Solutions of Equations of One Variable

Exercise Set 2.1, page 53

1. Use the Bisection method to find p_3 for $f(x) = \sqrt{x} - \cos x$ on $[0, 1]$.

SOLUTION: Using the Bisection method gives $a_1 = 0$ and $b_1 = 1$, so $f(a_1) = -1$ and $f(b_1) = 0.45970$. We have

$$p_1 = \frac{1}{2}(a_1 + b_1) = \frac{1}{2} \quad \text{and} \quad f(p_1) = -0.17048 < 0.$$

Since $f(a_1) < 0$ and $f(p_1) < 0$, we assign $a_2 = p_1 = 0.5$ and $b_2 = b_1 = 1$. Thus

$$f(a_2) = -0.17048 < 0, \quad f(b_2) = 0.45970 > 0, \quad \text{and} \quad p_2 = \frac{1}{2}(a_2 + b_2) = 0.75.$$

Since $f(p_2) = 0.13434 > 0$, we have $a_3 = 0.5$; $b_3 = p_3 = 0.75$ so that

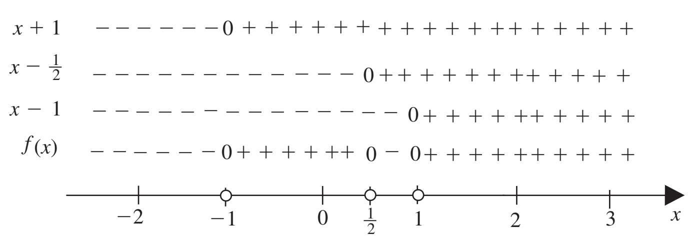
$$p_3 = \frac{1}{2}(a_3 + b_3) = 0.625.$$

2. a. Let $f(x) = 3(x+1)\left(x - \frac{1}{2}\right)(x-1)$. Use the Bisection method on the interval $[-2, 1.5]$ to find p_3.

SOLUTION: Since

$$f(x) = 3(x+1)\left(x - \frac{1}{2}\right)(x-1),$$

we have the following sign graph for $f(x)$:



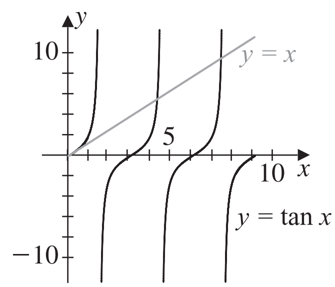
Thus, $a_1 = -2$, with $f(a_1) < 0$, and $b_1 = 1.5$, with $f(b_1) > 0$. Since $p_1 = -\frac{1}{4}$, we have $f(p_1) > 0$. We assign $a_2 = -2$, with $f(a_2) < 0$, and $b_2 = -\frac{1}{4}$, with $f(b_2) > 0$. Thus, $p_2 = -1.125$ and $f(p_2) < 0$. Hence, we assign $a_3 = p_2 = -1.125$ and $b_3 = -0.25$. Then $p_3 = -0.6875$.

8. **a.** Sketch the graphs of $y = x$ and $y = \tan x$.

b. Use the Bisection method to find an approximation to within 10^{-5} to the first positive value of x with $x = \tan x$.

SOLUTION:

a. The graphs of $y = x$ and $y = \tan x$ are shown in the figure. From the graph it appears that the graphs cross near $x = 4.5$.



b. Because $g(x) = x - \tan x$ has

$$g(4.4) \approx 1.303 > 0 \quad \text{and} \quad g(4.6) \approx -4.260 > 0,$$

the fact that g is continuous on $[4.4, 4.6]$ gives us a reasonable interval to start the bisection process. Using Algorithm 2.1 gives $p_{15} = 4.4934143$.

11. Let $f(x) = (x+2)(x+1)x(x-1)^3(x-2)$. To which zero of f does the Bisection method converge for the following intervals?

