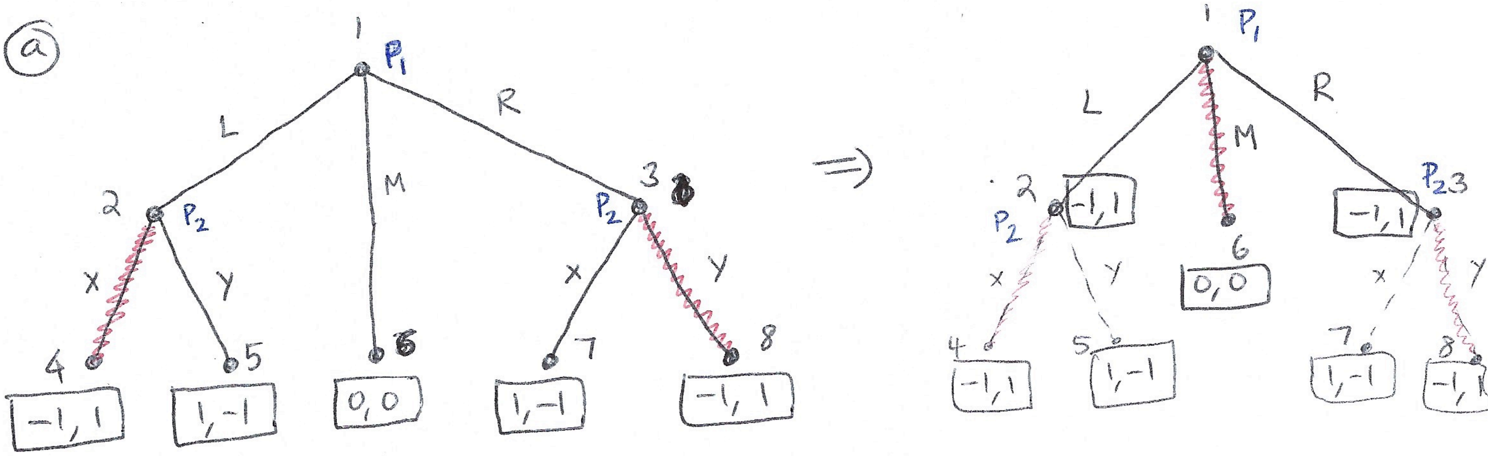
CS 5408: GAME THEORY FOR COMPUTING

HW-3 SOLUTIONS

Prob. 1

(a)



$$\boxed{\text{SPNE} \Rightarrow \quad P_1 : M, \quad P_2 : x/y}$$

(b) Python implementation.

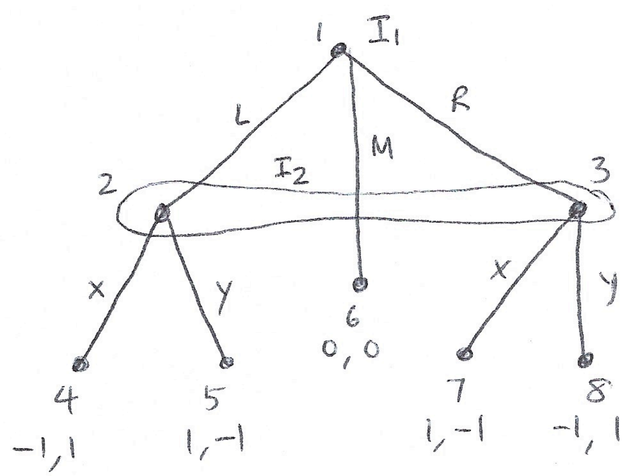
(c)

Let P_2's behavioral strategy in the only information set I_2 be denoted as

$x : \alpha, \quad y : 1-\alpha.$

i.e. $\mathbb{P}_2(x \mid I_2) = \alpha.$

This represents player index.



