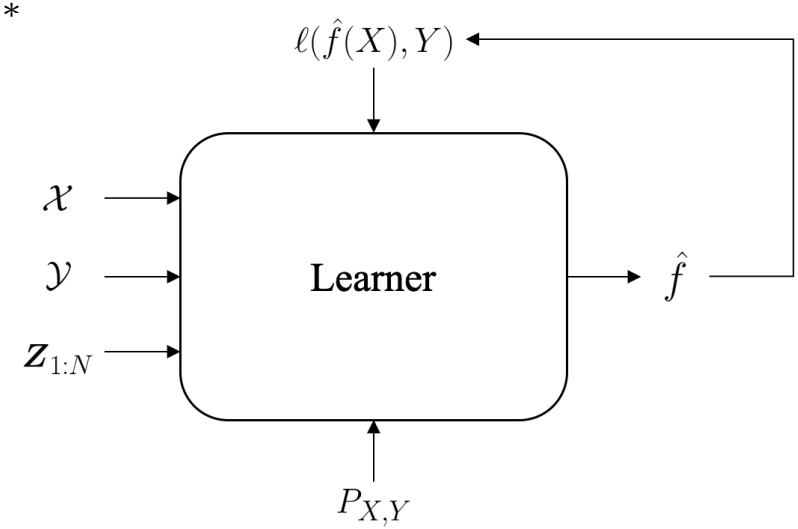
Topic 1: Learning

Learning: A Formal Introduction

- **Domain Set:** An arbitrary set $\mathcal{X}$ which we may wish to label.
- **Label Set:** Set of possible labels $\mathcal{Y} = f^*(\mathcal{X})$, where $f^* : \mathcal{X} \to \mathcal{Y}$ is the true labeling function.
- **Training Data:** A finite sequence of pairs in $\mathcal{X} \times \mathcal{Y}$, denoted as $\boldsymbol{z}_{1:N} = (z_1, \cdots, z_N) = \Big((x_1, y_1), \cdots, (x_N, y_N)\Big)$[1]
- **Learner's Output:** A prediction rule $f : \mathcal{X} \to \mathcal{Y}$ (also called a predictor, hypothesis, classifier) that predicts the labels of a new domain point from a predictor class $\mathcal{F}$. Technically, f is implemented using an algorithm $\mathcal{A}(\boldsymbol{z}_{1:N})$.
- **Data Generation Model:** A probability distribution $P_{X,Y}$ derived from a family of probability distributions $\mathcal{P}$, from which the training data $\boldsymbol{z}_{1:N}$ is generated.
- **Measure of Success:** Loss function $\ell : \mathcal{Y} \times \mathcal{Y} \to \mathbb{R}_+$ (also called risk, generalized error, true error) that quantifies how bad the prediction rule is, when compared to the true labeling function f^*

Goal: Choose the best predictor, i.e.

$$\hat{f} = \underset{f \in \mathcal{F}}{\text{minimize}}\ \mathbb{E}_{P_{X,Y}}\left[\ell(f(X), Y)\right]$$



[1] Sometimes, training data does not come with labels. In such a case, training data is $\boldsymbol{x}_{1:N} = (x_1, \cdots, x_N)$.

Example: Bias Estimation in Coin Tossing

- Goal: Given a (biased) coin with unknown probability θ of turning heads, determine θ as accurately as possible.
- Outcomes of N coin tosses: $\boldsymbol{x}_{1:N} = (x_1, \cdots, x_N)$, where $x_i = \begin{cases} 1, & \text{if Heads,} \\ 0, & \text{otherwise.} \end{cases}$
- Given θ, N and $\hat{\theta}_N(\cdot)$, we can partition $\{0,1\}^N$, the set of all N coin tosses, as

$$\text{Good Data:} \quad G_{N,\epsilon} \triangleq \left\{ \boldsymbol{x}_{1:N} \in \{0,1\}^N \;\middle|\; \left\| \hat{\theta}_N(\boldsymbol{x}_{1:N}) - \theta \right\| \leq \epsilon \right\}$$

$$\text{Bad Data:} \quad B_{N,\epsilon} \triangleq \left\{ \boldsymbol{x}_{1:N} \in \{0,1\}^N \;\middle|\; \left\| \hat{\theta}_N(\boldsymbol{x}_{1:N}) - \theta \right\| > \epsilon \right\}$$

