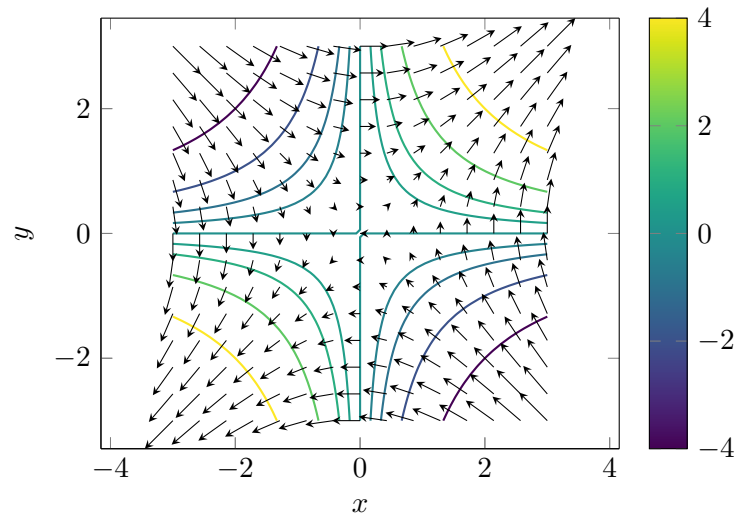


# Advanced Mathematics Exercises

**Exercise 1.** Let  $f$  be the scalar function defined by  $f(x, y) = xy$ . Sketch the contour lines and the vector field  $\nabla f$ .

**Solution:**

$$(\nabla f)(x, y) = \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right) = (y, x)$$



□

**Exercise 2.** Let

$$f(x, y) = \sin(\pi xy) \cdot e^{-\frac{x}{3}} \quad \text{and} \quad p = \left(1, \frac{1}{3}\right).$$

- (a) Compute  $\nabla f$  and  $(\nabla f)(p)$ . Use the special values of  $\sin$  and  $\cos$ .
- (b) Find the directions of maximum increase and decrease at  $p$ . You can give approximate values.
- (c) Give the direction of the contour line at  $p$ .
- (d) The equation of the tangent plane of the graph of  $f$  at  $(x_0, y_0)$  is

$$z = f(x_0, y_0) + (\nabla f)(x_0, y_0) \cdot (x - x_0, y - y_0).$$

Determine the equation of the tangent plane of  $z = f(x, y)$  at  $p$ . Give a normal vector of the plane.

- (e) Find the directional derivative of  $f(x, y)$  at  $p$  along the vector

$$v = \frac{1}{\sqrt{2}}(1, 1).$$

**Solution:**

(a)

$$\begin{aligned} \frac{\partial f}{\partial x} &= e^{-x/3} \left( \pi y \cos(\pi xy) - \frac{1}{3} \sin(\pi xy) \right) \\ \frac{\partial f}{\partial y} &= e^{-x/3} \pi x \cos(\pi xy) \\ \implies (\nabla f)(x, y) &= \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right) \\ &= e^{-x/3} \left( \pi y \cos(\pi xy) - \frac{1}{3} \sin(\pi xy), \pi x \cos(\pi xy) \right) \end{aligned}$$

With  $\sin(\frac{\pi}{3}) = \frac{\sqrt{3}}{2}$  and  $\cos(\frac{\pi}{3}) = \frac{1}{2}$ :

$$(\nabla f)(p) = e^{-1/3} \left( \frac{\pi - \sqrt{3}}{6}, \frac{\pi}{2} \right)$$

- (b) Maximum increase:  $(\nabla f)(p)$ , maximum decrease:  $-(\nabla f)(p)$
- (c) Contour lines are perpendicular to the gradient:

$$(x, y) \cdot (-y, x) = 0 \implies u = e^{-1/3} \left( -\frac{\pi}{2}, \frac{\pi - \sqrt{3}}{6} \right)$$

(d)

$$\begin{aligned} z = f(x, y) &= f(p) + (\nabla f)(p) \cdot (x - 1, y - \frac{1}{3}) \\ &= \frac{\sqrt{3}}{2} e^{-1/3} + e^{-1/3} \left( \frac{\pi - \sqrt{3}}{6}, \frac{\pi}{2} \right) \cdot (x - 1, y - \frac{1}{3}) \\ &= e^{-1/3} \left( \frac{\sqrt{3}}{2} + \frac{1}{6} \left( (\pi - \sqrt{3})x + 3\pi y + \sqrt{3} \right) \right) \end{aligned}$$

