Missouri University of Science & Technology — Department of Computer Science
Spring 2021 — **CS 6001: Algorithmic Game Theory**

Homework 1: Theory of Mechanism Design

Instructor: *Sid Nadendla* **Due:** *February 15, 2021*

Problem 1 Revelation Theorem *5 pts.*

In the class, we discussed revelation principle in the context of dominant strategy incentive compatibility (in short, DSIC), which is formally stated as

$$u_i(f(\theta_i, \boldsymbol{\theta}_{-i})) \geq u_i(f(\tau_i, \boldsymbol{\theta}_{-i})),$$

for all $\tau_i \in \Theta_i$, $\theta_i \in \Theta_i$, $\boldsymbol{\theta}_{-i} \in \Theta_{-i}$ and $i \in \mathcal{N}$.

However, in this problem, we will investigate another notion of incentive compatibility, which is based on Bayesian Nash equilibrium as defined below.

Definition 1. *Given a Bayesian game $\Gamma = \{\mathcal{N}, \Theta, \boldsymbol{p}, \mathcal{C}, \mathcal{U}\}$, a strategy profile $\{\pi_1^*, \cdots, \pi_N^*\} \in \Delta(\mathcal{C})$ is a **Bayes-Nash equilibrium** if, for all $i \in \mathcal{N}$, $\tau_i \in \mathcal{C}_i$ and $\theta_i \in \Theta_i$, we have*

$$\mathbb{E}_{\boldsymbol{\theta}_{-i}}\left[u_i(\pi_i^*, \boldsymbol{\pi}_{-i}^* | \theta_i, \boldsymbol{\theta}_{-i})\right] \ \geq \ \mathbb{E}_{\boldsymbol{\theta}_{-i}}\left[u_i(\tau_i, \boldsymbol{\pi}_{-i}^* | \theta_i, \boldsymbol{\theta}_{-i})\right].$$

Definition 2. *A mechanism $\mathcal{M} = (\mathcal{C}_1, \cdots, \mathcal{C}_N, x(\cdot))$ is **Bayesian incentive compatible (BIC)** if the strategy $\boldsymbol{c}^* = \{c_1^*(\theta_1), \cdots, c_N^*(\theta_N)\}$ at Bayesian Nash equilibrium has every player reporting their true types to $\mathcal{M}$.*

Suppose there exists a mechanism (direct or otherwise) $\mathcal{M}$ that implements a social-choice function f in Bayes-Nash equilibrium. Then, there always exists a Bayesian incentive compatible direct-revelation mechanism $\mathcal{M}^*$ that implements f with the same payoff as that of $\mathcal{M}$.

Problem 2 Arrow's Impossibility Theorems *3 pts.*

Consider a social choice setting with $\mathcal{A} = \{A, B, C\}$. Assume there are three agents $\{1, 2, 3\}$ whose preference profiles can be one of the three profiles:

$$\begin{aligned}
&(i) && 1: A \succ B \succ C, \quad 2: A \succ B \succ C, \quad 3: B \succ C \succ A \\
&(ii) && 1: A \succ B \succ C, \quad 2: B \succ A \succ C, \quad 3: B \succ C \succ A \\
&(iii) && 1: A \succ C \succ B, \quad 2: B \succ A \succ C, \quad 3: C \succ B \succ A
\end{aligned} \tag{1}$$

Prove that Agent 2 is a dictator of any social welfare function that is both unanimous and satisfies IIA.

Problem 3 Gibbard-Satterwaithe Theorem *7 pts.*

(a) Prove that a social choice function is incentive compatible if and only if it is monotone.

(b) Prove that any incentive compatible social choice function f on the set of alteratives $\mathcal{A}$, where $|\mathcal{A}| \geq 3$, is a dictatorship. (Hint: Use the result in (a)).

Problem 4 Quasi-Linear Mechanisms *5 pts.*

Prove that revealing truthful valuations is the dominating strategy for any Groves mechanism.