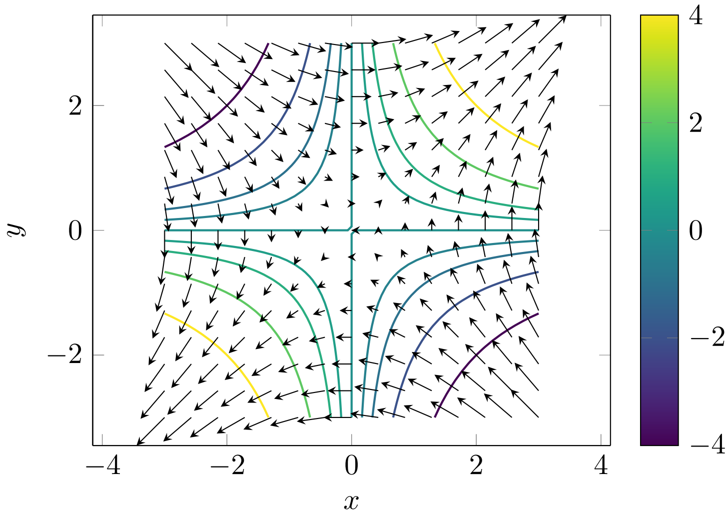
Advanced Mathematics Exercises

Exercise 1. *Let f be the scalar function defined by $f(x,y) = xy$. Sketch the contour lines and the vector field ∇f.*

Solution:

$$(\nabla f)(x,y) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = (y,x)$$



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Exercise 2. *Let*

$$f(x,y) = \sin(\pi xy) \cdot e^{-\frac{x}{3}} \quad \text{and} \quad p = \left(1, \frac{1}{3}\right).$$

(a) Compute ∇f and $(\nabla f)(p)$. Use the special values of sin *and* cos.

(b) Find the directions of maximum increase and decrease at p. You can give approximate values.

(c) Give the direction of the contour line at p.

(d) The equation of the tangent plane of the graph of f at (x_0, y_0) is

$$z = f(x_0, y_0) + (\nabla f)(x_0, y_0) \cdot (x - x_0, y - y_0).$$

Determine the equation of the tangent plane of $z = f(x,y)$ at p. Give a normal vector of the plane.

(e) Find the directional derivative of $f(x,y)$ at p along the vector

$$v = \frac{1}{\sqrt{2}}(1,1).$$

Solution:

(a)

$$\begin{aligned}
\frac{\partial f}{\partial x} &= e^{-x/3}\left(\pi y\cos(\pi xy) - \frac{1}{3}\sin(\pi xy)\right)\\
\frac{\partial f}{\partial y} &= e^{-x/3}\pi x\cos(\pi xy)\\
\implies (\nabla f)(x,y) &= \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)\\
&= e^{-x/3}\left(\pi y\cos(\pi xy) - \frac{1}{3}\sin(\pi xy),\ \pi x\cos(\pi xy)\right)
\end{aligned}$$

With $\sin(\frac{\pi}{3}) = \frac{\sqrt{3}}{2}$ and $\cos(\frac{\pi}{3}) = \frac{1}{2}$:

$$(\nabla f)(p) = e^{-1/3}\left(\frac{\pi - \sqrt{3}}{6}, \frac{\pi}{2}\right)$$

(b) Maximum increase: $(\nabla f)(p)$, maximum decrease: $-(\nabla f)(p)$

(c) Contour lines are perpendicular to the gradient:

$$(x,\ y) \cdot (-y,\ x) = 0 \implies u = e^{-1/3}\left(-\frac{\pi}{2}, \frac{\pi - \sqrt{3}}{6}\right)$$

(d)

$$\begin{aligned}
z = f(x,y) &= f(p) + (\nabla f)(p) \cdot (x - 1, y - \frac{1}{3})\\
&= \frac{\sqrt{3}}{2}e^{-1/3} + e^{-1/3}\left(\frac{\pi - \sqrt{3}}{6}, \frac{\pi}{2}\right) \cdot (x - 1,\ y - \frac{1}{3})\\
&= e^{-1/3}\left(\frac{\sqrt{3}}{2} + \frac{1}{6}\left((\pi - \sqrt{3})x + 3\pi y + \sqrt{3}\right)\right)
\end{aligned}$$

The surface $z = f(x, y)$ can be written as $F(x, y, z) = f(x, y) - z = 0$. Therefore,

$$\nabla F = (f_x,\ f_y,\ -1)$$

is normal to the surface:

$$\begin{aligned} n := (\nabla F)(p) &= (f_x(p),\ f_y(p),\ -1) \\ &= \left(\frac{\pi - \sqrt{3}}{6} e^{-1/3},\ \frac{\pi}{2} e^{-1/3},\ -1 \right) \end{aligned}$$

