be the coefficient of friction. Let us define the Coulomb friction cone K which is chosen as the isotropic second order cone

$$K = \{r \in \mathbb{R}^3 \mid \|r_{\mathsf{T}}\| \leqslant \mu r_{\mathsf{N}}\}. \tag{1}$$

Coulomb friction postulates

- for the sticking case that

$$u_{\mathsf{T}} = 0, \quad r \in K, \tag{2}$$

- and for the sliding case that

$$u_{\mathsf{T}} \neq 0, \quad \|r_{\mathsf{T}}\| = \mu r_{\mathsf{N}}, \quad r_{\mathsf{T}} = -\frac{u_{\mathsf{T}}}{\|u_{\mathsf{T}}\|}\|r_{\mathsf{T}}\|. \tag{3}$$

Disjunctive formulation of the frictional contact behavior

$$\begin{cases} r = 0 & \text{if } g_{\mathsf{N}} > 0 \quad \text{(no contact)} \\ r = 0, u_{\mathsf{N}} \geqslant 0 & \text{if } g_{\mathsf{N}} \leqslant 0 \quad \text{(take–off)} \\ r \in K, u = 0 & \text{if } g_{\mathsf{N}} \leqslant 0 \quad \text{(sticking)} \\ r \in \partial K, u_{\mathsf{N}} = 0, r_{\mathsf{T}} = -\dfrac{u_{\mathsf{T}}}{\|u_{\mathsf{T}}\|}\|r_{\mathsf{T}}\| & \text{if } g_{\mathsf{N}} \leqslant 0 \quad \text{(sliding)} \end{cases} \tag{4}$$

Signorini's condition and Coulomb's friction

Second Order Cone Complementarity (SOCCP) formulation

- Modified relative velocity $\tilde{u} \in \mathbb{R}^3$ (De Saxcé, 1992) defined by

$$\tilde{u} = u + \mu \|u_{\mathrm{T}}\| \mathrm{N}. \tag{5}$$

- Second-Order Cone Complementarity Problem (SOCCP)

$$K^\star \ni \tilde{u} \perp r \in K \tag{6}$$

if $g_{\mathrm{N}} \leqslant 0$ and $r = 0$ otherwise.
The set $K^\star$ is the dual convex cone to K defined by

$$K^\star = \{u \in \mathbb{R}^3 \mid r^\top u \geqslant 0, \quad \text{for all } r \in K\}. \tag{7}$$

(Acary and Brogliato, 2008; Acary et al., 2011)

Signorini's condition and Coulomb's friction

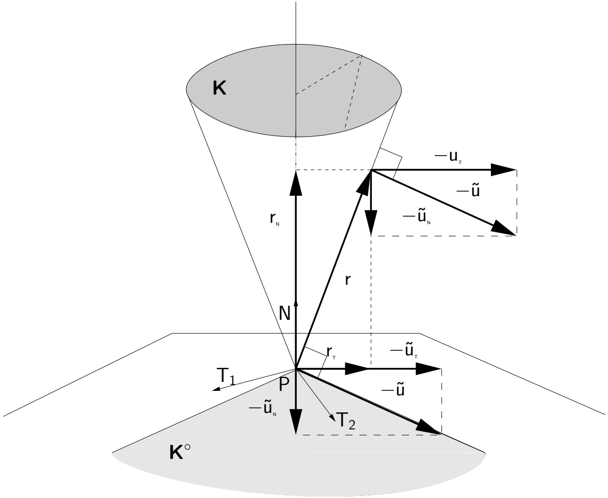


Figure: Coulomb's friction and the modified velocity $\tilde{u}$. The sliding case.

Discrete frictional contact problems

Problem 1 (General discrete frictional contact problem)

Given

- *a symmetric positive definite matrix* $M \in \mathbb{R}^{n \times n}$,
- *a vector* $f \in \mathbb{R}^n$,
- *a matrix* $H \in \mathbb{R}^{n \times m}$,
- *a vector* $w \in \mathbb{R}^m$,
- *a vector of coefficients of friction* $\mu \in \mathbb{R}^{n_c}$,

find three vectors $v \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ *and* $r \in \mathbb{R}^m$, *denoted by* $\mathrm{FC/I}(M, H, f, w, \mu)$ *such that*

$$\begin{cases} Mv = Hr + f \\ u = H^\top v + w \\ \tilde{u} = u + g(u) \\ K^\star \ni \tilde{u} \perp r \in K \end{cases} \tag{8}$$

with $g(u) = [[\mu^\alpha \|u_T^\alpha\| \mathrm{N}^\alpha]^\top, \alpha = 1 \dots n_c]^\top$. □

Discrete frictional contact problems

Wide range of applications

The problem is:

- is generic enough to include a large number of cases in practice,
- is really representative in the linear, or the linearized, case (Newton procedure),
- can be generalised to non-linear cases.

See for instance (Acary and Cadoux, 2013)

Origin of the linear relation $u = H^\top v + w$

- H is the contact configuration matrix (similar to the Jacobians of the constraints)
- w can contain
  - impact laws terms or prescribed velocity in velocity level formulations
  - displacements, or increments of displacements, in position level formulations

Discrete frictional contact problems

Origin of the linear relation $Mv = Hr + f$

- Time–discretization of the discrete dynamical mechanical system.
  Event–capturing or event–detecting time–stepping schemes
- Space discretization of the quasi–static problem of solids (FEM)
  (M is the tangent stiffness matrix !).
- Time–discretization and space discretization of the dynamic problem of solids.
  (FEM, MPM, PFEM, ...)
- Flexible or rigid multi-body Systems,
- Spectral methods, harmonic balance method, ...

Discrete frictional contact problems

Problem 2 (Reduced discrete frictional contact problem)

Given

- *a symmetric positive semi–definite matrix* $W \in \mathbb{R}^{m\times m}$,
- *a vector* $q \in \mathbb{R}^m$,
- *a vector* $\mu \in \mathbb{R}^{n_c}$ *of coefficients of friction,*

find two vectors $u \in \mathbb{R}^m$ *and* $r \in \mathbb{R}^m$, *denoted by* $\mathrm{FC/II}(W, q, \mu)$ *such that*

$$\begin{cases} u = Wr + q \\ \tilde{u} = u + g(u) \\ K^\star \ni \tilde{u} \perp r \in K \end{cases} \tag{9}$$

with $g(u) = [[\mu^\alpha \| u_T^\alpha \| \mathrm{N}^\alpha]^\top, \alpha = 1 \ldots n_c]^\top$. □

Relation with the general problem

$W = H^\top M^{-1} H$ and $q = H^\top M^{-1} f + w$.

