Topic 4: Games with Information Asymmetry

Outcomes & Objectives

- Be proficient in modeling information asymmetry in games
  - Imperfect Information
  - Bayesian Games
- Be proficient with incomplete extensive-form games.
  - Develop a solution concept (inspired from subgame perfect equilibrium) to solve Bayesian games in extensive-form.
- Be proficient in solving repeated games.
  - Investigate the effects of long-term strategic interactions, as opposed to short-term interactions.
  - Develop a solution concept which accounts for temporal dynamics (e.g. discounting behavior).

Observability: Perfect vs. Imperfect Information

Definition

A game where every agent can observe every other player's actions is called a ***perfect information game***.

Example: Chess

Imperfect Information: Player's actions are not observable!
Example: Poker

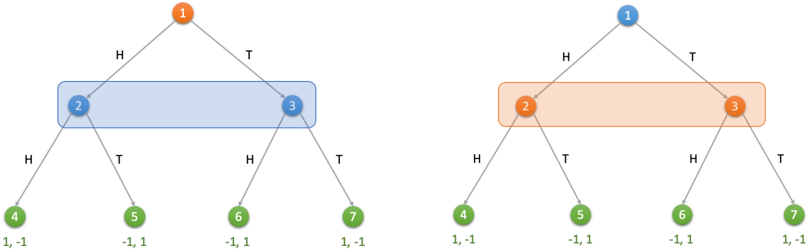
Games which are sequential, and which have chance events, but no secret information, are considered games of perfect information.

Example: Monopoly (uncertainty due to rolling dice.)

More on Imperfect Information Games...

Games with simultaneous moves are generally considered imperfect information games!

Matching Pennies with Simultaneous Moves:



Group all indistinguishable states into sets to disclose available information at each agent!

Nature's Role in Games

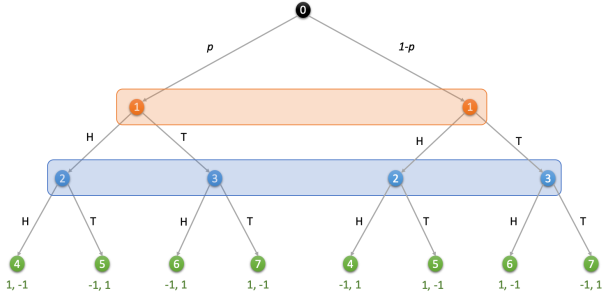
- Players play the left game with probability p,
- Players play the right game with probability $1 - p$,

Left Game (p)

Player 1 \ Player 2	Heads	Tails
Heads	1, -1	-1, 1
Tails	-1, 1	1, -1

Right Game ($1 - p$)

Player 1 \ Player 2	Heads	Tails
Heads	-1, 1	1, -1
Tails	1, -1	-1, 1



Agent Types: Complete vs. Incomplete Information

Sometimes, players may not know each others' types.

Such games are called incomplete-information (or Bayesian) games.

Definition

A ***Bayesian (or incomplete information game) game*** Γ is defined as a tuple $(\mathcal{N}, \Theta, \boldsymbol{p}, \mathcal{C}, \mathcal{U})$, where

- $\mathcal{N} = \{1, \cdots, N\}$ is the set of N players (agents),
- $\Theta = \{\Theta_1, \cdots, \Theta_N\}$, where Θ_i is the set of types of player i,
- $\boldsymbol{p} = \{p_1, \cdots, p_N\}$, where $p_i : \Theta_i \to \Delta(\Theta_{-i})$ is the conditional belief over the set of types of other players, given the type of player i,
- $\mathcal{C} = \mathcal{C}_1 \times \cdots \times \mathcal{C}_N$ is the strategy profile space, where $\mathcal{C}_i$ represents the set of strategic choices (actions) available at the i^{th} player,
- $\mathcal{U} = \{u_1, \cdots, u_N\}$ is the set of utility functions, where $u_i : \mathcal{C}_i \to \mathbb{R}$ represents the utility function at the i^{th} player.

Example: Competition in Job Markets

One final note...

How can we solve Bayesian games in normal-form?

In most game-theoretic settings, players does not have complete knowledge about other players and their utilities.

Examples:

- *Bargaining/Auctions/Contests:* Valuations of other players are unknown.
- *Markets:* Intellectual properties are dealt as a secret, which results in uncertain production costs about other players.
- *Signaling games:* The sender's intent behind sharing a signal is usually unknown to receivers.

and many more...

