12/02/19

LINEAR Equations

$a_{11} x_1 + \dots\dots a_{1n} x_n = b_1$

$\vdots$

$a_{m1} x_1 + \dots\dots a_{mn} x_n = b_m$

$$A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \\ a_{m1} & \dots & a_{mn} \end{pmatrix} \quad X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \quad b = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$$

$$AX = b$$

Model of Questions

(i) Gypotography

(ii) Traffic problem

(iii) Transportation problem

(iv) Electric circuit

(v) Economics - Market models.

I Gauss elimination method.

(i) Solve: $x_1 + 2x_2 + 3x_3 = 6$

$2x_1 - x_2 + 4x_3 = 5$

$3x_1 - x_2 - x_3 = 1.$

$R_2 \to R_2 - 2R_1$

$R_3 \to R_3 - 3R_1$

$(A:b)$ = Augmented matrix

$$(A:b) = \left(\begin{array}{ccc:c} 1 & 2 & 3 & 6 \\ 2 & -1 & 4 & 5 \\ 3 & -1 & -1 & 1 \end{array}\right) \sim \left(\begin{array}{ccc:c} 1 & 2 & 3 & 6 \\ 0 & -5 & -2 & -7 \\ 0 & -7 & -10 & -17 \end{array}\right)$$

we have to eliminate x_1, so we make coefficients of x_1 0 by applying a transformation

$R_2 \to \frac{R_2}{-5}$, $R_3 \to \frac{R_3}{-7}$

$$\begin{pmatrix} 1 & 2 & 3 & : & 6 \\ 0 & 1 & \frac{2}{5} & : & \frac{7}{5} \\ 0 & 1 & \frac{10}{7} & & 17/7 \end{pmatrix} \sim$$

~~$R_1 \to R_1 - 2R_2$~~

$$\cancel{\begin{pmatrix} 1 & 0 & 11/5 & : & 16/5 \\ 0 & 1 & \frac{2}{5} & : & \frac{7}{5} \\ 0 & 1 & 2 & & 17/7 \end{pmatrix}}$$

$R_3 \to R_3 - R_2$

$$\begin{pmatrix} 1 & 2 & 3 & : & 6 \\ 0 & 1 & \frac{2}{5} & : & \frac{7}{5} \\ 0 & 0 & \frac{36}{35} & : & \frac{36}{35} \end{pmatrix} \sim$$

$R_3 \to \frac{R_3(35)}{36}$

$$\begin{pmatrix} 1 & 2 & 3 & : & 6 \\ 0 & 1 & 2/5 & : & 7/5 \\ 0 & 0 & 1 & : & 1 \end{pmatrix}$$

Back substitution

$\underline{\underline{x_3 = 1.}}$

$x_2 + \frac{2}{5}x_3 = \frac{7}{5}$

$x_2 = \frac{7-2}{5} = 1. \Rightarrow \underline{\underline{x_2 = 1}}$

$x_1 + 2x_2 + 3x_3 = 6$

$x_1 + 2 + 3 = 6 \Rightarrow \underline{\underline{x_1 = 1.}}$

(ii) solve :

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 6 \\ 2x_1 - x_2 + 4x_3 &= 5 \\ 3x_1 + x_2 + 7x_3 &= 11 \end{aligned}$$

$$(A:b) = \begin{pmatrix} 1 & 2 & 3 & : & 6 \\ 2 & -1 & 4 & : & 5 \\ 3 & 1 & 7 & : & 11 \end{pmatrix}$$

$R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 3R_1$

$$\begin{pmatrix} 1 & 2 & 3 & : & 6 \\ 0 & -5 & -2 & : & -7 \\ 0 & -5 & -2 & & -7 \end{pmatrix}$$

$R_2 \to \frac{R_2}{-5}$ $R_3 \to \frac{R_3}{-5}$ $R_3 \to R_3 - R_2$

$$\left(\begin{array}{ccc:c} 1 & 2 & 3 & 6 \\ 0 & 1 & \frac{2}{5} & +\frac{7}{5} \\ 0 & 1 & \frac{+2}{5} & \frac{7}{5} \end{array}\right) \sim \left(\begin{array}{ccc:c} 1 & 2 & 3 & 6 \\ 0 & 1 & 2/5 & 7/5 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$R_3 \to R_3 - R_2$

$$\left(\begin{array}{ccc:c} 1 & 2 & 3 & 6 \\ 0 & 1 & \frac{2}{5} & \frac{4}{5} \\ 0 & 0 & -\frac{4}{35} & \frac{1}{5} \end{array}\right)$$