Claim 1

For any true value θ of the coin bias, given any $\epsilon, \delta > 0$, it suffices to collect $N \geq \frac{1}{2\epsilon^2} \log\left(\frac{2}{\delta}\right)$ samples to guarantee

$$\mathbb{P}_\theta\left(G_{N,\epsilon}\right) = \mathbb{P}_\theta\left(\left\| \hat{\theta}_N(\boldsymbol{x}_{1:N}) - \theta \right\| \leq \epsilon\right) > 1 - \delta.$$

Example: Bias Estimation in Coin Tossing (cont...)

Proof of Claim 1:

Main Essence of Learning

Our main wish is to learn something about a phenomenon of interest, via observing random samples of a quantity that is relevant to the phenomenon.

Two basic questions to ask:

- **Statistical Learning:** How many samples are needed to achieve a given accuracy (ϵ) with a given confidence (δ)?
- **Computational Learning:** How efficient is the learning algorithm?

Typical learning frameworks:

- Estimation (e.g. coin tossing)
- Prediction (e.g. classification)
- Clustering
- Representation (Feature) Learning
- Density Estimation...

All the frameworks can be broadly generalized into two learning problems:

- Concept Learning (Binary Outcomes)
- Function Learning (Generalized Outcomes)

Generalization 1: Concept Learning

- *Concept class:* A class $\mathscr{C}$ of subsets of $\mathcal{X}$
- *Unknown target concept:* $C^* \in \mathscr{C}$ picked by Nature
- Binary Label: $Y_i = \mathbb{1}_{\{X_i \in C^*\}}$
- The N feature-label pairs form the training set

$$Z_1 = (X_1, Y_1) = \left(X_1, \mathbb{1}_{\{X_1 \in C^*\}}\right), \cdots, Z_N = (X_N, Y_N) = \left(X_N, \mathbb{1}_{\{X_n \in C^*\}}\right).$$

The objective is to approximate target concept C^* as accurately as possible.

Examples: Classification

Problem 1: Concept Learning

- A concept learning problem is a triple $(\mathcal{X}, \mathcal{P}, \mathscr{C})$, where $\mathcal{X}$ is the feature space, $\mathcal{P}$ is a family of probability distributions on $\mathcal{X}$, and $\mathscr{C}$ is a concept class.
- A learning algorithm for $(\mathcal{X}, \mathcal{P}, \mathscr{C})$ is a sequence $\mathcal{A} = \{A_n\}_{n=1}^{\infty}$ of mappings $A_N : (\mathcal{X} \times \{0,1\})^N \to \mathscr{C}$.

Given a training set $\boldsymbol{Z}_{1:N} = (Z_1, \cdots, Z_N) \in \mathcal{Z}^N$ and a learning algorithm $\mathcal{A}$, the approximation to C^* is

$$\hat{C}_N = A_N(\boldsymbol{Z}_{1:N}) = A_N(Z_1, \cdots, Z_N) = A_N\left(\left(X_1, \mathbb{1}_{\{X_1 \in C^*\}}\right), \cdots, \left(X_N, \mathbb{1}_{\{X_N \in C^*\}}\right)\right).$$

Generalization 1: Concept Learning (cont...)

Two types of errors: (i) $X \in C^* \cap \hat{C}_N^c$, (ii) $X \in (C^*)^c \cap \hat{C}_N$.

Combining the two, misclassification happens when X lies in the symmetric difference

$$C^* \Delta \hat{C}_N = \left(C^* \cap \hat{C}_N^c\right) \cup \left((C^*)^c \cap \hat{C}_N\right).$$

Performance measure of $\mathcal{A}$: $L(C^*, \hat{C}_N) = \mathbb{P}(C^* \Delta \hat{C}_N) = \mathbb{P}(X \in C^* \Delta \hat{C}_N)$.

Good Algorithm $\Rightarrow L(C^*, \hat{C}_N) \to \infty$ as $N \to \infty$.

