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MA5710: Assignment-2

(2) Given: $\rho(x,t) = f\left(\frac{x - x_0}{t}\right)$ satisfies the conservation law as specified

To Prove: f must be the right inverse of j', ie,

$$j'(f(y)) = y \quad \forall\, y \in [j'(\rho_L), j'(\rho_R)]$$

Solution:

Given that,

$\rho(x,t) = f\left(\frac{x - x_0}{t}\right)$ satisfies — (i)

$$\frac{\partial \rho}{\partial t} + j'(\rho)\frac{\partial \rho}{\partial x} = 0$$

~~Ref Usin~~ Substituting for ρ using (i) gives

$$\frac{\partial}{\partial t}\left(f\left(\frac{x - x_0}{t}\right)\right) + j'\left(f\left(\frac{x - x_0}{t}\right)\right)\frac{\partial}{\partial x}\left(f\left(\frac{x - x_0}{t}\right)\right) = 0 \quad \text{— (ii)}$$

Substituting $u(x,t) = \left(\frac{x-x_0}{t}\right)$, we can get,

(A) $$\frac{\partial}{\partial t}\left(f\left(\frac{x-x_0}{t}\right)\right) = \frac{\partial f(u)}{\partial u}\frac{\partial u}{\partial t}$$

$$\Rightarrow \boxed{\frac{\partial}{\partial t}\left(f\left(\frac{x-x_0}{t}\right)\right) = -f'\left(\frac{x-x_0}{t}\right)\left(\frac{x-x_0}{t^2}\right)} \quad \text{(iii)}$$

(B) $$\frac{\partial}{\partial x}\left(f\left(\frac{x-x_0}{t}\right)\right) = \frac{\partial f(u)}{\partial u}\frac{\partial u}{\partial x}$$

$$\Rightarrow \boxed{\frac{\partial}{\partial x}\left(f\left(\frac{x-x_0}{t}\right)\right) = f'\left(\frac{x-x_0}{t}\right)\frac{1}{t}} \quad \text{(iv)}$$

Substituting (iii) & (iv) in (ii), gives,

$$\frac{\partial}{\partial t}\left(f\left(\frac{x-x_0}{t}\right)\right) + j'\left(f\left(\frac{x-x_0}{t}\right)\right)\frac{\partial}{\partial x}\left(f\left(\frac{x-x_0}{t}\right)\right) = 0$$

$$\Rightarrow -f'\left(\frac{x-x_0}{t}\right)\left(\frac{x-x_0}{t^2}\right) + j'\left(f\left(\frac{x-x_0}{t}\right)\right)f'\left(\frac{x-x_0}{t}\right)\frac{1}{t} = 0$$

$$\cancel{\Rightarrow f'\left(\frac{x-x_0}{t}\right)\left[-\left(\frac{x-x_0}{t^2}\right)\right.}$$

$$\Rightarrow \frac{1}{t}f'\left(\frac{x-x_0}{t}\right)\left[-\left(\frac{x-x_0}{t}\right) + j'\left(f\left(\frac{x-x_0}{t}\right)\right)\right] = 0$$

$$\Rightarrow \boxed{j'\left(f\left(\frac{x-x_0}{t}\right)\right) = \frac{x-x_0}{t}} \quad \text{(v)}$$

(Assuming $f'\left(\frac{x-x_0}{t}\right) \neq 0$)

Making it impossible to comment on j' $\Leftarrow$ (valid as $f'\left(\frac{x-x_0}{t}\right) = 0 \Rightarrow \partial/\partial t(f) = \frac{\partial f}{\partial x} = 0$)

Additionally, we know that,

$x'(t) = j'(p)$

$\Rightarrow x - x_0 = j'(p)\,t$

$$\Rightarrow \boxed{\frac{x - x_0}{t} = j'(p)} \quad \text{(vi)}$$

where $p = [p_l, p_r]$ depending on the value of x_0.

Now let $\frac{x - x_0}{t} = y$.

Substituting this (v) gives,

$$\boxed{j'(f(y)) = y \quad \forall\, y \in [j'(p_l), j'(p_r)]}$$

Hence, Proved.

(3) Given that, j' is continuous and monotonic on its domain. This would mean that there exists some unique function 'h' such that

$$j'(h(y)) = y \text{ and } h(j'(p)) = p \quad \text{(i)}$$

i.e., j' has ~~an inverse h~~ a unique inverse h.

But having solved Q2, we know that

$$j'(f(y)) = y \quad \text{(ii)}$$

From (i) & (ii), we can say that

$$h = f \text{ or } f \text{ is the inverse of } f' \quad \text{(iii)}$$

by using the uniqueness of inverse.

