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MA 5710 Assignment - 1

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① As we know, the Delay - Differential Equation is given as:

$$v + \frac{d}{dt} z_i(t+\tau) = v \ln\left\{ \rho_{max}\left[\frac{e}{\rho_{max}} + z_{i-1}(t) - z_i(t)\right]\right\}$$

for $2 \le i \le N$ with initial condⁿs:

$z_i(t) = 0$ for $t < 0$ $1 \le i \le N$

Also, $z_1(t) = \begin{cases} 0 & \text{for } t \le 0 \\ -vB(t) & \text{for } t > 0 \end{cases}$ (#)

Here, we are trying to find an explicit solution for ~~the~~ Delay-DE by substituting $(t-\tau)$ for t in the Delay - Differential Eqⁿ:

$$v + \frac{d}{dt} z_i(t) = v \ln\left\{ \rho_{max}\left[\frac{e}{\rho_{max}} + z_{i-1}(t-\tau) - z_i(t-\tau)\right]\right\}$$

$$\Rightarrow \frac{d}{dt} z_i(t) = v \ln\left\{ \rho_{max}\left[\frac{e}{\rho_{max}} + z_{i-1}(t-\tau) - z_i(t-\tau)\right]\right\} - v$$

$$\Rightarrow \frac{d}{dt} z_i(t) = v \ln\left[e + \rho_{max}(z_{i-1}(t-\tau) - z_i(t-\tau))\right] - v \ln e \quad (\because \ln e = 1)$$

$$* \Rightarrow \boxed{\frac{d}{dt} z_i(t) = v \ln\left[1 + \frac{\rho_{max}}{e}\left(z_{i-1}(t-\tau) - z_i(t-\tau)\right)\right]}$$

The eqn (*) is with initial condns

$$z_i(t-\tau) = 0 \quad \text{for } t < \tau$$

From (#), we also know that

$$z_1(t-\tau) = \begin{cases} 0 & \text{for } t \le \tau \\ -vB(t-\tau) & \text{for } t > \tau \end{cases}$$

Say, we want to find $z_2(t)$.
Let us consider the interval $t \in [0, \tau)$

$$\Rightarrow t-\tau \in [-\tau, 0)$$

$$\Rightarrow z_2(t-\tau) = 0$$

So for $z_2(t)$ in the interval $t \in [0, \tau)$, we can write the eqn (*) as:

$$\frac{d}{dt} z_2(t) = v \ln\left[1 + \frac{f_{max}}{e} z_1(t-\tau)\right] \quad \text{—(i)}$$

for $t \in [0, \tau]$, $t - \tau \in [-\tau, 0]$

$$\Rightarrow z_1(t-\tau) = 0$$

Substituting $z_1(t-\tau) = 0$ in eqn (i), we have

$$\frac{d}{dt} z_2(t) = v \ln\left[1 + \frac{f_{max}}{e}(0)\right]$$

$$\Rightarrow \frac{d}{dt} z_2(t) = 0$$

$$\Rightarrow \boxed{z_2(t) = k \quad \text{for } t \in [0, \tau)} \quad \text{—(ii)}$$

where k is some constant.
for continuity at $t = 0$, $z_2(t) = k = 0$ for $t \in [0, \tau)$

Now, if we want to find $z_2(t)$ in $t \in [\tau, 2\tau)$

~~$\Rightarrow t - \tau \in [0, \tau]$~~

$\Rightarrow t - \tau \in [0, \tau)$

We use $z_1(t-\tau)$ & $z_2(t-\tau)$ values from the interval $t \in [\tau, 2\tau]$

In $t \in [\tau, 2\tau)$

$$\frac{d}{dt} z_2(t) = v_{lim}\left[1 + \frac{f_{max}}{e}\left(z_1(t-\tau) - z_2(t-\tau)\right)\right]$$

$z_1(t-\tau)$ ~~& $z_2(t)$~~ is known as given

$z_2(t-\tau)$ for $t \in [\tau, 2\tau)$ is known from eqn (ii)

→ In this way, we can find $z_2(t)$ in the interval $t \in [\tau, 2\tau)$ by integrating.

→ For subsequent intervals, say $[2\tau, 3\tau)$, $[3\tau, 4\tau)$, ----, $[k\tau, (k+1)\tau)$ ...

we can use $z_2(t-\tau)$ from previous intervals & $z_1(t-\tau)$ as given.

→ Continuing these iterations can help us find $z_2(t)$ on any interval by using the values of $z_1(t)$ & $z_2(t)$ from the previous interval using integration.

→ Using this process recursively, will give us the function $z_2(t)$ for all $t \geq 0$.

