Missouri University of Science & Technology	Department of Computer Science
Fall 2024	**CS 5408: Game Theory for Computing**
Homework 1: Decision Theory	
Instructor: *Sid Nadendla*	**Due:** *September 3, 2024*

Problem 1 Lotteries, Preferences & Axioms *3 pts.*

Consider a choice experiment where an agent knows *a priori* the preference order on four lotteries f_1, f_2, f_3 and f_4. Let this preference ordering be $f_1 \succsim_E f_2 \succsim_E f_3 \succsim_E f_4$, which is evaluated based on some event $E \in \mathcal{E}$. Suppose that the agent also exhibits indifference between the following pairs of lotteries:

- $f_2 \sim_E 0.6f_1 + 0.4f_4$
- $f_3 \sim_E 0.2f_1 + 0.8f_4$

Let f and g denote two new random lotteries, which are defined as

- $f = 0.15f_1 + 0.50f_2 + 0.15f_3 + 0.20f_4$,
- $g = 0.25f_1 + 0.25f_2 + 0.25f_3 + 0.25f_4$.

Assuming that the first 8 preference axioms of decision theory (presented in slides 12-13 in *Topic 1: Decision Theory* lecture notes) are held true by the agent, prove that $g \succsim_E f$.

Solution:

Let the utilities of f_1, f_2, f_3 and f_4 be u_1, u_2, u_3 and u_4 respectively. Then,

$$u_2 = 0.6u_1 + 0.4u_4, \quad \text{and} \quad u_3 = 0.2u_1 + 0.8u_4. \tag{1}$$

If the utilities of lotteries f and g be u and v respectively, we have

$$\begin{aligned} u &= 0.15u_1 + 0.5u_2 + 0.15u_3 + 0.2u_4, \\ v &= 0.25u_1 + 0.25u_2 + 0.25u_3 + 0.25u_4. \end{aligned} \tag{2}$$

Substituting Equations (1) in Equations (2), we have

$$\begin{aligned} u &= 0.15u_1 + 0.5\Big(0.6u_1 + 0.4u_4\Big) + 0.15\Big(0.2u_1 + 0.8u_4\Big) + 0.2u_4 &= 0.48u_1 + 0.52u_4, \\ v &= 0.25u_1 + 0.25\Big(0.6u_1 + 0.4u_4\Big) + 0.25\Big(0.2u_1 + 0.8u_4\Big) + 0.25u_4 &= 0.45u_1 + 0.55u_4. \end{aligned} \tag{3}$$

Then, we have $u - v = 0.03(u_1 - u_4) \geq 0$, since $u_1 \geq u_2 \geq u_3 \geq u_4$ due to the preference ordering on f_1, f_2, f_3 and f_4.

In other words, we have $f \succsim_E g$. $\square$

Problem 2 Expected Utility Maximization *3 pts.*

A company must decide its investments between three mutually exclusive projects:

- Project P provides a net profit of \$50 million with a probability 0.75, and a net loss of \$10 million with probability 0.25.
- Project Q provides a net profit of \$100 million with a probability 0.6, and a net loss of \$40 million with probability 0.4.
- Project R provides a net profit of \$200 million with a probability 0.5, and a net loss of \$100 million with probability 0.5.

Suppose that the CEO of the corporation is *risk-seeking* and maximizes a <u>convex-increasing</u> utility. Then, determine his preferences over P, Q and R.

Solution:

Without any loss of generality, let $u(x)$ denote the utility of obtaining x million dollars. Note that, the losses incurred by the agent are represented by negative values of x. Then, we can compute the utilities of projects P, Q and R as

$$\begin{aligned} u_P &= 0.75 \cdot u(50) + 0.25 \cdot u(-10) \\ u_Q &= 0.6 \cdot u(100) + 0.4 \cdot u(-40) \\ u_R &= 0.5 \cdot u(200) + 0.5 \cdot u(-100) \end{aligned} \tag{4}$$

Given that the CEO is a risk-averse agent with a concave utility (i.e. we have $u(\lambda x + (1 - \lambda)y) \geq \lambda u(x) + (1-\lambda)u(y)$), we can find lower bounds to the utilities $u(-40)$, $u(-10)$, $u(50)$ and $u(100)$ in terms of $u(200)$ and $u(-100)$, as shown below.

