Missouri University of Science & Technology — Department of Computer Science

Fall 2024 — **CS 5408: Game Theory for Computing**

Homework 2: Basic Models & Solution Concepts

Instructor: *Sid Nadendla* — **Due:** *October 4, 2024*

Problem 1 Colonel Blotto Game *6 pts.*

A Colonel Blotto and his adversary, the Folk's Militia each try to occupy two posts by properly distributing their forces simultaneously. Assume that Colonel Blotto has three regiments and Militia has two regiments. If a given player has more regiments than the enemy at a post, then the player receives the enemy's regiments plus one (value for occupying the post). On the other hand, if the player has fewer regiments at a post than the enemy, the player loses one, plus the number of regiments he had at the post. A draw gives both sides zero. The total payoff is the sum of the payoffs at the two posts.

(a) Represent this game in normal-form, when both the players choose their strategies simultaneously.

(b) Find the pure strategy Nash equilibrium of this game.

Consider an alternative version of this game where (i) Colonel Blotto plays first, followed by Folk's Militia, and (ii) Folk's Militia can only observe which post has greater Colonel Blotto's regiments.

(c) Represent this new game in extensive-form.

(d) Transform the extensive representation of the new game into normal-form.

Solution:

(a) Let Blotto and Militia deploy $(b,\ 3-b)$ and $(m,\ 2-m)$ regiments in the two posts respectively. Let $U_i(b,m)$ and $V_i(b,m)$ denote the utilities at Blotto and Militia respectively, due to their own regiment deployments at the i^{th} post. Then,

$$U_1(b,m) = \begin{cases} m+1, & \text{if } b > m, \\ 0, & \text{if } b = m, \\ -b-1, & \text{otherwise} \end{cases} \quad \text{and} \quad U_2(b,m) = \begin{cases} (2-m)+1, & \text{if } 3-b > 2-m, \\ 0, & \text{if } 3-b = 2-m, \\ -(3-b)-1, & \text{otherwise} \end{cases} \tag{1}$$

Therefore, the utility matrix at Blotto is given by

$$U = U_1 + U_2 = \begin{array}{c|c|c|c|} & (0,2) & (1,1) & (2,0) \\ \hline (0,3) & 3 & 1 & 0 \\ \hline (1,2) & 1 & 2 & -1 \\ \hline (2,1) & -1 & 2 & 1 \\ \hline (3,0) & 0 & 1 & 3 \\ \hline \end{array} \tag{2}$$

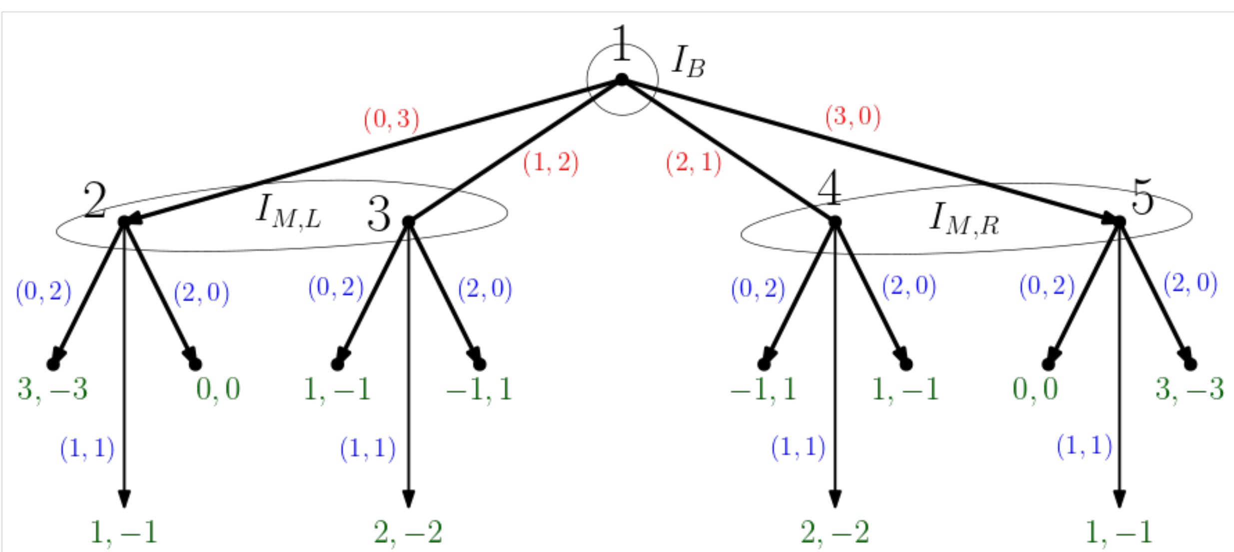
Given that this is a zero-sum game (since Blotto wins if Militia looses, and vice versa), we have

$$V = -U.$$

(b) Since no players' best responses align with each other as shown in the following matrix, this game has no PSNE.

