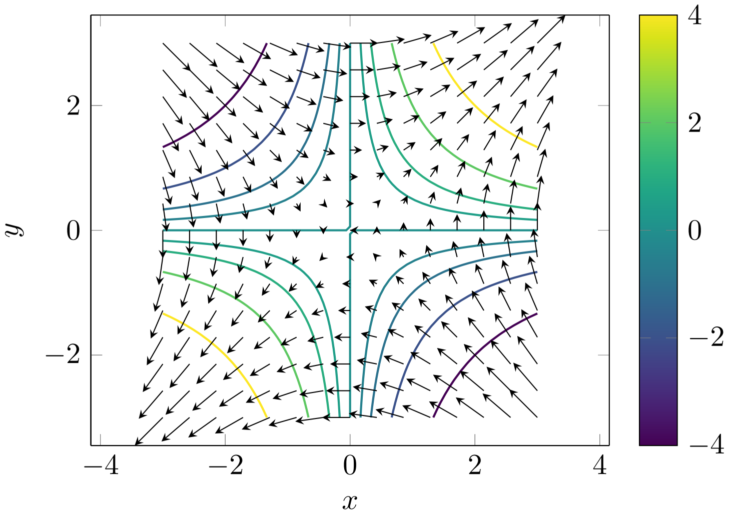
Advanced Mathematics Exercises

Exercise 1. *Let f be the scalar function defined by $f(x,y) = xy$. Sketch the contour lines and the vector field ∇f.*

Solution:

$$(\nabla f)(x,y) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = (y, x)$$



□

Exercise 2. *Let*

$$f(x,y) = \sin(\pi xy) \cdot e^{-\frac{x}{3}} \quad \text{and} \quad p = \left(1, \frac{1}{3}\right).$$

(a) Compute ∇f *and* $(\nabla f)(p)$*. Use the special values of* sin *and* cos.

(b) Find the directions of maximum increase and decrease at p*. You can give approximate values.*

(c) Give the direction of the contour line at p*.*

(d) The equation of the tangent plane of the graph of f *at* (x_0, y_0) *is*

$$z = f(x_0, y_0) + (\nabla f)(x_0, y_0) \cdot (x - x_0, y - y_0).$$

Determine the equation of the tangent plane of $z = f(x,y)$ *at* p*. Give a normal vector of the plane.*

(e) Find the directional derivative of $f(x,y)$ *at* p *along the vector*

$$v = \frac{1}{\sqrt{2}}(1,1).$$

Solution:

(a)

$$\begin{aligned}
\frac{\partial f}{\partial x} &= e^{-x/3}\left(\pi y\cos(\pi xy) - \frac{1}{3}\sin(\pi xy)\right)\\
\frac{\partial f}{\partial y} &= e^{-x/3}\pi x\cos(\pi xy)\\
\implies (\nabla f)(x,y) &= \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)\\
&= e^{-x/3}\left(\pi y\cos(\pi xy) - \frac{1}{3}\sin(\pi xy),\ \pi x\cos(\pi xy)\right)
\end{aligned}$$

With $\sin(\frac{\pi}{3}) = \frac{\sqrt{3}}{2}$ and $\cos(\frac{\pi}{3}) = \frac{1}{2}$:

$$(\nabla f)(p) = e^{-1/3}\left(\frac{\pi - \sqrt{3}}{6}, \frac{\pi}{2}\right)$$

(b) Maximum increase: $(\nabla f)(p)$, maximum decrease: $-(\nabla f)(p)$

(c) Contour lines are perpendicular to the gradient:

$$(x,\ y)\cdot(-y,\ x) = 0 \implies u = e^{-1/3}\left(-\frac{\pi}{2}, \frac{\pi - \sqrt{3}}{6}\right)$$

(d)

$$\begin{aligned}
z = f(x,y) &= f(p) + (\nabla f)(p)\cdot(x-1, y-\frac{1}{3})\\
&= \frac{\sqrt{3}}{2}e^{-1/3} + e^{-1/3}\left(\frac{\pi-\sqrt{3}}{6}, \frac{\pi}{2}\right)\cdot(x-1,\ y-\frac{1}{3})\\
&= e^{-1/3}\left(\frac{\sqrt{3}}{2} + \frac{1}{6}\left((\pi-\sqrt{3})x + 3\pi y + \sqrt{3}\right)\right)
\end{aligned}$$

The surface $z = f(x, y)$ can be written as $F(x, y, z) = f(x, y) - z = 0$. Therefore,

$$\nabla F = (f_x,\ f_y,\ -1)$$

is normal to the surface:

$$\begin{aligned} n := (\nabla F)(p) &= (f_x(p),\ f_y(p),\ -1) \\ &= \left(\frac{\pi - \sqrt{3}}{6} e^{-1/3},\ \frac{\pi}{2} e^{-1/3},\ -1 \right) \end{aligned}$$

