Missouri University of Science & Technology	Department of Computer Science
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Solutions to Homework 1: Decision Theory

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Problem 1 Lotteries, Preferences & Axioms *4 pts.*

Consider a choice experiment where an agent knows *a priori* the preference order on four lotteries f_1, f_2, f_3 and f_4. Let this preference ordering be $f_1 \succsim_E f_2 \succsim_E f_3 \succsim_E f_4$, which is evaluated based on some event $E \in \mathcal{E}$. Suppose that the agent also exhibits indifference between the following pairs of lotteries:

- $f_2 \sim_E 0.4f_1 + 0.6f_4$
- $f_3 \sim_E 0.3f_1 + 0.7f_4$

Let f and g denote two new lotteries, which are defined as

- $f = 0.2f_1 + 0.4f_2 + 0.2f_3 + 0.2f_4$,
- $g = 0.25f_1 + 0.25f_2 + 0.25f_3 + 0.25f_4$.

Assuming that the first 8 preference axioms of decision theory (presented in slides 12-13 in *Topic 1: Decision Theory* lecture notes) are held true by the agent, prove that $g \succsim_E f$.

Solution:

Substituting the indifferent lotteries for f_2 and f_3 in f and g respectively, we obtain

$$\begin{aligned} f &= 0.2f_1 + 0.4f_2 + 0.2f_3 + 0.2f_4, \\ &\sim_E 0.2f_1 + 0.4\Big(0.4f_1 + 0.6f_4\Big) + 0.2\Big(0.3f_1 + 0.7f_4\Big) + 0.2f_4 \\ &\sim_E 0.42f_1 + 0.58f_4, \\ \\ g &= 0.25f_1 + 0.25f_2 + 0.25f_3 + 0.25f_4 \\ &\sim_E 0.25f_1 + 0.25\Big(0.6f_1 + 0.4f_4\Big) + 0.25\Big(0.2f_1 + 0.8f_4\Big) + 0.25f_4 \\ &\sim_E 0.45f_1 + 0.55f_4. \end{aligned} \tag{1}$$

Since $f_1 \succsim_E f_4$, by the axiom of monotonicity (Axiom 4), assuming $\beta = 0.42$ and $\alpha = 0.45$, we have

$$g \ \sim_E \ 0.45f_1 + 0.55f_4 \ \succsim_E \ 0.42f_1 + 0.58f_4 \ \sim_E \ f. \tag{2}$$

$\square$

Problem 2 Expected Utility Maximization *4 pts.*

A company must decide its investments between three mutually exclusive projects:

- Project P provides a net profit of \$50 million with a probability 0.75, and a net loss of \$10 million with probability 0.25.
- Project Q provides a net profit of \$100 million with a probability 0.6, and a net loss of \$40 million with probability 0.4.
- Project R provides a net profit of \$200 million with a probability 0.5, and a net loss of \$100 million with probability 0.5.

Suppose that the CEO of the corporation is risk averse and maximizes a concave-increasing utility. Then, determine his preferences over P, Q and R.

Note: Bernoulli's logarithmic utility is not defined for losses, as they are negative quantities. Therefore, if you prefer working out with an example, one way out is to consider a shifted-logarithm $\log(x+a)$, where losses beyond a are treated as infinite in value. However, for full credits, you should prove this result for any concave-increasing utility function in this universe.

Solution:

Without any loss of generality, let $u(x)$ denote the utility of obtaining x million dollars. Note that, the losses incurred by the agent are represented by negative values of x. Then, we can compute the utilities of projects P, Q and R as

$$\begin{aligned} u_P &= 0.75 \cdot u(50) + 0.25 \cdot u(-10) \\ u_Q &= 0.6 \cdot u(100) + 0.4 \cdot u(-40) \\ u_R &= 0.5 \cdot u(200) + 0.5 \cdot u(-100) \end{aligned} \tag{3}$$

Given that the CEO is a risk-averse agent with a concave utility (i.e. we have $u(\lambda x + (1-\lambda)y) \geq \lambda u(x) + (1-\lambda)u(y)$), we can find lower bounds to the utilities $u(-40)$, $u(-10)$, $u(50)$ and $u(100)$ in terms of $u(200)$ and $u(-100)$, as shown below.

