Vector Spaces

V – nonempty set.
F – field F – restricted to $\mathbb{R}$
Addition and scalar multiplication.

Conditions:

1) let $v_1, v_2 \in V$, $v_1 + v_2 \in V$
2) let $v_1 + v_2 = v_2 + v_1$ $\forall$, $v_1, v_2 \in V$
3) $(v_1 + v_2) + v_3 = v_1 + (v_2 + v_3)$ $\forall$ $v_1, v_2, v_3 \in V$
4) $\exists$ $0 \in V$ such that $0 + v = v + 0$, $\forall$ $v \in V$ Identity element
5) $\forall$ $v \in V$, $\exists$ a $-v \in V$, such that $v + (-v)$ $= (-v) + v = 0$
6) for $v \in V$, $c \in R$, $cv \in V$
7) $c(v_1 + v_2) = cv_1 + cv_2$, $v_1, v_2 \in V$ $c \in R$
8) $(c_1 + c_2)v = c_1 v + c_2 v$ $v \in V$, $c_1, c_2 \in R$.
9) $(c_1 c_2)v = c_1(c_2 v)$, $v \in V$, $c_1, c_2 \in R$
10) $\exists$ $1 \in R$, $1 \cdot v = v \cdot 1 = v$ $\forall$ $v \in V$

V is a vectorspace over R(F)

eg(1) $V = R^n$ over R is a vector space

$\underline{v_1} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ $\overline{v_2} = \begin{pmatrix} y_1 \\ \vdots \\ y_2 \end{pmatrix}$

$v_1 + v_2 = \begin{pmatrix} x_1 + y_1 \\ \vdots \\ x_n + y_n \end{pmatrix}$

$cv_1 = \begin{pmatrix} cx_1 \\ \vdots \\ cx_n \end{pmatrix}$

R^n over $\mathbb{C}$ is not a vector space.

$\mathbb{C}$ over R is a vector space.

2) $M_{m\times n}$ → set of all $m\times n$ matrices with real entries.

$M_{m\times n} \to R$

$A = [a_{ij}]$ $A+B = [a_{ij} + b_{ij}]$

$B = [b_{ij}]$ $CA = [ca_{ij}]$

Square matrices is also a vector space

3) Set of ~~a~~ nonsingular ~~matrix~~ matrices order $M_{n\times n}$

$|M| \neq 0$ over R is not a vector space

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4) Set of all poly. of degree $\geq n$ over real no.

14.3.18 $V = R^2$ over R

$\oplus$ $*$

$v_1, v_2 \in R^2$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \oplus \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 + 2y_1 \\ x_2 + 2y_2 \end{pmatrix}$$

$$c * \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} cx_1 \\ cx_2 \end{pmatrix}$$

Checking conditions:

$v_1 + v_2 \in V$ ✓

$v_1 + (v_2 + v_3) \neq (v_1 + v_2) + v_3$

Addition many of the properties are not satisfied.

Q. R^2 where addition is defined as $v_1 + v_2 = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \end{pmatrix}$ but

$$c * v_1 = \begin{pmatrix} cx_1 \\ y_1 \end{pmatrix}$$

Q. Considering, V = set of all real valued continuous functions defined on $[-1,1]$ over R.

$f, g \in V$

$(f+g)x = f(x)+g(x)$
$(c*f)x = cf(x)$

$[f+(g+h)x] = f(x)+(g+h)x$
$= f(x)+g(x)+h(x)$
$= [f(x)+g(x)]+h(x)$

Associativity is satisfied.

$(f+(-f))x = f(x)+(-f)x$
$= f(x)+[-f(x)] = 0$

$((c_1+c_2)f)x = (c_1+c_2)f(x) = c_1 f(x)+c_2 f(x)$
$= (c_1 f+c_2 f)x \quad \forall x \in [-1,1]$

$\therefore$ is a vector space over R.