(e)

$$\begin{aligned} (\nabla f)(p) \cdot v &= e^{-1/3} \left(\frac{\pi - \sqrt{3}}{6},\ \frac{\pi}{2} \right) \cdot \frac{1}{\sqrt{2}}(1, 1) \\ &= \frac{e^{-1/3}}{\sqrt{2}} \left(\frac{\pi - \sqrt{3}}{6} + \frac{\pi}{2} \right) \\ &= \frac{e^{-1/3}}{\sqrt{2}} \cdot \frac{4\pi - \sqrt{3}}{6} \end{aligned}$$

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Exercise 3. *Compute curl and divergence of the vector field*

$$F(x, y, z) = (\sin(x)xy,\ ze^{-x},\ yz).$$

Solution:

$$\begin{aligned}
\frac{\partial F_x}{\partial y} &= \frac{\partial}{\partial y}(\sin(x)xy) = x\sin(x)\\
\frac{\partial F_x}{\partial z} &= \frac{\partial}{\partial z}(\sin(x)xy) = 0\\
\frac{\partial F_y}{\partial x} &= \frac{\partial}{\partial x}(ze^{-x}) = -ze^{-x}\\
\frac{\partial F_y}{\partial z} &= \frac{\partial}{\partial z}(ze^{-x}) = e^{-x}\\
\frac{\partial F_z}{\partial x} &= \frac{\partial}{\partial x}(yz) = 0\\
\frac{\partial F_z}{\partial y} &= \frac{\partial}{\partial y}(yz) = z
\end{aligned}$$

$$\begin{aligned}
\operatorname{curl} F = \nabla \times F &= \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z},\ \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x},\ \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\\
&= \left(z - e^{-x},\ 0 - 0,\ -ze^{-x} - x\sin(x)\right)\\
&= \left(z - e^{-x},\ 0,\ -ze^{-x} - x\sin(x)\right)
\end{aligned}$$

$$\begin{aligned}
\operatorname{div} F = \nabla \cdot F &= \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}\\
&= \frac{\partial}{\partial x}\big(\sin(x)xy\big) + \frac{\partial}{\partial y}(ze^{-x}) + \frac{\partial}{\partial z}(yz)\\
&= y\sin(x) + xy\cos(x) + 0 + y\\
&= y\sin(x) + xy\cos(x) + y
\end{aligned}$$

$\square$

Exercise 4. *Let*

$$f(x,y,z) = x^2yz^3 \quad \textit{and} \quad F(x,y,z) = (xz, -y^2, 2x^2y).$$

Give ∇f*,* $\nabla^2 f$*,* $\nabla \cdot F$*, and* $\nabla \times F$*.*

Solution:

$$\begin{aligned}
\nabla f &= \left(\frac{\partial f}{\partial x},\ \frac{\partial f}{\partial y},\ \frac{\partial f}{\partial z}\right) \\
&= \left(2xyz^3,\ x^2z^3,\ 3x^2yz^2\right)
\end{aligned}$$

$$\begin{aligned}
\nabla^2 f &= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \\
&= 2yz^3 + 0 + 6x^2yz \\
&= 2yz^3 + 6x^2yz
\end{aligned}$$

$$\begin{aligned}
\nabla \cdot F &= \frac{\partial}{\partial x}(xz) + \frac{\partial}{\partial y}(-y^2) + \frac{\partial}{\partial z}(2x^2y) \\
&= z - 2y
\end{aligned}$$

$$\begin{aligned}
\frac{\partial F_x}{\partial y} &= \frac{\partial}{\partial y}(xz) = 0 \\
\frac{\partial F_x}{\partial z} &= \frac{\partial}{\partial z}(xz) = x \\
\frac{\partial F_y}{\partial x} &= \frac{\partial}{\partial x}(-y^2) = 0 \\
\frac{\partial F_y}{\partial z} &= \frac{\partial}{\partial z}(-y^2) = 0 \\
\frac{\partial F_z}{\partial x} &= \frac{\partial}{\partial x}(2x^2y) = 4xy \\
\frac{\partial F_z}{\partial y} &= \frac{\partial}{\partial y}(2x^2y) = 2x^2
\end{aligned}$$

$$\begin{aligned}
\nabla \times F &= \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z},\ \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x},\ \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right) \\
&= \left(2x^2 - 0,\ x - 4xy,\ 0 - 0\right) \\
&= \left(2x^2,\ x - 4xy,\ 0\right)
\end{aligned}$$

$\square$

Exercise 5. *Compute differential forms:*

(a) *Let*

$$f(x, y, z) = \frac{x}{yz}$$

be a scalar function. Give the differential form df.
Hint:

$$df = F_1(x, y, z)\, dx + F_2(x, y, z)\, dy + F_3(x, y, z)\, dz.$$

(b) *Let*

$$\omega = x^2 \sin(y)\, dx + 2^x \cos(y)\, dy$$

be a differential form on $\mathbb{R}^2$. Give the 2-form $d\omega$.
Hint:

$$d\omega = \phi(x, y)\, dx \wedge dy.$$

(c) *Let*

$$\omega = x^2 \sin(y)\, dx + z^2 \cos(y)\, dy - xy^2\, dz$$

be a differential form on $\mathbb{R}^3$. Give the 2-form $d\omega$.
Hint:

$$d\omega = F_1(x, y, z)\, dy \wedge dz + F_2(x, y, z)\, dz \wedge dx + F_3(x, y, z)\, dx \wedge dy.$$

