Preserving Revenue in a Shrinking Market

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Abstract

We prove the first lower bounds on how much revenue can be preserved by truthful multi-item, multi-bidder auctions (for limited supply) when only a random fraction of the population participates in the market. Our bounds specifically concern the class of λ-auctions when bidders have valuation functions that satisfy natural conditions including winner monotonicity and submodularity of welfare. Our bounds hold in very general settings beyond combinatorial valuations as well – we allow for bidders' valuations to depend on what items other bidders receive. An important class of valuation functions for which our results hold is the class of gross-substitutes valuations. We construct an example that shows that when bidders have valuations violating winner monotonicity, only an exceedingly small fraction of the social surplus can be extracted as revenue by any auction. We additionally derive revenue-preservation guarantees when the mechanism designer places certain practically-motivated constraints on the auction, such as limiting the number of winners or placing bundling constraints that favor allocations that sell certain items together. Finally, we show how the mechanism designer can efficiently learn an auction that is robust to market shrinkage by leveraging practically-efficient routines for solving the winner determination problem.

1 Introduction

A shrinking market is a natural phase of products' and services' lifecycles. Current examples of great importance include media consumers—known as cord cutters—who cancel cable-TV subscriptions in favor of streaming services [1, 34], a thinning customer base for department stores due to online retailers like Amazon [16, 12], and reduced capacities for restaurants during the COVID-19 pandemic [44].

In this paper we study how mechanism design can help preserve revenue in this ubiquitous challenge of a shrinking market. We study the effect of a shrinking market on revenue in combinatorial auctions for limited supply. The seller has m indivisible items to allocate to a set S of n bidders. The bidders can express how much they value each possible bundle $b \subseteq \{1, \ldots, m\}$ of items. Combinatorial auctions have had wide reach in practice, from strategic sourcing to spectrum auctions to estate auctions. Cramton, Shoham, and Steinberg [11] provide a broad survey of various aspects of combinatorial auctions. The design of revenue-maximizing combinatorial auctions in multi-item, multi-bidder settings is an extremely elusive and difficult problem that has spurred a long and active line of research combining techniques from economics, artificial intelligence, and theoretical computer science. This is still largely an open question and a very active research area.

In this paper we start a new strand within that topic area, namely the study of shrinking markets. We introduce the first formal model of market shrinkage in multi-item settings, and prove

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the first guarantees for how much revenue can be preserved. Specifically, we show how much revenue can be preserved when only a random fraction of the set S of bidders participates.

1.1 Summary of the contributions of this paper

We present the first formal analysis of how much revenue can be preserved in a shrinking market, for multi-item settings. We prove the first known revenue-preservation guarantees when a random subset of the population participates in the market. Our results hold even in very general settings where bidders' valuations can depend on what items other bidders receive (allocational valuations). Externalities have been extensively studied in the auction literature [25, 17, 30, 31, 18] and have been empirically confirmed to exist in ad auctions [17]. The class of allocational valuations captures a larger scope of realistic bidder behavior than combinatorial valuations.

In Section 2 we provide a formal description of the problem setting, including formal definitions of the auction settings we consider. We precisely show how to reckon with subtleties that arise when auctions are run among a shrunken market of unknown size.

In Section 3 we provide examples that show that the revenue loss incurred by only a random fraction of bidders participating can be drastic. We show that, without further assumptions on valuation functions, there exist scenarios in which only an exponentially small (in the number of items) fraction of the revenue obtainable by even the vanilla Vickrey-Clarke-Groves (VCG) auction [48, 9, 19] on S can be guaranteed on a random subset of bidders, even if a large fraction of the market shows up. For example, if 50 items are for sale and each bidder shows up independently with 90% probability, our construction yields a maximum expected revenue of roughly 7% of the VCG revenue on S. If 100 items are for sale, at most 0.52% of the VCG revenue on S can be guaranteed.

In Section 4 we present our main bounds, which are of the form $\sup_{M\in\mathcal{M}} \mathbf{E}_{S_0\sim S}[\mathsf{Rev}_M(S_0)] \geq L_{\mathcal{M}} \cdot W(S)$. Here $W(S)$ is the maximum welfare of any allocation among bidders in S. $\mathcal{M}$ is the class of λ-auctions and $L_{\mathcal{M}}$ is the factor of revenue preserved. Our bounds on revenue preservation are compared to the very strong benchmark of $W(S)$, which is an upper bound on the revenue any auction can extract. We first establish bounds in the case that each bidder participates in the auction independently with probability p, and then show how to generalize our results to an arbitrary distribution over subsets of bidders. These bounds require that the bidders' valuation functions satisfy certain natural conditions including submodularity of welfare and winner monotonicity. The results of this section provide a positive contrast to the results of Section 3, where we showed that without any assumptions on the set of bidders the revenue loss can be arbitrarily bad. We then derive families of guarantees when the mechanism designer (1) limits the number of winners or (2) places bundling constraints on the items. Finally, we show how to generalize our bounds to the case of an arbitrary distribution over submarkets.

In Section 5 we show how to learn an auction from samples that is robust to market shrinkage. We show how to efficiently implement empirical revenue maximization for classes of sparse λ-auctions by leveraging practically-efficient routines for solving the winner determination problem. We then show how to use the paradigm of structural revenue maximization in order to balance the tradeoff between maximizing empirical revenue and preventing overfitting.

1.2 Related work

Shrinking markets. Shrinking markets have been studied by various researchers in the context of oil companies [47], cable TV [1, 34], labor markets [26], telecom markets [35], housing markets [27], and in combinatorial settings including a thinning customer base for department stores due to online retailers like Amazon [16, 12] and reduced capacities for restaurants during the COVID-19 pandemic [44]. Most of this existing research is extremely domain specific, and provides advisory content based on historical observations, data, and general economic knowledge. We introduce the first formal model of market shrinkage in multi-item settings, and prove the first known guarantees for how much revenue can be preserved in a shrinking market. Our bound on the revenue preserved in a shrinking market provides a positive contrast to recent work of Dobzinski and Uziely [13], who study the effect of market shrinkage on revenue loss. They show that even in the case of selling a single item to n buyers with known valuation distributions, the absence of a single buyer with a fixed "low" value can surprisingly result in a (multiplicative) revenue loss of $\frac{1}{e+1}$ (in expectation). We tackle the significantly more complex multi-item setting. Furthermore, our main results are prior-free (in that they are tailored to the specific set S of bidders and do not require bidders to come from a distribution) and thus provide a strong positive contrast to this negative result.

Revenue in combinatorial auctions We prove guarantees when the seller limits the number of winners and when the seller places bundling constraints on the items (for example, by enforcing that certain items must be sold together). Reasons for limiting the number of winners include: (1) avoiding the logistical hassle of having a large number of winners – this constraint is commonly used in sourcing auctions [23, 40, 38, 39] and (2) increasing competition to boost revenue, as is studied in Roughgarden et al. [36] (though in a different setting than ours). Kroer and Sandholm [29] show that even the vanilla VCG auction run with bundling constraints can yield significant revenue gains.

