CS 5408: GAME THEORY FOR COMPUTING

#1

HW-2 SOLUTIONS

Prob. 1

(a) Let us denote Colonel Blotto as "B" and Folk Militia as "M".

B has 3 regiments ⇒ B's strategies: (3,0), (2,1), (1,2), (0,3)

M has 2 regiments ⇒ M's strategies: (2,0), (1,1), (0,2)

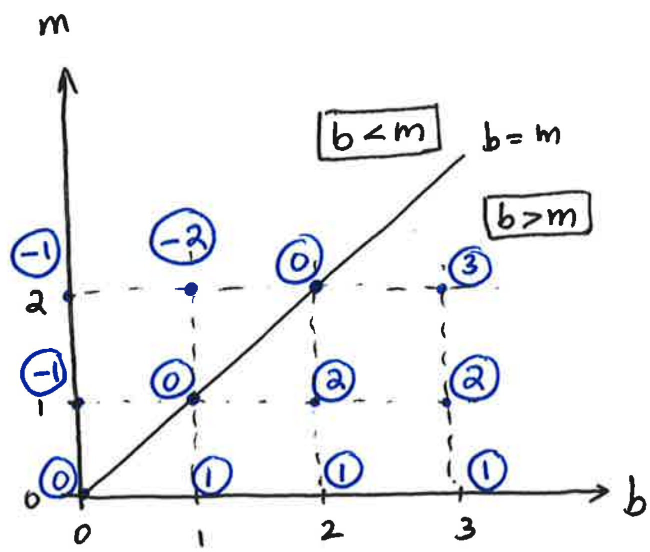
Say, $\{b, m\}$ represents a strategy profile where

- B deploys $(b, 3-b)$
- M deploys $(m, 2-m)$.

Let U_i denote the utility obtained by B due to deployments in post-i.

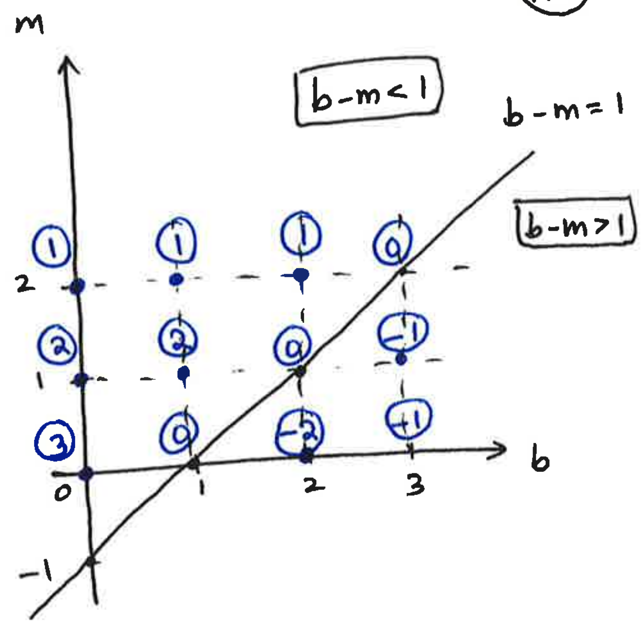
⇒

$$U_1(\{b,m\}) = \begin{cases} m+1; & \text{if } b > m \\ 0; & \text{if } b = m \\ -b-1; & \text{otherwise.} \end{cases}$$



and

$$V_2(\{b,m\}) = \begin{cases} (2-m)+1 = 3-m & ; \text{ if } 3-b > 2-m \text{ (or) } \boxed{b-m<1} \\ 0 & ; \text{ if } b-m=1 \\ -(3-b)-1 = -4+b & ; \text{ otherwise.} \end{cases}$$



$$\Rightarrow U = U_1 + U_2$$

=

	(0,2)	(1,1)	(2,0)
(0,3)	0+3 3	−1+2 1	−1+1 0
(1,2)	1+0 1	0+2 2	−2+1 −1
(2,1)	1−2 −1	2+0 2	0+1 1
(3,0)	1−1 0	2−1 1	3+0 3

lll^{ly}, let V_i represent the utility obtained by M due to deployments in post-i.

