Topic 3: Advanced Solution Concepts

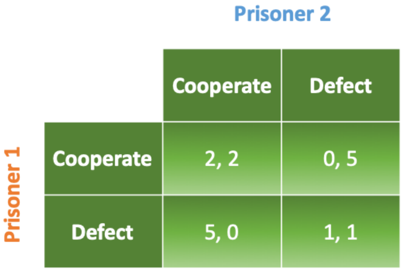
Outcomes & Objectives

- Be proficient in solving games with communication and/or multiple equilibria.
  - Model contracts to construct correlated equilibria, in order to improve Nash equilibria.
  - Use focal points to study agents' strategies in the presence of multiple equilibria.
  - Develop contracts based on social-choice functions, which are inspired from the notion of a focal point.
- Be proficient with refinements of Nash equilibrium.
  - Identify when/why agents may not follow Nash/correlated equilibria.
  - Develop the notion of ϵ-Nash equilibrium to capture the deviations from Nash equilibrium based on agents' satisficing behavior.
  - Study the effects of agents' errors in picking their strategies on equilibrium using the notion of Trembling-hand perfect equilibrium.
  - Model and analyze evolution of life using the notion of evolutionarily stable equilibrium based on simple rules of thumb.
- Be proficient in solving normal-form Bayesian games.

Revising Prisoner's Dilemma...

Two prisoners involved in the same crime are being interrogated simultaneously in separate rooms. They can either cooperate or defect with the interrogators.

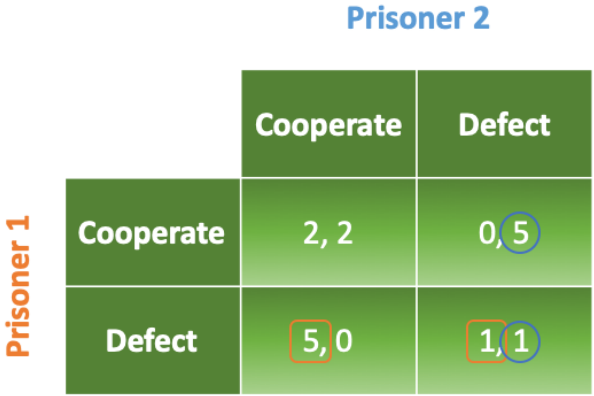
- $\mathcal{N} = \{P_1, P_2\}$
- $\mathcal{C} = \{C, D\} \times \{C, D\}$
- $\mathcal{U} = \{u_1, u_2\}$, where $u_i : \mathcal{C}_i \rightarrow \mathbb{R}$, as shown in the matrix below.



Prisoner 2

		Cooperate	Defect
Prisoner 1	Cooperate	2, 2	0, 5
	Defect	5, 0	1, 1

Revising Prisoner's Dilemma... (cont...)



		Prisoner 2	
		Cooperate	Defect
Prisoner 1	Cooperate	2, 2	0, 5
	Defect	5, 0	1, 1

- Nash equilibrium (D, D) is certainly inferior to (C, C).
- Can we improve the solution of this game?

Modified Prisoner's Dilemma

Assume the following:

- Players can talk to each other.
- Players can also sign binding contracts to coordinate their strategies.

Say, a lawyer (mediator) approaches with the following contract:

> *We, the undersigned, promise to choose (C, C) if this contract is signed by both the players. If it is signed by only one player, then he/she will choose D.*

This is a new game:

		Prisoner 2		
		Cooperate	**Defect**	**Sign**
Prisoner 1	**Cooperate**	2, 2	0, 5	0, 5
	Defect	5, 0	1, 1	1, 1
	Sign	5, 0	1, 1	2, 2

Modified Prisoner's Dilemma (cont...)

Prisoner 2

Prisoner 1

	Cooperate	Defect	Sign
Cooperate	2, 2	0, 5	0, 5
Defect	5, 0	1, 1	1, 1
Sign	5, 0	1, 1	2, 2

- Nash equilibria: (D, D) and (S, S)
- Nash equilibrium (S, S) is comparable to (C, C).