$$\begin{aligned} u(-40) &= u\left(\frac{1}{5} \times 200 + \frac{4}{5} \times (-100)\right) &> \frac{1}{5} \times u(200) + \frac{4}{5} \times u(-100) \\ u(-10) &= u\left(\frac{3}{10} \times 200 + \frac{7}{10} \times (-100)\right) &> \frac{3}{10} \times u(200) + \frac{7}{10} \times u(-100) \\ u(50) &= u\left(\frac{1}{2} \times 200 + \frac{1}{2} \times (-100)\right) &> \frac{1}{2} \times u(200) + \frac{1}{2} \times u(-100) \\ u(100) &= u\left(\frac{2}{3} \times 200 + \frac{1}{3} \times (-100)\right) &> \frac{2}{3} \times u(200) + \frac{1}{3} \times u(-100) \end{aligned} \tag{4}$$

Similarly, we can also evaluate lower bounds to the utilities $u(-10)$ and $u(50)$ in terms of $u(100)$ and $u(-40)$, as shown below.

$$\begin{aligned} u(-10) &= u\left(\frac{1}{4}\times 100 + \frac{3}{4}\times(-40)\right) &> \frac{1}{4}\times u(100) + \frac{3}{4}\times u(-40) \\ u(50) &= u\left(\frac{9}{14}\times 100 + \frac{5}{14}\times(-40)\right) &> \frac{9}{14}\times u(100) + \frac{5}{14}\times u(-40) \end{aligned} \tag{5}$$

Substituting the inequalities in Equation (4) in u_P and u_Q defined in Equation (3), we obtain

$$\begin{aligned} u_P &= 0.75\cdot u(50) + 0.25\cdot u(-10) \\ &> 0.75\Big(\frac{1}{2}\times u(200) + \frac{1}{2}\times u(-100)\Big) + 0.25\Big(\frac{3}{10}\times u(200) + \frac{7}{10}\times u(-100)\Big) \\ &= \frac{9}{20}\times u(200) + \frac{11}{20}\times u(-100) \\ u_Q &= 0.6\cdot u(100) + 0.4\cdot u(-40) \\ &> 0.6\Big(\frac{2}{3}\times u(200) + \frac{1}{3}\times u(-100)\Big) + 0.4\Big(\frac{1}{5}\times u(200) + \frac{4}{5}\times u(-100)\Big) \\ &= \frac{12}{25}\times u(200) + \frac{13}{25}\times u(-100) \end{aligned} \tag{6}$$

In such a case, the comparison in utilities of P and Q with respect to that of R can be evaluated as

$$\begin{aligned} u_P - u_R &> -\frac{1}{20}\times\Big(u(200) - u(-100)\Big) \\ u_Q - u_R &> -\frac{1}{25}\times\Big(u(200) - u(-100)\Big) \end{aligned} \tag{7}$$

A lower bound can be evaluated for u_P in a similar manner in terms of $u(100)$ and $u(-40)$ using Equation (5), as shown below:

$$\begin{aligned} u_P &= 0.75\cdot u(50) + 0.25\cdot u(-10) \\ &> 0.75\Big(\frac{1}{4}\times u(100) + \frac{3}{4}\times u(-40)\Big) + 0.25\Big(\frac{9}{14}\times u(100) + \frac{5}{14}\times u(-40)\Big) \\ &= \frac{39}{112}u(100) + \frac{73}{112}u(-40) \end{aligned} \tag{8}$$

Now, we can also compare u_P with u_Q by evaluating the bound of the difference

$$u_P - u_Q > -\frac{141}{560}\times\Big(u(100) - u(-40)\Big). \tag{9}$$

Note that the quantities on the right hand side of both inequalities in Equations (7) and (9) are negative, when the utility function is increasing with reward. In other words, the preferences over P, Q and R cannot be determined for a concave increasing utility, in general.

Note: However, if the CEO is risk-seeking with a convex increasing function, then we can show that the CEO can determine his/her preferences to be $R \succ Q \succ P$.

Problem 3 Expected Utility Maximization *4 pts.*

Suppose that Alice won a competition. As a reward, she was asked to choose one of the following two options:

Option 1: A laptop with probability 1.

Option 2: A tablet with probability 0.3, or a motorcycle with probability 0.7.

