Q1. a) Characteristics

$$Y_i \sim N(\mu_i, \tau^2)$$

$$\mu_i = f(x_i) = \alpha - \beta \gamma^{x_i}$$

$\alpha \sim N(0, \sigma_\alpha^2)$ $\alpha \in (1, \infty)$ Truncated Normal

$\beta \sim N(0, \sigma_\beta^2)$ $\beta \in (1, \infty)$ Truncated Normal

$\gamma \sim \text{Unif}(0,1)$

$\tau^2 \sim IG(a,b)$ Inverse Gamma. $\tau^2 (0, \infty)$

* ~~when α, β, τ^2 follow the co~~

* The parameters α, β, τ^2 obey the conjugacy property.

	Prior	Likelihood	Posterior (conditional)
α	Normal	Normal	Normal
β	Normal	Normal	Normal
τ^2	IG	Normal	IG.

b) $Y_i \sim N(\mu_i, \tau^2)$

$$p(y_1) = \frac{1}{\sqrt{2\pi\tau^2}} \exp\left(-\frac{(y_1 - \mu_1)^2}{2\tau^2}\right)$$

$$p(y_2) = \frac{1}{\sqrt{2\pi\tau^2}} \exp\left(-\frac{1 (y_2 - \mu_2)^2}{2\tau^2}\right)$$

$\vdots$

$$p(y_n) = \frac{1}{\sqrt{2\pi\tau^2}} \exp\left(-\frac{1 (y_n - \mu_n)^2}{2\tau^2}\right)$$

$$p(y_1 y_2 \dots y_n) = p(y_1) \times p(y_2) \times p(y_3) \dots p(y_n)$$

$$= \frac{1}{\sqrt{2\pi\psi^2}} \times \frac{1}{\sqrt{2\pi\psi^2}} \cdots \frac{1}{\sqrt{2\pi\psi^2}} \times \exp\left(\frac{-1}{2\psi^2} \left([(y_1-\mu_1)^2 + (y_2-\mu_2)^2 \cdots (y_n-\mu_n)^2] \right) \right)$$

$$= \frac{1}{(\sqrt{2\pi\psi^2})^n} \times \exp\left(\frac{-1}{2\psi^2} \Sigma (y_i - \mu_i)^2 \right)$$

$$L(y_1 y_2 \cdots y_n \mid \alpha, \beta, \gamma, x_i, \psi^2)$$

$$= \frac{1}{(\sqrt{2\pi\psi^2})^n} \exp\left(- \frac{\Sigma (y_i - (\alpha - \beta\gamma^{x_i})^2)}{2\psi^2} \right)$$

Q1. e) Joint Prior $\rightarrow$

$$p(\alpha, \beta, \gamma, \psi^2) = p(\alpha)\, p(\beta)\, p(\gamma)\, p(\psi^2)$$

$$p(\alpha) = \frac{1}{\sqrt{2\pi\sigma_\alpha^2}} \exp\left(\frac{-\alpha^2}{2\sigma^2} \right)$$

$$p(\beta) = \frac{1}{\sqrt{2\pi\sigma_\beta^2}} \exp\left(\frac{-\beta^2}{2\sigma_\beta^2} \right)$$

$$p(\gamma) = 1$$

$$p(\psi^2) = (\psi^2)^{-a-1} \exp\left(\frac{-b}{\psi^2} \right)$$

$$p(\alpha)\, p(\beta)\, p(\gamma)\, p(\psi^2) = \frac{1}{\sqrt{2\pi\sigma_\alpha^2}} \frac{1}{\sqrt{2\pi\sigma_\beta^2}} \exp\left(\frac{-\alpha^2}{2\sigma_\alpha^2} - \frac{\beta^2}{2\sigma_\beta^2} - \frac{b}{\psi^2} \right) \times (\psi^2)^{-a-1}$$

The hyper parameters σ_α^2, σ_β^2 should be such that enough samples can be generated from the right hand side of the normal distribution as we want α samples & β samples to be greater than 1.

So standard deviation should be seen that the tail of the normal distribution should not be less than 1 i.e. normal should not be too narrow.