The surface  $z = f(x, y)$  can be written as  $F(x, y, z) = f(x, y) - z = 0$ . Therefore,

$$\nabla F = (f_x, f_y, -1)$$

is normal to the surface:

$$\begin{aligned} n &:= (\nabla F)(p) = (f_x(p), f_y(p), -1) \\ &= \left( \frac{\pi - \sqrt{3}}{6} e^{-1/3}, \frac{\pi}{2} e^{-1/3}, -1 \right) \end{aligned}$$

(e)

$$\begin{aligned} (\nabla f)(p) \cdot v &= e^{-1/3} \left( \frac{\pi - \sqrt{3}}{6}, \frac{\pi}{2} \right) \cdot \frac{1}{\sqrt{2}} (1, 1) \\ &= \frac{e^{-1/3}}{\sqrt{2}} \left( \frac{\pi - \sqrt{3}}{6} + \frac{\pi}{2} \right) \\ &= \frac{e^{-1/3}}{\sqrt{2}} \cdot \frac{4\pi - \sqrt{3}}{6} \end{aligned}$$

□

**Exercise 3.** Compute curl and divergence of the vector field

$$F(x, y, z) = (\sin(x)xy, ze^{-x}, yz).$$

**Solution:**

$$\frac{\partial F_x}{\partial y} = \frac{\partial}{\partial y}(\sin(x)xy) = x \sin(x)$$

$$\frac{\partial F_x}{\partial z} = \frac{\partial}{\partial z}(\sin(x)xy) = 0$$

$$\frac{\partial F_y}{\partial x} = \frac{\partial}{\partial x}(ze^{-x}) = -ze^{-x}$$

$$\frac{\partial F_y}{\partial z} = \frac{\partial}{\partial z}(ze^{-x}) = e^{-x}$$

$$\frac{\partial F_z}{\partial x} = \frac{\partial}{\partial x}(yz) = 0$$

$$\frac{\partial F_z}{\partial y} = \frac{\partial}{\partial y}(yz) = z$$

$$\begin{aligned}\operatorname{curl} F &= \nabla \times F = \left( \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}, \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}, \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) \\ &= (z - e^{-x}, 0 - 0, -ze^{-x} - x \sin(x)) \\ &= (z - e^{-x}, 0, -ze^{-x} - x \sin(x))\end{aligned}$$

$$\begin{aligned}\operatorname{div} F &= \nabla \cdot F = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \\ &= \frac{\partial}{\partial x}(\sin(x)xy) + \frac{\partial}{\partial y}(ze^{-x}) + \frac{\partial}{\partial z}(yz) \\ &= y \sin(x) + xy \cos(x) + 0 + y \\ &= y \sin(x) + xy \cos(x) + y\end{aligned}$$

□

**Exercise 4.** *Let*

$$f(x, y, z) = x^2yz^3 \quad \text{and} \quad F(x, y, z) = (xz, -y^2, 2x^2y).$$

*Give  $\nabla f$ ,  $\nabla^2 f$ ,  $\nabla \cdot F$ , and  $\nabla \times F$ .*

**Solution:**

$$\begin{aligned} \nabla f &= \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right) \\ &= (2xyz^3, x^2z^3, 3x^2yz^2) \end{aligned}$$

$$\begin{aligned} \nabla^2 f &= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \\ &= 2yz^3 + 0 + 6x^2yz \\ &= 2yz^3 + 6x^2yz \end{aligned}$$

$$\begin{aligned} \nabla \cdot F &= \frac{\partial}{\partial x}(xz) + \frac{\partial}{\partial y}(-y^2) + \frac{\partial}{\partial z}(2x^2y) \\ &= z - 2y \end{aligned}$$

$$\frac{\partial F_x}{\partial y} = \frac{\partial}{\partial y}(xz) = 0$$

$$\frac{\partial F_x}{\partial z} = \frac{\partial}{\partial z}(xz) = x$$

$$\frac{\partial F_y}{\partial x} = \frac{\partial}{\partial x}(-y^2) = 0$$

$$\frac{\partial F_y}{\partial z} = \frac{\partial}{\partial z}(-y^2) = 0$$

$$\frac{\partial F_z}{\partial x} = \frac{\partial}{\partial x}(2x^2y) = 4xy$$

$$\frac{\partial F_z}{\partial y} = \frac{\partial}{\partial y}(2x^2y) = 2x^2$$

$$\begin{aligned} \nabla \times F &= \left( \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}, \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}, \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) \\ &= (2x^2 - 0, x - 4xy, 0 - 0) \\ &= (2x^2, x - 4xy, 0) \end{aligned}$$

□

**Exercise 5.** Compute differential forms:

(a) Let

$$f(x, y, z) = \frac{x}{yz}$$

be a scalar function. Give the differential form  $df$ .

