

Homework 4: Information Asymmetry and Dynamic Games

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Due: November 15, 2024

Problem 1 First-Price Auction

3 pts.

Consider a first-price sealed-bid auction with N bidders. Assume that each i^{th} bidder has a valuation $v_i \in [0, 1]$ and has a belief $v_j \sim U[0, 1]$ for all $j \neq i$, i.e. a uniform belief regarding the valuations of all the other players. Then, prove that the first-price sealed-bid auction has a Bayes-Nash equilibrium where every player adopts *strategic underbidding* by choosing

$$b_i^* = \left(\frac{N-1}{N}\right) v_i, \text{ for all } i = 1, \dots, N.$$

Solution:

Consider the i^{th} bidder without any loss of generality. Let all the other bidders choose a bid

$$b_j = \left(\frac{N-1}{N}\right) v_j, \quad \text{for all } j \neq i. \quad (1)$$

Note that the probability of i^{th} bidder being the winner is given by

$$\begin{aligned} \mathbb{P}(b_i = \max\{b_j\}_{j \neq i}) &= \prod_{j \neq i} \mathbb{P}(b_i \geq b_j) \\ &= \prod_{j \neq i} \mathbb{P}\left(b_i \geq \left(\frac{N-1}{N}\right) v_j\right) \quad (\text{from Equation (1)}) \\ &= \prod_{j \neq i} \mathbb{P}\left(v_j \leq \left(\frac{N}{N-1}\right) b_i\right) \\ &= \left(\left(\frac{N}{N-1}\right) b_i\right)^{N-1}. \end{aligned} \quad (2)$$

Therefore, the expected utility of the i^{th} bidder is given by

$$\begin{aligned} U_i &= \mathbb{P}(b_i = \max\{b_j\}_{j \neq i}) \cdot (v_i - b_i) + \mathbb{P}(b_i \neq \max\{b_j\}_{j \neq i}) \cdot 0 \\ &= \left(\left(\frac{N}{N-1}\right) b_i\right)^{N-1} \cdot (v_i - b_i) \quad (\text{from Equation (2)}). \end{aligned} \quad (3)$$

Since the i^{th} bidder wishes to maximize U_i by choosing an appropriate bid b_i , we perform

the first derivative test to compute the critical point, as follows.

$$\begin{aligned}\frac{dU_i}{db_i} &= \left(\frac{N}{N-1}\right)^N \cdot [(N-1)b_i^{N-2} \cdot v_i - N \cdot b_i^{N-1}] \\ &= \left(\frac{N}{N-1}\right)^N \cdot b_i^{N-2} \cdot [(N-1) \cdot v_i - N \cdot b_i] = 0.\end{aligned}\tag{4}$$

Since $\left(\frac{N}{N-1}\right)^N > 0$, we have two possibilities:

1. $b_i = 0$ – However, this is not desired as the i^{th} bidder is guaranteed to lose with this strategy.
2. $(N-1) \cdot v_i - N \cdot b_i = 0 \quad b_i = \left(\frac{N-1}{N}\right) v_i$. A quick second derivative test reveals that this possibility results in a maximum utility at the i^{th} player.

Since $b_i = \left(\frac{N-1}{N}\right) v_i$ is the best response to $b_j = \left(\frac{N-1}{N}\right) v_j$ for all $j \neq i$ regardless of the choice of i , this is a Nash equilibrium.

