CS 5408 : GAME THEORY FOR COMPUTING

HW-2 SOLUTIONS

Prob. 1

(a) Let us denote Colonel Blotto as "B"
and Folk Militia as "M".

B has 3 regiments $\Rightarrow$ B's strategies : $(3,0), (2,1), (1,2), (0,3)$

lly, M has 2 regiments $\Rightarrow$ M's strategies : $(2,0), (1,1), (0,2)$.

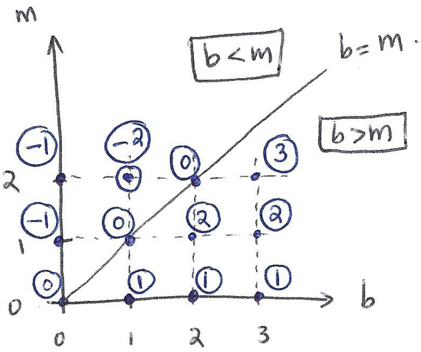
Say, (b, m) represents a strategy profile where
- B deploys $(b, 3-b)$
- M deploys $(m, 2-m)$.

Let U_i represent the utility obtained by B due to deployments in post-i.

$$\Rightarrow U_1(b,m) = \begin{cases} m+1 & \text{if } b > m \\ 0 & \text{if } b = m \\ -b-1 & \text{if } b < m. \end{cases}$$

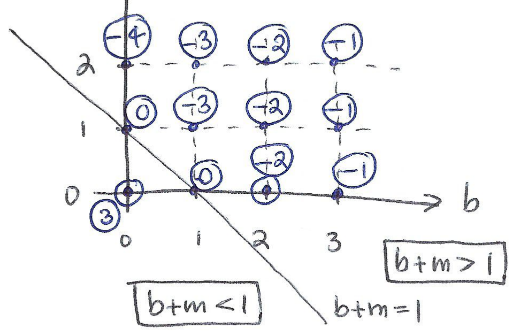
and

$$U_2(b,m) = \begin{cases} \underbrace{(2-m)+1}_{=3-m}, & \text{if } 3-b > 2-m, \text{ i.e. } \boxed{b+m<1} \\ 0, & \text{if } b+m = 1 \\ \underbrace{-(3-b)-1}_{=-4+b}, & \text{if } b+m > 1. \end{cases}$$



$U_1(b,m)$

$U_2(b,m)$



$\Rightarrow \quad U = U_1 + U_2 =$

	(0,2)	(1,1)	(2,0)
(0,3)	$3+0$ 3	$-1+0$ -1	$-1-4$ -5
(1,2)	$1+0$ 1	$0-3$ -3	$-2-3$ -5
(2,1)	$1-2$ -1	$2-2$ 0	$0-2$ -2
(3,0)	$1-1$ 0	$2-1$ 1	$3-1$ $2.$

|||ly, let V_i represent the utility obtained by M due to deployments in post-i.

$$\Rightarrow V_1(b,m) = \begin{cases} -m+1 & \text{if } b>m \\ 0 & \text{if } b=m \\ b+1 & \text{if } b<m \end{cases}, \quad V_2(b,m) = \begin{cases} -3+m & \text{if } b+m<1 \\ 0 & \text{if } b+m=1 \\ 4-b & \text{if } b+m>1. \end{cases}$$

$\Rightarrow V = V_1 + V_2 =$

	(0,2)	(1,1)	(2,0)
(0,3)	$0-3$ -3	$1+0$ 1	$1+4$ 5
(1,2)	$-1+0$ -1	$0+3$ 3	$2+3$ 5
(2,1)	$-1+2$ 1	$-2+2$ 0	$0+2$ 2
(3,0)	$-1+1$ 0	$-2+1$ -1	$-3+1$ $-2.$

#3

In summary, the bimatrix game is given by

	(0,2)	(1,1)	(2,0)
(0,3)	3, −3	−1, 1	−5, 5
(1,2)	1, −1	−3, 3	−5, 5
(2,1)	−1, 1	0, 0	−2, 2
(3,0)	0, 0	1, −1	2, −2

(b) The best responses of B are represented by circles, and those of M by squares.