Bayesian Games in Normal-Form

Definition

A ***Bayesian (or incomplete information game) game*** Γ is defined as a tuple $(\mathcal{N}, \Theta, \boldsymbol{p}, \mathcal{C}, \mathcal{U})$, where

- $\mathcal{N} = \{1, \cdots, N\}$ is the set of N players (agents),
- $\Theta = \{\Theta_1, \cdots, \Theta_N\}$, where Θ_i is the set of types of player i,
- $\boldsymbol{p} = \{p_1, \cdots, p_N\}$, where $p_i : \Theta_i \to \Delta(\Theta_{-i})$ is the conditional belief over the set of types of other players, given the type of player i,
- $\mathcal{C} = \mathcal{C}_1 \times \cdots \times \mathcal{C}_N$ is the strategy profile space, where $\mathcal{C}_i$ represents the set of strategic choices (actions) available at the i^{th} player,
- $\mathcal{U} = \{u_1, \cdots, u_N\}$ is the set of utility functions, where $u_i : \mathcal{C}_i \to \mathbb{R}$ represents the utility function at the i^{th} player.

Note: The label "*Bayesian games*" is coined because $p_i(\theta_{-i}|\theta_i)$ can be computed from prior probability distribution $p(\theta_i, \theta_{-i})$ using Bayes Rule, as shown below:

$$p_i(\theta_{-i}|\theta_i) = \frac{p(\theta_{-i}, \theta_i)}{\displaystyle\int p(\theta_{-i}, \theta_i) d\theta_{-i}}$$

Bayesian Nash Equilibrium (BNE)

Consider a game with finite types of agents:

- Let $\boldsymbol{\sigma}_i(\theta_i)$ denote the mixed strategy employed by Player i of type $\theta_i \in \Theta_i$.
- Expected utility of the i^{th} player of type θ_i is given by

$$U_i(\sigma_i, \sigma_{-i}, \theta_i) = \sum_{\theta_{-i} \in \Theta_{-i}} \left[p_i(\theta_{-i} | \theta_i) \sum_{c \in \mathcal{C}} \left(\prod_{j \in \mathcal{N}_{-i}} \sigma_j(c_j | \theta_j) \right) \sigma_i(c_i) u_i(c_i, c_{-i}(\theta_{-i}), \boldsymbol{\theta}) \right],$$

Definition

A ***Bayesian-Nash equilibrium*** is a strategy profile $\boldsymbol{\sigma} = \{\boldsymbol{\sigma}_1, \cdots, \sigma_N\} \in \Delta(\mathcal{C})$, if for all $i \in \mathcal{N}$ and for all $\theta_i \in \Theta_i$, we have

$$\sigma_i(\theta_i) \in \underset{\sigma_i \in \Delta(C_i)}{\text{arg max}} \ U_i(\sigma_i, \sigma_{-i}, \theta_i)$$

Theorem

There always exists a mixed-strategy BNE in any finite Bayesian game.

BNE in Second-Price Auctions

- Two players $\mathcal{N} = \{1, 2\}$.
- Players valuate the auctioned item as v_1 and v_2 respectively.
- However, the other players does not complete knowledge about valuations! Only know $p(v_{-i}|v_i) = \mathcal{U}[0, 1]$, a uniform distribution in the range $[0, 1]$.
- Utility of player i is

$$u_i(b_i, b_{-i}, v_i) = \begin{cases} v_i - b_{-i}, & \text{if } b_i > b_{-i} \\ \dfrac{v_i - b_{-i}}{2}, & \text{if } b_i = b_{-i} \\ 0, & \text{otherwise.} \end{cases}$$

As opposed to the complete information game,

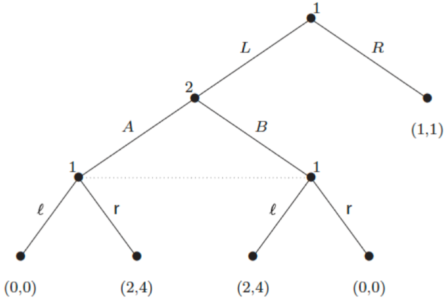
Theorem

There exists a ***unique*** Bayesian equilibrium in second-price auctions, which is the case when bidders choose bids equal to their valuations, i.e. $b_i^* = v_i$.