- Let $X \sim P$, and $(X, \mathbb{1}_{X_N \in C}) \sim P_C$ for any $C \in \mathscr{C}$.
- Since $\hat{C}_N$ is a random element in $\mathscr{C}$, the above convergence can only be achieved in a stochastic sense.
- Define "worst case" size of set of "bad" samples as

$$\phi_{\mathcal{A}}(N, \epsilon, P) = \sup_{C \in \mathscr{C}} P_C^N \left(L\Big(C, A_N(\boldsymbol{Z}_{1:N})\Big) > \epsilon \right)$$

- Since we do not know P, consider the worst case distribution as shown below:

$$\Phi_{\mathcal{A}}(N, \epsilon, \mathcal{P}) = \sup_{P \in \mathcal{P}} \phi_{\mathcal{A}}(N, \epsilon, P).$$

Generalization 1: Concept Learning (cont...)

Definition 1: PAC for Concept Learning

A learning algorithm $\mathcal{A} = \{A_N\}$ is probably approximately correct[a] (or PAC) to accuracy ϵ if

$$\lim_{N \to \infty} \Phi_{\mathcal{A}}(N, \epsilon, \mathcal{P}) = 0.$$

- We say that $\mathcal{A}$ is PAC, if it is PAC to accuracy ϵ for every $\epsilon > 0$.
- The concept class $\mathscr{C}$ is called PAC-learnable to accuracy ϵ w.r.t. P, if there exists an algorithm that is PAC to accuracy ϵ.
- Finally, we say that $\mathscr{C}$ is PAC-learnable, if there exists an algorithm that is PAC.

[a] D. Angluin. Queries and concept learning. Machine Learning, 2:319–342, 1988.

Equivalently, a learning algorithm $\mathcal{A} = \{A_N\}$ is PAC, if for any $\epsilon > 0$ and $\delta > 0$, there exists some $N^*(\epsilon, \delta) \in \mathbb{N}$ such that, for all $N \geq N^*(\epsilon, \delta)$, $C \in \mathscr{C}$ and $P \in \mathscr{P}$, we have

$$P_C^N \left(L\Big(C, A_N(\boldsymbol{Z}_{1:N})\Big) > \epsilon \right) \leq \delta.$$

Note: $N^*(\epsilon, \delta)$ is called the ***sample complexity*** of the learning algorithm $\mathcal{A}$.

Example: Axis-Parallel Rectangles

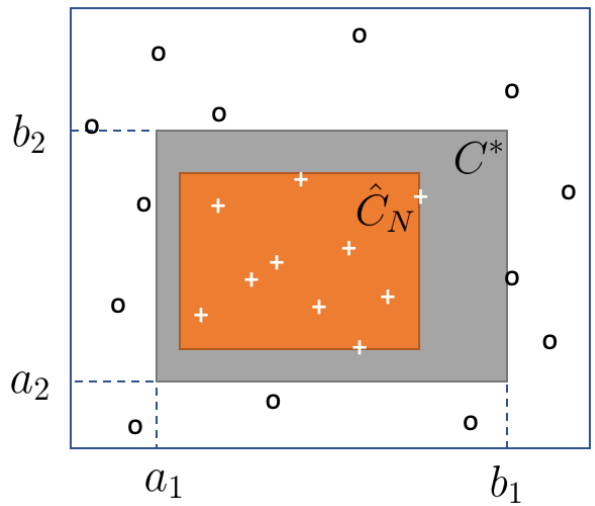
- Let $\mathcal{X} = [0,1]^2$ and $\mathscr{P}$ denote the set of all probability distributions on $\mathcal{X}$
- Let $\mathscr{C}$ denote the collection of all axis-parallel rectangles in $\mathcal{X}$, i.e., C is in $\mathscr{C}$ if it takes the form

$$C = [a_1, b_1] \times [a_2, b_2] = \left\{ (x_1, x_2) \in [0,1]^2 \;\middle|\; a_1 \leq x_1 \leq b_1,\ a_2 \leq x_2 \leq b_2 \right\},$$

for some $0 \leq a_1 \leq b_1 \leq 1$ and $0 \leq a_2 \leq b_2 \leq 1$.