$f(0)=1$

15.8.18 V = set of all $m \times n$ matrices with entries as real no. $\Downarrow$ vector space over

Let $A = [a_{ij}]$ $B = [b_{ij}]$

$A + B = [a_{ij} + b_{ij}]$

$cA = [ca_{ij}]$

$$0 = \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & & \\ 0 & & 0 \end{bmatrix}_{m \times n}$$

$-A = [-a_{ij}]$

$A + -A = 0$

$$c[A+B] = [c(a_{ij}+b_{ij})] = [ca_{ij} + cb_{ij}]$$

$$= c[a_{ij}] + c[b_{ij}]$$

$$= cA + cB$$

Set of all square $(n \times n)$ matrices real entries is a vector space

Set of all square "
(Singular matrices) $|A| = 0$ $\Longrightarrow$ Not a vector space as closure property is not satisfied.

let $A, B \in V$

$|A| = 0$ $|B| = 0$

$|A+B| \neq |A| + |B|$

Eg: V = set of twice differential fns defined on $\mathbb{R}$, such that $f(x) + f''(x) = 0$
(usual addition & scalar multiplication)

$(f+g)x = f(x) + g(x)$

Zero fn $0(x) = 0$

$$(f+(-f))x = f(x) + (-f)x$$
$$= f(x) + [-f(x)] = 0$$

$$(f+g)x + (f+g)''x = 0$$
$$f(x) + g(x) + f''(x) + g''(x) = 0$$

$$c\,f(x) + (c f'')x = c f(x) + c\,(f''(x))$$
$$= c\,[f(x) + f''(x)] = 0$$

$\therefore$ It is a vector space.

Subspace of a vector space.

V – a vector space

let $S \subset V$

and if S is a vector space.

then S is called a subspace.

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Let $V = R^2$.

$S = \{ (x, y) \in R^2 / y = x \}$

$= \{ (1,1), (2,2) \ (\frac{1}{2}, \frac{1}{2}) \ (\frac{-3}{2}, \frac{-3}{2}) \cdots \}$

$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$

$(x_1, x_1) + (x_2, x_2) = (x_1 + x_2, x_1 + x_2)$

$c(x_1, x_1) = (cx_1, cx_1)$

\# $v_1, v_2 \in S$

$v_1 + v_2 \in S$ if it is closed under

$cv_1 \in S$ additional & scalar multiplication.

$v_1, v_2 \in S$

$c_1, c_2 \in R$

$\boxed{c_1 v_1 + c_2 v_2 \in S}$

$S = \{ (x, y) \ / \ y = mx \}$

$(x_1, mx_1) \quad (y_1, y_1)$

$x_1 + y_1, + mx_1 + y_1$

It is a vector space

$S_2 = \{(x, y) / y = 2x + 3\}$
Not a vector space

$S_3 = \{(0,0)\}$ is a subspace of R^2
↳ Zero subspace or the trivial subspace

$R^3 \rightarrow$ plane passing through the origin.
$S \subset R^3$, $S = \{(x, y, z) / ax + by + cz = 0\}$

$v_1 = (x_1, y_1, z_1)$
$v_2 = (x_2, y_2, z_2)$
$v_1 + v_2 = (x_1 + x_2, y_1 + y_2, z_1 + z_2)$

$$a(x_1 + x_2) + b(y_1 + y_2) + c(z_1 + z_2) = (ax_1 + by_1 + cz_1) + (ax_2 + by_2 + cz_2) = 0 + 0$$

$\alpha v_1 = (\alpha x_1, \alpha y_1, \alpha z_1)$
$\cancel{a(\alpha x_1) + b(\alpha y_1) + c}$
$\alpha(ax_1 + ay_1 + cz_1) = 0$

$S \subset R^3$
$S = \{(x, y, z) / xyz = 0\} \Rightarrow$ Hence not a ~~subspace~~

$(x_1 + x_2)(y_1 + y_2)(z_1 + z_2) \neq 0$

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1) $S = \{ (x, y, z) \mid x^2 + y^2 = z^2 \}$

2) $S = \{ v = R^3 / v^T w_1 = v^T w_2 = 0 \}$
Where w_1 and w_2 are 2 fixed vectors in R^2

19.3.16 $V = C[a,b]$ - set of all real valued cont - fns. defined on $[a,b]$

1) $S \subset V$; set of all differentiable fns -
for any $f, g \in S$, $f+g \in S$
$\alpha f \in S$, $\alpha \in R$.
S is a subspace of V.

ii) let S be a subset $\rightarrow f$ is an odd fn
$f(-x) = -f(x) \quad g(-x) = -g(x)$

$$(f+g)(-x) = f(-x) + g(-x) = -f(x) + -g(x)$$
$$= -[f(x) + g(x)]$$
$$= -(f+g)(x)$$

$$(\alpha f)(-x) = \alpha[f(-x)] = \alpha[-f(x)] = -\alpha[f(x)]$$
$$= -(\alpha f)x$$

Eg: V - set of all polynomials of degrees $\leq n$ (with real coefficients)

$S \subset V$ polynomials with integers coeff. $\Rightarrow$ Not a subspace.