Our main results concern λ-auctions, which have been recently studied in the context of learning high-revenue auctions from data. Sandholm and Likhodedov [32, 33, 41] run experiments to design (generalizations of) λ-auctions with high expected revenue. Balcan, Sandholm, and Vitercik [5] prove tight bounds on the sample complexity of learning high-revenue λ-auctions.

Furthermore, catalyzed by the seminal work of Bulow and Klemperer [7], a recent line of work studies the *competition complexity* of auctions, and provides results that compare the optimal (expected) revenue to the revenue of simple mechanisms like VCG when the number of bidders is augmented [36, 8, 6, 14, 15]. While not directly related to our model, competition complexity results can be seen as tackling the "opposite" situation of a growing market. However, all existing work in this area uses the significantly different objective of expected revenue over buyer valuation distributions, and moreover has only tackled restrictive classes of valuation functions (such as unit-demand valuations and additive valuations).

Our results can also be viewed as a positive contrast to recent work by Collina and Weinberg [10], who study the computational hardness of approximation of revenue maximization in the gross-substitutes case (in the Bayesian setting, which is not the setting we study). They show that under standard complexity-theoretic assumptions, no polynomial-time algorithm can yield a $1/m^{1-\varepsilon}$-approximation to the optimal (expected) revenue. While our algorithms in Section 4 are not polynomial-time algorithms, they are practically efficient in various scenarios (for example, when λ-auctions are restricted to boost a constant number of allocations).

2 Problem formulation

The seller has m indivisible items to allocate among a set S of n bidders/buyers. Each buyer is described by her valuation function $v_i : \{0, \dots, n\}^{\{1,\dots,m\}} \to \mathbb{R}_{\geq 0}$ over allocations of the m items (index 0 denotes the seller). We call such valuation functions *allocational* valuations. Often a bidder's value only depends on the bundle of items she receives. We call such valuation functions $v_i : 2^{\{1,\dots,m\}} \to \mathbb{R}_{\geq 0}$ *combinatorial* valuations. For an allocation α, $W(\alpha) = \sum_{i=1}^{n} v_i(\alpha)$ denotes the welfare of α, and $W_{-i}(\alpha) = \sum_{j\neq i} v_j(\alpha)$ denotes the welfare of α when bidder i is absent. For a set of bidders S, $W(S) = \max_\alpha W(\alpha)$ denotes the welfare of an *efficient* allocation, that is, an allocation that maximizes welfare. We assume that if i receives no items under α, then $v_i(\alpha) = 0$.

The classical Vickrey-Clarke-Groves (VCG) auction [48, 9, 19] uses an efficient allocation α^*, and charges bidder i a payment of $\max_\alpha W_{-i}(\alpha) - W_{-i}(\alpha^*)$. It is well known that the VCG auction can yield low revenue in many scenarios. In this paper we focus on the class $\mathcal{M}$ of λ-auctions. A λ-auction is an incentive-compatible parameterized generalization of the VCG auction where the mechanism designer may specify additive boosts to specific allocations. λ-auctions were introduced by Jehiel, Meyer-ter-Vehn, and Moldovanu [24]. The class of λ-auctions is a rich class of auctions and has been studied towards designing high-revenue combinatorial auctions [32, 33, 41, 5]. Formally, a λ-auction run among n buyers is specified by a vector $\lambda \in \mathbb{R}^{(n+1)^m}$ indexed by the $(n+1)^m$ possible allocations. The overall allocation chosen is

$$\alpha^* = \operatorname*{argmax}_\alpha W(\alpha) + \lambda(\alpha)$$

and bidder i is charged a payment of

$$\max_\alpha (W_{-i}(\alpha) + \lambda(\alpha)) - (W_{-i}(\alpha^*) + \lambda(\alpha^*)).$$

For an auction M and a set of bidders $S' \subseteq S$, we denote by $\mathsf{Rev}_M(S')$ the sum of the payments made by bidders in S' when the seller runs M among bidders in S'. We write $S' \sim_p S$ to denote a subset S' that is sampled from S by including each bidder in S' independently with probability p. More generally, for a distribution D over 2^S, we write $S' \sim_D S$ to denote a random subset of S chosen according to D.

We are interested in what happens to the maximum revenue achievable when only a random fraction of the set S of bidders participates in the auction. For a mechanism class $\mathcal{M}$ and a distribution D supported on 2^S, this quantity is $\sup_{M \in \mathcal{M}} \mathbf{E}_{S_0 \sim_D S}[\mathsf{Rev}_M(S_0)]$.

Since a variable group of bidders of variable size can participate in the λ-auctions we run, it is important to formalize how to distinguish between allocations since the λ-auction adds "boosts" to allocations specifically. We assume that the mechanism designer knows the valuations of the bidders in S to begin with. So, each bidder can be thought of having an identity (for example, "the bidder who values apples at x and oranges at y", or "the bidder with valuation function v_4"), and the mechanism designer knows the identities/valuations $v_1, \ldots, v_n$ of all bidders in S. An allocation, formally, is a mapping from items to bidder identities. Traditionally λ-auctions assume that the number of bidders is fixed, so allocations are usually interpreted as mappings from items to the position/index of a given bidder. So, we, too, will consider λ-auctions run among all bidders, so an auction is parameterized by a $(n+1)^m$-dimensional vector that specifies boosts for allocations among all n identities in S. However, our auctions also have to be well defined in the case of a shrinking market that has only a subset of the bidders. We address that as follows.[1] If a λ-auction is run among a subset of bidder identities S_0 and chooses an allocation that allocates an item to a bidder identity not in S_0, we assume that the seller keeps that item.

The sequence of mechanism design and revelation in our setting is the same as in the standard mechanism design setting. Specifically, the mechanism design (computation) takes place before the bidders in the shrunken market are asked to reveal their valuations. This is important for incentive compatibility, that is, for motivating the bidders to reveal their true valuations. Indeed, λ-auctions are incentive compatible when that sequence is used. (If the design were allowed to be based on the revealed valuations, the auction might not be incentive compatible.) Because the designer does not know exactly which bidders are in the shrunken market S_0, the designer has uncertainty about the valuations of the bidders. He only knows that they belong to S.

[1]Another solution is to use *bundling-boosted-auctions* [4], though this would result in weaker revenue bounds.