Learning Algorithm:
Consider an intuitive algorithm $\mathcal{A} = \{A_N\}_{N=1}^{\infty}$ where, for each N, we have

$$\begin{aligned} \hat{C}_N &= A_N(\boldsymbol{Z}_{1:N}) \\ &= \text{smallest rectangle } C \in \mathscr{C} \text{ that contains all positive samples in } \boldsymbol{Z}_{1:N}. \end{aligned}$$



Connections to Computer Vision:

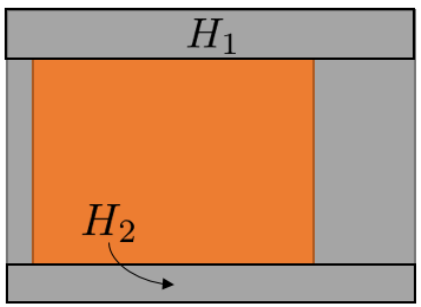
This problem is tangentially similar to estimating bounding boxes in images. The number of samples in $\mathcal{X}$ is similar to the resolution of an image.

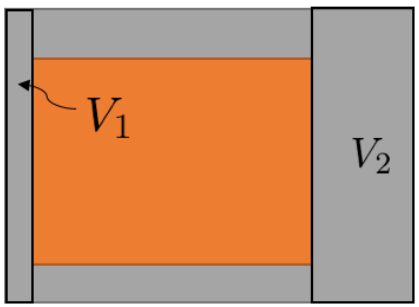
Example: Axis-Parallel Rectangles (cont...)

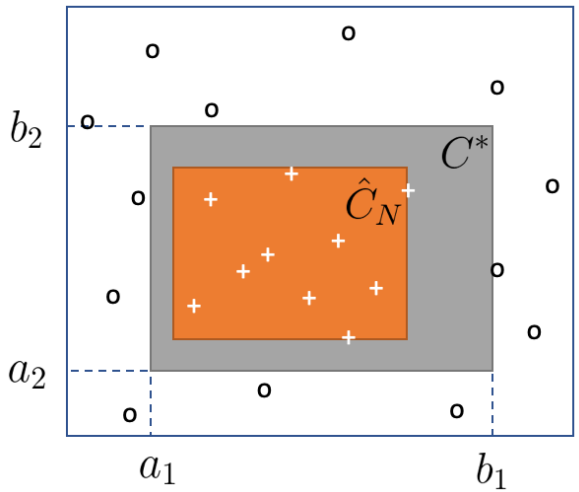
Theorem 1

The above tightest rectangle algorithm $\mathcal{A}$ is PAC, and therefore, the class $\mathscr{C}$ is PAC-learnable, since

$$\Phi_{\mathcal{A}}(N, \epsilon, \mathcal{P}) \leq 4\left(1 - \frac{\epsilon}{4}\right)^N .$$







Proof of Theorem 1:

Example: Axis-Parallel Rectangles (cont...)

Generalization 2: Function Learning

- *Function class:* A class $\mathcal{F}$ defined on $\mathcal{X}$
- *Target function:* $f^* \in \mathcal{F}$ picked by nature
- Real-valued output: $Y_i = f^*(X_i)$
- The N input-output pairs

$$Z_1 = (X_1, Y_1) = (X_1, f^*(X_1)), \cdots, Z_N = (X_N, Y_N) = (X_N, f^*(X_n)).$$

The objective is to approximate target function f^* as accurately as possible.

Examples: Estimation

Problem 2: Function Learning

- A function learning problem is a triple $(\mathcal{X}, \mathcal{P}, \mathcal{F})$, where $\mathcal{X}$ is the feature space, $\mathcal{P}$ is a family of probability distributions on $\mathcal{X}$, and $\mathcal{F}$ is a class of functions $f : \mathcal{X} \to [0, 1]$.
- A learning algorithm for $(\mathcal{X}, \mathcal{P}, \mathcal{F})$ is a sequence $\mathcal{A} = \{A_n\}_{n=1}^{\infty}$ of mappings $A_N : (\mathcal{X} \times \{0, 1\})^N \to \mathcal{F}$.

Given a training set $\boldsymbol{Z}_{1:N} = (Z_1, \cdots, Z_N) \in \mathcal{Z}^N$ and a learning algorithm $\mathcal{A}$, the approximation of f^* is

$$\hat{f}_N = A_N(\boldsymbol{Z}_{1:N}) = A_N\left(\left(X_1, \mathbb{1}_{\{X_1 \in C^*\}}\right), \cdots, \left(X_N, \mathbb{1}_{\{X_N \in C^*\}}\right)\right).$$

Generalization 2: Function Learning (cont...)