Option 3: A cell phone with probability 0.3, or a laptop with probability 0.2, or a motorcycle with probability 0.5.

Assuming that Alice maximizes her expected utility, if she prefers

$$\text{cell phone} \prec \text{tablet} \prec \text{laptop} \prec \text{motorcycle},$$

find the preference order over Options 1, 2 and 3.

Solution:

Let the utilities of cell phone, tablet, laptop and motorcycle be u_1, u_2, u_3 and u_4 respectively. Also, let the utilities of Option 1 and Option 2 be v_1 and v_2 respectively.

Then, we have

$$v_1 = u_3, \quad v_2 = 0.3u_2 + 0.7u_4, \quad \text{and} \quad v_3 = 0.3u_1 + 0.2u_3 + 0.5u_4. \tag{10}$$

Comparing v_2 and v_3, we obtain

$$v_2 - v_3 = 0.3\Big(u_2 - u_1\Big) + 0.2\Big(u_4 - u_3\Big) \geq 0. \tag{11}$$

Therefore, Alice prefers Option 2 over Option 3.

However, since $u_1 < u_2 < u_3 < u_4$, by the axiom of continuity (Axiom 5), there always exists two real numbers $\alpha, \beta \in (0, 1)$ such that

$$v_1 = u_3 = \alpha u_1 + (1-\alpha)u_4, \quad \text{and} \quad v_1 = u_3 = \beta u_2 + (1-\beta)u_4. \tag{12}$$

Therefore, we have

$$v_2 = 0.3u_2 + 0.7u_4, \quad \text{and} \quad v_3 = (0.3 + 0.2\alpha)u_1 + (0.7 - 0.2\alpha)u_4. \tag{13}$$

Using the axiom of monotonicity (Axiom 4) in Equations (10) and (13), we find that

- if $\alpha > (0.3 + 0.2\alpha)$ (or equivalently, $\alpha > 3/8$), Alice prefers Option 1 over Option 3,
- if $\alpha < 3/8$, Alice prefers Option 3 over Option 1.

Similarly, we also have the following:

- if $\beta > 0.3$, then Alice prefers Option 1 over Option 2,
- if $\beta < 0.3$, Alice prefers Option 2 over Option 1.

Combining all the conditions, we have the following preference orderings on a case-by-case basis:

- **CASE 1 ($\alpha < 3/8$ and $\beta < 0.3$):** Option 2 $\succ$ Option 3 $\succ$ Option 1
- **CASE 2 ($\alpha < 3/8$ and $\beta > 0.3$):** Cyclic preference
  (Option 1 $\succ$ Option 2 $\succ$ Option 3 $\succ$ Option 1)
- **CASE 3 ($\alpha > 3/8$ and $\beta < 0.3$):** Option 2 $\succ$ Option 1 $\succ$ Option 3
- **CASE 4 ($\alpha > 3/8$ and $\beta > 0.3$):** Option 1 $\succ$ Option 2 $\succ$ Option 3

$\square$

Problem 4 Limitations of EUM *4 pts.*

Daniel Ellsberg proposed the following thought-experiment[1] (known as *Ellsberg Paradox*) in 1961. An urn contains 90 balls, 30 of which are red. The other 60 are black or yellow, in unknown proportions. One ball will be drawn randomly from the urn. In this experiment, consider yourself as a decision maker.

(a) First, you must make a choice between Gamble A and Gamble B:

- **Gamble A:** You win $100 if the ball is red.
- **Gamble B:** You win $100 if the ball is black.

Which would you choose, and why?

(b) Next, you must make a choice between Gamble C and Gamble D:

- **Gamble C:** You win $100 if the ball is either red or yellow.
- **Gamble D:** You win $100 if the ball is either black or yellow.

Which would you choose, and why?

(c) Most people strongly prefer Gambles A and D over Gambles B and C respectively. Explain why this pattern of choices violates expected utility theory.

(d) Implement Ellsberg Paradox in Python using abstract classes within the Jupyter Notebook provided to you in your Gitlab repositories. Rename your notebook as "<last_name>_FS2021_CS5408_HW1_4d.ipynb".

Solution:

(a) and (b): Students are free to pick any one gamble and explain the latent rationality behind their choice.