Solving Imperfect Extensive Games...

What if, we have information sets in an extensive game?

Consider the following example:



Note that the subgame at Player 2's node is the smallest subgame!

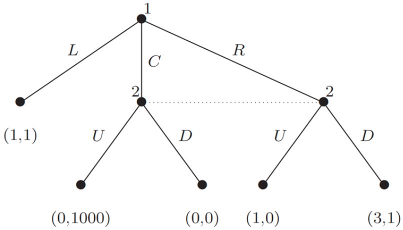
- Idea: Reduce this subgame into its strategic game and continue – *Inefficient*!
- Can we operate directly on the extensive-form representation?

Is Subgame Perfect Equilibrium Suitable?

What do we mean by a subgame in imperfect extensive games?
What if, we define a subforest (a collection of subgames) at each information set?

Example:

- Pure strategies: $P_1 \Rightarrow \{L, C, R\}$, $P_2 \Rightarrow \{U, D\}$
- PSNE: (L, U), (R, D)
- Can either of these equilibria be considered *subgame perfect*?
  - Left subtree – U dominates D
  - Right subtree – D dominates U
- But, R dominates C at Player 1
- So, (R, D) is subgame perfect!



Lesson: *The requirement that we need best responses in all subgames is too simplistic!*

Behavioral Strategies in Extensive Games

If the set of information sets at the i^{th} player is denoted as $\mathcal{I}_i$, then

- ***Pure strategies*** *in extensive-form games are choice tuples at a given player, where each entry is picked from one of his/her information sets.*

  Notation: $\boldsymbol{c}_i = (c_{i,j_1}, \cdots, c_{i,j_L}) \in \mathcal{C}_i$, where c_{i,j_ℓ} is the ℓ^{th} strategy in the j^{th} information set in $\mathcal{I}_i$.

- ***Mixed strategies*** *are lotteries on pure strategies.*

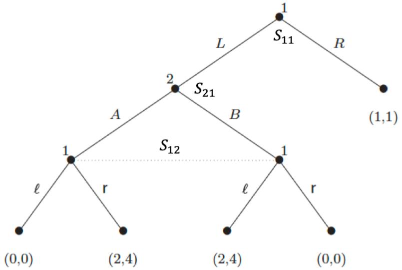
  Notation: $\boldsymbol{\sigma}_i \in \Delta(\mathcal{C}_i)$.

However, in extensive games, we can define another type of lottery, as shown below:

Definition

Given a extensive game $\Gamma = (\mathcal{N}, \mathcal{C}, G, \pi, P, \mathcal{I}, \mathcal{U})$, a ***behavioral strategy*** at the i^{th} player is a conditional lottery $\boldsymbol{\pi}_i \in \Delta(D_{i,s})$ on the choice set $D_{i,s}$ available within the state (node) s in a given information set at the i^{th} player.

Behavioral Strategies: An Example



- Information Sets: $\mathcal{I}_1 = \{S_{11}, S_{12}\}$, $\mathcal{I}_2 = \{S_{21}\}$
- Pure strategies: $\mathcal{C}_1 = \{(L, \ell), (L, r), (R, \ell), (R, r)\}$, $\mathcal{C}_2 = \{A, B\}$
- Mixed strategy: $\boldsymbol{\sigma}_1 = \{p_1, p_2, p_3, 1 - p_1 - p_2 - p_3\}$, $\boldsymbol{\sigma}_2 = \{q, 1 - q\}$
- Behavioral strategy for P_1: $\boldsymbol{\pi}_1 = \{\boldsymbol{\pi}_{11}, \boldsymbol{\pi}_{12}\}$, where
  - $\boldsymbol{\pi}_{11} = \boldsymbol{\pi}_1(S_{11}) = \{L : \alpha_{11}, R : 1 - \alpha_{11}\}$
  - $\boldsymbol{\pi}_{12} = \boldsymbol{\pi}_1(S_{12}) = \{\ell : \alpha_{12}, r : 1 - \alpha_{12}\}$
- Behavioral strategy for P_2: $\boldsymbol{\pi}_2(S_{21}) = \{A : \beta_{21}, B : 1 - \beta_{21}\}$.