Hint:

$$df = F_1(x, y, z) dx + F_2(x, y, z) dy + F_3(x, y, z) dz.$$

(b) Let

$$\omega = x^2 \sin(y) dx + 2^x \cos(y) dy$$

be a differential form on  $\mathbb{R}^2$ . Give the 2-form  $d\omega$ .

Hint:

$$d\omega = \phi(x, y) dx \wedge dy.$$

(c) Let

$$\omega = x^2 \sin(y) dx + z^2 \cos(y) dy - xy^2 dz$$

be a differential form on  $\mathbb{R}^3$ . Give the 2-form  $d\omega$ .

Hint:

$$d\omega = F_1(x, y, z) dy \wedge dz + F_2(x, y, z) dz \wedge dx + F_3(x, y, z) dx \wedge dy.$$

**Solution:**

(a)

$$df = \frac{1}{yz} dx - \frac{x}{y^2 z} dy - \frac{x}{yz^2} dz$$

(b)

$$\begin{aligned} \phi(x, y) &= \frac{\partial}{\partial x}(2^x \cos(y)) - \frac{\partial}{\partial y}(x^2 \sin(y)) \\ &= \ln(2) 2^x \cos(y) - x^2 \cos(y) \\ &= \cos(y) (\ln(2) 2^x - x^2) \end{aligned}$$

So:

$$\begin{aligned} d\omega &= \phi(x, y) dx \wedge dy \\ &= (\cos(y) (\ln(2) 2^x - x^2)) dx \wedge dy \end{aligned}$$

(c)

$$\begin{aligned} F_1(x, y, z) &= \frac{\partial}{\partial y}(-xy^2) - \frac{\partial}{\partial z}(z^2 \cos y) = -2xy - 2z \cos y \\ F_2(x, y, z) &= \frac{\partial}{\partial z}(x^2 \sin y) - \frac{\partial}{\partial x}(-xy^2) = 0 + y^2 = y^2 \\ F_3(x, y, z) &= \frac{\partial}{\partial x}(z^2 \cos y) - \frac{\partial}{\partial y}(x^2 \sin y) = 0 - x^2 \cos y = -x^2 \cos y \end{aligned}$$

So:

$$d\omega = (-2xy - 2z \cos y) dy \wedge dz + y^2 dz \wedge dx - (x^2 \cos y) dx \wedge dy.$$

□

**Exercise 6.** Let  $f(x, y) = \frac{1}{\sqrt{x^2+y^2}}$ . Compute  $\nabla f$ . Then express  $f$  and  $\nabla f$  in terms of the norm  $\|v\|$ , where  $v = (x, y)$ .

**Solution:**

$$(\nabla f)(x, y) = -\frac{1}{(x^2 + y^2)^{3/2}} (x, y)$$

With  $\|v\| = \sqrt{x^2 + y^2}$ :

$$\begin{aligned} f(v) &= \frac{1}{\|v\|} \\ (\nabla f)(v) &= -\frac{v}{\|v\|^3} \end{aligned}$$

□



**Exercise 7.** Let  $F(x, y, z) = (y^2, xz, 1)$  be a vector field. The following curves are given:

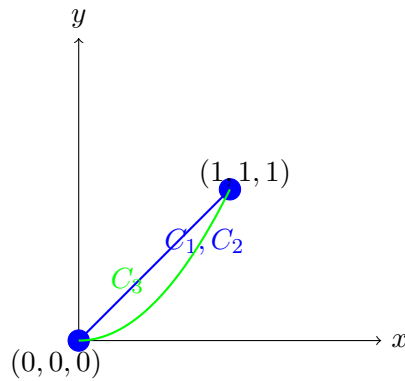
- $C_1 : r_1(t) = (t, t, t), \quad t \in [0, 1]$
- $C_2 : r_2(t) = (2t, 2t, 2t), \quad t \in [0, \frac{1}{2}]$
- $C_3 : r_3(t) = (t, t^2, t^3), \quad t \in [0, 1]$

Sketch the curves. Compute the line integrals

$$\int_{C_1} F \cdot dr, \quad \int_{C_2} F \cdot dr, \quad \int_{C_3} F \cdot dr.$$

Why do the first two line integrals coincide? Is  $F$  a conservative vector field?

