Missouri University of Science & Technology — Department of Computer Science

Fall 2024 — **CS 5408: Game Theory for Computing**

Homework 5: Mechanism Design

Instructor: *Sid Nadendla* — **Due:** *December 3, 2024*

Problem 1 Sponsored Search Auctions *10 pts.*

Search engines (e.g. Google, Yahoo and Bing) rely on auctions to decide which advertiser's links are shown, in what order and how they are charged. Sponsored search auctions play a significant role in Internet economics, as they continue to generate tens of billions of dollars every year. For example, Google ad revenue amounted to about \$39.58 billion in 2020, which is about 29.4% of the company's overall revenue. Initially, Google's engineers designed *Generalized Second-Price* (GSP) auctions in 2001, and developed Adwords in 2002. Later, Google hired economists to improve their business, who replaced GSP auctions with VCG auctions and other variants. The main difference is that, in GSP auctions, the ith slot is allocated to the ith highest bidder for the price of the $(i+1)^{th}$ highest bid in GSP auctions.

The goods for sale include k *slots* for sponsored links on a search results page. The bidders are the advertisers who have a standing bid on the keyword that was searched on. For example, Toyota, Honda and Volkswagon might be bidders on the keyword *sedan*, while Nikon and Canon might be bidders on the keyword *camera*. We define click-through-rate (CTR), denoted α_j at slot j, as the probability that the end user clicks on this slot. Without any loss of generality, let $\alpha_1 > \alpha_2 > \cdots > \alpha_k$. Also, assume that the ith advertiser has a private valuation v_i for each click on its link. For simplicity, let us make an unreasonable assumption that the CTR of a slot is independent of its occupant.

In this problem, we compare two mechanisms: (i) GSP auctions, and (ii) VCG auctions.

(a) Compute the valuation of ith advertiser in the jth slot, and the social welfare of both the auctions, assuming the bidders bid truthfully.

(b) Let A, B and C compete in a GSP auction with $k = 2$ slots having $\alpha_1 = 2/3$ and $\alpha_2 = 1/3$. If the advertisers' per-click valuations are $v_A = 10$, $v_B = 8$ and $v_C = 4$, show that A obtains a greater utility via bidding $b_A = 5$, as opposed to $b_A = 10$.

(c) Show that A prefers to bid $b_A = 10$, as opposed to $b_A = 5$ in a VCG auction, which is similar to that of (b).

(d) Compute the payments of all the three advertisers in a VCG auction in (c).

Solution:

(a) Note that the ith advertiser gets a reward v_i only when the user clicks the advertisement. Therefore, the valuation of the ith advertiser when his/her advertisement is placed in the jth slot is

$$V_{i,j} = \alpha_j \cdot v_i. \tag{1}$$

If $\beta_{i,j}$ is the auctioneer allocation variable, i.e. if

$$\beta_{i,j} = \begin{cases} 1, & \text{if } j^{th} \text{ slot is assigned to } i^{th} \text{ advertiser,} \\ 0, & \text{otherwise,} \end{cases} \tag{2}$$

then thee social welfare of all advertisers is given by

$$\Phi \;=\; \sum_{i=1}^{N}\sum_{j=1}^{k} \beta_{i,j} \cdot V_{i,j} \;=\; \sum_{i=1}^{N}\sum_{j=1}^{k} \beta_{i,j} \cdot \alpha_j \cdot v_i \tag{3}$$

□

(b) Given $\alpha_1 = 2/3$ and $\alpha_2 = 1/3$ and per-click rewards $v_A = 10$, $v_B = 8$ and $v_C = 4$, the valuations can be computed as

$$\begin{aligned} V_{A,1} &= \alpha_1 \times v_A &= \frac{20}{3} \\ V_{A,2} &= \alpha_2 \times v_A &= \frac{10}{3} \\ V_{B,1} &= \alpha_1 \times v_B &= \frac{16}{3} \\ V_{B,2} &= \alpha_2 \times v_B &= \frac{8}{3} \\ V_{C,1} &= \alpha_1 \times v_C &= \frac{8}{3} \\ V_{C,2} &= \alpha_2 \times v_C &= \frac{4}{3}. \end{aligned} \tag{4}$$

Assuming B and C bit truthfully, in a GSP auction, if $b_A = 10$, $b_B = 8$ and $b_C = 4$, we have $\beta_{A,1} = 1$, $\beta_{A,2} = 0$, $\beta_{B,1} = 0$, $\beta_{B,2} = 1$, $\beta_{C,1} = 0$, $\beta_{C,2} = 0$, i.e. A gets slot 1 and B gets slot 2. In other words, $p_A = V_{B,1} = 16/3$, $p_B = V_{C,2} = 4/3$ and $p_C = 0$. Consequently, we have

$$U_A = \beta_{A,1}V_{A,1} + \beta_{A,2}V_{A,2} - p_A = 20/3 - 16/3 = 4/3.$$

However, if $b_A = 5$, $b_B = 8$ and $b_C = 4$, we have $\beta_{A,1} = 0$, $\beta_{A,2} = 1$, $\beta_{B,1} = 1$, $\beta_{B,2} = 0$, $\beta_{C,1} = 0$, $\beta_{C,2} = 0$, i.e. A gets slot 1 and B gets slot 2. In other words, $p_A = V_{C,2} = 4/3$, $p_B = \alpha_1 b_A = 10/3$ and $p_C = 0$. Consequently, we have

$$U_A = \beta_{A,1}V_{A,1} + \beta_{A,2}V_{A,2} - p_A = 10/3 - 4/3 = 2.$$

Since $U_A(b_A = 5) = 2 > 4/3 = U_A(b_A = 10)$, A does not reveal its bid truthfully in this GSP auction. $\square$