3 Revenue loss can be drastic

At first glance it might appear that the expected revenue preserved by a mechanism M when each bidder participates independently with probability p should simply be $p \cdot \mathsf{Rev}_M(S)$ (or more if one thinks of revenue as having diminishing returns in the number of bidders). While this is true in some simple cases (for example, in the setting where a single item is for sale, a reserve price equal to the highest value $\bar{v}$ for the item among bidders in S generates an expected revenue of $p\bar{v}$), revenue can shrink by more than this. One reason for greater revenue loss is reduced competition among buyers. For example, suppose there are m items and $2m$ bidders, where bidder i for $1 \le i \le m$ has valuation $v_i(b) = c$ if $i \in b$ and $v_i(b) = 0$ otherwise, and bidder j for $m+1 \le j \le 2m$ has valuation $v_j(b) = c - \varepsilon/m$ if $i \in b$ and $v_j(b) = 0$ otherwise (bidders have combinatorial valuations in this example, so valuation functions only depend on the bundle of items received). The VCG auction will allocate item $i \in \{1, \dots, m\}$ to bidder i. The payment collected from bidder i will be $c - \varepsilon/m$, which is the second highest value for item i. The revenue from VCG is thus $mc - \varepsilon = W(S) - \varepsilon$. Now, suppose each bidder participates in the auction independently with probability p. The expected revenue can be computed by breaking it up across items and using linearity:

$$
\begin{aligned}
\mathbf{E}[\mathsf{Rev}_{VCG}(S_0)] &= \mathbf{E}\left[\sum_{i=1}^{m} \text{Revenue from item } i\right] \\
&= \sum_{i=1}^{m} \mathbf{E}[\text{Revenue from item } i] \\
&= \sum_{i=1}^{m} p^2(c - \varepsilon/m) \\
&= p^2(W(S) - \varepsilon).
\end{aligned}
$$

The third equality is due to the fact that VCG generates nonzero revenue from item i if and only if both bidders i and $m+i$ participate, since if at most one of them shows up there is no competition for that item.

Furthermore, we allow bidder valuations to depend on what items the other bidders receive, and in this case we can construct a simple example where a random fraction of bidders participating incurs a much more significant revenue loss. Suppose there is a distinguished set of k bidders (who can be viewed as unwealthy bidders) with negligibly low valuations for each bundle. All other bidders' (who can be viewed as wealthy bidders) valuation functions are defined to be above some threshold only on allocations that give a nonempty bundle to each of the k distinguished bidders (and zero otherwise). In other words, while the wealthy bidders would like to receive items, they are not willing to participate unless unwealthy bidders are also guaranteed items. Then, for any nontrivial fraction of the revenue to be preserved, all distinguished bidders must participate in the auction, which occurs with probability p^k. Thus, any auction can preserve revenue at most $p^k \cdot W(S)$. The number of distinguished bidders k can be taken to be as large as, for example, $m/2$. We now give a formal construction of the described example. Our construction satisfies the property that even the vanilla VCG auction extracts revenue nearly equal to the entire social surplus on the full set S of bidders.

Theorem 3.1. *For any $\varepsilon > 0$ there exists a set S of bidders with allocational valuations such that*

$$
\sup_{M \in \mathcal{M}} \mathop{\mathbf{E}}_{S_0 \sim_p S}[\mathsf{Rev}_M(S_0)] \le p^{m/2} \cdot (\mathsf{Rev}_{VCG}(S) + 2\varepsilon) + \varepsilon
$$

for any auction class $\mathcal{M}$.

Proof. For each item $1 \le i \le m/2$ we introduce two buyers with valuations $v_{i,1}$, $v_{i,2}$. For each item $m/2+1 \le j \le m$ we introduce a single buyer with valuation v_j. For $1 \le i \le m/2$ valuations $v_{i,1}$ are defined by $v_{i,1}(\alpha) = c$ if bidder $(i,1)$ is allocated item i and bidders $j = m/2+1, \ldots, m$ each receive at least one item, and $v_{i,1}(\alpha) = 0$ otherwise. Valuations $v_{i,2}$ are defined by $v_{i,2}(\alpha) = c - 2\varepsilon/m$ if bidder $(i,2)$ is allocated item i and bidders $j = m/2+1, \ldots, m$ each receive at least one item, and $v_{i,2}(\alpha) = 0$ otherwise. The only requirement on the valuations of bidders $j = m/2+1, \ldots, m$ is that $v_j(\alpha) \le 2\varepsilon/m$ for all α. The VCG auction would allocate item i to bidder $(i,1)$ for each $i = 1, \ldots, m/2$, and allocate the remaining $m/2$ items to bidders $j = m/2+1, \ldots, m$ such that each bidder j receives exactly one item. The welfare of this (efficient) allocation is at most $cm/2+\varepsilon$. The revenue obtained by VCG is at least $cm/2-\varepsilon = W(S)-2\varepsilon$. Let S^* denote the set of small-valuation bidders $j = m/2+1, \ldots, m$. If each bidder shows up independently with probability p, the expected revenue of any auction M is

$$\begin{aligned}
\mathbf{E}[\mathsf{Rev}_M(S_0)] &= \mathbf{E}[\mathsf{Rev}_M(S_0) \mid S^* \subseteq S_0] \cdot \Pr(S^* \subseteq S_0) + \mathbf{E}[\mathsf{Rev}_M(S_0) \mid S^* \nsubseteq S_0] \cdot \Pr(S^* \nsubseteq S_0) \\
&\le p^{m/2} \cdot \mathbf{E}[\mathsf{Rev}_M(S_0) \mid S^* \subseteq S_0] + \mathbf{E}[\mathsf{Rev}_M(S_0) \mid S^* \nsubseteq S_0] \\
&\le p^{m/2} \cdot \mathbf{E}[W(S_0) \mid S^* \subseteq S_0] + W(S^*) \\
&\le p^{m/2} \cdot W(S) + \varepsilon,
\end{aligned}$$

as desired. $\square$

The exponential revenue decay in the number of items means that even if the shrunken market is large in expectation, the revenue loss can be dramatic. For example, if 50 items are for sale and each bidder shows up independently with 90% probability, our construction shows that any auction can guarantee only at most roughly 7% of the VCG revenue on S. If 100 items are for sale, at most 0.52% of the VCG revenue on S can be guaranteed.

In the following section we present our main results. They provide a positive contrast to the negative example just outlined. Our results require conditions on the set of bidders and their valuations that preclude examples like the one above.

4 Main bounds on preserved revenue

Theorem 3.1 illustrates that when bidders have allocational valuations, the maximum expected revenue preserved when each bidder participates independently with probability p can be an exponentially small factor of the total social surplus. We now outline some important and natural conditions on the bidders' valuations for which we can prove positive guarantees on expected revenue preserved.

The following condition, which we call winner monotonicity, is a key ingredient in our main guarantees. A set S of bidders satisfies the *winner monotonicity* condition if the following holds: if bidder $i \in S$ is a winner according to the efficient allocation among bidders in S, bidder i is also a winner according to the efficient allocation among bidders in S' for any $S' \subset S$ that contains i.

Winner monotonicity is an instantiation of a more general axiom in decision theory called Sen's condition α [28]. A set of bidder valuations S is *submodular in bidders* if for any $S_1, S_2 \subseteq S$, $W(S_1)+W(S_2) \ge W(S_1 \cup S_2)+W(S_1 \cap S_2)$. An important class of (combinatorial) bidder valuations that satisfies submodularity in bidders is the class of *gross-substitute valuations*. A combinatorial valuation function $v : 2^{\{1,\ldots,m\}} \to \mathbb{R}_{\ge 0}$ satisfies the gross-substitutes property if for every price vector $\mathbf{p} \in \mathbb{R}^m$, if item $j^* \in b' \in \operatorname{argmax}_{b \subset \{1,\ldots,m\}} v(b) - \sum_{j \in b} \mathbf{p}_j$, then if $\mathbf{q} \in \mathbb{R}^m$ is such $\mathbf{q}_j \ge \mathbf{p}_j$ for all j and $\mathbf{q}_{j^*} = \mathbf{p}_{j^*}$, then there exists $b'' \ni j^*$ such that $b'' \in \operatorname{argmax}_{b \subset \{1,\ldots,m\}} v(b) - \sum_{j \in b} \mathbf{q}_j$.

