Missouri University of Science & Technology	Department of Computer Science
Fall 2023	**CS 5408: Game Theory for Computing**

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Homework 1: Decision Theory

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Instructor: *Sid Nadendla*	**Due:** *September 8, 2023*

Problem 1   Lotteries, Preferences & Axioms   *4 pts.*

Consider a choice experiment where an agent knows *a priori* the preference order on four lotteries f_1, f_2, f_3 and f_4. Let this preference ordering be $f_1 \succsim_E f_2 \succsim_E f_3 \succsim_E f_4$, which is evaluated based on some event $E \in \mathcal{E}$. Suppose that the agent also exhibits indifference between the following pairs of lotteries:

- $f_2 \sim_E 0.4f_1 + 0.6f_4$

- $f_3 \sim_E 0.3f_1 + 0.7f_4$

Let f and g denote two new lotteries, which are defined as

- $f = 0.2f_1 + 0.4f_2 + 0.2f_3 + 0.2f_4$,

- $g = 0.25f_1 + 0.25f_2 + 0.25f_3 + 0.25f_4$.

Assuming that the first 8 preference axioms of decision theory (presented in slides 12-13 in *Topic 1: Decision Theory* lecture notes) are held true by the agent, prove that $g \succsim_E f$.

Problem 2   Expected Utility Maximization   *4 pts.*

Suppose that Alice won a competition. As a reward, she was asked to choose one of the following two options:

<u>Option 1</u>: A laptop with probability 1.

<u>Option 2</u>: A tablet with probability 0.3, or a motorcycle with probability 0.7.

<u>Option 3</u>: A cell phone with probability 0.3, or a laptop with probability 0.2, or a motorcycle with probability 0.5.

Assuming that Alice maximizes her expected utility, if she prefers

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cell phone $\prec$ tablet $\prec$ laptop $\prec$ motorcycle,

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find the preference order over Options 1, 2 and 3.

Problem 3   Limitations of EUM   *4 pts.*

Daniel Ellsberg proposed the following thought-experiment[1] (known as *Ellsberg Paradox*) in 1961. An urn contains 90 balls, 30 of which are red. The other 60 are black or yellow, in unknown proportions. One ball will be drawn randomly from the urn. In this experiment, consider yourself as a decision maker.

(a) First, you must make a choice between Gamble A and Gamble B:

- **Gamble A:** You win $100 if the ball is red.
- **Gamble B:** You win $100 if the ball is black.

Which would you choose, and why?

(b) Next, you must make a choice between Gamble C and Gamble D:

- **Gamble C:** You win $100 if the ball is either red or yellow.
- **Gamble D:** You win $100 if the ball is either black or yellow.

Which would you choose, and why?

(c) Most people strongly prefer Gambles A and D over Gambles B and C respectively. Explain why this pattern of choices violates expected utility theory.

(d) Implement Ellsberg Paradox in Python using the template Jupyter Notebook labeled "HW1_3d_CS5408_FS2023.ipynb" provided to you in your Gitlab repositories.

Problem 4   Prospect Theory   *4 pts.*

Let your utility function for gains and losses be

$$u(x) = \begin{cases} x, & \text{if } x \geq 0 \\ \lambda x, & \text{if } x < 0, \end{cases}$$

where $\lambda > 0$ is an unknown parameter and your probability weighting function is $w(p) = p$. Consider the following two gambles:

- $P = \{\text{win } \$150 \text{ with probability } 0.5; \text{ loose } \$100 \text{ with probability } 0.5\}$

- $Q = \{\text{win } \$200 \text{ with probability } 0.5; \text{ loose } \$100 \text{ with probability } 0.5\}$

Suppose that you have the following preferences:

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[1]A similar experiment was also proposed by John Maynard Keynes in 1921.

- prefer getting nothing for sure over the gamble P,

- prefer the gamble Q over getting nothing for sure.

Then, what is the range of λ that is consistent with the choices above?

Problem 5 Domination   *4 pts.*

Suppose an agent is presented with a choice set $\mathcal{X} = \{\alpha, \beta, \gamma\}$, where the choice experiment can take the states $\Omega = \{t_1, t_2, t_3\}$. If the utilities at the agent are given as shown in the table below,

Decision	State t_1	State t_2	State t_3
α	4	1	-3
β	3	-2	4
γ	0	1	6

(a) Find the region in $\Delta(\Omega)$ in which α is optimal.

(b) Find the region in $\Delta(\Omega)$ in which β is optimal.

(c) Find the region in $\Delta(\Omega)$ in which γ is optimal.

(d) Implement this experiment in Python using the template Jupyter Notebook labeled "HW1_5d_CS5408_FS2023.ipynb" provided in your Gitlab repositories and validate your theoretical findings in (a)-(c).

Problem 6 St. Petersburg Paradox *(Extra Credit: 2 pt.)*

Model the choice experiment in St. Petersburg paradox formally as a lottery, i.e. clearly define the states, their corresponding probabilities, choices and a conditional distribution on the choice set given the state.

Problem 7 Allias Paradox   *(Extra Credit: 2 pt.)*

Prove that $1A \succ 1B$ and $2B \succ 2A$ violates expected utility maximization (EUM) framework.