$R_3 \to -\frac{R_3 (35)}{4}$

$$\left(\begin{array}{ccc:c} 1 & 2 & 3 & 6 \\ 0 & 1 & 2/5 & 4/5 \\ 0 & 0 & 1 & -7/4 \end{array}\right)$$

$x_2 + \frac{2}{5} x_3 = \frac{7}{5}$

Put $x_3 = k$

$x_2 = \frac{7}{5} - \frac{2}{5} k.$

$x_1 + 2x_2 + 3x_3 = 6$

$x_1 + 2\left(\frac{7}{5} - \frac{2}{5}k\right) + 3k = 6$

$x_1 + \frac{14}{5} - \frac{4}{5}k + 3k = 6$

$x_1 = \frac{4k - 15k}{5} - \frac{14}{5}$

$x_1 = -\frac{11k}{5} - \frac{14}{5}.$

$$\begin{pmatrix} -\frac{11k}{5} - \frac{14}{5} \\ \frac{7}{5} - \frac{2}{5}k \\ k \end{pmatrix}$$

Q

$3x_1 + 2x_2 + x_3 = 3$

$2x_1 + x_2 + x_3 = 0$

$4x_1 + 3x_2 + x_3 = 4.$

$R_1 \to \frac{R_1}{3}$

$$(A:b) = \left(\begin{array}{ccc:c} 3 & 2 & 1 & 3 \\ 2 & 1 & 1 & 0 \\ 4 & 3 & 1 & 4 \end{array}\right) \sim \left(\begin{array}{ccc:c} 1 & \frac{2}{3} & \frac{1}{3} & 1 \\ 2 & 1 & 1 & 0 \\ 4 & 3 & 1 & 4 \end{array}\right)$$

Scanned by CamScanner

$R_2 \longrightarrow R_2 - 2R_1$, $R_3 \longrightarrow R_3 - 4R_1$

$$\left(\begin{array}{ccc:c} 1 & \frac{2}{3} & \frac{1}{3} & 1 \\ 0 & -\frac{1}{3} & \frac{1}{3} & -2 \\ 0 & \frac{1}{3} & -\frac{1}{3} & 0 \end{array}\right)$$

$R_2 \rightarrow -3R_2$, $R_3 \longrightarrow 3R_3$.

$$\left(\begin{array}{ccc:c} 1 & \frac{2}{3} & \frac{1}{3} & 1 \\ 0 & 1 & -1 & 6 \\ 0 & 1 & -1 & 0 \end{array}\right)$$

$R_3 \longrightarrow R_3 - R_2$

$$\left(\begin{array}{ccc:c} 1 & \frac{2}{3} & \frac{1}{3} & 1 \\ 0 & 1 & -1 & 6 \\ 0 & 0 & 0 & -6 \end{array}\right)$$

There exists no solution for this system.

Echelon form of a Matrix

A matrix is said to be in an echelon form if the following conditions are satisfied.

(i) First non-zero elements in every row is unity

(ii) No. of zeroes before the first non-zero element should be in the increasing order.

(iii) If al all zero rows are there, they should appear towards the end.

Eg:
$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 1 & 5 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$
→ Echelon form.

If this 0 is changed to 1 it won't be in echelon form

We can change it to echelon form by interchanging the rows.

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \longrightarrow$$ not in an echelon form. but can made echelon by $R_3 \to R_3 - R_2$.

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \longrightarrow$$ not in an echelon form. but if interchange $R_2 \leftrightarrow R_3$

Rank of a matrix.

Order of the largest non-vanishing determinant.

(Rank = No. $r(A)$ = No. of non-zero rows. (in Echelon form).

Case I: $r(A) \neq r(A, b)$, no solution inconsistent

Eg: $x_1 + x_2 = 1$
$x_1 + x_2 = 3$

$$\left(\begin{array}{cc:c} 1 & 1 & 1 \\ 1 & 1 & 3 \end{array}\right) \sim \left(\begin{array}{cc:c} 1 & 1 & 1 \\ 0 & 0 & 2 \end{array}\right)$$

Case II: $r(A) = r(A, b) = r < n$ (n = no. of unknown).

infinitely many solution.