From the optimization point of view

Discrete frictional contact are complementarity problems / variational inequalities.

Finite dimensional Second-Order Cone Complementarity Problems (SOCCP)

$$K^\star \ni Wr + q + g(Wr + q) \perp r \in K \tag{10}$$

of more generally,

Variational Inequality (VI) (normal cone inclusion)

$$-(Wr + q + g(Wr + q)) \stackrel{\Delta}{=} -F(r) \in N_K(r). \tag{11}$$

Properties

- nonsmooth since $g()$ is nonsmooth
- nonmonotone since the mapping F is not monotone for large μ
- many possible reformulations such as nonsmooth equations $G(r) = 0$

From the optimization point of view

Important Remarks

- The variational inequality is NOT the optimality condition of a (convex) optimization problem.
- The problem is hard to solve efficiently and robustly at tight accuracy.
- Even harder if H is not full rank (constraints redundancy)
- Generic numerical methods for VI/CP exist and can be applied
- Numerous of existing methods for FC3D problems are adaptations of mathematical programming methods.

From the optimization point of view

Semismooth Newton methods for nonsmooth equations $G(r) = 0$.

Not just adaptations, but sometimes pioneering methods.

- The natural map F^{nat} associated with the VI (11)

$$F^{\text{nat}}(r) = r - P_K(r - F(r))$$

- Pioneering work of Alart and Curnier, 1991

$$\begin{cases} r_{\mathrm{N}} - P_{\mathbb{R}_+^{n_c}}(r_{\mathrm{N}} - \rho_{\mathrm{N}} u_{\mathrm{N}}) = 0, \\ r_{\mathrm{T}} - P_{D(\mu, r_{\mathrm{N},+} + \rho u_{\mathrm{N}})}(r_{\mathrm{T}} - \rho_{\mathrm{T}} u_{\mathrm{T}}) = 0, \end{cases}$$

- other SOCCP functions (Fisher-Bursmeister function)

From the optimization point of view

An optimization problem

$$
\begin{aligned}
\min_{v,u,r} \quad & \tilde{u}^\top r = u^\top r + \mu r_{\mathrm{N}} \|u_{\mathrm{T}}\| \stackrel{\Delta}{=} b(u,r) \\
\text{s.t.} \quad & Mv = Hr + f \\
& \tilde{u} = H^\top v + w + g(u) \in K^\star \\
& r \in K
\end{aligned}
\tag{12}
$$

$b(u,r)$ is the de Saxcé bi-potential.

- A solution of the discrete frictional contact problem is a solution of the optimization problem (12) with $b(u,r) = 0$
- A solution of the optimization problem (12) is a solution of the discrete frictional contact problem if $b(u,r) = 0$
- With constraints qualification, the problem has a solution.
- The problem is not convex and non smooth, may have a lot of local minima.

➜ In practice, finding a minimum is difficult, and a global minimum is not ensured.

The discrete frictional contact problem

An existence result via convex optimization

Existing numerical solvers

Benchmarking: Siconos/numerics and FCLIB

Interior Point Methods (IPM)

Conclusions & Perspectives

An existence result via convex optimization

PhD of F. Cadoux with C. Lemaréchal and J. Malick (Acary et al., 2011; Cadoux, 2009)

Let us introduce a slack variable

$$s^\alpha := \|u^\alpha_{\mathrm{T}}\|$$

New formulation of the modified velocity with $A \in \mathbb{R}^{m \times n_c}$

$$\tilde{u} := u + As \qquad (g(u) = As)$$

The problem $\mathrm{FC/I}(M, H, f, w, \mu)$ can be reformulated as

$$\begin{cases} Mv = Hr + f \\ \widetilde{u} = H^\top v + w + As \\ K^\star \ni \tilde{u} \perp r \in K \end{cases} \tag{13}$$

with

$$s^\alpha := \|u^\alpha_{\mathrm{T}}\| \tag{14}$$

An existence result via convex optimization

The problem (13) appears to be the KKT condition of

Primal problem

$$\begin{cases} \min \quad J(v) := \dfrac{1}{2} v^\top M v + f^\top v \\ H^\top v + w + As \in K^\star \end{cases} \tag{D_s}$$

Dual problem

$$\begin{cases} \min \quad J_s(r) := \dfrac{1}{2} r^\top W r - q_s^\top r \\ r \in K \end{cases} \tag{P_s}$$

with $q_s = q + As$

Interest

Two convex programs ➜ existence of solutions under feasibility conditions.

An existence result via convex optimization

Fixed point problem

Introducing

$$u(s) := \operatorname{argmin}_u(P_s) = \operatorname{argmin}_u(D_s)$$

practically computable by optimization software, and

$$F^\alpha(s) := \|u_T^\alpha(s)\|,$$

the incremental problem becomes a fixed point problem

$$F(s) = s$$

An existence result via convex optimization

Key assumption

$$\exists v \in \mathbb{R}^m \ : \ Hv + w \in \operatorname{int} K^\star \tag{15}$$

Using Assumption (15),

- the application $F : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ is well-defined, continuous and bounded
- apply Brouwer's theorem

Theorem 3

A fixed point exists

An existence result via convex optimization

Numerical validation of the key assumption

Solving a SOC linear program: find $x^\star \in \boldsymbol{R}$

$$\begin{array}{ll} \max\limits_{x} & x \\ \text{s.t.} & Hv + w - ax \in K^\star \end{array}$$

where $a = \mathrm{col}(\mathrm{N}^\alpha, \alpha \in [\![1, \mathrm{m}]\!]) \in \mathbb{R}^{\mathrm{m}}$.

If $x^\star > 0$, then the assumption is satisfied.

Numerical interest

The fixed point equation $F(s) = s$ can be tackled by

- fixed-point iterations
$$s \leftarrow F(s)$$
- Newton iterations
$$s \leftarrow \mathrm{Jac}[F](s) \backslash F(s)$$

The inner problem can be solved by QP solvers with SOC constraints (ADMM, IPM, AL, ...)

