Course MiniProject

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IIT Kanpur

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Outline

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1. Introduction
2. Algorithm and Proofs

Problem and Inspiration

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- To find median of an array

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- Let us improve partition function for our problem

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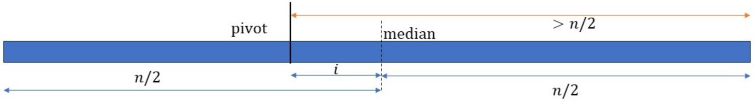

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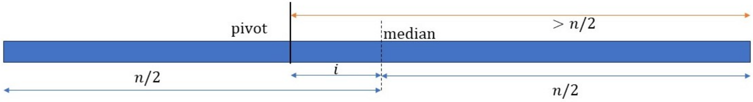

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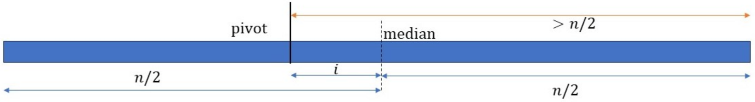

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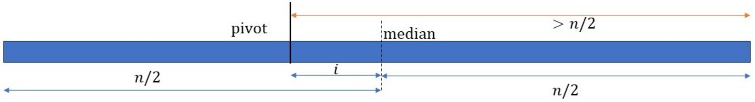

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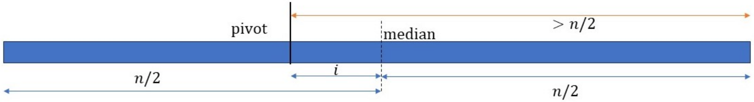

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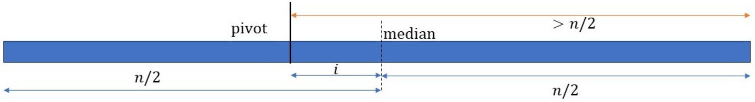

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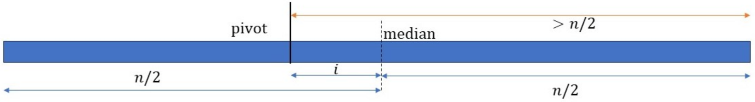

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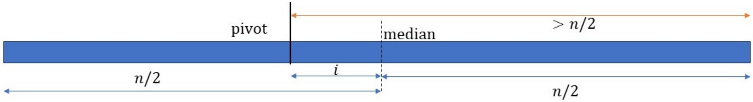

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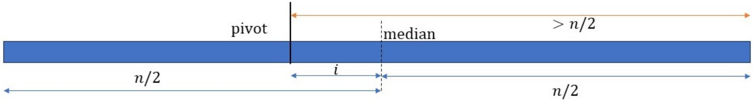

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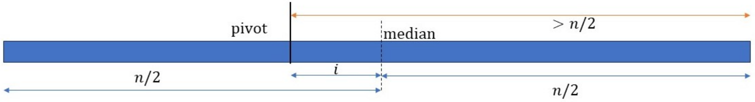

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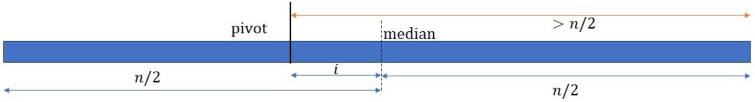

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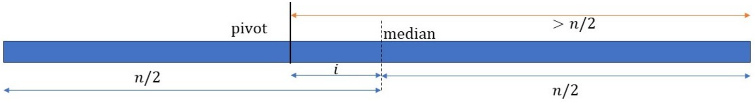

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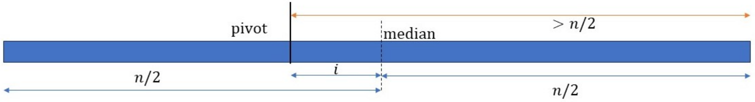

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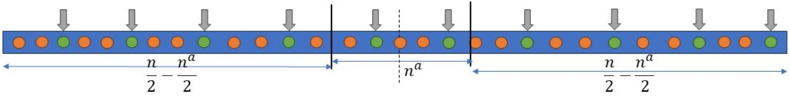

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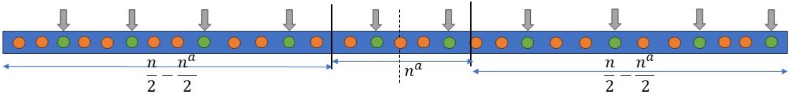

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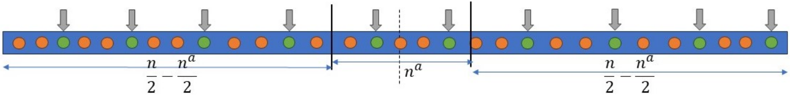