Case III: $r(A) = r(A, b) = r = n$

Unique solution.

Scanned by CamScanner

Q. For what values of 'a' does the following system have unique solution, many solution and no solution?

$x + 2y - 3z = 4$

$3x - y + 5z = 2$

$4x + y + (a^2 - 14)z = a + 2$

Ans:

$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 3 & -1 & 5 & 2 \\ 4 & 1 & (a^2-14) & a+2 \end{array}\right)$$

$R_2 \to R_2 - 3R_1$, $R_3 \to R_3 - 4R_1$

$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 0 & -7 & 14 & -10 \\ 0 & -7 & a^2-2 & a-14 \end{array}\right)$$

$R_2 \to \frac{R_2}{-7}$

$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 0 & 1 & -2 & 10/7 \\ 0 & -7 & a^2-2 & a-14 \end{array}\right)$$

$R_3 \to R_3 + 7R_2$

$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 0 & 1 & -2 & 10/7 \\ 0 & 0 & a^2-16 & a-4 \end{array}\right)$$

Case i): If $a^2 - 16 \neq 0$.

$r(A) = r(A,b) = 3$

i.e, system is consistent unique solution.

Case ii) If $a^2 - 16 = 0$

& $a \neq 4$.

$\therefore a = -4$.

$r(A) = 2$

$r(A,b) = 3$.

$r(A) \neq r(A,b)$

i.e, system is inconsistent

so, no solution.

Case iii) If $a^2 - 16 = 0$

& $a \neq -4$

$a = 4$.

$r(A) = r(A,b) = 2$

infinitely many solution.

Scanned by CamScanner

Q. Determine λ and μ, such that

$x+y+z=3$, $2x+y+3z=6$, $x+3y+\lambda z=\mu$

(i) no solution (ii) many infinite solution , (iii) unique.

Ans:

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 2 & 1 & 3 & 6 \\ 1 & 3 & \lambda & \mu \end{array}\right)$$

$R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - R_1$

$$= \left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & -1 & 1 & 0 \\ 0 & 2 & \lambda-1 & \mu-3 \end{array}\right)$$

$R_3 \to R_3 + 2R_2$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & (\lambda-1)+2 & (\mu-3) \end{array}\right)$$

$R_2 \to -R_2$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & \lambda+1 & \mu-3 \end{array}\right)_{3\times 4}$$

0

When we say rank (A) we consider

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & \lambda+1 & \mu \end{array}\right)$$

and when we say rank(A, b) we consider entire 3×4 matrix

Case i) Consider $\lambda+1=0$ and $\mu-3 \neq 0$

$\therefore\ r(A) = 2,\quad r(A,b) = 3$

$r(A) \neq r(A,b)$

No solution

Case ii) Consider $\lambda+1 \neq 0$. ~~and $\mu-3$~~

$r(A) = r(A,b) = 3$

$\therefore$ Unique solution

Case iii) Consider $\lambda+1=0$, $\lambda-3=0$

$r(A) = 2$, $r(A,b) = 2$

$r(A) = r(A,b) < n$

$\therefore$ Infinitely many solution.

Q Determine b_1, b_2, b_3 such that

$$2x + y + 7z = b_1$$
$$6x - 2y + 11z = b_2$$
$$2x - y + 3z = b_3$$

has many solutions.

Ans:

$$\left(\begin{array}{ccc|c} 2 & 1 & 7 & b_1 \\ 6 & -2 & 11 & b_2 \\ 2 & -1 & 3 & b_3 \end{array}\right)$$

$R_2 \to R_2 - 3R_1$ $\quad$ $R_3 \to R_3 - R_1$

$$\left(\begin{array}{ccc|c} 2 & 1 & 7 & b_1 \\ 0 & -5 & -10 & b_2 - 3b_1 \\ 0 & -2 & -4 & b_3 - b_1 \end{array}\right)$$

$R_1 \to \frac{R_1}{2}$

$$\left(\begin{array}{ccc|c} 1 & 1 & 7 & b_1/2 \\ 0 & -5 & -10 & b_2 - 3b_1 \\ 0 & -2 & -4 & b_3 - b_1 \end{array}\right)$$

$R_2 \to \frac{-R_2}{5}$

$$\left(\begin{array}{ccc|c} 1 & 1 & 7 & b_1/2 \\ 0 & 1 & 2 & \frac{-b_2 + 3b_1}{5} \\ 0 & -2 & -4 & b_3 - b_1 \end{array}\right)$$