On the left edge,
$$x_l = x_0 + j'(\rho_l)t$$

$$\Rightarrow j'(\rho_l) = \frac{x_l - x_0}{t}$$

$$\Rightarrow f(j'(\rho_l)) = f\left(\frac{x_l - x_0}{t}\right)$$

$$\Rightarrow \boxed{f\left(\frac{x_l - x_0}{t}\right) = \rho_l} \quad \text{(using (iii))}$$

At the right edge,
$$x_r = x_0 + j'(\rho_r)t$$

$$\Rightarrow j'(\rho_r) = \frac{x_r - x_0}{t}$$

$$\Rightarrow f(j'(\rho_r)) = f\left(\frac{x_r - x_0}{t}\right)$$

$$\Rightarrow \boxed{f\left(\frac{x_r - x_0}{t}\right) = \rho_r} \quad \text{(using (iii))}$$

(4) Given that, $j(\rho) = 4\rho(2-\rho)$
$$\Rightarrow j'(\rho) = 8(1-\rho)$$

& $$\rho_0(x) = \begin{cases} 1 & ; x \le 1 \\ 1/2 & ; 1 < x \le 3 \\ 3/2 & ; x > 3 \end{cases}$$

So,

$$j'(\rho_0(x)) = \begin{cases} 0 & ; x \le 1 \\ 4 & ; 1 < x \le 3 \\ -4 & ; x > 3 \end{cases}$$

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In order to find ~~out~~ the rarefaction wave solution emerging from $x=1$, we have to replace ρ_0 by ρ_0^ϵ under smooth approximation condition.

$$\rho_0^\epsilon(x) = \begin{cases} 1 & ; x_0 \le 1-\epsilon \\ \frac{3}{4} - \frac{(x_0-1)}{4\epsilon} & ; 1-\epsilon < x_0 \le 1+\epsilon \\ 1/2 & ; 1+\epsilon < x_0 \le 3 \\ 3/2 & ; x_0 > 3 \end{cases} \quad \text{— (CERO)}$$

So, using $j'(\rho) = 8(1-\rho)$, we get,

$$j'(\rho_0^\epsilon(x)) = \begin{cases} 0 & ; x_0 \le 1-\epsilon \\ 2\left(1+\frac{(x_0-1)}{\epsilon}\right) & ; 1-\epsilon < x_0 \le 1+\epsilon \\ 4 & ; 1+\epsilon < x_0 \le 3 \\ -4 & ; x_0 > 3 \end{cases}$$

where $0 < \epsilon < 2$

$$x(t) = x_0 + 2\left(1+\frac{(x_0-1)}{\epsilon}\right)t \qquad (1-\epsilon < x \le 1+\epsilon) \quad \text{— (*)}$$

$$\Rightarrow x = x_0 + 2t + \frac{2x_0 t}{\epsilon} - 2t/\epsilon$$

$$\Rightarrow x_0 = \frac{x - 2t(1 - 1/\epsilon)}{1+\frac{2t}{\epsilon}}$$

$$\Rightarrow x_0 = \frac{\epsilon x - 2t\epsilon + 2t}{\epsilon + 2t}$$

$$\Rightarrow \boxed{x_0 = \frac{\epsilon x - 2t(\epsilon-1)}{\epsilon+2t}} \quad \text{— (i)}$$

Now,

$$\rho^\epsilon(x(t),t) = \rho^\epsilon\left(x_0 + 2t\left(1 + \frac{x_0 - 1}{\epsilon}\right), t\right)$$

$$= \rho_0^\epsilon(x_0)$$

$$= \frac{3}{4} - \frac{(x_0 - 1)}{4\epsilon}$$

$$= \frac{3}{4} - \frac{1}{4\epsilon}\left(\frac{\epsilon x - 2t\epsilon + 2t}{\epsilon + 2t} - 1\right) \quad (\text{using (i)})$$

$$= \frac{3}{4} - \frac{1}{4\epsilon}\left(\frac{\epsilon x - (2t+1)\epsilon}{\epsilon + 2t}\right)$$

$$= \frac{3}{4} - \frac{x - 2t - 1}{(\epsilon + 2t)4}$$

$$\Rightarrow \boxed{\rho^\epsilon(x(t),t) = \frac{3}{4} - \frac{x - 2t - 1}{4(\epsilon + 2t)}} \quad (\text{using } \cancel{*}) \quad \text{(ii)}$$

Now applying the limit $x \to 0$ on $\rho^\epsilon(x(t),t)$, we get,

$$\lim_{x \to 0} \rho^\epsilon(x, t) = \rho(x,t) \qquad (\text{for } x_0 = 1)$$

$$\Rightarrow \rho(x,t) = \frac{3}{4} - \frac{x - 2t - 1}{4(2t)}$$

$$\Rightarrow \rho(x,t) = \frac{3}{4} - \frac{x-1}{8t} + \frac{2t}{4(2t)}$$

$$\Rightarrow \boxed{\rho(x,t) = 1 - \frac{(x-1)}{8t}} \text{ — (UNO)}$$

↳ Rarefaction waves emerging from $x_0 = 1$

Hence, Proved ✓

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(5) Rarefaction baselines and baselines for $x_0 > 3$ will start intersecting at

$$x = 1 + 4t = x_0 - 4t \quad \text{—— (i)}$$

The shock wave would remain stationary at $x = 3$ till $t = t_s$.

