Missouri University of Science & Technology — Department of Computer Science

Fall 2023 — **CS 5408: Game Theory for Computing**

Homework 1: Decision Theory

Instructor: *Sid Nadendla* — **Due:** *September 8, 2023*

Problem 1 Lotteries, Preferences & Axioms *4 pts.*

Consider a choice experiment where an agent knows *a priori* the preference order on four lotteries f_1, f_2, f_3 and f_4. Let this preference ordering be $f_1 \succsim_E f_2 \succsim_E f_3 \succsim_E f_4$, which is evaluated based on some event $E \in \mathcal{E}$. Suppose that the agent also exhibits indifference between the following pairs of lotteries:

- $f_2 \sim_E 0.4f_1 + 0.6f_4$
- $f_3 \sim_E 0.3f_1 + 0.7f_4$

Let f and g denote two new lotteries, which are defined as

- $f = 0.2f_1 + 0.4f_2 + 0.2f_3 + 0.2f_4$,
- $g = 0.25f_1 + 0.25f_2 + 0.25f_3 + 0.25f_4$.

Assuming that the first 8 preference axioms of decision theory (presented in slides 12-13 in *Topic 1: Decision Theory* lecture notes) are held true by the agent, prove that $g \succsim_E f$.

Problem 2 Expected Utility Maximization *4 pts.*

A company must decide its investments between three mutually exclusive projects:

- Project P provides a net profit of $50 million with a probability 0.75, and a net loss of $10 million with probability 0.25.
- Project Q provides a net profit of $100 million with a probability 0.6, and a net loss of $40 million with probability 0.4.
- Project R provides a net profit of $200 million with a probability 0.5, and a net loss of $100 million with probability 0.5.

Suppose that the CEO of the corporation is risk averse and maximizes a concave-increasing utility. Then, determine his preferences over P, Q and R.

Note: Utility functions have to be well-defined over all possible profits and losses. For instance, Bernoulli's logarithmic utility is not defined for losses, as they are negative quantities. Therefore, if you prefer working with Bernoulli's utility, then consider a shifted-logarithm $\log(x + a)$, where losses beyond a are treated as infinite in value. However, for full credits, you should prove this result for any concave-increasing utility function in this universe.

Problem 3 Expected Utility Maximization *4 pts.*

Suppose that Alice won a competition. As a reward, she was asked to choose one of the following two options:

Option 1: A laptop with probability 1.

Option 2: A tablet with probability 0.3, or a motorcycle with probability 0.7.

Option 3: A cell phone with probability 0.3, or a laptop with probability 0.2, or a motorcycle with probability 0.5.

Assuming that Alice maximizes her expected utility, if she prefers

$$\text{cell phone} \prec \text{tablet} \prec \text{laptop} \prec \text{motorcycle},$$

find the preference order over Options 1, 2 and 3.

Problem 4 Limitations of EUM *4 pts.*

Daniel Ellsberg proposed the following thought-experiment[1] (known as *Ellsberg Paradox*) in 1961. An urn contains 90 balls, 30 of which are red. The other 60 are black or yellow, in unknown proportions. One ball will be drawn randomly from the urn. In this experiment, consider yourself as a decision maker.

(a) First, you must make a choice between Gamble A and Gamble B:

- **Gamble A:** You win \$100 if the ball is red.
- **Gamble B:** You win \$100 if the ball is black.

Which would you choose, and why?

(b) Next, you must make a choice between Gamble C and Gamble D:

- **Gamble C:** You win \$100 if the ball is either red or yellow.
- **Gamble D:** You win \$100 if the ball is either black or yellow.

Which would you choose, and why?

(c) Most people strongly prefer Gambles A and D over Gambles B and C respectively. Explain why this pattern of choices violates expected utility theory.

(d) Implement Ellsberg Paradox in Python using the template Jupyter Notebook labeled "HW1_4d_CS5408_FS2023.ipynb" provided to you in your Gitlab repositories.

[1] A similar experiment was also proposed by John Maynard Keynes in 1921.

Problem 5 Prospect Theory *4 pts.*

Let your utility function for gains and losses be

$$u(x) = \begin{cases} x, & \text{if } x \geq 0 \\ \lambda x, & \text{if } x < 0, \end{cases}$$

where $\lambda > 0$ is an unknown parameter and your probability weighting function is $w(p) = p$. Consider the following two gambles:

- $P = \{$win \$150 with probability 0.5; loose \$100 with probability 0.5$\}$
- $Q = \{$win \$200 with probability 0.5; loose \$100 with probability 0.5$\}$

Suppose that you have the following preferences:

- prefer getting nothing for sure over the gamble P,
- prefer the gamble Q over getting nothing for sure.

Then, what is the range of λ that is consistent with the choices above?

Problem 6 Domination *4 pts.*

Suppose an agent is presented with a choice set $\mathcal{X} = \{\alpha, \beta, \gamma\}$, where the choice experiment can take the states $\Omega = \{t_1, t_2, t_3\}$. If the utilities at the agent are given as shown in the table below,

Decision	**State** t_1	**State** t_2	**State** t_3
α	4	1	-3
β	3	-2	4
γ	0	1	6

(a) Find the region in $\Delta(\Omega)$ in which α is optimal.

(b) Find the region in $\Delta(\Omega)$ in which β is optimal.

(c) Find the region in $\Delta(\Omega)$ in which γ is optimal.

(d) Implement this experiment in Python using the template Jupyter Notebook labeled "HW1_6d_CS5408_FS2023.ipynb" provided in your Gitlab repositories and validate your theoretical findings in (a)-(c).

Problem 7 St. Petersburg Paradox *(Extra Credit: 2 pt.)*

Model the choice experiment in St. Petersburg paradox formally as a lottery, i.e. clearly define the states, their corresponding probabilities, choices and a conditional distribution on the choice set given the state.

Problem 8 Allias Paradox *(Extra Credit: 2 pt.)*

Prove that $1A \succ 1B$ and $2B \succ 2A$ violates expected utility maximization (EUM) framework.