~~→ Knowledge of z_2 on $[-\tau, (k-1)\tau]$ can be used to compute z_3 on $[0, k\tau]$ & so on.~~

→ Knowledge of z_2 on $[-\tau, (k-1)\tau]$ can be used to compute z_3 on $[0, k\tau]$ & so on.

A step-by-step process is given as follows:

~~1) Say we want all z have N cars, we want~~

1) $z_1(t)$ is known over the desired interval $[0, k\tau]$

2) Say we have N cars. We want $z_i(t)$ for all $i = 2$ to N.

3) ~~First we find~~ Also, we know that

$$z_2(t) = 0 \quad \text{for } t < 0$$

(No perturbation difference exists before the instant of perturbation)

4) So we first find $z_2(t)$ in the interval $[0, \tau)$ using $z_1(t)$ in the interval $[0, \tau)$ which is known.

$$\frac{dz_2(t)}{dt} = v\tanh\left[1 + \frac{f_{max}}{e}\left(z_1(t-\tau) - \cancel{z_2(t-\tau)}^{\,0}\right)\right]$$

5) Using $z_2(t)$ in the interval $[0, \tau)$, we can find $z_2(t)$ in the interval $[\tau, 2\tau]$ using integration on the eq^n

$$\frac{dz_2(t)}{dt} = v\tanh\left[1 + \frac{f_{max}}{e}\left(z_1(t-\tau) - z_2(t-\tau)\right)\right]$$

6) Step (5) can be repeated to get $z_2(t)$ over all intervals $[2\tau, 3\tau)$, $[3\tau, 4\tau)$ ----- $[(k-1)\tau, k\tau)$

also.

Finally, we would have $z_2(t)$ for $t \in [0, k\tau)$

7) Now we can find $z_3(t)$ using integration on the eqn

$$\frac{dz_i(t)}{dt} = v_m\left\{1+\frac{f_{max}}{e}\left[z_{i-1}(t-\tau)-z_i(t-\tau)\right]\right\}$$

which becomes

$$\frac{dz_3(t)}{dt} = v_m\left\{1+\frac{f_{max}}{e}\left[z_2(t-\tau)-z_3(t-\tau)\right]\right\}$$

for $i=3$.

This is possible because we have $z_2(t)$ on all intervals of length τ over $t\in[0,k\tau]$

8) We can repeat step (7) again & again to get $z_4(t), z_5(t), \ldots, z_N(t)$

By the end, we'll have $z(t)$ for all the N cars.

~~A chart~~ The following chart depicts our method:

$t \to$ $z \downarrow$	$[-\tau, 0]$	$[0, \tau)$	------	$[k\tau, (k+1)\tau)$	...
z_1	0	given	given	given	
z_2	0	z_2 on $[0,\tau)$	----	z_2 on $[k\tau, (k+1)\tau)$	
$\vdots$					
z_i	0	z_i on $[0,\tau)$	...	z_i on $[k\tau, (k+1)\tau)$	
$\vdots$					
z_N	0	z_n on $[0,\tau)$	---	z_n on $[k\tau, (k+1)\tau]$	

Note that,

(i) We have N cars in the model
(ii) Time is broken into intervals of length τ
(iii) $z_i(t) = 0$ for all i when $t < 0$
(iv) The arrows in the chart depict the dependencies b/w various time intervals.
(v) Generally, $z_i(t)$ is determined from z_i & z_{i-1} from the previous interval $z_i(t-\tau)$ & $z_{i-1}(t-\tau)$

for examples,

$z_3(t)$ on the interval $[\tau, 2\tau]$ is determined by the values of z_2 & z_3 on the interval $[0, \tau)$

Limitations of the method: $z_1, z_2, \dots, z_N$ i from 1 to N.

(i) Integrations become more & more complex for the later time intervals & higher indices of i because these intervals are dependent on the previous ones.

This makes the use of the current method quite impractical

(ii) If we write an algorithm to automate the calculations, for N cars over k time intervals of length τ, the time complexity will at least be of the order $k * N$.

For large values of k & N, the computation time will be huge, making our ~~correct~~ current way of assessing the model highly inefficient.

Referencess used for this question:

(i) Prof. Sundar's Class Notes
(ii) The Textbook: Mathematical Modelling: A Case Studies Approach.

