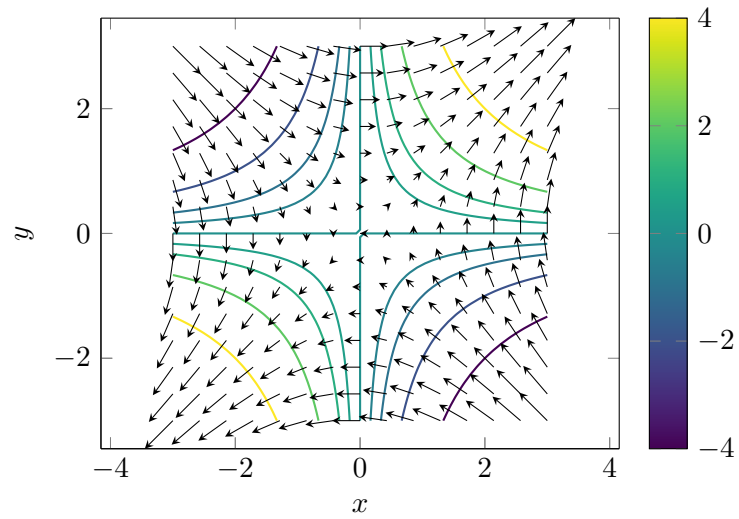# Advanced Mathematics Exercises

Exercise 1. *Let f be the scalar function defined by $f(x, y) = xy$. Sketch the contour lines and the vector field ∇f.*

Solution:

$$(\nabla f)(x, y) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right) = (y, x)$$

□

Exercise 2. *Let*
$$f(x, y) = \sin(\pi xy) \cdot e^{-\frac{x}{3}} \quad and \quad p = \left(1, \frac{1}{3}\right).$$

(a) Compute ∇f and $(\nabla f)(p)$. Use the special values of $\sin$ and $\cos$.

(b) Find the directions of maximum increase and decrease at p. You can give approximate values.

(c) Give the direction of the contour line at p.

(d) The equation of the tangent plane of the graph of f at (x_0, y_0) is
$$z = f(x_0, y_0) + (\nabla f)(x_0, y_0) \cdot (x - x_0, y - y_0).$$

Determine the equation of the tangent plane of $z = f(x, y)$ at p. Give a normal vector of the plane.

(e) Find the directional derivative of $f(x, y)$ at p along the vector
$$v = \frac{1}{\sqrt{2}}(1, 1).$$

Solution:

(a)
$$\frac{\partial f}{\partial x} = e^{-x/3} \left(\pi y \cos(\pi xy) - \frac{1}{3} \sin(\pi xy) \right)$$
$$\frac{\partial f}{\partial y} = e^{-x/3} \pi x \cos(\pi xy)$$
$$\implies (\nabla f)(x, y) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right)$$
$$= e^{-x/3} \left(\pi y \cos(\pi xy) - \frac{1}{3} \sin(\pi xy), \; \pi x \cos(\pi xy) \right)$$

With $\sin(\frac{\pi}{3}) = \frac{\sqrt{3}}{2}$ and $\cos(\frac{\pi}{3}) = \frac{1}{2}$:
$$(\nabla f)(p) = e^{-1/3} \left(\frac{\pi - \sqrt{3}}{6}, \; \frac{\pi}{2} \right)$$

(b) Maximum increase: $(\nabla f)(p)$, maximum decrease: $-(\nabla f)(p)$

(c) Contour lines are perpendicular to the gradient:
$$(x, \; y) \cdot (-y, \; x) = 0 \implies u = e^{-1/3} \left(-\frac{\pi}{2}, \; \frac{\pi - \sqrt{3}}{6} \right)$$