$$\begin{aligned} u(-40) &= u\left(\frac{1}{5} \times 200 + \frac{4}{5} \times (-100)\right) &< \frac{1}{5} \times u(200) + \frac{4}{5} \times u(-100) \\ u(-10) &= u\left(\frac{3}{10} \times 200 + \frac{7}{10} \times (-100)\right) &< \frac{3}{10} \times u(200) + \frac{7}{10} \times u(-100) \\ u(50) &= u\left(\frac{1}{2} \times 200 + \frac{1}{2} \times (-100)\right) &< \frac{1}{2} \times u(200) + \frac{1}{2} \times u(-100) \\ u(100) &= u\left(\frac{2}{3} \times 200 + \frac{1}{3} \times (-100)\right) &< \frac{2}{3} \times u(200) + \frac{1}{3} \times u(-100) \end{aligned} \tag{5}$$

Similarly, we can also evaluate lower bounds to the utilities $u(-10)$ and $u(50)$ in terms of $u(100)$ and $u(-40)$, as shown below.

$$\begin{aligned} u(-10) &= u\left(\frac{1}{4} \times 100 + \frac{3}{4} \times (-40)\right) &< \frac{1}{4} \times u(100) + \frac{3}{4} \times u(-40) \\ u(50) &= u\left(\frac{9}{14} \times 100 + \frac{5}{14} \times (-40)\right) &< \frac{9}{14} \times u(100) + \frac{5}{14} \times u(-40) \end{aligned} \tag{6}$$

Substituting the inequalities in Equation (5) in u_P and u_Q defined in Equation (4), we obtain

$$\begin{aligned}
u_P &= 0.75 \cdot u(50) + 0.25 \cdot u(-10) \\
&< 0.75\Big(\frac{1}{2} \times u(200) + \frac{1}{2} \times u(-100)\Big) + 0.25\Big(\frac{3}{10} \times u(200) + \frac{7}{10} \times u(-100)\Big) \\
&= \frac{9}{20} \times u(200) + \frac{11}{20} \times u(-100) \\
u_Q &= 0.6 \cdot u(100) + 0.4 \cdot u(-40) \\
&< 0.6\Big(\frac{2}{3} \times u(200) + \frac{1}{3} \times u(-100)\Big) + 0.4\Big(\frac{1}{5} \times u(200) + \frac{4}{5} \times u(-100)\Big) \\
&= \frac{12}{25} \times u(200) + \frac{13}{25} \times u(-100)
\end{aligned} \tag{7}$$

In such a case, the comparison in utilities of P and Q with respect to that of R can be evaluated as

$$\begin{aligned}
u_P - u_R &< -\frac{1}{20} \times \Big(u(200) - u(-100)\Big) < 0 \\
u_Q - u_R &< -\frac{1}{25} \times \Big(u(200) - u(-100)\Big) < 0
\end{aligned} \tag{8}$$

In other words, $R \succ P$ and $R \succ Q$.

Now, let us focus on the preference relationship between P and Q. A lower bound can be evaluated for u_P in a similar manner in terms of $u(100)$ and $u(-40)$ using Equation (6), as shown below:

$$\begin{aligned}
u_P &= 0.75 \cdot u(50) + 0.25 \cdot u(-10) \\
&< 0.75\Big(\frac{1}{4} \times u(100) + \frac{3}{4} \times u(-40)\Big) + 0.25\Big(\frac{9}{14} \times u(100) + \frac{5}{14} \times u(-40)\Big) \\
&= \frac{39}{112} u(100) + \frac{73}{112} u(-40)
\end{aligned} \tag{9}$$

Now, we can also compare u_P with u_Q by evaluating the bound of the difference

$$u_P - u_Q \ < \ -\frac{141}{560} \times \Big(u(100) - u(-40)\Big) < 0. \tag{10}$$

In other words, $Q \succ P$.

Combining all the preference relations together, we have $R \succ Q \succ P$. $\square$

Problem 3 Expected Utility Maximization *3 pts.*

Suppose that Alice won a competition. As a reward, she was asked to choose one of the following two options:

Option 1: A laptop with probability 1.

