Simplex method example for Linear Programming: ①

Maximize $Z = f(x, y) = 3x + 2y \rightarrow (1)$

subject to

$$2x + y \leq 18 \rightarrow (2)$$
$$2x + 3y \leq 42 \rightarrow (3)$$
$$3x + y \leq 24 \rightarrow (4)$$
$$x \geq 0,\ y \geq 0$$

1. Make a change of variables and normalize sign of the independent terms:

Old variable	New Variable
x	X_1
y	X_2

Since both $X_1 \geq 0$ and $X_2 \geq 0$ is given, no further action is required at this stage.

Inequality type	Variable which appears
$\geq$	− surplus + artificial
$=$	+ artificial
$\leq$	+ slack

For inequalities (2), (3), (4), add slack variables X_3, X_4, X_5, respectively. Now, the system of linear equations become

$$\left.\begin{aligned} 2X_1 + X_2 + X_3 &= 18 \\ 2X_1 + 3X_2 + X_4 &= 42 \\ 3X_1 + X_2 + X_5 &= 24 \end{aligned}\right\} \quad \begin{aligned} X_1 &\geq 0 \\ X_2 &\geq 0 \end{aligned}$$

Match the objective function to zero

(or) $Z - 3X_1 - 2X_2 - 0X_3 - 0X_4 - 0X_5 = 0$

Let $P_1, P_2, \ldots, P_5$ be coefficients of the decision variables of the original problem (ie; coeffts of X_1 and X_2), and the slack (X_3, X_4, X_5). Then, surplus and artificial variables are added in second step (eg; P_0 as constant term and the coeffts of rest of X_i variables).

$P_i \; \forall \; i = 1, 2, 3, 4, 5$ as

and constraints (in rows).

First row $\Rightarrow$ Objective function coefficients 3, 2, 0, 0, 0, and the last row $\Rightarrow$ objective function value and reduced costs $Z_j - C_j$.

Table 1 : Iteration 1

Base	C_b	P_0	P_1	P_2	P_3	P_4	P_5
			3	2	0	0	0
P_3	0	18	2	1	1	0	0
P_4	0	42	2	3	0	1	0
P_5	0	24	3	1	0	0	1
Z		0	-3	-2	0	0	0

<u>Last row</u>: $Z_j = \sum C_{bi} P_j \quad \forall \; i = 1, 2, \ldots, m$

Update Table:

In pivot row, each new value is calculated as

$$\text{New value} = \frac{\text{Previous value}}{\text{Pivot}}$$

Q1 How to identify the Pivot row?

- Divide P_0 by P_1 and find the minimum value.

$$\min\left\{\frac{18}{2}, \frac{42}{2}, \frac{24}{3}\right\} = \min\{9, 21, 8\} = 8$$

- So pivot row is P_5 row.

Q2 How to identify the Pivot column?

- The value of the Z row which is most negative is the pivot column.

Pivot column

		P_0	P_1				
P_3	...	18	2	...			
P_4	...	42	2	...			
P_5	0	24	3	1	0	0	1
Z		0	-3	-2	...		

Pivot row

After this

	c_b	P_0	P_1	P_2	P_3	P_4	P_5
New value in pivot row P_5	0	8	1	$\frac{1}{3}$	0	0	$\frac{1}{3}$

What about other rows?

New value = Prev. value − (Prev. value in pivot column * New value in pivot row)

For row P_3, we have:

column

c_b: New value $= 0 - 2 * 0 = 0$

P_0: New value $= 18 - 2 * 8 = 18 - 16 = 2$

P_1: New value $= 2 - 2 * 1 = 2 - 2 = 0$

P_2: New value $= 1 - 2 * \frac{1}{3} = \frac{1}{3}$

P_3: New value $= 1 - 2 * 0 = 1$

P_4: New value $= 0 - 2 * 0 = 0$

P_5: New value $= 0 - 2 * 0 = 0$

For row P_4, we have

Column

C_b : New value $= 0 - (2 * 0) = 0$

P_0 : New value $= 42 - (2 * 8) = 42 - 16 = 26$

P_1 : New value $= 2 - (2 * 1) = 0$

P_2 : New value $7/3$ (How?) $3 - \frac{2}{3} = 7/3$

P_3 : New value 0 (How?) $0 - 2 * 0 = 0$

P_4 : 1 (How?) $1 - 2 * 0 = 1$

P_5 : $-2/3$ (How?) $0 - 2 * \frac{1}{3} = -2/3$

For row P_1, we have values $3, 8, 1, 1/3, 0, 0, 1/3$

Table 2nd Iteration:

			3	2	0	0	0
Base	C_b	P_0	P_1	P_2	P_3	P_4	P_5
P_3	0	2	0	$1/3$	1	0	$-\frac{2}{3}$
P_4	0	26	0	$7/3$	0	1	$-2/3$
P_1	3	8	1	$1/3$	0	0	$1/3$
Z		24	0	-1	0	0	1

Pivot row

Pivot column

$2/\frac{1}{3} = 6$ (min)

$26/\frac{7}{3} = \frac{26 \times 3}{7} > 6$

$8/\frac{1}{3} = 24 >$

Table: 3rd iteration

			3	2	0	0	0
Base	C_b	P_0	P_1	P_2	P_3	P_4	P_5
P_2	2	6	0	1	3	0	−2
P_4	0	12	0	0	−7	1	4
P_1	3	6	1	0	−1	0	1
Z		30	0	0	3	0	−1

negative value not considered.

$6/-2 = -3 < 0$

$12/4 = 3$ (min)

$6/1 = 6$

Table: 4th iteration

			3	2	0	0	0
Base	C_b	P_0	P_1	P_2	P_3	P_4	P_5
P_2	2	12	0	1	$-1/2$	$1/2$	0
P_5	0	3	0	0	$-7/4$	$1/4$	1
P_1	3	3	1	0	$+3/4$	$-1/4$	0
Z		33	0	0	$5/4$	$1/4$	0

STOP here since all Z values are non-negative!

$$\left.\begin{array}{l} X_1 = 3 \\ X_2 = 12 \end{array}\right\}$$

$$Z_{max} = 3X_1 + 2X_2 = 3(3) + 2(12) = 9 + 24 = \boxed{33}$$