Equivalence between Mixed and Behavioral Strategies

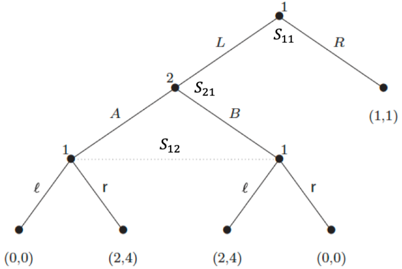
Theorem

In a game of perfect recall, for any mixed strategy, there is an outcome-equivalent behavioral strategy, and vice versa.

In the following example, we have

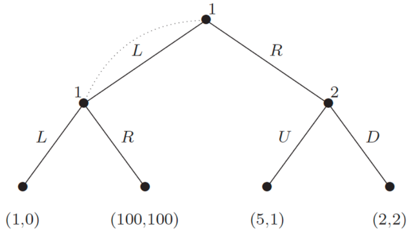
$$\boldsymbol{\sigma}_1 = \{(L,\ell):0.5,(R,\ell):0.5\} \equiv \boldsymbol{\pi}_1 = \{\boldsymbol{\pi}_{11} = \{L:0.5, R:0.5\},\ \boldsymbol{\pi}_{12} = \{\ell:1, r:0\}\}$$

since Player 2 believes that Player 1 does not play r in S_{12}, given $\boldsymbol{\sigma}_1$.



Extensive Games with Imperfect Recall

Behavioral and mixed strategies are incomparable in general.



- Pure strategies: $P_1 \Rightarrow \{L, R\}$, $P_2 \Rightarrow \{U, D\}$
- Mixed strategy for P_1: $(L : \pi, R : 1 - \pi))$ – once P_1 samples his/her mixed strategy, that strategy will be chosen in both nodes within the information state.
- Unique NE: (R, D)
- Behavioral strategy at P_1: $\{L : p, R : 1 - p\}$ (randomize afresh every time.)
  - $U_1(D) = p\,[p + 100(1 - p)] + (1 - p)2$
  - $\underset{p \in [0,1]}{\arg\max}\ U_1(D) = p^* = \dfrac{98}{198}$.
  - A new equilibrium in behavioral strategies: $\left\{ \left(\frac{98}{198}, \frac{100}{198} \right), (0, 1) \right\}$

Equilibrium in Perfect-Recall Games

Eliminate *nonsensical* NE using behavioral strategies!

Definition

A ***extensive-form Nash equilibrium*** is a mixed strategy Nash equilibrium $\boldsymbol{\sigma}$ that is equivalent to an assessment pair $(\boldsymbol{\pi}, \boldsymbol{\mu})$, where the behavioral strategy $\boldsymbol{\pi}$ is consistent with $\boldsymbol{\sigma}$ and a set of beliefs $\boldsymbol{\mu}$ according to Bayes' rule.

Can't we operate directly on the tree representation?

Definition

A ***behavioral equilibrium*** is a pair $(\boldsymbol{\pi}, \boldsymbol{\mu})$ which satisfies:

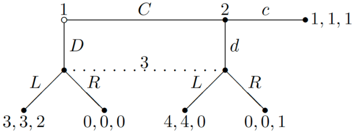
- **Sequential Rationality:** Given any alternative strategy $\boldsymbol{\pi}'_i$ at the i^{th} player and his/her belief $\boldsymbol{\mu}_{i,j_s}$ on the state j_s within an information set $\mathcal{I}_{i,j}$, we have

$$u_i(\boldsymbol{\pi}|\mathcal{I}_{i,j_s}, \boldsymbol{\mu}_{i,j_s}) \geq u_i(\boldsymbol{\pi}'_i, \boldsymbol{\pi}_{-i}|\mathcal{I}_{i,j_s}, \boldsymbol{\mu}_{i,j_s}), \text{ and}$$

- **Consistency:** Assuming that all the players picked a strategy $\boldsymbol{\pi}$ until reaching a state s, there exists a belief $\boldsymbol{\mu}(s)$ that is consistent with Bayes' rule.