(c) Let α denote the fraction of black balls amongst the remaining unknown 60 balls. Note that α can also be interpreted as the conditional probability of a ball being black, given that it is not a red ball. In other words, the urn has 30 red balls, 60α black balls and $60(1-\alpha)$ yellow balls.

If the agent follows EUM, then the utilities of gambles A, B, C and D are given by

$$\begin{aligned} u_A &= \mathbb{P}(\text{red}) \times \$100 + \mathbb{P}(\text{not red}) \times \$0 \\ &= \frac{30}{90} \times 100 + \frac{30}{90} \times 0 \\ &= \frac{100}{3}, \end{aligned} \tag{14}$$

[1] A similar experiment was also proposed by John Maynard Keynes in 1921.

$$\begin{aligned}
u_B &= \mathbb{P}(\text{black}) \times \$100 + \mathbb{P}(\text{not black}) \times \$0 \\
&= \frac{60\alpha}{90} \times 100 + \frac{90 - 60\alpha}{90} \times 0 \qquad (15)\\
&= \frac{200\alpha}{3},
\end{aligned}$$

$$\begin{aligned}
u_C &= \mathbb{P}(\text{red/yellow}) \times \$100 + \mathbb{P}(\text{not red/yellow}) \times \$0 \\
&= \frac{90 - 60\alpha}{90} \times 100 + \frac{60\alpha}{90} \times 0 \qquad (16)\\
&= \left(1 - \frac{2\alpha}{3}\right)100,
\end{aligned}$$

$$\begin{aligned}
u_D &= \mathbb{P}(\text{black/yellow}) \times \$100 + \mathbb{P}(\text{not black/yellow}) \times \$0 \\
&= \frac{60}{90} \times 100 + \frac{30}{90} \times 0 \qquad (17)\\
&= \frac{200}{3},
\end{aligned}$$

Given $A \succ B$ and $D \succ C$, if the agent follows EUM, we expect $u_A > u_B$ and $u_D > u_C$. In other words, we need

$$\begin{aligned}
u_A > u_B &\;\Rightarrow\; \frac{100}{3} > \frac{200\alpha}{3} &&\Rightarrow\; \alpha < \frac{1}{2}, \\
u_D > u_C &\;\Rightarrow\; \frac{200}{3} > \left(\frac{1 - 2\alpha}{3}\right)100 &&\Rightarrow\; \alpha > \frac{1}{2}.
\end{aligned} \qquad (18)$$

This is a contradiction! Therefore, any agent who exhibits preference orders $A \succ B$ and $D \succ C$ does not follow expected utility maximization. $\square$

Problem 5 Prospect Theory ***4 pts.***

Let your utility function for gains and losses be

$$u(x) = \begin{cases} x, & \text{if } x \geq 0 \\ \lambda x, & \text{if } x < 0, \end{cases}$$

where $\lambda > 0$ is an unknown parameter and your probability weighting function is $w(p) = p$. Consider the following two gambles:

- $P = \{$win \$150 with probability 0.5; loose \$100 with probability 0.5$\}$
- $Q = \{$win \$200 with probability 0.5; loose \$100 with probability 0.5$\}$

Suppose that you have the following preferences:

- prefer getting nothing for sure over the gamble P,
- prefer the gamble Q over getting nothing for sure.

Then, what is the range of λ that is consistent with the choices above?

Solution:

Given $w(p) = p$ and $u(x) = \begin{cases} x, & \text{if } x \geq 0, \\ \lambda x, & \text{otherwise,} \end{cases}$ let N denote the gamble wherein the agent gets nothing for sure. In other words, if the agent chooses N, he/she obtains a reward of \$0 with probability 1.

Then, the prospect theoretic utilities of gambles P, Q and N are given by

$$\begin{aligned} V_P &= 0.5 \times \$150 + 0.5\Big(\lambda(-\$100)\Big) &&= 75 - 50\lambda \\ V_Q &= 0.5 \times \$200 + 0.5\Big(\lambda(-\$100)\Big) &&= 100 - 50\lambda \\ V_N &= 1 \times \$0 &&= 0 \end{aligned} \tag{19}$$