(e)

$$\begin{aligned} (\nabla f)(p) \cdot v &= e^{-1/3} \left(\frac{\pi - \sqrt{3}}{6},\ \frac{\pi}{2} \right) \cdot \frac{1}{\sqrt{2}}(1, 1) \\ &= \frac{e^{-1/3}}{\sqrt{2}} \left(\frac{\pi - \sqrt{3}}{6} + \frac{\pi}{2} \right) \\ &= \frac{e^{-1/3}}{\sqrt{2}} \cdot \frac{4\pi - \sqrt{3}}{6} \end{aligned}$$

$\square$

Exercise 3. *Compute curl and divergence of the vector field*

$$F(x, y, z) = (\sin(x)xy,\ ze^{-x},\ yz).$$

Solution:

$$\begin{aligned}
\frac{\partial F_x}{\partial y} &= \frac{\partial}{\partial y}(\sin(x)xy) = x\sin(x)\\
\frac{\partial F_x}{\partial z} &= \frac{\partial}{\partial z}(\sin(x)xy) = 0\\
\frac{\partial F_y}{\partial x} &= \frac{\partial}{\partial x}(ze^{-x}) = -ze^{-x}\\
\frac{\partial F_y}{\partial z} &= \frac{\partial}{\partial z}(ze^{-x}) = e^{-x}\\
\frac{\partial F_z}{\partial x} &= \frac{\partial}{\partial x}(yz) = 0\\
\frac{\partial F_z}{\partial y} &= \frac{\partial}{\partial y}(yz) = z
\end{aligned}$$

$$\begin{aligned}
\operatorname{curl} F = \nabla \times F &= \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z},\ \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x},\ \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\\
&= \left(z - e^{-x},\ 0 - 0,\ -ze^{-x} - x\sin(x)\right)\\
&= \left(z - e^{-x},\ 0,\ -ze^{-x} - x\sin(x)\right)
\end{aligned}$$

$$\begin{aligned}
\operatorname{div} F = \nabla \cdot F &= \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}\\
&= \frac{\partial}{\partial x}\big(\sin(x)xy\big) + \frac{\partial}{\partial y}(ze^{-x}) + \frac{\partial}{\partial z}(yz)\\
&= y\sin(x) + xy\cos(x) + 0 + y\\
&= y\sin(x) + xy\cos(x) + y
\end{aligned}$$

$\square$

Exercise 4. *Let*

$$f(x,y,z) = x^2yz^3 \quad \textit{and} \quad F(x,y,z) = (xz, -y^2, 2x^2y).$$

Give ∇f*,* $\nabla^2 f$*,* $\nabla \cdot F$*, and* $\nabla \times F$*.*

Solution:

$$\begin{aligned} \nabla f &= \left(\frac{\partial f}{\partial x},\ \frac{\partial f}{\partial y},\ \frac{\partial f}{\partial z}\right) \\ &= \left(2xyz^3,\ x^2z^3,\ 3x^2yz^2\right) \end{aligned}$$

$$\begin{aligned} \nabla^2 f &= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \\ &= 2yz^3 + 0 + 6x^2yz \\ &= 2yz^3 + 6x^2yz \end{aligned}$$

$$\begin{aligned} \nabla \cdot F &= \frac{\partial}{\partial x}(xz) + \frac{\partial}{\partial y}(-y^2) + \frac{\partial}{\partial z}(2x^2y) \\ &= z - 2y \end{aligned}$$

$$\begin{aligned} \frac{\partial F_x}{\partial y} &= \frac{\partial}{\partial y}(xz) = 0 \\ \frac{\partial F_x}{\partial z} &= \frac{\partial}{\partial z}(xz) = x \\ \frac{\partial F_y}{\partial x} &= \frac{\partial}{\partial x}(-y^2) = 0 \\ \frac{\partial F_y}{\partial z} &= \frac{\partial}{\partial z}(-y^2) = 0 \\ \frac{\partial F_z}{\partial x} &= \frac{\partial}{\partial x}(2x^2y) = 4xy \\ \frac{\partial F_z}{\partial y} &= \frac{\partial}{\partial y}(2x^2y) = 2x^2 \end{aligned}$$

$$\begin{aligned} \nabla \times F &= \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z},\ \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x},\ \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right) \\ &= \left(2x^2 - 0,\ x - 4xy,\ 0 - 0\right) \\ &= \left(2x^2,\ x - 4xy,\ 0\right) \end{aligned}$$