Option 2: A tablet with probability 0.3, or a motorcycle with probability 0.7.

Option 3: A cell phone with probability 0.3, or a laptop with probability 0.2, or a motorcycle with probability 0.5.

Assuming that Alice maximizes her expected utility, if she prefers

$$\text{cell phone} \prec \text{tablet} \prec \text{laptop} \prec \text{motorcycle},$$

find the preference order over Options 1, 2 and 3.

Solution:

Let the utilities of cell phone, tablet, laptop and motorcycle be u_1, u_2, u_3 and u_4 respectively. Also, let the utilities of Option 1 and Option 2 be v_1 and v_2 respectively.

Then, we have

$$v_1 = u_3, \quad v_2 = 0.3u_2 + 0.7u_4, \quad \text{and} \quad v_3 = 0.3u_1 + 0.2u_3 + 0.5u_4. \tag{11}$$

Comparing v_2 and v_3, we obtain

$$v_2 - v_3 = 0.3\Big(u_2 - u_1\Big) + 0.2\Big(u_4 - u_3\Big) \geq 0. \tag{12}$$

Therefore, Alice prefers Option 2 over Option 3.

However, since $u_1 < u_2 < u_3 < u_4$, by the axiom of continuity (Axiom 5), there always exists two real numbers $\alpha, \beta \in (0, 1)$ such that

$$v_1 = u_3 = \alpha u_1 + (1-\alpha)u_4, \quad \text{and} \quad v_1 = u_3 = \beta u_2 + (1-\beta)u_4. \tag{13}$$

Therefore, we have

$$v_2 = 0.3u_2 + 0.7u_4, \quad \text{and} \quad v_3 = (0.3 + 0.2\alpha)u_1 + (0.7 - 0.2\alpha)u_4. \tag{14}$$

Using the axiom of monotonicity (Axiom 4) in Equations (11) and (14), we find that

- if $\alpha > (0.3 + 0.2\alpha)$ (or equivalently, $\alpha > 3/8$), Alice prefers Option 1 over Option 3,
- if $\alpha < 3/8$, Alice prefers Option 3 over Option 1.

Similarly, we also have the following:

- if $\beta > 0.3$, then Alice prefers Option 1 over Option 2,
- if $\beta < 0.3$, Alice prefers Option 2 over Option 1.

Combining all the conditions, we have the following preference orderings on a case-by-case basis:

- **CASE 1 ($\alpha < 3/8$ and $\beta < 0.3$):** Option 2 $\succ$ Option 3 $\succ$ Option 1
- **CASE 2 ($\alpha < 3/8$ and $\beta > 0.3$):** Cyclic preference
  (Option 1 $\succ$ Option 2 $\succ$ Option 3 $\succ$ Option 1)
- **CASE 3 ($\alpha > 3/8$ and $\beta < 0.3$):** Option 2 $\succ$ Option 1 $\succ$ Option 3
- **CASE 4 ($\alpha > 3/8$ and $\beta > 0.3$):** Option 1 $\succ$ Option 2 $\succ$ Option 3

□

Problem 4 Limitations of EUM *3 pts.*

Daniel Ellsberg proposed the following thought-experiment[1] (known as *Ellsberg Paradox*) in 1961. An urn contains 90 balls, 30 of which are red. The other 60 are black or yellow, in unknown proportions. One ball will be drawn randomly from the urn. In this experiment, consider yourself as a decision maker.

(a) First, you must make a choice between Gamble A and Gamble B:

- **Gamble A:** You win \$100 if the ball is red.
- **Gamble B:** You win \$100 if the ball is black.

Which would you choose, and why?

(b) Next, you must make a choice between Gamble C and Gamble D:

- **Gamble C:** You win \$100 if the ball is either red or yellow.
- **Gamble D:** You win \$100 if the ball is either black or yellow.

Which would you choose, and why?

(c) Most people strongly prefer Gambles A and D over Gambles B and C respectively. Explain why this pattern of choices violates expected utility theory.

Solution:

[1] A similar experiment was also proposed by John Maynard Keynes in 1921.

