Mathematical Preliminaries

Exercise Set 1.1, page 11

1. d. Show that the equation $x - (\ln x)^x = 0$ has at least one solution in the interval $[4, 5]$.

SOLUTION: It is not possible to algebraically solve for the solution x, but this is not required in the problem, we must show only that a solution exists. Let

$$f(x) = x - (\ln x)^x = x - \exp(x(\ln(\ln x))).$$

Since f is continuous on $[4, 5]$ with $f(4) \approx 0.3066$ and $f(5) \approx -5.799$, the Intermediate Value Theorem 1.11 implies that a number x must exist in $(4, 5)$ with $0 = f(x) = x - (\ln x)^x$.

4. c. Find intervals that contain a solution to the equation $x^3 - 2x^2 - 4x + 3 = 0$.

SOLUTION: Let $f(x) = x^3 - 2x^2 - 4x + 3$. The critical points of f occur when

$$0 = f'(x) = 3x^2 - 4x - 4 = (3x + 2)(x - 2);$$

that is, when $x = -\frac{2}{3}$ and $x = 2$. Relative maximum and minimum values of f can occur only at these values. There are at most three solutions to $f(x) = 0$, because $f(x)$ is a polynomial of degree three. Since $f(-2) = -5$ and $f\left(-\frac{2}{3}\right) \approx 4.48$; $f(0) = 3$ and $f(1) = -2$; and $f(2) = -5$ and $f(4) = 19$; solutions lie in the intervals $[-2, -2/3]$, $[0, 1]$, and $[2, 4]$.

5. a. Find $\max\limits_{0 \le x \le 1} |f(x)|$ when $f(x) = (2 - e^x + 2x)/3$.

SOLUTION: First note that $f'(x) = (-e^x + 2)/3$, so the only critical point of f occurs at $x = \ln 2$, which lies in the interval $[0, 1]$. The maximum for $|f(x)|$ must consequently be

$$\max\{|f(0)|, |f(\ln 2)|, |f(1)|\} = \max\{1/3, (2\ln 2)/3, (4 - e)/3\} = (2\ln 2)/3.$$

11. Find the second Taylor polynomial for $f(x) = e^x \cos x$ about $x_0 = 0$, and

a. Use $P_2(0.5)$ to approximate $f(0.5)$, find an upper bound for $|f(0.5) - P_2(0.5)|$, and compare this to the actual error.

b. Find a bound for the error $|f(x) - P_2(x)|$, for x in $[0, 1]$.

c. Approximate $\int_0^1 f(x)\,dx$ using $\int_0^1 P_2(x)\,dx$.

d. Find an upper bound for the error in part (c).

SOLUTION: Since

$$f'(x) = e^x(\cos x - \sin x), \quad f''(x) = -2e^x(\sin x), \quad \text{and} \quad f'''(x) = -2e^x(\sin x + \cos x),$$

we have $f(0) = 1$, $f'(0) = 1$, and $f''(0) = 0$. So

$$P_2(x) = 1 + x \quad \text{and} \quad R_2(x) = \frac{-2e^\xi(\sin\xi + \cos\xi)}{3!}x^3.$$

a. We have $P_2(0.5) = 1 + 0.5 = 1.5$ and

$$|f(0.5) - P_2(0.5)| \le \max_{\xi\in[0,0.5]} \left|\frac{-2e^\xi(\sin\xi + \cos\xi)}{3!}(0.5)^2\right| \le \frac{1}{3}(0.5)^2 \max_{\xi\in[0,0.5]} |e^\xi(\sin\xi + \cos\xi)|.$$

To maximize this quantity on $[0, 0.5]$, first note that $D_x e^x(\sin x + \cos x) = 2e^x \cos x > 0$, for all x in $[0, 0.5]$. This implies that the maximum and minimum values of $e^x(\sin x + \cos x)$ on $[0, 0.5]$ occur at the endpoints of the interval, and

$$e^0(\sin 0 + \cos 0) = 1 < e^{0.5}(\sin 0.5 + \cos 0.5) \approx 2.24.$$

Hence

$$|f(0.5) - P_2(0.5)| \le \frac{1}{3}(0.5)^3(2.24) \approx 0.0932.$$

b. A similar analysis to that in part (a) gives

$$|f(x) - P_2(x)| \le \frac{1}{3}(1.0)^3 e^1(\sin 1 + \cos 1) \approx 1.252.$$

c.

