Missouri University of Science & Technology — Department of Computer Science

Fall 2024 — **CS 5408: Game Theory for Computing**

Homework 4: Information Asymmetry and Dynamic Games

Instructor: *Sid Nadendla* — **Due:** *November 15, 2024*

Problem 1 First-Price Auction *3 pts.*

Consider a first-price sealed-bid auction with N bidders. Assume that each i^{th} bidder has a valuation $v_i \in [0,1]$ and has a belief $v_j \sim U[0,1]$ for all $j \neq i$, i.e. a uniform belief regarding the valuations of all the other players. Then, prove that the first-price sealed-bid auction has a Bayes-Nash equilibrium where every player adopts *strategic underbidding* by choosing

$$b_i^* = \left(\frac{N-1}{N}\right) v_i, \text{ for all } i = 1, \cdots, N.$$

Solution:

Consider the i^{th} bidder without any loss of generality. Let all the other bidders choose a bid

$$b_j = \left(\frac{N-1}{N}\right) v_j, \quad \text{for all } j \neq i. \tag{1}$$

Note that the probability of i^{th} bidder being the winner is given by

$$\begin{aligned}
\mathbb{P}\Big(b_i = \max\{b_j\}_{j\neq i}\Big) &= \prod_{j\neq i} \mathbb{P}\Big(b_i \geq b_j\Big) \\
&= \prod_{j\neq i} \mathbb{P}\left(b_i \geq \left(\frac{N-1}{N}\right) v_j\right) \qquad \text{(from Equation (1))} \\
&= \prod_{j\neq i} \mathbb{P}\left(v_j \leq \left(\frac{N}{N-1}\right) b_i\right) \\
&= \left(\left(\frac{N}{N-1}\right) b_i\right)^{N-1}.
\end{aligned} \tag{2}$$

Therefore, the expected utility of the i^{th} bidder is given by

$$\begin{aligned}
U_i &= \mathbb{P}\Big(b_i = \max\{b_j\}_{j\neq i}\Big) \cdot (v_i - b_i) + \mathbb{P}\Big(b_i \neq \max\{b_j\}_{j\neq i}\Big) \cdot 0 \\
&= \left(\left(\frac{N}{N-1}\right) b_i\right)^{N-1} \cdot (v_i - b_i) \quad \text{(from Equation (2))}.
\end{aligned} \tag{3}$$

Since the i^{th} bidder wishes to maximize U_i by choosing an appropriate bid b_i, we perform the first derivative test to compute the critical point, as follows.

$$\begin{aligned}\frac{dU_i}{db_i} &= \left(\frac{N}{N-1}\right)^N \cdot \left[(N-1)b_i^{N-2} \cdot v_i - N \cdot b_i^{N-1}\right] \\ &= \left(\frac{N}{N-1}\right)^N \cdot b_i^{N-2} \cdot [(N-1) \cdot v_i - N \cdot b_i] = 0.\end{aligned} \tag{4}$$

Since $\left(\frac{N}{N-1}\right)^N > 0$, we have two possibilities:

1. $b_i = 0$ – However, this is not desired as the i^{th} bidder is guaranteed to loose with this strategy.

2. $(N-1) \cdot v_i - N \cdot b_i = 0 \qquad b_i = \left(\dfrac{N-1}{N}\right) v_i$. A quick second derivative test reveals that this possibility results in a maximum utility at the i^{th} player.

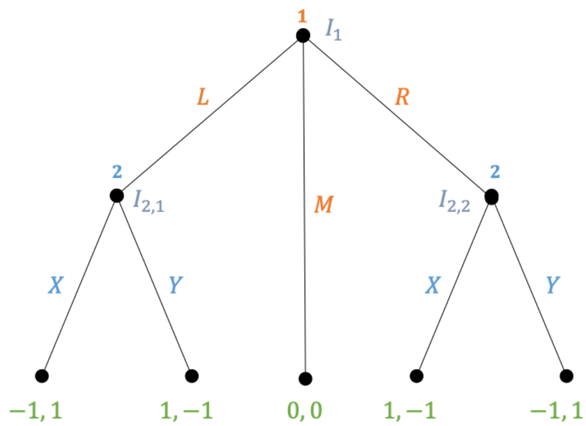
Since $b_i = \left(\dfrac{N-1}{N}\right) v_i$ is the best response to $b_j = \left(\dfrac{N-1}{N}\right) v_j$ for all $j \neq i$ regardless of the choice of i, this is a Nash equilibrium.

