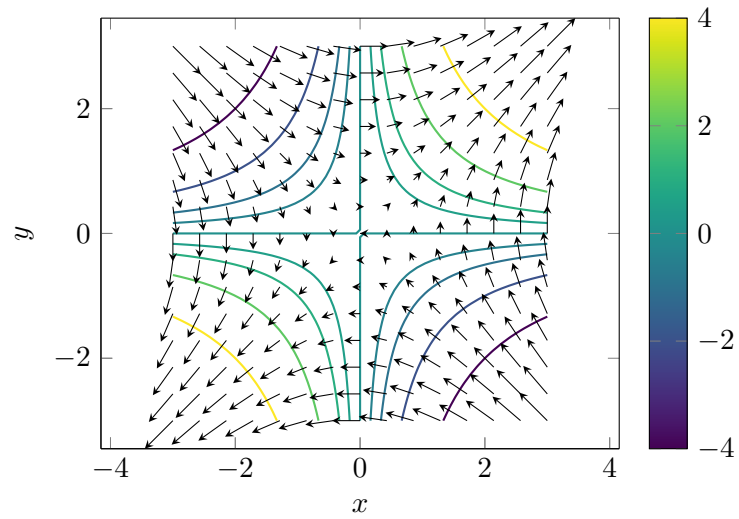


Advanced Mathematics Exercises

Exercise 1. Let f be the scalar function defined by $f(x, y) = xy$. Sketch the contour lines and the vector field ∇f .

Solution:

$$(\nabla f)(x, y) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right) = (y, x)$$



□

Exercise 2. Let

$$f(x, y) = \sin(\pi xy) \cdot e^{-\frac{x}{3}} \quad \text{and} \quad p = \left(1, \frac{1}{3}\right).$$

- (a) Compute ∇f and $(\nabla f)(p)$. Use the special values of \sin and \cos .
- (b) Find the directions of maximum increase and decrease at p . You can give approximate values.
- (c) Give the direction of the contour line at p .
- (d) The equation of the tangent plane of the graph of f at (x_0, y_0) is

$$z = f(x_0, y_0) + (\nabla f)(x_0, y_0) \cdot (x - x_0, y - y_0).$$

Determine the equation of the tangent plane of $z = f(x, y)$ at p . Give a normal vector of the plane.

- (e) Find the directional derivative of $f(x, y)$ at p along the vector

$$v = \frac{1}{\sqrt{2}}(1, 1).$$

Solution:

(a)

$$\begin{aligned} \frac{\partial f}{\partial x} &= e^{-x/3} \left(\pi y \cos(\pi xy) - \frac{1}{3} \sin(\pi xy) \right) \\ \frac{\partial f}{\partial y} &= e^{-x/3} \pi x \cos(\pi xy) \\ \implies (\nabla f)(x, y) &= \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right) \\ &= e^{-x/3} \left(\pi y \cos(\pi xy) - \frac{1}{3} \sin(\pi xy), \pi x \cos(\pi xy) \right) \end{aligned}$$

With $\sin(\frac{\pi}{3}) = \frac{\sqrt{3}}{2}$ and $\cos(\frac{\pi}{3}) = \frac{1}{2}$:

$$(\nabla f)(p) = e^{-1/3} \left(\frac{\pi - \sqrt{3}}{6}, \frac{\pi}{2} \right)$$

- (b) Maximum increase: $(\nabla f)(p)$, maximum decrease: $-(\nabla f)(p)$
- (c) Contour lines are perpendicular to the gradient:

$$(x, y) \cdot (-y, x) = 0 \implies u = e^{-1/3} \left(-\frac{\pi}{2}, \frac{\pi - \sqrt{3}}{6} \right)$$

(d)

$$\begin{aligned} z = f(x, y) &= f(p) + (\nabla f)(p) \cdot (x - 1, y - \frac{1}{3}) \\ &= \frac{\sqrt{3}}{2} e^{-1/3} + e^{-1/3} \left(\frac{\pi - \sqrt{3}}{6}, \frac{\pi}{2} \right) \cdot (x - 1, y - \frac{1}{3}) \\ &= e^{-1/3} \left(\frac{\sqrt{3}}{2} + \frac{1}{6} \left((\pi - \sqrt{3})x + 3\pi y + \sqrt{3} \right) \right) \end{aligned}$$