$$\int_0^1 f(x)\,dx \approx \int_0^1 (1 + x)\,dx = \left[x + \frac{x^2}{2}\right]_0^1 = \frac{3}{2}.$$

d. From part (b),

$$\int_0^1 |R_2(x)|\,dx \le \int_0^1 \frac{1}{3}e^1(\cos 1 + \sin 1)x^3\,dx = \int_0^1 1.252x^3\,dx = 0.313.$$

Since

$$\int_0^1 e^x \cos x\,dx = \left[\frac{e^x}{2}(\cos x + \sin x)\right]_0^1 = \frac{e}{2}(\cos 1 + \sin 1) - \frac{1}{2}(1 + 0) \approx 1.378,$$

the actual error is $|1.378 - 1.5| \approx 0.12$.

16. Use the error term of a Taylor polynomial to estimate the error involved in using $\sin x \approx x$ to approximate $\sin 1^\circ$.

SOLUTION: First we need to convert the degree measure for the sine function to radians. We have $180^\circ = \pi$ radians, so $1^\circ = \frac{\pi}{180}$ radians. Since $f(x) = \sin x$, $f'(x) = \cos x$, $f''(x) = -\sin x$, and $f'''(x) = -\cos x$, we have $f(0) = 0$, $f'(0) = 1$, and $f''(0) = 0$. The approximation $\sin x \approx x$ is given by $f(x) \approx P_2(x)$ and $R_2(x) = -\frac{\cos\xi}{3!}x^3$. If we use the bound $|\cos\xi| \le 1$, then

$$\left|\sin\left(\frac{\pi}{180}\right) - \frac{\pi}{180}\right| = \left|R_2\left(\frac{\pi}{180}\right)\right| = \left|\frac{-\cos\xi}{3!}\left(\frac{\pi}{180}\right)^3\right| \le 8.86 \times 10^{-7}.$$

22. Use the Intermediate Value Theorem 1.11 and Rolle's Theorem 1.7 to show that the graph of $f(x) = x^3 + 2x + k$ crosses the x-axis exactly once, regardless of the value of the constant k.

SOLUTION: For $x < 0$,we have $f(x) < 2x + k < 0$, provided that $x < -\frac{1}{2}k$. Similarly, for $x > 0$, we have $f(x) > 2x + k > 0$, provided that $x > -\frac{1}{2}k$. By Theorem 1.11, there exists a number c with $f(c) = 0$.

If $f(c) = 0$ and $f(c') = 0$ for some $c' \neq c$, then by Theorem 1.7, there exists a number p between c and c' with $f'(p) = 0$. However, $f'(x) = 3x^2 + 2 > 0$ for all x. This gives a contradiction to the statement that $f(c) = 0$ and $f(c') = 0$ for some $c' \neq c$. Hence there is exactly one number c with $f(c) = 0$.

27. In Example 3 it is stated that x we have $|\sin x| \leq |x|$. Use the following to verify this statement.

a. Show that for all $x \geq 0$ the function $f(x) = x - \sin x$ is non-decreasing, which implies that $\sin x \leq x$ with equality only when $x = 0$.

b. Use the fact that the sine function is odd to reach the conclusion.

SOLUTION: First observe that for $f(x) = x - \sin x$ we have $f'(x) = 1 - \cos x \geq 0$, because $-1 \leq \cos x \leq 1$ for all values of x.

a. The observation implies that $f(x)$ is non-decreasing for all values of x, and in particular that $f(x) > f(0) = 0$ when $x > 0$. Hence for $x \geq 0$, we have $x \geq \sin x$, and $|\sin x| = \sin x \leq x = |x|$.

b. When $x < 0$, we have $-x > 0$. Since $\sin x$ is an odd function, the fact (from part (a)) that $\sin(-x) \leq (-x)$ implies that $|\sin x| = -\sin x \leq -x = |x|$.