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SETUP

③ The density, ρ can be given as

$$\rho = \frac{1}{|x_i - x_{i-1}|}$$

$x_n \ldots\ldots x_2 \quad x_1$

(One car lies b/w the front posn of 2 cars)

$$\Rightarrow \boxed{\rho = \frac{1}{x_{i-1} - x_i}} \quad \text{(i)}$$

(If we maintain the order, $x_{i-1} > x_i$)

Given that accn is proportional to $(\dot{x}_{i-1}(t) - \dot{x}_i(t))$ & inversely proportional to the density, we can write the acceleration while factoring in the driver's reaction time as,

$$\boxed{\ddot{x}_i(t+\tau) = \lambda\left(\dot{x}_{i-1}(t) - \dot{x}_i(t)\right)\left(\frac{1}{\rho}\right)}$$

λ is some +ve priorly unknown proportionality constant

$$\Rightarrow \ddot{x}_i(t+\tau) = \lambda\left(\dot{x}_{i-1}(t) - \dot{x}_i(t)\right)\left(x_{i-1}(t) - x_i(t)\right)$$

$$\Rightarrow \ddot{x}_i(t+\tau) = \lambda \frac{d}{dt}\left[\frac{\left(x_{i-1}(t) - x_i(t)\right)^2}{2}\right]$$

$$\Rightarrow \boxed{\dot{x}_i(t+\tau) = \lambda \frac{\left(x_{i-1}(t) - x_i(t)\right)^2}{2} + \alpha}$$

$$\Rightarrow \boxed{v_i(t+\tau) = \frac{\lambda}{2}\left(x_{i-1}(t) - x_i(t)\right)^2 + \alpha} \quad \text{(ii)}$$

from (i), we know that $\rho = \dfrac{1}{x_{i-1} - x_i}$

$$\Rightarrow \boxed{x_{i-1} - x_i = \frac{1}{\rho}} \quad \text{(iii)}$$

Substituting (iii) in (ii), we get

$$\boxed{v(\rho) = \frac{\lambda}{2\rho^2} + \alpha} \text{ ——— (iv)}$$

(Only for $\rho \le \rho_{crit}$)

We also know that,
$v(\rho_{max})$ should be 0. (because at maxm density cars won't have any space to move)

$\Rightarrow v(\rho_{max}) = 0$

$\Rightarrow \dfrac{\lambda}{2(\rho_{max})^2} + \alpha = 0$

$$\Rightarrow \boxed{\alpha = \frac{-\lambda}{2(\rho_{max})^2}} \text{ ——— (v)}$$

We also know that $\boxed{\rho_{max} = \frac{1}{L}}$ ——— (vi) (All cars standing one after the other)

$L \rightarrow$ Car Length

Substituting (vi) in (v), we get,

$$\boxed{\alpha = -\frac{\lambda L^2}{2}} \text{ ——— (vii)}$$

Assuming that for some $\rho \le \rho_{crit}$, all cars have same speed $V = V_{max}$,
we apply the condition of continuity at $\rho = \rho_{crit}$.

$v(\rho_{crit}) = V_{max}$

$$\Rightarrow \frac{\lambda}{2(\rho_{crit})^2} + \alpha = V_{max}$$

$$\Rightarrow \frac{\lambda}{2(\rho_{crit})^2} - \frac{\lambda}{2(\rho_{max})^2} = V_{max} \qquad \left[\because \alpha = \frac{-\lambda}{2(\rho_{max})^2} \text{ from (v)}\right]$$

$$\Rightarrow \frac{\lambda}{2}\left[\frac{(\rho_{max})^2 - (\rho_{crit})^2}{(\rho_{max})^2 (\rho_{crit})^2}\right] = V_{max}$$

$$\Rightarrow \boxed{\lambda = \frac{2 v_{max} (\rho_{max})^2 (\rho_{crit})^2}{(\rho_{max})^2 - (\rho_{crit})^2}} \text{ —— (viii)}$$

Using (vi) & (viii) in (iv), we get

$$\cancel{v(\rho) = \frac{v_{max} (\rho_{max})^2 (\rho_{crit})^2}{(\rho_{max})^2 - (\rho_{crit})^2} \cdot \frac{1}{\rho^2} - \lambda}$$

$$v(\rho) = \frac{\lambda}{2\rho^2} - \frac{\lambda}{2(\rho_{max})^2}$$

$$\Rightarrow v(\rho) = \frac{\lambda}{2}\left(\frac{1}{\rho^2} - \frac{1}{(\rho_{max})^2}\right)$$

$$\Rightarrow v(\rho) = \frac{\lambda}{2}\left[\frac{(\rho_{max})^2 - (\rho)^2}{(\rho_{max})^2 (\rho)^2}\right]$$