Modified Prisoner's Dilemma 2

Say, a lawyer (mediator) approaches with the primary contract:

> *We, the undersigned, promise to choose (C, C) if this contract is signed by both the players. If it is signed by only one player, then he/she will choose D.*

The lawyer also approaches with a secondary contract:

> *We, the undersigned, promise to choose Strategy 'R' (stated below) if both contracts (primary and secondary) are signed by both the players. If it is signed by only one player, then he/she will choose D.*
> *Strategy 'R': Toss an unbiased coin, and choose (C, D) if the outcome is heads. Else, choose (D, C).*

This is a new game:

		Prisoner 2			
		Cooperate	**Defect**	**Sign 1**	**Sign 2**
Prisoner 1	**Cooperate**	2, 2	0, 5	0, 5	0, 5
	Defect	5, 0	1, 1	1, 1	1, 1
	Sign 1	5, 0	1, 1	2, 2	1, 1
	Sign 2	5, 0	1, 1	1, 1	2.5, 2.5

Modified Prisoner's Dilemma (cont...)

Prisoner 2

Prisoner 1

	Cooperate	Defect	Sign 1	Sign 2
Cooperate	2, 2	0, 5	0, 5	0, 5
Defect	5, 0	1, 1	1, 1	1, 1
Sign 1	5, 0	1, 1	2, 2	1, 1
Sign 2	5, 0	1, 1	1, 1	2.5, 2.5

- Nash equilibria: (D, D), (S_1, S_1) and (S_2, S_2)
- Nash equilibrium (S_2, S_2) is better than (S_1, S_1).

Games with Contracts

Definition

A ***correlated strategy*** $\tau_{\mathcal{S}}$ in a normal-form game $\Gamma = (\mathcal{N}, \mathcal{C}, \mathcal{U})$ is any probability distribution in the simplex $\Delta(\mathcal{C}_{\mathcal{S}})$ over a subset of players $\mathcal{S} \subseteq \mathcal{N}$, where $\mathcal{C}_{\mathcal{S}} = \times_{i \in \mathcal{S}} \mathcal{C}_i$.

Definition

Given a correlated strategy $\tau_{\mathcal{S}}$, the utility function ***allocation*** is the vector of utility functions that specifies the utility of correlated strategies to different players. In other words, for the i^{th} player, the utility is given by

$$U_i(\tau_{\mathcal{S}}) = \begin{cases} \displaystyle\sum_{c \in \mathcal{C}_{\mathcal{S}}} \tau_{\mathcal{S}}(c) u_i(c), & \text{if } i \in \mathcal{S} \\ u_i(c), & \text{otherwise.} \end{cases}$$

Definition

A ***contract*** can be mathematically represented by the vector $\boldsymbol{\tau} = (\tau_{\mathcal{S}})_{\mathcal{S} \subseteq \mathcal{N}}$ in the space $\times_{\mathcal{S} \subseteq \mathcal{N}} \Delta(\mathcal{C}_{\mathcal{S}})$.

Games with Contracts (cont...)

Note: Not all contracts are signed by every player.
For example, Prisoner 1 can refuse to sign a contract that commits players to implement (C, D).

Best payoff of the i^{th} player for the worst contract that other players can use against him/her:

Definition

The ***minimax value*** (or, ***security level***) for the i^{th} player in a game Γ is given by

$$v_i = \min_{\tau_{-i} \in \Delta(\mathcal{C}_{-i})} \left(\max_{c_i \in \mathcal{C}_i} \sum_{c_{-i} \in \mathcal{C}_{-i}} \tau_{-i}(c_{-i}) u_i(c_i, c_{-i}) \right)$$

Definition

A correlated strategy $\boldsymbol{\tau} \in \Delta(\mathcal{C})$ is ***individually rational*** if and only if

$$U_i(\boldsymbol{\tau}) \geq v_i, \text{ for all } i \in \mathcal{N}.$$

Correlated Equilibrium (CE)

Definition

A ***correlated equilibrium*** is a correlated strategy $\boldsymbol{\tau} \in \Delta(\mathcal{C})$ in a game Γ if

$$U_i(\boldsymbol{\tau}) \geq \sum_{c \in \mathcal{C}_{\mathcal{S}}} \tau_{\mathcal{S}}(c) u_i(c_i', c_{-i}), \text{ for all } c_i' \in \mathcal{C}_i, \text{ and for all } i \in \mathcal{N}.$$

Theorem

All Nash equilibria are correlated.