**Solution:**



$C_1$ :

$$\begin{aligned} r_1'(t) &= (1, 1, 1) \\ F(r_1(t)) &= (t^2, t^2, 1) \\ F(r_1(t)) \cdot r_1'(t) &= t^2 + t^2 + 1 = 2t^2 + 1 \\ \int_{C_1} F \cdot dr &= \int_0^1 (2t^2 + 1) dt = \left[ \frac{2t^3}{3} + t \right]_0^1 = \boxed{\frac{5}{3}} \end{aligned}$$

$C_2$ :

$$\begin{aligned} r_2'(t) &= (2, 2, 2) \\ F(r_2(t)) &= (4t^2, 4t^2, 1) \\ F(r_2(t)) \cdot r_2'(t) &= 4t^2 \cdot 2 + 4t^2 \cdot 2 + 1 \cdot 2 = 16t^2 + 2 \\ \int_{C_2} F \cdot dr &= \int_0^{1/2} (16t^2 + 2) dt = \left[ \frac{16t^3}{3} + 2t \right]_0^{1/2} = \boxed{\frac{5}{3}} \end{aligned}$$

$C_3$ :

$$\begin{aligned} r_3'(t) &= (1, 2t, 3t^2) \\ F(r_3(t)) &= (t^4, t^4, 1) \\ F(r_3(t)) \cdot r_3'(t) &= t^4 + 2t^5 + 3t^2 \\ \int_{C_3} F \cdot dr &= \int_0^1 (t^4 + 2t^5 + 3t^2) dt = \left[ \frac{t^5}{5} + \frac{t^6}{3} + t^3 \right]_0^1 = \boxed{\frac{23}{15}} \end{aligned}$$

$C_1$  and  $C_2$  represent the same geometric curve despite different parametrizations. The line integral is independent of parametrization; it depends only on the geometric path and endpoints.

To show:  $F$  is not conservative

Proof:

$$F \text{ conservative} \iff \nabla \times F = 0$$

$$\begin{aligned}\nabla \times F &= \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \times \begin{pmatrix} y^2 \\ xz \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} \frac{\partial(1)}{\partial y} - \frac{\partial(xz)}{\partial z} \\ \frac{\partial(y^2)}{\partial z} - \frac{\partial(1)}{\partial x} \\ \frac{\partial(xz)}{\partial x} - \frac{\partial(y^2)}{\partial y} \end{pmatrix} \\ &= \begin{pmatrix} 0 - x \\ 0 - 0 \\ z - 2y \end{pmatrix} \neq 0\end{aligned}$$

□

**Exercise 8.** Let  $C$  be the hypocycloid (astroid) given by

$$r(t) = (\cos^3(t), \sin^3(t)).$$

Find the length of  $C$ . Hint: Compute the length of  $C$  in the first quadrant ( $t \in [0, \frac{\pi}{2}]$ ) and multiply the result by 4.

**Solution:**

$$\begin{aligned} r'(t) &= (-3\cos^2(t)\sin(t), 3\sin^2(t)\cos(t)) \\ &= 3\sin(t)\cos(t)(-\cos(t), \sin(t)) \\ \implies \|r'(t)\| &= 3\sin(t)\cos(t)\sqrt{\cos^2(t) + \sin^2(t)} \\ &= 3\sin(t)\cos(t) \end{aligned}$$

$$\begin{aligned} l &= \int_C ds = 4 \int_0^{\frac{\pi}{2}} \|r'(t)\| dt \\ &= 12 \int_0^{\frac{\pi}{2}} \sin(t)\cos(t) dt \end{aligned}$$

Let  $u = \sin(t) \implies du = \cos(t) dt$ :

$$\begin{aligned} l &= 12 \int_0^1 u du \\ &= 12 \left[ \frac{1}{2} u^2 \right]_0^1 \\ &= 6 \left[ \sin^2(t) \right]_0^{\frac{\pi}{2}} \\ &= 6 \end{aligned}$$

□

**Exercise 9.** Compute the line integrals of the first and the second kind:

(a) Let  $C_1$  be the semi-circle given by  $r_1(t) = (3 \cos(t), 3 \sin(t))$ ,  $t \in [0, \pi]$ .

$$\int_{C_1} x^2 y \, ds$$

(b) Let  $C_2$  be given by the parametrization  $r_2(t) = (4t, 3t^2)$ ,  $t \in [0, 1]$ .

$$\int_{C_2} (x^2 y) \, dx - (x - yx) \, dy$$

**Solution:**

(a)

$$\begin{aligned} r_1'(t) &= 3(-\sin t, \cos t) \\ \|r_1'(t)\| &= 3\sqrt{\sin^2 t + \cos^2 t} = 3 \\ \implies ds &= 3 \, dt \\ \int_{C_1} x^2 y \, ds &= \int_0^\pi 27 \cos^2(t) \sin(t) \cdot 3 \, dt \\ &= 81 \int_0^\pi \cos^2(t) \sin(t) \, dt \end{aligned}$$