$a_0 + a_1 x + \cdots a_n x^n$

$b_0 + b_1 x + \cdots b_n x^n$

$p(x) = a_0 + a_1 x + \ldots a_n x^n$ $p(0) = 0 \Rightarrow$ is a subspace

V_1, V_2 be subspace of V

$V_1 \cup V_2$ need not be a subspace

$V_1 \cap V_2$ is always a subspace

When $V_1 \subset V_2$

& $V_2 \subset V_1$; $V_1 \cup V_2$ can be a subspace

$v_1 \ldots v_n \in V$

Set of all linear combinations of $v_1 \ldots v_n$

$\mathcal{L}(v_1 \ldots v_n)$

linear span of $v_1 \ldots v_n$

$\mathcal{L}(v_1 \ldots v_n) = \left\{ \sum_{i=1}^{n} \alpha_i v_i \mid \alpha_i \in F \right\}$ is a subspace of V.

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Proof: Let $\Sigma \alpha_i v_i$, $\Sigma \beta_i v_i \in \alpha(v_1 \dots v_n)$

$$\left.\begin{array}{l} \sum_{i=1}^{n} \alpha_i v_i + \Sigma \beta_i v_i = \sum_{i=1}^{n} (\alpha_i + \beta_i) v_i \\ c \, \Sigma \alpha_i v_i = \sum_{i=1}^{n} (c\alpha_i) v_i \end{array}\right\} \in \alpha(v_1 \dots v_n)$$

Basis of a vector space V

$S \subset V$ is called a basis

if i) S is a linearly independent set

ii) S ~~space~~ spans V

Basis of R^2:

$$B = \left\{ \overset{v_1}{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}, \overset{v_2}{\begin{pmatrix} 0 \\ 1 \end{pmatrix}} \right\}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = x \begin{pmatrix} 1 \\ 0 \end{pmatrix} + y \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$B' = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \underset{\alpha}{(x-y)} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \underset{\beta}{y} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$\alpha + \beta = x \qquad \alpha = x - y$

$\beta = y$

Dimensions = No. of vectors in a basis

R^3

$$B = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

\# $a_0 + a_1 x + \cdots a_n x^n$

$B = \{1, x, \ldots x^n\}$

Set of all polynomials

Dimension = $(n+1)$

\# Set of all 2×2 matrices.

~~Dimensions~~ Basis

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = a \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}_{v_1} + b \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{v_2} + c \underbrace{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}}_{v_3} + d \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}}_{v_4}$$

3.18

Eg: $S = \{(a, b, c, d) / b - 2c + d = 0\} \subset R^4$ is a subspace

$v_1 = \{a_1, b_1, c_1, d_1\}$ $\quad b_1 - 2c_1 + d_1 = 0$

$v_2 = (a_2, b_2, c_2, d_2)$ $\quad b_2 - 2c_2 + d_2 = 0$

$v_1 + v_2 = (a_1 + a_2, b_1 + b_2, c_1 + c_2, d_1 + d_2)$

$(b_1 + b_2) - 2(c_1 + c_2) + d_1 + d_2 = (b_1 - 2c_1 + d_1) + (b_2 - 2c_2 + d_2)$

$= 0$

$\alpha v_1 = (\alpha a_1, \alpha b_1, \alpha c_1, \alpha d_1) = \alpha b_1 - 2\alpha c_1 + \alpha d_1$

$= \alpha(b_1 - 2c_1 + d_1)$

$= 0$

∴ is a subspace.

Basis

Put $a = k_1$

$b = k_2$

$c = k_3$

$d = 2c - b$

$= 2k_3 - k_2$

$$\begin{pmatrix} k_1 \\ k_2 \\ k_3 \\ 2k_3 - k_2 \end{pmatrix}$$

$$\begin{pmatrix} k_1 \\ k_2 \\ k_3 \\ 2k_2 - k_3 \end{pmatrix} = k_1 \overset{k_1=1}{\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}} + k_2 \overset{k_2=1}{\begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}} + k_3 \overset{k_3=1}{\begin{pmatrix} 0 \\ 0 \\ 1 \\ 2 \end{pmatrix}}$$

$B = \{(1,0,0,0), (0,1,0,2), (0,0,1,-1)\}$

Dimension is 3.