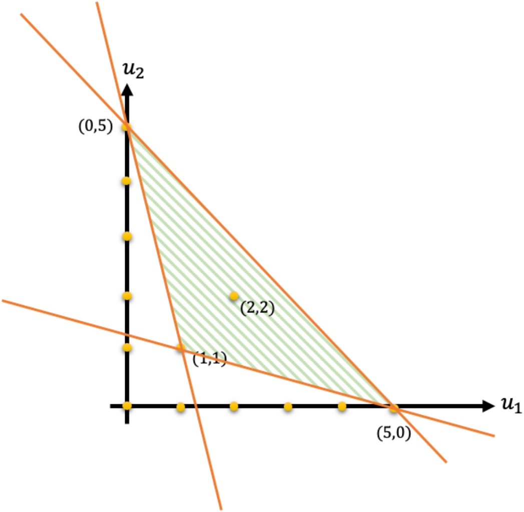
Theorem

All convex combinations of Nash equilibria are also correlated.

We can find the "best" correlated equilibrium via optimizing an appropriate objective (e.g. social welfare).

Graphical Interpretation of CE

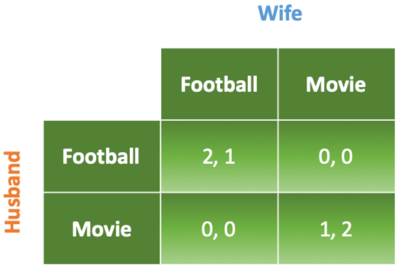
The set of correlated strategies is a convex hull of all the strategy profiles in $\mathcal{C}$.



Multiple Equilibria and Agents' Pick

Example: Battle of the Sexes

- $BR_H(W \rightsquigarrow F) = F$
- $BR_H(W \rightsquigarrow M) = M$
- $BR_W(H \rightsquigarrow F) = F$
- $BR_W(H \rightsquigarrow M) = M$
- **PSNE:** $(F, F), (M, M)$



		Wife	
		Football	Movie
Husband	Football	2, 1	0, 0
	Movie	0, 0	1, 2

*Thomas Schelling introduced the idea of a **focal point** which directs players to focus on one of the NE based on some structure outside the game's payoff representation.*

In the case of Battle-of-the-Sexes, we have the following focal points:

- Patriarchy $\Rightarrow (F, F)$
- Matriarchy $\Rightarrow (M, M)$

Focal Points and Social Choice Functions

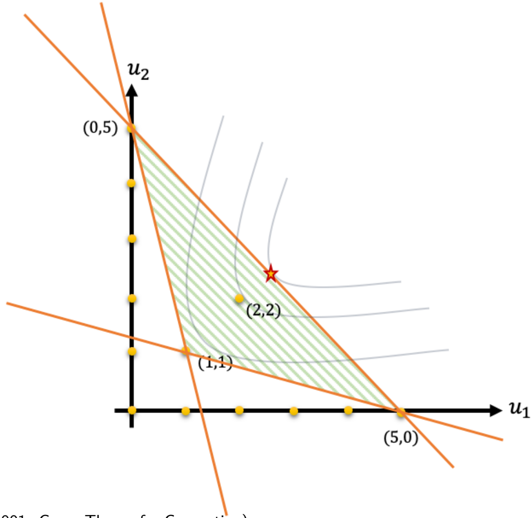
- Contracts $\Rightarrow$ Correlated Equilibria
- Correlated equilibria are at least as many as Nash equilibria.
- Social choice objective is usually picked based on the notion of a focal point which captures the dominant culture/convention within the player set.

Traffic Games: Driving conventions ensure safety at night on an undivided country road.

- U.K.: Drive on the left side of the road
- U.S.: Drive on the right side of the road

Graphical Interpretation of Social Choice Functions

The set of correlated strategies is a convex hull of all the strategy profiles in $\mathcal{C}$.



Social Contract vs. Social-Choice[1]

However, modeling cultural focal point with a static/idea social choice function has its own repercussions.

Definition

A ***social contract*** (also called a ***transcendental institution***) is a contract that is based on idealogies imposed on any given society.

Definition

A ***social choice*** is an evalution of the social state of affairs, which focuses on pairwise comparisons rather than identifying an ideal state of affairs.

Concerns with Social Contracts:

- Complete attention on what it identifies as perfect justice (fairness), rather than comparisons of justice and injustice.
- Concentrates on getting institutions right, instead of analyzing how actual societies emerge.

