Fall 2021

CS 5408 : Game Theory for Computing.

HW1 Solutions

Prob 1 :

Given the agent's preferences order $f_1 \succeq_E f_2 \succeq_E f_3 \succeq_E f_4$, and that the agent follows all the 9 preference axioms (i.e. agent is an expected utility maximizer),

we can define utilities u_1, u_2, u_3, u_4 for the lotteries f_1, f_2, f_3 and f_4 respectively s.t.

$$u_1 \geq u_2 \geq u_3 \geq u_4 .$$

Furthermore, we also have

$$\circledast \quad f_2 \sim_E 0.6 f_1 + 0.4 f_4 \Rightarrow u_2 = 0.6 u_1 + 0.4 u_4 \quad \text{— (1)}$$

$$\circledast \quad f_3 \sim_E 0.2 f_1 + 0.8 f_4 \Rightarrow u_3 = 0.2 u_1 + 0.8 u_4 \quad \text{— (2)}$$

If $f = 0.15 f_1 + 0.5 f_2 + 0.15 f_3 + 0.2 f_4$,

we have

$$u_f = 0.15 u_1 + 0.5 u_2 + 0.15 u_3 + 0.2 u_4 \quad \text{— (3)}$$

Substituting ①, ② in ③, we have

$$u_f = 0.15\, u_1 + \left[0.6\, u_1 + 0.4\, u_4\right] \cdot 0.5 + \left[0.2\, u_1 + 0.8\, u_4\right] \cdot 0.15 + 0.2 f_4$$

$$= 0.48\, u_1 + 0.52\, u_4 \quad \text{— ④}$$

lll^{ly}, $u_g = 0.25\, u_1 + 0.25\, u_2 + 0.25\, u_3 + 0.25\, u_4$

$$= 0.45\, u_1 + 0.55\, u_4 \quad \text{— ⑤}$$

Subtracting ④ and ⑤, we have

$$u_f - u_g = 0.03\, u_1 - 0.03\, u_4$$

$$= 0.03\left[u_1 - u_4\right].$$

Since $u_1 \geq u_4$, we also have $u_f - u_g \geq 0$.

$$\Rightarrow f \succeq_E g.$$

<u>Prob 2:</u>

P: $\xrightarrow{0.75}$ \$50M ; $\xrightarrow{0.25}$ −\$10M

Q: $\xrightarrow{0.6}$ \$100M ; $\xrightarrow{0.4}$ −\$40M

R: $\xrightarrow{0.5}$ \$200M ; $\xrightarrow{0.5}$ −\$100M.

Say, $u(x)$ is the utility obtained for getting a reward x.
(<u>NOTE</u>: One unit of x is \$1M).

$$\therefore \quad u_P = 0.75 \cdot u(50) + 0.25 \; u(-10) \quad \text{— (1a)}$$

$$u_Q = 0.6 \; u(100) + 0.4 \; u(-40) \quad \text{— (1b)}$$

$$u_R = 0.5 \; u(200) + 0.5 \; u(-100) \quad \text{— (1c)}$$

NOTE : R comprises of largest rewards / losses.

$\Rightarrow$ Any other reward / loss can be represented as a lottery of \$200M and −\$100M.

Since the agent is risk-averse,

$$u(\lambda x + (1-\lambda) y) \geq \lambda . u(x) + (1-\lambda) u(y)$$

$$\forall \; x, y \in \mathbb{R} \text{ and } \lambda \in [0,1].$$

Example :

$$u(50) = u\left(\tfrac{1}{2} \times 200 + \tfrac{1}{2}(-100)\right)$$

$$\geq \tfrac{1}{2} . u(200) + \tfrac{1}{2} \; u(-100) \quad \text{— (2)}$$

lll^{ly},

$$u(-10) = u\left(\tfrac{3}{10} \times 200 + \tfrac{7}{10} \times (-100)\right) \geq \tfrac{3}{10} . u(200) + \tfrac{7}{10} . u(-100) \quad \text{— (3)}$$

$$u(100) = u\left(\tfrac{2}{3} \times 200 + \tfrac{1}{3} \times (-100)\right) \geq \tfrac{2}{3} . u(200) + \tfrac{1}{3} . u(-100) \quad \text{— (4)}$$

$$u(-40) = u\left(\tfrac{1}{5} \times 200 + \tfrac{4}{5} \times (-100)\right) \geq \tfrac{1}{5} . u(200) + \tfrac{4}{5} . u(-100) \quad \text{— (5)}$$

