CS 5408 : GAME THEORY FOR COMPUTING

HW-5 SOLUTIONS

Prob 1

(a) Given the characteristic fn. game $\Gamma = (\mathcal{N}, v)$ where $\mathcal{N} = \{A, B, C, D\}$ and

total # representatives in C.

Party	# repres.
A	45
B	25
C	15
D	15

$$v(C) = \begin{cases} 10^{12} & \text{if } |C| \geq 51 \\ 0 & \text{otherwise.} \end{cases}$$

the marginal contributions in each permutation of N are given by

$\pi_1 = \{A, B, C, D\} \Rightarrow \Delta_1(A) = 0, \ \Delta_1(B) = 10^{12}, \ \Delta_1(C) = 0, \ \Delta_1(D) = 0$

$\pi_2 = \{A, B, D, C\} \Rightarrow \Delta_2(A) = 0, \ \Delta_2(B) = 10^{12}, \ \Delta_2(C) = 0, \ \Delta_2(D) = 0$

$\pi_3 = \{A, C, B, D\} \Rightarrow \Delta_3(A) = 0, \ \Delta_3(B) = 0, \ \Delta_3(C) = 10^{12}, \ \Delta_3(D) = 0$

$\pi_4 = \{A, C, D, B\} \Rightarrow \Delta_4(A) = 0, \ \Delta_4(B) = 0, \ \Delta_4(C) = 10^{12}, \ \Delta_4(D) = 0$

$\pi_5 = \{A, D, B, C\} \Rightarrow \Delta_5(A) = 0, \ \Delta_5(B) = 0, \ \Delta_5(C) = 0, \ \Delta_5(D) = 10^{12}$

$\pi_6 = \{A, D, C, B\} \Rightarrow \Delta_6(A) = 0, \ \Delta_6(B) = 0, \ \Delta_6(C) = 0, \ \Delta_6(D) = 10^{12}$

$\pi_7 = \{B, A, C, D\} \Rightarrow \Delta_7(A) = 10^{12}, \ \Delta_7(B) = 0, \ \Delta_7(C) = 0, \ \Delta_7(D) = 0$

$\pi_8 = \{B, A, D, C\} \Rightarrow \Delta_8(A) = 10^{12}, \ \Delta_8(B) = 0, \ \Delta_8(C) = 0, \ \Delta_8(D) = 0$

$\pi_9 = \{B, C, A, D\} \Rightarrow \Delta_9(A) = 10^{12}, \Delta_9(B) = 0, \Delta_9(C) = 0, \Delta_9(D) = 0$

$\pi_{10} = \{B, C, D, A\} \Rightarrow \Delta_{10}(A) = 0, \Delta_{10}(B) = 0, \Delta_{10}(C) = 0, \Delta_{10}(D) = 10^{12}$

$\pi_{11} = \{B, D, A, C\} \Rightarrow \Delta_{11}(A) = 10^{12}, \Delta_{11}(B) = 0, \Delta_{11}(C) = 0, \Delta_{11}(D) = 0$

$\pi_{12} = \{B, D, C, A\} \Rightarrow \Delta_{12}(A) = 0, \Delta_{12}(B) = 0, \Delta_{12}(C) = 10^{12}, \Delta_{12}(D) = 0$

$\pi_{13} = \{C, A, B, D\} \Rightarrow \Delta_{13}(A) = 10^{12}, \Delta_{13}(B) = 0, \Delta_{13}(C) = 0, \Delta_{13}(D) = 0$

$\pi_{14} = \{C, A, D, B\} \Rightarrow \Delta_{14}(A) = 10^{12}, \Delta_{14}(B) = 0, \Delta_{14}(C) = 0, \Delta_{14}(D) = 0$

$\pi_{15} = \{C, B, A, D\} \Rightarrow \Delta_{15}(A) = 10^{12}, \Delta_{15}(B) = 0, \Delta_{15}(C) = 0, \Delta_{15}(D) = 0$

$\pi_{16} = \{C, B, D, A\} \Rightarrow \Delta_{16}(A) = 0, \Delta_{16}(B) = 0, \Delta_{16}(C) = 0, \Delta_{16}(D) = 10^{12}$

$\pi_{17} = \{C, D, A, B\} \Rightarrow \Delta_{17}(A) = 10^{12}, \Delta_{17}(B) = 0, \Delta_{17}(C) = 0, \Delta_{17}(D) = 0$