Currently, an active research area in AI/ML literature.

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[1] Amartya Sen, *The Idea of Justice* (Cambridge, MA: Harvard University Press, 2009).

Do people play Nash Equilibrium?

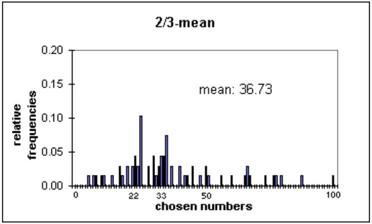
Consider the following *Keynesian Beauty Contest*:

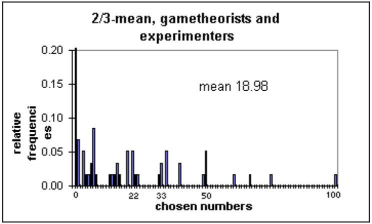
- N players are asked to choose any number between 0 to 100.
- The winner is the person whose choice is closest to two-thirds of the mean of the choices of all players.
- Winner gets a fixed prize of $20. In case of a tie, the prize is split equally amongst those who tie.

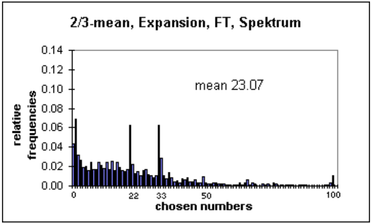
What would you choose?

What is the equilibrium of this game?

Experimental Observations in Keynesian Beauty Contests[2]







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[2] Derived from R. Nagel's "A Keynesian Beauty Contest in the Classroom", Available at http://w3.marietta.edu/ delemeeg/expernom/nagel.htm

Modeling Observed Equilibria...

Observed equilibria in experiments can be modeled broadly in the following ways:

- Model deviations from Nash Equilibrium as approximations:
  - ϵ-NE
  - Trembling perfect NE
- Define a new equilibrium notion based on simple rules-of-thumb:
  - Evolutionarily Stable Equilibrium
  - Behavioral Game Theory (e.g. Cognitive Hierarchy, Quantal Response Equilibrium)

ϵ-Nash Equilibrium

Definition

Given a fixed $\epsilon > 0$, any mixed strategy profile $\pi \in \Delta(\mathcal{C})$ is an ϵ-***Nash equilibrium*** to the game $\Gamma = (\mathcal{N}, \mathcal{C}, \mathcal{U})$ if

$$u_i(\pi_i, \pi_{-i}) \geq u_i(\pi_i', \pi_{-i}) - \epsilon$$

for all $\pi_i' \neq \pi_i$, and for all $i \in \mathcal{N}$.

Theorem

ϵ-Nash equilibrium exists in a finite game.

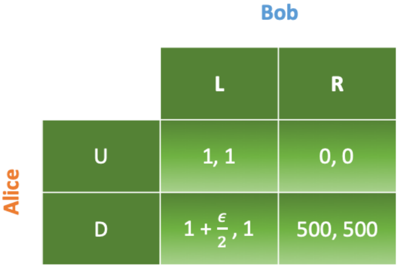
A computationally useful definition...

since computers represent real numbers using floating-point approximations.

- A ball $\mathcal{B}_\epsilon(c^*)$ centered around NE c^* consists of several ϵ-NEs.
- However, the opposite is not necessarily true!

ϵ-Nash Equilibrium: An Interesting Example

Consider the following game:



Bob

		L	R
Alice	U	1, 1	0, 0
	D	$1+\frac{\epsilon}{2}$, 1	500, 500

- NE: (D, R)
- ϵ-NE: (D, R), (U, L)

In other words, this is not a good solution concept!

Trembling-Hand Perfect Equilibrium (THPE)

Definition

A mixed-strategy $\pi \in \Delta(\mathcal{C})$ is a ***trembling-hand perfect equilibrium*** to the game $\Gamma = (\mathcal{N}, \mathcal{C}, \mathcal{U})$ if there exists a sequence $\pi^{(0)}, \pi^{(1)}, \cdots$ in the simplex $\Delta(\mathcal{C})$ such that

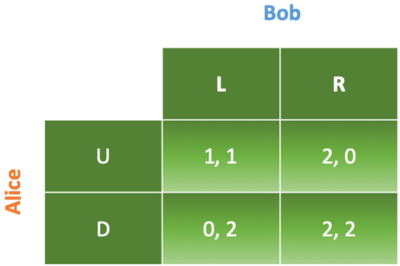
1. $\lim_{n \to \infty} \pi^{(n)} = \pi$,
2. π_i is the best response to $\pi_{-i}^{(k)}$, for all $i \in \mathcal{N}$ and for each k.