Eg. $S_1 = \{(a,b,c,d) / a = d,\ b = 2c\} \subset R^4$ is a subspace

$v_1 = (a_1, b_1, c_1, d_1)$ $\quad a_1 = d_1 \quad b_1 = 2c_1$

$v_2 = (a_2, b_2, c_2, d_2)$ $\quad a_2 = d_2 \quad b_2 = 2c_2$.

$v_1 + v_2 = (a_1 + a_2, b_1 + b_2, c_1 + c_2, d_1 + d_2)$

$a_1 + a_2 = d_1 + d_2$; $b_1 + b_2 = 2c_1 + 2c_2$.

$b = 2c$.

$b = 2k_2$.

Basis.

$a = k_1$

$b = 2k_2$.

$d = k_1$

$c = k_2$

$$\cancel{\begin{pmatrix} k_1 \\ 2k_2 \\ k_1 \\ k_2 \end{pmatrix}} \quad \begin{pmatrix} k_1 \\ 2k_2 \\ k_2 \\ k_1 \end{pmatrix} = k_1 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} + k_2 \begin{pmatrix} 0 \\ 2 \\ 1 \\ 0 \end{pmatrix}$$

Dimension = 2.

Q. Find the basis and dimension of solution space

$10x + 4y - 2z = 0$

$-17x + y + 2z - 3w = 0$

$x + y + z + w = 0$

$-34x + 10y - 10z + 8w = 0$

This is clearly a subspace of R^4

$x + y + z + w = 0$

$10x + 4y - 2z = 0$

$-17x + y + 2z - 3w = 0$

$-34x + 10y - 10z + 8w = 0$

$$\cancel{x = k_1} \quad \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 10 & 4 & -2 & 0 & 0 \\ -17 & 1 & 2 & -3 & 0 \\ -34 & 10 & -10 & 8 & 0 \end{array}\right]$$

$R_2 \to R_2 - 10R_1$

$R_3 \to R_3 + 17R_1$

$R_4 \to R_4 + 34R_1$

-10
34
8
0

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & -6 & -12 & -10 & 0 \\ 0 & 18 & 19 & 14 & 0 \\ 0 & 44 & 24 & 42 & 0 \end{array}\right]$$

$R_2 \to R_2/2 \quad R_4 \to R_4/2$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -3 & -6 & -5 \\ 0 & 18 & 19 & 14 \\ 0 & 22 & 12 & 21 \end{bmatrix}$$

$R_2 \to R_2 \div -3$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 5/3 \\ 0 & 18 & 19 & 14 \\ 0 & 22 & 12 & 21 \end{bmatrix}$$

* Dimension of a zero subspace is 0.

$R_3 \to R_3 - 18R_2$

19-18
14-19

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 5/3 \\ 0 & 0 & 1 & \\ & & & \end{bmatrix}$$

H.W.

$$x - 3y + 3z - w = 0$$
$$3x + y - 11z + 2w = 0$$
$$-2x + 5y - 4z + 5w = 0$$
$$4x - 7y + 2z + 3w = 0$$

Theorem: Let S be a linearly independent subset of the vector space V, let v be a vector in V such that v is not in the linear span of S.
Then, $S \cup \{v\}$ is also linearly independent.

Proof: $S \subset V$ S is linearly independent.
Let $S = \{v_1 \cdots v_n\}$
To prove

$$\alpha_1 v_1 + \cdots \alpha_n v_n + \alpha_{n+1} v = 0$$

if $\alpha_{n+1} \neq 0$,

$$v = \frac{-\alpha_1}{\alpha_{n+1}} v_1 \cdots \frac{-\alpha_n}{\alpha_{n+1}} v_n \in L(S)$$

this being a contradiction

$\Rightarrow \alpha_{n+1} = 0$
$\Rightarrow \alpha_1 v_1 + \cdots \alpha_n v_n = 0 \Rightarrow \alpha_i = 0$ for $i = 1, \cdots n$

Which implies.
$v_1 \cdots v_n, v$ is linearly independent.

Eg: R^2

$$\alpha_1\begin{pmatrix}1\\1\end{pmatrix}+\alpha_2\begin{pmatrix}1\\2\end{pmatrix}+\alpha_3\begin{pmatrix}2\\1\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$

$\alpha_1+\alpha_2+2\alpha_3=0$

$\alpha_1+2\alpha_2+\alpha_3=0$

This is a non trivial solution for $\alpha_1, \alpha_2, \alpha_3$.

linearly dependent.

Result: Any set of m vectors in R^n where $m>n$, is linearly dependent.

If a vector space V has a basis of n vectors then every other basis will also have to same no. of vectors.