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Conclusions & Perspectives

Existing numerical solution procedures

- VI based methods
- Nonsmooth Equations based methods
- Matrix block–splitting and projection based algorithms
- Proximal point algorithms
- Optimization based approaches

VI based methods

Variational Inequality (VI) reformulation

$$(9) \iff -F(r) := -(Wr + q + g(Wr + q)) \in N_K(r) \tag{16}$$

Standard methods

- Basic fixed point iterations with projection [FP-VI]

$$\mathrm{r_{k+1} \leftarrow P_K(r_k - \rho_k F(r_k))}$$

- with fixed $\rho_k = \rho$, we get the Uzawa Algorithm of Saxcé and Feng, 1998 with similarity with augmented Lagrangian methods(Wriggers, 2006) [FP-DS]
- Extragradient method [EG-VI]

$$\mathrm{r_{k+1} \leftarrow P_K(r_k - \rho_k F(P_K(r_k - \rho_k F(r_k))))}$$

Self-adaptive procedure for ρ_k [UPK]

$$\text{Armijo-like}: \; m_k \in \boldsymbol{N} \quad \text{such that} \left\{ \begin{array}{l} \rho_k = \rho 2^{m_k}, \\ \rho_k \|F(r_k) - F(\bar{r}_k)\| \leqslant \|r_k - \bar{r}_k\| \end{array} \right.$$

Nonsmooth Equations based methods

Nonsmooth Newton on $G(z) = 0$

$$z_{k+1} = z_k - \Phi^{-1}(z_k)(G(z_k)), \qquad \Phi(z_k) \in \partial G(z_k)$$

- Alart–Curnier Formulation (Alart and Curnier, 1991) [NSN-AC]

$$\begin{cases} r_{\mathrm{N}} - P_{\mathbf{R}^{n_c}_{+}}(r_{\mathrm{N}} - \rho_{\mathrm{N}} u_{\mathrm{N}}) = 0, \\ r_{\mathrm{T}} - P_{D(\mu, r_{\mathrm{N},+} + \rho u_{\mathrm{N}})}(r_{\mathrm{T}} - \rho_{\mathrm{T}} u_{\mathrm{T}}) = 0, \end{cases}$$

- Jean–Moreau Formulation [NSN-MJ]

$$\begin{cases} r_{\mathrm{N}} - P_{\mathbf{R}^{n_c}_{+}}(r_{\mathrm{N}} - \rho_{\mathrm{N}} u_{\mathrm{N}}) = 0, \\ r_{\mathrm{T}} - P_{D(\mu, r_{\mathrm{N},+})}(r_{\mathrm{T}} - \rho_{\mathrm{T}} u_{\mathrm{T}}) = 0, \end{cases}$$

- Direct normal map reformulation [NSN-NM]

$$r - P_K\left(r - \rho(u + g(u))\right) = 0$$

- Extension of Fischer-Burmeister function to SOCCP [NSN-FB]

$$\phi_{\mathrm{FB}}(x, y) = x + y - (x^2 + y^2)^{1/2}$$

Matrix block-splitting and projection based algorithms (Jean and Touzot, 1988; Moreau, 1994)

Block splitting algorithm with $W^{\alpha\alpha} \in \boldsymbol{R}^3$ [NSGS-*]

$$\left\{\begin{array}{l} u^{\alpha}_{i+1} - W^{\alpha\alpha} P^{\alpha}_{i+1} = q^{\alpha} + \sum\limits_{\beta<\alpha} W^{\alpha\beta} r^{\beta}_{i+1} + \sum\limits_{\beta>\alpha} W^{\alpha\beta} r^{\beta}_{i} \\[2ex] \widetilde{u}^{\alpha}_{i+1} = \left[u^{\alpha}_{\mathrm{N},i+1} + \mu^{\alpha}\, ||u^{\alpha}_{\mathrm{T},i+1}||, u^{\alpha}_{\mathrm{T},i+1}\right]^T \\[2ex] \mathbf{K}^{\alpha,*} \ni \widetilde{u}^{\alpha}_{i+1} \perp r^{\alpha}_{i+1} \in \mathbf{K}^{\alpha} \end{array}\right. \tag{17}$$

for all $\alpha \in \{1 \ldots m\}$.

Over-Relaxation [PSOR-*]

One contact point problem

- closed form solutions
- Any solver listed before.

Optimization based methods

- Alternating optimization problems (Panagiotopoulos et al.) [PANA-*]
- Successive approximation with Tresca friction (Haslinger et al.) [TRESCA-*]

$$\begin{cases} \theta = h(r_{\mathrm{N}}) \\ \min \ \frac{1}{2} r^\top W r + r^\top q \\ \quad \text{s.t.} \qquad r \in C(\mu, \theta) \end{cases} \tag{18}$$

where $C(\mu, \theta)$ is the cylinder of radius $\mu\theta$.

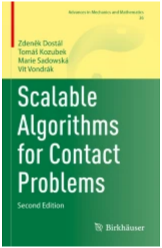
- Fixed point on the norm of the tangential velocity [A., Cadoux, Lemaréchal, Malick(2011)] [ACLM-*].

$$\begin{cases} s = \|u_{\mathrm{T}}\| \\ \min \ \frac{1}{2} r^\top W r + r^\top (q + \alpha s) \\ \quad \text{s.t.} \qquad r \in K \end{cases} \tag{19}$$

Fixed point or Newton Method on $F(s) = s$

Optimization based methods

Optimization, contact and huge-scale problems. (Dostál et al., 2023)



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siconos/numerics

siconos

Open source software for modelling and simulation of nonsmooth systems

siconos/numerics

Collection of C routines to solve FC3D problems in dense, sparse or block sparse versions:

- VI solvers: Fixed point, Extra-Gradient, Uzawa
- VI based projection/splitting algorithm: NSGS, PSOR
- Semismooth Newton methods
- Optimization based solvers. Panagiotopoulos, Tresca, SOCQP, ADMM
- Interior point methods, ...

Collection of routines for optimization and complementarity problems

- LCP solvers (iterative and pivoting (Lemke))
- Standard QP solvers (Projected Gradient (Calamai & Moré), Projected CG (Moré & Toraldo), active set technique)
- linear and nonlinear programming solvers.

FCLIB : a collection of discrete 3D Frictional Contact (FC) problems

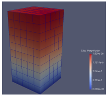
- Few mathematical results: existence, uniqueness, convergence, rate of convergence.
- Our inspiration: MCPLIB or CUTEst in Optimization.
- Without convergence proof, test your method on a large set of benchmarks shared by the community.

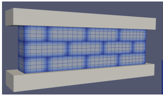
What is FCLIB ?

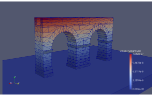
- A open source collection of Frictional Contact (FC) problems stored in a specific HDF5 format
- A open source light implementation of Input/Output functions in C Language to read and write problems (Python and Matlab coming soon)

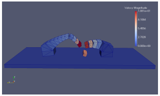
Goals of the project

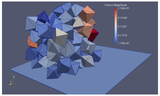
- Provide a standard framework for testing available and new algorithms for solving discrete frictional contact problems
- Share common formulations of problems in order to exchange data in a reproducible manner.