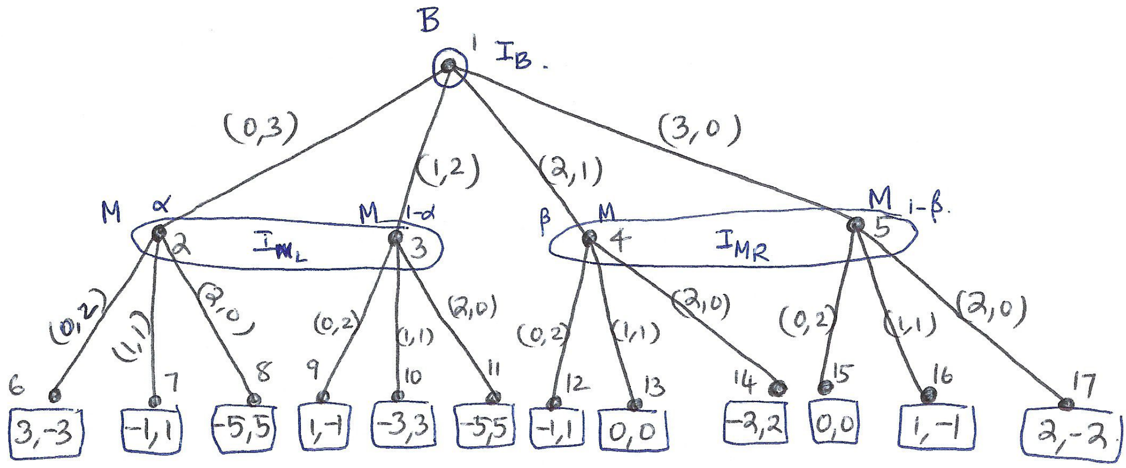
	(0,2)	(1,1)	(2,0)
(0,3)	3, −3	−1, 1	−5, $\boxed{5}$
(1,2)	1, −1	−3, 3	−5, $\boxed{5}$
(2,1)	−1, 1	0, 0	−2, $\boxed{2}$
(3,0)	0, $\boxed{0}$	1, −1	2, −2

⇒ No PSNE

(c), (d) are programming assignments.

(#4)

(e)



(f) B has one info. set, I_B

$\Rightarrow$ B's choices : $(0,3)$, $(1,2)$, $(2,1)$, $(3,0)$

M has 2 info. sets, I_{M_L} and I_{M_R}.

$\Rightarrow$ M's choices : $\{(0,2)_L, (0,2)_R\}$, $\{(0,2)_L, (1,1)_R\}$, $\{(0,2)_L, (2,0)_R\}$
$\{(1,1)_L, (0,2)_R\}$, $\{(1,1)_L, (1,1)_R\}$, $\{(1,1)_L, (2,0)_R\}$,
$\{(2,0)_L, (0,2)_R\}$, $\{(2,0)_L, (1,1)_R\}$, $\{(2,0)_L, (2,0)_R\}$.

~~Let α be the belief at M that the state is 2 given I_{M_L}. III^{ly}, β denote the belief at M that the state is 4 given I_{M_R}.~~

Then, we have the following bimatrix-game.

P.T.O

Let $(0,2)$ be x, $(1,1)$ be y and $(2,0)$ be z.

	(x_L, x_R)	(x_L, y_R)	(x_L, z_R)	(y_L, x_R)	(y_L, y_R)	(y_L, z_R)	(z_L, x_R)	(z_L, y_R)	(z_L, z_R)
0,3	3, −3	3, −3	3, −3	−1, 1	−1, 1	−1, 1	−5, 5	−5, 5	−5, 5
1,2	1, −1	1, −1	1, −1	−3, 3	−3, 3	−3, 3	−5, 5	−5, 5	−5, 5
2,1	−1, 1	0, 0	−2, 2	−1, 1	0, 0	−2, 2	−1, 1	0, 0	−2, 2
3,0	0, 0	1, −1	2, −2	0, 0	1, −1	2, −2	0, 0	1, −1	2, −2

Prob. 2

(a)

	L	C	R
~~U~~	~~1, 1~~	~~2, 0~~	~~2, 2~~
~~M~~	~~0, 3~~	~~1, 5~~	4, 4
D	2, 4	3, 6	3, 0

STEP 1: Since D dominates U, eliminate U.

STEP 2: Since C dominates R, eliminate R.

STEP 3: Since D dominates M, eliminate M.

STEP 4: Since C dominates L, eliminate L.

Solution:

	C
D	$\boxed{3, 6}$

Prob. 3

ⓐ

	r	p	s
R	0, 0	$-1, \boxed{1}$	$1, -1$
P	$1, -1$	0, 0	$-1, \boxed{1}$
S	$-1, \boxed{1}$	$1, -1$	0, 0

$\Rightarrow$ No PSNE.

ⓑ Let x denote the row player's mixed strategy, and y denote that of column player.