$R_3 \to R_3 + 2R_2$

$$\left(\begin{array}{ccc|c} 1 & 1 & 7 & b_1/2 \\ 0 & 1 & 2 & -b_2 + \frac{3b_1}{5} \\ 0 & 0 & 0 & b_3 - b_1 + 2\left(\frac{-b_2 + 3b_1}{5}\right) \end{array}\right)$$

$b_3 - b_1 + \frac{2}{5}(-b_2 + 3b_1) = 0$

$b_3 - b_1 - \frac{2}{5}b_2 + \frac{6}{5}b_1 = 0$

$+\frac{b_1}{5} - \frac{2}{5}b_2 + \frac{5}{5}b_3 = 0$

$+b_1 - 2b_2 + 5b_3 = 0$

II LU Decomposition Method

A - square matrix $\quad Ax = b.$

$A = LU.$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix}$$

$u_{11} = a_{11}, \quad u_{12} = a_{12}, \quad a_{13} = u_{13}$

$Ax = L(Ux) = b.$

$= Lz = b$

$= Ux = z.$

Eg:

$x_1 + 2x_2 + 3x_3 = 6$

$2x_1 + x_2 + 4x_3 = 5$

$3x_1 - x_2 - x_3 = 1.$

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & -1 & -1 \end{pmatrix} \cancel{\begin{pmatrix} 6 \\ 5 \\ 1 \end{pmatrix}} = \begin{pmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix}$$

$u_{11} = 1, \quad u_{12} = 2, \quad u_{13} = 3.$

$l_{21}u_{11} = 2.$

$l_{21} = 2$

$l_{21}u_{12} + u_{22} = -1$

$4 + u_{22} = -1$

$u_{22} = -5$

$l_{21}u_{13} + u_{23} = 4$

$6 + u_{23} = 4$

$u_{23} = -2.$

$l_{31}u_{11} = 3$

$l_{31} = 3$

$l_{31}u_{12} + l_{32}u_{22} = -1$

$3(2) + l_{32}(-5) = -1$

$6 - 5l_{32} = -1$

$-5l_{32} = -7$

$l_{32} = 7/5$

$l_{31}u_{13} + l_{32}u_{23} + u_{33} = -1$

$3(3) + \frac{7}{5}(-2) + u_{33} = -1$

$9 - \frac{14}{5} + u_{33} = -1$

$u_{33} = -1 - 9 + \frac{14}{5}$

$= \frac{-50+14}{5} = -\frac{36}{5}$

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & \frac{7}{5} & 1 \end{pmatrix} \left[\begin{pmatrix} 1 & 2 & 3 \\ 0 & -5 & -2 \\ 0 & 0 & \frac{-36}{5} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \right] = \begin{pmatrix} 6 \\ 5 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & \frac{7}{5} & 1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} 6 \\ 5 \\ 1 \end{pmatrix}$$

$z_1 = 6$, $2z_1 + z_2 = 5 \Rightarrow z_2 = -7$

$3z_1 + \frac{7}{5} z_2 + z_3 = 1.$

$3(6) + \frac{7z_2}{5} + z_3 = 1 \Rightarrow \frac{7z_2 + 5z_3}{5} = 17$

$7z_2 + 5z_3 = -85$

$$\begin{array}{l} \cancel{10} z_2 + 5z_3 = 25 \\ -7z_2 \mp 5z_3 = \pm 85 \\ \hline 3z_2 = 110 \\ z_2 = \frac{110}{3}. \end{array}$$

$3z_1 + \frac{7}{5} z_2 + z_3 = 1$

$3(6) + \frac{7}{5}(-7) + z_3 = 1$

$18 - \frac{49}{5} + z_3 = 1$

$z_3 = -17 + \frac{49}{5} = \underline{\underline{-\frac{36}{5}}}$

$$\begin{array}{ccc} U & X & Z \\ \begin{pmatrix} 1 & 2 & 3 \\ 0 & -5 & -2 \\ 0 & 0 & -36/5 \end{pmatrix} & \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = & \begin{pmatrix} 6 \\ -7 \\ -36/5 \end{pmatrix} \end{array}$$

$x_1 + 2x_2 + 3x_3 = 6 \Rightarrow \underline{\underline{x_1 = 1}}$

$-5x_2 - 2x_3 = -7 \Rightarrow \underline{\underline{x_2 = 1}}$

$x_3 \left(\frac{-36}{5}\right) = -\frac{36}{5} \Rightarrow \underline{\underline{x_3 = 1}}$

III. Gauss - Jordon elimination

$AX = b$ if A is non-singular

$X = A^{-1}b$

To find A^{-1},

$A = IA$

$\downarrow$

$I = B(A)$, $B = A^{-1}$.