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$$\Rightarrow v(\rho) = \frac{V_{max}(\rho_{crit})^2}{\rho^2}\left[\frac{(\rho_{max})^2 - \rho^2}{(\rho_{max})^2 - (\rho_{crit})^2}\right] \quad (1)$$

Flux: $j(\rho)$ = # No. of cars passing through a given point per unit

$$\Rightarrow j(\rho) = \rho v(\rho)$$

$$\Rightarrow j(\rho) = \frac{V_{max}(\rho_{crit})^2}{(\rho_{max})^2 - (\rho_{crit})^2}\left[\frac{(\rho_{max})^2 - \rho^2}{\rho^2}\right]\rho$$

$$\Rightarrow j(\rho) = \frac{V_{max}(\rho_{crit})^2}{(\rho_{max}) - (\rho_{crit})^2}\left[\frac{(\rho_{max})^2 - \rho^2}{\rho}\right].$$

for maxm $j(\rho)$ for $\rho > \rho_{crit}$, we use

$$\frac{d}{d\rho}(j(\rho)) = 0$$

$$\Rightarrow \frac{V_{max}(\rho_{crit})^2}{(\rho_{max})^2 - (\rho_{crit})^2}\frac{d}{d\rho}\left(\frac{(\rho_{max})^2 - \rho^2}{\rho}\right) = 0$$

$$\Rightarrow \frac{d}{d\rho}\left[\frac{(\rho_{max})^2}{\rho} - \rho\right] = 0$$

$$\Rightarrow -\frac{(\rho_{max})^2}{\rho^2} - 1 = 0$$

Not possible for real values of ρ.

$$\Rightarrow 1 + \frac{(\rho_{max})^2}{\rho^2} = 0 \Rightarrow \boxed{\left(\frac{\rho_{max}}{\rho}\right)^2 = -1}$$

~~$\Rightarrow \rho^2 = -(\rho_{max})^2$ (Which is not possible for real values of ρ_{max})~~

∴ We can conclude that $\rho_{opt} = \rho_{crit}$ as this gives us the max^m flux.

So
$$\boxed{\begin{aligned} v(\rho_{opt}) &= v(\rho_{crit}) = V_{max} \\ j(\rho_{opt}) &= j(\rho_{crit}) = (V_{max})(\rho_{crit}) \end{aligned}} \quad \text{— (X)}$$

Now we can find the difference b/w the true position & the unperturbed position

$$z_i(t) = x_i(t) - y_i(t)$$

At eq^m, displacement $y_i(t)$ is given as

$$\boxed{y_i(t) = vt - (i-1)(d+L)}$$ (distance b/w cars, length of car)

$$\Rightarrow \boxed{\dot{y}_i(t) = v}$$

At $\rho = \rho_{opt}$, $v = V_{max}$ from (X)

$$\Rightarrow \boxed{\dot{y}_i(t) = V_{max}}$$

from (i),

$$v_i(t+\tau) = \frac{\lambda}{2}\left(x_{i-1}(t) - x_i(t)\right)^2 + \alpha$$

$$\Rightarrow v_i(t+\tau) = \frac{\lambda}{2}\left[\left(x_{i-1}(t) - x_i(t)\right)^2 - L^2\right] \quad \left(\because \alpha = \frac{-\lambda L^2}{2}\right)$$

$$\Rightarrow \boxed{v_i(t+\tau) = \frac{V_{max}(\rho_{max})^2(\rho_{crit})^2}{(\rho_{max})^2 - (\rho_{crit})^2}\left[\left(x_{i-1}(t) - x_i(t)\right)^2 - L^2\right]} \quad \text{— (xi)}$$

$$\because \rho_{max} = 1/L$$

$$\Rightarrow \boxed{v_i(t+\tau) = \frac{V_{max}(\rho_{max})^2(\rho_{crit})^2}{(\rho_{max})^2 - (\rho_{crit})^2}\left[\left(x_{i-1}(t) - x_i(t)\right)^2 - \frac{1}{\rho_{max}^2}\right]} \quad \text{— (xii)}$$

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Setting $k = \dfrac{(\rho_{max})^2 (\rho_{crit})^2}{(\rho_{max})^2 - (\rho_{crit})^2}$

$$\boxed{v_i(t+\tau) = k\, v_{max}\left[(x_{i-1}(t) - x_i(t))^2 - \frac{1}{\rho_{max}^2}\right]}$$

(xiii)

PERTURBATION HAPPENS

Assuming the instance of perturbation, $t = t_0 = 0$.
The lead driver brakes for a short time & then accelerates to the original speed v ($= v_{max}$, in this case)

Speed of the front car,

$$v_1(t) = \begin{cases} v_{max} & \text{for } t < 0 \\ v_{max}(1 - b(t)) & \text{for } t \in [0, t_1] \end{cases}$$

Integrating this, we get

$$x_1(t) = \begin{cases} v_{max}\, t & t < 0 \\ v_{max}(t - B(t)) & t \in [0, t_1] \end{cases}$$

Perturbed Pos^n ~~Displacement~~ of the lead car.