Example: Selten's Horse

Induced Normal-Form Game:



Nash Equilibria:

- $NE_1 : \left\{D : 1,\ c : \left[\frac{1}{3}, 1\right],\ L : 1\right\}$
- $NE_2 : \left\{C : 1,\ c : 1,\ \sigma_3(R) \in \left[\frac{3}{4}, 1\right]\right\}$

Behavioral Equilibrium:

- NE_1 is not a behavioral equilibrium (violates sequential rationality at Player 2)
- NE_2 is sequentially rational. But, how about the beliefs in $\mathcal{I}_3$?
- Let $\sigma^\epsilon = \left\{\sigma_1^\epsilon(C) = 1 - \epsilon,\ \sigma_2^\epsilon(d) = \frac{2\epsilon}{1-\epsilon},\ \sigma_3^\epsilon(R) = \sigma_3(R) - \epsilon\right\}$, for a small ϵ.
- $\mu_{3,\ell} = \dfrac{\sigma_1^\epsilon(D)}{\sigma_1^\epsilon(D) + \sigma_1^\epsilon(C) \cdot \sigma_2^\epsilon(d)} = \dfrac{1}{3}.$

Sequential Equilibrium: A Refinement

Definition

An assessment pair $(\boldsymbol{\pi}, \boldsymbol{\mu})$ is a ***sequential equilibrium*** if

1. Given any alternative strategy $\boldsymbol{\pi}'_i$ at the i^{th} player and his/her belief $\boldsymbol{\mu}_{i,j_s}$ on the state j_s within an information set $\mathcal{I}_{i,j}$, we have
$$u_i(\boldsymbol{\pi}|\mathcal{I}_{i,j_s}, \boldsymbol{\mu}_{i,j_s}) \geq u_i(\boldsymbol{\pi}'_i, \boldsymbol{\pi}_{-i}|\mathcal{I}_{i,j_s}, \boldsymbol{\mu}_{i,j_s}),$$
2. **Consistency:** Assuming that all the players picked a strategy $\boldsymbol{\pi}$ until reaching a state s, there exists a belief $\boldsymbol{\mu}(s)$ that is consistent with Bayes' rule.
3. **Convergence:** There exists a sequence $\left\{ \left(\boldsymbol{\pi}^{(n)}, \boldsymbol{\mu}^{(n)} \right) \right\}_{n=1}^{\infty}$ such that $(\boldsymbol{\pi}, \boldsymbol{\mu}) = \lim_{n \to \infty} \left(\boldsymbol{\pi}^{(n)}, \boldsymbol{\mu}^{(n)} \right)$, where $\boldsymbol{\mu}_n$ is a belief that is consistent with the behavioral strategy $\boldsymbol{\pi}_n$, for all $n = 1, 2 \cdots$.

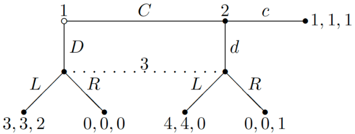
Theorem

- Every finite game of perfect recall has a sequential equilibrium.
- Every subgame-perfect equilibrium is a sequential equilibrium, but the converse is not true in general.

Example: Selten's Horse

Nash Equilibria:

- $NE_1 : \left\{D : 1,\ c : \left[\frac{1}{3}, 1\right],\ L : 1\right\}$
- $NE_2 : \left\{C : 1,\ c : 1,\ \sigma_3(R) \in \left[\frac{3}{4}, 1\right]\right\}$



Behavioral Equilibrium:

- NE_1 is not a behavioral equilibrium (violates sequential rationality at Player 2)
- NE_2 is sequentially rational with $\mu_{3,\ell} = \dfrac{1}{3}$.

Sequential Equilibrium:

- NE_1 is not a sequential equilibrium (violates sequential rationality at Player 2)
- For every equilibrium of type NE_2, there exists a sequential equilibrium with the following sequence:
  - $\sigma^\epsilon = \left\{\sigma_1^\epsilon(C) = 1 - \epsilon,\ \sigma_2^\epsilon(d) = 2\epsilon,\ \sigma_3^\epsilon(R) = \sigma_3(R) - \epsilon\right\}.$
  - $\mu_{3,\ell} = \dfrac{\sigma_1^\epsilon(D)}{\sigma_1^\epsilon(D) + \sigma_1^\epsilon(C)\sigma_2^\epsilon(d)} = \dfrac{1}{3 - 2\epsilon} \xrightarrow{\epsilon \to 0} \dfrac{1}{3}$

Perfect Bayesian Extensive-Form Games

- Let Θ_i denote the set of types of the i^{th} player with a prior belief p_i.
- Let $\boldsymbol{p} = \{p_i\}_{i \in \mathcal{N}}$ be the profile of prior beliefs.
- Perfect Bayesian equilibrium $\Rightarrow$ A generalization of *behavioral equilibrium*.