Then, why call it trembling hand perfect equilibrium?

- Since it is not just the best response to opponent's players, but also against small perturbations (trembles).

Trembling-Hand Perfect Equilibrium: An Example

Consider the following game:



		Bob	
		L	R
Alice	U	1, 1	2, 0
	D	0, 2	2, 2

- NE: $(U, L),\ (D, R)$
- THPE: (U, L)

- Let Alice choose a mixed strategy $(\epsilon, 1-\epsilon)$.
- $u_B(L) = 2 - \epsilon$, $u_B(R) = 2 - 2\epsilon$
- For small values of ϵ, Bob always plays L.
- Similarly, let Bob choose a strategy $(\delta, 1-\delta)$.
- $u_A(U) = 2 - \delta$, $u_A(D) = 2 - 2\delta$
- For small values of δ, Alice always plays U.

Evolutionary Game Theory

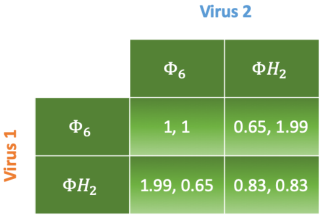
Study the evolution of a given species based on its interaction with other species...

Consider two species $\mathcal{N} = \{1, 2\}$.

- Species evolve via *mutating* their DNAs.
- Each species reproduces at a rate proportional to its DNA's *fitness*.
- Survival of a species is analogous to games...

Example: Mutations in a virus (Normalized payoffs based on replication rates in real experiments.)

- Phage Φ_6 infects cells and manufactures products needed for its own replicaiton
- Phage ΦH_2 replicates in bacterial hosts (though less effectively on its own), but takes advantage of chemicals produced by Φ_6
- NE: $(\Phi H_2, \Phi H_2)$



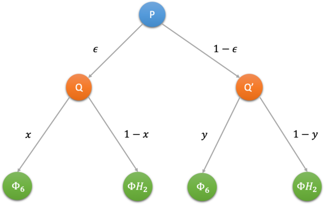
		Virus 2	
		Φ_6	ΦH_2
Virus 1	Φ_6	1, 1	0.65, 1.99
	ΦH_2	1.99, 0.65	0.83, 0.83

What does NE mean in this context?

Can we study evolution using traditional game theory?

Population Dynamics in Virus Cultures

- Suppose, in a monoculture P of the same species of viruses, a small fraction (say ϵ) of viruses in P (say Q) adopt x fraction of Φ_6.
- Let the remaining population in P (say Q') adopt NE, i.e., if y is fraction of Φ_6, we have $y = 0$.



Interactions of viruses in Q:

Virus in Q

Virus in Q

	Φ_6	ΦH_2
Φ_6	1, 1	0.65, 1.99
ΦH_2	1.99, 0.65	0.83, 0.83

Virus in Q'

Virus in Q

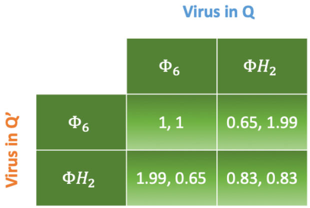
	Φ_6	ΦH_2
Φ_6	1, 1	0.65, 1.99
ΦH_2	1.99, 0.65	0.83, 0.83

Expected Utilities at Q:

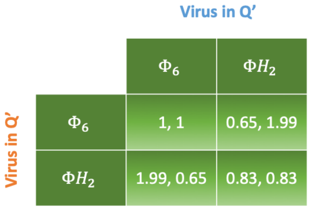
$$\begin{aligned} u_Q(Q) &= x\left[x + 0.65(1-x)\right] + (1-x)\left[1.99x + 0.83(1-x)\right] \\ u_Q(Q') &= x\left[y + 0.65(1-y)\right] + (1-x)\left[1.99y + 0.83(1-y)\right] = 0.83 - 0.18x \end{aligned}$$

Population Dynamics in Virus Cultures (cont...)