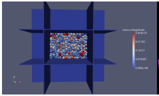

(a) Cubes_H8

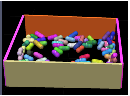

(b) LowWall_FEM


(c) Aqueduct_PR

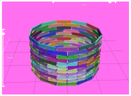

(d) Bridge_PR

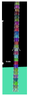

(e) 100_PR_Periobox


(f) 945_SP_Box_PL


(g) Capsules


(h) Chain


(i) KaplasTower


(j) BoxesStack

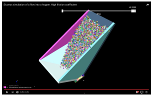

(k) Chute_1000, Chute_4000, Chute_local_problems

Figure: Illustrations of the FClib test problems from Siconos and LMGC90

Parameters of the simulation campaign

- More than 2500 problems
- Around 30 solvers with their variants
- More than 27000 runs between few seconds up to 400s.

Full error criteria

$$\text{error} = \frac{\|F^{\text{nat}}_{\text{vi-2}}(r)\|}{\|q\|}. \tag{20}$$

Performance profiles Dolan and Moré, 2002

- Given a set of problems $\mathcal{P}$
- Given a set of solvers $\mathcal{S}$
- A performance measure for each problem with a solver $t_{p,s}$ (cpu time, flops, ...)
- Compute the performance ratio

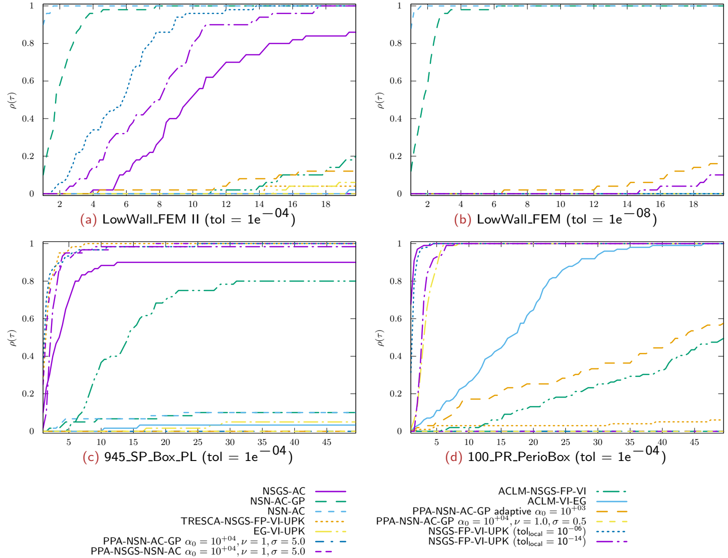
$$\tau_{p,s} = \frac{t_{p,s}}{\min\limits_{s \in \mathcal{S}} t_{p,s}} \geqslant 1 \tag{21}$$

- Compute the performance profile $\rho_s(\tau) : [1, +\infty] \to [0, 1]$ for each solver $s \in \mathcal{S}$

$$\rho_s(\tau) = \frac{1}{|\mathcal{P}|} \big| \{ p \in \mathcal{P} \mid \tau_{p,s} \leqslant \tau \} \big| \tag{22}$$

The value of $\rho_s(1)$ is the probability that the solver s will win over the rest of the solvers.

Comparisons by families of solvers



(a) LowWall_FEM II (tol = 1e−04)

(b) LowWall_FEM (tol = 1e−08)

(c) 945_SP_Box_PL (tol = 1e−04)

(d) 100_PR_PerioBox (tol = 1e−04)

Benchmarking : conclusions

Conclusions

1. No “Swiss–knife” solution : choose efficiency OR robustness
2. Newton–based solvers solve efficiently some problems, but robustness issues
3. First order iterative methods (VI, NSGS, PSOR) solves all the problems but very slowly
4. The rank of the H matrix (≈ratio number of contacts unknows/number of d.o.f) plays an important role on the robustness
5. Optimisation-based and proximal-point algorithm solvers are interesting but it is difficult to forecast theirs efficiencies.
6. Need for a second order method when H is rank-deficient (IPM?)

Mode details in Acary et al., 2018

The discrete frictional contact problem

An existence result via convex optimization

Existing numerical solvers

Benchmarking: Siconos/numerics and FCLIB

Interior Point Methods (IPM)

Conclusions & Perspectives

A Primer on IPM for convex QP

Convex QP

Consider the convex quadratic program in standard form:

$$\min_{x\in\mathbb{R}^n} \; f(x) := \tfrac{1}{2}x^T Qx + c^T x$$
$$\text{subject to} \;\; Ax = b, \quad x \geqslant 0,$$

where $Q \succeq 0$, $A \in \mathbb{R}^{m\times n}$.
First-order (KKT) conditions (existence of multipliers y, z):

$$\begin{aligned} Qx + c + A^T y - z = 0 \quad & \text{(stationarity)} \\ Ax - b = 0 \quad & \text{(primal feasibility)} \\ XZe = 0, \quad x \geqslant 0, \; z \geqslant 0 \quad & \text{(complementarity)} \end{aligned}$$

where $X = \mathrm{diag}(x)$, $Z = \mathrm{diag}(z)$, e is the vector of ones.

Interior-point idea

Enforce strict positivity $x > 0, z > 0$ and drive $XZe = \mu e$ to zero along the *central path* as $\mu \downarrow 0$.

A Primer on IPM for convex QP

Barrier formulation and Newton steps

Use a logarithmic barrier to keep $x > 0$:

$$\phi_\mu(x) = \tfrac{1}{2}x^T Qx + c^T x - \mu \sum_{i=1}^{n} \log x_i$$

with equality constraints $Ax = b$. For fixed $\mu > 0$ solve

$$\min_{x>0} \ \phi_\mu(x) \quad \text{s.t. } Ax = b.$$

The perturbed KKT conditions for $\mu > 0$ are:

$$\begin{aligned} Qx + c + A^T y - z &= 0, \\ Ax - b &= 0, \\ XZe &= \mu e. \end{aligned}$$

A Primer on IPM for convex QP

We linearize these equations at (x, y, z) to get the Newton system for $(\Delta x, \Delta y, \Delta z)$:

$$\begin{bmatrix} Q & A^T & -I \\ A & 0 & 0 \\ Z & 0 & X \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{bmatrix} = - \begin{bmatrix} r_d \\ r_p \\ r_c \end{bmatrix},$$

where

$$\begin{aligned} r_d &= Qx + c + A^T y - z, \\ r_p &= Ax - b, \\ r_c &= XZe - \mu e. \end{aligned}$$

This 3-block structure is the foundation of primal–dual interior-point methods. Take a damped Newton step and project back to positive orthant with step-length chosen to maintain $x > 0$.