Substituting 2 & 3 in 1a,

$$u_P \geqslant \frac{3}{4}\left[\frac{1}{2}\,u(200) + \frac{1}{2}\,u(-100)\right] + \frac{1}{4}\left[\frac{3}{10}u(200) + \frac{7}{10}u(-100)\right]$$

$$= \frac{9}{20}\cdot u(200) + \frac{11}{20}\,u(-100) \quad \text{— } 1a'$$

Substituting 4 and 5 in 1b,

$$u_Q \geqslant \frac{3}{5}\left[\frac{2}{3}\cdot u(200) + \frac{1}{3}\cdot u(-100)\right] + \frac{2}{5}\left[\frac{1}{5}\,u(200) + \frac{4}{5}\cdot u(-100)\right]$$

$$= \frac{12}{25}\cdot u(200) + \frac{13}{25}\cdot u(-100) \quad \text{— } 1b'$$

Rewriting 1a', 1b' and 1c equivalently with common denominators, we have

$$\left.\begin{aligned} u_P &\geqslant \frac{45}{100}\,u(200) + \frac{55}{100}\cdot u(-100) \\ u_Q &\geqslant \frac{48}{100}\,u(200) + \frac{52}{100}\cdot u(-100) \\ u_R &\geqslant \frac{50}{100}\cdot u(200) + \frac{50}{100}\cdot u(-100) \end{aligned}\right\} \text{— } 6$$

6 $\Rightarrow$

$$u_P - Q_R \geqslant -\frac{5}{100}\cdot u(200) + \frac{5}{100}\,u(-100)$$

$$= -0.05\left[u(200) - u(-100)\right]$$

$$u_Q - u_R \geqslant -0.02\left[u(200) - u(-100)\right]$$

P.T.O.

Assuming $u(200) \geq u(-100)$, we have

$u_P - u_R$ and $u_Q - u_R$ are both negative.

$\Rightarrow \quad R \succcurlyeq P$ and $R \succcurlyeq Q$.

Otherwise, $P \succcurlyeq R$ and $Q \succcurlyeq R$.

In order to compare P and Q, let us represent u_P in terms of $u(100)$ and $u(-40)$.

i.e. $$u(50) = u\left[\frac{9}{14} \times 100 + \frac{5}{14} \times (-40)\right]$$

$$\geq \frac{9}{14} \cdot u(100) + \frac{5}{14} \cdot u(-40)$$

and $$u(-10) \geq \frac{3}{14} \cdot u(100) + \frac{11}{14} \cdot u(-40).$$

$$\Rightarrow \quad u_P \geq \frac{3}{4}\left[\frac{9}{14} \cdot u(100) + \frac{5}{14} u(-40)\right] + \frac{1}{4}\left[\frac{3}{14} u(100) + \frac{11}{14} u(-40)\right]$$

$$= \frac{15}{28} \cdot u(100) + \frac{13}{28} u(-40)$$

$$= \frac{75}{140} \cdot u(100) + \frac{65}{140} u(-40)$$

<u>NOTE</u>: $$u_Q = \frac{84}{140} \cdot u(100) + \frac{56}{140} u(-40)$$

Assuming $u(100) \geq u(-40)$, then $u_Q - u_P \geq \frac{9}{140}\left[u(100) - u(-40)\right] \geq 0 \Rightarrow Q \succcurlyeq P$.

Otherwise, $P \succcurlyeq Q$.

Summary:

	$u(100) \geq u(-40)$	$u(100) < u(-40)$
$u(200) \geq u(-100)$	$R \succsim Q \succsim P$	$R \succsim P \succsim Q$
$u(200) < u(-100)$	$Q \succsim P \succsim R$	$P \succsim Q \succsim R$.

Prob. 3

(a) and (b) are agent-specific.
Student has freedom to choose any one rationality and make their decisions.

ⓒ Let α be the fraction of black balls amongst the remaining 60 balls in the urn.