(d)
$$z = f(x, y) = f(p) + (\nabla f)(p) \cdot (x - 1, y - \frac{1}{3})$$
$$= \frac{\sqrt{3}}{2} e^{-1/3} + e^{-1/3} \left(\frac{\pi - \sqrt{3}}{6}, \; \frac{\pi}{2} \right) \cdot (x - 1, \; y - \frac{1}{3})$$
$$= e^{-1/3} \left(\frac{\sqrt{3}}{2} + \frac{1}{6} \left((\pi - \sqrt{3})x + 3\pi y + \sqrt{3} \right) \right)$$

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The surface $z = f(x, y)$ can be written as $F(x, y, z) = f(x, y) - z = 0$. Therefore,

$$\nabla F = (f_x, \ f_y, \ -1)$$

is normal to the surface:

$$n := (\nabla F)(p) = (f_x(p), \ f_y(p), \ -1)$$
$$= \left(\frac{\pi - \sqrt{3}}{6} e^{-1/3}, \ \frac{\pi}{2} e^{-1/3}, \ -1 \right)$$

(e)

$$(\nabla f)(p) \cdot v = e^{-1/3} \left(\frac{\pi - \sqrt{3}}{6}, \ \frac{\pi}{2} \right) \cdot \frac{1}{\sqrt{2}} (1, 1)$$
$$= \frac{e^{-1/3}}{\sqrt{2}} \left(\frac{\pi - \sqrt{3}}{6} + \frac{\pi}{2} \right)$$
$$= \frac{e^{-1/3}}{\sqrt{2}} \cdot \frac{4\pi - \sqrt{3}}{6}$$

$\square$

Exercise 3. *Compute curl and divergence of the vector field*

$$F(x, y, z) = (\sin(x)xy,\ ze^{-x},\ yz).$$

Solution:

$$\frac{\partial F_x}{\partial y} = \frac{\partial}{\partial y}(\sin(x)xy) = x\sin(x)$$

$$\frac{\partial F_x}{\partial z} = \frac{\partial}{\partial z}(\sin(x)xy) = 0$$

$$\frac{\partial F_y}{\partial x} = \frac{\partial}{\partial x}(ze^{-x}) = -ze^{-x}$$

$$\frac{\partial F_y}{\partial z} = \frac{\partial}{\partial z}(ze^{-x}) = e^{-x}$$

$$\frac{\partial F_z}{\partial x} = \frac{\partial}{\partial x}(yz) = 0$$

$$\frac{\partial F_z}{\partial y} = \frac{\partial}{\partial y}(yz) = z$$

$$\text{curl } F = \nabla \times F = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z},\ \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x},\ \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right)$$

$$= (z - e^{-x},\ 0 - 0,\ -ze^{-x} - x\sin(x))$$

$$= (z - e^{-x},\ 0,\ -ze^{-x} - x\sin(x))$$

$$\text{div } F = \nabla \cdot F = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

$$= \frac{\partial}{\partial x}\left(\sin(x)xy \right) + \frac{\partial}{\partial y}(ze^{-x}) + \frac{\partial}{\partial z}(yz)$$

$$= y\sin(x) + xy\cos(x) + 0 + y$$

$$= y\sin(x) + xy\cos(x) + y$$

$\square$

Exercise 4. *Let*

$$f(x, y, z) = x^2yz^3 \quad and \quad F(x, y, z) = (xz, -y^2, 2x^2y).$$

Give ∇f, $\nabla^2 f$, $\nabla \cdot F$, *and* $\nabla \times F$.

Solution:

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right)$$
$$= \left(2xyz^3, \ x^2z^3, \ 3x^2yz^2 \right)$$

