2. Point Estimation.

let X be the Bernoulli random Variable with parameter p.
let $x_1, x_2 \dots x_n$ be the independent random sample of X

Probability Density Function for Bernoulli distribution with parameter p is

$$f(x) = p^{x}(1-p)^{1-x} \quad \text{where } x = 0, 1$$

likelihood function of sample is:

$$f(x_1, x_2, x_3 \dots x_n, p) = \prod_{i=1}^{n} f(x_i, p)$$

x_i are indepen

$$= \prod_{i=1}^{n} p^{x_i}(1-p)^{1-x_i} \qquad \because x_i = \sum_{i=1}^{n} x$$

$\Rightarrow$ Taking natural log on both side.

$$L(x_1, x_2, x_3 \dots x_n, p) = \sum x_i \log(p) + (n - \sum x_i)\log(1-p)$$

taking partial derivative.

$$\frac{\partial \log L[P]}{\partial p} = \frac{\partial}{\partial p} \Sigma x_i \log P + \frac{\partial}{\partial p} (n - \Sigma x_i) \log(1-P)$$

also setting it to 0.

$$= \frac{\Sigma x_i}{P} - \left(\frac{n - \Sigma x_i}{1-P} \right) = 0.$$

Multiplying by $P(1-P)$.

$$\Sigma x_i (1-p) - (n - \Sigma x_i) p = 0.$$

$$\Sigma x_i - p \Sigma x_i - np + p \Sigma x_i = 0$$

$$\Sigma x_i - np = 0.$$

$$\hat{p} = \frac{\Sigma x_i}{n} = \hat{x}_n$$

2- Probability Density Function with parameter P. is

$$f(x) = \binom{n}{x} p^x (1-p)^{n-x} \quad \text{where } x = 0, \dots n$$

∴ the likelihood function of the Sample is.

$$f(x_1 \dots x_m, P) = \prod_{i=1}^{m} f(x_i, P) \quad (x_i \text{ are independent})$$

$$= \prod_{i=1}^{m} \binom{n}{x_i} p^{x_i} (1-p)^{n-x_i}$$

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Taking natural log om B.S.

$$\log L(x_1 \ldots \ldots x_m, P) = \sum_{i=1}^{m} \log\binom{n}{x_i} + \sum_{i=1}^{m} x_i \log(P) + \left(mn - \sum_{i=1}^{m} x_i\right) \log(1-P)$$

Taking partial derivative and setting it to 0.

$$\frac{\partial L(x_1, \ldots x_m, P)}{\partial p} = \frac{\partial}{\partial p} \sum \log\binom{n}{x_i} + \frac{\partial}{\partial p} \sum x_i \log p + \frac{\partial}{\partial p} \left(mn - \sum_{i=1}^{m} x_i\right) \log(1-p) = 0$$

$$= 0 + \frac{\sum_{i=1}^{m} x_i}{P} - \frac{mn - \sum_{i=1}^{m} x_i}{1-P} = 0.$$

multiply by $P \cdot (1-P)$.

$$\sum_{i=1}^{m} x_i (1-P) - \left(mn - \sum_{i=1}^{m} x_i\right) P = 0.$$

$$\sum_{i=1}^{m} x_i - P \sum_{i=1}^{m} x_i - Pmn + P \sum_{i=1}^{m} x_i = 0.$$

$$\hat{P} = \frac{\sum_{i=1}^{m} x_i}{m \times n}.$$

using Formula

Problem $\hat{p} = \frac{3+6+2+0+0+3}{6 \times 10} = \frac{14}{60} = 0.2333$

3. let x be the uniform random Variable. with parameter a and b.
let $x_1, x_2 \ldots x_n$ be the independent random sample of x drawn from a uniform distribution $U(a,b)$ of sample size n

Probability distribution function for uniform distribution with parameter a and b is:

$$f(x) = \frac{1}{b-a} \text{ on } [a,b].$$

likelihood Function

$$f(x_1, \ldots x_n, a, b) = \begin{cases} \left(\frac{1}{b-a}\right)^n & \text{if all } x \text{ are in the interval } [a,b] \\ 0 & \text{otherwise.} \end{cases}$$

This can be maximized by making b-a as small as possible with restriction that the interval [a,b] must include all the data.

∴ MLE for pair (a,b) is.

$\hat{a} = \min(x_1 \ldots x_n)$, $\hat{b} = \max(x_1 \ldots x_n)$

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4. Probability Distiribution Function for: Known variance σ^2 and unknown μ mean with sample Size n.

$$f(x_i, \mu, \sigma^2) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{(x_i-\mu)^2}{2\sigma^2}\right]$$

$$= \prod_{i=1}^{n} \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{(x_i-\mu)^2}{2\sigma^2}\right].$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{\sum_{i=1}^{n}(x_i-\mu)^2}{2\sigma^2}\right].$$

Taking natural log on B.S.

$$L(\mu,\sigma) = \sum_{i=1}^{n}\left[-\log\sigma - \frac{1}{2}\log 2\pi - \frac{1}{2\sigma^2}(x_i-\mu)^2\right].$$

$$= -n\log\sigma - \frac{n}{2}\log 2\pi - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2$$