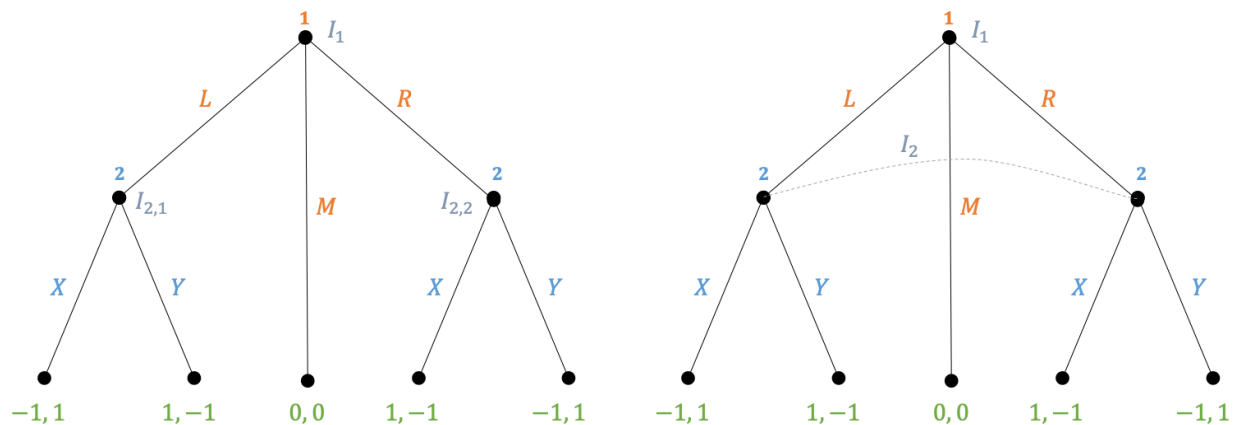
□

Remark: Therefore, in first-price auctions, the bidders strategically underbid at Nash equilibrium. This untruthful behavior is not desirable, especially since the principal does not receive payments that it truly deserves.

Problem 2 Imperfect Extensive Games

4 pts.

Consider the following modified matching pennies game, played in an extensive form, where Prisoner 1 plays first, followed by Prisoner 2. The main difference from the traditional matching pennies is that Player 1 can decide whether to play this game or not. If he decides not to play, both players get nothing.



- Find the subgame perfect equilibrium for this game, when Player 2 can perfectly observe Player 1's choices as in the left figure.
- Find behavioral equilibria for this game, when Player 2 cannot observe Player 1's choices as in the right figure.

Solution:

(a) Backward induction is used to compute subgame perfect Nash equilibrium, as shown in Figure 1. In the first stage, Nash equilibrium for subgames rooted at nodes 2 and 3 are first computed. At the end of the first stage, the values at nodes 2 and 3 are both updated to $(-1, 1)$. In the second stage, the Nash equilibrium for the subgame rooted at node 1 is evaluated and the value of node 1 is updated as $(0, 0)$.