$|||^{ly}$, assume that player 2 exhibits a belief

$$\mu = \mathbb{P}_2(2 \mid I_2) \quad \text{and} \quad 1-\mu = \mathbb{P}_2(3 \mid I_2)$$

This represents the node label present in I_2.

$\Rightarrow$ Player 2's expected utilities are

$$u_2(X \mid I_2) = \mathbb{P}_2(2 \mid I_2) \cdot (1) + \mathbb{P}_2(3 \mid I_2) \cdot (-1)$$
$$= \mu + (1-\mu)(-1) = 2\mu - 1.$$

$$u_2(Y \mid I_2) = \mathbb{P}_2(2 \mid I_2) \cdot (-1) + \mathbb{P}_2(3 \mid I_2) \cdot (1)$$
$$= -\mu + (1-\mu) = 1 - 2\mu.$$

$$\Rightarrow \quad u_2(I_2) = \mathbb{P}_2(X \mid I_2) \cdot u_2(X \mid I_2) + \mathbb{P}_2(Y \mid I_2) \cdot u_2(Y \mid I_2)$$
$$= \alpha \cdot (2\mu - 1) + (1-\alpha)(1-2\mu)$$
$$= (1-2\alpha)(1-2\mu)$$

$|||^{ly}$, P1's expected utilities are

$$u_1(L) = -1 \cdot \alpha + 1 \cdot (1-\alpha) = 1 - 2\alpha$$

$$u_1(R) = 1 \cdot \alpha + (-1)(1-\alpha) = 2\alpha - 1$$

$$u_1(M) = 0.$$

(#3)

P1's sequential rationality:

⊛ $\alpha > \frac{1}{2} \Rightarrow u_1(L) < u_1(M) < u_1(R)$

$\Rightarrow L \prec M \prec R$.

$\Rightarrow$ choose R.

⊛ $\alpha < \frac{1}{2} \Rightarrow u_1(L) > u_1(M) > u_1(R)$

$\Rightarrow L \succ M \succ R$

$\Rightarrow$ choose L.

⊛ $\alpha = \frac{1}{2} \Rightarrow u_1(L) = u_1(M) = u_1(R)$

$\Rightarrow L \simeq M \simeq R$

$\Rightarrow$ choose any strategy.

P2's sequential rationality

⊛ If $\mu > \frac{1}{2}$, $1-2\mu < 0$

$\Rightarrow u_2(I_2)$ is maximized when $\alpha = 1$.

⊛ If $\mu < \frac{1}{2}$, $1-2\mu > 0$

$\Rightarrow u_2(I_2)$ is maximized when $\alpha = 0$.

⊛ If $\mu = 1/2$, $u_2(I_2) = u_2(M) = 0$

$\Rightarrow$ choose any strategy.

P_2's Consistency

* P_1's sequential rationality $\Rightarrow$ if $\alpha < \frac{1}{2}$, P_1 chooses L
$\Rightarrow \mu = 1$.

But, this violates P_2's sequential rationality
since if $\mu > \frac{1}{2}$, $\alpha = 1$.

