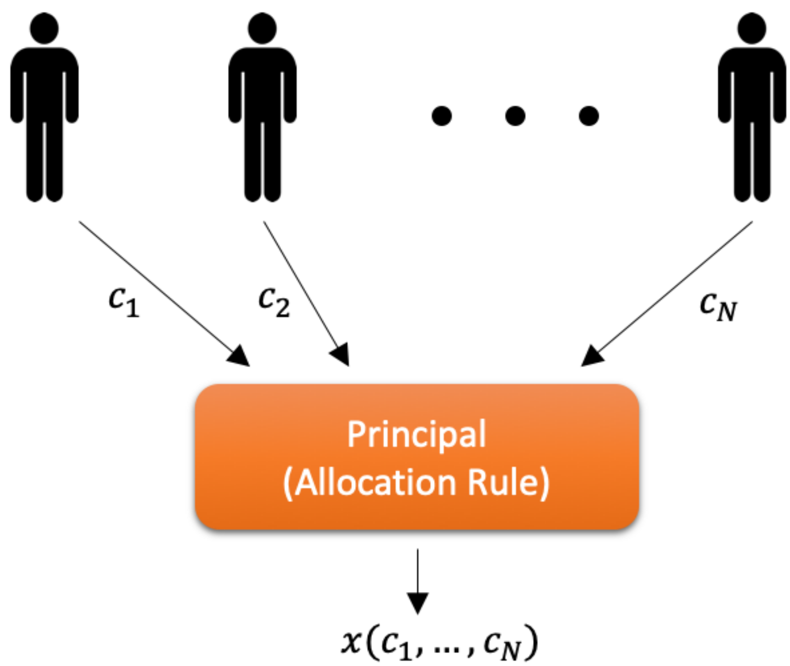
Mechanism Design

Implement an optimal system-wide solution to a decentralized optimization problem with self-interested agents with private preferences for different outcomes.



Examples:

- Auctions (One Seller, N Buyers)
- Reverse-Auctions (M Sellers, One Buyer)
- Bilateral Markets (M Sellers, N Buyers)
- Contracts
- Contests, Tournaments
- Voting Rules

1 Modeling Strategic Mechanisms

1.1 Notation

Definition 1. *A* **social choice function** $f : \Theta_1 \times \cdots \times \Theta_N \to \mathcal{O}$ *is a desired outcome* $f(\boldsymbol{\theta})$ *in the set of all outcomes* $\mathcal{O}$*, given the players' types* $\boldsymbol{\theta} \in \Theta_1 \times \cdots \times \Theta_N$*.*

Definition 2. *A* **mechanism** $\mathcal{M} = (\mathcal{C}_1, \cdots, \mathcal{C}_N, x(\cdot))$ *is a tuple that comprises of the set of choice strategies* $\mathcal{C}_i$ *available at* i^{th} *player, and an outcome rule* $x : \mathcal{C}_1 \times \cdots \times \mathcal{C}_N \to \mathcal{O}$*, such that* $x(\boldsymbol{c})$ *is the outcome implemented by the mechanism for choice profile* $\mathbf{c} = \{c_1, \cdots, c_N\}$*.*

Definition 3. *A mechanism* $\mathcal{M} = (\mathcal{C}_1, \cdots, \mathcal{C}_N, x(\cdot))$ **implements** *a social choice function* f *if*

$$x\left(c_1^*(\theta_1), \cdots, c_N^*(\theta_N)\right) = f(\boldsymbol{\theta}),$$

for all $\boldsymbol{\theta} \in \Theta_1 \times \cdots \times \Theta_N$*, where* $c_1^*(\theta_1), \cdots, c_N^*(\theta_N)$ *is the equilibrium of the game induced by* $\mathcal{M}$*.*

1.2 Direct and Indirect Revelation

Definition 4. $\mathcal{M} = (\mathcal{C}_1, \cdots, \mathcal{C}_N, x(\cdot))$ *is a* **direct revelation** *mechanism if the choice set at every player is restricted to its own type set, i.e.,*

$$\mathcal{C}_i = \Theta_i,$$

and has an outcome rule $x(\hat{\boldsymbol{\theta}})$ *based on revealed (reported) types* $\hat{\boldsymbol{\theta}} = \{\hat{\theta}_1, \cdots, \hat{\theta}_N\}$.