Q Solve :- $2x_1 - x_3 = 1$

$5x_1 + x_2 = 1$

$x_2 + 3x_3 = 1$.

Ans:
$$A = \begin{pmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{pmatrix}, \quad X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$AX = b.$

Now, $A = IA$

$$\begin{pmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} A.$$

$R_1 \to \frac{R_1}{2}$

$$\begin{pmatrix} 1 & 0 & -1/2 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} A.$$

$R_2 \to R_2 - 5R_1$

$$\begin{pmatrix} 1 & 0 & -1/2 \\ 0 & 1 & 5/2 \\ 0 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1/2 & 0 & 0 \\ -5/2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} A.$$

$R_3 \to R_3 - R_2$

$$\begin{pmatrix} 1 & 0 & -1/2 \\ 0 & 1 & 5/2 \\ 0 & 0 & 1/2 \end{pmatrix} = \begin{pmatrix} 1/2 & 0 & 0 \\ -5/2 & 1 & 0 \\ 5/2 & -1 & 1 \end{pmatrix} A$$

$R_3 \to 2R_3$

$$\begin{pmatrix} 1 & 0 & -1/2 \\ 0 & 1 & 5/2 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1/2 & 0 & 0 \\ -5/2 & 1 & 0 \\ 5 & -1 & 1 \end{pmatrix} A$$

$R_2 \to R_2 - \frac{5}{2} R_3$

$$\begin{pmatrix} 1 & 0 & -1/2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1/2 & 0 & 0 \\ -15 & 7/2 & -5/2 \\ 5 & -1 & 1 \end{pmatrix} A$$

$0 - \frac{1}{2} \quad \frac{1}{2} + \frac{5}{2} - \frac{6}{2} = 3$

$R_1 \to R_1 + \frac{R_3}{2}$.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & -1/2 & 1/2 \\ -15 & 7/2 & -5/2 \\ 5 & -1 & 1 \end{pmatrix} A.$$

Now, $X = A^{-1} b$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 & -1/2 & 1/2 \\ -15 & 7/2 & -5/2 \\ 5 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 - \frac{1}{2} + \frac{1}{2} \\ -15 + \frac{7}{2} - \frac{5}{2} \\ 5 - 1 + 1 \end{pmatrix} = \begin{pmatrix} 3 \\ -14 \\ 5 \end{pmatrix}$$

$\therefore \; \underline{\underline{x_1 = 3}}, \quad \underline{\underline{x_2 = -14}}, \quad \underline{\underline{x_3 = 5}}$

Scanned by CamScanner

System of Linear Homogeneous Equation

$AX = 0$

rank $(A|0) = r(A)$, i.e, to say that the system is consistent and a solution will always exist.

$\Rightarrow x_1, \ldots, x_n = 0$. trivial soln

$\Rightarrow r(A) = r < n$, non-trivial soln

- Let A be a square matrix

$|A| \neq 0$, only trivial solution.

$|A| = 0$, non-trivial solution exists.

Consider,

$AX = b$ — ① and $AX = 0$ — ②

X_p be any particular solution of ①

X_h be a solution to ②

$X = X_h + X_p$

$AX = A(X_h + X_p)$

$= AX_h + AX_p$

$= 0 + b = b$.

Solve: $x + y + z = 3$.

$2x + y + z = 4$.

$3x + 2y + 2z = 7$.