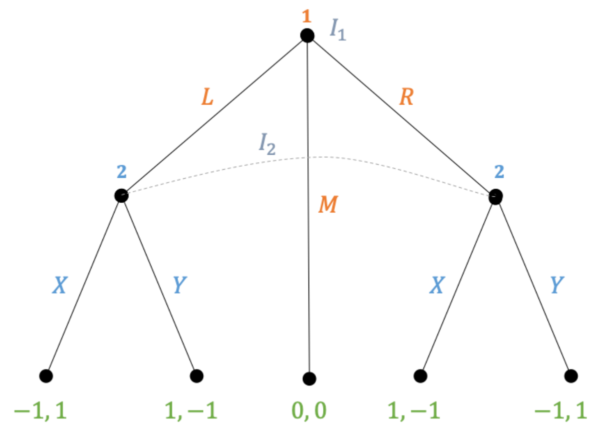
□

Remark: Therefore, in first-price auctions, the bidders strategically underbid at Nash equilibrium. This untruthful behavior is not desirable, especially since the principal does not receive payments that it truly deserves.

Problem 2 Imperfect Extensive Games *4 pts.*

Consider the following modified matching pennies game, played in an extensive form, where Prisoner 1 plays first, followed by Prisoner 2. The main difference from the traditional matching pennies is that Player 1 can decide whether to play this game or not. If he decides not to play, both players get nothing.





(a) Find the subgame perfect equilibrium for this game, when Player 2 can perfectly observe Player 1's choices as in the left figure.

(b) Find behavioral equilibria for this game, when Player 2 cannot observe Player 1's choices as in the right figure.

Solution:

(a) Backward induction is used to compute subgame perfect Nash equilibrium, as shown in Figure 1. In the first stage, Nash equilibrium for subgames rooted at nodes 2 and 3 are first computed. At the end of the first stage, the values at nodes 2 and 3 are both updated to $(-1, 1)$. In the second stage, the Nash equilibrium for the subgame rooted at node 1 is evaluated and the value of node 1 is updated as $(0, 0)$.