Examples:

1. First-Price Sealed-Bid Auction

2. Second-Price Sealed-Bid Auction

3. English Auction

4. Dutch Auction

2 Desired Properties of Mechanisms

Definition 5. *A mechanism* $\mathcal{M} = (\mathcal{C}_1, \cdots, \mathcal{C}_N, x(\cdot))$ *is* **individually rational** *if, for all agent types* $\boldsymbol{\theta} \in \Theta_1 \times \cdots \times \Theta_N$*, it implements a social choice function* f *such that*

$$u_i\left(f(\boldsymbol{\theta})\right) \geq \bar{u}_i(\boldsymbol{\theta}),$$

where $u_i\left(f(\boldsymbol{\theta})\right)$ *is the expected utility of* i^{th} *player averaged over a known distribution over other players' types* $\boldsymbol{\theta}_{-i}$*, and* $\bar{u}_i(\boldsymbol{\theta})$ *is the utility of the* i^{th} *player for not participating in* $\mathcal{M}$*.*

Definition 6. *A strategy* $c_i(\theta_i) \in \Theta_i$ *is a* **truthful revealation** *if* $c_i(\theta_i) = \theta_i$*, for all* $\theta_i \in \Theta_i$*.*

Definition 7. *A mechanism* $\mathcal{M} = (\mathcal{C}_1, \cdots, \mathcal{C}_N, x(\cdot))$ *is* **incentive compatible** *if the equilibrium strategy profile* $\boldsymbol{c}^* = \{c_1^*(\theta_1), \cdots, c_N^*(\theta_N)\}$ *has every player reporting their true types (preferences) to* $\mathcal{M}$*.*

Claim 2.1

In a first-price sealed-bid auction, each bidder bids

$$b_i = \left(\frac{N-1}{N}\right) v_i$$

at Nash equilibrium.

Claim 2.2

In a second-price sealed-bid auction, the strategy $b_i = v_i$ is a dominant strategy at every bidder.

3 Revelation Principle

Theorem 3.1

Suppose that $\boldsymbol{c}^*$ was an equilibrium of any (direct or indirect) mechanism $\mathcal{M}$. Then, there always exists a direct-revelation incentive-compatible (DRIC) mechanism $\mathcal{M}^*$ that is payoff-equivalent to $\mathcal{M}$.

4 Social Choice Theory

Consider a special type of mechanism where the agent type is based on the order of asymmetric and transitive preferences over the set of alternatives $\mathcal{A}$.

Example: Voting Methods

- Let $\mathcal{L}$ denote the set of linear orders of $\mathcal{A}$, i.e.

  $\mathcal{L}$ is isomorphic to the set of permutations on $\mathcal{A}$.

- The preferences of i^{th} agent $\pi_i : a \succ b$ means agent i with preference order $\pi_i \in \mathcal{L}$ ranks a over b, where $a, b \in \mathcal{A}$.

- A function $f : \mathcal{L}^N \to \mathcal{A}$ is called a social choice function.

- A function $F : \mathcal{L}^N \to \mathcal{L}$ is called a social welfare function.

Definition 8. *A social welfare function F satisfies **unanimity** if, for every $\pi_i : a \succ b$, then $\pi : a \succ b$, where $\pi = F(\pi_1, \cdots, \pi_N)$.*

Definition 9. *An agent i is a **dictator** in a social welfare function F, if for all $\pi_1, \cdots, \pi_N \in \mathcal{L}$, we have*

$$F(\pi_1, \cdots, \pi_N) = \pi_i.$$

Definition 10. *Given any pair of two alternatives* $a, b \in \mathcal{A}$*, a social welfare function* F *satisfies* ***independence of irrelevant alternatives (IIA)****, if for any two preference orders* π_i *and* τ_i *that ranks* $a \succ b$ *(denoted as* $\pi_i : a \succ b$ *and* $\tau_i : a \succ b$*) for all* i*, then*

$$\pi : a \succ b \quad \Rightarrow \quad \tau : a \succ b,$$

where $\pi = F(\pi_1, \cdots, \pi_N)$ *and* $\tau = F(\tau_1, \cdots, \tau_N)$.