$\sigma_\alpha = 5$ $\sigma_\beta = 2$

~~we want x to be distributed~~

Q2a) ~~Y & J are joint~~ $\cancel{P_r(Y = y^A) = ?}$

Aim: Generate R.V. belonging to a target distro $f_X(x)$. $f_X(\ )$

Assumption:

$$\boxed{f_X(y) \leq k\, q(y)}$$

where $q(y)$ is a bounding density.

we either accept or reject the samples generated based on the bounding condition.

$$\therefore \quad E = \begin{cases} 1 & \text{Accept} \\ 0 & \text{Reject} \end{cases} \text{ Bernoulli event}$$

So Accepting or rejecting the sample is now associated to this Bernoulli event.

$$P_r(E = 1 \mid Y = y) = \frac{f_X(y)}{k\, f_u(y)}$$

$$0 \leq \frac{f_X(y)}{k\, f_u(y)} \leq 1$$

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So, $E = \text{Ber}\left(\frac{f_x(y)}{k f_u(y)} \right)$

If $E=1$, Accept $Y = X$.

⇒ Y, E are jointly related on values of Y, E is either 0 or 1.

$(Y,E) \sim f_{Y,E}(y, e)$

$$f_{YE}(y,e) = \underbrace{f(y)}_{\text{marginal}} \times \underbrace{f(E \mid Y=y)}_{\text{Conditional}}$$

$$\boxed{\underbrace{f_{\cancel{YE}}(Y,E)}_{\text{Joint}} = f_u(y) \times \frac{f_x(y)}{k f_u(y)}}$$

Now we have Joint of Y & E, we can find $f(Y \mid E=1)$

$$f(y \mid E=1) = \frac{f_{joint}(y,e)}{f(e)} = \frac{f_{joint}(y,e)}{\Pr(E=1)}$$

$$= \frac{\cancel{f_u(y)} \times f_x(y) / k \cancel{f_u(y)}}{\Pr(E=1)}$$

$$= \frac{f_x(y)}{\Pr(E=1)} \propto \bar{f}_x(x).$$

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$Pr(E=1)$ is the acceptance probability we can get it from Joint distro.

$$Pr(E=1) = \int f_{joint}(y, E)\, dy$$

$$= \int f_u(y) \times \frac{f_x(y)}{k\, f_u(y)}\, dy$$

$$= \int_{-\infty}^{\infty} f_x(y)/k\, dy.$$

$$= \frac{1}{k} \int_{-\infty}^{\infty} f_x(y)\, dy \qquad = \boxed{\frac{1}{k}}$$

Q2 (b) In the previous derivation,

$$f(y \mid E=1) = \frac{f_x(y)}{Pr(E=1)} = \frac{f_x(y)}{1/k} = k\, f_x(y)$$
$$= k\, f_x(X)$$

$$\Rightarrow \alpha \quad f_x(\cancel{y})$$

This means the simulated R.V. has same shape as $f_x(X)$.

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Q2. c) Compare two cases when we have the posterior distro without normalizing factor & with normalizing factor.

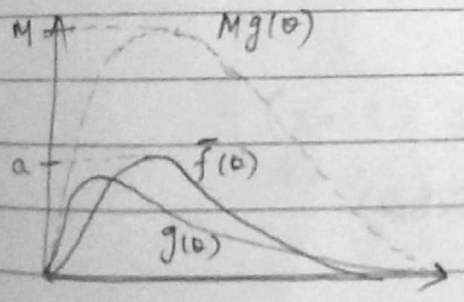
$$f(\theta) = \frac{f(y_1 \ldots y_n \mid \theta)\, g(\theta)}{f(y_1 \ldots y_n)} = k\, f(y_1 \ldots y_n \mid \theta)\, g(\theta)$$