□

Exercise 5. *Compute differential forms:*

(a) Let

$$f(x,y,z) = \frac{x}{yz}$$

be a scalar function. Give the differential form df.
Hint:

$$df = F_1(x,y,z)\,dx + F_2(x,y,z)\,dy + F_3(x,y,z)\,dz.$$

(b) Let

$$\omega = x^2 \sin(y)\,dx + 2^x \cos(y)\,dy$$

be a differential form on $\mathbb{R}^2$. Give the 2-form $d\omega$.
Hint:

$$d\omega = \phi(x,y)\,dx \wedge dy.$$

(c) Let

$$\omega = x^2 \sin(y)\,dx + z^2 \cos(y)\,dy - xy^2\,dz$$

be a differential form on $\mathbb{R}^3$. Give the 2-form $d\omega$.
Hint:

$$d\omega = F_1(x,y,z)\,dy \wedge dz + F_2(x,y,z)\,dz \wedge dx + F_3(x,y,z)\,dx \wedge dy.$$

Solution:

(a)

$$df = \frac{1}{yz}\,dx - \frac{x}{y^2 z}\,dy - \frac{x}{yz^2}\,dz$$

(b)

$$\begin{aligned}\phi(x,y) &= \frac{\partial}{\partial x}(2^x \cos(y)) - \frac{\partial}{\partial y}(x^2 \sin(y))\\ &= \ln(2)2^x \cos(y) - x^2 \cos(y)\\ &= \cos(y)\left(\ln(2)2^x - x^2\right)\end{aligned}$$

So:

$$\begin{aligned}d\omega &= \phi(x,y)\,dx \wedge dy\\ &= \left(\cos(y)\left(\ln(2)2^x - x^2\right)\right)\,dx \wedge dy\end{aligned}$$

(c)

$$\begin{aligned}F_1(x,y,z) &= \frac{\partial}{\partial y}(-xy^2) - \frac{\partial}{\partial z}(z^2 \cos y) = -2xy - 2z\cos y\\ F_2(x,y,z) &= \frac{\partial}{\partial z}(x^2 \sin y) - \frac{\partial}{\partial x}(-xy^2) = 0 + y^2 = y^2\\ F_3(x,y,z) &= \frac{\partial}{\partial x}(z^2 \cos y) - \frac{\partial}{\partial y}(x^2 \sin y) = 0 - x^2\cos y = -x^2 \cos y\end{aligned}$$

So:

$$d\omega = (-2xy - 2z\cos y)\,dy \wedge dz + y^2\,dz \wedge dx - (x^2 \cos y)\,dx \wedge dy.$$

$\square$

Exercise 6. *Let* $f(x,y) = \frac{1}{\sqrt{x^2+y^2}}$*. Compute* ∇f*. Then express* f *and* ∇f *in terms of the norm* $\|v\|$*, where* $v = (x,y)$*.*

Solution:

$$(\nabla f)(x,y) = -\frac{1}{(x^2+y^2)^{3/2}}\,(x,\ y)$$

With $\|v\| = \sqrt{x^2+y^2}$:

$$f(v) = \frac{1}{\|v\|}$$

$$(\nabla f)(v) = -\frac{v}{\|v\|^3}$$

□

Exercise 7. *Let $F(x, y, z) = (y^2, xz, 1)$ be a vector field. The following curves are given:*

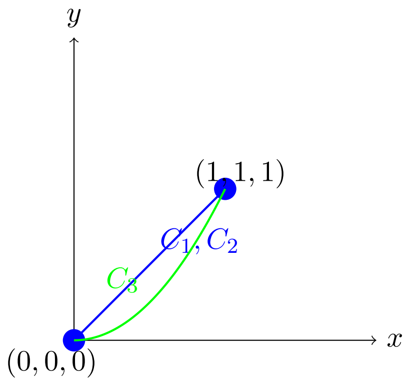
- $C_1 : r_1(t) = (t, t, t), \quad t \in [0, 1]$
- $C_2 : r_2(t) = (2t, 2t, 2t), \quad t \in [0, \frac{1}{2}]$
- $C_3 : r_3(t) = (t, t^2, t^3), \quad t \in [0, 1]$

Sketch the curves. Compute the line integrals

$$\int_{C_1} F \cdot dr, \quad \int_{C_2} F \cdot dr, \quad \int_{C_3} F \cdot dr.$$

Why do the first two line integrals coincide? Is F a conservative vector field?