Therefore, SPNE is $(P_1 : M, P_2 : X/Y)$. □

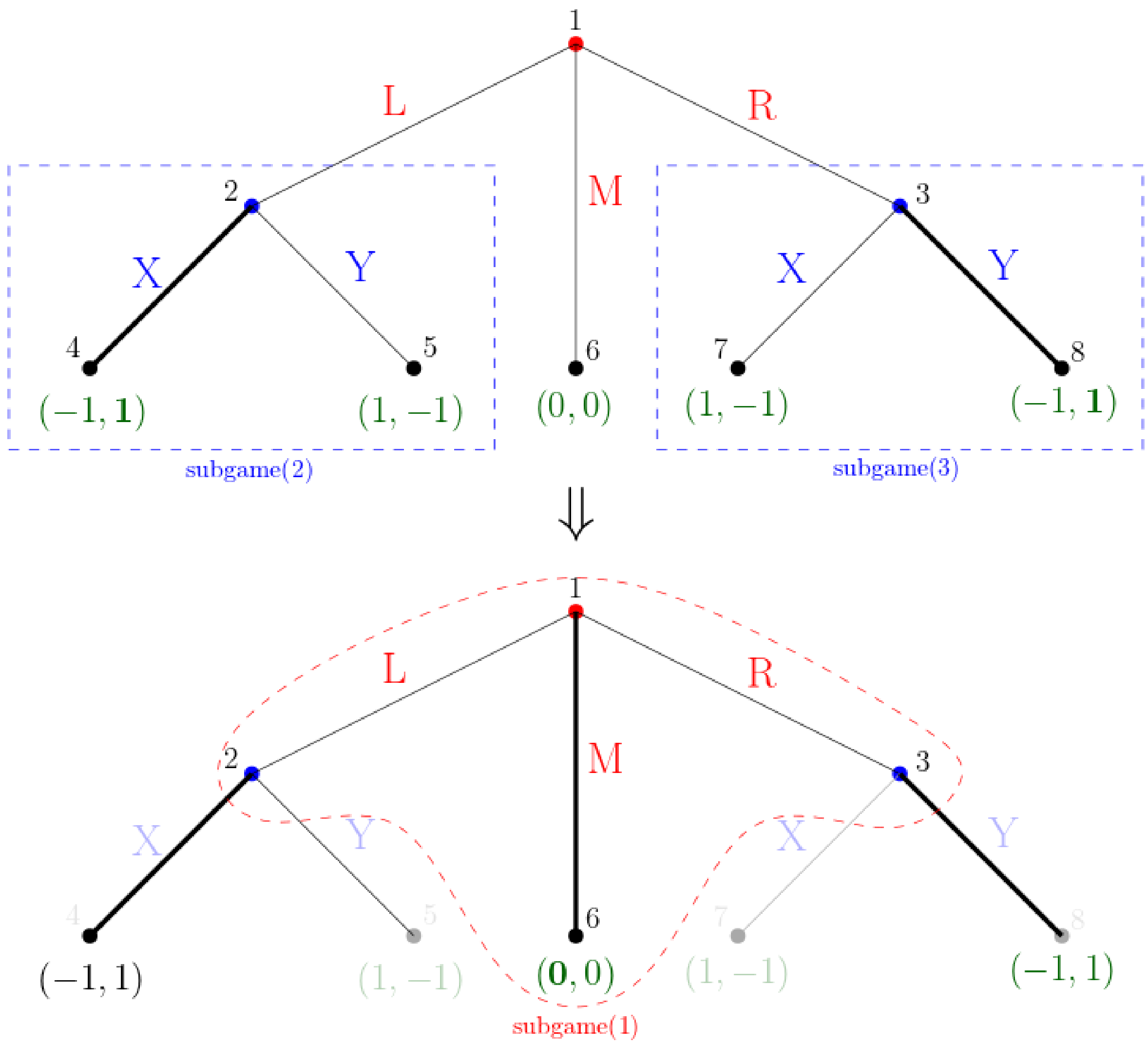


Figure 1: Stages of Backward Induction to compute SPNE

(b) Let P_2's behavioral strategy in I_2 be $\{X : \alpha, Y : 1-\alpha\}$. Also, assume that P_2 constructs a belief $\mu = \mathbb{P}(L|I_2)$ regarding being in the left node in I_2. Then, P_2's conditional expected utilities are given by

$$\begin{aligned} u_2(X|I_2) &= \mu \cdot 1 + (1-\mu)\cdot(-1) &= 2\mu - 1 \\ u_2(Y|I_2) &= \mu \cdot (-1) + (1-\mu)\cdot 1 &= 1 - 2\mu \end{aligned} \tag{5}$$

Therefore, the expected utility at P_2 due to the behavioral strategy $\{X : \alpha, Y : 1-\alpha\}$ is given by

$$u_2(I_2) = \alpha \cdot u_2(X|I_2) + (1-\alpha)\cdot u_2(Y|I_2) = (1-2\alpha)(1-2\mu). \tag{6}$$

Similarly, P_1's expected utilities are given by

$$\begin{aligned} u_1(L) &= \alpha \cdot (-1) + (1-\alpha) \cdot 1 &= 1 - 2\alpha, \\ u_1(R) &= \alpha \cdot 1 + (1-\alpha) \cdot (-1) &= 2\alpha - 1, \\ u_1(M) &= 0. \end{aligned} \tag{7}$$

Note that P_1's sequential rationality is satisfied by the following best-response strategy:

- If $\alpha > \dfrac{1}{2}$, then $u_1(L) < u_1(M) < u_1(R) \Rightarrow P_1$ chooses R.
- If $\alpha < \dfrac{1}{2}$, then $u_1(L) < u_1(M) < u_1(R) \Rightarrow P_1$ chooses L.
- If $\alpha = \dfrac{1}{2}$, then $u_1(L) = u_1(M) = u_1(R) \Rightarrow P_1$'s preference order is $L \sim M \sim R$.

Similarly, P_1's sequential rationality is satisfied by the following best-response strategy:

- If $\mu > \dfrac{1}{2}$, then $u_2(I_2)$ is maximized when $\alpha = 1$.
- If $\mu < \dfrac{1}{2}$, then $u_2(I_2)$ is maximized when $\alpha = 0$.
- If $\mu = \dfrac{1}{2}$, then $u_2(I_2) = u_2(M) = 0 \Rightarrow P_2$'s preference order is $X \sim Y$.

