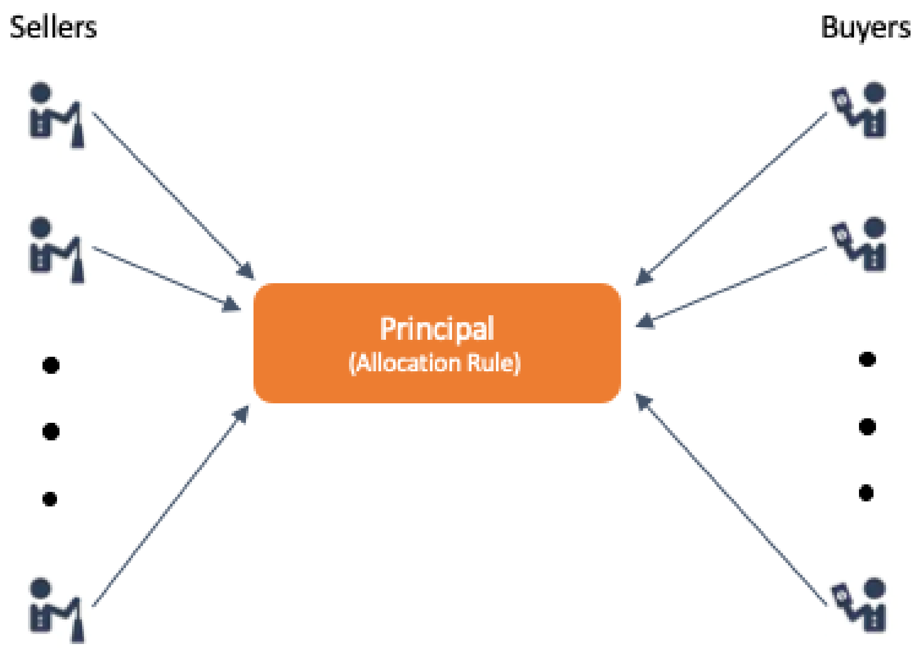
# Markets

Market $\Rightarrow$ $M$ sellers interact with $N$ buyers.




**Examples:**

- Auctions (1 Seller, N Buyers) – *Monopoly*
- Reverse-Auctions (M Sellers, 1 Buyer) – *Monopsony*
- Double Auctions (M Sellers, N Buyers) – *Market Exchange*
- Stable Matching (M Sellers, N Buyers) – *Oligopoly* (Mechanism without money)

# 1 Combinatorial Auctions

- Set of $M$ indivisible items: $\mathcal{M} = \{1, \cdots, M\}$
- Set of $N$ agents (bidders): $\mathcal{N} = \{1, \cdots, N\}$
- Bids placed by $i^{th}$ agent: $(\mathcal{S}_i, b_i)$,

  where $\mathcal{S}_i \subseteq \mathcal{M}$, and $b_i \in \mathbb{R}_+$

**Definition 1.** *Given a bundle of items $\mathcal{S} \in 2^{\mathcal{M}}$, the* ***valuation*** *$v_i(\mathcal{S})$ is the value that the $i^{th}$ bidder obtains if he/she receives $\mathcal{S}$, with the following properties:*

- *A valuation must have free disposal (monotonicity): $v(\mathcal{S}) \leq v(\mathcal{T})$, for every $S \subseteq T$.*
- *Valuation should be normalized: $v(\emptyset) = 0$.*

**Definition 2.** *An* ***allocation*** *of the items among bidders is $\boldsymbol{\mathcal{X}} = \{\mathcal{X}_1, \cdots, \mathcal{X}_N\}$, where $\mathcal{X}_i \cap \mathcal{X}_j = \emptyset$ for every $i \neq j$.*

**Definition 3.** *Given the valuation profile $v_1, \cdots, v_N$, a* ***socially efficient allocation*** *$\boldsymbol{\mathcal{X}}^*$ is the one that maximizes the sum of valuations at all the agents. In other words,*

$$\boldsymbol{\mathcal{X}}^* = \arg\max_{\boldsymbol{\mathcal{X}}} \sum_{i \in \mathcal{N}} v_i(\mathcal{X}_i).$$