Urn (90 balls) $\rightarrow$ 30 red balls.
Urn (90 balls) $\rightarrow$ $60 \cdot \alpha$ black balls.
Urn (90 balls) $\rightarrow$ $60(1-\alpha)$ yellow balls.

NOTE: α can be interpreted as the prob. of a ball being black, given that it is not red.

Given $A \succ B$ and $D \succ C$, if EUM holds, we have

$$u_A > u_B \quad \text{and} \quad u_D > u_C.$$

$$u_A = \mathbb{P}(\text{ball is red}) \times \$100 + \mathbb{P}(\text{ball is not red}) \cdot \times 0$$

$$= \frac{1}{3} \times \$100 = \$\frac{100}{3}.$$

$$u_B = \mathbb{P}(\text{Black}) \times \$100 + \mathbb{P}(\text{not black}) \times 0.$$

$$= \frac{60\alpha}{90} \times 100 = \$\frac{200\alpha}{3}.$$

$$u_C = \mathbb{P}(R \text{ or } Y) \cdot \times \$100 + \mathbb{P}(B) \times 0$$

$$= \left[1 - \frac{60\alpha}{90}\right] \times \$100 = \$\left[1 - \frac{2\alpha}{3}\right] \cdot 100$$

$$u_D = \mathbb{P}(B \text{ or } Y) \times \$100 + \mathbb{P}(R) \times 0$$

$$= \frac{2}{3} \times \$100 = \$\frac{200}{3}$$

$$u_A > u_B \Rightarrow \frac{100}{3} > \frac{200\alpha}{3} \Rightarrow \alpha < \frac{1}{2}.$$

$$u_D > u_C \Rightarrow \left[1 - \frac{2\alpha}{3}\right] 100 < \frac{200}{3} \Rightarrow \alpha > \frac{1}{2}.$$

This is a contradiction!

$\Rightarrow$ Any agent who chooses $A \succ B$ and $D \succ C$ does not follow expected utility maximization.

Prob. 4

Given $w(p) = p$ and $u(x) = \begin{cases} x \ ; & x \geq 0 \\ \lambda x \ ; & x < 0. \end{cases}$

Let N denote the gamble where the agent gets nothing for sure.

i.e. $N \xrightarrow{\text{prob. } 1} \$0$

IIIly,

$$P \begin{cases} \xrightarrow{0.5} \$150 \\ \xrightarrow{0.5} -\$100 \end{cases} \quad \text{and} \quad Q \begin{cases} \xrightarrow{0.5} \$200 \\ \xrightarrow{0.5} -\$100 \end{cases}$$

$\therefore$ The prospect-theoretic utilities of the gambles are

$$V_P = 0.5 \times \$150 + 0.5\left(\lambda \times (-\$100)\right)$$
$$= 75 - 50\lambda.$$

$$V_Q = 0.5 \times \$200 + 0.5\left(\lambda \times (-\$100)\right)$$
$$= 100 - 50\lambda$$

$$V_N = 1 \times 0 = 0.$$

Given $N \succ P$, $V_N > V_P \Rightarrow 75 - 50\lambda < 0$, — (A)

and $Q \succ N \Rightarrow V_Q > V_N \Rightarrow 100 - 50\lambda > 0$ — (B)

(A) $\Rightarrow \lambda > \frac{3}{2}$ and (B) $\Rightarrow \lambda < 2$.

Combining the two inequalities,

$$\boxed{\frac{3}{2} < \lambda < 2}.$$

Prob. 5

	t_1	t_2	t_3
α	4	1	-3
β	3	2	5
γ	0	1	6.

Let the prob. of states t_1, t_2, t_3 be p_1, p_2 and $p_3 = 1-p_1-p_2$ respectively.