Gul and Stacchetti [20] show that a set of bidders with gross-substitutes valuations is submodular in bidders and Guo [21] shows that a set of bidders that is submodular in bidders satisfies winner monotonicity:

$$\text{gross-substitutes} \overset{[20]}{\Longrightarrow} \text{submodularity in bidders} \overset{[21]}{\Longrightarrow} \text{winner monotonicity.}$$

We use the following basic fact about λ-auctions that shows that (in the full-information setting with no incentive-compatibility considerations) λ-auctions can always extract the maximum possible revenue equal to the social surplus $W(S)$.

Proposition 4.1. *For any set of bidders S, there exists a λ-auction, in the full-information setting with no incentive-compatibility constraints, with revenue equal to $W(S)$.*

Proof. Let α denote the efficient allocation among bidders in S and let α_{-i} denote the efficient allocation among bidders in $S \setminus \{i\}$. We show that the λ-auction with $\lambda(\alpha) = 0$, $\lambda(\alpha_{-i}) = W(\alpha) - W_{-i}(\alpha_{-i})$, and $\lambda(\beta) = -\infty$ for all other allocations β collects a payment of $v_i(\alpha)$ from each bidder, and thus extracts a revenue of $W(S)$. First, note that α is the overall allocation used since $W(\alpha) + \lambda(\alpha) \geq W(\alpha_{-i}) + \lambda(\alpha_{-i})$ for each i. To show that α_{-i} is the allocation used when bidder i is absent, observe that

$$\begin{aligned} W_{-i}(\alpha_{-i}) + \lambda(\alpha_{-i}) &= W(\alpha) \\ &\geq W(\alpha) - v_i(\alpha) \\ &= W_{-i}(\alpha) + \lambda(\alpha) \end{aligned}$$

and

$$\begin{aligned} W_{-i}(\alpha_{-i}) + \lambda(\alpha_{-i}) &= W(\alpha) \\ &\geq W(\alpha) - v_i(\alpha_{-j}) \\ &= W(\alpha) + v_j(\alpha_{-j}) - v_i(\alpha_{-j}) \\ &= W_{-i}(\alpha_{-j}) + W(\alpha) - W_{-j}(\alpha_{-j}) \\ &= W_{-i}(\alpha_{-j}) + \lambda(\alpha_{-j}) \end{aligned}$$

for any $j \neq i$. (We use the fact that if α allocates nothing to i, then $v_i(\alpha) = 0$.)

Thus the allocations used by this λ-auction are precisely the VCG allocations. The payment of bidder i is therefore $(W_{-i}(\alpha_{-i}) + \lambda(\alpha_{-i})) - (W_{-i}(\alpha) + \lambda(\alpha)) = W(\alpha) - (W(\alpha) - v_i(\alpha)) = v_i(\alpha)$, and so the total revenue is $W(\alpha)$. □

Computing and running the λ-auction given by Proposition 4.1 for the set S_0 of bidders after they have revealed their valuations would not be incentive compatible, as mentioned previously. However, the existence of λ-auctions that extract the full social surplus in the full-information setting will be useful in the proof of our main result.

We now present our main revenue bound when each bidder participates in the auction independently with probability p. We use the following result, which shows that for a set of bidder valuations that is submodular in bidders, the expected welfare when each bidder participates independently with probability p is at least $p \cdot W(S)$.

Lemma 4.2 ([22])**.** *If $f : 2^S \to \mathbb{R}_{\geq 0}$ is submodular and S_0 is sampled by including each element of S in S_0 independently with probability at least p, $\mathbf{E}[f(S_0)] \geq p \cdot f(S)$.*

For a set of bidders $S' \subseteq S$, let $\varphi(S') = \frac{1}{n}\sum_{i=1}^{n} W(S' \setminus \{i\})$. φ serves as a potential function in the proof of the following theorem and represents the average efficient welfare of S' when a uniformly random bidder in S drops out. Let $win(S') \subseteq S'$ denote the set of bidders who receive a nonempty bundle per the welfare-maximizing allocation for S' (that is, the set of winners in VCG). Let

$$\omega(S') = \bigcup_{i \in S'} win(S' \setminus \{i\}).$$

Theorem 4.3. *Let S be a set of $n \geq 2$ bidders with valuations such that $W : 2^S \to \mathbb{R}_{\geq 0}$ is submodular. Let $\gamma = \max_{S', i \in \omega(S')} \frac{\varphi(S' \setminus \{i\})}{\varphi(S')}$ and let $k = \max_{S'} |\omega(S')|$. The class $\mathcal{M}$ of λ-auctions satisfies*

$$\sup_{\lambda \in \mathcal{M}} \mathop{\mathbf{E}}_{S_0 \sim_p S}[\mathsf{Rev}_\lambda(S_0)] \geq \frac{p^2}{16 k^{1+\log_{1/\gamma}(4/p)}} \cdot W(S).$$

Proof. Define an equivalence relation $\equiv$ on subsets of S by $S_1 \equiv S_2$ if and only if $\omega(S_1) = \omega(S_2)$. If $S_1 \equiv S_2$, then $W(S_1) = W(S_2)$ and $\varphi(S_1) = \varphi(S_2)$, and furthermore the corresponding λ-auctions given by Proposition 4.1 are the same for both these sets. Call a set of bidders $S' \subset S$ *heavy* if $\varphi(S') > \frac{p}{4} W(S)$. Let ℓ denote the number of heavy equivalence classes, and let $\lambda_1, \ldots, \lambda_\ell$ denote the λ-auctions corresponding to each heavy equivalence class as defined in Proposition 4.1. Choose λ uniformly at random from $\{\lambda_1, \ldots, \lambda_\ell\}$. If S_0 is heavy, there is $\lambda \in \{\lambda_1, \ldots, \lambda_\ell\}$ such that $\mathsf{Rev}_\lambda(S_0) \geq W(S_0)$, so

$$\begin{aligned}
\mathop{\mathbf{E}}_{\lambda}[\mathsf{Rev}_\lambda(S_0)] &\geq \frac{1}{\ell} W(S_0) \\
&\geq \frac{1}{\ell} \varphi(S_0) \\
&> \frac{p/4}{\ell} W(S).
\end{aligned}$$