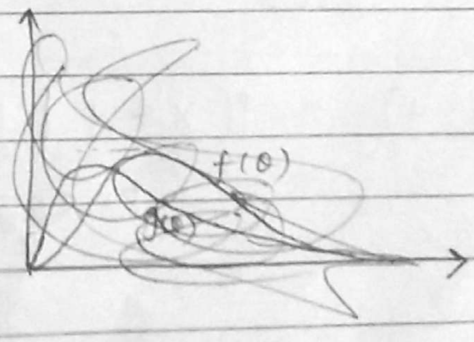
$$= k \bar{f}(\theta)$$

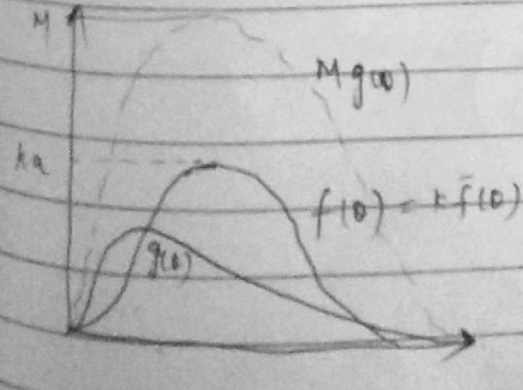
where $\bar{f}(\theta) = f(y_1 \ldots y_n \mid \theta)\, g(\theta)$

BOUNDING CONDITION:

Suppose we know a constant M which makes the prior distro $g(\theta)$ greater than both $f(\theta)$ & $\bar{f}(\theta)$.







Any Random number U drawn from $U[0,1]$ must be scaled to the level $M \times g(\theta)$.

This number will be accepted if falls below our target density.

$\therefore$ for this $\quad U \times Mg(\theta) <$ target

$$\underbrace{U \times Mg(\theta) < k\bar{f}(\theta)}$$

$\downarrow$

when the complete posterior is available

$$\underbrace{UMg(\theta) < \bar{f}(\theta)}$$

$\downarrow$

when only numerator (Likelihood $\times$ prior) is available

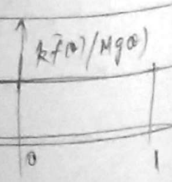
$X \sim g(\theta) \quad$ prior

$$\boxed{P(Y \le y) = P(X \le y \mid U\,Mg(\theta) \le k\bar{f}(\theta))}$$

$$= \frac{P(X \le y,\ U\,Mg(\theta) \le k\bar{f}(\theta))}{P\left(U \le \frac{k\bar{f}(\theta)}{Mg(\theta)}\right)}$$

$$P(A \mid B) = \frac{P(A,B)}{P(B)}$$

$$= \frac{\displaystyle\int_{-\infty}^{y} \int_{0}^{k\bar{f}(\theta)/Mg(\theta)} 1 \, du \ \underbrace{g(\theta)\, d\theta}_{\text{Prior}}}{\underbrace{\displaystyle\int_{-\infty}^{\infty} \int_{0}^{k\bar{f}(\theta)/Mg(\theta)} 1 \, du \ g(\theta)\, d(\theta)}_{\text{Marginals}}}$$



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$$= \frac{\displaystyle\int_{-\infty}^{y} u\Big|_0^{k\bar{f}(\theta)/Mg(\theta)} g(\theta)\, d\theta}{\displaystyle\int_{-\infty}^{\infty} u\Big|_0^{k\bar{f}(\theta)/Mg(\theta)} g(\theta)\, d\theta}$$

$$= \frac{\displaystyle\int_{-\infty}^{y} \frac{k\bar{f}(\theta)}{M\cancel{g(\theta)}} \times \cancel{g(\theta)}\, d\theta}{\displaystyle\int_{-\infty}^{\infty} \frac{k\bar{f}(\theta)}{M\cancel{g(\theta)}} \cancel{g(\theta)}\, d\theta}$$

$$= \frac{\displaystyle\int_{-\infty}^{y} \cancel{\frac{k}{M}}\, \bar{f}(\theta)\, d\theta}{\displaystyle\int_{-\infty}^{\infty} \cancel{\frac{k}{M}}\, \underbrace{\bar{f}(\theta)\, d\theta}_{1}} = \int_{-\infty}^{y} \bar{f}(\theta)\, d\theta$$

(we can see k gets cancelled)

Thus the generated R.V will have a distribution of $\bar{f}(\theta)$ i.e. $f(y_1, \dots y_n|\theta)\, g(\theta)$ where we don't need to know the proportionality constant.

Q2 d) 1) Suppose I have the following likelihood function & prior.