$\Rightarrow \quad u_\alpha = 4p_1 + 1 \cdot p_2 + (-3)(1-p_1-p_2)$

$\qquad = 7p_1 + 4p_2 - 3.$

$u_\beta = 3 \cdot p_1 + 2 \cdot p_2 + 5(1-p_1-p_2) = -2p_1 - 3p_2 + 5.$

$u_\gamma = 0 \cdot p_1 + 1 \cdot p_2 + 6(1-p_1-p_2) = -6p_1 - 5p_2 + 6$

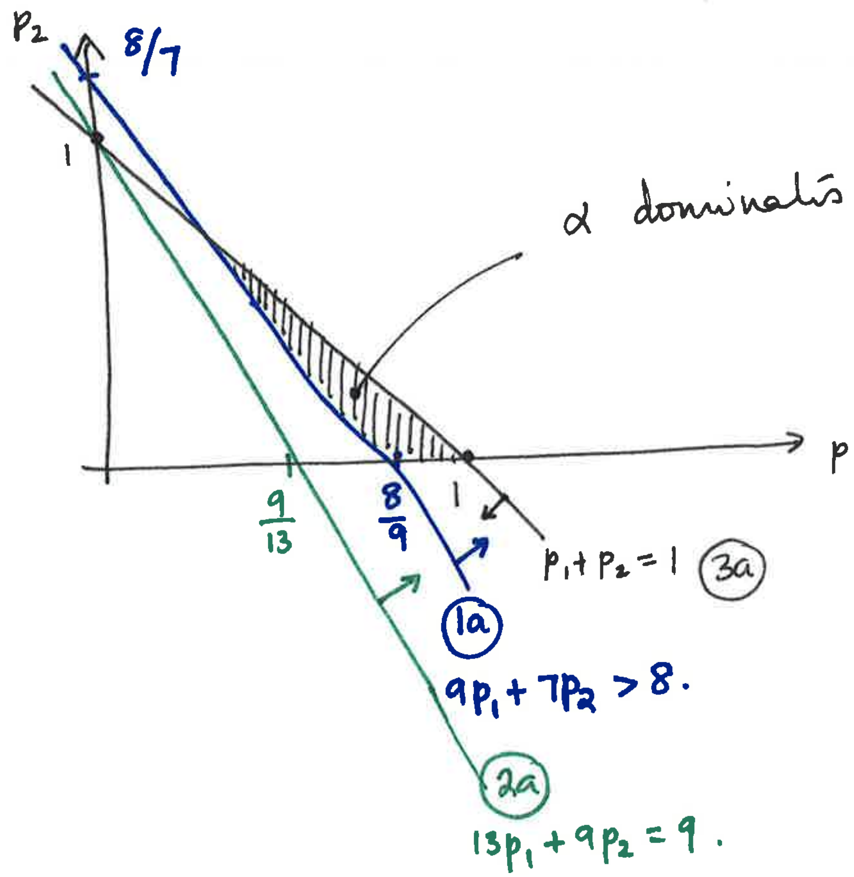
(a) $\quad \alpha \succ \beta$ and $\alpha \succ \gamma \Rightarrow u_\alpha > u_\beta$ and $u_\alpha > u_\gamma$

i.e. $7p_1 + 4p_2 - 3 > -2p_1 - 3p_2 + 5 \Rightarrow \boxed{9p_1 + 7p_2 > 8}$ — (1a) and

$$7p_1 + 4p_2 - 3 > -6p_1 - 5p_2 + 6 \Rightarrow \boxed{13p_1 + 9p_2 > 9} \quad \text{— (2a)}$$

In addition, we also need to ensure $p_1, p_2, p_3 > 0$

$$p_3 \geq 0 \Rightarrow 1 - p_1 - p_2 \geq 0 \Rightarrow \boxed{p_1 + p_2 \leq 1} \quad \text{— (3a)}$$



(b) $\beta \succ \alpha$ and $\beta \succ \gamma$.

$\Rightarrow u_\beta > u_\alpha$ and $u_\beta > u_\gamma$.