Let H denote the event that S_0 is heavy. Then,

$$\begin{aligned}
\mathop{\mathbf{E}}_{\lambda}[\mathop{\mathbf{E}}_{S_0}[\mathsf{Rev}_\lambda(S_0)]] &= \mathop{\mathbf{E}}_{S_0}[\mathop{\mathbf{E}}_{\lambda}[\mathsf{Rev}_\lambda(S_0)]] \\
&\geq \mathop{\mathbf{E}}_{S_0}[\mathop{\mathbf{E}}_{\lambda}[\mathsf{Rev}_\lambda(S_0) \mid H]] \cdot \Pr(H) \\
&\geq \frac{p/4}{\ell} W(S) \cdot \Pr(H),
\end{aligned}$$

and so there exists a λ-auction $\lambda^* \in \{\lambda_1, \ldots, \lambda_\ell\}$ that achieves this bound in expectation over S_0. We have

$$\begin{aligned}
\mathop{\mathbf{E}}_{S_0}[\varphi(S_0)] &= \frac{1}{n} \sum_{i=1}^{n} \mathop{\mathbf{E}}_{S_0}[W(S_0 \setminus \{i\})] \\
&= \mathop{\mathbf{E}}_{i \sim S}\big[\mathop{\mathbf{E}}_{S_0}[W(S_0 \setminus \{i\})]\big] \\
&\geq \frac{p}{2} W(S),
\end{aligned}$$

where in the final inequality we have used the fact that $|S| \geq 2$ and that S is submodular in bidders and so by Lemma 4.2 $\mathbf{E}_{S_0 \sim_p S}[W(S_0)] \geq pW(S)$. By Markov's inequality on the (nonnegative)

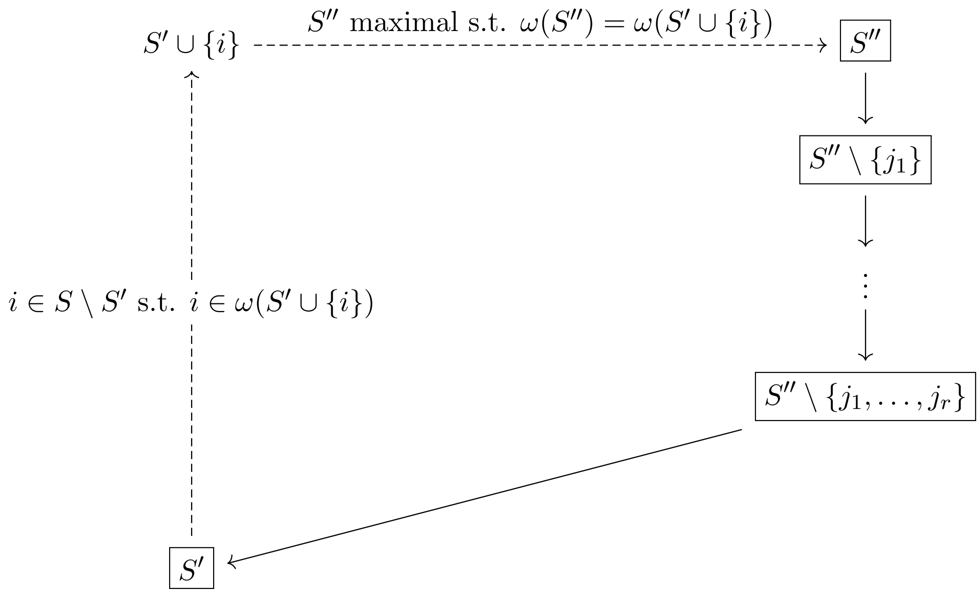

Figure 1: Illustration of the inductive step in Lemma 4.4. Boxed sets correspond to representative elements of equivalence classes in $\mathcal{T}$. Solid arrows represent directed edges in $\mathcal{T}$ from parent to child.

random variable $W(S) - \varphi(S_0)$,

$$\begin{aligned}\Pr(S_0 \text{ is heavy}) &\geq \frac{(p/2)W(S) - (p/4)W(S)}{W(S) - (p/4)W(S)} \\ &= \frac{p/4}{1 - p/4}.\end{aligned}$$

It remains to bound the number of heavy equivalence classes ℓ by $k^{1+\log_{1/\gamma}(4/p)}$. We do this by observing that the equivalence classes obey the following tree structure. Define tree $\mathcal{T}$ as follows: each node of $\mathcal{T}$ is labelled $(S', \omega(S'))$ for some subset $S' \subset S$. The root node is labelled $(S, \omega(S))$. The children of node $(S', \omega(S'))$ are given by $(S' \setminus \{i\}, \omega(S' \setminus \{i\}))$ for each $i \in \omega(S')$ such that $S' \setminus \{i\}$ is heavy. Winner monotonicity will allow us to show that this tree contains a node that represents every heavy equivalence class.

Lemma 4.4. *$\mathcal{T}$ contains all heavy equivalence classes.*

Proof. Let $S^* \subseteq S$ be a set of bidders that arises as a winner set, that is, $S^* = \omega(S'')$ for some $S'' \supseteq S^*$. The set $S' \supset S'' \supset S^*$ is *maximal* for S^* if $\omega(S') = S^*$ and $\omega(S' \cup \{i\}) \neq S^*$ for every $i \notin S'$. We show that for a given winner set of bidders S^*, there is a unique maximal set of bidders $S' \supseteq S^*$ such that $\omega(S') = S^*$. Initialize $S' = S^*$, and greedily add bidders from S to S' while $\omega(S') = S^*$ does not change. Due to winner monotonicity, if $i \notin \omega(S')$, then $i \notin \omega(S' \cup \{j\})$ for any bidder j. Hence, the order in which bidders are added by the greedy procedure does not matter, and therefore the final set S' is the unique maximal set for S^*. Let the representative element of each equivalence class $[(S', \omega(S'))]$ be the one such that S' is maximal for $\omega(S')$.

We prove the lemma by backwards induction on the size of the representative set S' of any equivalence class. The base case of $|S'| = n$ is immediate since the root $(S, \omega(S))$ is the only node for which the representative set has size n. For the inductive step suppose that $\mathcal{T}$ contains a node for every equivalence class for which the representative set is of size at least n'. Let $(S', \omega(S'))$

be the representative of an equivalence class with $|S'| = n' - 1$. Let $i \notin S'$ be a bidder such that $i \in \omega(S' \cup \{i\})$. Such an i exists due to winner monotonicity: if $i \in \omega(S)$, then $i \in \omega(S' \cup \{i\})$, since $S' \cup \{i\} \subset S$. Let S'' be the maximal set such that $\omega(S'') = \omega(S' \cup \{i\})$. We have $|S''| \geq |S' \cup \{i\}| > |S'|$, so by the induction hypothesis $\mathcal{T}$ contains a node labelled $(S'', \omega(S''))$. Now, there must exist a bidder $j_1 \in S'' \setminus S'$ such that $j_1 \in \omega(S'')$. If not, adding all the bidders in $S'' \setminus S'$ to S' would not introduce any new winners, that is,

$$\omega\left(S' \cup \left(S'' \setminus S'\right)\right) = \omega\left(S''\right) = \omega\left(S'\right),$$

which contradicts the maximality of S' for $\omega(S')$. Therefore, the node $(S'', \omega(S''))$ has a child

$$\left(S'' \setminus \{j_1\}, \omega\left(S'' \setminus \{j_1\}\right)\right)$$

($S'' \setminus \{j_1\}$ is maximal due to winner monotonicity). We may now find a bidder $j_2 \in S'' \setminus \{j_1\} \setminus S'$ such that $j_2 \in \omega(S'' \setminus \{j_1\} \setminus S')$ for the same reason as before. Continuing in this fashion yields a path from $(S'', \omega(S''))$ to $(S', \omega(S'))$, so $(S', \omega(S')) \in \mathcal{T}$, as desired. ☐

Each node of $\mathcal{T}$ has at most k children, and the depth of $\mathcal{T}$ is at most $\log_{1/\gamma}\left(\frac{W(S)}{(p/4)W(S)}\right) = \log_{1/\gamma}(4/p)$ since φ decreases by a factor of at least γ when passing from a parent node to a child node (and the tree is truncated at nodes that are not heavy). Hence the number of nodes in the tree is at most $k^{1+\log_{1/\gamma}(4/p)}$, which, combined with Lemma 4.4, completes the proof of Theorem 4.3. ☐

Remark. In the previous section we presented an example of (allocational) bidder valuations where the revenue loss incurred when each bidder participated independently with probability p was significant (Theorem 3.1). That example did not satisfy submodularity in bidders. Indeed, if S_1 and S_2 each contained half of the distinguished bidders S^* in our example and each contained sufficiently many high-valuation bidders, then $W(S_1 \cup S_2) > W(S_1) + W(S_2)$.