Let  $u = \cos(t)$ , then  $du = -\sin(t) \, dt$ .  $t = 0 \implies u = 1$ ;  $t = \pi \implies u = -1$ :

$$\begin{aligned} &= 81 \int_1^{-1} u^2 (-du) \\ &= 81 \int_{-1}^1 u^2 \, du \\ &= 81 \left[ \frac{u^3}{3} \right]_{-1}^1 \\ &= 81 \left( \frac{1}{3} - \frac{-1}{3} \right) = \boxed{54} \end{aligned}$$

(b)

$$\begin{aligned} r_2'(t) &= (4, 6t) \\ P(x, y) &= x^2 y, \quad Q(x, y) = -(x - yx) = yx - x \\ P(x(t), y(t)) &= (4t)^2 \cdot 3t^2 = 48t^4 \\ Q(x(t), y(t)) &= 3t^2 \cdot 4t - 4t = 12t^3 - 4t \end{aligned}$$

With  $dx = 4 \, dt$  and  $dy = 6t \, dt$ :

$$\begin{aligned} \int_{C_2} P \, dx + Q \, dy &= \int_0^1 (48t^4 \cdot 4 + (12t^3 - 4t) \cdot 6t) \, dt \\ &= \int_0^1 (264t^4 - 24t^2) \, dt \\ &= \left[ \frac{264t^5}{5} - \frac{24t^3}{3} \right]_0^1 \\ &= \frac{264}{5} - 8 = \boxed{\frac{224}{5}} \end{aligned}$$

□

**Exercise 10.** *Let*

$$\omega = x^2 \exp(xyz) \, dy \wedge dz$$

*be a 2-form on  $\mathbb{R}^3$ . Give the 3-form  $d\omega$ .*

*Hint:  $d\omega = f(x, y, z) \, dx \wedge dy \wedge dz$ , where  $f$  is a scalar function.*

**Solution:** Any smooth 2-form in  $\mathbb{R}^3$  can be written as:

$$\begin{aligned}\omega &= A(x, y, z) \, dy \wedge dz + B(x, y, z) \, dz \wedge dx + C(x, y, z) \, dx \wedge dy \\ &= x^2 e^{xyz} \, dy \wedge dz\end{aligned}$$

$$\implies A(x, y, z) = x^2 e^{xyz} \text{ and } B = C = 0$$

$$\begin{aligned}A_x(x, y, z) &= 2xe^{xyz} + x^2 yze^{xyz} \\ &= xe^{xyz}(2 + xyz)\end{aligned}$$

So:

$$d\omega = (xe^{xyz}(2 + xyz)) \, dx \wedge dy \wedge dz$$

□

**Exercise 11.** Let  $F$  be the vector field  $F(x, y) = (x^3 + xy^2, x^2y - y^5)$ .

- (a) Let  $C$  be the square curve with corner points  $(0, 0), (1, 0), (1, 1), (0, 1)$  and anti-clockwise orientation. Determine  $\int_C F \cdot dr$ .
- (b) Show that  $F$  has a potential  $\phi$ . How can you explain the result of part (a)?
- (c) Compute the line integral of  $F$  for a curve from  $(0, 0)$  to  $(1, 2)$ .

**Solution:**

- (a) 1.  $C_1$ : from  $(0, 0)$  to  $(1, 0)$  (bottom edge):

$$C_1 : r(t) = (x(t), y(t)) = (t, 0), \quad 0 \leq t \leq 1$$

$$dx = x'(t)dt = dt, \quad dy = y'(t)dt = 0, \quad y = 0:$$

$$\begin{aligned} \int_{C_1} F \cdot dr &= \int_0^1 P dx + Q dy = \int_0^1 P dx \\ &= \int_0^1 (t^3 + t \cdot 0^2) dt \\ &= \int_0^1 t^3 dt = \frac{1}{4} \end{aligned}$$

2.  $C_2$ : from  $(1, 0)$  to  $(1, 1)$  (right edge)

$$C_2 : r(t) = (x(t), y(t)) = (1, t), \quad 0 \leq t \leq 1$$

$$dx = 0, \quad dy = dt, \quad x = 1:$$

$$\begin{aligned} \int_{C_2} F \cdot dr &= \int_0^1 P dx + Q dy = \int_0^1 Q dy \\ &= \int_0^1 (t - t^5) dt \\ &= \left[ \frac{1}{2}t^2 - \frac{1}{6}t^6 \right]_0^1 = \frac{1}{3} \end{aligned}$$