$\Rightarrow$ Utility of the row player $u(x,y) = x^T U y$

where
$$U = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix}$$

$$\therefore \quad u(x,y) = [x_1 \quad x_2 \quad \underset{1-x_1-x_2}{\underset{\shortparallel}{x_3}}] \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \underset{1-y_1-y_2}{\underset{\shortparallel}{y_3}} \end{bmatrix}$$

$$= x_1 - x_2 - y_1 + y_2 - 3x_1 y_2 + 3x_2 y_1$$

$$\nabla u = \begin{bmatrix} \nabla_x u \\ \nabla_y u \end{bmatrix} = \begin{bmatrix} 1-3y_2 \\ -1+3y_1 \\ -1+3x_2 \\ 1-3x_1 \end{bmatrix} = 0 \Rightarrow \begin{array}{l} x_1 = x_2 = \frac{1}{3} \\ y_1 = y_2 = \frac{1}{3}. \end{array}$$

In other words, $x = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ and $y = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$

is a candidate solution.

But, why is this a saddle point?

NOTE: The second derivative test for n variables (for $n \geq 3$)

is as follows:

(*) Let H denote the Hessian matrix of f, i.e.

$$H = \nabla_x^2 f \quad \text{where} \quad x = (x_1, \dots, x_n).$$

Let $|H| \neq 0$.

(*) Let D_k = determinant of Hessian in variables $x_1, \dots, x_k$.

$$\text{i.e.} \quad D_k = \begin{vmatrix} \frac{\partial^2 f}{\partial x_1^2} & \cdots & \frac{\partial^2 f}{\partial x_k \partial x_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_1 \partial x_k} & \cdots & \frac{\partial^2 f}{\partial x_k^2} \end{vmatrix}$$

(a) If $D_k > 0 \quad \forall \, k = 1, \dots, n$, then minimum.

(b) If $(-1)^k \cdot D_k > 0 \quad \forall \, k = 1, \dots, n$, then maximum.

(c) Otherwise, saddle point.

In other words, $H = \begin{bmatrix} 0 & 0 & 0 & -3 \\ 0 & 0 & 3 & 0 \\ 0 & 3 & 0 & 0 \\ -3 & 0 & 0 & 0 \end{bmatrix}$

$\Rightarrow D_1 = 0, \quad D_2 = \begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix} = 0, \quad D_3 = \begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & 3 \\ 0 & 3 & 0 \end{vmatrix} = 0$

and $D_4 = |H| = 81 \Rightarrow$ SADDLE POINT

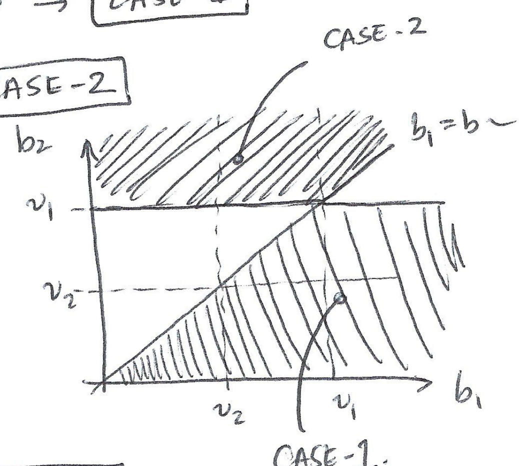
Prob. 4

(a) Player 1's BR:

Player 1's utility $u_1 = \begin{cases} v_1 - b_2 & \text{if } b_1 > b_2 \\ \frac{1}{2}(v_1 - b_2) & \text{if } b_1 = b_2 \\ 0 & \text{otherwise} \end{cases}$

$\Rightarrow$ Choose $b_1 > b_2$ if $v_1 - b_2 > 0 \rightarrow \boxed{\text{CASE-1}}$

Else, choose $b_1 < b_2 \rightarrow \boxed{\text{CASE-2}}$

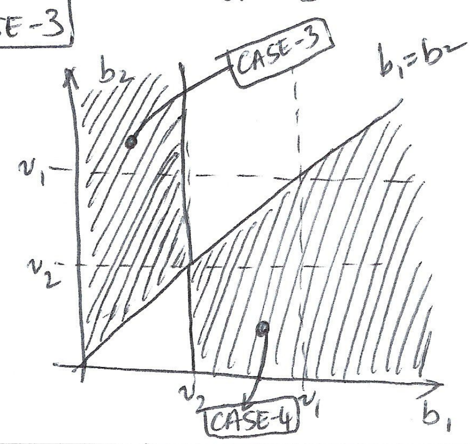


Player 2's BR:

Player 2's utility $u_2 = \begin{cases} v_2 - b_1 & \text{if } b_2 > b_1 \\ \frac{1}{2}(v_2 - b_1) & \text{if } b_2 = b_1 \\ 0 & \text{otherwise.} \end{cases}$

$\Rightarrow$ Choose $b_2 > b_1$ if $v_2 - b_1 > 0 \rightarrow \boxed{\text{CASE-3}}$

Else, choose $b_2 < b_1. \rightarrow \boxed{\text{CASE-4}}$



(b) NE ⇒ Intersection of both players' BR regions.