(c) Assuming B and C bit truthfully, in a VCG auction, if $b_A = 10$, $b_B = 8$ and $b_C = 4$, we have $\beta_{A,1} = 1$, $\beta_{A,2} = 0$, $\beta_{B,1} = 0$, $\beta_{B,2} = 1$, $\beta_{C,1} = 0$, $\beta_{C,2} = 0$, i.e. A gets slot 1 and B gets slot 2. In other words,

$$p_A = \Phi_{-A} - \Big(\Phi - \beta_{A,1}V_{A,1} - \beta_{A,2}V_{A,2}\Big) = (8 \times 2/3 + 4 \times 1/3) - (8 \times 1/3 + 0) = 4.$$

Consequently, we have

$$U_A = \beta_{A,1}V_{A,1} + \beta_{A,2}V_{A,2} - p_A = 20/3 - 4 = 8/3.$$

However, if $b_A = 5$, $b_B = 8$ and $b_C = 4$, we have $\beta_{A,1} = 0$, $\beta_{A,2} = 1$, $\beta_{B,1} = 1$, $\beta_{B,2} = 0$, $\beta_{C,1} = 0$, $\beta_{C,2} = 0$, i.e. A gets slot 1 and B gets slot 2. In other words,

$$p_A = \Phi_{-A} - \Big(\Phi - \beta_{A,1}V_{A,1} - \beta_{A,2}V_{A,2}\Big) = (8 \times 2/3 + 4 \times 1/3) - (8 \times 2/3) = 4/3.$$

Consequently, we have

$$U_A = \beta_{A,1}V_{A,1} + \beta_{A,2}V_{A,2} - p_A = 10/3 - 4/3 = 2.$$

Since $U_A(b_A = 5) = 2 < 8/3 = U_A(b_A = 10)$, A reveal bids truthfully in VCG auction. $\square$

(d) If $b_A = 10$, $b_B = 8$ and $b_C = 4$, the payments in VCG auction are computed as

$$\begin{aligned} p_A &= \Phi_{-A} - \Big(\Phi - \beta_{A,1}V_{A,1} - \beta_{A,2}V_{A,2}\Big) = (8 \times 2/3 + 4 \times 1/3) - (8 \times 1/3) \\ &= 4, \end{aligned}$$

$$\begin{aligned} p_B &= \Phi_{-B} - \Big(\Phi - \beta_{B,1}V_{B,1} - \beta_{B,2}V_{B,2}\Big) = (10 \times 2/3 + 4 \times 1/3) - (10 \times 2/3) \\ &= 4/3, \end{aligned}$$

$$\begin{aligned} p_C &= \Phi_{-C} - \Big(\Phi - \beta_{C,1}V_{C,1} - \beta_{C,2}V_{C,2}\Big) = (10 \times 2/3 + 8 \times 1/3) - (10 \times 2/3 + 8 \times 1/3) \\ &= 0. \end{aligned}$$

$\square$

Problem 2 Voting Rules ***10 pts.***

There were three candidates in 1998 Minnesota governatorial race: *Jesse Ventura* (denoted J, former professional wrestler and radio shock-jock), *Skip Humphrey* (denoted S, Minnesota Attorney General and a Democrat) and *Norm Coleman* (denoted N, St. Paul Mayor and a Republican). J was declared the winner of the election. Post-election surveys indicate (viewed as percentages) the voter preferences as follows:

Preference	**Percentage of Voters**
N $\succ S \succ J$	35%
S $\succ N \succ J$	28%
J $\succ N \succ S$	20%
J $\succ S \succ N$	17%

(a) Who won this election under Plurality rule? Explain.

(b) Who won this election under Borda count? Explain.

(c) Which candidate is ranked first by the largest number of voters? Which candidate is ranked last by the largest number of voters?

(d) Evaluate winners in pairwise contests, and show that the plurality rule does not satisfy Condorcet winner criterion?

Solution:

For simplicity, assume there are 100 voters in total.

(a) Plurality rule assigns 1 point to the top-preferred candidate. Then, N gets 35 points, S gets 28 points and J gets 37 points in total. Since J gets the highest number of points, J is the winning candidate according to Plurality rule.

(b) Borda count assigns 2 points to the top-preferred candidate and 1 point to the second-most preferred candidate. Consequently,

- N gets $35 \times 2 + 28 \times 1 + 20 \times 1 = 118$ points.
- S gets $35 \times 1 + 28 \times 2 + 17 \times 1 = 108$ points.
- J gets $20 \times 2 + 17 \times 2 = 74$ points.

Therefore, N is the winning candidate as per Borda count.

(c) The number of voters ranking N, S and J as top candidates are 35, 28 and 37 respectively. On the contrary, the number of voters ranking N, S and J as least preferred candidates are

17, 20 and 63 respectively. In other words, J is both the candidate with largest number of top votes and the least preferred votes.

(d) Following are the outcomes of pairwise elections:

- N vs S: N gets $35 + 20 = 55$ votes, S gets $28 + 17 = 45$ votes.
- N vs J: N gets $35 + 28 = 63$ votes, J gets $20 + 17 = 37$ votes.
- J vs S: J gets $20 + 17 = 37$ votes, S gets $35 + 28 = 63$ votes.

In other words, N wins both pairwise elections, S only wins one of the pairwise elections and J wins none. Therefore, N wins according to the Cordorcet's criterion.

Furthermore, J wins according to Plurality rule, but J does not win even a single pairwise election. Therefore, Plurality winner does not satisfy Cordorcet's criterion in this example.

$\square$