(a) and (b): Students are free to pick any one gamble and explain the latent rationality behind their choice.

(c) Let α denote the fraction of black balls amongst the remaining unknown 60 balls. Note that α can also be interpreted as the conditional probability of a ball being black, given that it is not a red ball. In other words, the urn has 30 red balls, 60α black balls and $60(1-\alpha)$ yellow balls.

If the agent follows EUM, then the utilities of gambles A, B, C and D are given by

$$\begin{aligned} u_A &= \mathbb{P}(\text{red}) \times \$100 + \mathbb{P}(\text{not red}) \times \$0 \\ &= \frac{30}{90} \times 100 + \frac{30}{90} \times 0 \\ &= \frac{100}{3}, \end{aligned} \tag{15}$$

$$\begin{aligned} u_B &= \mathbb{P}(\text{black}) \times \$100 + \mathbb{P}(\text{not black}) \times \$0 \\ &= \frac{60\alpha}{90} \times 100 + \frac{90-60\alpha}{90} \times 0 \\ &= \frac{200\alpha}{3}, \end{aligned} \tag{16}$$

$$\begin{aligned} u_C &= \mathbb{P}(\text{red/yellow}) \times \$100 + \mathbb{P}(\text{not red/yellow}) \times \$0 \\ &= \frac{90-60\alpha}{90} \times 100 + \frac{60\alpha}{90} \times 0 \\ &= \left(1 - \frac{2\alpha}{3}\right)100, \end{aligned} \tag{17}$$

$$\begin{aligned} u_D &= \mathbb{P}(\text{black/yellow}) \times \$100 + \mathbb{P}(\text{not black/yellow}) \times \$0 \\ &= \frac{60}{90} \times 100 + \frac{30}{90} \times 0 \\ &= \frac{200}{3}, \end{aligned} \tag{18}$$

Given $A \succ B$ and $D \succ C$, if the agent follows EUM, we expect $u_A > u_B$ and $u_D > u_C$. In other words, we need

$$\begin{aligned} u_A > u_B \quad &\Rightarrow \quad \frac{100}{3} > \frac{200\alpha}{3} &&\Rightarrow \quad \alpha < \frac{1}{2}, \\ u_D > u_C \quad &\Rightarrow \quad \frac{200}{3} > \left(\frac{1-2\alpha}{3}\right)100 &&\Rightarrow \quad \alpha > \frac{1}{2}. \end{aligned} \tag{19}$$

This is a contradiction! Therefore, any agent who exhibits preference orders $A \succ B$ and $D \succ C$ does not follow expected utility maximization. $\square$

Problem 5 Prospect Theory *3 pts.*

Let your utility function for gains and losses be

$$u(x) = \begin{cases} x, & \text{if } x \geq 0 \\ \lambda x, & \text{if } x < 0, \end{cases}$$

where $\lambda > 0$ is an unknown parameter and your probability weighting function is $w(p) = p$. Consider the following two gambles:

- $P = \{$win \$100 with probability 0.5; loose \$150 with probability 0.5$\}$
- $Q = \{$win \$200 with probability 0.5; loose \$100 with probability 0.5$\}$

Suppose that you have the following preferences:

- prefer getting nothing for sure over the gamble P,
- prefer the gamble Q over getting nothing for sure.

Then, what is the range of λ that is consistent with the choices above?

Solution:

Given $w(p) = p$ and $u(x) = \begin{cases} x, & \text{if } x \geq 0, \\ \lambda x, & \text{otherwise,} \end{cases}$ let N denote the gamble wherein the agent gets nothing for sure. In other words, if the agent chooses N, he/she obtains a reward of \$0 with probability 1.