	(0,2)	(1,1)	(2,0)
(0,3)	3, -3	1, -1	0, 0
(1,2)	1, -1	2, -2	-1, 1
(2,1)	-1, 1	2, -2	1, -1
(3,0)	0, 0	1, -1	3, -3

(c)



(d) Since there are two information sets $I_{M,L}$ and $I_{M,R}$ at the Militia, the new choices at Militia are found by picking one choice from $I_{M,L}$ and one from $I_{M,R}$ to obtain a *super-choice* of the form c_L, c_R. On the other hand, since Blotto only has one information set I_B, his choices in the transformed normal-form representation and extensive representation are the same. Therefore, we obtain the normal form game depicted on the right side of this page. □

$\mathrm{U}_{\mathrm{trans}}$ =

	(0,2),(0,2)	(0,2),(1,1)	(0,2),(2,0)	(1,1),(0,2)	(1,1),(1,1)	(1,1),(2,0)	(2,0),(0,2)	(2,0),(1,1)	(2,0),(2,0)
(0,3)	3	3	3	1	1	1	0	0	0
(1,2)	1	1	1	2	2	2	-1	-1	-1
(2,1)	-1	2	1	-1	2	1	-1	2	1
(3,0)	0	1	3	0	1	3	0	1	3

Problem 2 Iter. Elim. of Dominated Strategies *6 pts.*

Reduce the bimatrix game given below using Iterative Elimination of Dominated Strategies algorithm. Illustrate all the stages of the algorithm on this example very clearly.

	Left	Center	Right
Up	1,1	2,0	2,2
Middle	0,3	1,5	4,4
Down	2,4	3,6	3,0

Solution:

Eliminated strategies by row player is denoted in red, while that of column player is highlighted in blue.

	Left	Center	Right
~~Up~~	~~1,1~~	~~2,0~~	~~2,2~~
Middle	0,3	1,5	4,4
Down	2,4	3,6	3,0

P1: 'Down' dominates 'Up'

$\Rightarrow$

	Left	Center	~~Right~~
Middle	0,3	1,5	~~4,4~~
Down	2,4	3,6	~~3,0~~

P2: 'Center' dominates 'Right'

$\Downarrow$

	Left	Center
~~Middle~~	~~0,3~~	~~1,5~~
Down	2,4	3,6

P1: 'Down' dominates 'Middle'

$\Leftarrow$

	~~Left~~	Center
Down	~~2,4~~	3,6

P2: 'Center' dominates 'Left'

Therefore, IEDS algorithm terminates with a reduced matrix of $(D, C) \equiv (3, 6)$. $\square$

Problem 3 Rock-Paper-Scissors ***6 pts.***

Rock-Paper-Scissors is a well-known zero-sum game played by most kids, which is described as follows. Say, Alice and Bob simultaneously make a hand signal that represents one of their three pure strategies: *rock*, *paper* and *scissors*. The winner of the game is determined by the following rules:

- Rock *blunts* Scissors
- Scissors *cuts* Paper
- Paper *wraps* Rock

If both the players make the same signal, the result is a draw and obtain nothing. Therefore, Alice's payoff is given by the matrix

	Rock	Paper	Scissors
Rock	0	-1	1
Paper	1	0	-1
Scissors	-1	1	0

(a) Prove that there does not exist any pure-strategy equilibria for this game.

(b) Prove that the mixed-strategy minimax equilibrium for this game is $\left\{\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right\}$.

Solution:

(a) In the following figure, row player's best responses are denoted in red, while that of the column player in blue. Note that there is no choice profile where each player is playing a best response against their counterpart.