The surface $z = f(x, y)$ can be written as $F(x, y, z) = f(x, y) - z = 0$. Therefore,

$$\nabla F = (f_x, f_y, -1)$$

is normal to the surface:

$$\begin{aligned} n &:= (\nabla F)(p) = (f_x(p), f_y(p), -1) \\ &= \left(\frac{\pi - \sqrt{3}}{6} e^{-1/3}, \frac{\pi}{2} e^{-1/3}, -1 \right) \end{aligned}$$

(e)

$$\begin{aligned} (\nabla f)(p) \cdot v &= e^{-1/3} \left(\frac{\pi - \sqrt{3}}{6}, \frac{\pi}{2} \right) \cdot \frac{1}{\sqrt{2}} (1, 1) \\ &= \frac{e^{-1/3}}{\sqrt{2}} \left(\frac{\pi - \sqrt{3}}{6} + \frac{\pi}{2} \right) \\ &= \frac{e^{-1/3}}{\sqrt{2}} \cdot \frac{4\pi - \sqrt{3}}{6} \end{aligned}$$

□

Exercise 3. Compute curl and divergence of the vector field

$$F(x, y, z) = (\sin(x)xy, ze^{-x}, yz).$$

Solution:

$$\frac{\partial F_x}{\partial y} = \frac{\partial}{\partial y}(\sin(x)xy) = x \sin(x)$$

$$\frac{\partial F_x}{\partial z} = \frac{\partial}{\partial z}(\sin(x)xy) = 0$$

$$\frac{\partial F_y}{\partial x} = \frac{\partial}{\partial x}(ze^{-x}) = -ze^{-x}$$

$$\frac{\partial F_y}{\partial z} = \frac{\partial}{\partial z}(ze^{-x}) = e^{-x}$$

$$\frac{\partial F_z}{\partial x} = \frac{\partial}{\partial x}(yz) = 0$$

$$\frac{\partial F_z}{\partial y} = \frac{\partial}{\partial y}(yz) = z$$

$$\begin{aligned}\operatorname{curl} F &= \nabla \times F = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}, \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}, \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) \\ &= (z - e^{-x}, 0 - 0, -ze^{-x} - x \sin(x)) \\ &= (z - e^{-x}, 0, -ze^{-x} - x \sin(x))\end{aligned}$$

$$\begin{aligned}\operatorname{div} F &= \nabla \cdot F = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \\ &= \frac{\partial}{\partial x}(\sin(x)xy) + \frac{\partial}{\partial y}(ze^{-x}) + \frac{\partial}{\partial z}(yz) \\ &= y \sin(x) + xy \cos(x) + 0 + y \\ &= y \sin(x) + xy \cos(x) + y\end{aligned}$$

□

Exercise 4. *Let*

$$f(x, y, z) = x^2yz^3 \quad \text{and} \quad F(x, y, z) = (xz, -y^2, 2x^2y).$$

Give ∇f , $\nabla^2 f$, $\nabla \cdot F$, and $\nabla \times F$.

Solution:

$$\begin{aligned}\nabla f &= \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right) \\ &= (2xyz^3, x^2z^3, 3x^2yz^2)\end{aligned}$$

$$\begin{aligned}\nabla^2 f &= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \\ &= 2yz^3 + 0 + 6x^2yz \\ &= 2yz^3 + 6x^2yz\end{aligned}$$

$$\begin{aligned}\nabla \cdot F &= \frac{\partial}{\partial x}(xz) + \frac{\partial}{\partial y}(-y^2) + \frac{\partial}{\partial z}(2x^2y) \\ &= z - 2y\end{aligned}$$

$$\frac{\partial F_x}{\partial y} = \frac{\partial}{\partial y}(xz) = 0$$

$$\frac{\partial F_x}{\partial z} = \frac{\partial}{\partial z}(xz) = x$$

$$\frac{\partial F_y}{\partial x} = \frac{\partial}{\partial x}(-y^2) = 0$$

$$\frac{\partial F_y}{\partial z} = \frac{\partial}{\partial z}(-y^2) = 0$$

$$\frac{\partial F_z}{\partial x} = \frac{\partial}{\partial x}(2x^2y) = 4xy$$

$$\frac{\partial F_z}{\partial y} = \frac{\partial}{\partial y}(2x^2y) = 2x^2$$

$$\begin{aligned}\nabla \times F &= \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}, \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}, \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) \\ &= (2x^2 - 0, x - 4xy, 0 - 0) \\ &= (2x^2, x - 4xy, 0)\end{aligned}$$