Interactions of viruses in Q':



		Virus in Q	
		Φ_6	ΦH_2
Virus in Q’	Φ_6	1, 1	0.65, 1.99
	ΦH_2	1.99, 0.65	0.83, 0.83



		Virus in Q’	
		Φ_6	ΦH_2
Virus in Q’	Φ_6	1, 1	0.65, 1.99
	ΦH_2	1.99, 0.65	0.83, 0.83

Expected Utility of Q':

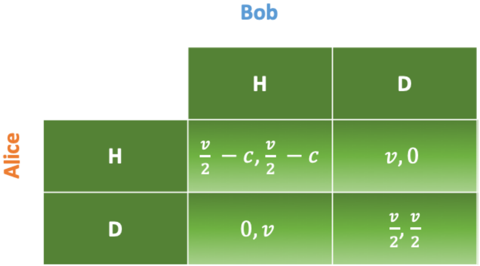
$$\begin{aligned} u_{Q'}(Q) &= y\left[x + 0.65(1-x)\right] + (1-y)\left[1.99x + 0.83(1-x)\right] = 0.83 + 1.16x \\ u_{Q'}(Q') &= y\left[y + 0.65(1-y)\right] + (1-y)\left[1.99y + 0.83(1-y)\right] = 0.83 \end{aligned}$$

Note that $u_Q(Q') < u_{Q'}(Q')$ for any positive x.

Q' is evolutionary stable, since future generations have more gain playing y when played by Q'.

Hawk-Dove Game

- Two players of same species.
- Personality traits: Hawk (Aggressive), Dove (Pacifist)
- Symmetric bi-matrix game, as shown below.



		Bob	
		H	**D**
Alice	**H**	$\frac{v}{2} - c, \frac{v}{2} - c$	$v, 0$
	D	$0, v$	$\frac{v}{2}, \frac{v}{2}$

- PSNE: $\begin{cases} (H,H) \text{ if } c < \frac{v}{2} \\ (H,D) \text{ and } (D,H) \text{ if } c > \frac{v}{2} \end{cases}$
- Symmetric game $\Rightarrow$ *Symmetric MSNE*[3] $= \left(\frac{v}{2c}, 1 - \frac{v}{2c}\right)$ when $c > \frac{v}{2}$.

  MSNE: $x^* = \underset{x \in (0,1)}{\text{arg max}}\ u_A(x, y^*)$, and $y^* = \underset{y \in (0,1)}{\text{arg max}}\ u_B(x^*, y)$

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[3] An equilibrium where both players employ the same strategy.

Population Dynamics in Hawk-Dove Game

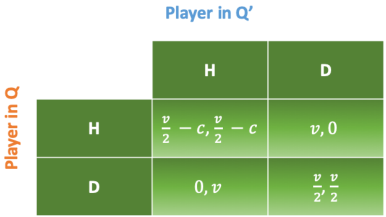
- Consider a species population P, whose interactions are modeled using Hawk-Dove game with $c > \frac{v}{2}$.
- Let a small fraction (say ϵ) of P (say Q) mutate and play the strategy $(x, 1-x)$.
- The remaining fraction in P (say Q') play MSNE, i.e., $(y, 1-y)$, where $y = \dfrac{v}{2c}$.

Interactions with Q:

Player in Q

Player in Q

	H	D
H	$\frac{v}{2} - c, \frac{v}{2} - c$	$v, 0$
D	$0, v$	$\frac{v}{2}, \frac{v}{2}$



Player in Q'

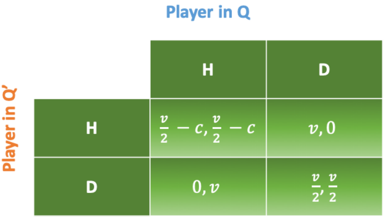
Player in Q

	H	D
H	$\frac{v}{2} - c, \frac{v}{2} - c$	$v, 0$
D	$0, v$	$\frac{v}{2}, \frac{v}{2}$

$$\begin{aligned} u_Q(Q) &= x\left[x\left(\frac{v}{2} - c\right) + (1-x)v\right] + (1-x)\left[(1-x)\frac{v}{2}\right] = \frac{v}{2} - cx^2 \\ u_Q(Q') &= x\left[y\left(\frac{v}{2} - c\right) + (1-y)v\right] + (1-x)\left[(1-y)\frac{v}{2}\right] = \frac{v}{2}\left(1 - \frac{v}{2c}\right) \end{aligned}$$

Population Dynamics in Hawk-Dove Game (cont...)

