Missouri University of Science & Technology	Department of Computer Science
Fall 2024	**CS 5408: Game Theory for Computing**

Homework 3: Coalitional Games

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Consider an electoral college with four parties A, B, C, and D with 45, 25, 15, and 15 representatives respectively, which can form coalitions to form a simple-majority government (at least 51 votes) and obtain control over a \$1 trillion budget. Assuming that the value of the winning coalition is

$$v(C) = \begin{cases} 10^{12}, & \text{if } |C| \geq 51, \\ \\ 0, & \text{otherwise,} \end{cases}$$

Problem 1 Fair Distribution ***9 pts.***

Find the Shapley value to fairly divide the total budget amongst all parties.

Solution:

This coalitional game can be modeled formally as $\Gamma = (\mathcal{N}, v)$, where $\mathcal{N} = \{A, B, C, D\}$ and

$$v(C) = \begin{cases} 10^{12}, & \text{if } C = \{A, B, C, D\}, \text{ or } \{A, B, C\}, \text{ or } \{A, B, D\}, \text{ or } \{A, C, D\}, \\ & \qquad \text{or } \{B, C, D\}, \text{ or } \{A, B\}, \text{ or } \{A, C\}, \text{ or } \{A, D\}. \\ \\ 0, & \text{otherwise.} \end{cases}$$

Since there are $|\mathcal{N}| = 4$ players, they can form a grand coalition in $4! = 24$ permutations. Given a permutation π, we compute the marginal contribution of player i, denoted as $\Delta_\pi(i)$, is given by

$$\Delta_\pi(i) = v(S_\pi(i) \cup \{i\}) - v(S_\pi(i)), \tag{1}$$

where $S_\pi(i)$ is thee set of all predecessors of the i^{th} player in π. Each of these values are computed and shown in Table 1.

Taking the empirical average of the marginal contributions for each player across all permutations, we obtain the Shapley values as shown below:

$$\begin{aligned} u_A &= \frac{1}{24} \sum_{i=1}^{24} \Delta_{\pi_i}(A) &= \frac{1}{2} \times 10^{12}, \\ u_B &= \frac{1}{24} \sum_{i=1}^{24} \Delta_{\pi_i}(B) &= \frac{1}{6} \times 10^{12}, \end{aligned}$$

Permutation	$\Delta_\pi(A)$	$\Delta_\pi(B)$	$\Delta_\pi(C)$	$\Delta_\pi(D)$
$\pi_1 = \{A, B, C, D\}$	0	10^{12}	0	0
$\pi_2 = \{A, B, D, C\}$	0	10^{12}	0	0
$\pi_3 = \{A, C, B, D\}$	0	0	10^{12}	0
$\pi_4 = \{A, C, D, B\}$	0	0	10^{12}	0
$\pi_5 = \{A, D, B, C\}$	0	0	0	10^{12}
$\pi_6 = \{A, D, C, B\}$	0	0	0	10^{12}
$\pi_7 = \{B, A, C, D\}$	10^{12}	0	0	0
$\pi_8 = \{B, A, D, C\}$	10^{12}	0	0	0
$\pi_9 = \{B, C, A, D\}$	10^{12}	0	0	0
$\pi_{10} = \{B, C, D, A\}$	0	0	0	10^{12}
$\pi_{11} = \{B, D, A, C\}$	10^{12}	0	0	0
$\pi_{12} = \{B, D, C, A\}$	0	0	10^{12}	0
$\pi_{13} = \{C, A, B, D\}$	10^{12}	0	0	0
$\pi_{14} = \{C, A, D, B\}$	10^{12}	0	0	0
$\pi_{15} = \{C, B, A, D\}$	10^{12}	0	0	0
$\pi_{16} = \{C, B, D, A\}$	0	0	0	10^{12}
$\pi_{17} = \{C, D, A, B\}$	10^{12}	0	0	0
$\pi_{18} = \{C, D, B, A\}$	0	10^{12}	0	0
$\pi_{19} = \{C, A, B, D\}$	10^{12}	0	0	0
$\pi_{20} = \{C, A, D, B\}$	10^{12}	0	0	0
$\pi_{21} = \{C, B, A, D\}$	10^{12}	0	0	0
$\pi_{22} = \{C, B, D, A\}$	0	0	10^{12}	0
$\pi_{23} = \{C, D, A, B\}$	10^{12}	0	0	0
$\pi_{24} = \{C, D, B, A\}$	0	10^{12}	0	0

Table 1: Marginal Contributions of Players under Permutations of Coalition Formation

$$\begin{aligned} u_C &= \frac{1}{24}\sum_{i=1}^{24}\Delta_{\pi_i}(C) &= \frac{1}{6}\times 10^{12}, \\ u_D &= \frac{1}{24}\sum_{i=1}^{24}\Delta_{\pi_i}(D) &= \frac{1}{6}\times 10^{12}. \end{aligned}$$

□

Problem 2 Coalition Stability ***9 pts.***

Find the core, i.e. the set of all stable coalitions, in the above game.

Solution:

Given any game $\Gamma = (\mathcal{N}, v)$, the core is given by

$$\mathbb{C} = \left\{ (u_1, \cdots, u_N) \;\middle|\; \sum_{i=1}^{N} u_i = v(\mathcal{N}), \; \sum_{i \in C} u_i \geq v(C) \text{ for all } C \subset \mathcal{N} \right\}.$$

Therefore, consider the following coalitions:

$$\begin{aligned} C_1 = \{A, B\} \quad &\Rightarrow \quad u_A + u_B \geq v(C_1) = 10^{12}, \\ C_2 = \{A, C\} \quad &\Rightarrow \quad u_A + u_C \geq v(C_2) = 10^{12}, \\ C_3 = \{A, B, C\} \quad &\Rightarrow \quad u_A + u_B + u_C \geq v(C_3) = 10^{12}, \end{aligned} \tag{2}$$

Note that, adding the first two inequalities gives

$$2u_A + u_B + u_C \geq 2 \times 10^{12}. \tag{3}$$

Substituting the third inequality in Equation (2), we obtain

$$2u_A + u_B + u_C \geq 10^{12} + u_A \tag{4}$$

Then, two cases arise:

CASE 1: $10^{12} + u_A \geq 2 \times 10^{12}$

In this case, we have $u_A \geq 10^{12}$. Consequently, $u_B + u_C + u_D = 0$. However, this violates the condition $u_B + u_C + u_D \geq 10^{12}$.

CASE 2: $10^{12} + u_A < 2 \times 10^{12}$

In this case, we have $u_A < 10^{12}$. Without any loss of generality, let $u_A = \epsilon \times 10^{12}$ for some $\epsilon \in [0, 1)$. Then, $u_B + u_C + u_D = (1 - \epsilon) \times 10^{12}$. However, this also violates the condition $u_B + u_C + u_D \geq 10^{12}$.

In other words, there exists no (u_A, u_B, u_C, u_D) that satisfies all stability conditions simultaneously. Therefore, the core is empty.

Alternative Proof: This game is equivalent to a simple game $\tilde{\Gamma} = \{\mathcal{N}, \tilde{v}\}$, where any winning coalition C (i.e. $|C| \geq 51$) gets a value of 1. Note that this game does not have a veto player since no one player can dictate the outcome of this game. Since $\tilde{\Gamma}$ is a simple game with no veto player, its core is empty. Multiplying a constant factor 10^{12} to all stability conditions in $\tilde{\Gamma}$, we find that the game Γ also has an empty core.

$\square$