4.1 Variations on our parameterized bound

The dependence of our bound on $\max_{S'} |\omega(S')|$ allows us to derive interesting families of bounds when the seller places practical constraints on the auction setting. The first is a constraint on the number of winners, and the second is a bundling constraint that favors allocations that sell certain items together. Reasons for limiting the number of winners include: (1) avoiding the logistical hassle of having a large number of winners – this constraint is commonly used in sourcing auctions [23, 40, 38, 39] and (2) increasing competition to boost revenue, as is studied in Roughgarden et al. [36] (though in a different setting than ours). Kroer and Sandholm [29] show that even the vanilla VCG auction run with bundling constraints can yield significant revenue gains compared to VCG with no bundling constraints.

4.1.1 Limiting the number of winners

Let $W^{n_0}(S)$ denote the maximum welfare of any allocation that allocates a nonempty bundle to at most n_0 bidders. The proofs of all previous theorems go through with this constraint taken into account, with parameters modified correspondingly. Proposition 4.1 holds with $W^{n_0}(S)$ for the class of λ-auctions subject to the constraint that at most n_0 bidders can win a nonempty bundle. In the following theorem statement, $\varphi(S') = \frac{1}{n}\sum_{i=1}^{n} W^{n_0}(S' \setminus \{i\})$.

Theorem 4.5. *Let S be a set of $n \geq 2$ bidders with valuations such that $W^{n_0} : 2^S \to \mathbb{R}_{\geq 0}$ is submodular. Let $\gamma = \max_{S', i \in \omega(S')} \frac{\varphi(S' \setminus \{i\})}{\varphi(S')}$. Let $\mathcal{M}$ be the class of λ-auctions run with the constraint that at most n_0 bidders may receive a nonempty bundle. Then,*

$$\sup_{\lambda \in \mathcal{M}} \mathop{\mathbf{E}}_{S_0 \sim_p S}[\mathsf{Rev}_\lambda(S_0)] \geq \frac{p^2}{16(2n_0)^{1+\log_{1/\gamma}(4/p)}} \cdot W^{n_0}(S).$$

Proof. Given a set $S' \subseteq S$ of bidders, if a bidder in $win(S')$ is removed, at most one new bidder can win a nonempty bundle per the efficient allocation, due to winner monotonicity and the limit on the number of bidders. Thus, $\max_{S'} |\omega(S')| \leq 2n_0$. The remainder of the proof follows from the same arguments used to prove Theorem 4.3. ☐

4.1.2 Bundling constraints

A *bundling* is a partition of the set of items $\{1, \ldots, m\}$. An allocation α *respects* a bundling ϕ if no two items in the same bundle according to ϕ are allocated to different buyers. For a set of bundlings Φ, the class of Φ-*boosted* λ-*auctions* consists of all λ-auctions satisfying $\lambda(\alpha) \geq 0$ for all α that respects a bundling in Φ and $\lambda(\alpha) = 0$ otherwise. Let $W^\Phi(S)$ denote the maximum welfare of any allocation that respects a bundling in Φ. Proposition 4.1 holds with $W^\Phi(S)$ for the class of Φ-boosted λ-auctions (the λ-auction constructed in Proposition 4.1 can be shifted by a constant additive factor to make all boosts nonnegative). In the following theorem statement, $\varphi(S') = \frac{1}{n}\sum_{i=1}^n W^\Phi(S' \setminus \{i\})$.

Theorem 4.6. *Let Φ be a set of bundlings. Let S be a set of $n \geq 2$ bidders with valuations such that $W^\Phi : 2^S \to \mathbb{R}_{\geq 0}$ is submodular. Let $\gamma = \max_{S', i \in \omega(S')} \frac{\varphi(S' \setminus \{i\})}{\varphi(S')}$. Let m_0 be the greatest number of bundles in any bundling in Φ. Let $\mathcal{M}$ be the class of Φ-boosted λ-auctions. Then,*

$$\sup_{\lambda \in \mathcal{M}} \mathop{\mathbf{E}}_{S_0 \sim_p S}[\mathsf{Rev}_\lambda(S_0)] \geq \frac{p^2}{16(2m_0)^{1+\log_{1/\gamma}(4/p)}} \cdot W^\Phi(S).$$

Proof. At most m_0 bidders can win a nonempty bundle of items, so $\max_{S'} |\omega(S')| \leq 2m_0$ by the same reasoning used to prove Theorem 4.5. The arguments used to prove Theorem 4.3 yield the desired bound. ☐

4.2 General distribution over submarkets

So far we have stated our bounds under the assumption that each bidder participates in the auction independently with probability p. Our proof easily generalizes to handle any distribution D over subsets of bidders since the only statistic of the distribution required is the expected welfare of a random subset of bidders $\mathbf{E}_{S_0 \sim_D S}[W(S_0)]$. When bidders participated independently with probability p, submodularity of the welfare function was required to ensure that $\mathbf{E}[W(S_0)] \geq pW(S)$. In the following more general bound, which is in terms of $\mathbf{E}[W(S_0)]$, we only need the more general condition of winner monotonicity.

Theorem 4.7. *Let S be a set of $n \geq 2$ bidders with valuations that satisfy winner monotonicity. Let D be a distribution supported on 2^S with $\mathbf{E}_{S_0 \sim_D S}[W(S_0)] = \mu \cdot W(S)$. Let $\gamma = \max_{S', i \in \omega(S')} \frac{\varphi(S' \setminus \{i\})}{\varphi(S')}$ and let $k = \max_{S'} |\omega(S')|$. The class M of λ-auctions satisfies*

$$\sup_{\lambda \in \mathcal{M}} \mathop{\mathbf{E}}_{S_0 \sim_D S}[\mathsf{Rev}_\lambda(S_0)] \geq \frac{\eta\mu}{k^{1+\log_{1/\gamma}(1/\eta\mu)}} \left(\frac{\mu - 2\eta\mu}{2(1-\eta\mu)}\right) \cdot W(S)$$

for all $0 \leq \eta \leq 1/2$.