Note that the problem of finding a socially efficient allocation is a linear-integer program.

$$
\begin{aligned}
\underset{X}{\text{maximize}} \quad & \sum_{i\in\mathcal{N}} \sum_{\mathcal{S}\subseteq\mathcal{M}} x_{i,\mathcal{S}} \cdot v_i(\mathcal{S}) \\
\text{subject to} \quad & 1.\ \sum_{\mathcal{S}\subseteq\mathcal{M}} x_{i,\mathcal{S}} \leq 1,\ \forall\, i \in \mathcal{N} \\
& 2.\ \sum_{i\in\mathcal{N},\mathcal{S}|j\in\mathcal{S}} x_{i,\mathcal{S}} \leq 1,\ \forall\, j \in \mathcal{M} \\
& 3.\ x_{i,\mathcal{S}} \in \{0,1\},\ \forall\, i \in \mathcal{N},\ \mathcal{S} \subseteq \mathcal{M}.
\end{aligned}
\tag{P1}
$$

where the matrix $X$ is a compilation of the allocation variables $x_{i,S}$, which is identical in spirit to our original notation for allocation $\boldsymbol{\mathcal{X}}^*$.

## 1.1 Single Minded Case

**Definition 4.** *A valuation $v$ is called **single-minded** if there exists a bundle $\mathcal{S}^*$ and a value $v^*$ such that*

$$
v(\mathcal{S}) = \begin{cases} v^*, & \textit{for all } S \supseteq S^* \\ 0, & \textit{otherwise.} \end{cases}
$$

*A single minded bid is the pair $(\mathcal{S}^*, v^*)$.*

**Proposition 1.1**

The decision problem of whether the optimal allocation has a social welfare of at least $k$ is NP-Complete.

### 1.1.1 Approximation

**Proposition 1.2**

Approximating the optimal allocation among single-minded bidders to within a factor better than $O(\sqrt{M})$ is NP-Complete.

### 1.1.2 Greedy Approach

GREEDY-SINGLE-MINDED-AUCTION$(\boldsymbol{S}^*, \boldsymbol{v}^*, )$

// Order agents according to their bids...

Reorder the agents such that $\frac{v_1^*}{\sqrt{|\mathcal{S}_1^*|}} \geq \cdots \geq \frac{v_N^*}{\sqrt{|\mathcal{S}_N^*|}}$

// Initialize the set of winners and payments

$\mathcal{W} = 0$, $\boldsymbol{p} = \mathbf{0}$

**for** $i = 1$ **to** $N$

    **if** $\mathcal{S}_i^* \cap \left( \bigcup_{j \in \mathcal{W}} \mathcal{S}_j^* \right) = \emptyset$

        $\mathcal{W} = \mathcal{W} \cup \{i\}$

**for** $i \in \mathcal{W}$

    $j = \arg\min_{k \in \mathcal{N}} \{\mathcal{S}_k^* \cap \mathcal{S}_i^* \neq \emptyset$ and $\mathcal{S}_{k'}^* \cap \mathcal{S}_i^* = \emptyset,\ \forall k' < k, k' \neq i.\}$

    **if** $j \neq \emptyset$

        $p_i = \frac{v_j^*}{\sqrt{|\mathcal{S}_j^*| / |\mathcal{S}_i^*|}}$

**return** $\mathcal{W}, \boldsymbol{p}$

**Proposition 1.3**

Let $OPT$ be an allocation with maximum value $\sum_{i \in OPT} v_i^*$, and let $\mathcal{W}$ denote the output of GREEDY-SINGLE-MINDED-AUCTION, then

$$\sum_{i \in OPT} v_i^* \leq \sqrt{M} \sum_{i \in \mathcal{W}} v_i^*.$$

# 2 Profit Maximizing Mechanismss

So far, we designed mechanisms that maximized social welfare (*a.k.a.* surplus, which is sum of valuations of all players).

