

Homework 1: Decision Theory

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Due: September 13, 2021

Problem 1 Lotteries, Preferences & Axioms 4 pts.

Consider a choice experiment where an agent is presented with four lotteries f_1 , f_2 , f_3 and f_4 . Assume that the agent's preference ordering be $f_1 \succsim_E f_2 \succsim_E f_3 \succsim_E f_4$, which is evaluated based on some event $E \in \mathcal{E}$. Suppose that the agent also exhibits indifferences between the following pairs of lotteries:

- $f_2 \sim_E 0.6f_1 + 0.4f_4$
- $f_3 \sim_E 0.2f_1 + 0.8f_4$

Assuming that the agent satisfies all the 9 preference axioms of decision theory (presented in slides 12-13 in *Topic 1: Decision Theory* lecture notes), if lotteries f and g are defined as

- $f = 0.15f_1 + 0.50f_2 + 0.15f_3 + 0.20f_4$,
- $g = 0.25f_1 + 0.25f_2 + 0.25f_3 + 0.25f_4$,

prove that $f \succsim_E g$.

Problem 2 Expected Utility Maximization 4 pts.

A company must decide its investments between three mutually exclusive projects:

- Project P provides a net profit of \$50 million with a probability 0.75, and a net loss of \$10 million with probability 0.25.
- Project Q provides a net profit of \$100 million with a probability 0.6, and a net loss of \$40 million with probability 0.4.
- Project R provides a net profit of \$200 million with a probability 0.5, and a net loss of \$100 million with probability 0.5.

Suppose that the CEO of the corporation has expected utility preferences and is risk averse (i.e. concave utility function, e.g. Bernoulli's logarithmic utility.). Can you determine his preferences over P , Q and R ?

[Hint: Note that logarithms are not defined for the losses, as they are negative quantities. Therefore, if you prefer working out with an example, one way out is to consider a shifted-logarithm $\log(x + a)$, where losses beyond a are treated as infinite in value. However, for full credits, you should prove this result for any concave utility function in this universe.]

Problem 3 Limitations of EUM

4 pts.

Daniel Ellsberg proposed the following thought-experiment¹ (known as *Ellsberg Paradox*) in 1961. An urn contains 90 balls, 30 of which are red. The other 60 are black or yellow, in unknown proportions. One ball will be drawn randomly from the urn. In this experiment, consider yourself as a decision maker.

(a) First, you must make a choice between Gamble A and Gamble B:

- **Gamble A:** You win \$100 if the ball is red.
- **Gamble B:** You win \$100 if the ball is black.

Which would you choose, and why?

(b) Next, you must make a choice between Gamble C and Gamble D:

- **Gamble C:** You win \$100 if the ball is either red or yellow.
- **Gamble D:** You win \$100 if the ball is either black or yellow.

Which would you choose, and why?

(c) Most people strongly prefer Gambles A and D over Gambles B and C respectively. Explain why this pattern of choices violates expected utility theory.

(d) Implement Ellsberg Paradox in Python using abstract classes within the Jupyter Notebook provided to you in your Gitlab repositories. Rename your notebook as “<last_name>_FS2021_CS5408_HW1_3d.ipynb”.

Problem 4 Prospect Theory

4 pts.

Let your utility function for gains and losses be

$$u(x) = \begin{cases} x, & \text{if } x \geq 0 \\ \lambda x, & \text{if } x < 0, \end{cases}$$

where $\lambda > 0$ is an unknown parameter and your probability weighting function is $w(p) = p$. Consider the following two gambles:

- $P = \{\text{win \$150 with probability 0.5; lose \$100 with probability 0.5}\}$
- $Q = \{\text{win \$200 with probability 0.5; lose \$100 with probability 0.5}\}$

¹A similar experiment was also proposed by John Maynard Keynes in 1921.

Suppose that you have the following preferences:

- prefer getting nothing for sure over the gamble P ,
- prefer the gamble Q over getting nothing for sure.

Then, what is the range of λ that is consistent with the choices above?

Problem 5 Domination

4 pts.

Suppose an agent is presented with a choice set $\mathcal{X} = \{\alpha, \beta, \gamma\}$, where the choice experiment can take the states $\Omega = \{t_1, t_2, t_3\}$. If the utilities at the agent are given as shown in the table below,

Decision	State t_1	State t_2	State t_3
α	4	1	-3
β	3	2	5
γ	0	1	6

- Find the region in $\Delta(\Omega)$ in which α is optimal.
- Find the region in $\Delta(\Omega)$ in which β is optimal.
- Find the region in $\Delta(\Omega)$ in which γ is optimal.
- Implement this experiment in Python using the Jupyter Notebooks provided in your Gitlab repositories and validate your theoretical findings in (a)-(c). Rename your Jupyter Notebook as “<last_name>_FS2021_CS5408_HW1_5d.ipynb”.

Problem 6 St. Petersburg Paradox (*Extra Credit: 1 pt.*)

Model the choice experiment in St. Petersburg paradox formally as a lottery, i.e. clearly define the states, their corresponding probabilities, choices and a conditional distribution on the choice set given the state.

Problem 7 Allias Paradox

(Extra Credit: 2 pt.)

Prove that $1A \succ 1B$ and $2B \succ 2A$ violates expected utility maximization (EUM) framework.