□

Exercise 5. Compute differential forms:

(a) Let

$$f(x, y, z) = \frac{x}{yz}$$

be a scalar function. Give the differential form df .

Hint:

$$df = F_1(x, y, z) dx + F_2(x, y, z) dy + F_3(x, y, z) dz.$$

(b) Let

$$\omega = x^2 \sin(y) dx + 2^x \cos(y) dy$$

be a differential form on \mathbb{R}^2 . Give the 2-form $d\omega$.

Hint:

$$d\omega = \phi(x, y) dx \wedge dy.$$

(c) Let

$$\omega = x^2 \sin(y) dx + z^2 \cos(y) dy - xy^2 dz$$

be a differential form on \mathbb{R}^3 . Give the 2-form $d\omega$.

Hint:

$$d\omega = F_1(x, y, z) dy \wedge dz + F_2(x, y, z) dz \wedge dx + F_3(x, y, z) dx \wedge dy.$$

Solution:

(a)

$$df = \frac{1}{yz} dx - \frac{x}{y^2 z} dy - \frac{x}{yz^2} dz$$

(b)

$$\begin{aligned} \phi(x, y) &= \frac{\partial}{\partial x}(2^x \cos(y)) - \frac{\partial}{\partial y}(x^2 \sin(y)) \\ &= \ln(2) 2^x \cos(y) - x^2 \cos(y) \\ &= \cos(y) (\ln(2) 2^x - x^2) \end{aligned}$$

So:

$$\begin{aligned} d\omega &= \phi(x, y) dx \wedge dy \\ &= (\cos(y) (\ln(2) 2^x - x^2)) dx \wedge dy \end{aligned}$$

(c)

$$\begin{aligned} F_1(x, y, z) &= \frac{\partial}{\partial y}(-xy^2) - \frac{\partial}{\partial z}(z^2 \cos y) = -2xy - 2z \cos y \\ F_2(x, y, z) &= \frac{\partial}{\partial z}(x^2 \sin y) - \frac{\partial}{\partial x}(-xy^2) = 0 + y^2 = y^2 \\ F_3(x, y, z) &= \frac{\partial}{\partial x}(z^2 \cos y) - \frac{\partial}{\partial y}(x^2 \sin y) = 0 - x^2 \cos y = -x^2 \cos y \end{aligned}$$

So:

$$d\omega = (-2xy - 2z \cos y) dy \wedge dz + y^2 dz \wedge dx - (x^2 \cos y) dx \wedge dy.$$

□

Exercise 6. Let $f(x, y) = \frac{1}{\sqrt{x^2+y^2}}$. Compute ∇f . Then express f and ∇f in terms of the norm $\|v\|$, where $v = (x, y)$.

Solution:

$$(\nabla f)(x, y) = -\frac{1}{(x^2 + y^2)^{3/2}} (x, y)$$

With $\|v\| = \sqrt{x^2 + y^2}$:

$$\begin{aligned} f(v) &= \frac{1}{\|v\|} \\ (\nabla f)(v) &= -\frac{v}{\|v\|^3} \end{aligned}$$

□

Exercise 7. Let $F(x, y, z) = (y^2, xz, 1)$ be a vector field. The following curves are given:

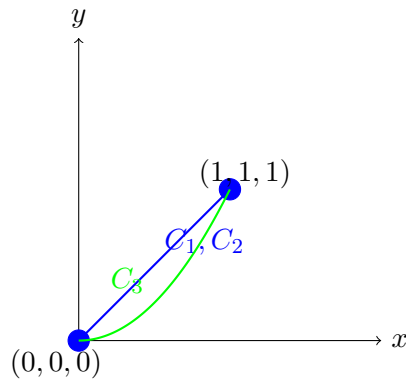
- $C_1 : r_1(t) = (t, t, t), \quad t \in [0, 1]$
- $C_2 : r_2(t) = (2t, 2t, 2t), \quad t \in [0, \frac{1}{2}]$
- $C_3 : r_3(t) = (t, t^2, t^3), \quad t \in [0, 1]$

Sketch the curves. Compute the line integrals

$$\int_{C_1} F \cdot dr, \quad \int_{C_2} F \cdot dr, \quad \int_{C_3} F \cdot dr.$$

Why do the first two line integrals coincide? Is F a conservative vector field?