3.  $C_3$ : from  $(1, 1)$  to  $(0, 1)$  (top edge):

$$C_3 : r(t) = (x(t), y(t)) = (1 - t, 1), \quad 0 \leq t \leq 1$$

$$dx = -dt, \quad dy = 0, \quad y = 1:$$

$$\begin{aligned} \int_{C_3} F \cdot dr &= \int_0^1 P dx + Q dy = \int_0^1 P dx \\ &= \int_0^1 ((1 - t)^3 + (1 - t)) (-dt) \\ &= - \int_0^1 (1 - t)^3 dt - \int_0^1 (1 - t) dt \\ &= - \left[ -\frac{1}{4}(1 - t)^4 \right]_0^1 - \left[ t - \frac{1}{2}t^2 \right]_0^1 \\ &= -\frac{1}{4} - \frac{1}{2} = -\frac{3}{4} \end{aligned}$$

4.  $C_4$ : from  $(0, 1)$  to  $(0, 0)$  (left edge):

$$C_4 : r(t) = (x(t), y(t)) = (0, 1 - t), \quad 0 \leq t \leq 1$$

$$dx = 0, \quad dy = -dt, \quad x = 0:$$

$$\begin{aligned} \int_{C_4} F \cdot dr &= \int_0^1 P dx + Q dy = \int_0^1 Q dy \\ &= \int_0^1 (-(1-t)^5) (-dt) \\ &= \left[ -\frac{1}{6}(1-t)^6 \right]_0^1 = \frac{1}{6} \end{aligned}$$

$$\int_C F \cdot dr = \frac{1}{4} + \frac{1}{3} - \frac{3}{4} + \frac{1}{6} = 0$$

(b)

$$\begin{aligned} \phi_1(x, y) &= \frac{1}{4}x^4 + \frac{1}{2}x^2y^2 + h_1(y) \\ \phi_2(x, y) &= \frac{1}{2}x^2y^2 - \frac{1}{6}y^6 + h_2(x) \\ \implies \phi(x, y) &= \frac{1}{4}x^4 + \frac{1}{2}x^2y^2 - \frac{1}{6}y^6 \end{aligned}$$

Or use *Theorem 1.4.15* with *Remark 1.4.16*:

$$\begin{aligned} \frac{\partial P}{\partial y} &= \frac{\partial}{\partial y}(x^3 + xy^2) = 2xy \\ \frac{\partial Q}{\partial x} &= \frac{\partial}{\partial x}(x^2y + y^5) = 2xy \end{aligned}$$

$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$  and  $\mathbb{R}^2$  is simply connected  $\wedge$   $F$  is continuously differentiable  
 $\implies F$  is conservative and  $\int_C F \cdot dr = 0$ , which explains the result of part (a).

(c)

$$\phi(x, y) = \frac{1}{4}x^4 + \frac{1}{2}x^2y^2 - \frac{1}{6}y^6$$

Let  $\gamma$  be any path from  $(0, 0)$  to  $(1, 2)$ . Since  $F = \nabla\phi$ , we have:

$$\int_{\gamma} F \cdot dr = \phi(1, 2) - \phi(0, 0)$$

with

$$\begin{aligned} \phi(1, 2) &= \frac{1}{4}1^4 + \frac{1}{2}1^22^2 - \frac{1}{6}2^6 = -\frac{101}{12} \\ \phi(0, 0) &= 0 + 0 - 0 = 0 \end{aligned}$$

Therefore:

$$\int_{(0,0)}^{(1,2)} F \cdot dr = -\frac{101}{12}$$

□

**Exercise 12.** Let  $r = (x, y, z)$  and suppose  $F$  is the vector field

$$F(r) = -\frac{c}{\|r\|^3}r$$

where  $c$  is a real constant.

- (a) Show that  $\phi(r) = \frac{c}{\|r\|}$  is a potential of  $F$ .
- (b) Show that  $\nabla^2\phi = 0$  for  $r \neq 0$ .
- (c) Let  $C$  be a curve from  $p$  to  $q$ . Determine  $\int_C F(r) dr$ .
- (d) What can be said about  $\text{curl } F$ ?

Hint: Write  $\phi$  as a function in  $x, y, z$  using  $\|r\| = \sqrt{x^2 + y^2 + z^2}$ .