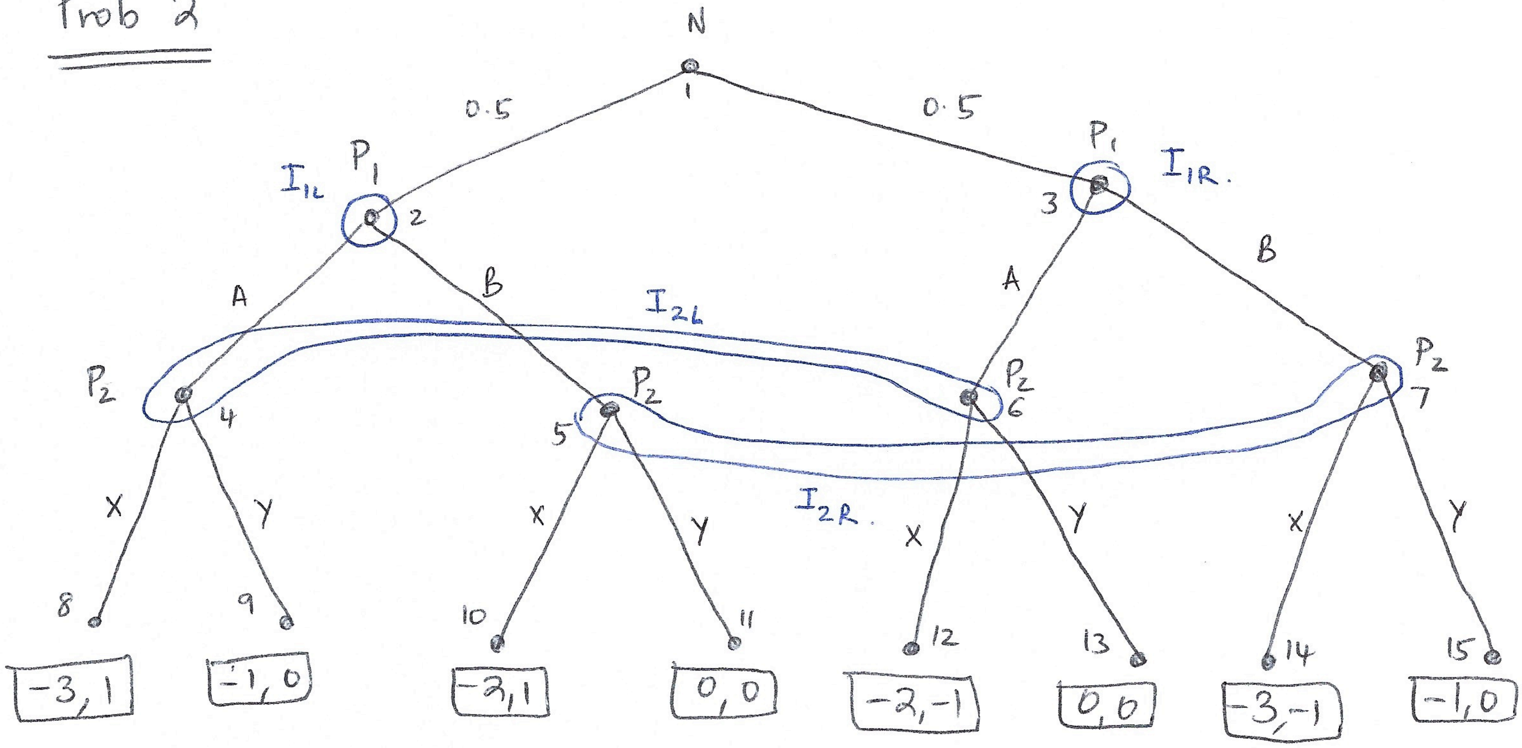
* P_1's sequential rationality $\Rightarrow$ if $\alpha > \frac{1}{2}$, P_1 chooses R
$\Rightarrow \mu = 0$.

Also, a violation to P_2's sequential rationality
since if $\mu < \frac{1}{2}$, $\alpha = 0$.

$\Rightarrow$ Behavioral equilibrium:

* P_1 chooses M.
* P_2 chooses $\{X : \frac{1}{2}, Y : \frac{1}{2}\}$ with $\mu = \frac{1}{2}$.

