CS 5408 : GAME THEORY FOR COMPUTING

HW-4 SOLUTIONS

Problem 1

(a) Given CTRs $\alpha_1, \ldots, \alpha_k$ where

$$\alpha_j = \mathbb{P}(\text{end user clicks the } j^{th} \text{ slot}).$$

and per-click valuations $v_1, \ldots, v_n$ for n advertisers,

the valuation of i^{th} advertiser regarding j^{th} slot

$$\boxed{V_{ij} = \alpha_j \cdot v_i}$$

Let β_{ij} denote the allocation variable at the i^{th} advertiser regarding j^{th} slot.

i.e. $$\beta_{ij} = \begin{cases} 1 ; & j^{th} \text{ slot is assigned to } i^{th} \text{ advertiser} \\ 0 ; & \text{otherwise.} \end{cases}$$

Then, social welfare $$\boxed{\Phi = \sum_{i=1}^{n} \sum_{j=1}^{k} \beta_{ij} \cdot V_{ij}}$$

(b) Given $\alpha_1 = 2/3$ and $\alpha_2 = 1/3$ for the two slots, and per-click valuations $v_A = 10$, $v_B = 8$ and $v_C = 4$,

the valuations are

$$V_{A,1} = \alpha_1 . v_A = \frac{2}{3} \times 10 = \frac{20}{3}$$

$$V_{A,2} = \alpha_2 . v_A = \frac{1}{3} \times 10 = \frac{10}{3}.$$

$$V_{B,1} = \alpha_1 . v_B = \frac{2}{3} \times 8 = \frac{16}{3}$$

$$V_{B,2} = \alpha_2 . v_B = \frac{1}{3} \times 8 = \frac{8}{3}$$

$$V_{C,1} = \alpha_1 . v_C = \frac{2}{3} \times 4 = \frac{8}{3}$$

$$V_{C,2} = \alpha_2 . v_C = \frac{1}{3} \times 4 = \frac{4}{3}$$

Assuming B and C bids truthfully,

in GSP, if $b_A = 10$, $b_B = 8$, $b_C = 4$ are the bids, $\beta_{A,1} = 1$, $\beta_{A,2} = 0$, $\beta_{B,1} = 0$, $\beta_{B,2} = 1$, $\beta_{C,1} = \beta_{C,2} = 0$.

i.e. A gets slot 1, B gets slot 2.

P.T.O.

$$\Rightarrow p_A = V_{B,1} = \frac{16}{3}, \quad p_B = V_{C,2} = \frac{4}{3},$$

$$p_C = 0.$$

$\therefore$ Utilities are given as

$$U_A = \beta_{A,1} \cdot V_{A,1} + \beta_{A,2} V_{A,2} - p_A$$
$$= 1 \cdot \frac{20}{3} + 0 \cdot \frac{10}{3} - \frac{16}{3}$$
$$= \frac{20}{3} - \frac{16}{3} = \frac{4}{3}.$$

However, if $b_A = 5$, $b_B = 8$ and $b_C = 4$, the allocation turns out to be

$$\beta_{A,1} = 0, \quad \beta_{A,2} = 1, \quad \beta_{B,1} = 1, \quad \beta_{B,2} = 0,$$
$$\beta_{C,1} = \beta_{C,2} = 0.$$

and the payments are

$$p_A = V_{C,2} = \frac{4}{3}, \quad p_B = \alpha_1 \cdot b_A = \frac{2}{3} \times 5 = \frac{10}{3}, \quad p_C = 0.$$

$\therefore$ Utilities are given by

$$U_A = V_{A,1} \cdot \beta_{A,1} + V_{A,2} \cdot \beta_{A,2} - p_A = \frac{10}{3} - \frac{4}{3} = 2.$$

Since $U_A(b_A = 5) = 2 > \frac{4}{3} = U_A(b_A = 10)$,

A does not bid truthfully in GSP auctions.

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(c) In VCG auctions,

if $b_A = 10$, $b_B = 8$ and $b_c = 4$,

then $p_A = \phi_{-A} - \underbrace{(\phi - \beta_{A,1} V_{A,1} - \beta_{A,2} V_{A,2})}_{\text{welfare of the remaining players.}}$

$$= \left(8 \times \frac{2}{3} + 4 \times \frac{1}{3}\right) - \left(8 \times \frac{1}{3} + 0\right)$$

$$= \frac{20}{3} - \frac{8}{3} = 4.$$

$$\Rightarrow U_A = \frac{20}{3} - 4 = \frac{8}{3}.$$

But, if $b_A = 5$, $b_B = 8$ and $b_c = 4$,

$$p_A = \left(8 \times \frac{2}{3} + 4 \times \frac{1}{3}\right) - \left(8 \times \frac{2}{3}\right) = \frac{20}{3} - \frac{16}{3} = 4/3.$$

$$\Rightarrow U_A = \frac{10}{3} - \frac{4}{3} = 2.$$

Since $U_A(b_A = 10) = \frac{8}{3} > 2 = U_A(b_A = 5)$,

A bids $b_A = 10$ and stays truthful in a VCG auction.

P.T.O.

(d) If $b_A = 10$, $b_B = 8$, $b_c = 4$,

$$p_A = \left(8 \times \frac{2}{3} + 4 \times \frac{1}{3}\right) - \left(8 \times \frac{1}{3}\right) = 4$$

$$p_B = \left(10 \times \frac{2}{3} + 4 \times \frac{1}{3}\right) - \left(10 \times \frac{2}{3}\right) = \frac{4}{3}$$

$$p_c = \left(10 \times \frac{2}{3} + 8 \times \frac{1}{3}\right) - \left(10 \times \frac{2}{3} + 8 \times \frac{1}{3}\right) = 0.$$

Prob. 2

(a)

Plurality rule assigns 1 point to top-preferred candidate.

For simplicity, we assume there are 100 voters in total.

PLURALITY COUNTS

CANDIDATE	# VOTES	# POINTS
N	35	$35 \times 1 = 35$
S	28	$28 \times 1 = 28$
J	37	$37 \times 1 = 37$

Since J gets highest points, J wins.

(b) By Borda count, N has the max. points.

$\Rightarrow$ N wins.

BORDA COUNTS

CANDIDATE	# POINTS
N	$35 \times 2 + (28+20) \times 1 = 118$
S	$28 \times 2 + (35+17) \times 1 = 108$
J	$(20+17) \times 2 + 0 = 74$

(c)

CANDIDATE	# VOTERS RANKING FIRST	# VOTERS RANKING LAST
N	35	17
S	28	20
J	37	63

$\therefore$ J is ranked both as top candidate as well as last candidate by most # of voters.

(d)

Pairwise contests	# VOTERS
N vs. S	N: 35+20 = 55, S: 28+17 = 45
N vs J	N: 35+28 = 63, J: 20+17 = 37
S vs J	S: 35+28 = 63, J: 20+17 = 37

$\therefore$ # wins for N = 2

wins for S = 1

wins for J = 0.

$\Rightarrow$ By Condorcet's criterion, N wins.

Since J wins by Plurality vote, plurality rule does not follow Condorcet's criterion.