Ans: solving homogeneous system,

$R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 3R_1$

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ 3 & 2 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & -1 \\ 0 & -1 & -1 \end{pmatrix}$$

e system

ill always

2

$R_2 \to -R_2$

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad R_3 \to R_3 + R_2$$

$AX = 0$

$y + z = 0$

$y = -z$

Let $z = k$

$\therefore y = -k.$

$x + y + z = 0 \Rightarrow x = -(y+z) = -(0) = 0$

$$X_h = \begin{pmatrix} 0 \\ -k \\ k \end{pmatrix}$$

$1 + 2 - k + k = 3$

$1 + 2 = 3.$

$$X_p = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$$

$2 + 2 - k + k = 4.$

→ through trial and error pick one solution

$$X = \begin{pmatrix} 1 \\ 2-k \\ k \end{pmatrix}, \; k \in \mathbb{R}$$

substituting in the ~~soln~~ equations,

$$X = \begin{pmatrix} 0 \\ -k_1 \\ k_1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1-k_1 \\ 1+k_1 \end{pmatrix}, \; k_1 \in \mathbb{R}.$$

consider $AX = b$

$$a_{11} x_1 + a_{12} \cdots\cdots + a_{1n} x_n = b_1$$

$$\vdots$$

$$a_{m1} x_1 + a_{m2} + \cdots\cdots + a_{mn} x_n = b_m$$

$$\begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} x_1 + \begin{pmatrix} a_{12} \\ \vdots \\ a_{m2} \end{pmatrix} x_2 + \cdots + \begin{pmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{pmatrix} x_n = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b.$

$\sum_{j=1}^{n} a_j x_j = b.$

$a_1 = \begin{pmatrix}1\\0\end{pmatrix} \quad a_2 = \begin{pmatrix}0\\1\end{pmatrix}$

$a_1x_1 + a_2x_2 = b.$

$\begin{pmatrix}1\\0\end{pmatrix}b_1 + \begin{pmatrix}0\\1\end{pmatrix}b_2 = \begin{pmatrix}b_1\\b_2\end{pmatrix}$

R_3 — vectors in R_3

$$a_1 = \begin{pmatrix}1\\2\\1\end{pmatrix} \quad a_2 = \begin{pmatrix}1\\1\\1\end{pmatrix} \quad a_3 = \begin{pmatrix}0\\1\\1\end{pmatrix} \quad b = \begin{pmatrix}2\\4\\3\end{pmatrix}$$

$a_1x_1 + a_2x_2 + a_3x_3 = b.$

$$\begin{pmatrix}1\\2\\1\end{pmatrix}x_1 + \begin{pmatrix}1\\1\\1\end{pmatrix}x_2 + \begin{pmatrix}0\\1\\1\end{pmatrix}x_3 = \begin{pmatrix}2\\4\\3\end{pmatrix}$$

$x_1 + x_2 = 2$

$2x_1 + x_2 + x_3 = 4$

$x_1 + x_2 + x_3 = 3.$

$$\left(\begin{array}{ccc:c}1 & 1 & 0 & 2\\2 & 1 & 1 & 4\\1 & 1 & 1 & 3\end{array}\right) \sim \left(\begin{array}{ccc:c}1 & 1 & 1 & 3\\2 & 1 & 1 & 4\\1 & 1 & 0 & 2\end{array}\right)$$

$R_2 \to R_2 - 2R_1, \quad R_3 \to R_3 - R_1$

$$\left(\begin{array}{ccc:c}1 & 1 & 1 & 3\\0 & -1 & -1 & -2\\0 & 0 & -1 & -1\end{array}\right) \sim \left(\begin{array}{ccc:c}1 & 1 & 1 & 3\\0 & 1 & 1 & 2\\0 & 0 & 1 & 1\end{array}\right)$$

$\therefore x_3 = 1$

$x_2 = 1$

$x_1 = 1.$

$r(A) = 3$, max. no. of linearly independant rows

$=$ row rank.

Rank(A) = max. no. of linearly independant rows
= row rank
= order of the largest non-vanishing determinant
= no. of non-zero rows (Echelon form)

column rank = Max. no of linearly independant column matrix

For a matrix, $A_{m \times n}$.

$$\boxed{r(A_{m\times n}) \leq \min\{m, n\}}$$

Eg: $\begin{pmatrix}1\\2\\3\end{pmatrix}\begin{pmatrix}2\\1\\4\end{pmatrix}\begin{pmatrix}3\\3\\7\end{pmatrix}$

$$A = \begin{pmatrix}1 & 2 & 3\\ 2 & 1 & 4\\ 3 & 3 & 7\end{pmatrix} \sim \begin{pmatrix}1 & 2 & 3\\ 0 & -3 & -2\\ 0 & -3 & -2\end{pmatrix} \sim \begin{pmatrix}1 & 0 & 3\\ 0 & 1 & 2/3\\ 0 & 0 & 0\end{pmatrix}$$

$r(A) = 2 < 3.$ ← The.