Then, the prospect theoretic utilities of gambles P, Q and N are given by

$$\begin{aligned} V_P &= 0.5 \times \$100 + 0.5\Big(\lambda(-\$150)\Big) &&= 50 - 75\lambda \\ V_Q &= 0.5 \times \$200 + 0.5\Big(\lambda(-\$100)\Big) &&= 100 - 50\lambda \\ V_N &= 1 \times \$0 &&= 0 \end{aligned} \tag{20}$$

Given that $N \succ P$ and $Q \succ N$, we expect

$$\begin{aligned} V_N > V_P \quad &\Rightarrow \quad 0 > 50 - 75\lambda \quad &&\Rightarrow \quad \lambda > \frac{2}{3} \\ V_Q > V_N \quad &\Rightarrow \quad 100 - 50\lambda > 0 \quad &&\Rightarrow \quad \lambda < 2 \end{aligned} \tag{21}$$

respectively. Combining the above two inequalities, we have $\dfrac{2}{3} < \lambda < 2$. □

Decision	**State t_1**	**State t_2**	**State t_3**
α	4	1	-3
β	3	-2	4
γ	0	1	6

Problem 6 Domination *3 pts.*

Suppose an agent is presented with a choice set $\mathcal{X} = \{\alpha, \beta, \gamma\}$, where the choice experiment can take the states $\Omega = \{t_1, t_2, t_3\}$. If the utilities at the agent are given as shown in the table below,

(a) Find the region in $\Delta(\Omega)$ in which α is optimal.

(b) Find the region in $\Delta(\Omega)$ in which β is optimal.

(c) Find the region in $\Delta(\Omega)$ in which γ is optimal.

Solution:

Let the probability of states t_1, t_2 and t_3 be denoted as p_1, p_2 and $p_3 = 1-p_1-p_2$ respectively. Then, the expected utilities of α, β and γ are given by

$$\begin{aligned} u_\alpha &= 4 \cdot p_1 + 1 \cdot p_2 + (-3)(1-p_1-p_2) &&= 7p_1 + 4p_2 - 3 \\ u_\beta &= 3 \cdot p_1 + (-2) \cdot p_2 + 4(1-p_1-p_2) &&= -p_1 - 6p_2 + 4 \\ u_\gamma &= 0 \cdot p_1 + 1 \cdot p_2 + 6(1-p_1-p_2) &&= -6p_1 - 5p_2 + 6 \end{aligned} \tag{22}$$

respectively.

(a) **Optimality of α:** In order to find the region where α is optimal, we need to evaluate the inequalities: $u_\alpha > u_\beta$ and $u_\alpha > u_\gamma$. In other words, we expect

$$\begin{aligned} u_\alpha > u_\beta &\Rightarrow 7p_1 + 4p_2 - 3 > -p_1 - 6p_2 + 4 &&\Rightarrow 8p_1 + 10p_2 > 7 \\ u_\alpha > u_\gamma &\Rightarrow 7p_1 + 4p_2 - 3 > -6p_1 - 5p_2 + 6 &&\Rightarrow 13p_1 + 9p_2 > 9. \end{aligned} \tag{23}$$

In addition, we also need to ensure $p_1 \geq 0$, $p_2 \geq 0$, $p_3 \geq 0$ (or, $p_1 + p_2 \leq 1$).

In other words, the region on the simplex where α is the optimal choice is given by

$$\mathbb{R}_\alpha = \left\{ (p_1, p_2) \in [0,1]^2 \;\middle|\; 8p_1 + 10p_2 > 7,\ 13p_1 + 9p_2 > 9,\ \text{and } p_1 + p_2 \leq 1 \right\}. \tag{24}$$

(b) **Optimality of β:** In order to find the region where β is optimal, we need to evaluate the inequalities: $u_\beta > u_\alpha$ and $u_\beta > u_\gamma$. In other words, we expect

$$\begin{aligned} u_\beta > u_\alpha &\Rightarrow -p_1 - 6p_2 + 4 > 7p_1 + 4p_2 - 3 &&\Rightarrow 8p_1 + 10p_2 < 7 \\ u_\beta > u_\gamma &\Rightarrow -p_1 - 6p_2 + 4 > -6p_1 - 5p_2 + 6 &&\Rightarrow 5p_1 - p_2 > 2. \end{aligned} \tag{25}$$

In addition, we also need to ensure $p_1 \geq 0$, $p_2 \geq 0$, $p_3 \geq 0$ (or, $p_1 + p_2 \leq 1$).

