

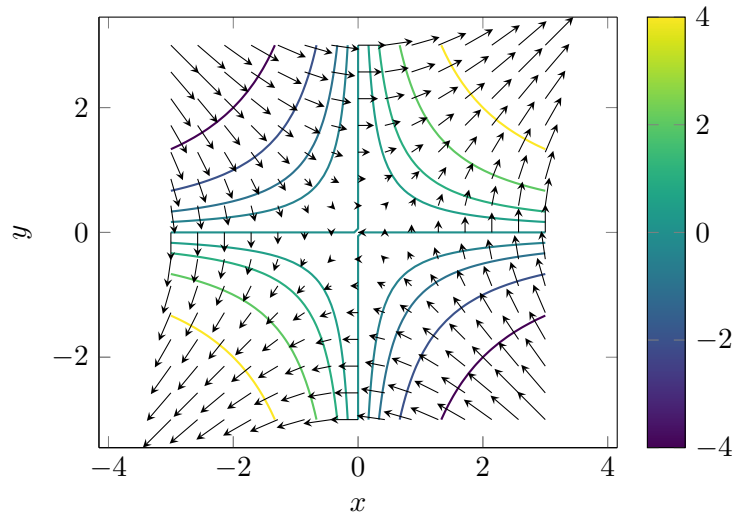
Advanced Mathematics Exercises

1 CW 40

Exercise 1.1. Let f be the scalar function defined by $f(x, y) = xy$. Sketch the contour lines and the vector field ∇f .

Solution:

$$(\nabla f)(x, y) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right) = (y, x)$$



□

Exercise 1.2. Let

$$f(x, y) = \sin(\pi xy) \cdot e^{-\frac{x}{3}} \quad \text{and} \quad p = \left(1, \frac{1}{3}\right).$$

- (a) Compute ∇f and $(\nabla f)(p)$. Use the special values of \sin and \cos .
- (b) Find the directions of maximum increase and decrease at p . You can give approximate values.
- (c) Give the direction of the contour line at p .
- (d) The equation of the tangent plane of the graph of f at (x_0, y_0) is

$$z = f(x_0, y_0) + (\nabla f)(x_0, y_0) \cdot (x - x_0, y - y_0).$$

Determine the equation of the tangent plane of $z = f(x, y)$ at p . Give a normal vector of the plane.

- (e) Find the directional derivative of $f(x, y)$ at p along the vector

$$v = \frac{1}{\sqrt{2}}(1, 1).$$

Solution:

(a)

$$\begin{aligned}\frac{\partial f}{\partial x} &= e^{-x/3} \left(\pi y \cos(\pi xy) - \frac{1}{3} \sin(\pi xy) \right) \\ \frac{\partial f}{\partial y} &= e^{-x/3} \pi x \cos(\pi xy) \\ \Rightarrow (\nabla f)(x, y) &= \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right) \\ &= e^{-x/3} \left(\pi y \cos(\pi xy) - \frac{1}{3} \sin(\pi xy), \pi x \cos(\pi xy) \right)\end{aligned}$$

With $\sin(\frac{\pi}{3}) = \frac{\sqrt{3}}{2}$ and $\cos(\frac{\pi}{3}) = \frac{1}{2}$:

$$(\nabla f)(p) = e^{-1/3} \left(\frac{\pi - \sqrt{3}}{6}, \frac{\pi}{2} \right)$$

(b) Maximum increase: $(\nabla f)(p)$, maximum decrease: $-(\nabla f)(p)$

(c) Since when do contour lines have a direction? Is the question asking for the tangent vector to the contour line at p ?

(d)

$$\begin{aligned}z = f(x, y) &= f(p) + (\nabla f)(p) \cdot (x - 1, y - \frac{1}{3}) \\ &= \frac{\sqrt{3}}{2} e^{-1/3} + e^{-1/3} \left(\frac{\pi - \sqrt{3}}{6}, \frac{\pi}{2} \right) \cdot (x - 1, y - \frac{1}{3}) \\ &= e^{-1/3} \left(\frac{\sqrt{3}}{2} + \frac{1}{6} ((\pi - \sqrt{3})x + 3\pi y + \sqrt{3}) \right)\end{aligned}$$

The surface $z = f(x, y)$ can be written as $F(x, y, z) = f(x, y) - z = 0$. Therefore,

$$\nabla F = (f_x, f_y, -1)$$

is normal to the surface:

$$\begin{aligned}n := (\nabla f)(p) &= (f_x(p), f_y(p), -1) \\ &= \left(\frac{\pi - \sqrt{3}}{6} e^{-1/3}, \frac{\pi}{2} e^{-1/3}, -1 \right)\end{aligned}$$

(e)

$$\begin{aligned}(\nabla f)(p) \cdot v &= e^{-1/3} \left(\frac{\pi - \sqrt{3}}{6}, \frac{\pi}{2} \right) \cdot \frac{1}{\sqrt{2}} (1, 1) \\ &= \frac{e^{-1/3}}{\sqrt{2}} \left(\frac{\pi - \sqrt{3}}{6} + \frac{\pi}{2} \right) \\ &= \frac{e^{-1/3}}{\sqrt{2}} \cdot \frac{4\pi - \sqrt{3}}{6}\end{aligned}$$