Proof. The proof is nearly identical to that of Theorem 4.3. The main modification is that $S' \subset S$ is heavy if $\varphi(S') \geq \eta\mu \cdot W(S)$. Then, $\mathbf{E}_{S_0 \sim_D S}[\varphi(S_0)] \geq \frac{\mu}{2} W(S)$. Markov's inequality yields $\Pr(S_0 \text{ is heavy}) \geq \frac{\mu/2-\eta}{1-\eta}$. □

Versions of Theorems 4.5 and Theorems 4.6 for general distributions can be similarly obtained.

5 How to choose an auction

The proof of Theorem 4.3 involved choosing a random λ-auction, but that process can be made deterministic to give the mechanism designer a λ-auction that achieves the revenue-loss lower bound. The mechanism designer could search over the set of auctions $\{\lambda_1, \ldots, \lambda_\ell\}$ considered in the randomization that takes place in the proof of Theorem 4.3 to find the one that yields the desired expectation threshold (though this would be a highly-inefficient procedure).

A more natural way for the mechanism designer to arrive at a mechanism is to learn from samples, which ensures that the mechanism designer uses the auction that (nearly) optimizes the expected preserved revenue, which could be significantly higher than what our lower bounds guarantee. In this section we assume that bidders have combinatorial, rather than allocational, valuations (that is, a bidder's valuation function only depends on the bundle she receives).

Learning a high-revenue λ-auction would require a number of samples on the order of $(n+1)^m$ [5]. However, for sparse λ-auctions that are restricted to boost only a constant number of allocations, we can perform sample and computationally efficient learning while satisfying a similar guarantee to the ones derived for the entire class.

Let $\Gamma \subseteq \{1, \ldots, (n+1)^m\}$ be a set of allocations. The class of Γ-*boosted* λ-*auctions* consists of all λ-auctions satisfying $\lambda(\alpha) \geq 0$ for all $\alpha \in \Gamma$ and $\lambda(\alpha) = 0$ for all $\alpha \notin \Gamma$ (and can be specified by vectors in $\mathbb{R}^{|\Gamma|}$). Γ-boosted λ-auctions were introduced by Balcan, Sandholm, and Vitercik [5]. Let $W^\Gamma(S) = \max_{\alpha \in \Gamma} W(\alpha)$. We may derive a guarantee for this class of auctions analogous to Theorem 4.3. In the following theorem statement, $\varphi(S') = \frac{1}{n}\sum_{i=1}^{n} W^\Gamma(S' \setminus \{i\})$.

Theorem 5.1. *Let S be a set of $n \geq 2$ bidders with valuations such that $W^\Gamma : 2^S \to \mathbb{R}_{\geq 0}$ is submodular. Let $\gamma = \max_{S', i \in \omega(S')} \frac{\varphi(S' \setminus \{i\})}{\varphi(S')}$ and let k be the maximum number of winners in any allocation in Γ. The class $\mathcal{M}$ of Γ-boosted λ-auctions satisfies*

$$\sup_{\lambda \in \mathcal{M}} \mathop{\mathbf{E}}_{S_0 \sim_p S}[\mathsf{Rev}_\lambda(S_0)] \geq \frac{p^2}{16(2k)^{1+\log_{1/\gamma}(4/p)}} \cdot W^\Gamma(S).$$

5.1 Algorithm for learning an auction from samples

We now give an algorithm that the mechanism designer can use to compute a Γ-boosted λ-auction that nearly achieves an expected revenue of $\sup_{\lambda \in \mathcal{M}} \mathbf{E}_{S_0 \sim_p S}[\mathsf{Rev}_\lambda(S_0)]$. Our algorithm leverages practically-efficient routines for solving winner determination, which is a generalization of the problem of computing welfare-maximizing allocations. Our algorithm fits the paradigm of *sample-based automated mechanism design* [32, 33, 41, 5]. The mechanism designer samples several subsets of bidders according to D, and computes the Γ-boosted λ-auction that maximizes empirical revenue over the samples.

While computing the empirical-revenue-maximizing auction is NP-hard in general, since winner determination is NP-hard, winner determination can be efficiently solved in practice [37, 42, 43]. Furthermore, when bidders have gross-substitutes valuations, winner determination can be solved in polynomial time [11]. The run-time of our algorithm is exponential only in $|\Gamma|$ but polynomial in all other problem parameters (including the run-time required to solve winner determination with m items and n bidders).

Theorem 5.2. *Let $\mathcal{M}$ be the class of Γ-boosted λ-auctions. A $\hat{\lambda} \in \mathcal{M}$ such that*

$$\mathop{\mathbf{E}}_{S_0 \sim S}[\mathsf{Rev}_{\hat{\lambda}}(S_0)] \geq \sup_{\lambda \in \mathcal{M}} \mathop{\mathbf{E}}_{S_0 \sim S}[\mathsf{Rev}_{\lambda}(S_0)] - \varepsilon$$

with probability at least $1-\delta$ can be computed in $N(\min\{m,n\}+1)w(m,n) + (Nn|\Gamma|)^{O(|\Gamma|)}$ time, where $w(m,n)$ is the time required to solve winner determination for n buyers with valuations over m items and $N = O\left(\frac{|\Gamma|\ln(n|\Gamma|)}{\varepsilon^2}\ln(\frac{1}{\delta})\right)$.

Proof. The algorithm is empirical revenue maximization. The mechanism designer samples $S_1, \ldots, S_N$ independently and identically according to distribution D on 2^S. (We assume for simplicity that sampling according to D can be done in a computationally efficient manner. If bidders participate independently with probability p, then the mechanism designer simply needs to flip N coins of bias p for each of the n bidders in S.) The auction used will be the one that maximizes empirical revenue

$$\hat{\lambda} = \mathop{\mathrm{argmax}}_{\lambda \in \mathbb{R}_{\geq 0}^{|\Gamma|}} \frac{1}{N} \sum_{t=1}^{N} \mathsf{Rev}_{\lambda}(S_t).$$

Balcan, Sandholm, and Vitercik [5] show that

$$N = O\left(\frac{|\Gamma|\ln(n|\Gamma|)}{\varepsilon^2}\ln\left(\frac{1}{\delta}\right)\right)$$

samples suffice to guarantee that the expected revenue of $\hat{\lambda}$ is ε-close to optimal with probability at least $1-\delta$ over the draw of $S_1, \ldots, S_N$.

We now determine the computational complexity of empirical revenue maximization. Our algorithm exploits similar geometric intuition that was used by Balcan, Sandholm, and Vitercik [5] to derive the above sample complexity guarantee. A similar approach was used by Balcan, Prasad, and Sandholm to compute empirical revenue maximizing two-part tariff structures [3] and to efficiently design prior-free combinatorial auctions [4].

For each $1 \leq t \leq N$ let α^t denote the efficient allocation among bidders in S_t. For each $1 \leq t \leq N$ and each $i \in win(S_t)$ let α^t_{-i} denote the efficient allocation among bidders in $S_t \setminus \{i\}$. For $i \notin win(S_t)$, $\alpha^t_{-i} = \alpha^t$. Determining these allocations requires at most $N + N \cdot \min\{m,n\}$ calls to the winner determination routine, since $|win(S_t)| \leq m$. The allocation used by any Γ-boosted λ-auction on S_t is in $\Gamma \cup \{\alpha^t\}$, and the allocation used to determine the payment by bidder $i \in S_t$ by any Γ-boosted λ-auction on S_t is in $\Gamma \cup \{\alpha^t_{-i}\}$.