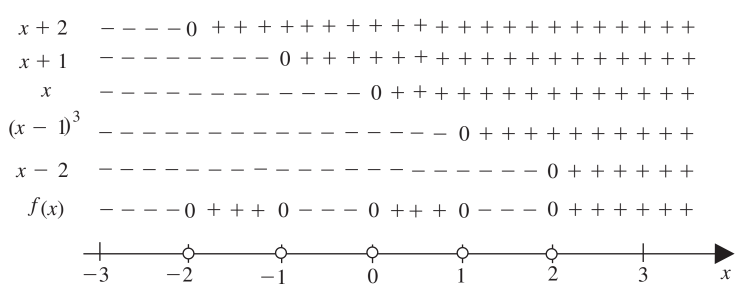
a. $[-3, 2.5]$

c. $[-1.75, 1.5]$

SOLUTION: Since

$$f(x) = (x+2)(x+1)x(x-1)^3(x-2),$$

we have the following sign graph for $f(x)$.



a. The interval $[-3, 2.5]$ contains all 5 zeros of f. For $a_1 = -3$, with $f(a_1) < 0$, and $b_1 = 2.5$, with $f(b_1) > 0$, we have $p_1 = (-3 + 2.5)/2 = -0.25$, so $f(p_1) < 0$. Thus we assign $a_2 = p_1 = -0.25$, with $f(a_2) < 0$, and $b_2 = b_1 = 2.5$, with $f(b_1) > 0$.

Hence $p_2 = (-0.25 + 2.5)/2 = 1.125$ and $f(p_2) < 0$. Then we assign $a_3 = 1.125$, with $f(a_3) < 0$, and $b_3 = 2.5$, with $f(b_3) > 0$. Since $[1.125, 2.5]$ contains only the zero 2, the method converges to 2.

c. The interval $[-1.75, 1.5]$ contains the zeros -1, 0, 1. For $a_1 = -1.75$, with $f(a_1) > 0$, and $b_1 = 1.5$, with $f(b_1) < 0$, we have $p_1 = (-1.75 + 1.5)/2 = -0.125$ and $f(p_1) < 0$. Then we assign $a_2 = a_1 = -1.75$, with $f(a_1) > 0$, and $b_2 = p_1 = -0.125$, with $f(b_2) < 0$. Since $[-1.75, -0.125]$ contains only the zero -1, the method converges to -1.

14. Use the Bisection Algorithm to find an approximation to $\sqrt{3}$ that is accurate to within 10^{-4}.

SOLUTION: The function defined by $f(x) = x^2 - 3$ has $\sqrt{3}$ as its only positive root. Applying the Bisection method to this function on the interval $[1, 2]$ gives $\sqrt{3} \approx p_{14} = 1.7320$. Using a smaller starting interval would decrease the number of iterations that are required.

15. A trough of water of length $L = 10$ feet has a cross section in the shape of a semicircle with radius $r = 1$ foot. When filled with water to within a distance h of the top, the volume $V = 12.4$ ft^3 of the water is given by the formula

$$12.4 = 10\left[0.5\pi - \arcsin h - h\left(1 - h^2\right)^{1/2}\right]$$

Determine the depth of the water to within 0.01 feet.

SOLUTION: Applying the Bisection Algorithm on the interval $[0, 1]$ to the function

$$f(h) = 12.4 - 10\left[0.5\pi - \arcsin h - h\left(1 - h^2\right)^{1/2}\right]$$

gives $h \approx p_{13} = 0.1617$, so the depth is $r - h \approx 1 - 0.1617 = 0.838$ feet.

18. Use Theorem 2.1 to find a bound for the number of iterations needed to approximate a solution to the equation $x^3 + x - 4 = 0$ on the interval $[1, 4]$ to an accuracy of 10^{-3}.

SOLUTION: First note that the particular equation plays no part in finding the bound; all that is needed is the interval and the accuracy requirement. To find an approximation that is accurate to within 10^{-3}, we need to determine the number of iterations n so that

$$|p - p_n| < \frac{b - a}{2^n} = \frac{4 - 1}{2^n} < 0.001; \quad \text{that is,} \quad 3 \times 10^3 < 2^n.$$

As a consequence, a bound for the number of iterations is $n \geq 12$. Applying the Bisection Algorithm gives $p_{12} = 1.3787$.