$\pi_{18} = \{C, D, B, A\} \Rightarrow \Delta_{18}(A) = 0, \Delta_{18}(B) = 10^{12}, \Delta_{18}(C) = 0, \Delta_{18}(D) = 0$

$\pi_{19} = \{D, A, B, C\} \Rightarrow \Delta_{19}(A) = 10^{12}, \Delta_{19}(B) = 0, \Delta_{19}(C) = 0, \Delta_{19}(D) = 0$

$\pi_{20} = \{D, A, C, B\} \Rightarrow \Delta_{20}(A) = 10^{12}, \Delta_{20}(B) = 0, \Delta_{20}(C) = 0, \Delta_{20}(D) = 0$

$\pi_{21} = \{D, B, A, C\} \Rightarrow \Delta_{21}(A) = 10^{12}, \Delta_{21}(B) = 0, \Delta_{21}(C) = 0, \Delta_{21}(D) = 0$

$\pi_{22} = \{D, B, C, A\} \Rightarrow \Delta_{22}(A) = 0, \Delta_{22}(B) = 0, \Delta_{22}(C) = 10^{12}, \Delta_{22}(D) = 0$

$\pi_{23} = \{D, C, A, B\} \Rightarrow \Delta_{23}(A) = 10^{12}, \Delta_{23}(B) = 0, \Delta_{23}(C) = 0, \Delta_{23}(D) = 0.$

$\pi_{24} = \{D, C, B, A\}. \Rightarrow \Delta_{24}(A) = 0, \Delta_{24}(B) = 10^{12}, \Delta_{24}(C) = 0, \Delta_{24}(D) = 0$

$\therefore$ The Shapley values are given by

$$u_A = \frac{1}{4!}\sum_{j=1}^{24} \Delta_j(A) = \frac{1}{24}\left(12\times 10^{12}\right) = \frac{1}{2}\times 10^{12}.$$

$$u_B = \frac{1}{4!}\sum_{j=1}^{24} \Delta_j(B) = \frac{1}{24}\left(4\times 10^{12}\right) = \frac{1}{6}\times 10^{12}$$

$$u_C = \frac{1}{4!}\sum_{j=1}^{24} \Delta_j(C) = \frac{1}{24}\left(4\times 10^{12}\right) = \frac{1}{6}\times 10^{12}.$$

$$u_D = \frac{1}{4!}\sum_{j=1}^{24} \Delta_j(D) = \frac{1}{24}\left(4\times 10^{12}\right) = \frac{1}{6}\times 10^{12}.$$

(b) Core of $\Gamma = (N, v)$ $\Rightarrow$ the set of all stable utility profiles that satisfy

$\circledast$ $u_A + u_B + u_C + u_D = 10^{12}$

$\circledast$ $\sum_{i\in G} u_i \geq v(G)$ for all $G \subset N$.

Consider two of these conditions:

$u_A + u_B \geq 10^{12}$ and $u_A + u_C \geq 10^{12}$

$\Rightarrow$ Summing them, we get $2u_A + u_B + u_C \geq 2.10^{12}$ — (1)

Also, consider the inequality $u_A + u_B + u_C \geq 10^{12}$,

which is equivalent to $2u_A + u_B + u_C \geq 10^{12} + u_A$. — (2)

Comparing inequalities 1 and 2, we have two cases.
(esp. the RHS)

CASE-1: $10^{12} + u_A \geq 2 \cdot 10^{12}$

In such a case, $u_A = 10^{12}$.

$\Rightarrow u_B + u_C + u_D = 0$.

But, this violates one of the conditions of core,
which is $u_B + u_C + u_D \geq 10^{12}$.

CASE-2: $10^{12} + u_A < 2 \cdot 10^{12} \Rightarrow u_A < 10^{12}$.

Say, $u_A = \varepsilon \times 10^{12}$ where $\varepsilon \in [0, 1)$.

$\Rightarrow u_B + u_C + u_D = (1-\varepsilon) \times 10^{12} < 10^{12}$

This also violates the condition $u_B + u_C + u_D \geq 10^{12}$.

$\Rightarrow$ There exists no u_A, u_B, u_C, u_D that satisfies all the stability conditions.

$\Rightarrow$ The core is empty.

NOTE: If we normalize the utilities by 10^{12}, the game Γ reduces into a simple game.
Since there is no veto player in this game, the core has to be empty.