21.8.18 Eg: $P_n(x)$ - set of all polynomials of degree atmost n with real coefficients

$a_0+a_1x+\cdots a_nx^n$

$B=\{1, x, x^2, \cdots x^n\}$

dimension $= n+1$

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$a_0 + a_1 x = \alpha(1+x) + \beta(1-x)$

$B = \{1, x\}$
$B = \{1+x, 1-x\}$

set of all symmetric matrices $(n \times n)$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{31} & a_{23} & a_{33} \end{pmatrix}$$

$$a_{11}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + a_{12}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \cdots$$

Dimension = 6 $\left(\frac{3 \times 4}{2} = 6\right)$

For $n \times n$ matrix $= \frac{n(n+1)}{2}$

Linear Transformation.

$T = V \longrightarrow W$, V and W are vector spaces.

1) $\underbrace{T(v_1 + v_2)}_{\text{happening in } V} = \underbrace{T(v_1) + T(v_2)}_{\text{happening in space of } W}$

2) $T(\alpha v) = \alpha T(v)$

If these conditions are satisfied we call T as the linear transformation of V ~~in W~~ into W.

Examples: $T : R^3 \rightarrow R^2$.

$T(x_1, x_2, x_3) = (x_1 - x_2, x_2 + x_3)$

let $v_1 = (x_1, x_2, x_3)$

$v_2 = (y_1, y_2, y_3)$

$v_1 + v_2 = (x_1 + y_1, x_2 + y_2, x_3 + y_3)$

$T(v_1 + v_2) = (x_1 + y_1 - (x_2 + y_2), x_2 + y_2 + x_3 + y_3)$

$= (x_1 - x_2 + y_1 - y_2, x_2 + x_3 + y_2 + y_3)$

$T(v_1) + T(v_2) = (x_1 - x_2, x_2 + x_3) + (y_1 - y_2, y_2 + y_3)$

$T(\alpha v_1) = T(\alpha x_1, \alpha x_2, \alpha x_3)$

$= (\alpha x_1 - \alpha x_2, 2x_2 + \alpha x_3)$

$= \alpha(x_1 - x_2, x_2 + x_3) = \alpha T(v_1)$

Both conditions are satisfied.

So the T defined is a linear transformation

Ex: $T = R^3 \to R^2$.

$T(x_1, x_2, x_3) = (2x_1, 3x_2)$

It is a linear transformation

Ex: $T = R^3 \to R^3$

$T(x_1, x_2, x_3) = (x_1 + x_2, x_2 + x_3, x_3 + x_1)$

$T(x_1, x_2, x_3) = (2x_1, 3x_2, 4x_3)$

Ex: $T = R^2 \to R^3$.

or $5x_1, 3x_1$

$T(x_1, x_2) = (x_1 - x_2, 2x_2, 0)$

Examples which cannot be linear transformation

1] $T : R^2 \to R^2$.

$T(x_1, x_2) = (x_1 + 1, 2x_2)$

$(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2)$

$T(v_1 + v_2) = (x_1 + y_1 + 1, 2(x_2 + y_2))$

$T(v_1) + T(v_2) = (x_1 + y_1 + 2, \text{—})$

Doesn't match

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a] $T: R^2 \to R^2$

$T(x_1, x_2) = (x_1^2, x_2^2)$

~~$T(v_1) + T(v_2) =$~~

$T(v_1 + v_2) = ((x_1+y_1)^2, (x_2+y_2)^2)$

$T(v_1) + T(v_2) = (x_1^2 + y^2, x_2^2 + y_2^2)$

⇒ linear transformations ~~that~~
cannot have square terms or constants.

→ linear transformation for the set of all continous functions:

$T: c(a,b) \to c(a,b)$ (with v written above the first $c(a,b)$ and w above the second)

$T(f) = f'$

$T(f+g) = (f+g)' = f' + g' = T(f) + T(g)$

$T(\alpha f) = (\alpha f)' = \alpha f' = \alpha T(f)$

This is clearly a linear transformation.

→ $T = c[a,b] \to R$.

