DIFFERENTIAL EQUATION

An equation involving derivatives or differentials of one or more dependant variables with respect to one or more independant variable is called a differential equation.

i) $e^x dx + e^y dy = 0 \rightarrow \frac{dy}{dx} = \frac{e^x}{e^y}$ [O.D.E] ; degree 1, order 1

ii) $\frac{d^2x}{dt^2} + n^2 x = 0$ [O.D.E] ; degree = 1, order = 1

iii) $y = x\frac{dy}{dx} + \frac{x}{\frac{dy}{dx}}$ [O.D.E] ; degree = 2, order = 1

iv) $\frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\left(\frac{d^2y}{dx^2}\right)} = c$ [O.D.E] ; degree = 2, order = 2.

dependant considered constants

v) $\frac{dx}{dt} - wy = a\cos pt$ [O.D.E] ; degree = 1, order = 1

independant

vi) $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2u$ [P.D.E] ; degree = 1

independant variable.

vii) $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 x}{\partial t^2}$ [P.D.E]

Ordinary Differential Equation (O.D.E)

A differential equation involving derivatives with respect to a single independant variable is called an O.D.E

Note: $y = f(x)$ — y dependant, x independant; $z = f(x, y)$ — z dependant, x, y independant

Partial differential equation (P.D.E)

A differential equation involving partial derivatives with respect to more than one independant variable is called partial differential equation

Order of a differential equation.

The order of the highest derivative involved in a differential is called the order of the differential equation.

Degree of a differential equation.

The degree of a differential equation is the degree of the highest derivative which occurs in it, after the differential equation has been made free from radicals and fractions as far as the derivatives are concerned.

Linear and Non-linear differential

A differential equation is called linear if every dependant variable and every derivative occurs in the first degree. No products of dependant variable and/or derivatives occur.

A differential equation which is not linear is called a non-linear differential equation.

Eg: (i) $\frac{d^3y}{dx^3} + 4x\left(\frac{dy}{dx}\right)^2 = y\frac{d^2y}{dx^2} + e^y$ Non-linear

(ii) $\left(\frac{d^2y}{dx^2}\right)^3 + \frac{dy}{dx} = \sin x$ Non-linear

(iii) $y''' + y^2 + e^{y'} = 0$ Non-linear

(iv) $xy\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right)^2 - y\frac{dy}{dx} = 0$ Non-linear

Formation of a differential equation.

Formation of a differential equations a geometric or physical problem into mathematical symbols.

Eg: 1. Form the differential equation of simple harmonic motion given by

i) $x = A\cos(nt+\alpha)$.

$$\frac{dx}{dt} = -A\sin(nt+\alpha)(n).$$

$$\frac{d^2x}{dt^2} = -n^2A\cos(nt+\alpha)$$

$$\frac{d^2x}{dt^2} = -n^2x \Rightarrow \frac{d^2x}{dt^2} + n^2x = 0$$

Eg 2. Obtain the differential equation of all circles of radius a and centre (h,k).

$$(x-h)^2 + (y-k)^2 = a^2 \quad \text{—— (1)}$$

$$2(x-h) + 2(y-k)\frac{dy}{dx} = 0 \Rightarrow (x-h) + (y-k)\frac{dy}{dx} = 0.$$

$$1 + (y-k)\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0$$

$$(y-k) = \frac{-\left[1+\left(\frac{dy}{dx}\right)^2\right]}{\frac{d^2y}{dx^2}} \quad \text{—— (2)}$$

$$(x-h) = -(y-k)\frac{dy}{dx} \qquad \text{[substituting (2)]}$$

$$= \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]\left(\frac{dy}{dx}\right)}{\frac{d^2y}{dx^2}} \quad \text{—— (3)}$$

Substituting (2) and (3) in (1)

$$\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^2\left(\frac{dy}{dx}\right)^2}{\left(\frac{d^2y}{dx^2}\right)^2} + \frac{\left(1+\left(\frac{dy}{dx}\right)^2\right)^2}{\left(\frac{d^2y}{dx^2}\right)^2} = a^2.$$

$$\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^2\left[\left(\frac{dy}{dx}\right)^2+1\right]}{\left(\frac{d^2y}{dx^2}\right)^2} = \cancel{\left(\frac{d^2y}{dx^2}\right)^2} a^2.$$

$$\left[\left(\frac{dy}{dx}\right)^2+1\right]^3 = \left(\frac{d^2y}{dx^2}\right)^2 a^2$$