Substituting $x = 3$ & $t = t_s$ in (i), we get

$$3 = 1 + 4t_s$$

$$\Rightarrow \boxed{t_s = \frac{1}{2}} \quad \text{—— (ii)}$$

The shock wave would move differently after $t_s = 1/2$.

Let its trajectory be represented as $\sigma(t)$.

The shock speed after $t_s = 1/2$ is given by using the rarefaction density from Q4, ie,

$$\rho(x,t) = 1 - \frac{(x-1)}{8t}.$$

So, shock speed after $t_s = 1/2$ is

$$\frac{d\sigma}{dt} = \frac{j_2 - j_1}{\rho_2 - \rho_1}.$$

$$= \frac{j\left(\frac{3}{2}\right) - j\left(1 - \frac{(\sigma-1)}{8t}\right)}{\frac{3}{2} - \left(1 - \frac{(\sigma-1)}{8t}\right)}$$

$$= \frac{3 - 4\left(1 - \frac{(\sigma-1)}{8t}\right)\left(1 + \frac{(\sigma-1)}{8t}\right)}{\frac{1}{2} + \frac{\sigma-1}{8t}}$$ (Using $j(\rho) = 4\rho(2-\rho)$)

$$\Rightarrow \frac{dv}{dt} = \frac{64t^2 - 4(8t-(v-1))(8t+(v-1))}{8t(4t+v-1)}$$

$$\Rightarrow \frac{dv}{dt} = \frac{192t^2 - 4(64t^2-(v-1)^2)}{8t(4t+v-1)}$$

$$\Rightarrow \frac{dv}{dt} = \frac{4(v-1)^2 - 64t^2}{8t(4t+v-1)}$$

$$\Rightarrow \frac{dv}{dt} = \frac{((v-1)-4t)\cancel{((v-1)+4t)}}{2t\cancel{(4t+v-1)}}$$

$$\Rightarrow \frac{dv}{dt} = \frac{v-1}{2t} - 2$$

$$\Rightarrow \boxed{\frac{dv}{dt} - \frac{v}{2t} = -2\left(1+\frac{1}{4t}\right)} \quad \text{— (iii)}$$

This is in the form of a linear Bernoulli differential equation.

Let $P(t) = -\frac{v}{2t}$ & $Q(t) = -2\left(1+\frac{1}{4t}\right)$

Integrating factor, $IF = e^{\int P(t)\,dt}$

$$= e^{\int -\frac{dt}{2t}}$$

$$= e^{\ln(1/\sqrt{t})}$$

$$\Rightarrow \boxed{IF = \frac{1}{\sqrt{t}}} \quad \text{— (iv)}$$

To find the solution, we proceed as

$$(IF)\,v = \int (IF)(Q(t))\,dt + c$$

$$\Rightarrow \frac{v}{\sqrt{t}} = \int \frac{-2}{\sqrt{t}}\left(1 + \frac{1}{4t}\right)dt + c$$

$$\Rightarrow \frac{v}{\sqrt{t}} = -4\sqrt{t} + \frac{1}{\sqrt{t}} + c$$

$$\Rightarrow \boxed{v(t) = -4t + 1 + c\sqrt{t}} \quad \text{— (v)}$$

We know that $v\left(\frac{1}{2}\right) = 3$

$$\Rightarrow -2 + 1 + \frac{c}{\sqrt{2}} = 3$$

$$\Rightarrow \boxed{c = 4\sqrt{2}} \quad \text{— (vi)}$$

Substituting (vi) in (v), we get

$$\boxed{v(t) = 1 + 4(\sqrt{2t} - t)} \quad \text{— (vii). — (DOS)}$$

But, keep in mind that this trajectory is only valid when the shock wave reaches the end of rarefaction waves, i.e. $v = 1$.

Using,

$v(t) = 1$

$\Rightarrow 1 + 4(\sqrt{2t} - t) = 1$

$\Rightarrow t = 0$ or $\boxed{t = 2}$ — (viii)

Hence, after $t = 2$, the shock wave goes on to follow a different trajectory. The left density is now

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given by $\rho_l = 1$ & right density by $\rho_r = 3/2$ at the intersection of $x_0 \le 1$ & $x_0 > 3$ baselines.