Prob 2



Let Behavioral Strategies be

$$I_{1L}: \{A : \alpha, B : 1-\alpha\}, \quad I_{1R}: \{A : \beta, B : 1-\beta\}$$

$$I_{2L}: \{X : p, Y : 1-p\}, \quad I_{2R}: \{X : q, Y : 1-q\}$$

and beliefs at player 2 be denoted as

$$I_{2L}: \{4 : \mu, 6 : 1-\mu\}$$

$$I_{2R}: \{5 : \theta, 7 : 1-\theta\}.$$

Then, P2's consistency $\Rightarrow$ $\mu = \dfrac{0.5\alpha}{0.5\alpha + 0.5\beta} = \dfrac{\alpha}{\alpha+\beta}$

$$\theta = \frac{0.5(1-\alpha)}{0.5(1-\alpha)+0.5(1-\beta)} = \frac{1-\alpha}{2-\alpha-\beta}.$$

P_2's expected utilities

$$u_2(X \mid I_{2L}) = \mu \cdot 1 + (1-\mu)(-1) = 2\mu - 1$$

$$u_2(Y \mid I_{2L}) = \mu \cdot 0 + (1-\mu) \cdot 0 = 0.$$

$$u_2(X \mid I_{2R}) = \theta \cdot 1 + (1-\theta)(-1) = 2\theta - 1$$

$$u_2(Y \mid I_{2R}) = \theta \cdot 0 + (1-\theta) \cdot 0 = 0.$$

$$\Rightarrow u_2(I_{2L}) = \mathbb{P}_2(X \mid I_{2L}) \cdot u_2(X \mid I_{2L}) + \mathbb{P}_2(Y \mid I_{2L}) \cdot u_2(Y \mid I_{2L})$$

$$= p \cdot (2\mu - 1) + (1-p) \cdot 0 = p \cdot (2\mu - 1)$$

$$u_2(I_{2R}) = q \cdot (2\theta - 1) + (1-q) \cdot 0 = q(2\theta - 1)$$

Illy, **P_1's expected utilities**

$$u_1(I_{1L}) = \alpha\left[(-3)p + (-1)(1-p)\right] + (1-\alpha)\left[q \cdot (-2) + (1-q) \cdot 0\right]$$

$$= -2q + \alpha(2q - 2p - 1)$$

$$u_1(I_{1R}) = \beta\left[p(-2) + (1-p) \cdot 0\right] + (1-\beta)\left[q \cdot (-3) + (1-q)(-1)\right]$$

$$= -2q - 1 + \beta\left[2q - 2p + 1\right]$$

P_2's sequential rationality

$\underline{\underline{I_{2L}}}$: $u_2(I_{2L}) = p(2\mu - 1)$ is maximized when

- ⊛ $p = 0$ if $\mu < \frac{1}{2}$.
- ⊛ $p = 1$ if $\mu > \frac{1}{2}$
- ⊛ choose any p if $\mu = \frac{1}{2}$.

$\underline{\underline{I_{2R}}}$: $u_2(I_{2R}) = q(2\theta - 1)$ is maximized when

- ⊛ $q = 0$ when $\theta < \frac{1}{2}$.
- ⊛ $q = 1$ if $\theta > \frac{1}{2}$
- ⊛ choose any q if $\theta = \frac{1}{2}$.

P_1's sequential rationality

- ⊛ If $p = 0, q = 0$, P_1 chooses $\alpha = 0, \beta = 1$
  i.e. (BA, YY).
- ⊛ If $p = 0, q = 1$, P_1 chooses $\alpha = 1, \beta = 1$
  i.e (AA, YX)
- ⊛ If $p = 1, q = 0$, P_1 chooses $\alpha = 0, \beta = 0$
  i.e. (BB, XY).
- ⊛ If $p = 1, q = 1$, P_1 chooses $\alpha = 0, \beta = 1$
  i.e. (BA, XX).

#8

Amongst the four possible candidates,

(i) (BA, XY) and (ii) (BA, XX)

are separating strategies profiles.

Note that

(i) (BA, YY) will be an equilibrium if

$$\mu = \frac{\alpha}{\alpha + \beta} = 0 \quad \text{and} \quad \theta = \frac{1-\alpha}{2-\alpha-\beta} = 1.$$

But $\theta = 1$ is a violation since P_2 would choose $q = 0$ only if $\theta < \frac{1}{2}$.

