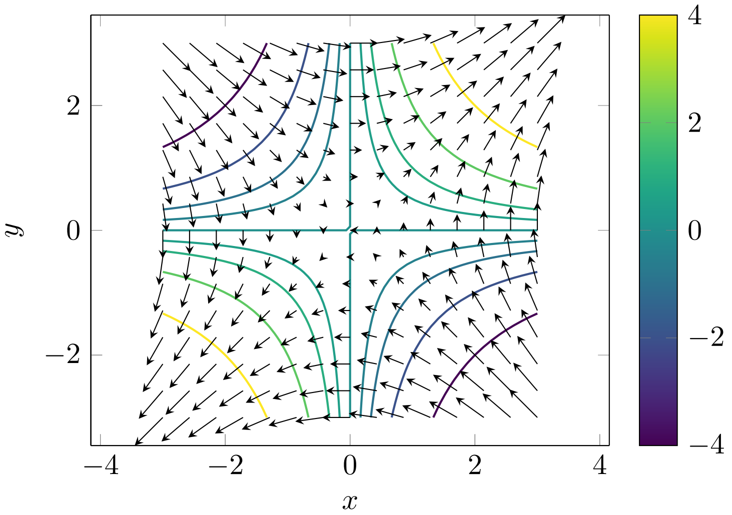
Advanced Mathematics Exercises

Exercise 1. *Let f be the scalar function defined by $f(x, y) = xy$. Sketch the contour lines and the vector field ∇f.*

Solution:

$$(\nabla f)(x, y) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = (y, x)$$



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Exercise 2. *Let*

$$f(x,y) = \sin(\pi xy) \cdot e^{-\frac{x}{3}} \quad \textit{and} \quad p = \left(1, \frac{1}{3}\right).$$

(a) Compute ∇f *and* $(\nabla f)(p)$*. Use the special values of* sin *and* cos.

(b) Find the directions of maximum increase and decrease at p*. You can give approximate values.*

(c) Give the direction of the contour line at p*.*

(d) The equation of the tangent plane of the graph of f *at* (x_0, y_0) *is*

$$z = f(x_0, y_0) + (\nabla f)(x_0, y_0) \cdot (x - x_0, y - y_0).$$

Determine the equation of the tangent plane of $z = f(x,y)$ *at* p*. Give a normal vector of the plane.*

(e) Find the directional derivative of $f(x,y)$ *at* p *along the vector*

$$v = \frac{1}{\sqrt{2}}(1,1).$$

Solution:

(a)

$$\begin{aligned}
\frac{\partial f}{\partial x} &= e^{-x/3}\left(\pi y \cos(\pi xy) - \frac{1}{3}\sin(\pi xy)\right) \\
\frac{\partial f}{\partial y} &= e^{-x/3}\pi x \cos(\pi xy) \\
\implies (\nabla f)(x,y) &= \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) \\
&= e^{-x/3}\left(\pi y \cos(\pi xy) - \frac{1}{3}\sin(\pi xy),\ \pi x \cos(\pi xy)\right)
\end{aligned}$$

With $\sin(\frac{\pi}{3}) = \frac{\sqrt{3}}{2}$ and $\cos(\frac{\pi}{3}) = \frac{1}{2}$:

$$(\nabla f)(p) = e^{-1/3}\left(\frac{\pi - \sqrt{3}}{6}, \frac{\pi}{2}\right)$$

(b) Maximum increase: $(\nabla f)(p)$, maximum decrease: $-(\nabla f)(p)$

(c) Contour lines are perpendicular to the gradient:

$$(x,\ y) \cdot (-y,\ x) = 0 \implies u = e^{-1/3}\left(-\frac{\pi}{2}, \frac{\pi - \sqrt{3}}{6}\right)$$

(d)

$$\begin{aligned}
z = f(x,y) &= f(p) + (\nabla f)(p) \cdot (x - 1, y - \frac{1}{3}) \\
&= \frac{\sqrt{3}}{2}e^{-1/3} + e^{-1/3}\left(\frac{\pi - \sqrt{3}}{6}, \frac{\pi}{2}\right) \cdot (x - 1,\ y - \frac{1}{3}) \\
&= e^{-1/3}\left(\frac{\sqrt{3}}{2} + \frac{1}{6}\left((\pi - \sqrt{3})x + 3\pi y + \sqrt{3}\right)\right)
\end{aligned}$$

The surface $z = f(x,y)$ can be written as $F(x,y,z) = f(x,y) - z = 0$. Therefore,

$$\nabla F = (f_x,\ f_y,\ -1)$$

is normal to the surface:

$$\begin{aligned} n := (\nabla F)(p) &= (f_x(p),\ f_y(p),\ -1) \\ &= \left(\frac{\pi - \sqrt{3}}{6} e^{-1/3},\ \frac{\pi}{2} e^{-1/3},\ -1 \right) \end{aligned}$$