$T(f) = \int_a^b f(x)\,dx$

$T(f+g) = \int_a^b (f+g)x\,dx$

$$= \int_a^b [f(x)+g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

Properties of linear transformation

→ $T(-v) = -T(v)$

→ $T(0) = 0$ element of W

$T(0)$ belongs to V, 0 belongs to W

$$T(0) = T(v+(-v))$$
$$= T(v) + T(-v)$$
$$= T(v) - T(v)$$
$$= 0$$

~~$T(x_1, x_2) = (x_1 - x_2, x_1 + x_2)$~~

~~$T(x_1, x_1)$~~

$T : R^2 \to R^2$

$T(x_1, x_2) = (x_1 - x_2, x_1 - x_2)$

$T(x_1, x_1) = (0, 0)$

→ Dealing with linear independence.

$T = V \to W$

$v_1 \cdots v_n$ are linearly independent.

$T(v_1) \cdots T(v_n)$

$T : R^2 \to R^2$

$T(x_1, x_2) = (x_1, 0)$

$(1, 0)$ $(0, 1)$

$T(1,0) = (1,0)$
$T(0,1) = (0,0)$
linearly dependent
Any set containing 0 is linearly dependent.

→ $T : V \to W$
$w_1 \ldots w_n$ li
$w_i = T(v_i)$
$v_1 \ldots v_n$ is linearly independent.

Proof:
$\alpha_1 v_1 + \ldots \alpha_n v_n = 0$
$T(\alpha_1 v_1 + \ldots \alpha_n v_n) = T(0)$
$\cancel{\alpha_i (t} = \alpha_1 T(v_1) + \ldots \alpha_n T(v_n)$
$= 0$
$\Rightarrow \alpha_1 w_1 + \ldots \alpha_n w_n = 0$
$w_1 \ldots w_n$ are given as li
which implies $\Rightarrow \alpha_i = 0$ for $i = 1$ to n

27/8/18 V, W Vectorspace.

$T : V \to W$

Kernal (T) = Nullspace (T)

$= \{ v \in V \,/\, T(v) = 0 \}$

Image (T) = Range (T)

$= \{ w \in W \,/\, \exists\, v \in V \quad T(v) = w \}$

$= \{ T(v) \,/\, v \in V \}$

Theorem: Ker (T) and Image (T) are subspaces of V and W respectively.

Proof: let $v_1, v_2 \in \text{Ker}(T)$

$T(v_1) = 0 \quad \& \quad T(v_2) = 0$

Consider $T(v_1 + v_2) = T(v_1) + T(v_2) = 0$
which implies $v_1 + v_2 \in \text{Ker}(T)$

Consider, $T(\alpha v_1) = \alpha T(v_1) = 0 \qquad \alpha v_1 \in \text{Ker}(T)$

~~∴ It is closed~~

Hence $\text{Ker}(T)$ is a subspace.

To prove $Img(T)$ is a subspace

Let $w_1\ w_2 \in Im(T)$

$\exists\ v_1, v_2 \in V$ such that $T(v_1) = w_1\ T(v_2) = w_2$.

$$w_1 + w_2 = T(v_1) + T(v_2)$$
$$= T(v_1 + v_2)$$

$$w_1 + w_2 \in Im(T)$$

Hence closed under addition

$$\alpha w_1 = \alpha T(v_1) = T(\alpha v_1)$$
$$\alpha w_1 \in Im(T)$$

Examples. Consider a linear transformation $T: R^2 \longrightarrow R^3$

$T(x, y) = (x+y, x-2y, 3x+y)$

$$Ker(T) = \left\{(x, y) \mid T(x, y) = \begin{pmatrix}0\\0\\0\end{pmatrix}\right\}.$$
$$= \left\{x, y) \mid \begin{matrix} x+y=0 \\ x-2y=0 \\ 3x+y=0 \end{matrix}\right\}$$

$$= \{(0,0)\} = \{0\}$$

Dimension of $\{0\} = 0$

$$\text{Im}(T) = \{ (a,b,c) \,/\, T(x,y) = (a,b,c) \}$$

$$\Longrightarrow \begin{aligned} x+y &= a \\ x-2y &= b \\ 3x+y &= c \end{aligned}$$

$$\left(\begin{array}{cc|c} 1 & 1 & a \\ 1 & -2 & b \\ 3 & 1 & c \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 1 & a \\ 0 & -3 & b-a \\ 0 & -2 & c-3a \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 1 & a \\ 0 & 1 & \frac{b-a}{-3} \\ 0 & -2 & c-3a \end{array}\right)$$

$$\sim \left(\begin{array}{cc|c} 1 & 1 & a \\ 0 & 1 & \frac{b-a}{-3} \\ 0 & 0 & c-3a+2a \end{array}\right)$$