For each t and each pair of allocations $\alpha, \alpha' \in \Gamma \cup \{\alpha^t\}$, let $H(t,\alpha,\alpha')$ denote the hyperplane

$$\sum_{i \in S_t} v_i(\alpha) + \lambda(\alpha) = \sum_{i \in S_t} v_i(\alpha') + \lambda(\alpha').$$

For each t, each $i \in S_t$, and each pair of allocations $\alpha, \alpha' \in \Gamma \cup \{\alpha^t_{-i}\}$, let let $H_{-i}(t,\alpha,\alpha')$ denote the hyperplane

$$\sum_{j \in S_t \setminus \{i\}} v_j(\alpha) + \lambda(\alpha) = \sum_{j \in S_t \setminus \{i\}} v_j(\alpha') + \lambda(\alpha').$$

Let $\mathcal{H}$ denote the collection of these hyperplanes. The total number of such hyperplanes is at most $N(|\Gamma|+1)^2 + Nn(|\Gamma|+1)^2$. It is a basic combinatorial fact that $\mathcal{H}$ partitions $\mathbb{R}^{|\Gamma|}$ into at most

$|\mathcal{H}|^{|\Gamma|} \leq (N(n+1)(|\Gamma|+1)^2)^{|\Gamma|}$ regions. Each region is a convex polytope that is the intersection of at most $|\mathcal{H}|$ halfspaces. Representations of these regions as $0/1$ constraint-vectors of length $|\mathcal{H}|$ can be computed in $poly(|\mathcal{H}|^{|\Gamma|})$ time using standard techniques [46]. Empirical revenue is linear as a function of λ in each region, since the allocations used by λ are constant within as λ varies in a given region. Thus, the auction maximizes empirical revenue within a given region can be found by solving a linear program that involves $|\Gamma|$ variables and at most $|\mathcal{H}|$ constraints, which can be done in $poly(|\mathcal{H}|, |\Gamma|)$ time. □

One special case is when all allocations in Γ are given the *same* boost. Then, the parameter space is $\mathbb{R}$, the number of relevant regions (subinteverals of $\mathbb{R}$) is $O(Nn|\Gamma|^2)$, and the algorithm in Theorem 5.2 has a run-time of $O(N \min\{m, n\} w(m, n) + Nn|\Gamma|^2)$. Mixed bundling auctions [24, 45] are an instance of this with $|\Gamma| = n$.

5.2 Structural revenue maximization

Let $\Gamma_1 \subset \Gamma_2$ be collections of allocations, and let $\mathcal{M}_1$ and $\mathcal{M}_2$ denote the classes of Γ_1-boosted λ-auctions and Γ_2-boosted λ-auctions, respectively. Suppose the mechanism designer has drawn some number of samples N, and observes that the empirical-revenue-maximizing auction λ_2 over $\mathcal{M}_2$ yields slightly higher revenue than the empirical-revenue-maximizing auction λ_1 over $\mathcal{M}_1$, but λ_2 assigns nonzero boosts to significantly more allocations than λ_1 and is much more complex to describe. $\mathcal{M}_2$ is a richer auction class than $\mathcal{M}_1$, so it always yields higher empirical revenue, but there is the risk that it overfits to the samples. *Structural revenue maximization* allows the mechanism designer to precisely choose between such auctions by quantifying the tradeoff between empirical revenue maximization and overfitting [2, 5]. Instead of choosing λ_2 by default, the mechanism designer should choose λ_k, $k \in \{1, 2\}$ that maximizes empirical revenue minus a regularization term $\varepsilon_{\mathcal{M}_k}(N, \delta)$. The correct regularization term is precisely the error term

$$\varepsilon_{\mathcal{M}_k}(N, \delta) = O\left(\frac{|\Gamma_k| \ln(n|\Gamma_k|)}{\varepsilon^2} \ln\left(\frac{1}{\delta}\right)\right)$$

in the generalization guarantee for Γ-boosted λ-auctions obtained by Balcan, Sandholm, and Vitercik [5], which is fine-tuned to the intrinsic complexity of the auction class. Structural revenue maximization can be especially useful to the mechanism designer when there is a limit on the number of samples he can draw (due to a run-time constraint, for example). In this case, he may run the exact same geometric algorithm given in Theorem 5.2 with the modified objective of empirical revenue minus the regularizer described above. In particular, the algorithm may be run over the entire class of λ-auctions, and the mechanism designer effectively learns the best set Γ of allocations to boost in order to guarantee high expected revenue while also generalizing well with high confidence.

6 Conclusions and future research

We proved the first-known lower bounds on how much revenue can be preserved by truthful (combinatorial) auctions (for limited supply) when only a random fraction of the population submits bids. Our bounds were for the class of λ-auctions when bidders have valuation functions that satisfy a natural winner-monotonicity criterion. While one's first instinct might be that the expected revenue preserved by a mechanism when each bidder participates independently with probability p should be at least a p fraction of the revenue obtained on the entire set of bidders, we gave a simple example where market shrinkage resulted in a more-significant decrease in competition. That example yielded an expected revenue of only p^2 times the revenue over the entire set. We then constructed a more dramatic example that showed that when bidders have allocational valuations that violate winner monotonicity, only an exponentially small fraction of the social surplus can be extracted as revenue by any auction (while the vanilla VCG auction extracts nearly the entire social surplus on the full market). In contrast to this negative example, we proved a lower bound on preserved-revenue when bidders have valuations that satisfy conditions like welfare submodularity and winner monotonicity. We additionally derived revenue-preservation guarantees when the mechanism designer places certain practically-motivated constraints on the auction, such as limiting the number of winners or placing bundling constraints that stipulate that certain items must be sold together. Finally, we showed how the mechanism designer can efficiently learn an auction that is robust to market shrinkage by leveraging practically-efficient routines for solving winner determination. We described how to use structural revenue maximization in order to balance the tradeoff between maximizing revenue and preventing overfitting.

Our work is the first to formally study the problem of preserving revenue in a shrinking market. It spurs several open questions and new interesting research directions. First, our bounds exploited combinatorial structure imposed on S by winner monotonicity. Can similar ideas be used to derive results for more general valuation classes? How close are our bounds to tight? Another interesting, and seemingly more difficult, setting is the one where the mechanism designer does not know the distribution D over 2^S beforehand. Can he still arrive at an auction that is robust to the shrinking market? More broadly, what is the "least" amount of information about the distribution over submarkets the mechanism designer needs to design a robust auction? For example, if the mechanism designer knows that a uniformly random subset of S of some fixed size participates, he need not know that size beforehand (since he can run empirical revenue maximization on samples after observing how many bidders have shown up, with no barrier to incentive compatibility). Alternately, if the mechanism designer knows that each bidder participates independently with probability p, but does not know p, is it still possible to design a robust auction? Finally, it would be interesting to extend our techniques to understand market shrinkage in other settings including objectives beyond revenue, other auction classes, and unlimited supply.

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