Missouri University of Science & Technology	Department of Computer Science
Fall 2023	**CS 5408: Game Theory for Computing**

Solutions to HW 6: Advanced Solution Concepts (Extra Credit)

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Problem 1 Correlated Equilibrium *5 pts.*

Every day, Alice drives from her house in the north to her job in the south along route 66. Bob drives from his house in the west to his job in the east along route 99. The two roads intersect in the middle of nowhere, where there are currently no stop-signs or traffic-lights. Alice and Bob like to get to work without delays, but they are averse to crashing. This means that they prefer to drive through the intersection without stopping if they can get through safely. Stopping and then proceeding safely through the intersection is preferred by both to crashing. The situation is summarized in the normal-form game below, where Alice is the row-player and Bob is the column-player.

	Go	Stop
Go	-100, -100	10, 5
Stop	5, 10	5, 5

(a) Find all pure-strategy and mixed-strategy Nash equilibria of the game.

(b) Show how the introduction of a traffic light can be modeled as a correlated strategy.

(c) Design and show a contract based on traffic lights, whose correlated equilibrium improves the social welfare (sum of utilities of all players) of the game.

Solution:

(a) PSNE can be found as (Go, Stop) and (Stop, Go) by evaluating best responses of both players (red for row player, and blue for column player) as follows:

	Go	Stop
Go	-100, -100	10, 5
Stop	5, 10	5, 5

For mixed strategies, let p and q denote the probability of row and column players choosing Go respectively. Then, the expected utility of row and column players is

$$\begin{aligned} U(p,q) &= q \cdot \Big[p \cdot (-100) + (1-p) \cdot 5\Big] + (1-q) \cdot \Big[p \cdot 10 + (1-p) \cdot 5\Big] \\ &= -110pq + 5p + 5 \end{aligned} \tag{1}$$

and

$$
\begin{aligned}
V(p,q) &= p \cdot \Big[q \cdot (-100) + (1-q) \cdot 5\Big] + (1-p) \cdot \Big[q \cdot 10 + (1-q) \cdot 5\Big] \\
&= -110pq + 5q + 5
\end{aligned} \tag{2}
$$

respectively.

Note that both U and V can be maximized simultaneously if $\dfrac{\partial U}{\partial p} = 0$ and $\dfrac{\partial V}{\partial q} = 0$. Note that, in this case, the second derivative test results in a negative-definite Hessian matrix of $\begin{bmatrix} 0 & -110 \\ -110 & 0 \end{bmatrix}$, over all $p \in [0,1]$ and $q \in [0,1]$.

In other words, $p = q = \dfrac{5}{110} = \dfrac{1}{22}$.

(b) In the case of a traffic light, if Alice gets a green signal along Route 66, Bob gets a red on Route 99, or vice versa. In other words, Alice and Bob will choose either (Go, Stop) or (Stop, Go) choice profiles with some probability. Formally, this is a correlated strategy defined as (Go ,Go): 0, (Go, Stop): π, (Stop, Go): $1-\pi$, and (Stop, Stop): 0, for some $\pi \in [0,1]$.

(c) The general traffic rules are that if the light turns red, the driver stops. Of course, if the opposite player does not want to follow the traffic rules, the driver who follows the rules will always Go. If we consider the aforementioned as the contract, the corresponding updated game can be modeled as

	Go	Stop	Sign
Go	-100, -100	10, 5	-100, -100
Stop	5, 10	5, 5	5, 10
Sign	-100, -100	10, 5	7.5, 7.5

The consequence is that one of the correlated equilibrium is (Sign, Sign), which maximizes the social welfare of both players.