(ii) (BA, XX) will be an equilibrium if

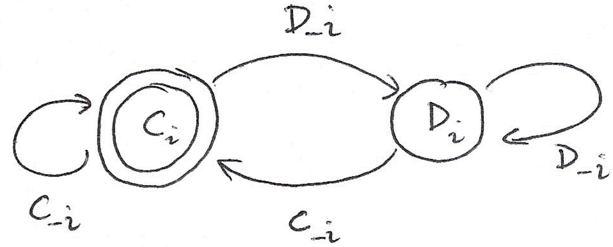
$$\mu = \frac{\alpha}{\alpha + \beta} = 0 \quad \text{and} \quad \theta = \frac{1-\alpha}{2-\alpha-\beta} = 1.$$

But $\mu = 0$ is a violation since P_2 would choose $p = 1$ only if $\mu > \frac{1}{2}$.

FYI, (AA, YX) and (BB, XY) are both valid behavioral (pooling) equilibrium with $\mu = \frac{1}{2}$, $\theta = \frac{1}{2}$.

Prob. 3

Tit-for-tat can be summarized by the following four classes of strategy profiles, given some time T :



CASE-ⓐ :

P_i : $C_1 \quad C_2 \quad \cdots \quad C_T \quad \cdots$

P_{-i} : $C_1 \quad C_2 \quad \cdots \quad C_T \quad \cdots$

follows Tit-for-Tat

CASE-ⓑ :

P_i : $C_1 \quad C_2 \quad \cdots \quad C_{T-1} \quad D_T \quad D_{T+1} \quad \cdots$

P_{-i} : $C_1 \quad C_2 \quad \cdots \quad C_{T-1} \quad C_T \quad D_{T+1} \quad \cdots$

follows Tit-for-Tat

CASE-ⓒ :

P_i : $C_1 \quad C_2 \quad \cdots \quad C_{T-1} \quad D_T \quad C_{T+1} \quad C_{T+2} \quad \cdots$

P_{-i} : $C_1 \quad C_2 \quad \cdots \quad C_{T-1} \quad C_T \quad D_{T+1} \quad C_{T+2} \quad \cdots$

follows Tit-for-Tat.

CASE-ⓓ :

P_i : $C_1 \quad C_2 \quad \cdots \quad C_{T-1} \quad D_T \quad \cdots \quad D_{T+k-1} \quad C_{T+k} \quad \cdots$

P_{-i} : $C_1 \quad C_2 \quad \cdots \quad C_{T-1} \quad C_T \quad D_{T+1} \quad \cdots \quad D_{T+k-1} \quad D_{T+k} \quad C_{T+k+1} \quad \cdots$

follows Tit-for-Tat.

for some $k = 1, \ldots$

Now, let us construct the discounted utilities for each of the above four classes.

P.T.O.

$$u_i^a = u_i\left[\left(c_t^i, c_t^{-i}\right)_{t=1}^{\infty}\right] = \sum_{t=1}^{\infty} \beta^{t-1} \cdot 2$$

$$u_i^b = \sum_{t=1}^{T-1} \beta^{t-1} \cdot 2 + \beta^{T-1} \cdot 3 + \sum_{t=T+1}^{\infty} \beta^{t-1} \cdot 1$$

$$u_i^c = \sum_{t=1}^{T-1} \beta^{t-1} \cdot 2 + \beta^{T-1} \cdot 3 + \beta^{T} \cdot 0 + \sum_{t=T+2}^{\infty} \beta^{t-1} \cdot 2$$

$$u_i^d = \sum_{t=1}^{T-1} \beta^{t-1} \cdot 2 + \beta^{T-1} \cdot 3 + \sum_{t=T+1}^{T+k-1} \beta^{t-1} \cdot 1 + \beta^{T+k-1} \cdot 0 + \sum_{t=T+k+1}^{\infty} \beta^{t-1} \cdot 2$$

Note that, given P_{-i} follows Tit-for-Tat strategy, the only case which seems as if P_i follows Tit-for-Tat is case-(a).