Performance of $\mathcal{A}$: $L_P(\hat{f}_N, f^*) = \mathbb{E}_P\left[\left|\hat{f}_N - f^*\right|^2\right] = \int_{\mathcal{X}} |\hat{f}_N(x) - f^*(x)|^2 P(dx)$

Remark: Concept learning is a special case of function learning.

- Define "worst case" size of set of "bad" samples as

$$\phi_{\mathcal{A}}(N, \epsilon, P) = \sup_{f \in \mathcal{F}} P_f^N \left(L\Big(A_N(\boldsymbol{Z}_{1:N}), f\Big) > \epsilon \right)$$

- Since we do not know P, consider the worst case distribution as shown below:

$$\Phi_{\mathcal{A}}(N, \epsilon, \mathcal{P}) = \sup_{P \in \mathcal{P}} \gamma_{\mathcal{A}}(N, \epsilon, P).$$

Definition 2: PAC for Function Learning

A learning algorithm $\mathcal{A} = \{A_N\}$ is probably approximately correct (or PAC) to accuracy ϵ if

$$\lim_{N \to \infty} \Phi_{\mathcal{A}}(N, \epsilon, \mathcal{P}) = 0.$$

- The function class $\mathcal{F}$ is called PAC-learnable to accuracy ϵ w.r.t. P, if there exists an algorithm that is PAC to accuracy ϵ.
- Finally, $\mathcal{F}$ is called PAC-learnable, if there exists an algorithm that is PAC.

Limitations of Model-Based Approaches

- We assume $C^* \in \mathscr{C}$ (or equivalently, $f^* \in \mathcal{F}$) $\Rightarrow$ Fit data regarding a well-studied phenomenon to some ***a priori*** **known hypothesis class**
- Labels $y = \mathbf{1}_{x \in C^*}$ (or equivalently, $y = f^*(x)$) are assumed to be **noiseless**.

Such limitations will lead us naturally towards a new framework called ***model-agnostic learning*** (also called model-free learning).

The main goal is to find the best possible hypothesis (concept/function) within a chosen hypothesis class $\mathcal{F}$.

Problem 3: Model-Agnostic Learning

A model-agnostic learning problem is a tuple $(\mathcal{X}, \mathcal{Y}, \mathcal{P}, \mathcal{F})$, where

- Sets: $\mathcal{X}$ (input feature space), $\mathcal{Y}$ (label space) and $\mathcal{U}$[a] (hypothesis space)
- A class $\mathcal{P}$ of probability distributions on $\mathcal{Z} = \mathcal{X} \times \mathcal{Y}$.
- A class $\mathcal{F}$ of functions $f : \mathcal{X} \to \mathcal{U}$.

A learning algorithm for $(\mathcal{X}, \mathcal{Y}, \mathcal{U}, \mathcal{P}, \mathcal{F}, \ell)$ is a sequence of mappings $\mathcal{A} = \{A_N\}_{N=1}^{\infty}$, where $A_N : \mathcal{Z}^N \to \mathcal{F}$.

[a] $\mathcal{U} \neq \mathcal{Y}$, since the true-hypothesis labels are corrupted by noise.

PAC Learnability for Model-Agnostic Learning

- Given a learning algorithm $\mathcal{A} = \{A_N\}_{N=1}^{\infty}$ with $A_N : \mathcal{Z}^N \to \mathcal{F}$, if $\hat{f}_N = A_N(\boldsymbol{Z}_{1:N})$, then the performance can be measured as

$$L_P(\hat{f}_N) = \mathbb{E}_P\left[\ell(Y, \hat{f}_N(X))\right] = \int_{\mathcal{X}\times\mathcal{Y}} \ell(y, \hat{f}_(x))P(dx, dy)$$

- **Minimum risk:** $L_P^*(\mathcal{F}) = \inf_{f\in\mathcal{F}} L_P(f)$ for an induced function class $\mathcal{F}$.