**Solution:**

- (a) To prove:  $F = \nabla\phi$

$$(\nabla\phi)(r) = \left( \frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z} \right) = \left( -\frac{c}{\|r\|^3}x, -\frac{c}{\|r\|^3}y, -\frac{c}{\|r\|^3}z \right) = -\frac{c}{\|r\|^3}r$$

- (b) Let  $r \neq 0$ :

$$\begin{aligned} (\nabla^2\phi)(r) &= \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} \\ &= -\frac{c}{\|r\|^3} \left( \left(1 - \frac{3x^2}{\|r\|^2}\right) + \left(1 - \frac{3y^2}{\|r\|^2}\right) + \left(1 - \frac{3z^2}{\|r\|^2}\right) \right) \\ &= -\frac{c}{\|r\|^3} \left( 3 - \frac{3(x^2 + y^2 + z^2)}{\|r\|^2} \right) \\ &= -\frac{c}{\|r\|^3} (3 - 3) = 0 \end{aligned}$$

- (c) Since  $\phi$  is a potential of  $F$  with  $\nabla\phi = F$ , *Proposition 1.4.12* holds:

$$\int_C F(r) dr = \phi(q) - \phi(p) = \frac{c}{\|q\|} - \frac{c}{\|p\|}$$

- (d) With *Theorem 1.4.15*, since  $F$  has a potential,  $\text{curl } F = 0$ .

□



**Exercise 13.** Find the work done by the force field

$$F(x, y, z) = \left( -\frac{1}{2}x, -\frac{1}{2}y, \frac{1}{4} \right)$$

on a particle as it moves along the spiral given by

$$r(t) = (\cos(t), \sin(t), t), \quad t \in [0, 3\pi].$$

**Solution:**

$$\begin{aligned} W &= \int_C F \, dr \\ &= \int_0^{3\pi} F(r(t)) \cdot r'(t) \, dt \\ &= \int_0^{3\pi} \left( -\frac{1}{2} \cos(t), -\frac{1}{2} \sin(t), \frac{1}{4} \right) \cdot (-\sin(t), \cos(t), 1) \, dt \\ &= \int_0^{3\pi} \left( \frac{1}{2} \sin(t) \cos(t) - \frac{1}{2} \sin(t) \cos(t) + \frac{1}{4} \right) \, dt \\ &= \int_0^{3\pi} \frac{1}{4} \, dt \\ &= \left[ \frac{1}{4} t \right]_0^{3\pi} \\ &= \frac{3}{4} \pi \end{aligned}$$

□

**Exercise 14.** Let  $F$  be the vector field given by

$$F(x, y) = (y^3 + 1, 3xy^2 + 1)$$

and let  $C$  be the semicircular path from  $(0, 0)$  to  $(2, 0)$ .

(a) Express  $\int_C F dr$  as an integral in one variable.

(b) Find a potential of  $F$  by integration. Then use the potential to compute  $\int_C F dr$ .

**Solution:**

(a) Let  $r(t)$  go from  $(2, 0)$  to  $(0, 0)$  as  $t$  goes from  $0$  to  $\pi$ :

$$\begin{aligned} C : r(t) &= (x(t), y(t)), \quad t \in [0, \pi] \\ &= (1 + \cos t, \sin t) \end{aligned}$$

$$\begin{aligned} \int_C F dr &= - \int_0^\pi F(r(t)) \cdot r'(t) dt \\ &= - \int_0^\pi (1 + \sin^3 t, 1 + 3(1 + \cos t) \sin^2 t) \cdot (-\sin t, \cos t) dt \\ &= - \int_0^\pi (-\sin t - \sin^4 t + \cos t + 3 \sin^2 t \cos t + 3 \sin^2 t \cos^2 t) dt \end{aligned}$$

(b) Let  $r(t) = (tx, ty)$ ,  $t \in [0, 1]$ :

$$\begin{aligned} \phi(x, y) &= \int_0^1 F(r(t)) \cdot r'(t) dt \\ &= \int_0^1 (t^3 y^3 + 1, 3t^3 xy^2 + 1) \cdot (x, y) dt \\ &= \int_0^1 (t^3 y^3 x + x + 3t^3 xy^3 + y) dt \\ &= \left[ \frac{1}{4} t^4 (y^3 x + 3xy^3) + t(x + y) \right]_0^1 \\ &= xy^3 + x + y \end{aligned}$$

$$\nabla(xy^3 + x + y) = (y^3 + 1, 3xy^2 + 1) = F(x, y) \implies \phi \text{ is a potential of } F$$

□

**Exercise 15.** Check if  $F(x, y, z) = (y^2z, 2xyz, xy^2)$  is conservative.

**Solution:** To prove:  $\nabla \times F = 0$

$$\begin{aligned}\nabla \times F &= \left( \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \\ &= (2xy - 2xy, y^2 - y^2, 2yz - 2yz) \\ &= 0\end{aligned}$$