19. Define the sequence $\{p_n\}$ by $p_n = \displaystyle\sum_{k=1}^{n} \frac{1}{k}$. Show that $\displaystyle\lim_{n\to\infty}(p_n - p_{n-1}) = 0$, even though the sequence $\{p_n\}$ diverges.

SOLUTION: Since $p_n - p_{n-1} = 1/n$, we have $\lim_{n\to\infty}(p_n - p_{n-1}) = 0$. However, p_n is the nth partial sum of the divergent *harmonic* series. The harmonic series is the classic example of a series whose terms go to zero, but not rapidly enough to produce a convergent series. There are many proofs of divergence of this series, any calculus text should give at least two. One proof will simply analyze the partial sums of the series and another is based on the Integral Test.

The point of the problem is not the fact that this particular sequence diverges, it is that a test for an approximate solution to a root based on the condition that $|p_n - p_{n-1}|$ is small should always be suspect. Consecutive terms of a sequence might be close to each other, but not sufficiently close to the actual solution you are seeking.

Exercise Set 2.2, page 63

5. The following methods are proposed to compute $21^{1/3}$. Rank them in order, based on their apparent speed of convergence, assuming $p_0 = 1$.

a. $p_n = \dfrac{20p_{n-1} + 21/p_{n-1}^2}{21}$

b. $p_n = p_{n-1} - \dfrac{p_{n-1}^3 - 21}{3p_{n-1}^2}$

c. $p_n = p_{n-1} - \dfrac{p_{n-1}^4 - 21p_{n-1}}{p_{n-1}^2 - 21}$

d. $p_n = \left(\dfrac{21}{p_{n-1}}\right)^{1/2}$

SOLUTION: **a.** Since

$$p_n = \frac{20p_{n-1} + 21/p_{n-1}^2}{21}, \quad \text{we have} \quad g(x) = \frac{20x + 21/x^2}{21} = \frac{20}{21}x + \frac{1}{x^2},$$

and $g'(x) = \dfrac{20}{21} - \dfrac{2}{x^3}$. Thus, $g'\left(21^{1/3}\right) = \dfrac{20}{21} - \dfrac{2}{21} = 0.857$.

b. Since

$$p_n = p_{n-1} - \frac{p_{n-1}^3 - 21}{3p_{n-1}^2}, \quad \text{we have} \quad g(x) = x - \frac{x^3 - 21}{3x^2} = x - \frac{1}{3}x + \frac{7}{x^2} = \frac{2}{3}x + \frac{7}{x^2}$$

and $g'(x) = \dfrac{2}{3} - \dfrac{14}{x^3}$. Thus, $g'\left(21^{1/3}\right) = \dfrac{2}{3} - \dfrac{2}{3} = 0$.

c. Since

$$p_n = p_{n-1} - \frac{p_{n-1}^4 - 21p_{n-1}}{p_{n-1}^2 - 21},$$

we have

$$g(x) = x - \frac{x^4 - 21x}{x^2 - 21} = \frac{x^3 - 21x - x^4 + 21x}{x^2 - 21} = \frac{x^3 - x^4}{x^2 - 21}$$

and

$$\begin{aligned} g'(x) =& \frac{\left(x^2 - 21\right)\left(3x^2 - 4x^3\right) - \left(x^3 - x^4\right)2x}{\left(x^2 - 21\right)^2} = \frac{3x^4 - 63x^2 - 4x^5 + 84x^3 - 2x^4 + 2x^5}{\left(x^2 - 21\right)^2} \\ =& \frac{-2x^5 + x^4 + 84x^3 - 63x^2}{\left(x^2 - 21\right)^2}. \end{aligned}$$

Thus $g'\left(21^{1/3}\right) = 5.706 > 1$.