Solution:



C_1:

$$\begin{aligned} r_1'(t) &= (1, 1, 1) \\ F(r_1(t)) &= (t^2, t^2, 1) \\ F(r_1(t)) \cdot r_1'(t) &= t^2 + t^2 + 1 = 2t^2 + 1 \\ \int_{C_1} F \cdot dr &= \int_0^1 (2t^2 + 1)\, dt = \left[\frac{2t^3}{3} + t \right]_0^1 = \boxed{\frac{5}{3}} \end{aligned}$$

C_2:

$$\begin{aligned} r_2'(t) &= (2, 2, 2) \\ F(r_2(t)) &= (4t^2, 4t^2, 1) \\ F(r_2(t)) \cdot r_2'(t) &= 4t^2 \cdot 2 + 4t^2 \cdot 2 + 1 \cdot 2 = 16t^2 + 2 \\ \int_{C_2} F \cdot dr &= \int_0^{1/2} (16t^2 + 2)\, dt = \left[\frac{16t^3}{3} + 2t \right]_0^{1/2} = \boxed{\frac{5}{3}} \end{aligned}$$

C_3:

$$\begin{aligned} r_3'(t) &= (1, 2t, 3t^2) \\ F(r_3(t)) &= (t^4, t^4, 1) \\ F(r_3(t)) \cdot r_3'(t) &= t^4 + 2t^5 + 3t^2 \\ \int_{C_3} F \cdot dr &= \int_0^1 (t^4 + 2t^5 + 3t^2)\, dt = \left[\frac{t^5}{5} + \frac{t^6}{3} + t^3 \right]_0^1 = \boxed{\frac{23}{15}} \end{aligned}$$

C_1 and C_2 represent the same geometric curve despite different parametrizations. The line integral is independent of parametrization; it depends only on the geometric path and endpoints.

To show: F is not conservative
Proof:

$$F \text{ conservative} \iff \nabla \times F = 0$$

$$\begin{aligned}
\nabla \times F &= \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \times \begin{pmatrix} y^2 \\ xz \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} \frac{\partial(1)}{\partial y} - \frac{\partial(xz)}{\partial z} \\ \frac{\partial(y^2)}{\partial z} - \frac{\partial(1)}{\partial x} \\ \frac{\partial(xz)}{\partial x} - \frac{\partial(y^2)}{\partial y} \end{pmatrix} \\
&= \begin{pmatrix} 0 - x \\ 0 - 0 \\ z - 2y \end{pmatrix} \neq 0
\end{aligned}$$

$\square$

Exercise 8. *Let C be the hypocycloid (astroid) given by*

$$r(t) = (\cos^3(t), \sin^3(t)).$$

Find the length of C. Hint: *Compute the length of C in the first quadrant ($t \in [0, \frac{\pi}{2}]$) and multiply the result by 4.*

Solution:

$$\begin{aligned} r'(t) &= \Big(-3\cos^2(t)\sin(t),\ 3\sin^2(t)\cos(t)\Big) \\ &= 3\sin(t)\cos(t)\,(-\cos(t),\ \sin(t)) \\ \implies \|r'(t)\| &= 3\sin(t)\cos(t)\sqrt{\cos^2(t)+\sin^2(t)} \\ &= 3\sin(t)\cos(t) \end{aligned}$$

$$\begin{aligned} l = \int_C ds &= 4\int_0^{\frac{\pi}{2}} \|r'(t)\|\ dt \\ &= 12\int_0^{\frac{\pi}{2}} \sin(t)\cos(t)\,dt \end{aligned}$$

Let $u = \sin(t) \implies du = \cos(t)\,dt$:

$$\begin{aligned} l &= 12\int_0^1 u\,du \\ &= 12\left[\frac{1}{2}u^2\right]_0^1 \\ &= 6\left[\sin^2(t)\right]_0^{\frac{\pi}{2}} \\ &= 6 \end{aligned}$$

□

Exercise 9. *Compute the line integrals of the first and the second kind:*

(a) Let C_1 be the semi-circle given by $r_1(t) = (3\cos(t), 3\sin(t))$, $\quad t \in [0, \pi]$.

$$\int_{C_1} x^2 y \, ds$$

(b) Let C_2 be given by the parametrization $r_2(t) = (4t, 3t^2)$, $\quad t \in [0, 1]$.

$$\int_{C_2} (x^2 y)\, dx - (x - yx)\, dy$$

Solution:

(a)

$$\begin{aligned}
r_1'(t) &= 3(-\sin t, \cos t) \\
\|r_1'(t)\| &= 3\sqrt{\sin^2 t + \cos^2 t} = 3 \\
&\implies ds = 3\, dt \\
\int_{C_1} x^2 y\, ds &= \int_0^{\pi} 27\cos^2(t)\sin(t) \cdot 3\, dt \\
&= 81 \int_0^{\pi} \cos^2(t)\sin(t)\, dt
\end{aligned}$$