$$\Rightarrow \quad u_i^a \geq u_i^b \quad \text{— (1)}, \qquad u_i^a \geq u_i^c \quad \text{— (2)} \qquad \text{and} \qquad u_i^a \geq u_i^d. \quad \text{— (3)}$$

Inequality (1) $\Rightarrow$

$$\sum_{t=1}^{\infty} \beta^{t-1} \cdot 2 \geq \sum_{t=1}^{T-1} \beta^{t-1} \cdot 2 + \beta^{T-1} \cdot 3 + \sum_{t=T+1}^{\infty} \beta^{t-1} \cdot 1 \quad \text{— (1b)}$$

Simplifying (1b), we obtain

$$\sum_{t=T}^{\infty} \beta^{t-1} \cdot 2 \geq 3 \cdot \beta^{T-1} + \sum_{t=T+1}^{\infty} 1 \cdot \beta^{t-1}$$

$$\Rightarrow \quad 2 \cdot \frac{\beta^{T-1}}{1-\beta} \geq 3 \cdot \beta^{T-1} + \frac{\beta^{T}}{1-\beta}$$

$$\Rightarrow \quad 2 \geq 3(1-\beta) + \beta \quad (\text{or}) \quad 2\beta \geq 1 \quad (\text{or}) \quad \boxed{\beta \geq 1/2}$$

III$^{\text{ly}}$, inequality (2) $\Rightarrow$

$$\sum_{t=1}^{\infty} 2 \cdot \beta^{t-1} \geq \sum_{t=1}^{T-1} 2 \cdot \beta^{t-1} + 3 \cdot \beta^{T-1} + \sum_{t=T+2}^{\infty} 2 \cdot \beta^{t-1}$$

$$\Rightarrow \quad 2 \cdot \beta^{T-1} + 2 \cdot \beta^{T} \geq 3 \cdot \beta^{T-1}$$

$$\Rightarrow \quad 2 + 2\beta \geq 3 \quad \Rightarrow \quad \boxed{\beta \geq \frac{1}{2}}$$

Inequality (3) $\Rightarrow$

$$\sum_{t=1}^{\infty} 2 \cdot \beta^{t-1} \geq \sum_{t=1}^{T-1} 2 \cdot \beta^{t-1} + 3 \cdot \beta^{T-1} + \sum_{t=T+1}^{T+k-1} 1 \cdot \beta^{t-1} + 0 \cdot \beta^{T+k-1} + \sum_{t=T+k+1}^{\infty} 2 \cdot \beta^{t-1}$$

$$\Rightarrow \quad \sum_{t=T}^{T+k} 2 \cdot \beta^{t-1} \geq 3 \cdot \beta^{T-1} + \sum_{t=T+1}^{T+k-1} 1 \cdot \beta^{t-1} \quad \text{—— (3b)}$$

P.T.O.

Inequality 3b $\Rightarrow$

$$\frac{2.\beta^{T-1}.(1-\beta^{k+1})}{1-\beta} \geq 3.\beta^{T-1} + \frac{\beta^{T}(1-\beta^{k-1})}{1-\beta}$$

$$\Rightarrow \quad 2(1-\beta^{k+1}) \geq 3.(1-\beta) + \beta(1-\beta^{k-1})$$

$$\Rightarrow \quad 1 - 2\beta - \beta^{k} + 2.\beta^{k+1} \leq 0$$

$$\Rightarrow \quad (1-2\beta) - \beta^{k}(1-2\beta) \leq 0$$

$$\Rightarrow \quad (1-2\beta)(1-\beta^{k}) \leq 0$$

Since $\beta \in [0,1]$, the above inequality holds true only if $1-2\beta \leq 0 \Rightarrow \boxed{\beta \geq \frac{1}{2}}$

$\therefore$ Tit-for-tat is a NE only when $\beta \geq \frac{1}{2}$.