There are 3 vectors. If $r(A) = 3$, then l.i
but $r(A) < 3$ hence l. dependant.

Eg: 3 vectors in R^2

$\begin{pmatrix}1\\2\end{pmatrix} \begin{pmatrix}2\\1\end{pmatrix} \begin{pmatrix}3\\2\end{pmatrix}$

$\begin{pmatrix}1 & 2\\ 2 & 1\\ 3 & 2\end{pmatrix}_{2\times 3}$

Here there are 3 vectors, so for these 3 to be l.i, $r(A) = 3$.

But the rank of this matrix can have is 2 because

rank(A) $\leq$ min {2, 3)

rank A $\leq$ 2.

R^m – n vectors $n > m$

linearly dependant.

Vector Spaces

V - nonempty collection of elements addition, scalar multiplication. V is a vector space over R.

R, real numbers, $c \in R$.

(i) for any $v_1, v_2 \in V$, $v_1 + v_2 \in V$

(ii) v_1, v_2, v_3, $(v_1 + v_2) + v_3 = v_1 + (v_2 + v_3)$

(iii) for any $v \in V$, $v_1 + 0 = 0 + v_1 = v_1$

(iv) for any $v_1 \in V$, $-v_1 \in V$, $v_1 + (-v_1) = (-v_1) + v_1 = 0$.

(v) $v_1, v_2 \in V$, $v_1 + v_2 = v_2 + v_1$

(vi) $c \in R$, $v_1 \in V$, $cv_1 \in V$

(vii) $c_1, c_2 \in R$ $(c_1 + c_2) v_1 = c_1 v_1 + c_2 v_1$.

$c_1 \in R_1$, $v_1, v_2 \in V$ $c_1(v_1 + v_2) = c_1 v + c_2 v_2$.

(viii) $c_1 \in R_1$, $v_1, v_2 \in V$, $c_1(v_1 + v_2) = c_1 v_1 + c_2 v_2$

(ix) $c_1 c_2 \in R$ $c_1(c_2 v_1) = (c_1 c_2) v_1$

$v_1 \in V$

(x) $\exists\ 1 \in R$, $1 \cdot v_1 = v_1$ $v_1 \in V$.

Eg: (1) $V = R^n$ over R.

$v_1 = (x_1, \ldots\ldots x_n)$ where $x_i \in R$.

$v_2 = (y_1, \ldots\ldots, y_n)$

$v_1 + v_2 = (x_1 + y_1 \ldots\ldots x_n + y_n)$

$cv_1 = (cx_1 + \ldots\ldots cx_n)$

$v_3 = (z_1, z_2 \ldots\ldots, z_n)$

$(v_1 + v_2) + v_3 = ((x_1 + y_1) + z_1 \ldots\ldots (x_n + y_n) + z_n)$

$= (x_1 + (y_1 + z_1) \ldots\ldots x_n + (y_n + z_n))$

$= v_1 + (v_2 + v_3)$

$0 = (0, \ldots, 0)$

$v_1 = (x_1, \ldots x_n)$

$-v_1 = (-x_1, \ldots -x_n) \in R^n$

$cv = (cx_1, \ldots, cx_n), \; cx_i \in R$

$\in R^n$

Eg: R^2 over R, addition,

$(x_1, y_1) + (x_2, y_2)$

$= (x_1 + 2x_2, y_1 + 2y_2)$

$c(x_1, y_1) = (cx_1, cy_1)$

Is R^2 over R, a vector space.?

$\int \cos^4 x \, . dx$

$\int \left(\frac{\cos 2x + 1}{2}\right) . dx$

$2\cos^2 x \quad 1 + \cos 2x$

$= \frac{1}{4} \int (\cos 2x + 1)^2$

$= \frac{1}{4} \int \cos^2 2x + 1 + 2\cos 2x$

$= \frac{1}{4} \int \frac{\cos 4x + 1}{2} + 1 + \cos 2x$

$= \frac{1}{8} \int (\cos 4x + 3 + 2\cos 2x) . dx$

$= \frac{1}{8} \left[\frac{\sin 4x}{4} + 3x + \frac{2 \sin 2x}{2} \right]$

$= \frac{\sin 4x}{32} + \frac{3x}{8} + \frac{\sin 2x}{8}$

Scanned by CamScanner