Now, P_2's consistency is guaranteed if

- If $\alpha < \dfrac{1}{2}$, then P_1 chooses $L \Rightarrow \mu = 1$.

  But, this is a <u>violation</u> to P_2's sequential rationality since P_2 chooses $\alpha = 1$ if $\mu > \frac{1}{2}$.
- If $\alpha > \dfrac{1}{2}$, then P_1 chooses $R \Rightarrow \mu = 0$.

  But, this is a <u>violation</u> to P_2's sequential rationality since P_2 chooses $\alpha = 0$ if $\mu < \frac{1}{2}$.

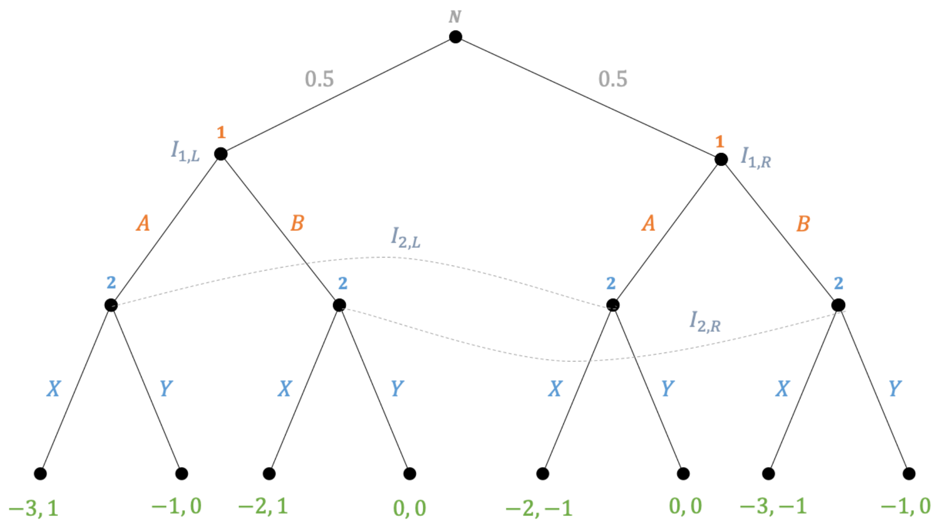
This leads us to the behavioral equilibrium, which is

- P_1 chooses M,
- P_2 chooses $\{X : \frac{1}{2}, Y : \frac{1}{2}\}$, with $\mu = \frac{1}{2}$.

$\square$

Problem 3 Perfect Bayesian Equilibrium *3 pts.*

Prove that there is no separating equilibrium in the following two-player signaling game (as depicted in the figure below), where the player set is $\mathcal{N} = \{1, 2\}$, the choice sets at the corresponding players are $\mathcal{C}_1 = \{A, B\}$ and $\mathcal{C}_2 = \{X, Y\}$ respectively. Assume that Player 1 can take two types $\{L, R\}$, and Player 2's belief about Player 1's type is uniformly distributed across types.



Solution:

Let the pure strategy at Player 1 be denoted by two letters, where the first letter corresponds to the strategy chosen in the information set $I_{1,L}$ and the second letter represents the strategy chosen in the information set $I_{1,R}$. For example, a pure strategy AB means that the sender chooses A in $I_{1,L}$ and B in $I_{1,R}$.

Note that Player 1 only has two separating strategies: AB and BA. Let us consider each of these strategies on a case-by-case basis:

Case 1 (AB): Since this is a separating strategy, the receiver clearly knows the information set in which he/she is in. For example, if the receiver observes a signal A, then he/she is on the left node of the information set $I_{2,L}$. In such a case, the receiver will choose X since $u_2(X|AB, I_{2,L}) = 1 > 0 = u_2(Y|AB, I_{2,L})$. Similarly, in $I_{1,R}$, if the sender chooses B, the receiver will always choose Y since $u_2(Y|AB, I_{2,R}) = 0 > -1 = u_2(X|AB, I_{2,R})$. In other words, the receiver's best response to AB is XY. However, sequential rationality is satisfied if the sender's best response to XY is also AB. However, if the receiver always chooses X in $I_{2,L}$ and Y in $I_{2,R}$, then the sender will always choose B in $I_{1,L}$ since $u_1(B|XY, I_{1,L}) = 0 > -3 = u_1(A|XY, I_{1,L})$. In other words, sequential rationality is violated for the separating strategy AB.