To find maximum value take partial derivative of the above function with μ and setting it to 0

$$\frac{\partial L(\mu,\sigma)}{\partial\mu} = \frac{1}{\sigma^2}\sum_{i=1}^{n}(x_i-\mu) = 0.$$

$$= \sum_{i=1}^{n} (x_i - \mu) = 0.$$

$$= \sum_{x=1}^{n} x_i - \mu \sum_{i=1}^{n} 1 = 0.$$

$$\therefore \hat{\mu} = \frac{\sum x_i}{n} = \bar{x}$$

5. Probability distribution. Function. for. Normal distribution for know mean μ and unknown Variance σ^2 with sample of size n.

$$f(x_i|\mu,\sigma^2) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{(x_i-\mu)^2}{2\sigma^2}\right].$$

$$= \prod_{i=1}^{n} \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{\sum_{i=1}^{n}(x_i-\mu)^2}{2\sigma^2}\right]$$

Taking natural logarithm on the B.S.

$$L(\mu,\sigma^2) = -n\log\sigma - \frac{n}{2}\log 2\pi - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2$$

To maximize value ~~can be~~ we take partial derivative with respect to σ^2 and setting it to 0.

$$\frac{\partial L}{\partial \sigma^2} = \frac{-n}{2\sigma^2} + \left(\sum \frac{(x_i-\mu)^2}{2}\right)\frac{1}{(\sigma^2)^2} = 0.$$

$\sigma^2 \rightarrow L(\mu, \sigma^2)$ is maximized when.

$$\sigma^2 = \sum \frac{(x_i - \bar{x})^2}{n} = \overline{(x_i - \bar{x})^2}$$

$$\hat{\sigma} = \sqrt{\overline{(x_i-\bar{x})^2}}.$$

6. let data $x_1, x_2 \ldots x_n$ are drawn from a $N(\mu, \sigma^2)$ distribution. where μ and σ are unknown.

Probability Density function for normal distribution is.

$$f(x_i \mid \mu, \sigma^2) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{(x_i-\mu)^2}{2\sigma^2}\right]$$

$\because$ x_i are independent their joint pdf. is the product of the individual pdf

$$= \left(\frac{1}{\sigma\sqrt{2\pi}}\right)^n \exp\left[\frac{\sum_{i=1}^{n}(x_i-\mu)^2}{2\sigma^2}\right].$$

Taking log on B.S.

$$L(\mu,\sigma) = -n\log\sigma - \frac{n}{2}\cdot\log 2\pi - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2$$

$\because$ $L[\mu,\sigma]$ is a function of the two unknown variable μ, σ.

→ Mean of the Data.

To maximize the value of $\hat{\mu}$ take partial derivative of Eq 1 w.r.t μ and setting it 0.

$$\frac{\partial L(\mu,\sigma)}{\partial \mu} = \sum_{i=1}^{n}\frac{(x_i-\mu)}{\sigma^2} = 0$$

$$= \sum_{i=1}^{n} x_i = n\mu$$

$$\boxed{\hat{\mu} = \frac{\sum_{i=1}^{n} x_i}{n} = \bar{x}}$$ – the mean of the data

Variance of the data.

to maximum value of $\hat{\sigma}^2$ take partial derivative of Eq(1) w.r.t σ.

$$\frac{\partial L(\mu,\sigma)}{\partial \sigma} = \frac{-n}{\sigma} + \sum_{i=1}^{m} \frac{(x_i-\mu)^2}{\sigma^3} = 0.$$

$$\boxed{\tilde{\sigma}^2 = \frac{\sum_{i=1}^{n} (x_i-\mu)^2}{n.}}$$

2. Coin and thumbt

1. MLE for the thumbtack and coin will be same.

No of heads = 60 } Total = 100 trials.
No of trails = 40 }

tossing a thumtack.

Probability distribution function of Binomial distribution.

$$P(D_1 | \theta_1) = \theta_1^{\alpha_H} (1-\theta_1)^{\alpha_T}.$$

$$\theta_1 = \arg \max_{\theta_1} \log \left(\theta^{\alpha_H} (1-\theta_1)^{\alpha_T}\right).$$

To maximime value of θ do partial derivative and set it to 0.

$$= \frac{\partial}{\partial \theta_1} \left[\alpha_H \log \theta_1 + \alpha_T \log (1-\theta_1)\right] = 0.$$

$$= \alpha_H \frac{\partial}{\partial \theta_1} \log \theta_1 + \alpha + \frac{\partial}{\partial \theta_1} \log (1-\theta_1) = 0.$$

$$= \frac{\alpha_H}{\theta_1} - \frac{\alpha_T}{1-\theta_1} = 0.$$

$$(1-\theta_1)\alpha_H - \theta_1\alpha_T = 0.$$

$$\alpha_H - \theta_1\alpha_H - \theta_1\alpha_T = 0$$

$$\alpha_H - \theta_1(\alpha_H + \alpha_T) = 0$$

$$\hat{\theta}_1 = \frac{\alpha_H}{\alpha_H + \alpha_T}.$$

$\alpha_H = 60$ $\qquad$ $\alpha_T = 40$

$$\theta_1 = \frac{60}{60+40} = \frac{60}{100} = \frac{3}{5} = 0.6.$$

$\therefore$ MLE for the coin and thumbtack will be same.