$p(\theta) \sim \text{dbeta}(a,b) \; \{a \neq 0,\; b \neq 1\}$

$p(x_1, \dots x_n|\theta) \sim \text{Binomial}(n, \theta)$

Posterior $P(\theta | x_1, \dots x_n) = \dfrac{P(x_1, \dots x_n | \theta)\, P(\theta)}{P(x_1, \dots x_n)}$

$= \text{dbeta}(\theta, a + \Sigma x_i, b + n - \Sigma x_i)$

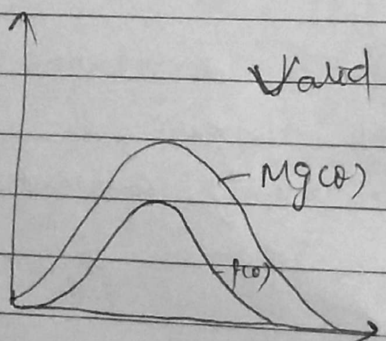
If we wish to sample from our prior distro, it can't be done analytically!

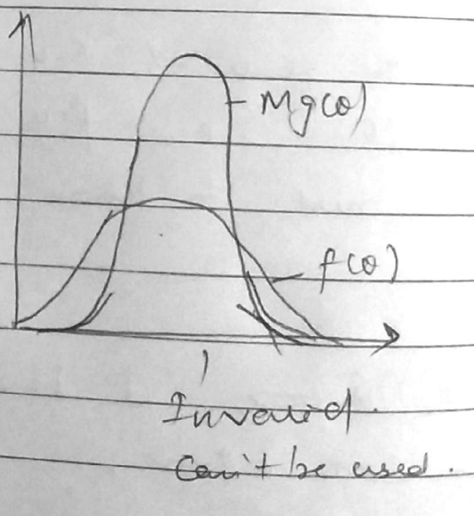
Because for a Beta density the sampling function doesn't exist (Just like normal density)

The integral over density of beta is not possible analytically.

2) It has to be ensured that $Mg(\theta) \geq f(\theta)$ over all the support of $f(\theta)$.

The tails of $g(\theta)$ has to be somehow cover the tails of $f(\theta)$.





Q3. Cauchy

$$f(x, 0, 1) = \frac{1}{\pi(1+x^2)}$$

Normal

$$f(x \mid 0, 1) = \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-x^2}{2}\right)$$

Acceptance Probability =>

$$\Pr(EB=1 \mid Y=y) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \Big/ k \times \left(\frac{1}{\pi(1+x^2)}\right)$$

$\therefore$ Acceptance Probability $= 1/k$

for $k = 1.5$,

$$= \frac{10^2}{15\,3} = \boxed{\frac{2}{3}}$$

Q4 d) Transition Matrix

$$P = \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{bmatrix}$$

where,

$\pi = \{\pi_1, \pi_2, \pi_3\}$ $S = \{1, 2, 3\}$

The stationary distro π must satisfy the following equations -

$$\pi_1 P_{11} + \pi_2 P_{21} + \pi_3 P_{31} = \pi_1$$
$$\pi_1 P_{12} + \pi_2 P_{22} + \pi_3 P_{32} = \pi_2$$
$$\pi_1 P_{13} + \pi_2 P_{23} + \pi_3 P_{33} = \pi_3$$

which is equiv to

$$P^T \pi = P$$
$$(P^T - \lambda I)\pi = 0$$

~~so~~, there must one value of λ that is equal to 1 for sure.

Q5. a) $y_i \sim N(\mu_i, \tau^2)$

$$L_y(\alpha, \beta, \gamma, \tau^2) = \frac{1}{(\sqrt{2\pi\tau^2})^n} \exp\left(-\frac{\sum_{i=1}^{n}(y_i - \mu_i)^2}{2\tau^2}\right)$$

$\downarrow$

Likelihood

$$\text{Prior}(\alpha, \beta, \gamma, \tau^2) = P(\alpha)\, P(\beta)\, P(\gamma)\, P(\tau^2)$$
$$\propto \exp\left(\frac{-\alpha^2}{2\sigma_\alpha^2}\right) \exp\left(\frac{-\beta^2}{2\sigma_\beta^2}\right) (\tau^2)^{-a-1} \exp\left(\frac{-b}{\tau^2}\right)$$

Joint $\Rightarrow$ $L_y(\alpha, \beta, \gamma, \tau^2, x_i) \times \text{Prior}(\alpha, \beta, \gamma, \tau^2)$