  i.e., given any algorithm $\mathcal{A}$, we have $0 \leq L_P^*(\mathcal{F}) \leq L_P(\hat{f}_N) \leq 1$.
- Given any $\epsilon > 0$, let the worst case probability of getting a bad sample be

$$\Phi_{\mathcal{A}}(N, \epsilon) = \sup_{P\in\mathcal{P}} P^N\left(L_P(\hat{f}_N) > L_P^*(\mathcal{F}) + \epsilon\right)$$

Definition 3: PAC for Model-Agnostic Learning

A learning algorithm $\mathcal{A} = \{A_N\}$ for a problem $(\mathcal{X}, \mathcal{Y}, \mathcal{U}, \mathcal{P}, \mathcal{F}, \ell)$ is PAC to accuracy ϵ if

$$\lim_{N\to\infty} \Phi_{\mathcal{A}}(N, \epsilon) = 0.$$

- An algorithm that is PAC to accuracy ϵ for every $\epsilon > 0$ is said to be PAC.
- Finally, a learning problem $(\mathcal{X}, \mathcal{Y}, \mathcal{U}, \mathcal{P}, \mathcal{F}, \ell)$ is *model-agnostically learnable* if there exists an algorithm for it, which is PAC.

Empirical Risk Minimization

But, we do not always know the input distribution $P \in \mathcal{P}$.

- $L_P(f)$ is unknown. Can we replace this with some surrogate?
- **ERM Algorithm:** $\hat{f}_N = \arg\min_{f \in \mathcal{F}} L_N(f) \triangleq \frac{1}{N} \sum_{i=1}^{N} \ell\big(Y_i, f(X_i)\big)$

Theorem 2: Fundamental Theorem of Learning

Consider an agnostic learning problem $(\mathcal{X}, \mathcal{P}, \mathcal{F})$ and let $\delta > 0$. For any $P \in \mathcal{P}$, the ERM algorithm satisfies

$$L_P(\hat{f}_N) \leq L_P^*(\mathcal{F}) + 8\sqrt{\frac{V(\mathcal{F}) \log(n+1)}{n}} + \sqrt{\frac{2\log(1/\delta)}{n}}$$

with probability at least $1 - \delta$.

Moreover, there is a universal constant C so that for any distribution P on $\mathcal{Z}$ and $\delta \in (0, 1)$, the ERM algorithm satisfies

$$L_P(\hat{f}_N) \leq L_N(\hat{f}_N) + C\sqrt{\frac{V(\mathcal{F})}{n}} + \sqrt{\frac{2\log(1/\delta)}{n}}$$

with probability at least $1 - \delta$.

Vapnik-Chervonenkis Dimension

Definition 4: Shattering

Let $\mathscr{C}$ be a class of (measurable) subsets of some space $\mathcal{Z}$. We say that a finite set $S = \{z_1, , \cdots, z_N\} \subset \mathcal{Z}$ is shattered by $\mathscr{C}$, if for every subset $S' \subseteq S$, there exists some $C \in \mathscr{C}$ such that $S' = S \cap C$.

i.e. $S = \{z_1, , \cdots, z_N\} \subset \mathcal{Z}$ is shattered by $\mathscr{C}$ if, for any binary N-tuple $b = (b_1, \cdots, b_N) \in \{0, 1\}^N$, there exists some $C \in \mathscr{C}$ such that

$$\left(\mathbb{1}_{\{z_1 \in C\}}, \cdots, \mathbb{1}_{\{z_N \in C\}}\right) = b.$$

N^{th} Shatter Coefficient of $\mathscr{C}$: $$\mathbb{S}_N(\mathscr{C}) = \sup_{S \subset \mathcal{Z};\ |S|=n} \left\{S \cap C \;\middle|\; C \in \mathscr{C}\right\}.$$

Definition 5: VC Dimension

The Vapnik-Chervonenkis dimension (or the VC dimension) of $\mathscr{C}$ is

$$V(\mathscr{C}) = \sup\left\{|S|, \text{ such that } S \text{ is shattered by } \mathscr{C}\right\}.$$

If $V(\mathscr{C}) < \infty$, we say that $\mathscr{C}$ is a VC class (of sets).

In other words,

$$V(\mathscr{C}) = \sup\ \left\{n \in \mathcal{N} \;\middle|\; \mathbb{S}_n(\mathscr{C}) = 2^n\right\}.$$

Vapnik-Chervonenkis Dimension (cont...)