Therefore, SPNE is $(P_1 : M, P_2 : X/Y)$. □

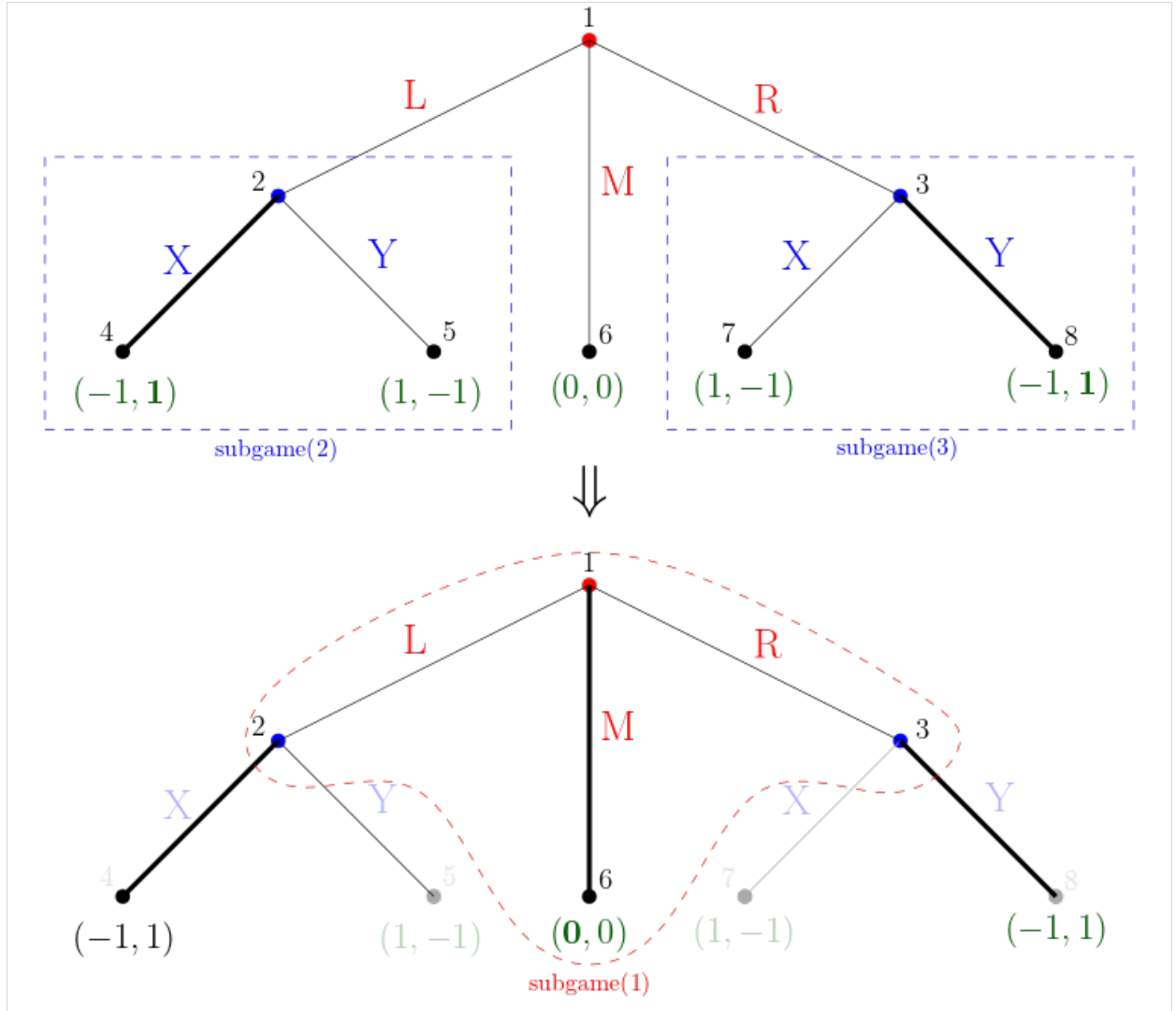


Figure 1: Stages of Backward Induction to compute SPNE

(b) Let P_2 's behavioral strategy in I_2 be $\{X : \alpha, Y : 1 - \alpha\}$. Also, assume that P_2 constructs a belief $\mu = \mathbb{P}(L|I_2)$ regarding being in the left node in I_2 . Then, P_2 's conditional expected utilities are given by

$$\begin{aligned} u_2(X|I_2) &= \mu \cdot 1 + (1 - \mu) \cdot (-1) = 2\mu - 1 \\ u_2(Y|I_2) &= \mu \cdot (-1) + (1 - \mu) \cdot 1 = 1 - 2\mu \end{aligned} \tag{5}$$

Therefore, the expected utility at P_2 due to the behavioral strategy $\{X : \alpha, Y : 1 - \alpha\}$ is given by

$$u_2(I_2) = \alpha \cdot u_2(X|I_2) + (1 - \alpha) \cdot u_2(Y|I_2) = (1 - 2\alpha)(1 - 2\mu). \tag{6}$$

Similarly, P_1 's expected utilities are given by

$$\begin{aligned} u_1(L) &= \alpha \cdot (-1) + (1 - \alpha) \cdot 1 = 1 - 2\alpha, \\ u_1(R) &= \alpha \cdot 1 + (1 - \alpha) \cdot (-1) = 2\alpha - 1, \\ u_1(M) &= 0. \end{aligned} \tag{7}$$

Note that P_1 's sequential rationality is satisfied by the following best-response strategy:

- If $\alpha > \frac{1}{2}$, then $u_1(L) < u_1(M) < u_1(R) \Rightarrow P_1$ chooses R .
- If $\alpha < \frac{1}{2}$, then $u_1(L) < u_1(M) < u_1(R) \Rightarrow P_1$ chooses L .
- If $\alpha = \frac{1}{2}$, then $u_1(L) = u_1(M) = u_1(R) \Rightarrow P_1$'s preference order is $L \sim M \sim R$.

Similarly, P_1 's sequential rationality is satisfied by the following best-response strategy:

- If $\mu > \frac{1}{2}$, then $u_2(I_2)$ is maximized when $\alpha = 1$.
- If $\mu < \frac{1}{2}$, then $u_2(I_2)$ is maximized when $\alpha = 0$.
- If $\mu = \frac{1}{2}$, then $u_2(I_2) = u_2(M) = 0 \Rightarrow P_2$'s preference order is $X \sim Y$.

Now, P_2 's consistency is guaranteed if

- If $\alpha < \frac{1}{2}$, then P_1 chooses $L \Rightarrow \mu = 1$.

But, this is a violation to P_2 's sequential rationality since P_2 chooses $\alpha = 1$ if $\mu > \frac{1}{2}$.