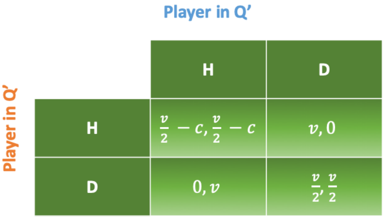
Interactions with Q':



Player in Q

Player in Q'

	H	D
H	$\frac{v}{2} - c, \frac{v}{2} - c$	$v, 0$
D	$0, v$	$\frac{v}{2}, \frac{v}{2}$



Player in Q'

Player in Q'

	H	D
H	$\frac{v}{2} - c, \frac{v}{2} - c$	$v, 0$
D	$0, v$	$\frac{v}{2}, \frac{v}{2}$

$$\begin{aligned} u_{Q'}(Q) &= y\left[x\left(\frac{v}{2} - c\right) + (1-x)v\right] + (1-y)\left[(1-x)\frac{v}{2}\right] = \frac{v}{2}\left(1 - \frac{v}{2c}\right) + \frac{v(v-1)}{2}x \\ u_{Q'}(Q') &= y\left[y\left(\frac{v}{2} - c\right) + (1-y)v\right] + (1-y)\left[(1-y)\frac{v}{2}\right] = \frac{v}{2}\left(1 - \frac{v}{2c}\right) \end{aligned}$$

In this example, $u_Q(Q') = u_{Q'}(Q')$ for any x. However, we have $u_{Q'}(Q) > u_Q(Q)$.

Although** $u_Q(Q') = u_{Q'}(Q')$ **for any positive** x**, we have** $u_{Q'}(Q) > u_Q(Q)$**.

Q' is evolutionary stable, since future generations have more gain playing x by Q'.

Evolutionarily Stable Strategies (ESS)

- Consider a symmetric, two-player game with utility matrices A and B, where $A_{i,j} = B_{j,i}$.
- Suppose a majority of population P (denoted Q') playing a mixed strategy $\boldsymbol{x}$ invades a small population of mutants Q whose mixed strategy is $\boldsymbol{y}$.

Definition

A mixed strategy $\boldsymbol{x}$ is ***evolutionarily stable***, if for any "mutant" strategy $\boldsymbol{y}$, we have

(a) $\boldsymbol{y}^T A \boldsymbol{x} < \boldsymbol{x}^T A \boldsymbol{x}$

(b) Or, if $\boldsymbol{y}^T A \boldsymbol{x} = \boldsymbol{x}^T A \boldsymbol{x}$, then $\boldsymbol{y}^T A \boldsymbol{y} < \boldsymbol{x}^T A \boldsymbol{y}$

Theorem

If a strategy S is evolutionarily stable, then (S, S) is a Nash equilibrium.

ESS: Examples

Virus Game:
The PSNE $(\Phi H_2, \Phi H_2)$ is an ESS.

Hawk-Dove Game:
The mixed strategy $\left(\dfrac{v}{2c}, 1 - \dfrac{v}{2c}\right)$ defined when $c > \frac{v}{2}$, is an ESS.

- If $\boldsymbol{y} = (1, 0)$, then $\boldsymbol{y}^T A \boldsymbol{y} = \frac{v}{2} - c < \boldsymbol{x}^T A \boldsymbol{y} = x\left(\frac{v}{2} - c\right)$.
- If $\boldsymbol{y} = (0, 1)$, then $\boldsymbol{y}^T A \boldsymbol{y} = \frac{v}{2} < \boldsymbol{x}^T A \boldsymbol{y} = xv + (1 - x)\frac{v}{2}$.

Rock-Paper-Scissors: Observed in the lizard species called *Uta Stansburia*.

Claim: The unique Nash equilibrium in Rock-Paper-Scissors, $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ is not a ESS.

One final note...

How can we solve Bayesian games in normal-form?

In most game-theoretic settings, players does not have complete knowledge about other players and their utilities.

Examples:

- *Bargaining/Auctions/Contests:* Valuations of other players are unknown.
- *Markets:* Intellectual properties are dealt as a secret, which results in uncertain production costs about other players.
- *Signaling games:* The sender's intent behind sharing a signal is usually unknown to receivers.

and many more...