Full Newton step for primal–dual system

The perturbed KKT conditions for $\mu > 0$ are:

$$\begin{aligned} Qx + c + A^T y - z &= 0, \\ Ax - b &= 0, \\ XZe &= \mu e. \end{aligned}$$

We linearize these equations at (x, y, z) to get the Newton system for $(\Delta x, \Delta y, \Delta z)$:

$$\begin{bmatrix} Q & A^T & -I \\ A & 0 & 0 \\ Z & 0 & X \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{bmatrix} = - \begin{bmatrix} r_d \\ r_p \\ r_c \end{bmatrix},$$

where

$$\begin{aligned} r_d &= Qx + c + A^T y - z, \\ r_p &= Ax - b, \\ r_c &= XZe - \mu e. \end{aligned}$$

This 3-block structure is the foundation of primal–dual interior-point methods.

Illustration of the central path

Definition: The *central path* is the trajectory of strictly feasible points $(x(\mu), y(\mu), z(\mu))$ satisfying

$$Qx(\mu) + c + A^T y(\mu) - z(\mu) = 0, \quad Ax(\mu) = b, \quad X(\mu)Z(\mu)e = \mu e, \ \mu > 0.$$

As $\mu \to 0$, the path converges to the optimal solution (x^*, y^*, z^*).

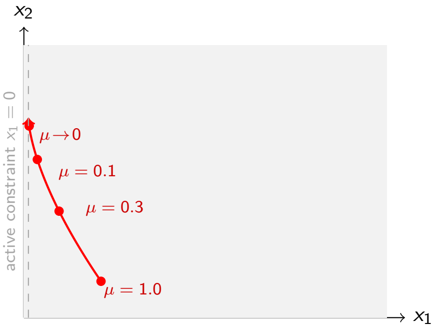
Geometric intuition:

- Each μ defines a barrier problem that smooths the feasible region.
- The central path traces the minimizers of these smoothed problems.
- Interior-point methods follow this path using Newton directions.

Interpretation: As μ decreases, the minimizer moves smoothly along the red curve toward the boundary optimum.

Central path in the positive orthant (boundary optimum)

The central path for a problem with $x \geqslant 0$ approaches the boundary smoothly as $\mu \to 0$. The iterates remain strictly positive until convergence.



Interpretation: As $\mu \downarrow 0$, $x_1(\mu) \to 0$ and the iterates approach the boundary while maintaining strict positivity, ensuring well-defined Newton steps.

Primal–dual interior-point method & practical notes

Primal–dual methods work with both primal and dual variables and maintain perturbed complementarity:

$$XZe = \sigma\mu e, \quad \mu = \frac{x^T z}{n}, \quad 0 < \sigma \leqslant 1.$$

A common predictor–corrector algorithm (Mehrotra) computes a predictor (affine-scaling) step, estimates a σ, then corrects to stay close to the central path. Key practical points:

- Use sparse symmetric linear algebra to solve the Newton/KKT system efficiently.
- Choose step-lengths to keep iterates positive; use Mehrotra predictor-corrector for fast practical convergence.
- Convergence: polynomial worst-case complexity (typically 30–60 iterations for large QPs in practice).

Interior-point methods follow the central path using Newton steps on a barrier-regularized KKT system; primal–dual variants maintain both primal and dual feasibility and usually reach high accuracy in few iterations.

Interior Point Methods for frictional contact

PhD thesis of Hoang Minh Nguyen(2025), with Paul Armand

Perturbation of the complementarity condition with a barrier parameter τ

Original problem	Perturbed problem
$\begin{gathered} Mv + f = H^\top r \\ Hv + w + se = \tilde{u} \\ s = \|\tilde{u}_\mathrm{T}\| \\ \tilde{u} \circ r = 0 \\ (\tilde{u}, r) \in K^2 \end{gathered}$	$\begin{gathered} Mv + f = H^\top r \\ Hv + w + se = \tilde{u} \\ s = \|\tilde{u}_\mathrm{T}\| \\ \tilde{u} \circ r = 2\tau e \\ (\tilde{u}, r) \in \mathrm{int}(K^2) \end{gathered} \qquad (23)$

Convex case (s fixed)

IPM is able to solve very accurately and efficiently the problem with a given $s := \|u_\mathrm{T}\|$ even when H is rank-deficient (see Acary et al., 2023b).
Special care of sparse linear systems and conditioning with Nesterov-Todd scaling.

➜ Extension to general frictional contact problems: nonsmooth interior point method

Nonsmooth Interior-Point Method (NIPM)

Slater's assumption (SA) $\exists v \in \mathbb{R}^m$ such that $Hv + w \in \mathrm{int}(K)$

Theorem 4

1. *Under SA, for each $\tau > 0$, the perturbed problem (23) has a solution $(v_\tau, \tilde{u}_\tau, r_\tau, s_\tau)$*
2. *Under SA, there exists an analytic central path $\{(v_\tau, \tilde{u}_\tau, r_\tau, s_\tau) : \tau > 0\}$, which converges to a solution of the original problem*

Proof: Fixed point Brouwer's Theorem and curve selection lemma in semi-algebraic analysis.

Main theoretical outcome

- Alternative proof of solution existence for $\mathrm{FC/I}(M, H, f, w, \mu)$
- The central path is not necessarily unique !

Nonsmooth Interior-Point Method (NIPM) - Linearization

Iterations and Jacobian matrix

$$G := \begin{bmatrix} Mv + f - H^\top r \\ Hv + w - \tilde{u} + se \\ s - \|\tilde{u}_\mathrm{T}\| \\ \tilde{u} \circ r \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2\tau e \end{bmatrix} \qquad J := \begin{bmatrix} M & -H^\top & 0 & 0 \\ H & 0 & -I & e \\ 0 & 0 & -L & 1 \\ 0 & \tilde{U} & R & 0 \end{bmatrix}$$

where $L = \begin{pmatrix} 0 & \partial\|\tilde{u}_\mathrm{T}\|^\top \end{pmatrix}$, with $\partial\|\tilde{u}_\mathrm{T}\| = \begin{cases} \dfrac{\tilde{u}_\mathrm{T}}{\|\tilde{u}_\mathrm{T}\|} & \text{if } \tilde{u}_\mathrm{T} \neq 0 \\ d \in \mathbb{B} & \text{if } \tilde{u}_\mathrm{T} = 0 \end{cases}$ (unit ball $\mathbb{B}$)