Conditional for ~~exp~~ α. $\rightarrow$

$$\pi(\alpha \mid \beta, \gamma, \tau^2, y_i) = \frac{\pi(\alpha, \beta, \gamma, \tau^2 \mid y_i)}{\pi(\gamma, \beta, \tau^2 \mid y_i)}$$

All the terms in the joint having only α.

$$= \exp\left(-\frac{\sum(y_i - \mu_i)^2}{2\tau^2} - \frac{\alpha^2}{2\sigma_\alpha^2}\right)$$

$$\sum(y_i - \mu_i)^2 \Rightarrow \sum(y_i - \alpha + \beta\gamma^{x_i})^2$$
$$= \sum(-\alpha + y_i + \beta\gamma^{x_i})^2$$
$$= \sum\left(\alpha^2 + (y_i + \beta\gamma^{x_i})^2 - 2\alpha(y_i + \beta\gamma^{x_i})\right)$$
$$= \alpha^2 n + \sum_{i=1}^{n}(y_i + \beta\gamma^{x_i})^2 - 2\alpha\sum(y_i + \beta\gamma^{x_i})$$

$$\rightarrow = \exp\left(-\frac{\left(\alpha^2 n + \sum_{i=1}^{n}(y_i + \beta\gamma^{x_i})^2 - 2\alpha\sum(y_i + \beta\gamma^{x_i})\right)}{2\tau^2} - \frac{\alpha^2}{2\sigma_\alpha^2}\right)$$

$$= \exp\left(\frac{-\left[\sigma_\alpha^2 \alpha^2 n - 2\alpha\sigma_\alpha^2 \Sigma(y_i + \beta\gamma^{x_i}) + \sigma_\alpha^2 \Sigma(y_i+\beta\gamma^{x_i})^2 + \tau^2\alpha^2\right]}{2\tau^2\sigma_\alpha^2}\right)$$

$$= \exp\left(\frac{-\left[\alpha^2(\sigma_\alpha^2 n + \tau^2) - 2\alpha\,\sigma_\alpha^2 \Sigma(y_i+\beta\gamma^{x_i}) + \sigma_\alpha^2 \Sigma(y_i+\beta\gamma^{x_i})^2\right]}{2\tau^2\sigma_\alpha^2}\right)$$

$$= \exp\left(\frac{-\left[\alpha^2 - \frac{2\alpha\,\sigma_\alpha^2 \Sigma(y_i+\beta\gamma^{x_i})}{(\sigma_\alpha^2 n + \tau^2)} + \frac{\sigma_\alpha^2 \Sigma(y_i+\beta\gamma^{x_i})^2}{(\sigma_\alpha^2 n + \tau^2)}\right]}{\left(\frac{2\tau^2\sigma_\alpha^2}{\sigma_\alpha^2 n + \tau^2}\right)}\right)$$

$$= \exp\left(\frac{-\left[\left(\alpha - \left(\frac{\sigma_\alpha^2 \Sigma(y_i+\beta\gamma^{x_i})}{\sigma_\alpha^2 n + \tau^2}\right)\right)^2\right]}{\left(2\frac{\tau^2\sigma_\alpha^2}{\sigma_\alpha^2 n + \tau^2}\right)}\right) \exp\left(\left(\quad\right)\right)$$

Not relevant = constant

$$= \sim N\left(\frac{\sigma_\alpha^2 \Sigma(y_i+\beta\gamma^{x_i})}{\tau^2 + n\sigma_\alpha^2}, \frac{\tau^2\sigma_\alpha^2}{\sigma_\alpha^2 n + \tau^2}\right)$$

$$\pi(\beta \mid \alpha, \gamma, \psi^2, y_i) = L_a(\alpha, \beta, \gamma, \psi^2) \times prior(\beta)$$

$$= \exp\left(\frac{\Sigma(y_i - \mu_i)^2}{2\psi^2} - \frac{\beta^2}{2\sigma_\beta^2} \right)$$

$$\Sigma(y_i - \mu_i)^2 = \left(\Sigma(\beta\gamma^{x_i}) + (y_i - \alpha) \right)^2$$