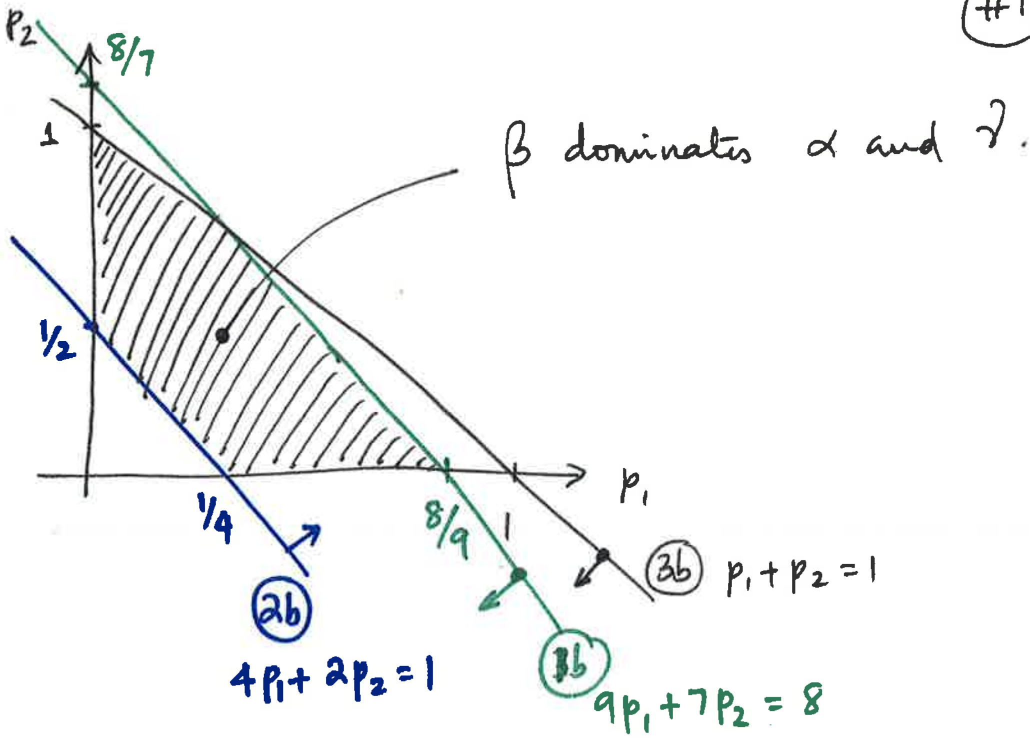
i.e.

$$-2p_1 - 3p_2 + 5 > 7p_1 + 4p_2 - 3 \Rightarrow \boxed{9p_1 + 7p_2 < 8} \quad \text{— (1b)}$$

$$-2p_1 - 3p_2 + 5 > -6p_1 - 5p_2 + 6 \Rightarrow \boxed{4p_1 + 2p_2 > 1} \quad \text{— (2b)}$$

$$p_3 > 0 \Rightarrow \boxed{p_1 + p_2 \leq 1} \quad \text{— (3b)}$$

P.T.O.

