Series

$\{a_n\}$ — Sequence of real numbers.

Series $\Rightarrow \sum_{n=1}^{\infty} a_n$

Example : ① $1 + \frac{1}{2} + \frac{1}{3} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n}$

② $1 + \frac{1}{4} + \frac{1}{9} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^2}$.

Partial Sums : $S_n = a_1 + \cdots + a_n$

— n^{th} partial sum of $\sum_{n=1}^{\infty} a_n$

Progressions

① Arithmetic progression

$a_1 = a, \quad a_2 = a + d, \quad a_3 = a + 2d,$

$\ldots, \quad a_n = a + (n-1)d, \quad \ldots$

$$\sum_{n=1}^{N} a_n = \sum_{n=1}^{N} \left[a + (n-1)d\right] = \frac{N}{2}\left[2a + (N-1)d\right]$$

— Prove by induction.

Example : ① $\sum_{n=1}^{N} n = \frac{N(N+1)}{2}$

② $\sum_{n=1}^{N} (2n-1) = N^2.$

② Geometric Progression

$a_1 = a \ , \ a_2 = a.q \ , \ a_3 = a.q^2,$

$\dots, \ a_n = a.q^{n-1}, \ \dots$

$$\sum_{n=1}^{N} a.q^{n-1} = \begin{cases} \frac{a(q^N-1)}{q-1} & \text{if } q \neq 1. \\ Na & \text{if } q = 1. \end{cases}$$

Example : $\sum_{n=1}^{N} 2^{n-1} = \frac{2^N-1}{2-1} = 2^N-1$

③ Telescopic Series : $\sum_{k=1}^{n} (a_k - a_{k-1}) = a_n - a_0$

Example : $\sum_{k=1}^{n-1} \frac{1}{k(k+1)} = \sum_{k=1}^{n-1} \left(\frac{1}{k} - \frac{1}{k+1} \right)$ 1/..

4. Arithmetico-geometric progression

$$\sum_{n=1}^{N} \left[a + (n-1)d \right] \cdot q^{n-1}$$

$$= \frac{a - \left[a + (N-1)d \right] \cdot q^{N}}{1-q} + \frac{d \cdot q \left(1 - q^{N-1} \right)}{1-q^2}$$

$$\forall \; q \neq 1.$$

Some important series

(1) $$\sum_{n=1}^{N} n = \frac{N(N+1)}{2}$$

(2) $$\sum_{n=1}^{N} n^2 = \frac{N(N+1)(2N+1)}{6}$$

(3) $$\sum_{n=1}^{N} n^3 = \left[\frac{N(N+1)}{2} \right]^2$$

(4) $$\sum_{n=1}^{N} n^4 = \frac{N(N+1)(2N+1)(3N^2+3N-1)}{30}$$

(5) $$\sum_{n=1}^{N} (2n-1) = N^2$$

(6) $$\sum_{n=1}^{N} (2k-1)^2 = \frac{N(4N^2-1)}{3}$$

(7) $$\sum_{n=1}^{N} n! \cdot n = (N+1)! - 1$$

Taylor Series (about $x = 0$)

$$f(x) = f(0) + \frac{f'(0)}{1!} x + \cdots + \frac{f^{(n)}(x)}{n!} x^n + \cdots$$

Example: $$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

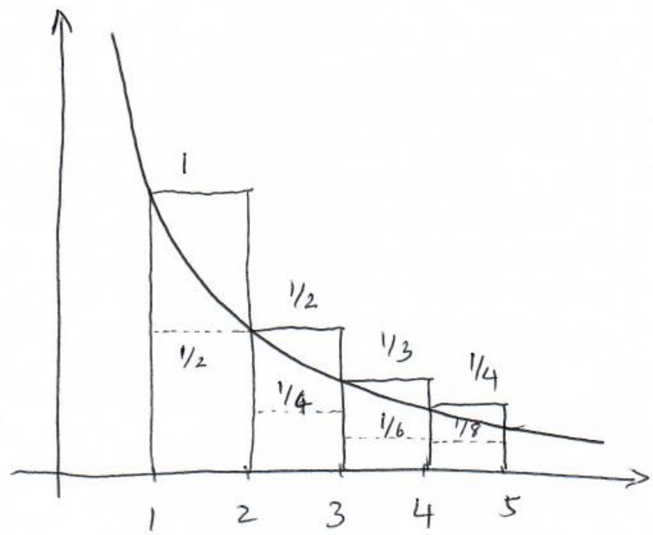
$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

$$\int_0^\infty \frac{1}{x}\,dx \longrightarrow \text{does not converge.}$$

$$\int_1^n \frac{1}{x}\,dx = \log n$$

$$\leq 1 + \frac{1}{2} + \cdots + \frac{1}{n}.$$



and $$\log n \geqslant \frac{1}{2}\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right)$$

$$\therefore \frac{1}{2} S_n \leq \log n \leq S_n \quad \forall\ n \geqslant 1$$

where $S_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$.

$$\Rightarrow \log n = \Theta(S_n)$$

$$\Longleftrightarrow S_n = \Theta(\log n)$$

Convergence of infinite Series

$$\lim_{n \to \infty} S_n = S \implies \text{Convergent}$$

otherwise, divergent.