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$$E\left[\text{elements from } B \text{ with rank} < \left(\frac{n}{2} \pm \frac{n^a}{2}\right) \text{ in } A\right]$$

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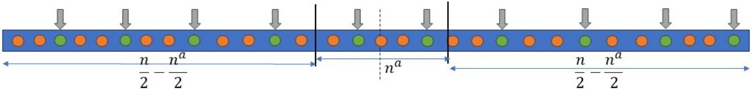

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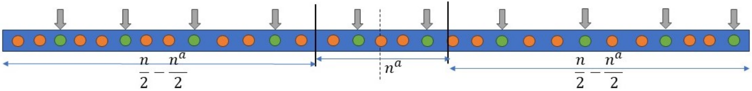

- Expected number of elements in B with $< \frac{n}{2} \pm \frac{n^a}{2}$ in $A =$

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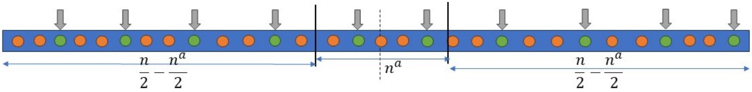

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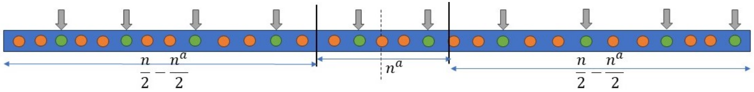

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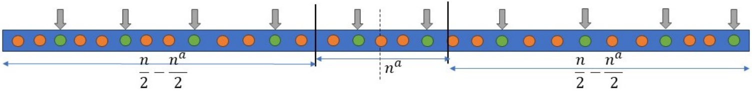

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- Sort C to find median: $C[\frac{n}{2} - x_1 + 1]$

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Input: Array A with n elements
Output: Median of A
$k \leftarrow n^b$;
$t \leftarrow \frac{k}{2n^{1-a}}$;
Select a multi set B of k elements from A r.u.i.;
Sort B;
$p_1 \leftarrow B[\frac{k}{2} - t]$;
$p_2 \leftarrow B[\frac{k}{2} + t]$;
$(A_{\text{new}}, x_1, x_2) \leftarrow$ partition(A, p_1, p_2); //x_1, x_2 ranks of pivots
$C \leftarrow A_{\text{new}}[x_1 : x_2]$;
Sort C;
if $x_1 \leq \frac{n}{2} \leq x_2$ **then**
  return $C[\frac{n}{2} - x_1 + 1]$;
else
  $Median \leftarrow$ Compute the exact median by $O(n)$ deterministic algorithm;
  return $Median$;
end

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- By Union Theorem,

$$P(E_1 \cup E_2) \leq P(E_1) + P(E_2)$$

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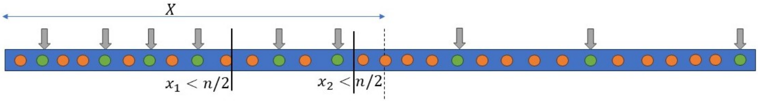

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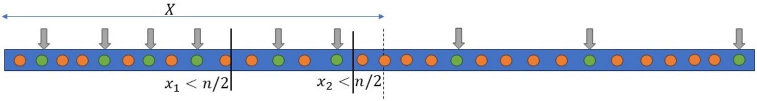

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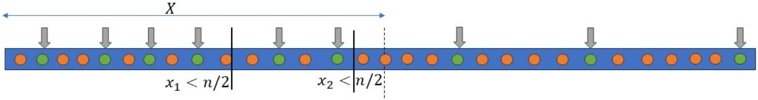

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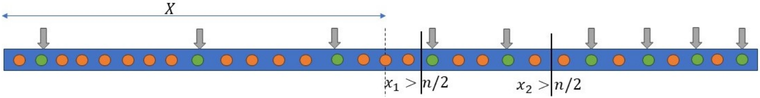

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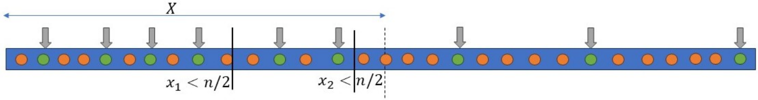

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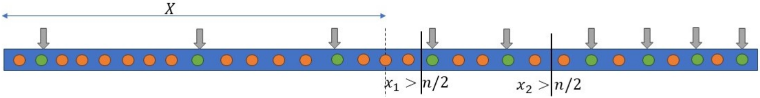

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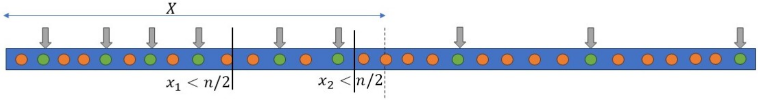