Given that $N \succ P$ and $Q \succ N$, we expect

$$\begin{aligned} V_N > V_P \quad &\Rightarrow \quad 0 > 75 - 50\lambda \quad &&\Rightarrow \quad \lambda > \frac{3}{2} \\ V_Q > V_N \quad &\Rightarrow \quad 100 - 50\lambda > 0 \quad &&\Rightarrow \quad \lambda < 2 \end{aligned} \tag{20}$$

respectively. Combining the above two inequalities, we have $\dfrac{3}{2} < \lambda < 2$. $\square$

Problem 6 Domination *4 pts.*

Suppose an agent is presented with a choice set $\mathcal{X} = \{\alpha, \beta, \gamma\}$, where the choice experiment can take the states $\Omega = \{t_1, t_2, t_3\}$. If the utilities at the agent are given as shown in the table below,

Decision	**State** t_1	**State** t_2	**State** t_3
α	4	1	-3
β	3	-2	4
γ	0	1	6

(a) Find the region in $\Delta(\Omega)$ in which α is optimal.

(b) Find the region in $\Delta(\Omega)$ in which β is optimal.

(c) Find the region in $\Delta(\Omega)$ in which γ is optimal.

(d) Implement this experiment in Python using the Jupyter Notebooks provided in your Gitlab repositories and validate your theoretical findings in (a)-(c). Rename your Jupyter Notebook as
"<last_name>_FS2021_CS5408_HW1_6d.ipynb".

Solution:

Let the probability of states t_1, t_2 and t_3 be denoted as p_1, p_2 and $p_3 = 1-p_1-p_2$ respectively. Then, the expected utilities of α, β and γ are given by

$$\begin{aligned} u_\alpha &= 4 \cdot p_1 + 1 \cdot p_2 + (-3)(1 - p_1 - p_2) &&= 7p_1 + 4p_2 - 3 \\ u_\beta &= 3 \cdot p_1 + (-2) \cdot p_2 + 4(1 - p_1 - p_2) &&= -p_1 - 6p_2 + 4 \\ u_\gamma &= 0 \cdot p_1 + 1 \cdot p_2 + 6(1 - p_1 - p_2) &&= -6p_1 - 5p_2 + 6 \end{aligned} \tag{21}$$

respectively.

(a) **Optimality of α:** In order to find the region where α is optimal, we need to evaluate the inequalities: $u_\alpha > u_\beta$ and $u_\alpha > u_\gamma$. In other words, we expect

$$\begin{aligned} u_\alpha > u_\beta \;&\Rightarrow\; 7p_1 + 4p_2 - 3 > -p_1 - 6p_2 + 4 &&\Rightarrow\; 8p_1 + 10p_2 > 7 \\ u_\alpha > u_\gamma \;&\Rightarrow\; 7p_1 + 4p_2 - 3 > -6p_1 - 5p_2 + 6 &&\Rightarrow\; 13p_1 + 9p_2 > 9. \end{aligned} \tag{22}$$

In addition, we also need to ensure $p_1 \geq 0$, $p_2 \geq 0$, $p_3 \geq 0$ (or, $p_1 + p_2 \leq 1$).

In other words, the region on the simplex where α is the optimal choice is given by

$$\mathbb{R}_\alpha = \left\{(p_1, p_2) \in [0,1]^2 \;\middle|\; 8p_1 + 10p_2 > 7,\ 13p_1 + 9p_2 > 9,\ \text{and } p_1 + p_2 \leq 1\right\}. \tag{23}$$

(b) **Optimality of β:** In order to find the region where β is optimal, we need to evaluate the inequalities: $u_\beta > u_\alpha$ and $u_\beta > u_\gamma$. In other words, we expect

$$\begin{aligned} u_\beta > u_\alpha &\;\Rightarrow\; -p_1 - 6p_2 + 4 > 7p_1 + 4p_2 - 3 &&\Rightarrow\; 8p_1 + 10p_2 < 7 \\ u_\beta > u_\gamma &\;\Rightarrow\; -p_1 - 6p_2 + 4 > -6p_1 - 5p_2 + 6 &&\Rightarrow\; 5p_1 - p_2 > 2. \end{aligned} \tag{24}$$