Now, by the Rankine-Hugoniot condⁿ, we can determine the shockspeed as follows

$$s = \frac{dX}{dt} = \frac{j_2 - j_1}{\rho_2 - \rho_1} = \frac{j(3/2) - j(1)}{3/2 - 1}$$

$$\Rightarrow \boxed{\frac{dX}{dt} = \frac{3-4}{1/2} = -2} \quad (\text{using } j(\rho) = 4\rho(2-\rho))$$

$$\Rightarrow \boxed{X = X_0 - 2t} \text{ — (ix)} \quad (\text{Here, } X \text{ is the Shockwave trajectory})$$

At $t=2$, $X=1$ from (viii)

$$\Rightarrow 1 = X_0 - 4$$
$$\Rightarrow \boxed{X_0 = 5} \text{ — (x)}$$

Substituting (x) in (ix), we get, the trajectory of the shockwave is

$$\boxed{X = 5 - 2t \text{ for } t > 2} \text{ — (xi)} \quad (\text{ANS})$$

Hence, we have successfully worked out that the ~~density to the right~~ left density & right density of the shockwave at $t=2$, $x=1$ are 1 & $3/2$ respectively.

Hence for $t > 2$ (Proved)

$$\rho(x,t) = \begin{cases} 1 & ; \; x \le 5-2t \\ \frac{3}{2} & ; \; x > 5-2t \end{cases}$$

(Q4) (Q4) (Q5) (Q5)

(6) Using the equations, CERO, UNO, DOS & TRES, we can write the complete explicit solution as:

$0 \le t \le 1/2$: $$\rho(x,t) = \begin{cases} 1 & ; x \le 1 \\ 1 - \frac{x-1}{8t} & ; 1 < x \le 1+4t \\ 1/2 & ; 1+4t < x \le 3 \\ 3/2 & ; x > 3 \end{cases}$$

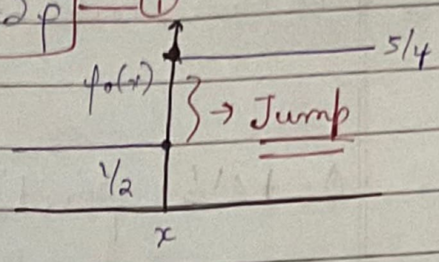
$1/2 < t \le 2$: $$\rho(x,t) = \begin{cases} 1 & ; x \le 1 \\ 1 - \frac{x-1}{8t} & ; 1 < x \le 1 + 4(\sqrt{2t} - t) \\ 3/2 & ; x > 1 + 4(\sqrt{2t} - t) \end{cases}$$

$t > 2$: $$\rho(x,t) = \begin{cases} 1 & ; x \le 5 - 2t \\ 3/2 & ; x > 5 - 2t \end{cases}$$

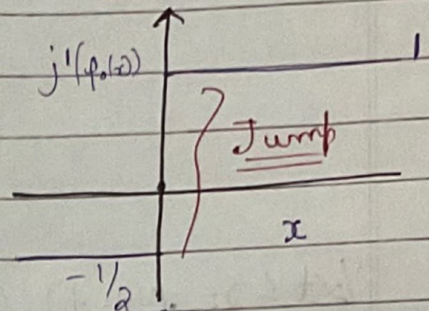
7. Given that,

$j(\rho) = 2\rho - \rho^2 \Rightarrow \boxed{j'(\rho) = 2 - 2\rho}$ — (i)

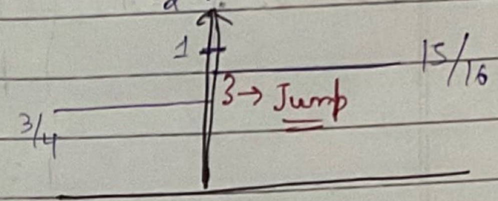
& $\rho_0(x) = \begin{cases} 1/2 & x \le 0 \\ 5/4 & x > 0 \end{cases}$



So $j'(\rho_0) = \begin{cases} 1 & ; x \le 0 \\ -1/2 & ; x > 0 \end{cases}$



& $j(\rho_0(x)) = \begin{cases} 3/4 & x \le 0 \\ 15/16 & x > 0 \end{cases}$



We know that the characteristics associated with this situation are

$$x'(t) = j'(\rho(x, t))$$

Hence, here, the characteristics are forced to intersect because of the discontinuity in ρ & j functions. So, there must be a shock where the characteristics converge.

Employing the Rankine - Hugoniot condition gives

$$s = \frac{dx}{dt} = \frac{j_2 - j_1}{\rho_2 - \rho_1}$$

$$\Rightarrow s = \frac{j(5/4) - j(1/2)}{5/4 - 1/2} \qquad (\rho_2 = 5/4, \ \rho_1 = 1/2)$$

$$\Rightarrow s = \frac{15/16 - 3/4}{3/4} \Rightarrow s = 5/4 - 1 \Rightarrow \boxed{s = 1/4} \rightarrow \text{Shock Speed.}$$