# Numerical Analysis (10th ed)

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# Chapter 6

# Initial-Value Problems for Ordinary Differential Equations

## Chapter 6.1: Linear Systems of Equations*

In this section we review the fact that any linear system of equations can be written in the form $Ax = b$ where A is the coefficient matrix, $x$ is the solution vector, and $b$ is the constant vector. We saw that this form can be represented by the augmented matrix $[A \mid b]$. Elementary row operations are then used to find the solution of the system. Legal row operations consist of swapping rows, multiplying rows by a constant, or adding rows.

Gaussian elimination with backward substitution was introduced as a method that, using the legal row operations, reduces the augmented matrix to row echelon form. That is where the coefficient matrix is upper triangular. Backward substitution is then used to obtain the solutions to the system. It is important to note that this method does NOT require entries of 1 along the diagonal.

Another method, Gauss-Jordan elimination, was discussed in the exercises. This method is very popular since it goes one step further than the previous method in that the legal row operations are used to reduced the augmented matrix to reduced row echelon form. That is the coefficient matrix is a diagonal matrix. form. Although entries of 1 along the diagonal are not required, for smaller systems, it is often useful. However, keep in mind that in doing so, you are adding to the number of computations needed. With larger systems, this will add to the cost of finding the solution as indicated in the texts discussion of operation counts.

## Chapter 6.2: Pivoting Strategies*

Needless to say, because we are using finite digit arithmetic, round-off error can be a huge issue when solving systems numerically. This section addresses those issues with several different pivoting strategies. The development of those strategies is thoroughly discussed in the text so we will simply summarize them here.

**PARTIAL PIVOTING:**

Errors can arise when the pivot element $a_{kk}^{(k)}$, the diagonal entry at the $k^{th}$ step, is small relative to the absolute value of the other entries below it in the matrix. The strategy is to determine the largest entry in absolute value on or below the diagonal and swap the two corresponding rows so that that entry becomes the new pivot element. This is stated mathematically as follows: we need to determine the smallest $p \geq k$ such that $\left|a_{pk}^{(k)}\right| = \max_{k \leq i \leq n}\left|a_{ik}^{(k)}\right|$ and perform the row swap $\left(E_k\right) \leftrightarrow \left(E_p\right)$, if $p \neq k$.

**SCALED PARTIAL PIVOTING:**

This method is used when the first method is not applicable but there are larger elements to the right of the curent pivot element $a_{kk}$. This method places the element that is largest relative to the entries in its row as the new pivot element by swapping the corresponding columns. This is stated mathematically as follows: we need to determine the smallest integer $p \geq i$ such that $\frac{|a_{pi}|}{s_p} = \max_{i \leq k \leq n} \frac{|a_{ki}|}{s_k}$ where $s_i = \max_{1 \leq j \leq n} |a_{ij}|$ and perform the row swap $(E_k) \leftrightarrow (E_p)$.

**COMPLETE PIVOTING:**
Complete (or maximal) pivoting at the kth step searches all the entries $a_{ij}$, for $i = k, k+1, \ldots, n$ and $j = k, k+1, \ldots, n$, to find the entry with the largest magnitude. Both row and column interchanges are performed to bring this entry to the pivot position. The max |*| of all row entries considered must be moved to pivot position. This strategy requires numerous comparisons at each stage and is therefore, only for systems where accuracy is essential and the amount of execution time needed for this method can be justified.

# Chapter 6.3: Linear Algebra and Matrix Inversion

Euler's method is the simplest of the methods. However, it is seldom used in practice due to lack of accuracy, The objective is to obtain approximations to the well-posed initial-value problem $y'(t) = f(t, y),\ a \leq t \leq b,\ y(a) = \alpha$. As we stated in the first section, approximations to $y$ will be generated at equally spaced specified points called **mesh points** in the interval [a,b]. Once we find the approximate solutions at these points, approximations at other points can be obtained by interpolation.

# Chapter 6.4: The Determinant of a Matrix

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# Chapter 6.5: Matrix Factorization*

In this section we discuss how matrix factorization can be used to solve the system $Ax = b$. Two methods that are developed in the section are the LU decomposition, used when no row interchange has been employed. The other method, used when row interchanges have been made, involves a permutation matrix.

The LU decomposition takes the system $Ax = b \rightarrow L(Ux) = b$. We let $y = Ux$.The system Ly = $b$ is solved for $y$ and then the system $Ux = y$ is solved for $x$. From a practical standpoint, this factorization is useful only when row interchanges are not required to control the round-off error resulting from the use of finitedigit arithmetic.

When a row interchange is required, we decompose the matrix $A$ into $A = P^t LU$ where $P$ is the permutation matrix that interchanges the rows.

# Chapter 6.6: Special Types of Matrices*

In this section, we summarize the special types of matrices. Given a matrix A

- A is symmetric if $A = A^t$
- A is singular if $\det A = 0$
- A is diagonally dominant if $|a_{ii}| \geq \sum_{j=1}^{n} |a_{ij}|\ , j \neq i$
- A is strictly diagonally dominant if $|a_{ii}| > \sum_{j=1}^{n} |a_{ij}|\ , j \neq i$. The matrix A always has an LU decomposition.
- A is positive definite if $x^t Ax > 0$ for every n-dimensional vector $x \neq 0$. The matrix A always has special factorizations $LL^t$ and $LDL^t$.
- A tridiagonal matrix occur often and have special factorizations. Refer to the text.