Here $B(t) = \begin{cases} 0 & \text{for } t < 0 \\ \int_0^t b(s)\,ds & \text{for } t \in [0, t_1] \end{cases}$

Now, we want to find the difference

$z_1(t) =$ (True Pos^n of the lead car) $-$ (Unperturbed / Equilibrium of the lead car)

$\Rightarrow z_1(t) = x_1(t) - y_1(t)$

$y_1(t) = v_{max}\, t$ for all t

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$$\Rightarrow \boxed{z_1(t) = \begin{cases} 0 & t<0 \\ -V_{max}\,B(t) & \text{for } t \in [0, t_1) \end{cases}}$$

IMPACT ON THE FOLLOWING CARS

$$z_i(t) = x_i(t) - y_i(t) \qquad i \geq 2$$

$$= \begin{cases} 0 & t \leq 0 \\ x_i(t) - (V_{max}\,t - (i-1)(d+L)) & t \in [0, t_1] \end{cases}$$

Note that, for eqm situation the posn of the i^{th} car is given as
$y_i(t) = V_{max}\,t - (i-1)(d+L)$
d = distance b/w cars
L = Length of cars

$$\Rightarrow z_i(t) = \begin{cases} 0 & t \leq 0 \\ x_i(t) - V_{max}\,t + (i-1)(d+L) & t \in [0, t_1] \end{cases} \qquad i \geq 2$$

$$\cancel{\Rightarrow z_i(t) = \begin{cases} 0 & \\ x_i(t) - V_{max}\,t + \frac{(i-1)}{\rho_{crit}} & \end{cases}}$$

$$\Rightarrow \boxed{z_i(t) = \begin{cases} 0 & t \leq 0 \\ x_i(t) - V_{max}\,t + \frac{(i-1)}{\rho_{crit}} & t \in [0, t_1] \end{cases}}$$

This is because $\rho_{crit} = \frac{1}{L+d}$

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$$\dot{z}_i(t) = \begin{cases} 0 & t \le 0 \\ \dot{x}_i(t) - v_{max} & t \in [0, t_1] \end{cases}$$

for $t > 0$,

$$\dot{z}_i(t) = \dot{x}_i(t) - v_{max}$$

$$\Rightarrow \dot{z}_i(t+\tau) = \dot{x}_i(t+\tau) - v_{max}$$

$$\Rightarrow \dot{z}_i(t+\tau) = k v_{max}\left[(x_{i-1}(t) - x_i(t))^2 - \frac{1}{\rho_{max}^2}\right] - v_{max}$$

$$\boxed{\Rightarrow v_{max} + \frac{d}{dt}(z_i(t+\tau)) = k v_{max}\left[(x_{i-1}(t) - x_i(t))^2 - \frac{1}{\rho_{max}^2}\right]} \quad \text{(xi)}$$

~~for $t > 0$~~
~~x_i & $z_{i-1}(t)$~~

for $t > 0$, from eqn (iv)

$$z_{i-1}(t) = x_{i-1}(t) - v_{max}t + \frac{(i-2)}{\rho_{crit}}$$

$$\& \; z_i(t) = x_i(t) - v_{max}t + \frac{(i-1)}{\rho_{crit}}$$

$$z_{i-1}(t) - z_i(t) = x_{i-1}(t) - x_i(t) - \frac{1}{\rho_{crit}}$$

$$\boxed{\Rightarrow x_{i-1}(t) - x_i(t) = z_{i-1}(t) - z_i(t) + \frac{1}{\rho_{crit}}} \quad \text{(xii)}$$

Substituting (xvi) in (xv), we get

$$V_{max} + \frac{d}{dt}\left(z_i(t+\tau)\right) = k V_{max}\left[\left(z_{i-1}(t) - z_i(t)\right) \cdot f \frac{1}{\rho(rit)} - \frac{1}{\rho_{max}^2}\right]$$

DELAY DIFFERENTIAL EQUATION.

for $2 \le i < N$ with initial condⁿs $z_i(t) = 0$ for $t < 0$

No. of cars

$1 \le i < N$.

$$\& \; z_1(t) = \begin{cases} 0 & t < 0 \\ -V_{max} B(t) & t \in [0, t_1] \end{cases}$$