- If $\alpha > \frac{1}{2}$, then P_1 chooses $R \Rightarrow \mu = 0$.

But, this is a violation to P_2 's sequential rationality since P_2 chooses $\alpha = 0$ if $\mu < \frac{1}{2}$.

This leads us to the behavioral equilibrium, which is

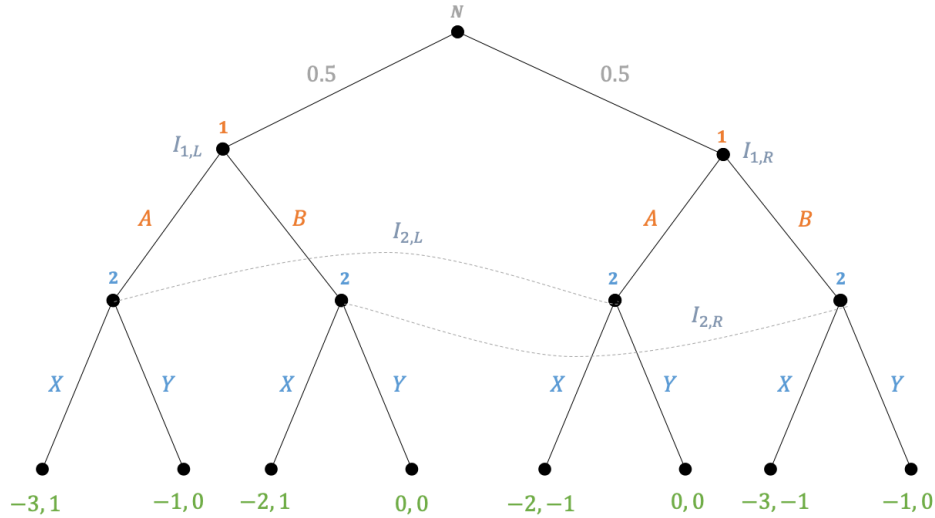
- P_1 chooses M ,
- P_2 chooses $\{X : \frac{1}{2}, Y : \frac{1}{2}\}$, with $\mu = \frac{1}{2}$.

□

Problem 3 Perfect Bayesian Equilibrium

3 pts.

Prove that there is no separating equilibrium in the following two-player signaling game (as depicted in the figure below), where the player set is $\mathcal{N} = \{1, 2\}$, the choice sets at the corresponding players are $\mathcal{C}_1 = \{A, B\}$ and $\mathcal{C}_2 = \{X, Y\}$ respectively. Assume that Player 1 can take two types $\{L, R\}$, and Player 2's belief about Player 1's type is uniformly distributed across types.



Solution:

Let the pure strategy at Player 1 be denoted by two letters, where the first letter corresponds to the strategy chosen in the information set $I_{1,L}$ and the second letter represents the strategy chosen in the information set $I_{1,R}$. For example, a pure strategy AB means that the sender chooses A in $I_{1,L}$ and B in $I_{1,R}$.

Note that Player 1 only has two separating strategies: AB and BA . Let us consider each of these strategies on a case-by-case basis:

Case 1 (AB): Since this is a separating strategy, the receiver clearly knows the information set in which he/she is in. For example, if the receiver observes a signal A , then he/she is on the left node of the information set $I_{2,L}$. In such a case, the receiver will choose X since $u_2(X|AB, I_{2,L}) = 1 > 0 = u_2(Y|AB, I_{2,L})$. Similarly, in $I_{1,R}$, if the sender chooses B , the receiver will always choose Y since $u_2(Y|AB, I_{2,R}) = 0 > -1 = u_2(X|AB, I_{2,R})$. In other words, the receiver's best response to AB is XY . However, sequential rationality is satisfied if the sender's best response to XY is also AB . However, if the receiver always chooses X in $I_{2,L}$ and Y in $I_{2,R}$, then the sender will always choose B in $I_{1,L}$ since $u_1(B|XY, I_{1,L}) = 0 > -3 = u_1(A|XY, I_{1,L})$. In other words, sequential rationality is violated for the separating strategy AB .