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$
$$= 2yz^3 + 0 + 6x^2yz$$
$$= 2yz^3 + 6x^2yz$$

$$\nabla \cdot F = \frac{\partial}{\partial x}(xz) + \frac{\partial}{\partial y}(-y^2) + \frac{\partial}{\partial z}(2x^2y)$$
$$= z - 2y$$

$$\frac{\partial F_x}{\partial y} = \frac{\partial}{\partial y}(xz) = 0$$
$$\frac{\partial F_x}{\partial z} = \frac{\partial}{\partial z}(xz) = x$$
$$\frac{\partial F_y}{\partial x} = \frac{\partial}{\partial x}(-y^2) = 0$$
$$\frac{\partial F_y}{\partial z} = \frac{\partial}{\partial z}(-y^2) = 0$$
$$\frac{\partial F_z}{\partial x} = \frac{\partial}{\partial x}(2x^2y) = 4xy$$
$$\frac{\partial F_z}{\partial y} = \frac{\partial}{\partial y}(2x^2y) = 2x^2$$

$$\nabla \times F = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}, \ \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}, \ \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right)$$
$$= \left(2x^2 - 0, \ x - 4xy, \ 0 - 0 \right)$$
$$= \left(2x^2, \ x - 4xy, \ 0 \right)$$

$\square$

Exercise 5. *Compute differential forms:*

(a) Let
$$f(x, y, z) = \frac{x}{yz}$$
be a scalar function. Give the differential form df.
Hint:
$$df = F_1(x, y, z)\, dx + F_2(x, y, z)\, dy + F_3(x, y, z)\, dz.$$

(b) Let
$$\omega = x^2 \sin(y)\, dx + 2^x \cos(y)\, dy$$
be a differential form on $\mathbb{R}^2$. Give the 2-form $d\omega$.
Hint:
$$d\omega = \phi(x, y)\, dx \wedge dy.$$

(c) Let
$$\omega = x^2 \sin(y)\, dx + z^2 \cos(y)\, dy - xy^2\, dz$$
be a differential form on $\mathbb{R}^3$. Give the 2-form $d\omega$.
Hint:
$$d\omega = F_1(x, y, z)\, dy \wedge dz + F_2(x, y, z)\, dz \wedge dx + F_3(x, y, z)\, dx \wedge dy.$$

Solution:

(a)
$$df = \frac{1}{yz}\, dx - \frac{x}{y^2 z}\, dy - \frac{x}{yz^2}\, dz$$

(b)
$$\begin{aligned}
\phi(x, y) &= \frac{\partial}{\partial x}(2^x \cos(y)) - \frac{\partial}{\partial y}(x^2 \sin(y)) \\
&= \ln(2) 2^x \cos(y) - x^2 \cos(y) \\
&= \cos(y)\left(\ln(2) 2^x - x^2\right)
\end{aligned}$$

So:
$$\begin{aligned}
d\omega &= \phi(x, y)\, dx \wedge dy \\
&= \left(\cos(y)\left(\ln(2) 2^x - x^2\right)\right) dx \wedge dy
\end{aligned}$$

(c)
$$\begin{aligned}
F_1(x, y, z) &= \frac{\partial}{\partial y}(-xy^2) - \frac{\partial}{\partial z}(z^2 \cos y) = -2xy - 2z \cos y \\
F_2(x, y, z) &= \frac{\partial}{\partial z}(x^2 \sin y) - \frac{\partial}{\partial x}(-xy^2) = 0 + y^2 = y^2 \\
F_3(x, y, z) &= \frac{\partial}{\partial x}(z^2 \cos y) - \frac{\partial}{\partial y}(x^2 \sin y) = 0 - x^2 \cos y = -x^2 \cos y
\end{aligned}$$

So:
$$d\omega = (-2xy - 2z \cos y)\, dy \wedge dz + y^2\, dz \wedge dx - (x^2 \cos y)\, dx \wedge dy.$$

$\square$

Exercise 6. *Let* $f(x, y) = \frac{1}{\sqrt{x^2+y^2}}$. *Compute* ∇f. *Then express* f *and* ∇f *in terms of the norm* $\|v\|$, *where* $v = (x, y)$.