	Rock	Paper	Scissors
Rock	0, 0	-1, 1	1, -1
Paper	1, -1	0, 0	-1, 1
Scissors	-1, 1	1, -1	0, 0

(b) Let $\boldsymbol{x}$ and $\boldsymbol{y}$ denote the mixed strategies at the row and column players respectively. Then, if the utility matrix for the row player is given by

$$U = \begin{bmatrix} 0 & \text{-1} & 1 \\ 1 & 0 & \text{-1} \\ \text{-1} & 1 & 0 \end{bmatrix}, \tag{3}$$

then the expected utility at the row player is $u(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{x}^T U \boldsymbol{y}$. Without any loss of generality, let $\boldsymbol{x} = [x_1 \ x_2 \ 1 - x_1 - x_2]^T$ and $\boldsymbol{y} = [y_1 \ y_2 \ 1 - y_1 - y_2]^T$. Then,

$$u(\boldsymbol{x}, \boldsymbol{y}) = x_1 - x_2 - y_1 + y_2 - 3x_1 y_2 + 3x_2 y_1. \tag{4}$$

Then, any critical point $(\boldsymbol{x}^*, \boldsymbol{y}^*)$ satisfies

$$\nabla u(\boldsymbol{x}^*, \boldsymbol{y}^*) = \begin{bmatrix} \dfrac{\partial u}{\partial x_1} \\[2ex] \dfrac{\partial u}{\partial x_2} \\[2ex] \dfrac{\partial u}{\partial y_1} \\[2ex] \dfrac{\partial u}{\partial y_2} \end{bmatrix} = \begin{bmatrix} 1 - 3y_2^* \\[2ex] -1 + 3y_1^* \\[2ex] -1 + 3x_2^* \\[2ex] 1 - 3x_1^* \end{bmatrix} = 0 \quad \Rightarrow \quad \boldsymbol{x}^* = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) \text{ and } \boldsymbol{y}^* = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right). \tag{5}$$

Second Derivative Test: There are two methods to check if a given critical point is indeed a saddle point based on Hessian matrix H of a function u defined on N variables (when $N \geq 3$).

- **Method 1:** Let $H(\boldsymbol{x}, \boldsymbol{y}) = \nabla^2 u(\boldsymbol{x}, \boldsymbol{y})$ denote the Hessian matrix of u, with $\det H \neq 0$. Let D_k be the determinant of Hessian formed by the first k variables, i.e.

$$D_k(\boldsymbol{x}, \boldsymbol{y}) = \begin{vmatrix} \dfrac{\partial^2 u}{\partial x_1^2} & \cdots & \dfrac{\partial^2 u}{\partial x_k \partial x_1} \\[2ex] \vdots & \ddots & \vdots \\[2ex] \dfrac{\partial^2 u}{\partial x_1 \partial x_k} & \cdots & \dfrac{\partial^2 u}{\partial x_k^2} \end{vmatrix}.$$

  Then,

  1. If $D_k(\boldsymbol{x}^*, \boldsymbol{y}^*) > 0$ for all $k = 1, \cdots, N$, then $(\boldsymbol{x}^*, \boldsymbol{y}^*)$ is the **minimum** of $u(\boldsymbol{x}, \boldsymbol{y})$.
  2. If $(-1)^k D_k(\boldsymbol{x}^*, \boldsymbol{y}^*) > 0$ for all $k = 1, \cdots, N$, then $(\boldsymbol{x}^*, \boldsymbol{y}^*)$ is the **maximum** of $u(\boldsymbol{x}, \boldsymbol{y})$.
  3. Otherwise, $(\boldsymbol{x}^*, \boldsymbol{y}^*)$ is a **saddle point** of $u(\boldsymbol{x}, \boldsymbol{y})$.

  In the case of RPS game,

$$H(\boldsymbol{x}^*, \boldsymbol{y}^*) = \begin{bmatrix} 0 & 0 & 0 & -3 \\ 0 & 0 & 3 & 0 \\ 0 & 3 & 0 & 0 \\ -3 & 0 & 0 & 0 \end{bmatrix} \quad \Rightarrow \quad D_1 = 0, \ D_2 = 0, D_3 = 0 \text{ and } D_4 = 81.$$

  Hence, $(\boldsymbol{x}^*, \boldsymbol{y}^*)$ is a saddle point!