Case 1 (BA): Since $u_2(X|BA, I_{1,L}) = 1 > 0 = u_2(Y|BA, I_{1,L})$ and $u_2(Y|BA, I_{1,R}) = 0 > -1 = u_2(X|BA, I_{1,R})$, the receiver's best response to BA is XY . However, sequential

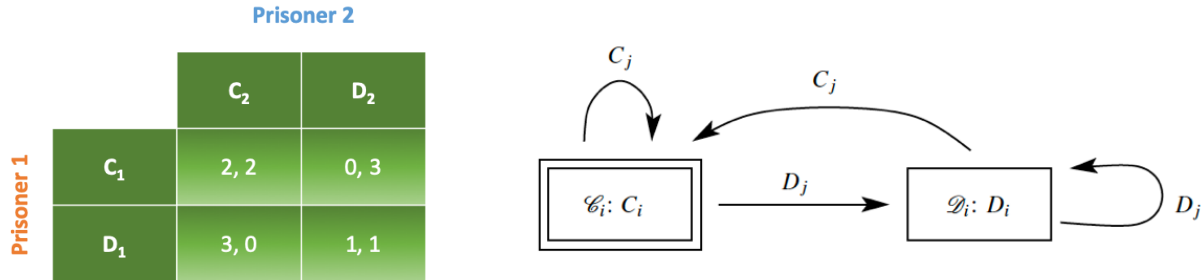
rationality is satisfied if the sender's best response to XY is also BA . However, in $I_{1,R}$, the sender always chooses B since $u_1(B|XY, I_{1,R}) = -1 > -2 = u_1(A|XY, I_{1,R})$. This is a violation of the sequential rationality condition, too.

In other words, since separating strategies violate sequential rationality condition, this game does not have a separating equilibrium. \square

Problem 4 Repeated Games

3 pts.

Consider the following repeated prisoner's dilemma game, where players play the game over an infinite time horizon. Prove that Tit-for-Tat strategy (given below) is a Nash equilibrium to this game, only when the discounting factor $\beta \geq \frac{1}{2}$.



Solution:

Assuming that Player $-i$ follows Tit-for-Tat, Player i 's responses can be summarized by the following four classes of strategy profile sequences of length T :

- CASE (a):

$$\begin{aligned} \text{Player } i : & C_i^{(1)} \quad C_i^{(2)} \quad \dots \quad C_i^{(T)} \quad \dots \\ \text{Player } -i : & C_{-i}^{(1)} \quad C_{-i}^{(2)} \quad \dots \quad C_{-i}^{(T)} \quad \dots \end{aligned}$$

- CASE (b):

$$\begin{aligned} \text{Player } i : & C_i^{(1)} \quad C_i^{(2)} \quad \dots \quad C_i^{(T-1)} \quad D_i^{(T)} \quad D_i^{(T+1)} \quad \dots \\ \text{Player } -i : & C_{-i}^{(1)} \quad C_{-i}^{(2)} \quad \dots \quad C_{-i}^{(T-1)} \quad C_{-i}^{(T)} \quad D_{-i}^{(T+1)} \quad \dots \end{aligned}$$

- CASE (c):

$$\begin{aligned} \text{Player } i : & C_i^{(1)} \quad C_i^{(2)} \quad \dots \quad C_i^{(T-1)} \quad D_i^{(T)} \quad C_i^{(T+1)} \quad C_i^{(T+2)} \quad \dots \\ \text{Player } -i : & C_{-i}^{(1)} \quad C_{-i}^{(2)} \quad \dots \quad C_{-i}^{(T-1)} \quad C_{-i}^{(T)} \quad D_{-i}^{(T+1)} \quad C_{-i}^{(T+2)} \quad \dots \end{aligned}$$

- CASE (d):

$$\begin{aligned} \text{Player } i : & C_i^{(1)} \quad \dots \quad C_i^{(T-1)} \quad D_i^{(T)} \quad D_i^{(T+1)} \quad \dots \quad D_i^{(T+k-1)} \quad C_i^{(T+k)} \quad \dots \\ \text{Player } -i : & C_{-i}^{(1)} \quad \dots \quad C_{-i}^{(T-1)} \quad C_{-i}^{(T)} \quad D_{-i}^{(T+1)} \quad \dots \quad D_{-i}^{(T+k-1)} \quad D_{-i}^{(T+k)} \quad C_{-i}^{(T+k+1)} \quad \dots \end{aligned}$$