Let $u = \cos(t)$, then $du = -\sin(t)\, dt$. $t = 0 \implies u = 1$; $t = \pi \implies u = -1$:

$$\begin{aligned}
&= 81 \int_1^{-1} u^2(-du) \\
&= 81 \int_{-1}^{1} u^2\, du \\
&= 81 \left[\frac{u^3}{3}\right]_{-1}^{1} \\
&= 81 \left(\frac{1}{3} - \frac{-1}{3}\right) = \boxed{54}
\end{aligned}$$

(b)

$$\begin{aligned}
r_2'(t) &= (4, 6t) \\
P(x, y) = x^2 y, \quad Q(x, y) &= -(x - yx) = yx - x \\
P(x(t), y(t)) &= (4t)^2 \cdot 3t^2 = 48t^4 \\
Q(x(t), y(t)) &= 3t^2 \cdot 4t - 4t = 12t^3 - 4t
\end{aligned}$$

With $dx = 4\, dt$ and $dy = 6t\, dt$:

$$\begin{aligned}
\int_{C_2} P\, dx + Q\, dy &= \int_0^1 \left(48t^4 \cdot 4 + (12t^3 - 4t) \cdot 6t\right) dt \\
&= \int_0^1 \left(264t^4 - 24t^2\right) dt \\
&= \left[\frac{264t^5}{5} - \frac{24t^3}{3}\right]_0^1 \\
&= \frac{264}{5} - 8 = \boxed{\frac{224}{5}}
\end{aligned}$$

□

Exercise 10. *Let*

$$\omega = x^2 \exp(xyz)\, dy \wedge dz$$

be a 2-form on $\mathbb{R}^3$. *Give the 3-form* $d\omega$.

Hint: $d\omega = f(x,y,z)\, dx \wedge dy \wedge dz$, *where* f *is a scalar function.*

Solution: Any smooth 2-form in $\mathbb{R}^3$ can be written as:

$$\begin{aligned} \omega &= A(x,y,z)\, dy \wedge dz + B(x,y,z)\, dz \wedge dx + C(x,y,z)\, dx \wedge dy \\ &= x^2 e^{xyz}\, dy \wedge dz \end{aligned}$$

$$\begin{aligned} \implies A(x,y,z) &= x^2 e^{xyz} \text{ and } B = C = 0 \\ A_x(x,y,z) &= 2xe^{xyz} + x^2 yze^{xyz} \\ &= xe^{xyz}(2 + xyz) \end{aligned}$$

So:

$$d\omega = \left(xe^{xyz}(2 + xyz)\right)\, dx \wedge dy \wedge dz$$

□

Exercise 11. *Let F be the vector field $F(x,y) = (x^3 + xy^2, x^2y - y^5)$.*

(a) Let C be the square curve with corner points $(0,0), (1,0), (1,1), (0,1)$ and anti-clockwise orientation. Determine $\int_C F \cdot dr$.

(b) Show that F has a potential ϕ. How can you explain the result of part (a)?

(c) Compute the line integral of F for a curve from $(0,0)$ to $(1,2)$.

Solution:

(a) 1. C_1: from $(0,0)$ to $(1,0)$ (bottom edge):

$$C_1 : r(t) = (x(t), y(t)) = (t, 0), \ 0 \leq t \leq 1$$

$dx = x'(t)dt = dt$, $dy = y'(t)dt = 0$, $y = 0$:

$$\begin{aligned}\int_{C_1} F \cdot dr = \int_0^1 P\,dx + Q\,dy &= \int_0^1 P\,dx \\ &= \int_0^1 \left(t^3 + t \cdot 0^2\right)\,dt \\ &= \int_0^1 t^3\,dt = \frac{1}{4}\end{aligned}$$

2. C_2: from $(1,0)$ to $(1,1)$ (right edge)

$$C_2 : r(t) = (x(t), y(t)) = (1, t), \ 0 \leq t \leq 1$$

$dx = 0$, $dy = dt$, $x = 1$:

$$\begin{aligned}\int_{C_2} F \cdot dr = \int_0^1 P\,dx + Q\,dy &= \int_0^1 Q\,dy \\ &= \int_0^1 (t - t^5)\,dt \\ &= \left[\tfrac{1}{2}t^2 - \tfrac{1}{6}t^6\right]_0^1 = \frac{1}{3}\end{aligned}$$