$$\sim \left(\begin{array}{cc|c} 1 & 1 & a \\ 0 & 1 & \frac{a-b}{3} \\ 0 & 1 & \frac{3a-c}{2} \end{array}\right)$$

$$\sim \left(\begin{array}{cc|c} 1 & 1 & a \\ 0 & 1 & \frac{a-b}{3} \\ 0 & 0 & \frac{3a-c}{2} - \left(\frac{a-b}{3}\right) \end{array}\right)$$

$$\frac{3a-c}{2} - \left(\frac{a-b}{3}\right) = 0$$

$$3(3a-c) - 2(a-b) = 0$$

$$7a + 2b - 3c = 0$$

$$\text{Im}(T) = \{ (a,b,c) \,/\, 7a+2b-3c = 0 \}$$

let $c = k_1$

$b = k_2$

$a = \frac{1}{7}(3k_1 - 2k_2)$

$$\begin{pmatrix} \frac{1}{7}(3k_1 - 2k_2) \\ k_2 \\ k_1 \end{pmatrix}$$

$k_1 = 7$, $k_2 = 0$ $\qquad$ $k_1 = 0$, $k_2 = 7$

$$\begin{pmatrix} 3 \\ 0 \\ 7 \end{pmatrix} \qquad \begin{pmatrix} -2 \\ 7 \\ 0 \end{pmatrix}$$

Dimension $\text{Im}(T) = 2$

Dimension of $\ker(T)$ + Dimension of image(T)

$= 0 + 2 = 2 \rightarrow$ dimension of R^2.

Theorem: When

$V \rightarrow W$ where V, W vectorspace over R.

Dimension of $\ker(T)$ + Dimension of $\text{Im}(T)$ = dimension of V

Nullity (T) + Rank (T) = $\dim V$

RANK - NULLITY THEOREM

Eg:- Consider a linear transformation from $T: M_{2\times 2} \to M_{2\times 2}$.

$$T\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a+b & b+c \\ c+d & a+d \end{bmatrix}$$

$$\text{Ker}(T) = \{ \cancel{(a,b,c,d)} \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| \begin{array}{l} a+b=0 \\ b+c=0 \\ c+d=0 \\ a+d=0 \end{array} \right\}$$

$$\text{Basis for ker}(T) = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$$

$a = -b$

$b = -a$

$c = -b = a$

$d = -a$

$$\begin{bmatrix} a & -a \\ a & -a \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$$

Dimension Ker $(T) = 1$

$$\text{Im}(T) = \left\{ \begin{bmatrix} x & y \\ z & w \end{bmatrix} \middle| T\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} x & y \\ z & w \end{bmatrix} \right\}$$

$a+b = x$

$b+c = y$

$c+d = z$

$a+d = w$.

$$\left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & x \\ 0 & 1 & 1 & 0 & y \\ 0 & 0 & 1 & 1 & z \\ 1 & 0 & 0 & 1 & w \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & x \\ 0 & 1 & 1 & 0 & y \\ 0 & 0 & 1 & 1 & z \\ 0 & 0 & 1 & 1 & w-x+y \end{array}\right)$$

$$\sim \left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & x \\ 0 & 1 & 1 & 0 & y \\ 0 & 0 & 1 & 1 & z \\ 0 & 0 & 0 & 0 & w-x+y-z \end{array}\right)$$

$w - x + y - z = 0$

$w = k_1$

$z = k_2$

$y = k_3$

$x = k_3 - k_2 + k_1$

$$\begin{pmatrix} k_3 - k_2 + k_1 \\ k_3 \\ k_2 \\ k_1 \end{pmatrix}$$

$$\begin{bmatrix} k_3 - k_2 + k_1 & k_3 \\ k_2 & k_1 \end{bmatrix} = k_3 \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + k_2 \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix} + k_1 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Dimension of $\operatorname{Im}(T) = 3$

Dimension of $\operatorname{Ker}(T) = 1$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Dimension $(M_{2\times 2}) = 4$

Dimension Ker $T = 1$ Dimension $\operatorname{Im}(T) = 3$

<u>Inverse of a L.T.</u>

$T : V \to W$,

T is one-one and onto, then T^{-1} exists

$T^{-1} : W \to V$, is a linear transformation

To prove $T^{-1}(w_1 + w_2) = T^{-1}(w_1) + T^{-1}(w_2)$

$T^{-1}(\alpha w_1) = \alpha T^{-1}(w_1)$

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Proof: $w_1, w_2 \in W$

$\exists\ v_1, v_2 \in V$ such that $T(v_1) = w_1$, $T(v_2) = w_2$

$\Rightarrow v_1 = T^{-1}(w_1) \quad v_2 = T^{-1}(w_2)$

$T^{-1}(w_1 + w_2) = v_1 + v_2$

$= T^{-1}(w_1) + T^{-1}(w_2)$

$\alpha w_1 = T(\alpha v_1)$

3|18 To find inverse

$T: R^3 \longrightarrow R^3$

$T(x, y, z) = (2x, 4x-y, 2x+3y-z)$

Show that T is invertible & find the T^{-1}

→ To prove T is one-one.