As a consequence, for all real numbers x we have $|\sin x| \leq |x|$.

29. Suppose $f \in C[a, b]$, that x_1 and x_2 are in $[a, b]$.

a. Show that a number ξ exists between x_1 and x_2 with

$$f(\xi) = \frac{f(x_1) + f(x_2)}{2} = \frac{1}{2}f(x_1) + \frac{1}{2}f(x_2).$$

b. Suppose that c_1 and c_2 are positive constants. Show that a number ξ exists between x_1 and x_2 with

$$f(\xi) = \frac{c_1 f(x_1) + c_2 f(x_2)}{c_1 + c_2}.$$

c. Give an example to show that the result in part (b) does not necessarily hold when c_1 and c_2 have opposite signs with $c_1 \neq -c_2$.

SOLUTION:

a. The number

$$\frac{1}{2}(f(x_1) + f(x_2))$$

is the average of $f(x_1)$ and $f(x_2)$, so it lies between these two values of f. By the Intermediate Value Theorem 1.11 there exist a number ξ between x_1 and x_2 with

$$f(\xi) = \frac{1}{2}(f(x_1) + f(x_2)) = \frac{1}{2}f(x_1) + \frac{1}{2}f(x_2).$$

b. Let $m = \min\{f(x_1), f(x_2)\}$ and $M = \max\{f(x_1), f(x_2)\}$. Then $m \le f(x_1) \le M$ and $m \le f(x_2) \le M$, so

$$c_1 m \le c_1 f(x_1) \le c_1 M \quad \text{and} \quad c_2 m \le c_2 f(x_2) \le c_2 M.$$

Thus

$$(c_1 + c_2)m \le c_1 f(x_1) + c_2 f(x_2) \le (c_1 + c_2)M$$

and

$$m \le \frac{c_1 f(x_1) + c_2 f(x_2)}{c_1 + c_2} \le M.$$

By the Intermediate Value Theorem 1.11 applied to the interval with endpoints x_1 and x_2, there exists a number ξ between x_1 and x_2 for which

$$f(\xi) = \frac{c_1 f(x_1) + c_2 f(x_2)}{c_1 + c_2}.$$

c. Let $f(x) = x^2 + 1$, $x_1 = 0$, $x_2 = 1$, $c_1 = 2$, and $c_2 = -1$. Then for all values of x,

$$f(x) > 0 \quad \text{but} \quad \frac{c_1 f(x_1) + c_2 f(x_2)}{c_1 + c_2} = \frac{2(1) - 1(2)}{2 - 1} = 0.$$

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4. c. Find the largest interval in which p^* must lie to approximate $\sqrt{2}$ with relative error at most 10^{-4}.

SOLUTION: We need

$$\frac{\left|p^* - \sqrt{2}\right|}{\left|\sqrt{2}\right|} \le 10^{-4}, \quad \text{so} \quad \left|p^* - \sqrt{2}\right| \le \sqrt{2} \times 10^{-4};$$

that is,

$$-\sqrt{2} \times 10^{-4} \le p^* - \sqrt{2} \le \sqrt{2} \times 10^{-4}.$$

This implies that p^* must be in the interval $\left(\sqrt{2}(0.9999), \sqrt{2}(1.0001)\right)$.

7. a. Use three-digit rounding arithmetic to compute

$$\frac{\frac{13}{14} - \frac{6}{7}}{2e - 5.4},$$

and determine the absolute and relative errors.

SOLUTION: Using three-digit rounding arithmetic gives $\frac{13}{14} = 0.929$, $\frac{6}{7} = 0.857$, and $e = 2.72$. So

$$\frac{13}{14} - \frac{6}{7} = 0.0720 \quad \text{and} \quad 2e - 5.4 = 5.44 - 5.40 = 0.0400.$$

Hence

$$\frac{\frac{13}{14} - \frac{6}{7}}{2e - 5.4} = \frac{0.0720}{0.0400} = 1.80.$$

The correct value is approximately 1.9535, so the absolute and relative errors to three digits are

$$|1.80 - 1.9535| = 0.154 \quad \text{and} \quad \frac{|1.80 - 1.9535|}{1.9535} = 0.0786,$$

respectively.