Solution:

$$(\nabla f)(x, y) = -\frac{1}{(x^2 + y^2)^{3/2}} (x, \ y)$$

With $\|v\| = \sqrt{x^2 + y^2}$:

$$f(v) = \frac{1}{\|v\|}$$

$$(\nabla f)(v) = -\frac{v}{\|v\|^3}$$

$\square$

Exercise 7. Let $F(x, y, z) = (y^2, xz, 1)$ be a vector field. The following curves are given:
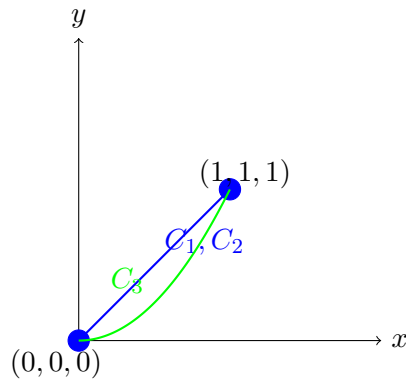
- $C_1 : r_1(t) = (t, t, t), \quad t \in [0, 1]$

- $C_2 : r_2(t) = (2t, 2t, 2t), \quad t \in [0, \frac{1}{2}]$

- $C_3 : r_3(t) = (t, t^2, t^3), \quad t \in [0, 1]$

Sketch the curves. Compute the line integrals

$$\int_{C_1} F \cdot dr, \quad \int_{C_2} F \cdot dr, \quad \int_{C_3} F \cdot dr.$$

Why do the first two line integrals coincide? Is F a conservative vector field?

Solution:

C_1:

$$r_1'(t) = (1, 1, 1)$$
$$F(r_1(t)) = (t^2, t^2, 1)$$
$$F(r_1(t)) \cdot r_1'(t) = t^2 + t^2 + 1 = 2t^2 + 1$$
$$\int_{C_1} F \cdot dr = \int_0^1 (2t^2 + 1)\, dt = \left[\frac{2t^3}{3} + t \right]_0^1 = \boxed{\frac{5}{3}}$$

C_2:

$$r_2'(t) = (2, 2, 2)$$
$$F(r_2(t)) = (4t^2, 4t^2, 1)$$
$$F(r_2(t)) \cdot r_2'(t) = 4t^2 \cdot 2 + 4t^2 \cdot 2 + 1 \cdot 2 = 16t^2 + 2$$
$$\int_{C_2} F \cdot dr = \int_0^{1/2} (16t^2 + 2)\, dt = \left[\frac{16t^3}{3} + 2t \right]_0^{1/2} = \boxed{\frac{5}{3}}$$

C_3:

$$r_3'(t) = (1, 2t, 3t^2)$$
$$F(r_3(t)) = (t^4, t^4, 1)$$
$$F(r_3(t)) \cdot r_3'(t) = t^4 + 2t^5 + 3t^2$$
$$\int_{C_3} F \cdot dr = \int_0^1 (t^4 + 2t^5 + 3t^2)\, dt = \left[\frac{t^5}{5} + \frac{t^6}{3} + t^3 \right]_0^1 = \boxed{\frac{23}{15}}$$

C_1 and C_2 represent the same geometric curve despite different parametrizations. The line integral is independent of parametrization; it depends only on the geometric path and endpoints.

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To show: F is not conservative
Proof:

$$F \text{ conservative} \iff \nabla \times F = 0$$

$$\nabla \times F = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \times \begin{pmatrix} y^2 \\ xz \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\partial(1)}{\partial y} - \frac{\partial(xz)}{\partial z} \\ \frac{\partial(y^2)}{\partial z} - \frac{\partial(1)}{\partial x} \\ \frac{\partial(xz)}{\partial x} - \frac{\partial(y^2)}{\partial y} \end{pmatrix}$$

$$= \begin{pmatrix} 0 - x \\ 0 - 0 \\ z - 2y \end{pmatrix} \neq 0$$

$\square$

Exercise 8. *Let C be the hypocycloid (astroid) given by*

$$r(t) = (\cos^3(t), \sin^3(t)).$$

Find the length of C. Hint: *Compute the length of C in the first quadrant ($t \in [0, \frac{\pi}{2}]$) and multiply the result by 4.*