Let $\exists\ v_1, v_2$ show that $T(v_1) = T(v_2)$

$v_1 = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} \quad v_2 = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} \quad (2x_1, 4x_1 - y_1, 2x_1 + 3y_1 - z_1)$

$= (2x_2, 4x_2 - y_2, 2x_2 + 3y_2 - z_2)$

$2x_1 = 2x_2 \qquad \cancel{4x_1} - y_1 = \cancel{4x_2} - y_2$

$\Rightarrow x_1 = x_2 \qquad \Rightarrow y_1 = y_2$

$\cancel{2x_1} + \cancel{3y_1} - z_1 = \cancel{2x_2} + \cancel{3y_2} - z_2$

$\Rightarrow z_1 = z_2$

To prove its auto.

$T(x, y, z) = (r, s, t)$

$(2x, 4x - y, 2x + 3y - z) = (r, s, t)$

$2x = r$

$y = 2r - s$

$z = 4r - 3s - t$

$T^{-1}(r, s, t) = \left(\frac{r}{2}, 2r - s, 7r - 3s - t\right)$

If $T = V \to W$

T is one-one $\Rightarrow$ Ker $T\{0\}$

$\text{Ker}(T) = \{0\} \Rightarrow T$ is one-one.

let $T(v_1) = T(v_2)$

$T(v_1) - T(v_2) = 0$

$T(v_1 - v_2) = 0 \in W$

$v_1 - v_2 = 0 \in V$

Relation b/w linear transformation

$T: R^3 \longrightarrow R^3$

$T(x, y, z) = (2x, 4x-y, 2x+3y-z)$

$$= \begin{pmatrix} 2 & 0 & 0 \\ 4 & -1 & 0 \\ 2 & 3 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = A\underline{X}$$

eg: $T: R^2 \to R^3$

$T(x, y) = (2x+y, 3x-y, x+y)$

$X = \begin{pmatrix} x \\ y \end{pmatrix}$

$$T(\underline{X}) = \begin{pmatrix} 2 & 1 \\ 3 & -1 \\ 1 & 1 \end{pmatrix}_{3\times 2} \begin{pmatrix} x \\ y \end{pmatrix}_{2\times 1}$$

$R^n \to R^m$ $\begin{pmatrix} \end{pmatrix}_{m \times n}$

Inner Product

function defined by $V \times V \rightarrow \mathbb{R}$ satisfying the following.

1) $\langle v, v \rangle \geq 0 \quad \forall\, v \in V \Rightarrow$ Positivity

$\langle v, v \rangle = 0 \Rightarrow v = 0$

2) $\langle cv_1, v_2 \rangle = c\langle v_1, v_2 \rangle \Rightarrow$ Homogenity.

3) $\langle v_1, v_2 \rangle = \langle v_2, v_1 \rangle \Rightarrow$ symmetry

4) $\langle v_1 + v_2, v_3 \rangle = \langle v_1, v_3 \rangle + \langle v_2, v_3 \rangle$ Additivity.

$v_1 \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$

$cv_1 \begin{pmatrix} cx_1 \\ cy_1 \end{pmatrix}$

$\langle v_1, v_2 \rangle = x_1x_2 + y_1y_2$.

$\langle v_1 + v_2, v_3 \rangle = (x_1 + x_2)x_3 + (y_1 + y_2)y_3$.

$v_1 + v_2 = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \end{pmatrix}$

$v_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \quad v_2 \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$

1) $\langle v_1, v_2 \rangle = x_1x_2 + 4y_1y_2$.

V = set of all continuous real valued function on $[0,1]$

$f, g \in V$

Define

$$\langle f, g \rangle = \int_0^1 f(x) \cdot g(x)\, dx$$

$$\langle cf, g \rangle = \int_0^1 (cf) x\, g(x)\, dx = \int_0^1 c f(x) g(x)\, dx.$$

$$= c \langle f, g \rangle$$