3. C_3: from $(1,1)$ to $(0,1)$ (top edge):

$$C_3 : r(t) = (x(t), y(t)) = (1-t, 1), \ 0 \leq t \leq 1$$

$dx = -dt$, $dy = 0$, $y = 1$:

$$\begin{aligned}\int_{C_3} F \cdot dr = \int_0^1 P\,dx + Q\,dy &= \int_0^1 P\,dx \\ &= \int_0^1 \left((1-t)^3 + (1-t)\right)\,(-dt) \\ &= -\int_0^1 (1-t)^3\,dt - \int_0^1 (1-t)\,dt \\ &= -\left[-\tfrac{1}{4}(1-t)^4\right]_0^1 - \left[t - \tfrac{1}{2}t^2\right]_0^1 \\ &= -\frac{1}{4} - \frac{1}{2} = -\frac{3}{4}\end{aligned}$$

4. C_4: from $(0,1)$ to $(0,0)$ (left edge):

$$C_4 : r(t) = (x(t), y(t)) = (0, 1-t),\ 0 \le t \le 1$$

$dx = 0$, $dy = -dt$, $x = 0$:

$$\begin{aligned}\int_{C_4} F \cdot dr = \int_0^1 P\,dx + Q\,dy &= \int_0^1 Q\,dy \\ &= \int_0^1 \left(-(1-t)^5\right)(-dt) \\ &= \left[-\tfrac{1}{6}(1-t)^6\right]_0^1 = \frac{1}{6}\end{aligned}$$

$$\int_C F \cdot dr = \frac{1}{4} + \frac{1}{3} - \frac{3}{4} + \frac{1}{6} = 0$$

(b)

$$\begin{aligned}\phi_1(x,y) &= \frac{1}{4}x^4 + \frac{1}{2}x^2y^2 + h_1(y) \\ \phi_2(x,y) &= \frac{1}{2}x^2y^2 - \frac{1}{6}y^6 + h_2(x) \\ \implies \phi(x,y) &= \frac{1}{4}x^4 + \frac{1}{2}x^2y^2 - \frac{1}{6}y^6\end{aligned}$$

Or use *Theorem 1.4.15* with *Remark 1.4.16*:

$$\begin{aligned}\frac{\partial P}{\partial y} &= \frac{\partial}{\partial y}(x^3 + xy^2) = 2xy \\ \frac{\partial Q}{\partial x} &= \frac{\partial}{\partial}(x^2y + y^5) = 2xy\end{aligned}$$

$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \wedge \mathbb{R}^2$ is simply connected $\wedge$ F is continuously differentiable
$\implies$ F is conservative and $\int_C F \cdot dr = 0$, which explains the result of part (a).

(c)

$$\phi(x,y) = \frac{1}{4}x^4 + \frac{1}{2}x^2y^2 - \frac{1}{6}y^6$$

Let γ be any path from $(0,0)$ to $(1,2)$. Since $F = \nabla\phi$, we have:

$$\int_\gamma F \cdot dr = \phi(1,2) - \phi(0,0)$$

with

$$\begin{aligned}\phi(1,2) &= \frac{1}{4}1^4 + \frac{1}{2}1^2 2^2 - \frac{1}{6}2^6 = -\frac{101}{12} \\ \phi(0,0) &= 0 + 0 - 0 = 0\end{aligned}$$

Therefore:

$$\int_{(0,0)}^{(1,2)} F \cdot dr = -\frac{101}{12}$$

$\square$

Exercise 12. *Let $r = (x, y, z)$ and suppose F is the vector field*

$$F(r) = -\frac{c}{\|r\|^3} r$$

where c is a real constant.

(a) Show that $\phi(r) = \frac{c}{\|r\|}$ is a potential of F.

(b) Show that $\nabla^2 \phi = 0$ for $r \neq 0$.

(c) Let C be a curve from p to q. Determine $\int_C F(r)\, dr$.

(d) What can be said about $\operatorname{curl} F$*?*

Hint: *Write ϕ as a function in x, y, z using $\|r\| = \sqrt{x^2 + y^2 + z^2}$.*

Solution:

(a) To prove: $F = \nabla \phi$

$$(\nabla \phi)(r) = \left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z} \right) = \left(-\frac{c}{\|r\|^3} x, -\frac{c}{\|r\|^3} y, -\frac{c}{\|r\|^3} z \right) = -\frac{c}{\|r\|^3} r$$

(b) Let $r \neq 0$:

$$\begin{aligned}
(\nabla^2 \phi)(r) &= \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} \\
&= -\frac{c}{\|r\|^3} \left(\left(1 - \frac{3x^2}{\|r\|^2} \right) + \left(1 - \frac{3y^2}{\|r\|^2} \right) + \left(1 - \frac{3z^2}{\|r\|^2} \right) \right) \\
&= -\frac{c}{\|r\|^3} \left(3 - \frac{3(x^2 + y^2 + z^2)}{\|r\|^2} \right) \\
&= -\frac{c}{\|r\|^3} (3 - 3) = 0
\end{aligned}$$

(c) Since ϕ is a potential of F with $\nabla \phi = F$, *Proposition 1.4.12* holds:

$$\int_C F(r)\, dr = \phi(q) - \phi(p) = \frac{c}{\|q\|} - \frac{c}{\|p\|}$$

(d) With *Theorem 1.4.15*, since F has a potential, *curl* $F = 0$.