In other words, the region on the simplex where β is the optimal choice is given by

$$\mathbb{R}_\beta = \left\{ (p_1, p_2) \in [0,1]^2 \;\middle|\; 8p_1 + 10p_2 < 7,\; 5p_1 - p_2 > 2, \text{ and } p_1 + p_2 \leq 1 \right\}. \tag{26}$$

(c) **Optimality of γ:** In order to find the region where γ is optimal, we need to evaluate the inequalities: $u_\gamma > u_\alpha$ and $u_\gamma > u_\beta$. In other words, we expect

$$\begin{aligned} u_\gamma > u_\alpha \quad &\Rightarrow \quad -6p_1 - 5p_2 + 6 > 7p_1 + 4p_2 - 3 \quad \Rightarrow \quad 13p_1 + 9p_2 < 9 \\ u_\gamma > u_\beta \quad &\Rightarrow \quad -6p_1 - 5p_2 + 6 > -p_1 - 6p_2 + 4 \quad \Rightarrow \quad 5p_1 - p_2 < 2. \end{aligned} \tag{27}$$

In addition, we also need to ensure $p_1 \geq 0$, $p_2 \geq 0$, $p_3 \geq 0$ (or, $p_1 + p_2 \leq 1$).

In other words, the region on the simplex where γ is the optimal choice is given by

$$\mathbb{R}_\gamma = \left\{ (p_1, p_2) \in [0,1]^2 \;\middle|\; 13p_1 + 9p_2 < 9,\; 5p_1 - p_2 < 2, \text{ and } p_1 + p_2 \leq 1 \right\}. \tag{28}$$

$\square$

Problem 7 Allias Paradox *(Extra Credit: 3 pts)*

(a) Model the choice experiment in Allias paradox formally as a lottery, i.e. clearly define the states, their corresponding probabilities, choices and a conditional distribution on the choice set given the state.

(b) Prove that $1A \succ 1B$ and $2B \succ 2A$ violates expected utility maximization (EUM) framework.

Solution:

(a) In Allias paradox, each agent is expected to choose one lottery in each experiment. In other words, the choice set is given by

$$\mathcal{C} = \left\{ \; AA, \; AB, \; BA, \; BB \; \right\}, \tag{29}$$

where the first letter in the choice ij corresponds to picking $1i$ in experiment 1 and the second letter corresponds to picking $2j$ in experiment 2.

The state of each experiment is the actual outcome in each choice within each experiment, which is

$$\boldsymbol{s}_k = (s_{1A}, s_{1B}, s_{2A}, s_{2B}), \tag{30}$$

where $s_{1A} \in \{\$1M\}$, $s_{1B} \in \{\$0, \$1M, \$5M\}$, $s_{2A} \in \{\$0, \$1M\}$ and $s_{2B} \in \{\$0, \$5M\}$.

The state uncertainty is characterized by the four marginal probability distributions $\pi_{1A} = \{1\}$, $\pi_{1B} = \{0.89, 0.01, 0.1\}$, $\pi_{2A} = \{0.89, 0.11\}$ and $\pi_{2B} = \{0.9.0.1\}$.

If the agent were to employ a lottery on $\mathcal{C}$, they need to choose a probability vector $\boldsymbol{\pi} = \{\pi_{AA}, \pi_{AB}, \pi_{BA}, \pi_{BB}\}$ as his/her lottery, where

$$\pi_{ij} = \mathbb{P}(\text{agent picks } 1i \text{ in experiment 1, and } 2j \text{ in experiment 2}).$$

□

(b) Let $u(x)$ denote the utility of winning x million dollars. In other words, if the agent prefers $1A \succ 1B$, we have

$$u(1) > 0.89u(1) + 0.01u(0) + 0.1u(5) \quad \Rightarrow \quad 0.11u(1) > 0.01u(0) + 0.1u(5). \tag{31}$$

Similarly, if the agent prefers $2B \succ 2A$, we have

$$0.9u(0) + 0.1u(5) > 0.89u(0) + 0.11u(1) \quad \Rightarrow \quad 0.11u(1) < 0.01u(0) + 0.1u(5). \tag{32}$$

Note that it is impossible to satisfy Equations (31) and (32) simultaneously using any feasible utility function. Therefore, we have a violation to expected utility maximization. □