In other words, IIA $\Rightarrow$ the social preference between any two alternatives does not depend on voters' preferences about other irrelevant alternatives.

Claim 4.1: Pairwise Neutrality

Let $\boldsymbol{\pi} = \{\pi_1, \cdots, \pi_N\}$ and $\boldsymbol{\tau} = \{\tau_1, \cdots, \tau_N\}$ denote two preference profiles such that for every agent i, $\pi_i : a \succ b$ and $\tau_i : c \succ d$ holds true. Then, given an unanimous and IIA social welfare function F,

$$\pi : a \succ b \Rightarrow \tau : c \succ d,$$

where $\pi \ = F(\pi_1, \cdots, \pi_N)$ and $\tau \ = F(\tau_1, \cdots, \tau_N)$.

Theorem 4.1: Arrow

Every social welfare function over a set of more than 2 alternatives that satisfies unanimity and independence of irrelevant alternatives is a dictatorship.

Definition 11. *A social choice function f can be **strategically manipulated** by the i^{th} agent, if for some profile $\boldsymbol{\pi} = \{\pi_1, \cdots, \pi_N\} \in \mathcal{L}^N$ where $\pi_i : a \succ b$, and some $\tau_i \in \mathcal{L}$, we have $a = f(\tau_i, \boldsymbol{\pi}_{-i})$, and $b = f(\pi_i, \boldsymbol{\pi}_{-i})$, where $\boldsymbol{\pi}_{-i} = \{\pi_1, \cdots, \pi_{i-1}, \pi_{i+1}, \cdots, \pi_N\}$.*

Definition 12. *A social choice function f is **monotone** if $f(\pi_i, \boldsymbol{\pi}_{-i}) = a \neq a' = f(\tau_i, \boldsymbol{\pi}_{-i})$ implies that $\pi_i : a \succ a'$ and $\tau_i : a' \succ a$.*

Claim 4.2

A social choice function is incentive compatible if and only if it is monotone.

Definition 13. *The i^{th} agent is a **dictator** in a social choice function f, if for any profile $\boldsymbol{\pi} = \{\pi_1, \cdots, \pi_N\}$ such that $\pi_i : a \succ b$ for all $b \neq a$, we have $f(\boldsymbol{\pi}) = a$.*

*Furthermore, f is called a **dictatorship** if there exists a dictator in it.*

Theorem 4.2: Gibbard-Satterthwaite

Let f be an incentive-compatible social choice function on the set of alternatives $\mathcal{A}$, where $|A| \geq 3$. Then, f is a dictatorship.

In other words, manipulation is inevitable in voting!
Then, can we make manipulation difficult?

5 Quasi-Linear Mechanisms

- GS theorem $\Rightarrow$ Cannot design incentive-compatible social-choice functions

- Solution: Modify the model…
  - Monetary incentives and/or penalties (prices)
  - Restricted domain of preferences (e.g. single-peaked preferences)

Current focus: Introduce money into the mechanism model...

Formally, the outcome of a quasi-linear mechanism is a tuple $\boldsymbol{x} = \{k, t_1, \cdots, t_N\}$, where $k \in \mathcal{K}$ are the set of allocations, and $t_i \in \mathbb{R}$ denotes the money received by player i from the principal.

Definition 14. *The utility of i^{th} agent in a quasi-linear mechanism is of the form*

$$u_i(x, \theta_i) = v_i(k, \theta_i) + t_i, \tag{1}$$

where $v_i(k, \theta_i)$ is the value of allocation k at the i^{th} agent.