Definition

A pair $(\boldsymbol{\pi}, \boldsymbol{\mu})$ is a ***perfect Bayesian equilibrium*** if

1. The mixed strategy profile $\boldsymbol{\pi}$ is ***sequentially rational***, given $\boldsymbol{\mu}$.
2. There exists a belief system $\boldsymbol{\mu}$ that is ***consistent*** with the mixed strategy profile $\boldsymbol{\pi}$ and the prior belief about agents' type $\boldsymbol{p}$.

Theorem

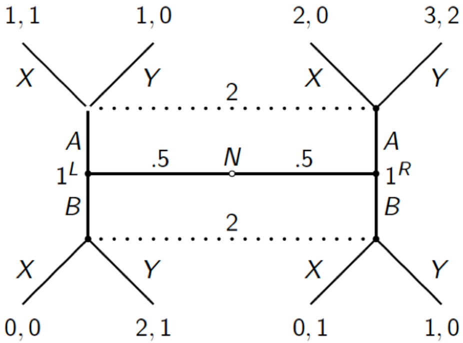
Every finite Bayesian extensive game has a perfect Bayesian equilibrium.

Theorem

Every perfect Bayesian equilibrium is a Nash equilibrium.

Signaling Games

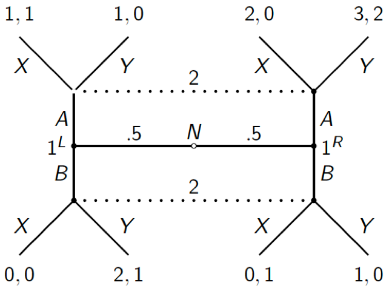
Consider the following sender-receiver (signaling) game, where the sender is characterized by one of the two types.



Sender's strategies:

- **Pooling Strategies:** AA, BB – *Sender does not reveal its type*
- **Separating Strategies:** AB, BA – *Sender reveals its type*

Signaling Games (cont...)



Two perfect Bayesian equilibria: Proof will be provided in a separate handout.

- ***Pooling Equilibrium:*** (AA, YX) when $\mu(L|A) = 0.5$ and $\mu(L|B) \leq 0.5$
  - How did we compute $\mu(L|B)$ if Player 1 plays AA?
  - Note: X is the best response to B only when $\mu(L|B) \leq 0.5$
- ***Separating Equilibrium:*** (BA, YY) when $\mu(L|A) = 0$ and $\mu(L|B) = 1$.

Pooling in e-Bay markets: *Buyers do not trust sellers who always signal high quality products, regardless of their true type.*

Repeated Games

Repeated interactions stimulate agents to track players' reputation over time and design strategies either to retaliate, or to act prosocially.

- Why participate in free crowdsourcing platforms (e.g. Wikipedia, Google's Crowdsource) even though workers do not get paid?
- Why look after neighbor's house when they are away?

Two types:

- Finite-Horizon Repeated Games – similar to extensive-form games
- Infinite-Horizon Repeated Games – no outcome nodes in the game!

Our Focus: ***Infinitely repeated games***

How to define choices and utilities in an infinitely repeated game?

Choices in Infinitely Repeated Games

Definition

Assuming that the players only observe strategy profiles at the end of each repetition stage, any ***choice in an infinitely repeated game*** is of the form

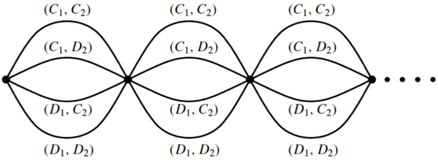
$$\boldsymbol{c} = \{\boldsymbol{c}_1, \boldsymbol{c}_2, \cdots, \boldsymbol{c}_k, \cdots\} \in \mathcal{C}_\infty,$$

where $\boldsymbol{c}_i \in \mathcal{C}_1 \times \cdots \times \mathcal{C}_N$ is the strategy profile chosen in the i^{th} iteration.