Linear system and Mehrotra's-Like algorithm

$$J\,d = -G + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2\sigma\tau e \end{bmatrix}$$

predictor-corrector scheme $\sigma \in (0, 0.5)$: centralization parameter

Stopping test

$$\max\left\{ \|Hv + w - \tilde{u}\|_\infty,\ \|Mv + f - H^\top r\|_\infty, |s - \|\tilde{u}_\mathrm{T}\||,\ \|\tilde{u} \circ r\| \right\} \leqslant \text{ tol}$$

Nonsmooth Interior-Point Method (NIPM)

Algorithm Mehrotra type Nonsmooth Interior Point Method (NIPM)

Parameters: Choose a starting point (v, r, u, s) such that $(r, u) \in \mathrm{int}(\mathcal{K}^2)$. Choose $\eta_1 \in (0, 1)$, $\eta_2 \geqslant 1$, $\eta_3 \geqslant 1$, $\gamma_1 \in (0, 1)$, $\gamma_2 \in (0, 1 - \gamma_1)$, $c_1 \geqslant 1$, $c_2 \in (0, 1)$, $\mathtt{tol} > 0$ and set $\gamma = 0.99$;

1: **if** the stopping criterion is satisfied **then** return (v, r, u, s) as solution of (??);
2: Set $\tau \leftarrow u^\top r/n$;
3: **if** $\|s - \ell(u)\|_2 \geqslant c_1 \|u \circ r\|_2$ **then** set $\sigma \leftarrow 0.998$, $d_a^u \circ d_a^r \leftarrow 0$ and go to 10,
4: **else** compute $d_a = (d_a^v, d_a^r, d_a^u, d_a^s)$ solution of

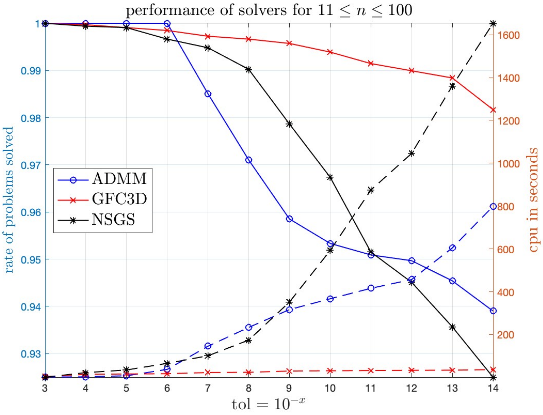
$$J(v, r, u, s)d_a = -G(v, r, u, s);$$

5: Find the greatest $\alpha_a \in (0, 1]$ such that $(r, u) + \alpha_a(d_a^r, d_a^u) \in \mathcal{K}^2$;
6: Set $\tau_a \leftarrow (u + \alpha_a d_a^u)^\top (r + \alpha_a d_a^r)/n$;
7: **if** $\tau > \eta_1$ **then** set $\beta \leftarrow \max\{1, \eta_2 \alpha_a^2\}$ **else** set $\beta \leftarrow \eta_3$;
8: Set $\sigma \leftarrow 0.998 \min\{1, (\tau_a/\tau)^\beta\}$;
9: **if** $\sigma \geqslant c_2$ **then** set $d_a^u \circ d_a^r \leftarrow 0$;
10: Compute $d = (d^v, d^r, d^u, d^s)$ solution of

$$J(v, r, u, s)d = -G(v, r, u, s) + \begin{bmatrix} 0 \\ 0 \\ -d_a^u \circ d_a^r + \sigma\tau e \\ 0 \end{bmatrix} \tag{24}$$

11: Find the greatest $\alpha \in (0, 1]$ such that $(r, u) + \alpha(d^r, d^u) \in (1 - \gamma)(r, u) + \mathcal{K}^2$;
12: Set $\gamma \leftarrow \gamma_1 + \alpha\gamma_2$;
13: Set $(v, r, u, s) \leftarrow (v, r, u, s) + \alpha(d^v, d^r, d^u, d^s)$ and goto 1.

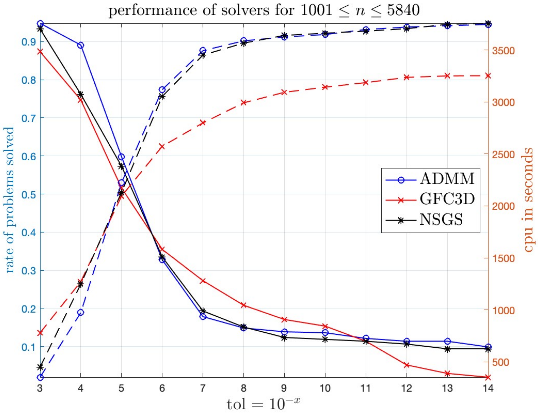
Nonsmooth Interior-Point Method (NIPM)



Problem's size $11 \leqslant n \leqslant 100$

IPM (GFC3D) outperforms NSGS and ADMM in terms of efficiency and robustness

Nonsmooth Interior-Point Method (NIPM)

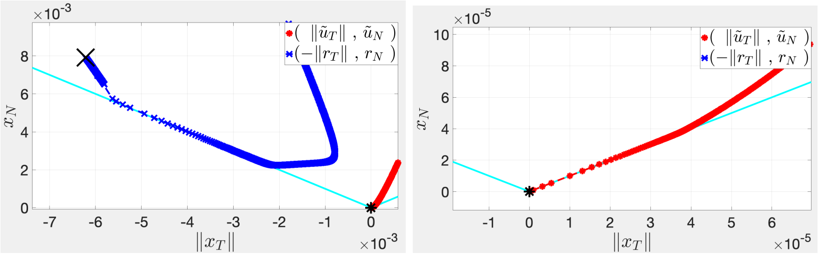


Problem's size $1000 \leqslant n$

IPM (GFC3D) suffers from robustness

Nonsmooth Interior-Point Method (NIPM) - failures

Failure #1: A special shape of the central path



Solution: $r^{\times} \in \mathrm{int}(K)$, $\tilde{u}^{*} = 0$ (sticking)

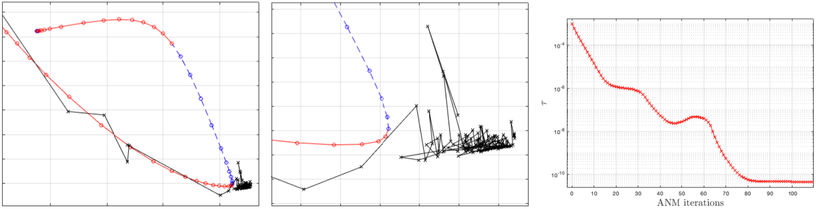
This shape of the central path can cause iterates to get stuck on the boundary, which is not the correct position for the solution.