$$\Rightarrow V_1(\{b,m\}) = \begin{cases} -m-1 & \text{if } b>m \\ 0 & \text{if } b=m \\ b+1 & \text{if } b<m \end{cases}, \quad V_2(\{b,m\}) = \begin{cases} -3+m & \text{if } b-m<1 \\ 0 & \text{if } b-m=1 \\ 4-b & \text{otherwise.} \end{cases}$$

$$\Rightarrow V = V_1 + V_2 = -U.$$

<u>NOTE</u>: This is a zero-sum game.

In Summary, the normal-form (bi-matrix) game is

B \ M	(0,2)	(1,1)	(2,0)
(0,3)	3, −3	1, −1	0, 0
(1,2)	1, −1	2, −2	−1, 1
(2,1)	−1, 1	2, −2	1, −1
(3,0)	0, 0	1, −1	3, −3

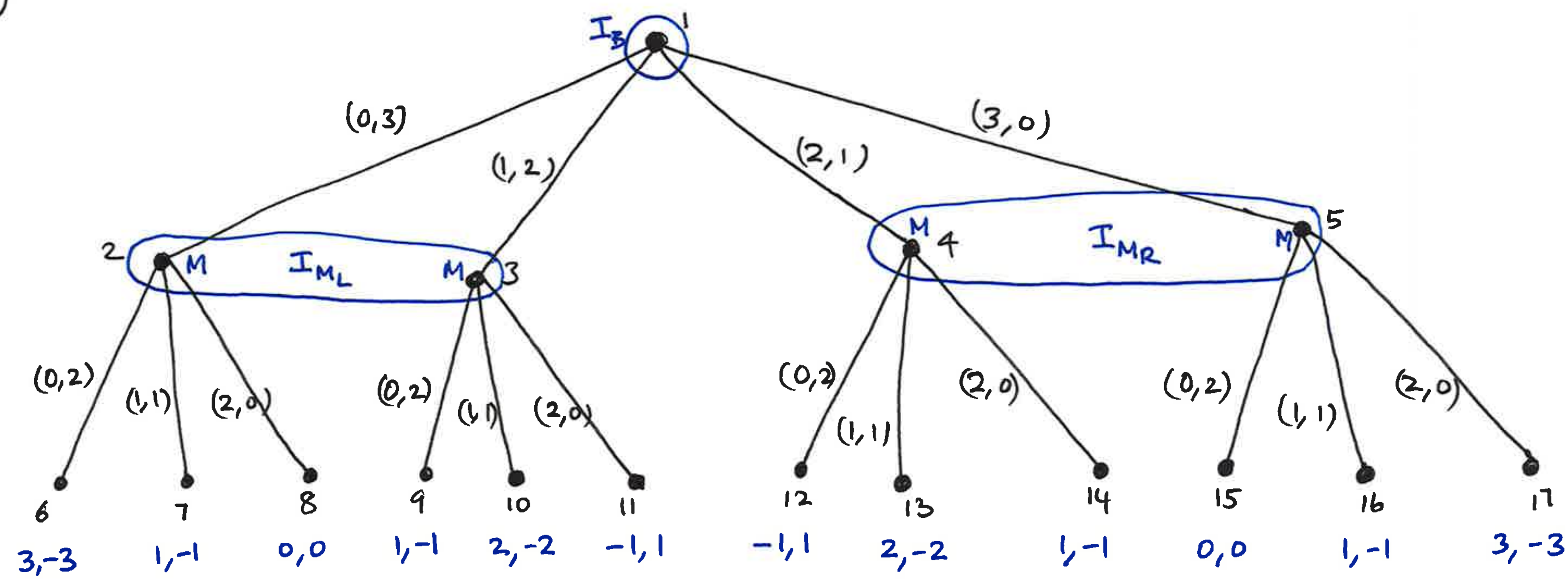
(b) The best responses of B are represented by <u>circles</u>, and those ~~of~~ of M by <u>squares</u>.