$\square$

Exercise 13. *Find the work done by the force field*

$$F(x, y, z) = \left(-\frac{1}{2}x, -\frac{1}{2}y, \frac{1}{4}\right)$$

on a particle as it moves along the spiral given by

$$r(t) = (\cos(t), \sin(t), t), \quad t \in [0, 3\pi].$$

Solution:

$$\begin{aligned}
W &= \int_C F\, dr \\
&= \int_0^{3\pi} F(r(t)) \cdot r'(t)\, dt \\
&= \int_0^{3\pi} \left(-\tfrac{1}{2}\cos(t), -\tfrac{1}{2}\sin(t), \tfrac{1}{4}\right) \cdot (-\sin(t), \cos(t), 1)\, dt \\
&= \int_0^{3\pi} \left(\tfrac{1}{2}\sin(t)\cos(t) - \tfrac{1}{2}\sin(t)\cos(t) + \tfrac{1}{4}\right)\, dt \\
&= \int_0^{3\pi} \tfrac{1}{4}\, dt \\
&= \left[\tfrac{1}{4}t\right]_0^{3\pi} \\
&= \frac{3}{4}\pi
\end{aligned}$$

$\square$

Exercise 14. *Let F be the vector field given by*

$$F(x,y) = (y^3+1, 3xy^2+1)$$

and let C be the semicircular path from $(0,0)$ to $(2,0)$.

(a) Express $\int_C F\,dr$ as an integral in one variable.

(b) Find a potential of F by integration. Then use the potential to compute $\int_C F\,dr$.

Solution:

(a) Let $r(t)$ go from $(2,0)$ to $(0,0)$ as t goes from 0 to π:

$$\begin{aligned} C : r(t) &= (x(t), y(t)),\ t \in [0,\pi] \\ &= (1+\cos t, \sin t) \end{aligned}$$

$$\begin{aligned} \int_C F\,dr &= -\int_0^\pi F(r(t)) \cdot r'(t)\,dt \\ &= -\int_0^\pi (1+\sin^3 t, 1+3(1+\cos t)\sin^2 t) \cdot (-\sin t, \cos t)\,dt \\ &= -\int_0^\pi (-\sin t - \sin^4 t + \cos t + 3\sin^2 t\cos t + 3\sin^2 t\cos^2 t)\,dt \end{aligned}$$

(b) Let $r(t) = (tx, ty)$, $t \in [0,1]$:

$$\begin{aligned} \phi(x,y) &= \int_0^1 F(r(t)) \cdot r'(t)\,dt \\ &= \int_0^1 (t^3y^3+1,\ 3t^3xy^2+1) \cdot (x,\ y)\,dt \\ &= \int_0^1 (t^3y^3x + x + 3t^3xy^3 + y)\,dt \\ &= \Big[\tfrac{1}{4}t^4(y^3x+3xy^3) + t(x+y)\Big]_0^1 \\ &= xy^3 + x + y \end{aligned}$$

$$\nabla(xy^3+x+y) = (y^3+1,\ 3xy^2+1) = F(x,y) \implies \phi \text{ is a potential of } F$$

□

Exercise 15. *Check if* $F(x,y,z) = (y^2z, 2xyz, xy^2)$ *is conservative.*

Solution: To prove: $\nabla \times F = 0$

$$\begin{aligned}
\nabla \times F &= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \\
&= \Big(2xy - 2xy, y^2 - y^2, 2yz - 2yz\Big) \\
&= 0
\end{aligned}$$

$\square$

Exercise 16. *Let $D \subset \mathbb{R}^2$ be the triangle area given by $0 \le x \le y$ and $0 \le y \le 1$. Evaluate the following double integral:*

$$\int_D \int y^2 e^{-xy}\, dx\, dy$$

Solution:

$$\begin{aligned}
\int_D \int y^2 e^{-xy}\, dx\, dy &= \int_0^1 \left(\int_0^y y^2 e^{-xy} dx \right) dy \\
&= \int_0^1 \left(y^2 \left[-\frac{1}{y} e^{-xy} \right]_{x=0}^{x=y} \right) dy \\
&= \int_0^1 \left(-y e^{-y^2} + y \right) dy \\
&= \left[\tfrac{1}{2} e^{-y^2} + \tfrac{1}{2} y^2 \right]_0^1 \\
&= \frac{1}{2} \left(e^{-1} - 1 + (1 - 0) \right) \\
&= \frac{1}{2e}
\end{aligned}$$