Nonsmooth Interior-Point Method (NIPM) - failures

Failure #2: Non-monotone parameterization of the central path

- Red-blue curve: Central path $\tau \to r(\tau)$ calculated by Asymptotic Numerical Method (ANM). Red: τ decreases. Blue: τ increases

- **Black** curve: the path of NIPM iterates



➜ Asymptotic Numerical Method (ANM): algorithm based on the computation of series to perform accurate numerical continuations of parameterized non-linear problems

Nonsmooth Interior-Point Method (NIPM) and ANM

Reformulation as bilinear problem

Let us write the perturbed problem (23) under the form

$$F(x,\tau) = G(x) - \tau \begin{pmatrix} 0 \\ 0 \\ 2e \\ 0 \end{pmatrix} = 0, \tag{25}$$

where $x = (v, r, u, s)$ and takes advantage of the fact that $G(x)$ can be written as a sum of linear term, a constant term and bilinear function as

$$L(x) + b + Q(x,x) \tag{26}$$

with

$$x = \begin{pmatrix} v \\ u \\ r \\ s \end{pmatrix}, \quad L = \begin{pmatrix} M & -H^\top & & 0 \\ -H & 0 & I & -E \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad Q(x,\tilde{x}) = \begin{pmatrix} 0 \\ 0 \\ u \circ \tilde{r} \\ s \bullet \tilde{s} - l(u) \bullet l(\tilde{u}) \end{pmatrix} \tag{27}$$

and

$$F = \begin{pmatrix} 0 & 0 & 2e & 0 \end{pmatrix}^\top, \quad b = \begin{pmatrix} -f & -w & 0 & 0 \end{pmatrix}^\top, \quad \lambda = \tau. \tag{28}$$

Nonsmooth Interior-Point Method (NIPM) and ANM

Principles of Asymptotic Numerical method (ANM)

The principle of ANM is to start from an initial known vector (x_0, τ_0) such that $F(x_0, \tau_0) = 0$, then to calculate a solution of (25) under the form of truncated series

$$x(t) = \sum_{k=0}^{N} x_k a^k \quad \tau(t) = \sum_{k=0}^{N} \tau_k a^k,$$

➜ In our particular case, bilinear function $\implies$ constant Jacobian for a computation of all the terms of the series.

$$J(x) = \begin{pmatrix} M & -H^\top & & 0 \\ -H & 0 & I & -E \\ 0 & U & R & 0 \\ 0 & 0 & -A & 2\,\mathrm{diag}(s) \end{pmatrix} \tag{29}$$

with

$$A = 2\,\mathrm{diag}(\begin{bmatrix} 0 & u_\alpha^\top \end{bmatrix}, \alpha \in \{1 \ldots m\}). \tag{30}$$

Nonsmooth Interior-Point Method (NIPM) and ANM

Principles of Asymptotic Numerical method (ANM)

Zeroth order ($x_0 a^0$). For $k = 0$, the substitution of (**??**) in (**??**) gives

$$R(x_0, \lambda_0) = L(x_0) + b + Q(x_0, x_0) - \lambda_0 F = 0 \tag{31}$$

First order ($x_1 a^1$).

$$\begin{cases} J(\bar{x}_1) = F \\ \lambda_1^2 = \dfrac{s^2}{1 + \|\bar{x}_1\|^2} \\ x_1 = \lambda_1 \bar{x}_1 \end{cases} \tag{32}$$

Higher orders ($x_k a^k$, $k \geqslant 2$).

$$\begin{cases} J\bar{x}_k = -\displaystyle\sum_{i=1}^{k-1} Q(x_i, x_{k-i}) \\ \lambda_k = -\dfrac{\lambda_1 x_1^\top \bar{x}_k}{s^2} \\ x_k = \dfrac{\lambda_k}{\lambda_1} x_1 + \bar{x}_k. \end{cases} \tag{33}$$

Nonsmooth Interior-Point Method (NIPM) and ANM

Principles of Asymptotic Numerical method (ANM)

Radius of convergence The radius of convergence for a user tolerance tol is approximated as follows:

$$a_{\max} = \left(\mathrm{tol}\frac{\|x_1\|}{\|x_N\|}\right)^{1/(N-1)} \tag{34}$$

ANM is able to calculate the central path with very tight tolerance ($\leqslant 10^{-14}$)

Nonsmooth Interior-Point Method (NIPM) and ANM

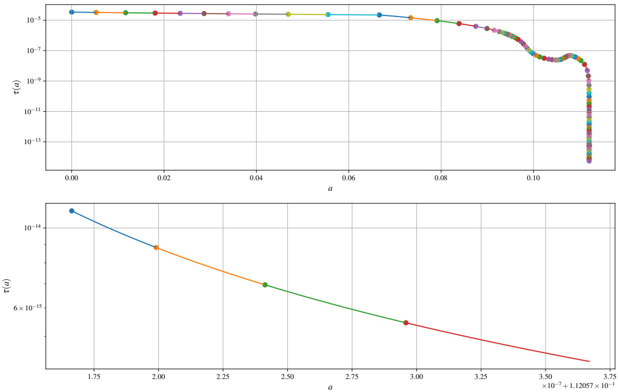


Figure: Values of τ_k w.r.t a for IPM-ANM for solving a problem with eleven contact points with accuracy 10^{-14}. Each marker indicates a new computation of an asymptotic sum. Bottom graph: a zoom on the last 4 ANM-IPM iterations.

Nonsmooth Interior-Point Method (NIPM) and ANM

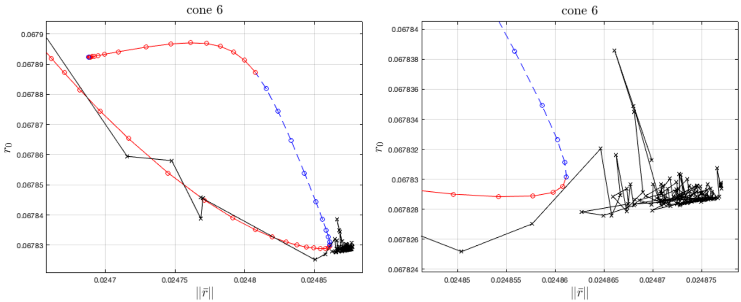


Figure: Behavior of Algorithm 1 when solving PrimitiveSoup-ndof-6000-nc-1087-183. The projection of the central path in the cone 6 is represented by the red and blue curve with small circles. The dotted blue part of the curve corresponds to ANM iterations for which τ increases. The black curve represents the path of the interior point iterates. We can see that as τ decreases, the iterates of Algorithm 1 follow the central trajectory, but that as τ increases, the algorithm is no longer able to generate correct directions to continue following this trajectory.