- **Method 2:** Another option is to investigate the eigenvalues of $H(\boldsymbol{x}^*, \boldsymbol{y}^*)$. Let $\lambda_1, \cdots, \lambda_N$ denote the eigenvalues of $H(\boldsymbol{x}^*, \boldsymbol{y}^*)$. Then,

    1. If $\lambda_i > 0$ for all $i = 1, \cdots, N$, then $(\boldsymbol{x}^*, \boldsymbol{y}^*)$ is a **local minimum** of $u(\boldsymbol{x}, \boldsymbol{y})$,
    2. If $\lambda_i < 0$ for all $i = 1, \cdots, N$, then $(\boldsymbol{x}^*, \boldsymbol{y}^*)$ is a **local maximum** of $u(\boldsymbol{x}, \boldsymbol{y})$,
    3. If $\lambda_i \neq 0$ for all $i = 1, \cdots, N$, but have both positive and negative values, then $(\boldsymbol{x}^*, \boldsymbol{y}^*)$ is a **saddle point** of $u(\boldsymbol{x}, \boldsymbol{y})$,
    4. If $H(\boldsymbol{x}^*, \boldsymbol{y}^*)$ is singular, then the degenerate critical point can be a *monkey saddle point*. In such a case, the eigenvalue changes its sign as it crosses the critical point.

In the case of RPS game, a simple way to compute the eigenvalues of H is to compute the eigenvalues of H^2 and evaluate their square-root. In other words,

$$H^2(\boldsymbol{x}^*, \boldsymbol{y}^*) = \begin{bmatrix} 0 & 0 & 0 & -3 \\ 0 & 0 & 3 & 0 \\ 0 & 3 & 0 & 0 \\ -3 & 0 & 0 & 0 \end{bmatrix}^2 = \begin{bmatrix} 9 & 0 & 0 & 0 \\ 0 & 9 & 0 & 0 \\ 0 & 0 & 9 & 0 \\ 0 & 0 & 0 & 9 \end{bmatrix}.$$

In other words, the eigenvalues of $H(\boldsymbol{x}^*, \boldsymbol{y}^*)$ is ± 3. Therefore, $(\boldsymbol{x}^*, \boldsymbol{y}^*)$ is a saddle point! $\square$

Problem 4 Second-Price Auction *6 pts.*

Consider a second-price sealed-bid auction with two bidders (players), where players 1 and 2 value the object being auctioned as v_1 and v_2 respectively, such that $v_1 > v_2 > 0$. Assuming that the i^{th} player submits a sealed bid b_i, he/she obtains the object at a price equal to the other player's bid, say b_{-i}, and hence receives the net payoff $v_i - b_{-i}$. Since the other player does not obtain the object, he/she receives the payoff of zero. If there is a tie, then the tie is broken by choosing the winner amongst the two bidders with equal probabilities.

(a) Find the best response functions of both the bidders.

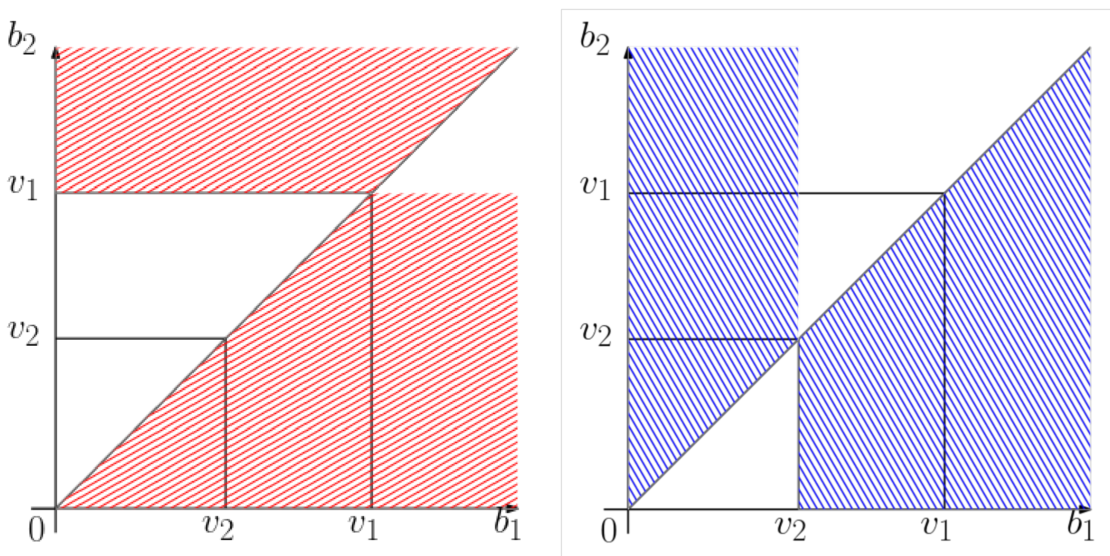
(b) Find all the pure-strategy Nash equilibria in this game.