Solution:

$$r'(t) = \left(-3\cos^2(t)\sin(t),\ 3\sin^2(t)\cos(t)\right)$$
$$= 3\sin(t)\cos(t)\left(-\cos(t),\ \sin(t)\right)$$
$$\implies \|r'(t)\| = 3\sin(t)\cos(t)\sqrt{\cos^2(t) + \sin^2(t)}$$
$$= 3\sin(t)\cos(t)$$

$$l = \int_C ds = 4\int_0^{\frac{\pi}{2}} \|r'(t)\|\ dt$$
$$= 12\int_0^{\frac{\pi}{2}} \sin(t)\cos(t)\ dt$$

Let $u = \sin(t) \implies du = \cos(t)\ dt$:

$$l = 12\int_0^1 u\ du$$
$$= 12\left[\frac{1}{2}u^2\right]_0^1$$
$$= 6\left[\sin^2(t)\right]_0^{\frac{\pi}{2}}$$
$$= 6$$

$\square$

Exercise 9. *Compute the line integrals of the first and the second kind:*

(a) Let C_1 be the semi-circle given by $r_1(t) = (3\cos(t), 3\sin(t))$, $\quad t \in [0, \pi]$.

$$\int_{C_1} x^2 y \, ds$$

(b) Let C_2 be given by the parametrization $r_2(t) = (4t, 3t^2)$, $\quad t \in [0, 1]$.

$$\int_{C_2} (x^2 y) \, dx - (x - yx) \, dy$$

Solution:

(a)

$$r_1'(t) = 3(-\sin t, \cos t)$$
$$\|r_1'(t)\| = 3\sqrt{\sin^2 t + \cos^2 t} = 3$$
$$\implies ds = 3 \, dt$$
$$\int_{C_1} x^2 y \, ds = \int_0^\pi 27 \cos^2(t) \sin(t) \cdot 3 \, dt$$
$$= 81 \int_0^\pi \cos^2(t) \sin(t) \, dt$$

Let $u = \cos(t)$, then $du = -\sin(t) \, dt$. $t = 0 \implies u = 1$; $t = \pi \implies u = -1$:

$$= 81 \int_1^{-1} u^2(-du)$$
$$= 81 \int_{-1}^1 u^2 \, du$$
$$= 81 \left[\frac{u^3}{3} \right]_{-1}^1$$
$$= 81 \left(\frac{1}{3} - \frac{-1}{3} \right) = \boxed{54}$$

(b)

$$r_2'(t) = (4, 6t)$$
$$P(x,y) = x^2 y, \quad Q(x,y) = -(x - yx) = yx - x$$
$$P(x(t), y(t)) = (4t)^2 \cdot 3t^2 = 48t^4$$
$$Q(x(t), y(t)) = 3t^2 \cdot 4t - 4t = 12t^3 - 4t$$

With $dx = 4 \, dt$ and $dy = 6t \, dt$:

$$\int_{C_2} P \, dx + Q \, dy = \int_0^1 \left(48t^4 \cdot 4 + (12t^3 - 4t) \cdot 6t \right) dt$$
$$= \int_0^1 \left(264t^4 - 24t^2 \right) dt$$
$$= \left[\frac{264t^5}{5} - \frac{24t^3}{3} \right]_0^1$$
$$= \frac{264}{5} - 8 = \boxed{\frac{224}{5}}$$

$\square$

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Exercise 10. *Let*

$$\omega = x^2 \exp(xyz)\, dy \wedge dz$$

be a 2-form on $\mathbb{R}^3$. Give the 3-form $d\omega$.

    Hint: $d\omega = f(x, y, z)\, dx \wedge dy \wedge dz$, where f is a scalar function.