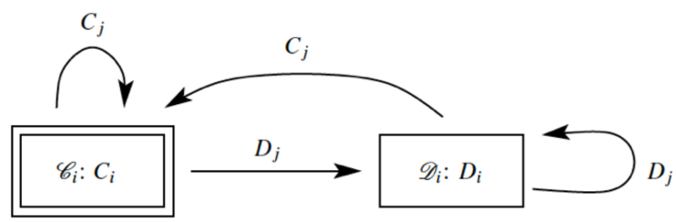
Case 1 (BA): Since $u_2(X|BA, I_{1,L}) = 1 > 0 = u_2(Y|BA, I_{1,L})$ and $u_2(Y|BA, I_{1,R}) = 0 > -1 = u_2(X|BA, I_{1,R})$, the receiver's best response to BA is XY. However, sequential rationality is satisfied if the sender's best response to XY is also BA. However, in $I_{1,R}$, the sender always chooses B since $u_1(B|XY, I_{1,R}) = -1 > -2 = u_1(A|XY, I_{1,R})$. This is a violation of the sequential rationality condition, too.

In other words, since separating strategies violate sequential rationality condition, this game does not have a separating equilibrium. □

Problem 4 Repeated Games *3 pts.*

Consider the following repeated prisoner's dilemma game, where players play the game over an infinite time horizon. Prove that Tit-for-Tat strategy (given below) is a Nash equilibrium to this game, only when the discounting factor $\beta \geq \frac{1}{2}$.

		Prisoner 2	
		C_2	D_2
Prisoner 1	C_1	2, 2	0, 3
	D_1	3, 0	1, 1



Solution:

Assuming that Player $-i$ follows Tit-for-Tat, Player i's responses can be summarized by the following four classes of strategy profile sequences of length T:

- **CASE (a):**

$$\begin{array}{rlllll} \text{Player} & i: & C_i^{(1)} & C_i^{(2)} & \cdots & C_i^{(T)} & \cdots \\ \text{Player} & -i: & C_{-i}^{(1)} & C_{-i}^{(2)} & \cdots & C_{-i}^{(T)} & \cdots \end{array}$$

- **CASE (b):**

$$\begin{array}{rllllllll} \text{Player} & i: & C_i^{(1)} & C_i^{(2)} & \cdots & C_i^{(T-1)} & D_i^{(T)} & D_i^{(T+1)} & \cdots \\ \text{Player} & -i: & C_{-i}^{(1)} & C_{-i}^{(2)} & \cdots & C_{-i}^{(T-1)} & C_{-i}^{(T)} & D_{-i}^{(T+1)} & \cdots \end{array}$$

- **CASE (c):**

$$\begin{array}{rlllllllll} \text{Player} & i: & C_i^{(1)} & C_i^{(2)} & \cdots & C_i^{(T-1)} & D_i^{(T)} & C_i^{(T+1)} & C_i^{(T+2)} & \cdots \\ \text{Player} & -i: & C_{-i}^{(1)} & C_{-i}^{(2)} & \cdots & C_{-i}^{(T-1)} & C_{-i}^{(T)} & D_{-i}^{(T+1)} & C_{-i}^{(T+2)} & \cdots \end{array}$$

- **CASE (d):**

$$\begin{array}{rlllllllllll} \text{Player} & i: & C_i^{(1)} & \cdots & C_i^{(T-1)} & D_i^{(T)} & D_i^{(T+1)} & \cdots & D_i^{(T+k-1)} & C_i^{(T+k)} & & \cdots \\ \text{Player} & -i: & C_{-i}^{(1)} & \cdots & C_{-i}^{(T-1)} & C_{-i}^{(T)} & D_{-i}^{(T+1)} & \cdots & D_{-i}^{(T+k-1)} & D_{-i}^{(T+k)} & C_{-i}^{(T+k+1)} & \cdots \end{array}$$