3

$y = a\sin x + b\cos x + x\sin x.$

$\frac{dy}{dx} = a\cos x + (-b\sin x) + x\cos x + \sin x.$

$\frac{d^2y}{dx^2} = -a\sin x - b\cos x + (-x\sin x) + \cos x + \cos x.$

$\frac{d^2y}{dx^2} = -y + 2\cos x.$

$y = Ae^{2x} + Be^{-2x}$

$\frac{dy}{dx} = Ae^{2x}(2) + Be^{-2x}(-2)$

$\frac{d^2y}{dx^2} = A(4)e^{2x} - 2Be^{-2x}(-2) = 4Ae^{2x} + 4Be^{-2x}$

$= 4(Ae^{2x} + Be^{-2x})$

$\frac{d^2y}{dx^2} = 4y \Rightarrow \frac{d^2y}{dx^2} - 4y = 0$

eq: 2

$(x-h)^2 + (y-k)^2 = a^2$ — ①

$2(x-h) + (2)(y-k)\frac{dy}{dx} = 0$

$(x-h) + (y-k)\frac{dy}{dx} = 0$ — ②

$1 + (y-k)\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0$ — ③

$(y-k)\frac{d^3y}{dx^3} + \left(\frac{dy}{dx}\right)\left(\frac{d^2y}{dx^2}\right) + 2\left(\frac{dy}{dx}\right)\left(\frac{d^2y}{dx^2}\right) = 0$ — ④

Substitute the value of $(y-k)$ from ③ to ④

$(y-k) = \cancel{-\frac{(x-h)}{dy/dx}} \quad \frac{-\left[\left(\frac{dy}{dx}\right)^2 + 1\right]}{d^2y/dx^2}$

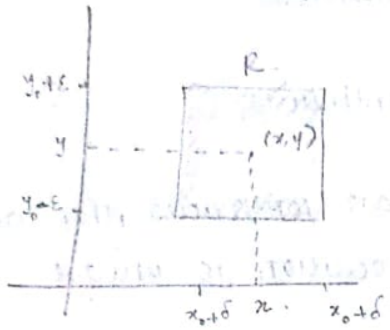
$-\frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]}{\frac{d^2y}{dx^2}} \frac{d^3y}{dx^3} + 3\left(\frac{dy}{dx}\right)\left(\frac{d^2y}{dx^2}\right) = 0$

$\Rightarrow 3y'y''^2 - (1 + y'^2)y''' = 0$

Existence and Uniqueness Theorem first order O.D.E

Theorem 1: Existence Theorem

Suppose that $f(x,y)$ is a continuous defined in some region R
$$R = \{ (x,y) : x_0 - \delta < x < x_0 + \delta , \quad y_0 - \varepsilon < y < y_0 + \varepsilon \}$$
containing the point (x_0, y_0). Then there exists a number δ_1 (possibly smaller than δ) so that a solution $y = f(x)$ to the general first order O.D.E $y' = f(x,y)$, $y(x_0) = y_0$ is defined for $x_0 - \delta_1 < x < x_0 + \delta_1$.



If $f(x,y)$ is continuous in a given region then there exists a solution within that region.

Eg: $y' = x - y + 1$, $y(1) = 2$

$$f(x,y) = x - y + 1$$

The function is continuous everywhere, hence according to existence theorem, there exists a solution,

$$x_0 = 1, \quad y_0 = 2$$

Here δ is our choice as there is no restriction laid down by the function.

However if $f(x,y) = \dfrac{x-y+1}{x-5}$, the function is not continuous at $x = 5$ so the δ can be chosen from 1 to 4.

Theorem 2 : Uniqueness Theorem

Suppose that $f(x,y)$ and $\frac{\partial f(x,y)}{\partial y}$ are continuous function defined on a range R as defined in theorem 1 then there exists a number δ_2 (possibly smaller than δ_1) so that the solution $y = f(x)$ to the general 1^{st} order O.D.E $y' = f(x,y)$ with initial condition $y(x_0) = y_0$ whose existence was garanteed by theorem 1 is the unique solution to general 1^{st} order differential equation. for $x_0 - \delta_2 < x < x_0 + \delta_2$

Eg: ~~$f(x,y) = x$~~ $y' = x - y + 1$

$$f(x,y) = x - y + 1 \quad , \text{continuous}$$

$$\frac{\partial f(x,y)}{\partial y} = -1 \quad , \text{continuous}$$

Since $f(x,y)$ and $\frac{\partial f(x,y)}{\partial y}$ are continuous, then according to the uniqueness theorem, the solution is unique.

Solution of a differential equation

A solution of a differential equation is a functional relation between the variables involved, which satisfy the given equations.

General solution

The solution of a differential equation in which the number of orbitary constants is equal to the order of the differential equation is called the general/ complete solution.

Particular solution

If particular values are given to the arbitrary constants in the general solution, then the solution so obtained is called general solution.

Linearly independant solution.

Two solutions $y_1(x)$ and $y_2(x)$ of the differential equation

$$\frac{\partial^2 y}{\partial x^2} + a_1(x)\frac{\partial y}{\partial x} + a_2(x)\, y = 0 \quad \text{—— (1)}$$

are said to be linearly independant if $c_1 y_1 + c_2 y_2 = 0$ such that $c_1 = 0$, and $c_2 = 0$. If c_1 and c_2 are non-zero then the two solutions y_1 and y_2 are said to be linearly dependant.