(e)

$$\begin{aligned} (\nabla f)(p) \cdot v &= e^{-1/3} \left(\frac{\pi - \sqrt{3}}{6},\ \frac{\pi}{2} \right) \cdot \frac{1}{\sqrt{2}}(1,1) \\ &= \frac{e^{-1/3}}{\sqrt{2}} \left(\frac{\pi - \sqrt{3}}{6} + \frac{\pi}{2} \right) \\ &= \frac{e^{-1/3}}{\sqrt{2}} \cdot \frac{4\pi - \sqrt{3}}{6} \end{aligned}$$

$\square$

Exercise 3. *Compute curl and divergence of the vector field*

$$F(x, y, z) = (\sin(x)xy,\ ze^{-x},\ yz).$$

Solution:

$$\begin{aligned}
\frac{\partial F_x}{\partial y} &= \frac{\partial}{\partial y}(\sin(x)xy) = x\sin(x) \\
\frac{\partial F_x}{\partial z} &= \frac{\partial}{\partial z}(\sin(x)xy) = 0 \\
\frac{\partial F_y}{\partial x} &= \frac{\partial}{\partial x}(ze^{-x}) = -ze^{-x} \\
\frac{\partial F_y}{\partial z} &= \frac{\partial}{\partial z}(ze^{-x}) = e^{-x} \\
\frac{\partial F_z}{\partial x} &= \frac{\partial}{\partial x}(yz) = 0 \\
\frac{\partial F_z}{\partial y} &= \frac{\partial}{\partial y}(yz) = z
\end{aligned}$$

$$\begin{aligned}
\operatorname{curl} F = \nabla \times F &= \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z},\ \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x},\ \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right) \\
&= \left(z - e^{-x},\ 0 - 0,\ -ze^{-x} - x\sin(x)\right) \\
&= \left(z - e^{-x},\ 0,\ -ze^{-x} - x\sin(x)\right)
\end{aligned}$$

$$\begin{aligned}
\operatorname{div} F = \nabla \cdot F &= \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \\
&= \frac{\partial}{\partial x}\big(\sin(x)xy\big) + \frac{\partial}{\partial y}(ze^{-x}) + \frac{\partial}{\partial z}(yz) \\
&= y\sin(x) + xy\cos(x) + 0 + y \\
&= y\sin(x) + xy\cos(x) + y
\end{aligned}$$

□

Exercise 4. *Let*

$$f(x,y,z) = x^2yz^3 \quad \text{and} \quad F(x,y,z) = (xz, -y^2, 2x^2y).$$

Give ∇f, $\nabla^2 f$, $\nabla \cdot F$, *and* $\nabla \times F$.

Solution:

$$\begin{aligned}\nabla f &= \left(\frac{\partial f}{\partial x},\ \frac{\partial f}{\partial y},\ \frac{\partial f}{\partial z}\right)\\ &= \left(2xyz^3,\ x^2z^3,\ 3x^2yz^2\right)\end{aligned}$$

$$\begin{aligned}\nabla^2 f &= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}\\ &= 2yz^3 + 0 + 6x^2yz\\ &= 2yz^3 + 6x^2yz\end{aligned}$$

$$\begin{aligned}\nabla \cdot F &= \frac{\partial}{\partial x}(xz) + \frac{\partial}{\partial y}(-y^2) + \frac{\partial}{\partial z}(2x^2y)\\ &= z - 2y\end{aligned}$$

$$\begin{aligned}\frac{\partial F_x}{\partial y} &= \frac{\partial}{\partial y}(xz) = 0\\ \frac{\partial F_x}{\partial z} &= \frac{\partial}{\partial z}(xz) = x\\ \frac{\partial F_y}{\partial x} &= \frac{\partial}{\partial x}(-y^2) = 0\\ \frac{\partial F_y}{\partial z} &= \frac{\partial}{\partial z}(-y^2) = 0\\ \frac{\partial F_z}{\partial x} &= \frac{\partial}{\partial x}(2x^2y) = 4xy\\ \frac{\partial F_z}{\partial y} &= \frac{\partial}{\partial y}(2x^2y) = 2x^2\end{aligned}$$

$$\begin{aligned}\nabla \times F &= \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z},\ \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x},\ \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\\ &= \left(2x^2 - 0,\ x - 4xy,\ 0 - 0\right)\\ &= \left(2x^2,\ x - 4xy,\ 0\right)\end{aligned}$$

$\square$

Exercise 5. *Compute differential forms:*

(a) Let

$$f(x, y, z) = \frac{x}{yz}$$

be a scalar function. Give the differential form df.
Hint:

$$df = F_1(x, y, z)\, dx + F_2(x, y, z)\, dy + F_3(x, y, z)\, dz.$$

(b) Let

$$\omega = x^2 \sin(y)\, dx + 2^x \cos(y)\, dy$$

be a differential form on $\mathbb{R}^2$. Give the 2-form $d\omega$.
Hint:

$$d\omega = \phi(x, y)\, dx \wedge dy.$$

(c) Let

$$\omega = x^2 \sin(y)\, dx + z^2 \cos(y)\, dy - xy^2\, dz$$

be a differential form on $\mathbb{R}^3$. Give the 2-form $d\omega$.
Hint:

$$d\omega = F_1(x, y, z)\, dy \wedge dz + F_2(x, y, z)\, dz \wedge dx + F_3(x, y, z)\, dx \wedge dy.$$