Nonsmooth Interior-Point Method (NIPM) and ANM

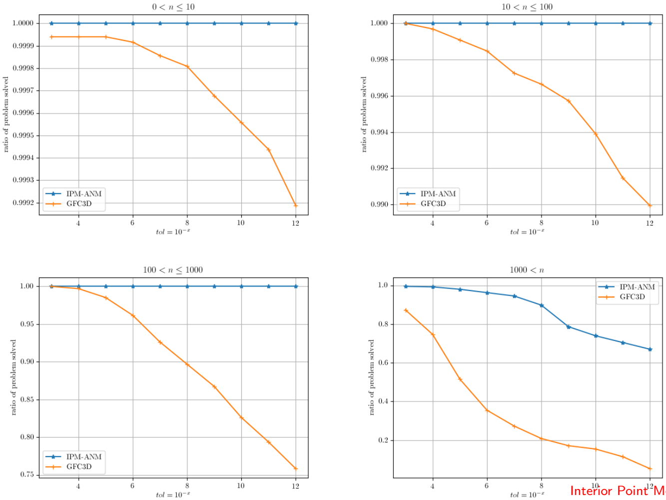
Takeway

The central path is not monotonically parameterized and sometimes slides on the boundary of feasible domains. ➜ Need for a precise and fully functional continuation method: ANM

Nonsmooth Interior-Point Method (NIPM)

Moderate size problems

IPM with ANM is robust (but slow for large problems)



Conclusions & Perspectives

Conclusions

- Further research is still needed for an robust AND efficient solver.
- IPM and ANM numerical method provides a robust solver.
- Coupling with other physical phenomena to obtain a monolithic variational inequality :
  - (non associated) plasticity (Acary et al., 2023a; Guillet et al., 2025)
  - fracture with cohesive zone model (Collins-Craft et al., 2022)
  - damage mechanics.

Open software and data collections.

- Siconos/Numerics. A open source collection of solvers. https://github.com/siconos/siconos
- FCLIB: a open collection of discrete 3D Frictional Contact (FC) problems https://github.com/FrictionalContactLibrary contribute ...

Use and contribute ...

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Acary, V. and B. Brogliato (2008). *Numerical methods for nonsmooth dynamical systems. Applications in mechanics and electronics.* English. Lecture Notes in Applied and Computational Mechanics 35. Berlin: Springer. xxi, 525 p.

Acary, V. and F. Cadoux (2013). "Recent Advances in Contact Mechanics, Stavroulakis, Georgios E. (Ed.)". In: vol. 56. Lecture Notes in Applied and Computational Mechanics. Springer Verlag. Chap. Applications of an existence result for the Coulomb friction problem.

Acary, V. et al. (2011). "A formulation of the linear discrete Coulomb friction problem via convex optimization". In: *ZAMM - Journal of Applied Mathematics and Mechanics / Zeitschrift für Angewandte Mathematik und Mechanik* 91.2, pp. 155–175. ISSN: 1521-4001. DOI: 10.1002/zamm.201000073. URL: http://dx.doi.org/10.1002/zamm.201000073.

Acary, Vincent, Franck Bourrier, and Benoit Viano (Feb. 2023a). "Variational approach for nonsmooth elasto-plastic dynamics with contact and impacts". In: *Computer Methods in Applied Mechanics and Engineering* 414, p. 116156. DOI: 10.1016/j.cma.2023.116156. URL: https://inria.hal.science/hal-03978387.

Acary, Vincent, Maurice Brémond, and Olivier Huber (June 2018). "On solving contact problems with Coulomb friction: formulations and numerical comparisons". In: *Advanced Topics in Nonsmooth Dynamics - Transactions of the European Network for Nonsmooth Dynamics*. Ed. by Remco Leine, Vincent Acary, and Olivier Brüls. Springer International Publishing, pp. 375–457. DOI: 10.1007/978-3-319-75972-2_10. URL: https://hal.inria.fr/hal-01878539.

Acary, Vincent et al. (Apr. 2023b). "Second order cone programming for frictional contact mechanics using interior point algorithm". working paper or preprint. In revision. URL: https://hal.science/hal-03913568.

Alart, P. and A. Curnier (1991). "A mixed formulation for frictional contact problems prone to Newton like solution method". In: *Computer Methods in Applied Mechanics and Engineering* 92.3, pp. 353–375.

Cadoux, F. (2009). "Analyse convexe et optimisation pour la dynamique non–regulière". PhD thesis. Université Joseph Fourier, Grenoble I.

Collins-Craft, Nicholas Anton, Franck Bourrier, and Vincent Acary (Oct. 2022). "On the formulation and implementation of extrinsic cohesive zone models with contact". In: *Computer Methods in Applied Mechanics and Engineering* 400, p. 115545. DOI: 10.1016/j.cma.2022.115545. URL: https://hal.science/hal-03371667.

De Saxcé, G. (1992). "Une généralisation de l'inégalité de Fenchel et ses applications aux lois constitutives". In: *Comptes Rendus de l'Académie des Sciences* t 314,série II, pp. 125–129.

Dolan, E.D. and J.J. Moré (2002). "Benchmarking optimization software with performance profiles". In: *Mathematical Programming* 91.2, pp. 201–213.

Dostál, Zdeněk et al. (2023). *Scalable algorithms for contact problems*. Vol. 36. Springer.

Guillet, Louis et al. (2025). "Implicit Material Point Method for non-associated plasticity of geomaterials". working paper or preprint. URL: https://hal.science/hal-05070887.

Jean, M. and G. Touzot (1988). "Implementation of unilateral contact and dry friction in computer codes dealing with large deformations problems". In: *J. Méc. Théor. Appl.* 7.1, pp. 145–160.

Moreau, J.J. (1994). "Some numerical methods in multibody dynamics: Application to granular materials". In: *European Journal of Mechanics - A/Solids* supp.4, pp. 93–114.

Saxcé, G. De and Z.-Q. Feng (1998). "The bipotential method: A constructive approach to design the complete contact law with friction and improved numerical algorithms". In: *Mathematical and Computer Modelling* 28.4. Recent Advances in Contact Mechanics, pp. 225–245. ISSN: 0895-7177. DOI: https://doi.org/10.1016/S0895-7177(98)00119-8. URL: http://www.sciencedirect.com/science/article/pii/S0895717798001198.

Wriggers, P. (2006). *Computational Contact Mechanics*. Second. originally published by John Wiley & Sons Ltd., 2002. Springer Verlag.