Solution:



C_1 :

$$\begin{aligned} r_1'(t) &= (1, 1, 1) \\ F(r_1(t)) &= (t^2, t^2, 1) \\ F(r_1(t)) \cdot r_1'(t) &= t^2 + t^2 + 1 = 2t^2 + 1 \\ \int_{C_1} F \cdot dr &= \int_0^1 (2t^2 + 1) dt = \left[\frac{2t^3}{3} + t \right]_0^1 = \boxed{\frac{5}{3}} \end{aligned}$$

C_2 :

$$\begin{aligned} r_2'(t) &= (2, 2, 2) \\ F(r_2(t)) &= (4t^2, 4t^2, 1) \\ F(r_2(t)) \cdot r_2'(t) &= 4t^2 \cdot 2 + 4t^2 \cdot 2 + 1 \cdot 2 = 16t^2 + 2 \\ \int_{C_2} F \cdot dr &= \int_0^{1/2} (16t^2 + 2) dt = \left[\frac{16t^3}{3} + 2t \right]_0^{1/2} = \boxed{\frac{5}{3}} \end{aligned}$$

C_3 :

$$\begin{aligned} r_3'(t) &= (1, 2t, 3t^2) \\ F(r_3(t)) &= (t^4, t^4, 1) \\ F(r_3(t)) \cdot r_3'(t) &= t^4 + 2t^5 + 3t^2 \\ \int_{C_3} F \cdot dr &= \int_0^1 (t^4 + 2t^5 + 3t^2) dt = \left[\frac{t^5}{5} + \frac{t^6}{3} + t^3 \right]_0^1 = \boxed{\frac{23}{15}} \end{aligned}$$

C_1 and C_2 represent the same geometric curve despite different parametrizations. The line integral is independent of parametrization; it depends only on the geometric path and endpoints.

To show: F is not conservative

Proof:

$$F \text{ conservative} \iff \nabla \times F = 0$$

$$\begin{aligned}\nabla \times F &= \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \times \begin{pmatrix} y^2 \\ xz \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} \frac{\partial(1)}{\partial y} - \frac{\partial(xz)}{\partial z} \\ \frac{\partial(y^2)}{\partial z} - \frac{\partial(1)}{\partial x} \\ \frac{\partial(xz)}{\partial x} - \frac{\partial(y^2)}{\partial y} \end{pmatrix} \\ &= \begin{pmatrix} 0 - x \\ 0 - 0 \\ z - 2y \end{pmatrix} \neq 0\end{aligned}$$

□

Exercise 8. Let C be the hypocycloid (astroid) given by

$$r(t) = (\cos^3(t), \sin^3(t)).$$

Find the length of C . Hint: Compute the length of C in the first quadrant ($t \in [0, \frac{\pi}{2}]$) and multiply the result by 4.

Solution:

$$\begin{aligned} r'(t) &= (-3\cos^2(t)\sin(t), 3\sin^2(t)\cos(t)) \\ &= 3\sin(t)\cos(t)(-\cos(t), \sin(t)) \\ \implies \|r'(t)\| &= 3\sin(t)\cos(t)\sqrt{\cos^2(t) + \sin^2(t)} \\ &= 3\sin(t)\cos(t) \end{aligned}$$

$$\begin{aligned} l &= \int_C ds = 4 \int_0^{\frac{\pi}{2}} \|r'(t)\| dt \\ &= 12 \int_0^{\frac{\pi}{2}} \sin(t)\cos(t) dt \end{aligned}$$