**What if the principal is an agent (auctioneer)?**

- Bidder Set: $\mathcal{N} = \{1, \cdots, N\}$
- Valuation: $\boldsymbol{v} = \{v_1, \cdots, v_N\}$
- Bid: $\boldsymbol{b} = \{b_1, \cdots, b_N\}$
- Allocation: $\boldsymbol{x} = \{x_1, \cdots, x_N\}$
- Price: $\boldsymbol{p} = \{p_1, \cdots, p_N\}$

**Definition 5.** *The auctioneer's* ***profit*** *is defined as*

$$\textit{Profit} = \sum_{i \in \mathcal{N}} p_i - c(\boldsymbol{x}), \tag{1}$$

*where* $c(\boldsymbol{x})$ *denotes the inherent cost in producing the outcome* $\boldsymbol{x}$.

**Goal:** A truthful mechanism that maximizes profit!

## 2.1 Single-Item, Single-Bidder Auction

- Profit maximization is not trivial! Why?
- For now, assume $c(\boldsymbol{x}) = 0$ for any allocation $\boldsymbol{x}$.
- Vickery auction for one bidder:

- Profit-maximizing auction for one bidder:

**Definition 6.** *A mechanism is **truthful** in expecation if and only if, for all $i$, $b_i$ and $\boldsymbol{b}_{-i}$, the $i^{th}$ agent's expected utility for bidding his/her true valuation $v_i$ is at least their expected utility for bidding any other value, i.e.,*

$$u_i(v_i, \boldsymbol{b}_{-i}) \geq u_i(b_i, \boldsymbol{b}_{-i}), \quad \forall i, b_i,\ \boldsymbol{b}_{-i}. \tag{2}$$

**Theorem 2.1**

A mechanism is truthful in expectation if and only if, for any agent $i$ and any fixed choice of bids by the other agents $\boldsymbol{b}_{-i}$,

1. $x_i(b_i, \boldsymbol{b_{-i}})$ is monotonically increasing,
2. $p_i(b_i, \boldsymbol{b_{-i}}) = b_i \cdot x_i(b_i, \boldsymbol{b_{-i}}) - \int_0^{b_i} x_i(z, \boldsymbol{b_{-i}}) dz$

(Converse proof for HW assignment)

Note: If the allocation $x_i(b_i, \boldsymbol{b_{-i}})$ is deterministic, then there exists a threshold $t_i(\boldsymbol{b_{-i}})$ such that

$$x_i(b_i, \boldsymbol{b_{-i}}) = \begin{cases} 1, & \text{if } b_i \geq t_i(\boldsymbol{b_{-i}}), \\ 0, & \text{otherwise.} \end{cases} \tag{3}$$

Then, the corresponding price for any $b_i > t_i$ is given by

$$p_i(b_i, \boldsymbol{b_{-i}}) = b_i - \int_{t_i}^{b_i} dz = t_i \tag{4}$$

## 2.2 Bayesian-Optimal Mechanisms

- Assume valuation $v_i$ is drawn independently at random from a cumulative distribution $F_i$ (i.e., $F_i(z) = \mathbb{P}(v_i \leq z)$).
- Let the corresponding density function be denoted as $f_i(z) = \dfrac{dF_i(z)}{dz}$.

**Definition 7.** *The virtual valuation of the $i^{th}$ bidder with valuation $v_i$ is defined as*

$$\phi_i(v_i) = v_i - \frac{1 - F_i(v_i)}{f_i(v_i)}. \tag{5}$$

**Lemma 2.1**

Given a bid profile $\boldsymbol{b}_{-i}$ for all players except the $i^{th}$ bidder, the expected payment of the $i^{th}$ bidder in a truthful mechanism is given by

$$\mathbb{E}_{b_i}\left[p_i(b_i, \boldsymbol{b}_{-i})\right] = \mathbb{E}_{b_i}\left[\phi_i(b_i)x_i(b_i, \boldsymbol{b}_{-i})\right]. \tag{6}$$