□

**Exercise 16.** Let  $D \subset \mathbb{R}^2$  be the triangle area given by  $0 \leq x \leq y$  and  $0 \leq y \leq 1$ . Evaluate the following double integral:

$$\int_D \int y^2 e^{-xy} dx dy$$

**Solution:**

$$\begin{aligned} \int_D \int y^2 e^{-xy} dx dy &= \int_0^1 \left( \int_0^y y^2 e^{-xy} dx \right) dy \\ &= \int_0^1 \left( y^2 \left[ -\frac{1}{y} e^{-xy} \right]_{x=0}^{x=y} \right) dy \\ &= \int_0^1 \left( -y e^{-y^2} + y \right) dy \\ &= \left[ \frac{1}{2} e^{-y^2} + \frac{1}{2} y^2 \right]_0^1 \\ &= \frac{1}{2} \left( e^{-1} - 1 + (1 - 0) \right) \\ &= \frac{1}{2e} \end{aligned}$$

□

**Exercise 17.** Compute the integral

$$\int_D \int (x^2 + y^2)^{10} dx dy$$

over the upper circular area

$$D = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1 \wedge y \geq 0\}.$$

**Solution:** The corresponding domain with respect to the polar coordinates  $r$  and  $\phi$  is

$$D^* = [0, 1] \times [0, \pi]$$

$$\begin{aligned} \int_D \int (x^2 + y^2)^{10} dx dy &= \int_0^1 \int_0^\pi (r^2 \cos^2 \phi + r^2 \sin^2 \phi)^{10} r d\phi dr \\ &= \int_0^1 \int_0^\pi (r^2 (\cos^2 \phi + \sin^2 \phi))^{10} r d\phi dr \\ &= \int_0^1 \int_0^\pi r^{21} d\phi dr \\ &= \int_0^1 \left( r^{21} [\phi]_{\phi=0}^{\phi=\pi} \right) dr \\ &= \pi \left[ \frac{1}{22} r^{22} \right]_{r=0}^{r=1} = \frac{\pi}{22} \end{aligned}$$

□

**Exercise 18.** Give the volume of a cylinder around the  $x$ -axis with radius 2, beginning at  $x = 1$  and ending at  $x = 5$ .

**Solution:** Transform  $(x, y, z)$  to cylindrical coordinates  $(r, \phi, z)$ :

$$\begin{cases} x = r \cos \phi \\ y = r \sin \phi \\ z = z \end{cases}$$

$$J = \frac{\partial(x, y, z)}{\partial(r, \phi, z)} = \begin{bmatrix} \cos \phi & -r \sin \phi & 0 \\ \sin \phi & r \cos \phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\det(J) = \begin{vmatrix} \cos \phi & -r \sin \phi \\ \sin \phi & r \cos \phi \end{vmatrix} \cdot 1 = r$$

$$\implies dx \wedge dy \wedge dz = r dr \wedge d\phi \wedge dz$$

Move the cylinder in space so that its axis aligns with the  $z$ -axis from  $z = 0$  to  $z = 4$ :

$$\begin{aligned} V &= \int_0^2 \int_0^{2\pi} \int_0^4 r dz d\phi dr \\ &= 4 \int_0^2 \int_0^{2\pi} r d\phi dr \\ &= 8\pi \int_0^2 r dr \\ &= 8\pi \left[ \frac{1}{2} r^2 \right]_0^2 \\ &= 16\pi \end{aligned}$$

□

**Exercise 19.** Compute the volume between the plane defined by

$$z = 2x + 3y$$

and the triangle

$$D = \{(x, y) \mid x \in [1, 2] \wedge y \in [0, x]\}$$

in the  $xy$ -plane.

**Solution:**

$$\begin{aligned} V &= \iint_D z \, dA \\ &= \int_1^2 \int_0^x (2x + 3y) \, dy \, dx \\ &= \int_1^2 \left[ 2xy + \frac{3}{2}y^2 \right]_{y=0}^{y=x} dx \\ &= \int_1^2 (2x^2 + \frac{3}{2}x^2) \, dx \\ &= \left[ \frac{2}{3}x^3 + \frac{1}{2}x^3 \right]_1^2 \\ &= \left[ \frac{7}{6}x^3 \right]_1^2 \\ &= \frac{56}{6} - \frac{7}{6} \\ &= \frac{49}{6} \end{aligned}$$

□

**Exercise 20.** Determine the volume  $V$  of the portion of a ball of radius 3 that is defined by polar angles  $\theta \in [0, \frac{\pi}{6}]$  and arbitrary azimuth angles  $\varphi$ .

Remark:  $V$  is called a spherical sector or a spherical cone.

**Solution:**    TODO

□