Bayesian Games in Normal-Form

Definition

A ***Bayesian (or incomplete information game) game*** Γ is defined as a tuple $(\mathcal{N}, \Theta, \boldsymbol{p}, \mathcal{C}, \mathcal{U})$, where

- $\mathcal{N} = \{1, \cdots, N\}$ is the set of N players (agents),
- $\Theta = \{\Theta_1, \cdots, \Theta_N\}$, where Θ_i is the set of types of player i,
- $\boldsymbol{p} = \{p_1, \cdots, p_N\}$, where $p_i : \Theta_i \to \Delta(\Theta_{-i})$ is the conditional belief over the set of types of other players, given the type of player i,
- $\mathcal{C} = \mathcal{C}_1 \times \cdots \times \mathcal{C}_N$ is the strategy profile space, where $\mathcal{C}_i$ represents the set of strategic choices (actions) available at the i^{th} player,
- $\mathcal{U} = \{u_1, \cdots, u_N\}$ is the set of utility functions, where $u_i : \mathcal{C}_i \to \mathbb{R}$ represents the utility function at the i^{th} player.

Note: The label "*Bayesian games*" is coined because $p_i(\theta_{-i}|\theta_i)$ can be computed from prior probability distribution $p(\theta_i, \theta_{-i})$ using Bayes Rule, as shown below:

$$p_i(\theta_{-i}|\theta_i) = \frac{p(\theta_{-i}, \theta_i)}{\displaystyle\int p(\theta_{-i}, \theta_i) d\theta_{-i}}$$

Bayesian Nash Equilibrium (BNE)

Consider a game with finite types of agents:

- Let $\boldsymbol{\sigma}_i(\theta_i)$ denote the mixed strategy employed by Player i of type $\theta_i \in \Theta_i$.
- Expected utility of the i^{th} player of type θ_i is given by

$$U_i(\sigma_i, \sigma_{-i}, \theta_i) = \sum_{\theta_{-i} \in \Theta_{-i}} \left[p_i(\theta_{-i} | \theta_i) \sum_{c \in \mathcal{C}} \left(\prod_{j \in \mathcal{N}_{-i}} \sigma_j(c_j | \theta_j) \right) \sigma_i(c_i) u_i(c_i, c_{-i}(\theta_{-i}), \boldsymbol{\theta}) \right],$$

Definition

A ***Bayesian-Nash equilibrium*** is a strategy profile $\boldsymbol{\sigma} = \{\boldsymbol{\sigma}_1, \cdots, \sigma_N\} \in \Delta(\mathcal{C})$, if for all $i \in \mathcal{N}$ and for all $\theta_i \in \Theta_i$, we have

$$\sigma_i(\theta_i) \in \underset{\sigma_i \in \Delta(C_i)}{\text{arg max}} \; U_i(\sigma_i, \sigma_{-i}, \theta_i)$$

Theorem

There always exists a mixed-strategy BNE in any finite Bayesian game.

BNE in Second-Price Auctions

- Two players $\mathcal{N} = \{1, 2\}$.
- Players valuate the auctioned item as v_1 and v_2 respectively.
- However, the other players does not complete knowledge about valuations! Only know $p(v_{-i}|v_i) = \mathcal{U}[0, 1]$, a uniform distribution in the range $[0, 1]$.
- Utility of player i is

$$u_i(b_i, b_{-i}, v_i) = \begin{cases} v_i - b_{-i}, & \text{if } b_i > b_{-i} \\ \dfrac{v_i - b_{-i}}{2}, & \text{if } b_i = b_{-i} \\ 0, & \text{otherwise.} \end{cases}$$

As opposed to the complete information game,

Theorem

There exists a ***unique*** Bayesian equilibrium in second-price auctions, which is the case when bidders choose bids equal to their valuations, i.e. $b_i^* = v_i$.

Summary

- *Correlated Equilibria:* How to improve NE in games using communication and contracts?
- *Focal Point:* How do players differentiate multiple equilibria?
- *Modeling Deviations:* How to model observed deviations from NE?
    - ϵ Nash equilibrium
    - Trembling hand perfect equilibrium
- *Evolutionary game theory:* Can we explain evolution of life using game theory?
- *Bayesian Equilibrium:* How can we find equilibria in an Bayesian game, when agent types are unknown?