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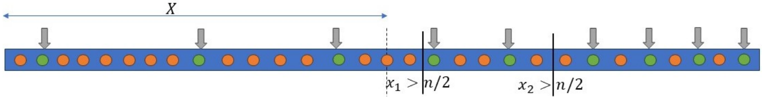

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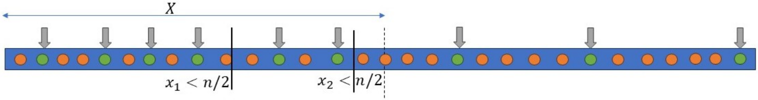

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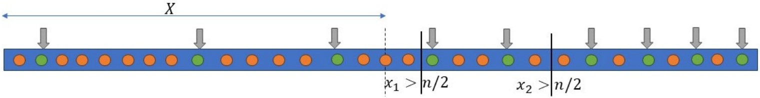

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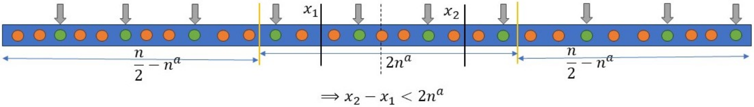

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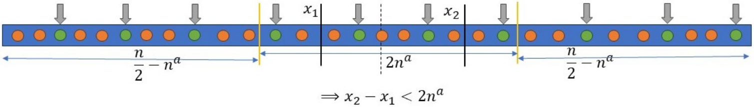

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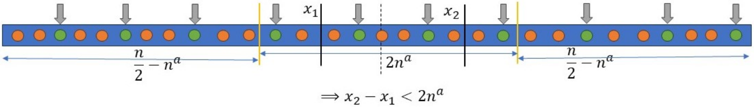

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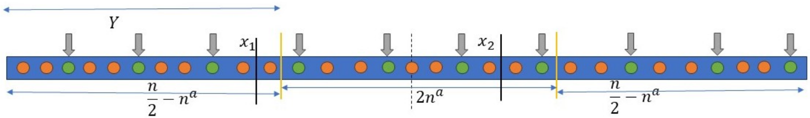

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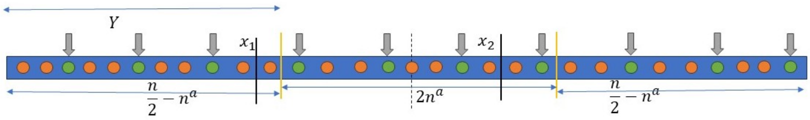

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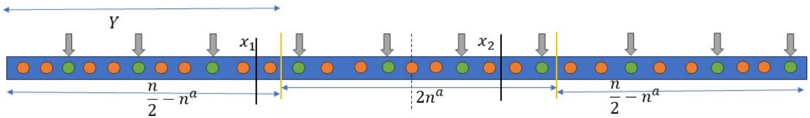

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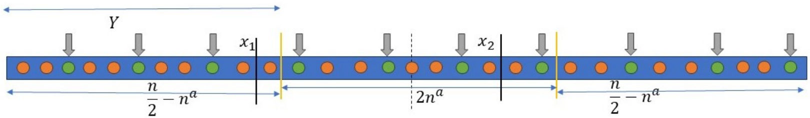

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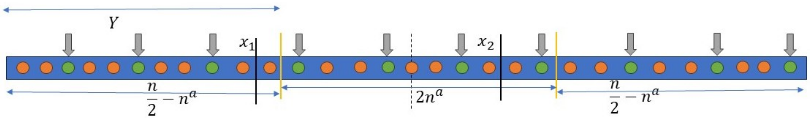

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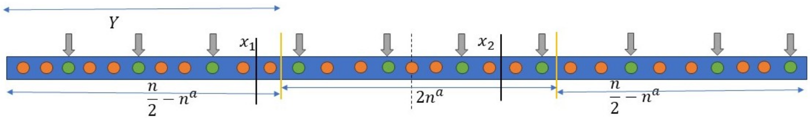

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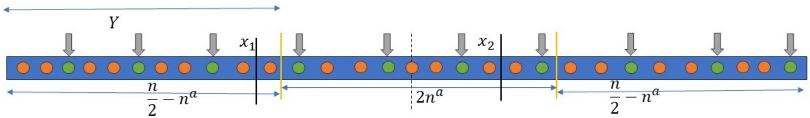

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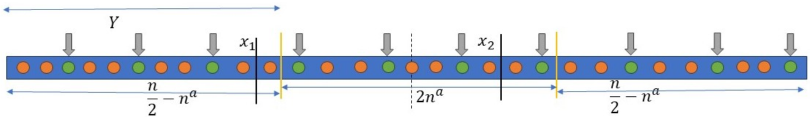

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