Let $u = \sin(t) \implies du = \cos(t) dt$:

$$\begin{aligned} l &= 12 \int_0^1 u du \\ &= 12 \left[\frac{1}{2} u^2 \right]_0^1 \\ &= 6 \left[\sin^2(t) \right]_0^{\frac{\pi}{2}} \\ &= 6 \end{aligned}$$

□

Exercise 9. Compute the line integrals of the first and the second kind:

(a) Let C_1 be the semi-circle given by $r_1(t) = (3 \cos(t), 3 \sin(t))$, $t \in [0, \pi]$.

$$\int_{C_1} x^2 y \, ds$$

(b) Let C_2 be given by the parametrization $r_2(t) = (4t, 3t^2)$, $t \in [0, 1]$.

$$\int_{C_2} (x^2 y) \, dx - (x - yx) \, dy$$

Solution:

(a)

$$\begin{aligned} r_1'(t) &= 3(-\sin t, \cos t) \\ \|r_1'(t)\| &= 3\sqrt{\sin^2 t + \cos^2 t} = 3 \\ \implies ds &= 3 \, dt \\ \int_{C_1} x^2 y \, ds &= \int_0^\pi 27 \cos^2(t) \sin(t) \cdot 3 \, dt \\ &= 81 \int_0^\pi \cos^2(t) \sin(t) \, dt \end{aligned}$$

Let $u = \cos(t)$, then $du = -\sin(t) \, dt$. $t = 0 \implies u = 1$; $t = \pi \implies u = -1$:

$$\begin{aligned} &= 81 \int_1^{-1} u^2 (-du) \\ &= 81 \int_{-1}^1 u^2 \, du \\ &= 81 \left[\frac{u^3}{3} \right]_{-1}^1 \\ &= 81 \left(\frac{1}{3} - \frac{-1}{3} \right) = \boxed{54} \end{aligned}$$

(b)

$$\begin{aligned} r_2'(t) &= (4, 6t) \\ P(x, y) &= x^2 y, \quad Q(x, y) = -(x - yx) = yx - x \\ P(x(t), y(t)) &= (4t)^2 \cdot 3t^2 = 48t^4 \\ Q(x(t), y(t)) &= 3t^2 \cdot 4t - 4t = 12t^3 - 4t \end{aligned}$$

With $dx = 4 \, dt$ and $dy = 6t \, dt$:

$$\begin{aligned} \int_{C_2} P \, dx + Q \, dy &= \int_0^1 (48t^4 \cdot 4 + (12t^3 - 4t) \cdot 6t) \, dt \\ &= \int_0^1 (264t^4 - 24t^2) \, dt \\ &= \left[\frac{264t^5}{5} - \frac{24t^3}{3} \right]_0^1 \\ &= \frac{264}{5} - 8 = \boxed{\frac{224}{5}} \end{aligned}$$

□

Exercise 10. *Let*

$$\omega = x^2 \exp(xyz) \, dy \wedge dz$$

be a 2-form on \mathbb{R}^3 . Give the 3-form $d\omega$.

Hint: $d\omega = f(x, y, z) \, dx \wedge dy \wedge dz$, where f is a scalar function.

Solution: TODO

□

Exercise 11. Let F be the vector field $F(x, y) = (x^3 + xy^2, x^2y - y^5)$.

- (a) Let C be the square curve with corner points $(0, 0), (1, 0), (1, 1), (0, 1)$ and anti-clockwise orientation. Determine $\int_C F \cdot dr$.
- (b) Show that F has a potential ϕ . How can you explain the result of part (a)?
- (c) Compute the line integral of F for a curve from $(0, 0)$ to $(1, 2)$.

Solution: TODO

□

Exercise 12. Let $r = (x, y, z)$ and suppose F is the vector field

$$F(r) = -\frac{c}{\|r\|^3}r$$

where c is a real constant.

- (a) Show that $\phi(r) = \frac{c}{\|r\|}$ is a potential of F .
- (b) Show that $\nabla^2\phi = 0$ for $r \neq 0$.
- (c) Let C be a curve from p to q . Determine $\int_C F(r) dr$.
- (d) What can be said about $\text{curl } F$?

Hint: Write ϕ as a function in x, y, z using $\|r\| = \sqrt{x^2 + y^2 + z^2}$.

Solution: TODO

□