Definition 15. *The social function $f(\theta_1, \cdots, \theta_N)$ is also a tuple*

$$\{k(\theta_1, \cdots, \theta_N), t_1(\theta_1, \cdots, \theta_N), \cdots, t_N(\theta_1, \cdots, \theta_N)\}.$$

Definition 16. *A social function* $f = \{k, t_1, \cdots, t_N\}$ *is* ***allocatively efficient*** *if, for each* $\boldsymbol{\theta} \in \boldsymbol{\Theta}$*, we have*

$$k(\boldsymbol{\theta}) \in \arg\max_{k \in \mathcal{K}} \sum_{i=1}^{N} v_i(k, \theta_i) \tag{2}$$

Definition 17. *A social function* $f = \{k, t_1, \cdots, t_N\}$ *is* ***strongly budget balanced*** *if, for each* $\boldsymbol{\theta} \in \boldsymbol{\Theta}$*, we have*

$$\sum_{i=1}^{N} t_i(\boldsymbol{\theta}) = 0, \tag{3}$$

and ***weakly budget balanced*** *if, for each* $\boldsymbol{\theta} \in \boldsymbol{\Theta}$*, we have*

$$\sum_{i=1}^{N} t_i(\boldsymbol{\theta}) \leq 0. \tag{4}$$

Claim 5.1

If there are two or more players, no social choice function in a quasi-linear mechanism is a dictatorship.

Definition 18. *A social function* $f = \{k, t_1, \cdots, t_N\}$ *is* ***ex-post efficient*** *if, for any* $\boldsymbol{\theta} \in \boldsymbol{\Theta}$*, we have*

$$\sum_{i=1}^{N} u_i(f(\boldsymbol{\theta}), \theta_i) \geq \sum_{i=1}^{N} u_i(x, \theta_i), \tag{5}$$

for all $i \in \mathcal{N}$*, and any allocation* $x \in \mathcal{X}$*.*

Claim 5.2

A social choice function $f = (k, t_1, \cdots, t_N)$ is ex-post efficient in quasi-linear environment if and only if it is allocatively efficient and budget balanced.

Definition 19. *A **Groves mechanism** is one whose allocation tuple $\boldsymbol{x} = \{k, t_1, \cdots, t_N\}$ is of the form*

$$\begin{aligned} k^*(\hat{\boldsymbol{\theta}}) &= \arg\max_{k \in \mathcal{K}} \sum_{i \in \mathcal{N}} v_i(k, \hat{\theta}_i), \\ t_i(\hat{\boldsymbol{\theta}}) &= \sum_{j \neq i} v_j(k^*, \hat{\theta}_j) - h_i(\hat{\boldsymbol{\theta}}_{-i}), \end{aligned} \tag{6}$$

where $\hat{\boldsymbol{\theta}} = \{\hat{\theta}_1, \cdots, \hat{\theta}_N\} = s(\boldsymbol{\theta})$ is the profile of revealed types, which may not be the same as the true profile $\boldsymbol{\theta}$.

Theorem 5.1

Groves mechanisms are allocatively efficient and strategy-proof for agents with quasi-linear preferences.

- Groves mechanism is allocatively efficient by the definition of $k^*(\hat{\boldsymbol{\theta}})$.
- HW-1: Prove that Groves mechanisms are strategy-proof for agents with quasi-linear preferences.

Note that the converse is also true!

Theorem 5.2

The Groves mechanisms are the only allocatively efficient and strategy-proof mechanisms for agents with quasi-linear preferences and general valuation functions, amongst all direct-revelation mechanisms.

Definition 20. *A* ***Clarke (Pivotal) mechanism*** *is a Groves mechanism where*

$$h_i(\hat{\boldsymbol{\theta}}_{-i}) = \sum_{j \neq i} v_j(k^*_{-i}(\hat{\boldsymbol{\theta}}_{-i}), \hat{\theta}_j), \tag{7}$$

where the allocation rule $k^*_{-i}(\hat{\boldsymbol{\theta}}_{-i})$ *is defined as*

$$k^*_{-i}(\hat{\boldsymbol{\theta}}_{-i}) = \arg\max_{k \in \mathcal{K}} \sum_{j \neq i} v_j(k, \hat{\theta}_j) \tag{8}$$

Theorem 5.3

Clarke mechanisms are individually rational and ex-post efficient.

Example 1: Clarke-Groves mechanism for single-item auctions.

This auction was first described academically by Vickery in 1961. Hence, Clarke-Groves mechanisms are also called *Vickery-Clarke-Groves (in short, VCG) mechanisms*.

Example 2: VCG mechanisms for multi-item auctions.