Solution: Any smooth 2-form in $\mathbb{R}^3$ can be written as:

$$\begin{aligned}
\omega &= A(x, y, z)\, dy \wedge dz + B(x, y, z)\, dz \wedge dx + C(x, y, z)\, dx \wedge dy \\
&= x^2 e^{xyz}\, dy \wedge dz
\end{aligned}$$

$$\begin{aligned}
\implies\ A(x, y, z) &= x^2 e^{xyz} \text{ and } B = C = 0 \\
A_x(x, y, z) &= 2xe^{xyz} + x^2 yz e^{xyz} \\
&= xe^{xyz}(2 + xyz)
\end{aligned}$$

So:

$$d\omega = \left(xe^{xyz}(2 + xyz)\right)\, dx \wedge dy \wedge dz$$

$\square$

Exercise 11. *Let F be the vector field $F(x, y) = (x^3 + xy^2, x^2y - y^5)$.*

(a) *Let C be the square curve with corner points $(0, 0), (1, 0), (1, 1), (0, 1)$ and anti-clockwise orientation. Determine $\int_C F \cdot dr$.*

(b) *Show that F has a potential ϕ. How can you explain the result of part (a)?*

(c) *Compute the line integral of F for a curve from $(0, 0)$ to $(1, 2)$.*

Solution:

(a)
$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}(x^3 + xy^2) = 2xy$$
$$\frac{\partial Q}{\partial x} = \frac{\partial}{\partial}(x^2y + y^5) = 2xy$$

$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \wedge \mathbb{R}^2$ is simply connected $\wedge F$ is continuously differentiable
$\implies F$ is conservative and $\int_C F \cdot dr = 0$.

(b)
$$F \text{ conservative} \overset{\text{Theorem 1.4.15}}{\implies} F \text{ has a potential } \phi$$

(c) We need to find ϕ such that:
$$\frac{\partial \phi}{\partial x} = P = x^3 + xy^2, \quad \frac{\partial \phi}{\partial y} = Q = x^2y - y^5.$$

Integrate P with respect to x:
$$\phi(x, y) = \int (x^3 + xy^2) \, dx = \frac{1}{4}x^4 + \frac{1}{2}x^2y^2 + h(y),$$

where $h(y)$ is an unknown function of y only.

$$\frac{\partial \phi}{\partial y} = \frac{\partial}{\partial y}\left(\frac{1}{4}x^4 + \frac{1}{2}x^2y^2 + h(y)\right) = x^2y + h'(y).$$

With $\frac{\partial \phi}{\partial y} = Q = x^2y - y^5$:
$$x^2y + h'(y) = x^2y - y^5 \implies h'(y) = -y^5$$
$$h(y) = \int -y^5 \, dy = -\frac{1}{6}y^6 + C$$

WLOG, take $C = 0$:
$$\phi(x, y) = \frac{1}{4}x^4 + \frac{1}{2}x^2y^2 - \frac{1}{6}y^6$$

Let $\gamma = $ any path from $(0, 0)$ to $(1, 2)$. Since $F = \nabla \phi$, we have:
$$\int_\gamma F \cdot dr = \phi(1, 2) - \phi(0, 0)$$

with
$$\phi(1, 2) = \frac{1}{4}1^4 + \frac{1}{2}1^2 2^2 - \frac{1}{6}2^6 = -\frac{101}{12}$$
$$\phi(0, 0) = 0 + 0 - 0 = 0$$

Therefore:
$$\int_{(0,0)}^{(1,2)} F \cdot dr = -\frac{101}{12}$$

$\square$

Exercise 12. *Let* $r = (x, y, z)$ *and suppose* F *is the vector field*

$$F(r) = -\frac{c}{\|r\|^3} r$$

where c *is a real constant.*

(a) *Show that* $\phi(r) = \frac{c}{\|r\|}$ *is a potential of* F.

(b) *Show that* $\nabla^2 \phi = 0$ *for* $r \neq 0$.

(c) *Let* C *be a curve from* p *to* q. *Determine* $\int_C F(r) \, dr$.

(d) *What can be said about* $\operatorname{curl} F$?

   Hint: *Write* ϕ *as a function in* x, y, z *using* $\|r\| = \sqrt{x^2 + y^2 + z^2}$.

Solution: TODO   □