□

Exercise 17. *Compute the integral*

$$\int_D \int (x^2 + y^2)^{10} \, dx \, dy$$

over the upper circular area

$$D = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1 \wedge y \geq 0\}.$$

Solution: The corresponding domain with respect to the polar coordinates r and ϕ is

$$D^* = [0, 1] \times [0, \pi]$$

$$\begin{aligned}
\int_D \int (x^2 + y^2)^{10} \, dx \, dy &= \int_0^1 \int_0^\pi (r^2 \cos^2 \phi + r^2 \sin^2 \phi)^{10} r \, d\phi \, dr \\
&= \int_0^1 \int_0^\pi (r^2 (\cos^2 \phi + \sin^2 \phi))^{10} r \, d\phi \, dr \\
&= \int_0^1 \int_0^\pi r^{21} \, d\phi \, dr \\
&= \int_0^1 \left(r^{21} \left[\phi\right]_{\phi=0}^{\phi=\pi} \right) dr \\
&= \pi \left[\frac{1}{22} r^{22} \right]_{r=0}^{r=1} = \frac{\pi}{22}
\end{aligned}$$

□

Exercise 18. *Give the volume of a cylinder around the x-axis with radius 2, beginning at $x = 1$ and ending at $x = 5$.*

Solution: Transform (x, y, z) to cylindrical coordinates (r, ϕ, z):

$$\begin{cases} x = r\cos\phi \\ y = r\sin\phi \\ z = z \end{cases}$$

$$J = \frac{\partial(x, y, z)}{\partial(r, \phi, z)} = \begin{bmatrix} \cos\phi & -r\sin\phi & 0 \\ \sin\phi & r\cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\det(J) = \begin{vmatrix} \cos\phi & -r\sin\phi \\ \sin\phi & r\cos\phi \end{vmatrix} \cdot 1 = r$$

$$\implies dx \wedge dy \wedge dz = r\, dr \wedge d\phi \wedge dz$$

Move the cylinder in space so that its axis aligns with the z-axis from $z = 0$ to $z = 4$:

$$\begin{aligned} V &= \int_0^2 \int_0^{2\pi} \int_0^4 r\, dz\, d\phi\, dr \\ &= 4 \int_0^2 \int_0^{2\pi} r\, d\phi\, dr \\ &= 8\pi \int_0^2 r\, dr \\ &= 8\pi \left[\frac{1}{2} r^2\right]_0^2 \\ &= 16\pi \end{aligned}$$

□

Exercise 19. *Compute the volume between the plane defined by*

$$z = 2x + 3y$$

and the triangle

$$D = \{(x, y) \mid x \in [1, 2] \wedge y \in [0, x]\}$$

in the xy-plane.

Solution:

$$\begin{aligned}
V &= \iint_D z \, dA \\
&= \int_1^2 \int_0^x (2x + 3y) \, dy \, dx \\
&= \int_1^2 \left[2xy + \frac{3}{2}y^2 \right]_{y=0}^{y=x} dx \\
&= \int_1^2 (2x^2 + \frac{3}{2}x^2) \, dx \\
&= \left[\frac{2}{3}x^3 + \frac{1}{2}x^3 \right]_1^2 \\
&= \left[\frac{7}{6}x^3 \right]_1^2 \\
&= \frac{56}{6} - \frac{7}{6} \\
&= \frac{49}{6}
\end{aligned}$$

□

Exercise 20. *Determine the volume V of the portion of a ball of radius 3 that is defined by polar angles $\theta \in \left[0, \frac{\pi}{6}\right]$ and arbitrary azimuth angles φ.*

Remark: *V is called a spherical sector or a spherical cone.*

Solution:

$$\begin{aligned}
V &= \iiint_{[0,3]\times[0,\frac{\pi}{6}]\times[-\pi,\pi]} dV \\
&= \int_0^{2\pi} \int_0^{\pi/6} \int_0^3 r^2 \sin\theta \, dr \, d\theta \, d\varphi \\
&= 2\pi \int_0^{\pi/6} \int_0^3 r^2 \sin\theta \, dr \, d\theta \\
&= 2\pi \left[\tfrac{1}{3} r^3\right]_0^3 \int_0^{\pi/6} \sin\theta \, d\theta \\
&= -18\pi \left[\cos\theta\right]_0^{\pi/6} \\
&= 9\pi(2 - \sqrt{3})
\end{aligned}$$

□