B \ M	(0,2)	(1,1)	(2,0)
(0,3)	3, −3	1, −1	0, $\boxed{0}$
(1,2)	1, −1	2, −2	−1, $\boxed{1}$
(2,1)	−1, $\boxed{1}$	2, −2	1, −1
(3,0)	0, $\boxed{0}$	1, −1	3, −3

⟹ <u>No PSNE.</u>

(c), (d) are programming assignments.

(e)



I_B
1
(0,3)
(1,2)
(2,1)
(3,0)
2
M
I_{M_L}
M
3
M
4
I_{M_R}
M
5
(0,2)
(1,1)
(2,0)
(0,2)
(1,1)
(2,0)
(0,2)
(1,1)
(2,0)
(0,2)
(1,1)
(2,0)
6
7
8
9
10
11
12
13
14
15
16
17
3,-3
1,-1
0,0
1,-1
2,-2
-1,1
-1,1
2,-2
1,-1
0,0
1,-1
3,-3

#4

(f) B has one info set I_B $\Rightarrow$ B's choices: $(0,3), (1,2), (2,1), (3,0)$

M has two info. sets I_{M_L} and I_{M_R}

$\Rightarrow$ M's choices: $\{(0,2)_L, (0,2)_R\}$, $\{(0,2)_L, (1,1)_R\}$, $\{(0,2)_L, (2,0)_R\}$,
$\{(1,1)_L, (0,2)_R\}$, $\{(1,1)_L, (1,1)_R\}$, $\{(1,1)_L, (2,0)_R\}$,
$\{(2,0)_L, (0,2)_R\}$, $\{(2,0)_L, (1,1)_R\}$, $\{(2,0)_L, (2,0)_R\}$.

Let $(0,2)$ be x, $(1,1)$ be y and $(2,0)$ be z. Then, we have the foll. bimatrix game.

	(x_L,x_R)	(x_L,y_R)	(x_L,z_R)	(y_L,x_R)	(y_L,y_R)	(y_L,z_R)	(z_L,x_R)	(z_L,y_R)	(z_L,z_R)
$(0,3)$	3,-3	3,-3	3,-3	1,-1	1,-1	1,-1	0,0	0,0	0,0
$(1,2)$	1,-1	1,-1	1,-1	2,-2	2,-2	2,-2	-1,1	-1,1	-1,1
$(2,1)$	-1,1	2,-2	1,-1	-1,1	2,-2	1,-1	-1,1	2,-2	+1,-1
$(3,0)$	0,0	1,-1	3,-3	0,0	1,-1	3,-3	0,0	1,-1	3,-3.

Prob. 2

(a)

	~~L~~	C	~~R~~
~~U~~	1, 1	2, 0	2, 2
~~M~~	0, 3	1, 5	4, 4
D	2, 4	3, 6	3, 0

STEP 1: Since D dominates U, eliminate U.

STEP 2: Since C dominates R, eliminate R.

STEP 3: Since D dominates M, eliminate M.

STEP 4: Since C dominates L, eliminate L.

Solution:

	C
D	$\boxed{3, 6}$

Prob. 3

(a)

	r	p	s
R	0, 0	−1, $\boxed{1}$	1, −1
P	1, −1	0, 0	−1, $\boxed{1}$
S	−1, $\boxed{1}$	1, −1	0, 0

$\Rightarrow$ No PSNE.

(b) Let x denote the row player's mixed strategy, and y denote that of column player.

$\Rightarrow$ Utility of the row player $u(x,y) = x^T U y$

where
$$U = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix}$$

$$\therefore \quad u(x,y) = [x_1 \quad x_2 \quad \underset{1-x_1-x_2}{\overset{\shortparallel}{x_3}}] \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \underset{1-y_1-y_2}{\overset{}{y_3}} \end{bmatrix}$$

$$= x_1 - x_2 - y_1 + y_2 - 3x_1y_2 + 3x_2y_1$$

$$\nabla u = \begin{bmatrix} \nabla_x u \\ \\ \nabla_y u \end{bmatrix} = \begin{bmatrix} 1-3y_2 \\ -1+3y_1 \\ -1+3x_2 \\ 1-3x_1 \end{bmatrix} = 0 \Rightarrow \begin{array}{l} x_1 = x_2 = \frac{1}{3} \\ y_1 = y_2 = \frac{1}{3}. \end{array}$$

In other words, $x = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ and $y = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$

is a candidate solution.