Solution:

(a)

Player 1's Best Response: Note that

$$u_1 = \begin{cases} v_1 - b_2, & \text{if } b_1 > b_2, \\ \dfrac{1}{2}\left(v_1 - b_2\right), & \text{if } b_1 = b_2, \\ 0, & \text{otherwise.} \end{cases} \quad \Rightarrow \quad \text{P1 chooses } b_1 > b_2 \text{ if } v_1 - b_2 > 0. \text{ Else, P1 chooses } b_1 < b_2.$$



(a) Player 1's Best Response

(b) Player 2's Best Response

Figure 1: Best Response Regions on $b_1 - b_2$ plane

Player 2's Best Response: Note that

$$u_2 = \begin{cases} v_2 - b_1, & \text{if } b_1 < b_2, \\ \frac{1}{2}\left(v_2 - b_1\right), & \text{if } b_1 = b_2, \\ 0, & \text{otherwise.} \end{cases} \quad \Rightarrow \quad \text{P1 chooses } b_1 < b_2 \text{ if } v_2 - b_1 > 0. \text{ Else, P1 chooses } b_1 > b_2.$$

(b) NE is the region where both the players play best responses against each other. In other words, the intersection of both best response regions is the set of all NE for a second-price auction.

□

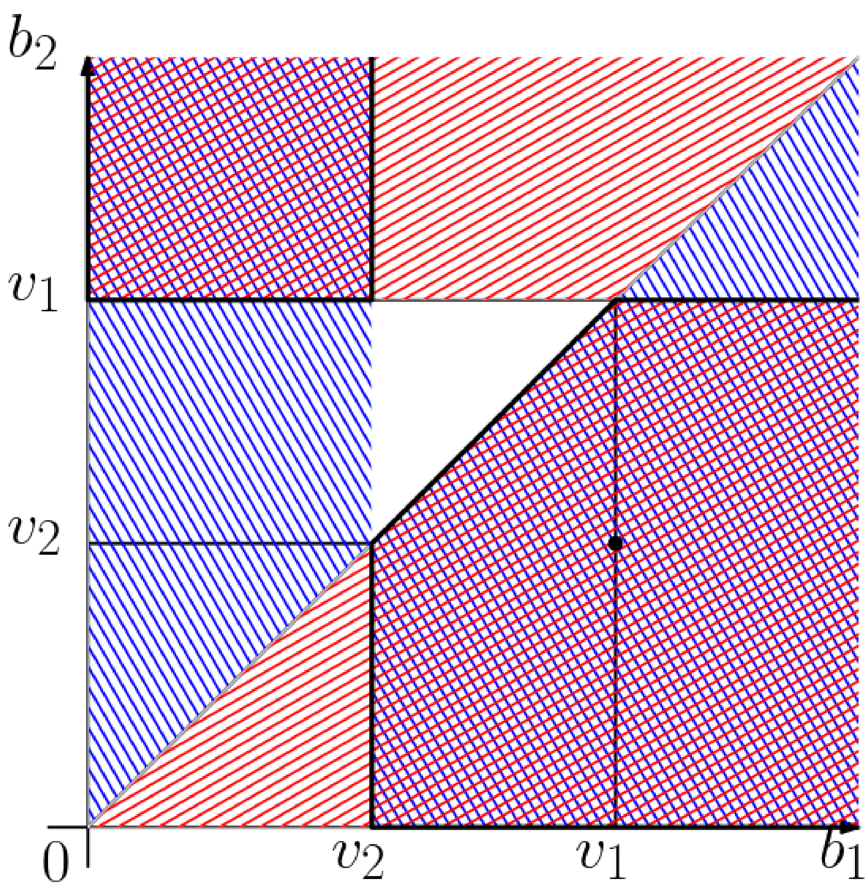


Figure 2: Set of all Nash equilibria on $b_1 - b_2$ plane, highlighted by double-hatched region. Note that the disk marker represents a NE that corresponds to truthful revelations.