Definition 6: VC Dimension of $\mathcal{F}$

Let $\mathcal{F}$ be a class of functions $f : \mathcal{Z} \to \{0, 1\}$. We say that a finite set $S = \{z_1, \cdots, z_n\} \subset \mathcal{Z}$ is shattered by $\mathcal{F}$ if it is shattered by the class $\mathcal{C}_{\mathcal{F}} = \{C_f : f \in \mathcal{F}\}$, where $C_f = \{z \in Z \mid f(z) = 1\}$.

Example 1: Semi-Infinite Intervals

Let $\mathcal{Z} = \mathbb{R}$ and $\mathscr{C}$ is a class of all intervals of the form (∞, t). Then, $V(\mathscr{C}) = 1$.

Example 2: Closed Intervals

Let $\mathcal{Z} = \mathbb{R}$ and $\mathscr{C}$ is a class of all intervals of the form (s, t). Then, $V(\mathscr{C}) = 2$.

Example 3: Closed Half-Spaces

Let $\mathcal{Z} = \mathbb{R}^2$ and $\mathscr{C}$ is a class of all closed half-spaces, i.e. the sets of the form $\{z = (z_1, z_2) \in \mathbb{R}^2 \mid w_1 z_1 + w_2 z_2 \geq b\}$ for all choices of w_1, w_2, b such that $(w_1, w_2) \neq (0, 0)$. Then, $V(\mathscr{C}) = 3$.

Example 4: Axis-Parallel Rectangles

Let $\mathcal{Z} = \mathbb{R}^2$ and $\mathscr{C}$ is a class of all axis-parallel rectangles, i.e. the sets of the form $C = [a_1, b_1] \times [a_2, b_2]$ for all $a_1, a_2, b_1, b_2 \in \mathbb{R}$. Then, $V(\mathscr{C}) = 4$.

No-Free-Lunch Theorem

$$\mathbb{P}\left[L_P(\hat{f}_N) \leq L_N(\hat{f}_N) + C\sqrt{\frac{V(\mathcal{F})}{n}} + \sqrt{\frac{2\log(1/\delta)}{n}}\right] \geq 1 - \delta.$$

The converse of Theorem 2 is also true, i.e. *there is no universal learner*.

Theorem 3: No-Free-Lunch Theorem

Let $\mathcal{A} = \{A_N\}_{N \in \mathbb{N}}$ be any binary classification algorithm with $0-1$ binary loss on a domain $\mathcal{X}$. Let the training set be of size $m \leq \frac{|\mathcal{X}|}{2}$. Then, there always exists a distribution P over $\mathcal{X} \times \{0, 1\}$ such that

1. there exists a function $f : \mathcal{X} \to \{0, 1\}$ such that $L_P(f) = 0$,
2. with probability at least $1/7$ over some training set $S \sim P^m$, we have $L_P(A_m(S)) \geq 1/8$.

Interpretation: Design ML algorithm that performs well on a specific task on a practical data distribution space.

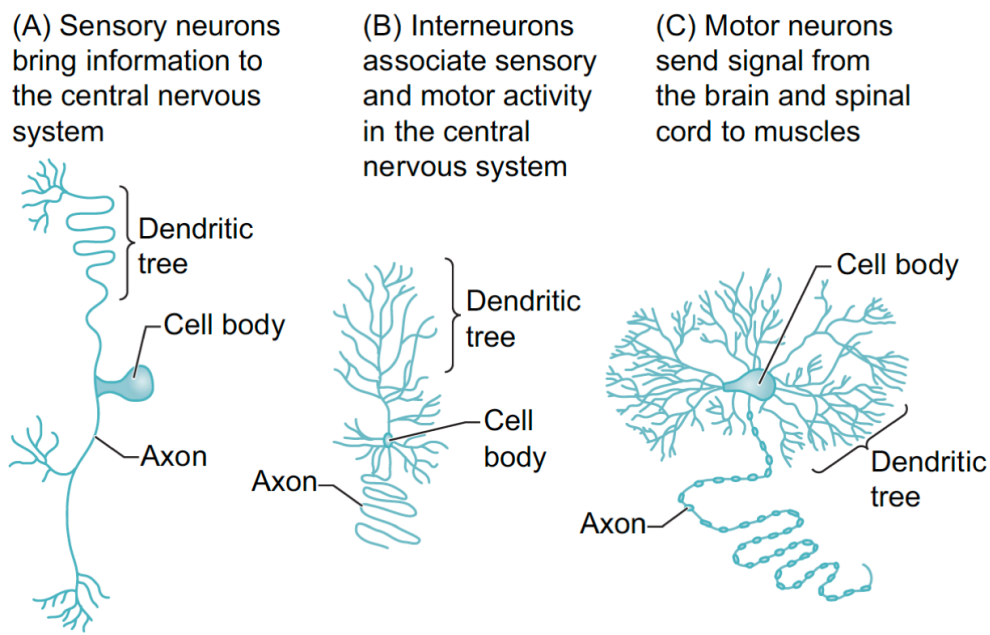
Perceptron[2]: A Linear Threshold Unit (LTU)