□

Exercise 1.3. Compute curl and divergence of the vector field

$$F(x, y, z) = (\sin(x)xy, ze^{-x}, yz).$$

Solution:

$$\frac{\partial F_x}{\partial y} = \frac{\partial}{\partial y}(\sin(x)xy) = x \sin(x)$$

$$\frac{\partial F_x}{\partial z} = \frac{\partial}{\partial z}(\sin(x)xy) = 0$$

$$\frac{\partial F_y}{\partial x} = \frac{\partial}{\partial x}(ze^{-x}) = -ze^{-x}$$

$$\frac{\partial F_y}{\partial z} = \frac{\partial}{\partial z}(ze^{-x}) = e^{-x}$$

$$\frac{\partial F_z}{\partial x} = \frac{\partial}{\partial x}(yz) = 0$$

$$\frac{\partial F_z}{\partial y} = \frac{\partial}{\partial y}(yz) = z$$

$$\begin{aligned} \operatorname{curl} F &= \nabla \times F = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}, \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}, \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) \\ &= (z - e^{-x}, 0 - 0, -ze^{-x} - x \sin(x)) \\ &= (z - e^{-x}, 0, -ze^{-x} - x \sin(x)) \end{aligned}$$

$$\begin{aligned} \operatorname{div} F &= \nabla \cdot F = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \\ &= \frac{\partial}{\partial x}(\sin(x)xy) + \frac{\partial}{\partial y}(ze^{-x}) + \frac{\partial}{\partial z}(yz) \\ &= y \sin(x) + xy \cos(x) + 0 + y \\ &= y \sin(x) + xy \cos(x) + y \end{aligned}$$

□

Exercise 1.4. *Let*

$$f(x, y, z) = x^2yz^3 \quad \text{and} \quad F(x, y, z) = (xz, -y^2, 2x^2y).$$

Give ∇f , $\nabla^2 f$, $\nabla \cdot F$, and $\nabla \times F$.

Solution:

$$\begin{aligned} \nabla f &= \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right) \\ &= (2xyz^3, x^2z^3, 3x^2yz^2) \end{aligned}$$

$$\begin{aligned} \nabla^2 f &= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \\ &= 2yz^3 + 0 + 6x^2yz \\ &= 2yz^3 + 6x^2yz \end{aligned}$$

$$\begin{aligned} \nabla \cdot F &= \frac{\partial}{\partial x}(xz) + \frac{\partial}{\partial y}(-y^2) + \frac{\partial}{\partial z}(2x^2y) \\ &= z - 2y \end{aligned}$$

$$\begin{aligned}
\frac{\partial F_x}{\partial y} &= \frac{\partial}{\partial y}(xz) = 0 \\
\frac{\partial F_x}{\partial z} &= \frac{\partial}{\partial z}(xz) = x \\
\frac{\partial F_y}{\partial x} &= \frac{\partial}{\partial x}(-y^2) = 0 \\
\frac{\partial F_y}{\partial z} &= \frac{\partial}{\partial z}(-y^2) = 0 \\
\frac{\partial F_z}{\partial x} &= \frac{\partial}{\partial x}(2x^2y) = 4xy \\
\frac{\partial F_z}{\partial y} &= \frac{\partial}{\partial y}(2x^2y) = 2x^2 \\
\nabla \times F &= \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}, \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}, \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) \\
&= (2x^2 - 0, x - 4xy, 0 - 0) \\
&= (2x^2, x - 4xy, 0)
\end{aligned}$$

□

Exercise 1.5. Compute differential forms:

(a) Let

$$f(x, y, z) = \frac{x}{yz}$$

be a scalar function. Give the differential form df .

Hint:

$$df = F_1(x, y, z) dx + F_2(x, y, z) dy + F_3(x, y, z) dz.$$

(b) Let

$$\omega = x^2 \sin(y) dx + 2^x \cos(y) dy$$

be a differential form on \mathbb{R}^2 . Give the 2-form $d\omega$.

Hint:

$$d\omega = \phi(x, y) dx \wedge dy.$$

(c) Let

$$\omega = x^2 \sin(y) dx + z^2 \cos(y) dy - xy^2 dz$$

be a differential form on \mathbb{R}^3 . Give the 2-form $d\omega$.

Hint:

$$d\omega = F_1(x, y, z) dy \wedge dz + F_2(x, y, z) dz \wedge dx + F_3(x, y, z) dx \wedge dy.$$

Solution:

(a)

$$df = \frac{1}{yz} dx - \frac{x}{y^2 z} dy - \frac{x}{yz^2} dz.$$

(b)

$$\begin{aligned}
\phi(x, y) &= \frac{\partial}{\partial x}(2^x \cos(y)) - \frac{\partial}{\partial y}(x^2 \sin(y)) \\
&= \ln(2) \cos(y) - x^2 \cos(y) \\
&= \cos(y) (\ln(2) - x^2)
\end{aligned}$$

So:

$$\begin{aligned}d\omega &= \phi(x, y) \, dx \wedge dy \\ &= \left(\cos(y) \left(\ln(2) - x^2 \right) \right) \, dx \wedge dy\end{aligned}$$

(c)

$$F_1(x, y, z) = \frac{\partial}{\partial y}(-xy^2) - \frac{\partial}{\partial z}(z^2 \cos y) = -2xy - 2z \cos y$$

$$F_2(x, y, z) = \frac{\partial}{\partial z}(x^2 \sin y) - \frac{\partial}{\partial x}(-xy^2) = 0 + y^2 = y^2$$

$$F_3(x, y, z) = \frac{\partial}{\partial x}(z^2 \cos y) - \frac{\partial}{\partial y}(x^2 \sin y) = 0 - x^2 \cos y = -x^2 \cos y$$

So:

$$d\omega = (-2xy - 2z \cos y) \, dy \wedge dz + y^2 \, dz \wedge dx - (x^2 \cos y) \, dx \wedge dy.$$

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