Each of these classes of strategy profile sequences generates the following discounted utilities at Player i:

$$
\begin{aligned}
u_i^{(a)} &= \sum_{t=1}^{\infty} \beta^{t-1} \cdot 2 \\
u_i^{(b)} &= \sum_{t=1}^{T-1} \beta^{t-1} \cdot 2 \ + \ \beta^{T-1} \cdot 3 \ + \sum_{t=T+1}^{\infty} \beta^{t-1} \cdot 1 \\
u_i^{(c)} &= \sum_{t=1}^{T-1} \beta^{t-1} \cdot 2 \ + \ \beta^{T-1} \cdot 3 \ + \ \beta^{T} \cdot 0 \ + \sum_{t=T+2}^{\infty} \beta^{t-1} \cdot 2 \\
u_i^{(d)} &= \sum_{t=1}^{T-1} \beta^{t-1} \cdot 2 \ + \ \beta^{T-1} \cdot 3 \ + \sum_{t=T+1}^{T+k-1} \beta^{t-1} \cdot 1 \ + \ \beta^{T+k-1} \cdot 0 \ + \sum_{t=T+k+1}^{\infty} \beta^{t-1} \cdot 2
\end{aligned}
\tag{8}
$$

Note that Case (a) corresponds to Tit-for-Tat strategy at Player i. In other words, Tit-for Tat at Player i is the best response to Tit-for Tat at Player $-i$ if

$$u_i^{(a)} \geq u_i^{(b)}, \tag{9}$$

$$u_i^{(a)} \geq u_i^{(c)}, \tag{10}$$

$$u_i^{(a)} \geq u_i^{(d)}. \tag{11}$$

Substituting Equation (1) in inequalities (2)-(4), we obtain

$$
\begin{aligned}
& \sum_{t=1}^{\infty} \beta^{t-1} \cdot 2 \ \geq \ \sum_{t=1}^{T-1} \beta^{t-1} \cdot 2 \ + \ \beta^{T-1} \cdot 3 \ + \sum_{t=T+1}^{\infty} \beta^{t-1} \cdot 1 \\
\Rightarrow \quad & \sum_{t=T}^{\infty} \beta^{t-1} \cdot 2 \ \geq \ \beta^{T-1} \cdot 3 \ + \sum_{t=T+1}^{\infty} \beta^{t-1} \cdot 1 \\
\Rightarrow \quad & \sum_{t=T+1}^{\infty} \beta^{t-1} \ \geq \ \beta^{T-1} \\
\Rightarrow \quad & \beta^{T} \cdot \frac{1}{1-\beta} \ \geq \ \beta^{T-1} \\
\Rightarrow \quad & \beta^{T-1}(2\beta - 1) \ \geq \ 0, \quad \text{or} \quad \beta \geq \frac{1}{2},
\end{aligned}
\tag{2a}
$$

$$
\begin{aligned}
& \sum_{t=1}^{\infty} \beta^{t-1} \cdot 2 \ \geq \ \sum_{t=1}^{T-1} \beta^{t-1} \cdot 2 \ + \ \beta^{T-1} \cdot 3 \ + \sum_{t=T+2}^{\infty} \beta^{t-1} \cdot 2 \\
\Rightarrow \quad & \beta^{T-1} \cdot (2\beta - 1) \ \geq \ 0, \quad \text{or} \quad \beta \geq \frac{1}{2},
\end{aligned}
\tag{3a}
$$

$$\begin{aligned}
\sum_{t=1}^{\infty}\beta^{t-1}\cdot 2 \;&\geq\; \sum_{t=1}^{T-1}\beta^{t-1}\cdot 2 \;+\; \beta^{T-1}\cdot 3 \;+\; \sum_{t=T+1}^{T+k-1}\beta^{t-1}\cdot 1 \;+\; \sum_{t=T+k+1}^{\infty}\beta^{t-1}\cdot 2 \\
\Rightarrow \qquad 2\beta^{T-1}\cdot\frac{1-\beta^{k+1}}{1-\beta} \;&\geq\; 3\beta^{T-1}+\beta^{T}\cdot\frac{1-\beta^{k-1}}{1-\beta} \\
\Rightarrow \quad 1-2\beta-\beta^{k}+2\beta^{k+1} \;&\leq\; 0 \\
\Rightarrow \qquad (1-2\beta)(1-\beta^{k}) \;&\leq\; 0, \quad \text{or} \quad \beta \geq \frac{1}{2}.
\end{aligned} \tag{4a}$$

Since Inequalities (2a)-(4a) all hold true when $\beta \geq \frac{1}{2}$, Tit-for-Tat is a best response strategy for Player i against a Tit-for-Tat strategy adopted by Player $-i$. Since the analysis holds true for both $i = 1, 2$, Tit-for-Tat is a Nash equilibrium if $\beta \geq \frac{1}{2}$.

$\square$