Solution:

(a)

$$df = \frac{1}{yz}\, dx - \frac{x}{y^2 z}\, dy - \frac{x}{yz^2}\, dz$$

(b)

$$\begin{aligned}\phi(x, y) &= \frac{\partial}{\partial x}(2^x \cos(y)) - \frac{\partial}{\partial y}(x^2 \sin(y)) \\ &= \ln(2)2^x \cos(y) - x^2 \cos(y) \\ &= \cos(y) \left(\ln(2)2^x - x^2\right)\end{aligned}$$

So:

$$\begin{aligned}d\omega &= \phi(x, y)\, dx \wedge dy \\ &= \left(\cos(y) \left(\ln(2)2^x - x^2\right)\right)\, dx \wedge dy\end{aligned}$$

(c)

$$\begin{aligned}F_1(x, y, z) &= \frac{\partial}{\partial y}(-xy^2) - \frac{\partial}{\partial z}(z^2 \cos y) = -2xy - 2z \cos y \\ F_2(x, y, z) &= \frac{\partial}{\partial z}(x^2 \sin y) - \frac{\partial}{\partial x}(-xy^2) = 0 + y^2 = y^2 \\ F_3(x, y, z) &= \frac{\partial}{\partial x}(z^2 \cos y) - \frac{\partial}{\partial y}(x^2 \sin y) = 0 - x^2 \cos y = -x^2 \cos y\end{aligned}$$

So:

$$d\omega = (-2xy - 2z \cos y)\, dy \wedge dz + y^2\, dz \wedge dx - (x^2 \cos y)\, dx \wedge dy.$$

□

Exercise 6. *Let* $f(x,y) = \frac{1}{\sqrt{x^2+y^2}}$. *Compute* ∇f. *Then express* f *and* ∇f *in terms of the norm* $\|v\|$, *where* $v = (x,y)$.

Solution:

$$(\nabla f)(x,y) = -\frac{1}{(x^2+y^2)^{3/2}}\,(x,\ y)$$

With $\|v\| = \sqrt{x^2+y^2}$:

$$f(v) = \frac{1}{\|v\|}$$

$$(\nabla f)(v) = -\frac{v}{\|v\|^3}$$

$\square$

Exercise 7. *Let* $F(x,y,z) = (y^2, xz, 1)$ *be a vector field. The following curves are given:*

- $C_1 : r_1(t) = (t,t,t), \quad t \in [0,1]$
- $C_2 : r_2(t) = (2t,2t,2t), \quad t \in [0,\frac{1}{2}]$
- $C_3 : r_3(t) = (t,t^2,t^3), \quad t \in [0,1]$

Sketch the curves. Compute the line integrals

$$\int_{C_1} F \cdot dr, \quad \int_{C_2} F \cdot dr, \quad \int_{C_3} F \cdot dr.$$

Why do the first two line integrals coincide? Is F a conservative vector field?

Solution: TODO □

Exercise 8. *Let C be the hypocycloid (astroid) given by*

$$r(t) = (\cos^3(t), \sin^3(t)).$$

Find the length of C. Hint: *Compute the length of C in the first quadrant ($t \in [0, \frac{\pi}{2}]$) and multiply the result by 4.*

Solution:

$$\begin{aligned}
r'(t) &= \Big(-3\cos^2(t)\sin(t),\ 3\sin^2(t)\cos(t)\Big) \\
&= 3\sin(t)\cos(t)\,(-\cos(t),\ \sin(t)) \\
\implies \|r'(t)\| &= 3\sin(t)\cos(t)\sqrt{\cos^2(t)+\sin^2(t)} \\
&= 3\sin(t)\cos(t)
\end{aligned}$$

$$\begin{aligned}
l = \int_C ds &= 4\int_0^{\frac{\pi}{2}} \|r'(t)\|\ dt \\
&= 12\int_0^{\frac{\pi}{2}} \sin(t)\cos(t)\,dt
\end{aligned}$$

Let $u = \sin(t) \implies du = \cos(t)\,dt$:

$$\begin{aligned}
l &= 12\int_0^1 u\,du \\
&= 12\left[\frac{1}{2}u^2\right]_0^1 \\
&= 6\left[\sin^2(t)\right]_0^{\frac{\pi}{2}} \\
&= 6
\end{aligned}$$

□

Exercise 9. *Compute the line integrals of the first and the second kind:*

(a) Let C_1 be the semi-circle given by $r_1(t) = (3\cos(t), 3\sin(t)), \quad t \in [0, \pi].$

$$\int_{C_1} x^2 y \, ds$$

(b) Let C_2 be given by the parametrization $r_2(t) = (4t, 3t^2), \quad t \in [0, 1].$

$$\int_{C_2} (x^2 y) \, dx - (x - yx) \, dy$$

Solution: TODO □