$$= \Sigma \beta^2 \gamma^{2x_i} + (\Sigma(y_i - \alpha))^2 + 2\beta \Sigma \gamma^{x_i}(y_i - \alpha)$$

$$= \beta^2 \Sigma \gamma^{2x_i} + \Sigma(y_i - \alpha)^2 + 2\beta \Sigma \gamma^{x_i}(y_i - \alpha)$$

$$= \beta^2 + \frac{\Sigma(y_i - \alpha)^2}{\Sigma \gamma^{2x_i}} + 2\beta \Sigma(y_i - \alpha)\gamma^{-x_i}$$

$$= \exp\left(- \frac{(\quad)}{2\psi^2} - \frac{\beta^2}{\sigma_\beta^2} \right)$$

$$= \exp\left(- \frac{\left[\beta^2(\sigma_\beta^2 + \psi^2) + 2\beta\sigma_\beta^2 \Sigma(y_i - \alpha)\gamma^{-x_i} + \sigma_\beta^2 \frac{\Sigma(y_i - \alpha)^2}{\Sigma \gamma^{2x_i}} \right]}{2\psi^2\sigma_\beta^2} \right)$$

$$= \exp\left(- \frac{\left[\beta^2 + \frac{2\beta\sigma_\beta^2 \Sigma(y_i - \alpha)\gamma^{-x_i}}{\sigma_\beta^2 + \psi^2} + \frac{\sigma_\beta^2 \Sigma(y_i - \alpha)^2}{\Sigma \gamma^{2x_i}(\sigma_\beta^2 + \psi^2)} \right]}{\frac{2\psi^2\sigma_\beta^2}{\sigma_\beta^2 + \psi^2}} \right)$$

$$= \exp\left(- \frac{\left[\left(\beta - \frac{\sigma_\beta^2 (\Sigma(\alpha - y_i)\gamma^{-x_i})}{\psi^2 + \sigma_\beta^2} \right)^2 \right]}{2 \frac{\psi^2 \sigma_\beta^2}{\psi^2 + \sigma_\beta^2}} \right)$$

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$$\approx N\left(\frac{\sigma_\beta^2\ \Sigma(\cancel{\alpha}-y_i)\gamma^{-x_i}}{\sigma_\beta^2+\tau^2},\ \frac{\tau^2\sigma_\beta^2}{\tau^2+\sigma_\beta^2}\right)$$

$$\pi(\gamma|\alpha,\beta,\tau^2) = L_y(\alpha,\beta,\gamma,\tau^2) \times p(\gamma)$$

$$= L_y() \times 1$$

$$= \exp\left(-\frac{\left[\beta^2\Sigma\gamma^{2x_i} + 2\beta\,\Sigma\gamma^{x_i}(y_i-\alpha) + \Sigma(y_i-\alpha)^2\right]}{2\tau^2}\right)$$

$$= \exp\left(-\frac{\left[\Sigma(\gamma^{x_i})^2\beta^2 + 2\,\Sigma\gamma^{x_i}(y_i-\alpha)\beta + \Sigma(y_i-\alpha)^2\right]}{2\tau^2}\right)$$

$\downarrow$

$\boxed{\text{Not a recognizable kernel}}$

$$\pi(\tau^2|\alpha,\beta,\gamma) = L_y(\ ) \times p(\tau^2)$$

$$= \frac{1}{(\sqrt{2\pi\tau^2})^n}\exp\left(-\frac{1}{2}\frac{\Sigma(y_i-\mu_i)^2}{\tau^2}\right) \times (\tau^2)^{-a-1}\exp\left(\frac{-b}{\tau^2}\right)$$

$$= \frac{(\tau^2)^{-a-1}}{(\sqrt{2\pi\tau^2})^n}\exp\left(-\frac{\overbrace{\Sigma(y_i-\mu_i)^2}^{k}}{2\tau^2} - \frac{b}{\tau^2}\right)$$

$$\approx (\tau^2)^{-a-n-1}\exp\left(-\left(\frac{k+2b}{2\tau^2}\right)\right)$$

$$= (\tau^2)^{-a-n-1}\exp\left(\frac{-(k+2b)/2}{\tau^2}\right)$$

$$\approx IG\left(a+n,\ \frac{2b+\Sigma(y_i-\mu_i)^2}{2}\right)$$