**Theorem 2.2**

The expected profit of any truthful mechanism $\mathcal{M}$ with independent bidder valuations is equal to its expected virtual surplus, i.e.

$$\mathbb{E}_{\boldsymbol{v}}\left[\sum_{i=1}^{N} p_i(\boldsymbol{v})\right] = \mathbb{E}_{\boldsymbol{v}}\left[\sum_{i=1}^{N} \phi_i(v_i)x_i(\boldsymbol{v}) - c(\boldsymbol{x}(\boldsymbol{v}))\right]. \tag{7}$$

Given the bids $\boldsymbol{b}$ and joint distribution $\boldsymbol{F} = F_1 \times \cdots \times F_N$, Myerson's optimal auction can be summarized as follows:

MYERSONOPTIMALAUCTION$(\boldsymbol{b}, \boldsymbol{F})$

Compute virtual bids $b'_i = \phi_i(b_i)$.
$(\boldsymbol{x}', \boldsymbol{p}') = VCG(\boldsymbol{b}')$
$\boldsymbol{x} = \boldsymbol{x}'$
**for** $i = 1$ **to** $N$
    $p_i = \phi_i^{-1}(p'_i)$
**return** $\boldsymbol{x}, \boldsymbol{p}$

# 3 Stable Matching

One of the most successful mechanisms without money...

**Applications:**

- Matching residents to medical schools
- Kidney exchange

**Notation:**

- Two types of agents: Men and Women
- Set of men: $\mathcal{M}$, Set of women: $\mathcal{W}$.
- Each $m \in \mathcal{M}$ has a preference order $\pi_m$ over $\mathcal{W}$.
- Each $w \in \mathcal{W}$ has a preference order $\tau_w$ over $\mathcal{M}$.
- If an agent prefers to stay single, include a dummy agent in the other set $\Rightarrow |\mathcal{M}| = |\mathcal{W}| = n$.

**Definition 8.** *A* ***matching*** $\mu$ *is an assignment of men to women such that each man is assigned to at most one woman and vice versa.*
*Let* $\mu(m)$ *denote the woman assigned to* $m$*, and* $\mu(w)$ *denote the man assigned to* $w$*.*

**Definition 9.** *A matching is said to be **unstable** if there exists any two men $m$, $m'$ and two women $w$, $w'$ such that:*

- *$m$ is matched to $w$,*
- *$m'$ is matched to $w'$,*
- *$\pi_m : w' \succ w$ and $\tau_{w'} : m \succ m'$*

*In such a case, the pair $(m, w')$ is called a **blocking pair**.*

**Definition 10.** *A matching is said to be **stable** if there are no blocking pairs.*

DEFERREDACCEPTANCE$(\boldsymbol{\pi}, \boldsymbol{\tau})$

Initialize $\mu = \emptyset$, // Tentative matching
Initialize $R = \boldsymbol{\pi}$ // Set of non-rejected women
**while** $\mu(m) = \emptyset$
    $m$ attempts to match with her favorite $w \in R(m)$
    **if** $\mu(w) = \emptyset$
        $\mu(m) = w$, $\mu(w) = m$
    **elseif** $\mu(w) = m'$
        **if** $\tau_w : m \succ m'$
            $R(m) = R(m) - \{w\}$ // $\mu(m') = \emptyset$
        **else**
            $R(m) = R(m) - \{w\}$
**return** $\mu$.

**Theorem 3.1**

The deferred acceptance algorithm terminates in a stable matching after at most $n^2$ iterations, where $n$ is the number of agents on each side.

**Definition 11.** *A matching $\mu$ is **male-optimal** if there is no stable matching $\nu$ such that $\pi_m : \nu(m) \succ \mu(m)$ for at least one $m \in \mathcal{M}$.*

**Theorem 3.2**

The stable matching proposed by (male-proposed) deferred acceptance algorithm is male-optimal.

**Theorem 3.3**

The male-proposed deferred acceptance algorithm is strategy-proof for the males.

(FYI, it is not strategy-proof for the females... this is your HW problem.)