But, why is this a saddle point?

<u>NOTE</u>: The second derivative test for n variables (for $n \geq 3$)

is as follows:

* Let H denote the Hessian matrix of f, i.e.

$$H = \nabla_x^2 f \quad \text{where} \quad x = (x_1, \cdots, x_n).$$

Let $|H| \neq 0$.

* Let D_k = determinant of Hessian in variables $x_1, \cdots, x_k$.

i.e. $$D_k = \begin{vmatrix} \frac{\partial^2 f}{\partial x_1^2} & \cdots & \frac{\partial^2 f}{\partial x_k \partial x_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_1 \partial x_k} & \cdots & \frac{\partial^2 f}{\partial x_k^2} \end{vmatrix}$$

(a) If $D_k > 0 \ \forall \ k = 1, \cdots, n$, then <u>minimum</u>.

(b) If $(-1)^k \cdot D_k > 0 \ \forall \ k = 1, \cdots, n$, then <u>maximum</u>.

(c) Otherwise, <u>saddle point</u>.

In other words, $H = \begin{bmatrix} 0 & 0 & 0 & -3 \\ 0 & 0 & 3 & 0 \\ 0 & 3 & 0 & 0 \\ -3 & 0 & 0 & 0 \end{bmatrix}$

$\Rightarrow D_1 = 0, \quad D_2 = \begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix} = 0, \quad D_3 = \begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & 3 \\ 0 & 3 & 0 \end{vmatrix} = 0$

and $D_4 = |H| = 81 \Rightarrow$ SADDLE POINT

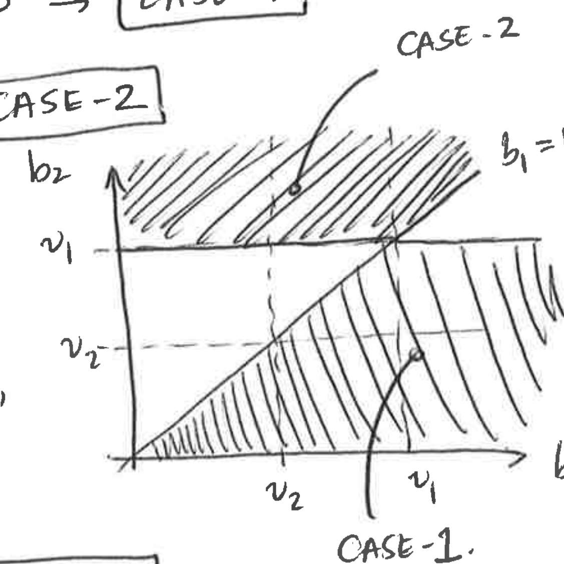
Prob. 4

(a) Player 1's BR:

Player 1's utility $u_1 = \begin{cases} v_1 - b_2 & \text{if } b_1 > b_2 \\ \frac{1}{2}(v_1 - b_2) & \text{if } b_1 = b_2 \\ 0 & \text{otherwise.} \end{cases}$

$\Rightarrow$ Choose $b_1 > b_2$ if $v_1 - b_2 > 0 \rightarrow$ $\boxed{\text{CASE-1}}$