$$y = \sigma(\boldsymbol{w}^T \boldsymbol{x}) = \sigma\left(\sum_{i=0}^{d} w_i x_i\right) \in \{-1, +1\},$$

where $x_0 = 1$ (bias neuron), w_0 is the threshold, and σ is the *activation* function.

Multi-Output Perceptron: $\boldsymbol{y} = \sigma(W\boldsymbol{x}) \in \{-1, +1\}^K$, for K binary classes.

In nature, NN layers take diverse structural forms for various functionalities



Source: Marie T. Banich , Rebecca J. Compton, "Cognitive Neuroscience," Cambridge University Press, April 2018.

[2] Frank Rosenblatt, "The Perceptron: A Probabilistic Model for Information Storage and Organization in the Brain," *Psychological Review*, vol. 65, no. 6, pp: 386, 1958.

Feedforward Neural Networks (NNs)

Definition 7: Feedforward Neural Network

A feedforward neural network is a directed acyclic graph $\mathcal{G} = (V, E)$, and a weight function $w : E \to \mathbb{R}$. Each node is a single neuron (LTU) which is modeled by an activation function $\sigma : \mathbb{R} \to \mathbb{R}$.

Let the set of nodes be decomposed into a union of disjoint nodes, i.e.

$$V = \bigcup_{\ell=0}^{L} V_\ell,$$

such that each edge $e \in E$ connects some node in $V_{\ell-1}$ to V_ℓ for some $\ell = 1, \cdots, L$.

Let $\boldsymbol{f}_\ell(\boldsymbol{x})$ denote the output of V_ℓ, when the NN is fed with some input $\boldsymbol{x}$. Then,

$$\boldsymbol{f}_\ell(\boldsymbol{x}) = \sigma_{ell}\Big(W_\ell \cdot \boldsymbol{f}_{\ell-1}(\boldsymbol{x})\Big)$$

In other words,

$$\mathcal{F} = \left\{ \boldsymbol{x} \to \sigma_L\Big(W_L \cdot \sigma_{L-1}\big(\cdots \sigma_1(W_1\boldsymbol{x})\cdots\big)\Big) \;\middle|\; ||W_\ell|| \leq B \text{ for all } \ell = 1, \cdots, L \right\}$$

Multilayer Feedforward NNs and Universal Approximation

Definition 8: Universal Approximator

A class of functions $\mathcal{F}$ is a universal approximator over a compact set S, if for every continuous function g and target accuracy $\epsilon > 0$, there exists $f \in \mathcal{F}$ with

$$\sup_{x \in S} |f(x) - g(x)| \leq \epsilon.$$

Let $\mathcal{F}_{\sigma,d}$ denote the set of all multilayer feedforward NNs $\mathcal{F}$, that is restricted to a d-dimensional input and uses $\sigma(\cdot)$ as an activation function in all layers.

Theorem 4: Hornik, Stinchcombe and White, 1989

Suppose $\sigma : \mathbb{R} \to \mathbb{R}$ denote a continuous sigmoidal function such that

$$\lim_{z \to -\infty} \sigma(z) = 0, \quad \text{and} \quad \lim_{z \to \infty} \sigma(z) = 1.$$

Then, for any $d \in \mathbb{R}$, $\mathcal{F}_{\sigma,d}$ is universal.