Each of these classes of strategy profile sequences generates the following discounted utilities at Player i :

$$\begin{aligned}
u_i^{(a)} &= \sum_{t=1}^{\infty} \beta^{t-1} \cdot 2 \\
u_i^{(b)} &= \sum_{t=1}^{T-1} \beta^{t-1} \cdot 2 + \beta^{T-1} \cdot 3 + \sum_{t=T+1}^{\infty} \beta^{t-1} \cdot 1 \\
u_i^{(c)} &= \sum_{t=1}^{T-1} \beta^{t-1} \cdot 2 + \beta^{T-1} \cdot 3 + \beta^T \cdot 0 + \sum_{t=T+2}^{\infty} \beta^{t-1} \cdot 2 \\
u_i^{(d)} &= \sum_{t=1}^{T-1} \beta^{t-1} \cdot 2 + \beta^{T-1} \cdot 3 + \sum_{t=T+1}^{T+k-1} \beta^{t-1} \cdot 1 + \beta^{T+k-1} \cdot 0 + \sum_{t=T+k+1}^{\infty} \beta^{t-1} \cdot 2
\end{aligned} \tag{8}$$

Note that Case (a) corresponds to Tit-for-Tat strategy at Player i . In other words, Tit-for-Tat at Player i is the best response to Tit-for-Tat at Player $-i$ if

$$u_i^{(a)} \geq u_i^{(b)}, \tag{9}$$

$$u_i^{(a)} \geq u_i^{(c)}, \tag{10}$$

$$u_i^{(a)} \geq u_i^{(d)}. \tag{11}$$

Substituting Equation (1) in inequalities (2)-(4), we obtain

$$\begin{aligned}
&\sum_{t=1}^{\infty} \beta^{t-1} \cdot 2 \geq \sum_{t=1}^{T-1} \beta^{t-1} \cdot 2 + \beta^{T-1} \cdot 3 + \sum_{t=T+1}^{\infty} \beta^{t-1} \cdot 1 \\
\Rightarrow &\sum_{t=T}^{\infty} \beta^{t-1} \cdot 2 \geq \beta^{T-1} \cdot 3 + \sum_{t=T+1}^{\infty} \beta^{t-1} \cdot 1 \\
\Rightarrow &\sum_{t=T+1}^{\infty} \beta^{t-1} \geq \beta^{T-1} \\
\Rightarrow &\beta^T \cdot \frac{1}{1-\beta} \geq \beta^{T-1} \\
\Rightarrow &\beta^{T-1}(2\beta - 1) \geq 0, \quad \text{or } \beta \geq \frac{1}{2},
\end{aligned} \tag{2a}$$

$$\begin{aligned}
&\sum_{t=1}^{\infty} \beta^{t-1} \cdot 2 \geq \sum_{t=1}^{T-1} \beta^{t-1} \cdot 2 + \beta^{T-1} \cdot 3 + \sum_{t=T+2}^{\infty} \beta^{t-1} \cdot 2 \\
\Rightarrow &\beta^{T-1} \cdot (2\beta - 1) \geq 0, \quad \text{or } \beta \geq \frac{1}{2},
\end{aligned} \tag{3a}$$

$$\begin{aligned}
\sum_{t=1}^{\infty} \beta^{t-1} \cdot 2 &\geq \sum_{t=1}^{T-1} \beta^{t-1} \cdot 2 + \beta^{T-1} \cdot 3 + \sum_{t=T+1}^{T+k-1} \beta^{t-1} \cdot 1 + \sum_{t=T+k+1}^{\infty} \beta^{t-1} \cdot 2 \\
\Rightarrow 2\beta^{T-1} \cdot \frac{1 - \beta^{k+1}}{1 - \beta} &\geq 3\beta^{T-1} + \beta^T \cdot \frac{1 - \beta^{k-1}}{1 - \beta} \\
\Rightarrow 1 - 2\beta - \beta^k + 2\beta^{k+1} &\leq 0 \\
\Rightarrow (1 - 2\beta)(1 - \beta^k) &\leq 0, \quad \text{or } \beta \geq \frac{1}{2}.
\end{aligned} \tag{4a}$$

Since Inequalities (2a)-(4a) all hold true when $\beta \geq \frac{1}{2}$, Tit-for-Tat is a best response strategy for Player i against a Tit-for-Tat strategy adopted by Player $-i$. Since the analysis holds true for both $i = 1, 2$, Tit-for-Tat is a Nash equilibrium if $\beta \geq \frac{1}{2}$.

□