Consider the following infinitely repeated prisoner's dilemma:

		Prisoner 2	
		C_2	D_2
Prisoner 1	C_1	2, 2	0, 3
	D_1	3, 0	1, 1

$\Rightarrow$



- ***Defection Strategy:*** $\boldsymbol{c}_i = (D_1, D_2)$, for all $i = 1, 2, \cdots$
- ***Grim (Trigger) Strategy:*** At the j^{th} player, we have

$$c_{i,j} = \begin{cases} D_j, & \text{if } c_{t,-j} = D_{-j} \\ C_j, & \text{otherwise} \end{cases}, \text{ for all } i = t+1, t+2, t+3, \text{ for any } t = 1, 2, \cdots.$$

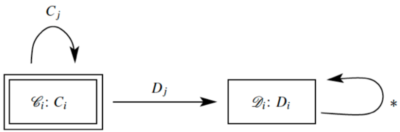
Representing Choices as Finite Machines

- Uncountably infinite strategy spaces ⇒ High strategic complexity
- **Finite automata** ⇒ tractable way to study infinitely repeated choices.
- ***Moore machine:*** Current strategy at a given player is a function of his current state, which in turn is computed using a transition function of the player's previous state and the strategy profile in the previous iteration.

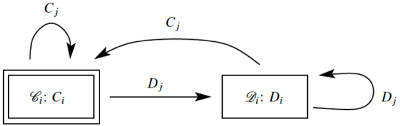
$$s_{i,t} = h\left(s_{i,t-1}, \boldsymbol{c}_{t-1}\right)$$

Examples:

- ***Grim (Trigger) Strategy:*** Both players start playing C



- ***Tit-for-Tat (TfT):*** Both players start playing C



Average Utilities

How should we define choice utilities in an infinitely repeated game?

Definition

Given an infinite sequence of one-stage utilities $u_{i,1}, u_{i,2}, \cdots$ at the i^{th} player, the ***average utility*** of the i^{th} player is defined as

$$\bar{u}_i = \lim_{k \to \infty} \frac{1}{k} \sum_{j=1}^{k} u_{i,j}.$$

Claim

If the choice is represented as a Moore machine with the longest cycle T, then the ***average utility*** of the i^{th} player can be computed as

$$\bar{u}_i = \frac{1}{T} \sum_{j=1}^{T} u_{i,j}.$$

Discounted Utilities

What if, the players build frustration with time?

Definition

Given an infinite sequence of one-stage utilities $u_{i,1}, u_{i,2}, \cdots$ at the i^{th} player, and a discounting factor $\beta \in [0,1]$, the ***discounted utility*** of the i^{th} player is defined as

$$\bar{u}_i = \lim_{k \to \infty} \frac{1}{k} \sum_{j=1}^{k} \beta^{j-1} u_{i,j}.$$

Claim

If the choice is represented as a Moore machine with the longest cycle T, then the ***discounted utility*** of the i^{th} player can be computed as

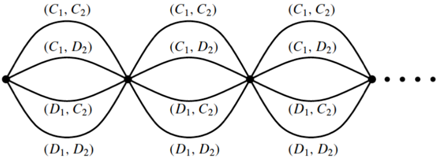
$$\bar{u}_i = \frac{1}{(1-\beta)T} \sum_{j=1}^{T} \beta^{j-1} u_{i,j}.$$

Equilibrium: Repeated Prisoner's Dilemma

Prisoner 2

Prisoner 1	C_2	D_2
C_1	2, 2	0, 3
D_1	3, 0	1, 1

$\Rightarrow$



Claim

$(Grim, Grim)$ is a Nash equilibria in repeated Prisoner's Dilemma with β-discounted utilities, when $\beta \geq \frac{1}{2}$.

- Assume P_{-i} plays C_{-i} for the first T times.
- Let P_i choose C_i for the first $T-1$ times and then choose D_i at time T.
- Then, P_i's best response utility is

$$\begin{aligned} u_i &= \sum_{t=1}^{T-1} \beta^t 2 + 3\beta^T + \sum_{t=T+1}^{\infty} \beta^t 1 \\ &= 2\frac{1-\beta^T}{1-\beta} + 3\beta^T + \beta^{T+1}\frac{1}{1-\beta} \end{aligned}$$

Equilibrium: Repeated Prisoner's Dilemma

But, Grim includes the possibility where C_i can be played against C_{-i} forever!

Is C_i a best response to C_{-i} as well?