In addition, we also need to ensure $p_1 \geq 0$, $p_2 \geq 0$, $p_3 \geq 0$ (or, $p_1 + p_2 \leq 1$).
In other words, the region on the simplex where β is the optimal choice is given by

$$\mathbb{R}_\beta = \left\{(p_1, p_2) \in [0,1]^2 \;\middle|\; 8p_1 + 10p_2 < 7,\ 5p_1 - p_2 > 2,\ \text{and } p_1 + p_2 \leq 1\right\}. \tag{25}$$

(c) **Optimality of γ:** In order to find the region where γ is optimal, we need to evaluate the inequalities: $u_\gamma > u_\alpha$ and $u_\gamma > u_\beta$. In other words, we expect

$$\begin{aligned} u_\gamma > u_\alpha &\;\Rightarrow\; -6p_1 - 5p_2 + 6 > 7p_1 + 4p_2 - 3 &&\Rightarrow\; 13p_1 + 9p_2 < 9 \\ u_\gamma > u_\beta &\;\Rightarrow\; -6p_1 - 5p_2 + 6 > -p_1 - 6p_2 + 4 &&\Rightarrow\; 5p_1 - p_2 < 2. \end{aligned} \tag{26}$$

In addition, we also need to ensure $p_1 \geq 0$, $p_2 \geq 0$, $p_3 \geq 0$ (or, $p_1 + p_2 \leq 1$).
In other words, the region on the simplex where γ is the optimal choice is given by

$$\mathbb{R}_\gamma = \left\{(p_1, p_2) \in [0,1]^2 \;\middle|\; 13p_1 + 9p_2 < 9,\ 5p_1 - p_2 < 2,\ \text{and } p_1 + p_2 \leq 1\right\}. \tag{27}$$

Problem 7 St. Petersburg Paradox *(Extra Credit: 2 pt.)*

Model the choice experiment in St. Petersburg paradox formally as a lottery, i.e. clearly define the states, their corresponding probabilities, choices and a conditional distribution on the choice set given the state.

Solution:

In Saint Petersburg paradox experiment, each agent is expected to choose their stopping time, i.e. the number of iterations they wish to play at the beginning of the experiment. In other words, the choice set is given by

$$\mathcal{C} = \{1, 2, \cdots\}. \tag{28}$$

If the agent were to employ a lottery on $\mathcal{C}$, they need to choose a probability vector $\boldsymbol{\pi} = \{\pi_1, \pi_2, \cdots\}$ as his/her lottery, where $\pi_n = \mathbb{P}(\text{agent picks his stopping time as } n)$. Note that, since

$$\sum_{n=1}^{\infty} \pi_n = 1 \tag{29}$$

is an infinite sum, $\boldsymbol{\pi}$ is always a *sparse* vector.

The state of this experiment at iteration k (for any $k \in \mathcal{C}$) is determined by the outcome of the sequence of coin tosses, i.e.

$$\boldsymbol{s}_k = (s_1, \cdots, s_k), \tag{30}$$

where $s_i \in \{H, T\}$ is the outcome of the coin toss at the i^{th} iteration. Consequently, we have $\mathbb{P}(s_i = H) = \mathbb{P}(s_i = T) = \dfrac{1}{2}$. In other words, the state uncertainty at the k^{th} iteration is given by

$$\mathbb{P}(\boldsymbol{s}_k = \boldsymbol{s}) = \left(\frac{1}{2}\right)^k, \tag{31}$$

for any $\boldsymbol{s} \in \{H, T\}^k$. $\square$

Problem 8 Allias Paradox ***(Extra Credit: 2 pt.)***

Prove that $1A \succ 1B$ and $2B \succ 2A$ violates expected utility maximization (EUM) framework.

Solution:

Let $u(x)$ denote the utility of winning x million dollars. In other words, if the agent prefers $1A \succ 1B$, we have

$$u(1) > 0.89u(1) + 0.01u(0) + 0.1u(5) \quad \Rightarrow \quad 0.11u(1) > 0.01u(0) + 0.1u(5). \tag{32}$$

Similarly, if the agent prefers $2B \succ 2A$, we have

$$0.9u(0) + 0.1u(5) > 0.89u(0) + 0.11u(1) \quad \Rightarrow \quad 0.11u(1) < 0.01u(0) + 0.1u(5). \tag{33}$$

Note that it is impossible to satisfy Equations (32) and (33) simultaneously using any feasible utility function. Therefore, we have a violation to expected utility maximization. $\square$