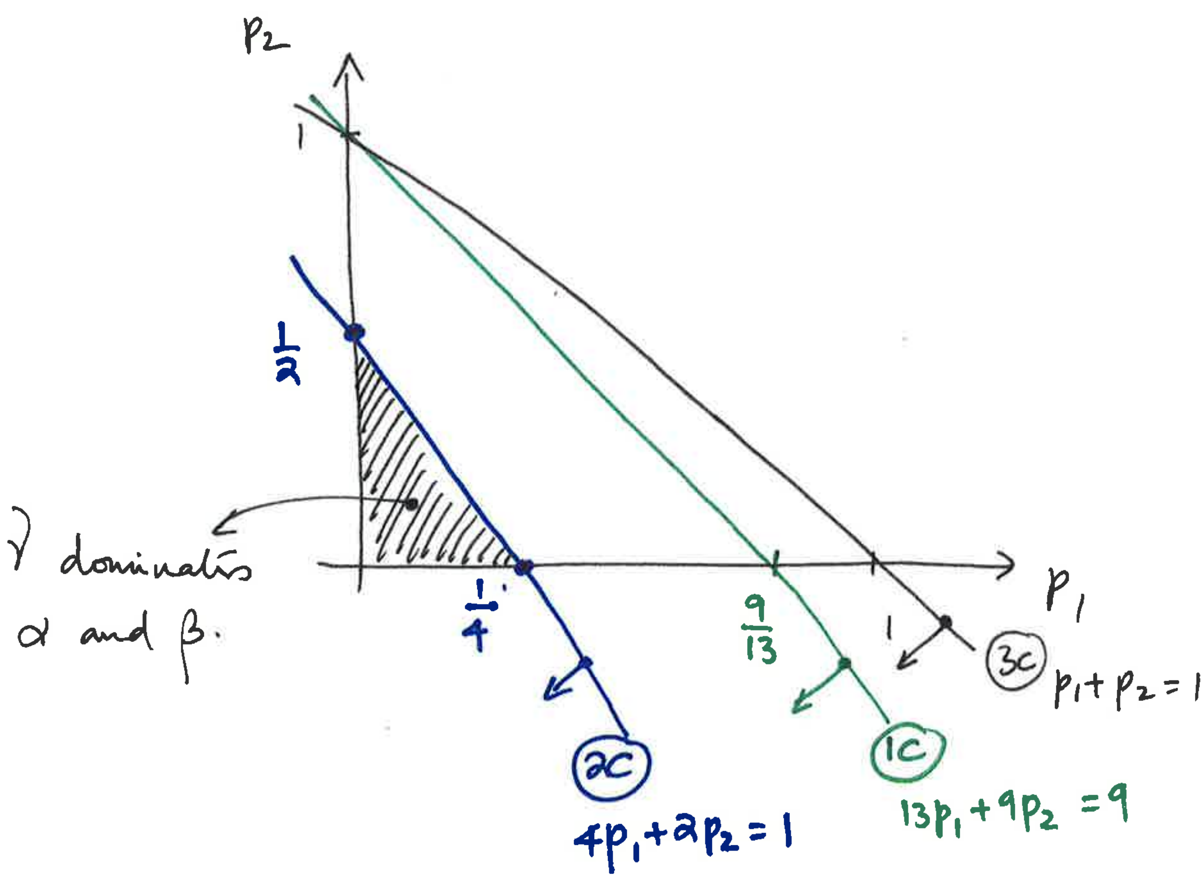


(c)

$$\gamma > \alpha \Rightarrow -6p_1 - 5p_2 + 6 > 7p_1 + 4p_2 - 3 \Rightarrow \boxed{13p_1 + 9p_2 < 9} \text{ — (1c)}$$

$$\gamma > \beta \Rightarrow -6p_1 - 5p_2 + 6 > -2p_1 - 3p_2 + 5 \Rightarrow \boxed{4p_1 + 2p_2 < 1} \text{ — (2c)}$$

$$p_3 > 0 \Rightarrow \boxed{p_1 + p_2 \le 1} \text{ — (3c)}$$



Prob. 6

Choices $\Rightarrow$ stopping time $\mathcal{C} = \{1, 2, \ldots\}$

Say, the agent employs a lottery on $\mathcal{C}$.

Let $\pi = \{\pi_1, \pi_2, \ldots\}$ denote this lottery.

$$\therefore \sum_{i=1}^{\infty} \pi_i = 1$$ (NOTE: Since there are countably infinite choices, this is an infinite sum)

$\Rightarrow$ π is always a sparse vector.

State of St. Petersberg paradox is determined by the outcome of coin toss sequence, i.e.

$$s_{1:n} = (s_1, \ldots s_n).$$

where $s_i \in \{H, T\}$

and $\mathbb{P}(s_i = H) = \mathbb{P}(s_i = T) = 1/2$.

$$\therefore \mathbb{P}(s_{1:n}) = \left(\frac{1}{2}\right)^n \longrightarrow \text{state uncertainty.}$$

Prob. 7

Let $u(x)$ denote the utility of winning x where 1 unit of x = \$1M.

$$1A \succ 1B \Rightarrow u(1) > 0.89\,u(1) + 0.01\,u(0) + 0.1\,u(5)$$

$$\Rightarrow 0.11\,u(1) > 0.01\,u(0) + 0.1\,u(5) \quad \text{—— (1)}$$

$$2B \succ 2A \Rightarrow 0.89\,u(0) + 0.11\,u(1) < 0.9\,u(0) + 0.1\,u(5)$$

$$\Rightarrow 0.11\,u(1) < 0.01\,u(0) + 0.1\,u(5) \quad \text{—— (2)}$$

Note that (1) & (2) contradict with each other since no $u(x)$ satisfies (1) & (2) simultaneously

$\Rightarrow$ An agent who simultaneously prefers

$1A \succ 1B$ and $2B \succ 2A$ violates EUT.