- Note that if P_i continued to play C_i for all $t \geq T$, P_i's best response utility is

$$\begin{aligned} u_i &= \sum_{t=1}^{T-1} \beta^t 2 + 2\beta^T + \sum_{t=T+1}^{\infty} \beta^t 2 \\ &= 2\frac{1-\beta^T}{1-\beta} + 2\beta^T + \beta^{T+1}\frac{2}{1-\beta} \end{aligned}$$

- C_i is the best response to C_{-i} only when

$$2\beta^T + \beta^{T+1}\frac{2}{1-\beta} \geq 3\beta^T + \beta^{T+1}\frac{1}{1-\beta}, \text{ for any } T.$$

- Upon simplification, Grim is the best response to Grim only when $\beta \geq \frac{1}{2}$.

In other words, both players should be patient enough for $(Grim, Grim)$ to be a Nash equilibrium!

Enforceability and Feasibility

Is there an easier way to validate, if a machine tuple is NE?

For that, we need to define two properties of utility profiles:

Definition

Given the minimax value of the i^{th} player as $v_i = \min_{\boldsymbol{c}^{(-i)}} \max_{\boldsymbol{c}^{(i)}} u_i(\boldsymbol{c}^{(i)}, \boldsymbol{c}^{(-i)})$, the utility profile $u = \{u_1, \cdots, u_n\}$ is ***enforceable***, if $u_i \geq v_i$ holds true for all $i \in \mathcal{N}$.

Definition

A utility profile $u = \{u_1, \cdots, u_n\}$ is ***feasible*** if there exists a lottery $\alpha \in \Delta(\mathcal{C}_\infty)$ such that, for all i, we have

$$u_i = \sum_{\boldsymbol{c} \in \mathcal{C}_\infty} \alpha_{\boldsymbol{c}} u_i(\boldsymbol{c}).$$

Folk's Theorem

Folk's theorem is actually a class of theorems, which characterizes equilibria in different types of infinitely repeated games...

Theorem

Consider any n-player normal-form game Γ, which has an average utility profile $\boldsymbol{u} = \{u_1, u_2, \cdots, u_n\}$, when repeated over an infinite time-horizon.

- If $\boldsymbol{u}$ is the utility profile for any Nash equilibrium $\boldsymbol{c}^*$ of the infinitely repeated Γ, then $\boldsymbol{u}$ is enforceable.
- If u is both feasible and enforceable, then u is the utility profile for some Nash equilibrium $\boldsymbol{c}^*$ of the infinitely repeated Γ.

Theorem

Consider any n-player normal-form game Γ, which has an discounted utility profile $\boldsymbol{u} = \{u_1, u_2, \cdots, u_n\}$ with some $\beta \in [0, 1]$, when repeated over an infinite time-horizon.

- If $\boldsymbol{u}$ is the utility profile for any Nash equilibrium $\boldsymbol{c}$ of the infinitely repeated Γ, then $\boldsymbol{u}$ is enforceable.
- If u is both feasible and enforceable, then u is the utility profile for some Nash equilibrium of the infinitely repeated Γ.

Bounded Rationality in Repeated Games

- Best response analysis $\Rightarrow$ Uncountably infinite comparisons...
- Can we define preference orders on Moore machines?

Definition

Given two machine tuples $(M_1, \cdots, M_N)$ and $(M_1', \cdots, M_N')$, we define a ***preference order*** at the i^{th} player as $(M_1, \cdots, M_N) \succeq_i (M_1', \cdots, M_N')$, if

$$(u_i(M_1, \cdots, M_N), -|M_i|) \succeq_L \Big(u_i(M_1', \cdots, M_N'), -|M_i'|\Big)$$

where $\succeq_L$ defines a lexicographical order in $\mathbb{R}^2$.

Definition

The tuple $(M_i, \boldsymbol{M}_{-i})$ is said to be a ***Nash-Rubinstein equilibrium*** in a repeated game, if

$$(M_i, \boldsymbol{M}_{-i}) \succeq_i (M_i', \boldsymbol{M}_{-i}),$$

for any M_i', for all $i = 1, \cdots, N$.

Summary

- *Stackelberg Games:* How to define equilibria in leader-follower games?
- *Perfect Extensive Games:* How to solve perfect extensive games via the notion of subgame perfect equilibrium?
- *Imperfect Extensive Games:* Subgame perfect equilibrium is no longer sufficient!

  Then, how?
  - Behavioral Equilibrium
  - Sequential Equilibrium
- *Perfect Bayesian Games:* What if, there are chance nodes (due to unknown agent types) in the game?
- *Repeated Games:* How to define choices and utilities in an infinitely repeated game, and solve it?