Example: (1) $1 + \frac{1}{2} + \frac{1}{3} + \cdots = \sum_{i=1}^{\infty} \frac{1}{i}$

is divergent.

(2) $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ Converges since $S_n = 1 - \frac{1}{n+1}$

$$\lim_{n \to \infty} S_n = 1.$$

(3) $\forall\ 0 < q < 1, \quad \sum_{n=0}^{\infty} q^n \longrightarrow \frac{1}{1-q}$

Necessary Condition

Theorem: If $\sum_{n=1}^{\infty} a_n$ converges, then $a_n \longrightarrow 0$.

Proof: $S_{n+1} - S_n = a_{n+1} \longrightarrow S - S = 0.$

Note that this is not sufficient!

Cauchy Convergence test

Necessary and Sufficient Condition

Theorem: Suppose $a_n \geq 0 \ \forall n$.

Then $\sum_{n=1}^{\infty} a_n$ converges if and only if

~~S_n is bounded above~~

$\forall \varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$|S_{n+p} - S_n| = |a_{n+1} + a_{n+2} + \dots + a_{n+p}| < \varepsilon$$

$$\forall\ n > N \text{ and } p \geq 1.$$

i.e. S_n is a Cauchy sequence.

Counter-example:

$$\sum_{n=1}^{\infty} \frac{1}{n} \geq 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \dots$$

$$S_{2^k} = 1 + \frac{1}{2} + 2 \cdot \frac{1}{4} + 4 \cdot \frac{1}{8} + \dots + 2^{k-1} \cdot \frac{1}{2^k}$$

$$= 1 + \frac{k}{2} \quad \forall k.$$

$\longrightarrow$ No Convergence!

Theorem If $\sum_{n=1}^{\infty} |a_n|$ converges,

then $\sum_{n=1}^{\infty} a_n$ converges.

Comparison test

(*) If $0 \le a_n \le b_n$ $\forall\ n \geqslant k$ for some k,

Convergence of $\sum_{n=1}^{\infty} b_n \Rightarrow$ Convergence of $\sum_{n=1}^{\infty} a_n$.

(*) ~~Suppose~~ $\cancel{\lim \frac{a_n}{b_n} = L}$.

Theorem: If $a_n \geqslant 0$ and $a_{n+1} \le a_n$ $\forall\ n$,

then $\sum_{n=1}^{\infty} a_n$ converges $\Longleftrightarrow \sum_{k=0}^{\infty} 2^k \cdot a_{2^k}$ converges.

Proof: Let $S_n = \sum_{i=1}^{n} a_i$ and $T_k = \sum_{i=0}^{k} 2^i a_{2^i}$

T_k converges $\Rightarrow$ Choose k s.t. $2^k \geqslant n$ for a fixed n.

$$\begin{aligned} \therefore\ S_n &= a_1 + \cdots + a_n \\ &\le a_1 + (a_2 + a_3) + \cdots + (a_{2^k} + \cdots + a_{2^{k+1}-1}) \\ &\le a_1 + 2a_2 + \cdots + 2^k \cdot a_{2^k} = T_k. \end{aligned}$$

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Suppose S_n converges.

$$\Rightarrow \quad S_n = a_1 + \cdots + a_n$$

$$\geq a_1 + a_2 + (a_3 + a_4) + \cdots + \left(a_{2^{k-1}+1} + \cdots + a_{2^k}\right)$$

$$\geq \frac{1}{2} a_1 + a_2 + 2a_4 + \cdots + 2^{k-1} \cdot a_{2^k}$$

$$= \frac{1}{2} T_k .$$

Since T_k is bounded from above,

T_k converges.

<u>Example</u>: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ → converges if $p > 1$.

→ diverges if $p \leq 1$.

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(\log n)^p}$ → converges if $p > 1$.

→ diverges if $p \leq 1$.

Binomial expansion

$$(1+x)^n = \binom{n}{0} + \binom{n}{1} x + \cdots + \binom{n}{i} x^i + \cdots + \binom{n}{n} x^n.$$

Example: $x = 1 \Rightarrow \boxed{2^n = \sum_{i=1}^{n} \binom{n}{i}}$

$$x = -1 \Rightarrow 0 = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \cdots + (-1)^n \binom{n}{n}$$

$$\cancel{\Rightarrow \sum_{i=1} \binom{n}{2i-1}}$$

$$\Rightarrow \binom{n}{0} + \binom{n}{2} + \cdots = 2^{n-1}$$

$$\binom{n}{1} + \binom{n}{3} + \cdots = 2^{n-1}.$$

$$\boxed{\sum_{i=0}^{m} \binom{n+i}{n} = \binom{n+m+1}{n+1}}$$