Solution:

(a)

$$df = \frac{1}{yz}\, dx - \frac{x}{y^2 z}\, dy - \frac{x}{yz^2}\, dz$$

(b)

$$\begin{aligned}
\phi(x, y) &= \frac{\partial}{\partial x}(2^x \cos(y)) - \frac{\partial}{\partial y}(x^2 \sin(y)) \\
&= \ln(2) 2^x \cos(y) - x^2 \cos(y) \\
&= \cos(y) \left(\ln(2) 2^x - x^2 \right)
\end{aligned}$$

So:

$$\begin{aligned}
d\omega &= \phi(x, y)\, dx \wedge dy \\
&= \left(\cos(y) \left(\ln(2) 2^x - x^2 \right) \right)\, dx \wedge dy
\end{aligned}$$

(c)

$$\begin{aligned}
F_1(x, y, z) &= \frac{\partial}{\partial y}(-xy^2) - \frac{\partial}{\partial z}(z^2 \cos y) = -2xy - 2z \cos y \\
F_2(x, y, z) &= \frac{\partial}{\partial z}(x^2 \sin y) - \frac{\partial}{\partial x}(-xy^2) = 0 + y^2 = y^2 \\
F_3(x, y, z) &= \frac{\partial}{\partial x}(z^2 \cos y) - \frac{\partial}{\partial y}(x^2 \sin y) = 0 - x^2 \cos y = -x^2 \cos y
\end{aligned}$$

So:

$$d\omega = (-2xy - 2z \cos y)\, dy \wedge dz + y^2\, dz \wedge dx - (x^2 \cos y)\, dx \wedge dy.$$

$\square$

Exercise 6. *Let* $f(x,y) = \frac{1}{\sqrt{x^2+y^2}}$. *Compute* ∇f. *Then express* f *and* ∇f *in terms of the norm* $\|v\|$, *where* $v = (x,y)$.

Solution:

$$(\nabla f)(x,y) = -\frac{1}{2(x^2+y^2)^{3/2}} \left(2x + y^2,\ x^2 + 2y\right)$$

With $\|v\| = \sqrt{x^2+y^2}$:

$$f(v) = \frac{1}{\|v\|}$$

$$(\nabla f)(v) = -\frac{1}{2\,\|v\|^3} \left(2x + y^2,\ x^2 + 2y\right)$$

□

Exercise 7. *Let $F(x,y,z) = (y^2, xz, 1)$ be a vector field. The following curves are given:*

- $C_1 : r_1(t) = (t,t,t), \quad t \in [0,1]$
- $C_2 : r_2(t) = (2t,2t,2t), \quad t \in [0,\frac{1}{2}]$
- $C_3 : r_3(t) = (t,t^2,t^3), \quad t \in [0,1]$

Sketch the curves. Compute the line integrals

$$\int_{C_1} F \cdot dr, \quad \int_{C_2} F \cdot dr, \quad \int_{C_3} F \cdot dr.$$

Why do the first two line integrals coincide? Is F a conservative vector field?

Solution: TODO □

Exercise 8. *Let C be the hypocycloid (astroid) given by*

$$r(t) = (\cos^3(t), \sin^3(t)).$$

Find the length of C. Hint: *Compute the length of C in the first quadrant ($t \in [0, \frac{\pi}{2}]$) and multiply the result by 4.*

Solution:

$$\begin{aligned} r'(t) &= \Big(-3\cos^2(t)\sin(t),\ 3\sin^2(t)\cos(t)\Big) \\ &= 3\sin(t)\cos(t)\,(-\cos(t),\ \sin(t)) \\ \implies \|r'(t)\| &= 3\sin(t)\cos(t)\sqrt{\cos^2(t)+\sin^2(t)} \\ &= 3\sin(t)\cos(t) \end{aligned}$$

$$\begin{aligned} l = \int_C ds &= 4\int_0^{\frac{\pi}{2}} \|r'(t)\|\ dt \\ &= 12\int_0^{\frac{\pi}{2}} \sin(t)\cos(t)\,dt \end{aligned}$$

Let $u = \sin(t) \implies du = \cos(t)\,dt$:

$$\begin{aligned} l &= 12\int_0^1 u\,du \\ &= 12\left[\frac{1}{2}u^2\right]_0^1 \\ &= 6\left[\sin^2(t)\right]_0^{\frac{\pi}{2}} \\ &= 6 \end{aligned}$$

$\square$

Exercise 9. *Compute the line integrals of the first and the second kind:*

(a) Let C_1 be the semi-circle given by $r_1(t) = (3\cos(t), 3\sin(t)), \quad t \in [0, \pi]$.

$$\int_{C_1} x^2 y \, ds$$

(b) Let C_2 be given by the parametrization $r_2(t) = (4t, 3t^2), \quad t \in [0, 1]$.

$$\int_{C_2} (x^2 y) \, dx - (x - yx) \, dy$$

Solution: TODO □