Else, choose $b_1 < b_2 \longrightarrow$ $\boxed{\text{CASE-2}}$

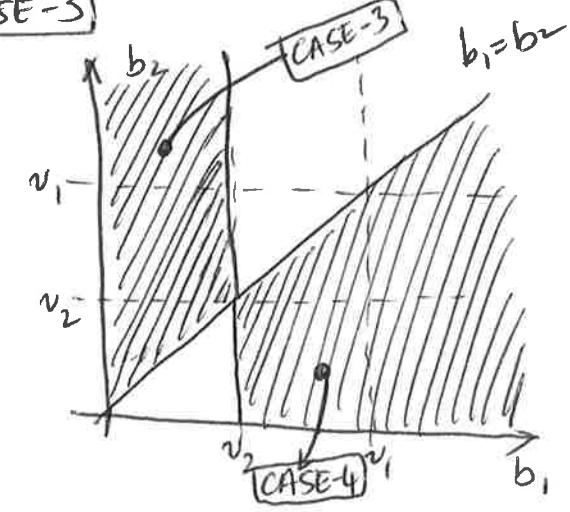


Player 2's BR:

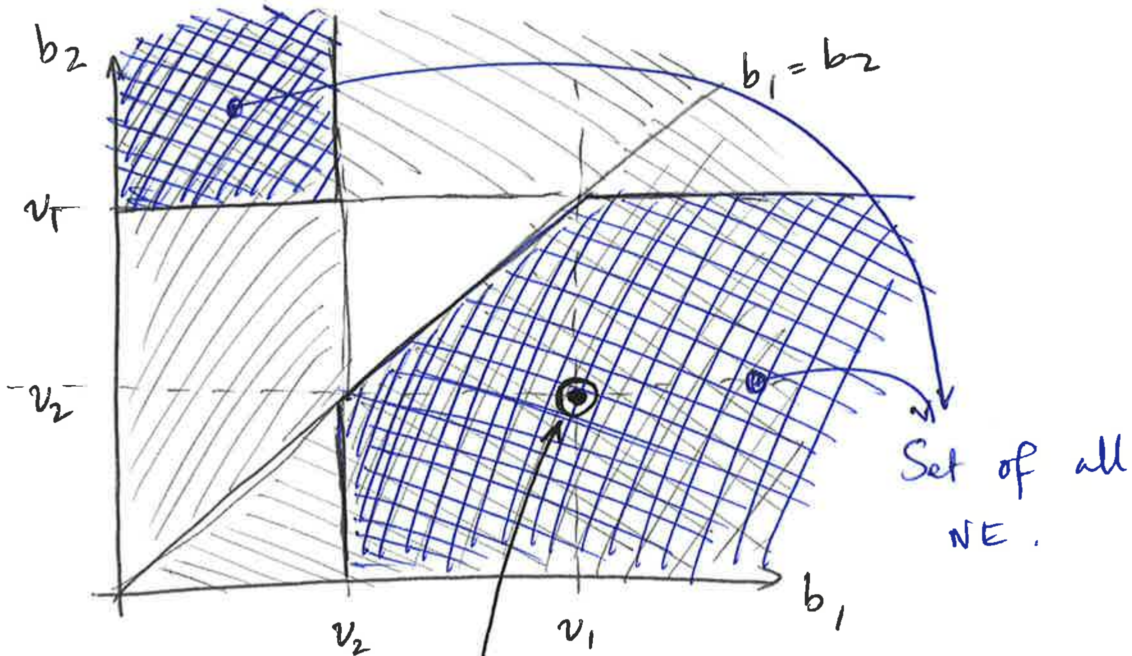
Player 2's utility $u_2 = \begin{cases} v_2 - b_1 & \text{if } b_2 > b_1 \\ \frac{1}{2}(v_2 - b_1) & \text{if } b_2 = b_1 \\ 0 & \text{otherwise.} \end{cases}$

$\Rightarrow$ Choose $b_2 > b_1$ if $v_2 - b_1 > 0 \rightarrow$ $\boxed{\text{CASE-3}}$

Else, choose $b_2 < b_1$. $\longrightarrow$ $\boxed{\text{CASE-4}}$



(b) NE $\Rightarrow$ Intersection of both players' BR regions.



NOTE: If Player i cannot observe v_{-i} and assumes that $v_{-i} \sim \mathcal{U}[0,1]$, then, this is the Bayes' Nash equilibrium.