Note:- If $y_1(x)$ and $y_2(x)$ are any two solutions of eqn ① then their linear combination $c_1y_1 + c_2y_2$ where c_1 and c_2 are constants is also solution of ①.

Eg ① Show that $x^2 + 4y = 0$ is a solution of $\left(\frac{dy}{dx}\right)^2 + x\left(\frac{dy}{dx}\right) - y = 0$. — ①

$$x^2 + 4y = 0$$

$$2x + 4\left(\frac{dy}{dx}\right) = 0 \Rightarrow x + 2\frac{dy}{dx} = 0$$

$$-\frac{x}{2} = +\left(\frac{dy}{dx}\right). — ②$$

Substituting ② in ①

$$L.H.S = \frac{x^2}{4} + x\left(-\frac{x}{2}\right) + \frac{x^2}{4}.$$

$$= \frac{x^2}{2} - \frac{x^2}{2}$$

$$= 0 = R.H.S$$

$\therefore$ L.H.S = R.H.S, hence $x^2 + 4y = 0$ is a solution of ①.

② $y = \frac{c}{x} + d$, is a solution of $\frac{d^2y}{dx^2} + \frac{2}{x}\left(\frac{dy}{dx}\right) = 0$

$$\frac{dy}{dx} = -\frac{c}{x^2}$$

$$\frac{d^2y}{dx^2} = -\frac{c(-2)}{x^3} = \frac{2c}{x^3}$$

$$L.H.S = \frac{2c}{x^3} + \frac{2}{x}\left(\frac{-c}{x^2}\right) = \frac{2c}{x^3} - \frac{2c}{x^3} = 0 = R.H.S.$$

$\therefore$ L.H.S = R.H.S, hence $y = \frac{c}{x} + d$ is the solution

(Q3) Find the differential equation whose set of independent solutions are e^x, $x.e^x$

Ans:

$$y = c_1 e^x + c_2 x.e^x$$

$$\frac{dy}{dx} = c_1 e^x + c_2 (x e^x + e^x). = \underbrace{c_1 e^x + c_2 x e^x}_{y} + c_2 e^x.$$

$$\frac{d^2y}{dx^2} = c_1 e^x + c_2 [x e^x + e^x + e^x]$$

$$\frac{d^2y}{dx^2} = c_1 e^x + c_2 x.e^x + c_2 2e^x.$$

$$\frac{d^2y}{dx^2} = y + 2\left(\frac{dy}{dx} - y\right) = 2\frac{dy}{dx} - y$$

The required differential equation :

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 0.$$

(Q4) $(y-x)\,dy - (y^2 - x^2)\,dx = 0$; $y = -(1+x)$

Ans :

$$\frac{dy}{dx} = \frac{y^2 - x^2}{y - x} = y + x. \Rightarrow \frac{dy}{dx} - y - x = 0 \quad \text{--- (1)}$$

Now, $y = -(1+x)$.

$$\frac{dy}{dx} = -1$$

$$\text{L.H.S} = \frac{dy}{dx} - y - x = -\cancel{1} + \cancel{1} + \cancel{x} - \cancel{x}$$

$$= 0 = \text{R.H.S}$$

$\therefore$ L.H.S = R.H.S

$y = -(1+x)$ is the solution of the given D.E

(Q5) $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 0$; $y = e^{3x}(A + Bx)$

$$\frac{dy}{dx} = e^{3x}[B] + (A + Bx)e^{3x}(3)$$

$$= Be^{3x} + 3Ae^{3x} + 3Be^{3x}.x$$

$$\frac{d^2y}{dx^2} = 3Be^{3x} + 9Ae^{3x} + 3B[e^{3x} + xe^{3x}(3)]$$

$$\frac{d^2y}{dx^2} = 3Be^{3x} + 9Ae^{3x} + 3Be^{3x} + 9Bxe^{3x}.$$

$$= 6Be^{3x} + 9Ae^{3x} + 9Bxe^{3x} = 6Be^{3x} + 9(e^{3x})(A+Bx)$$

$$L.H.S = \frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y$$

$$= 6Be^{3x} + 9Ae^{3x} + 9Bxe^{3x} - 6(Be^{3x} + 3Ae^{3x} + 3Bxe^{3x}) + 9y$$

$$= \cancel{6Be^{3x}} + 9Ae^{3x} + 9Bxe^{3x} - \cancel{6Be^{3x}} - 18Ae^{3x} - 18Bxe^{3x} + 9y.$$

$$= -9Ae^{3x} - 9Bxe^{3x} + 9(e^{3x})A + 9e^{3x}Bx$$

$$= 0 = R.H.S$$

$\therefore$ L.H.S = R.H.S

Hence $y = e^{3x}(A+Bx)$ is a solution of the given D.E.

Q6. $xy' - 3y + 3 = 0$, $y(0) = 0$, $y(1) = 1$, $y(0) = 1$.

$$x\frac{dy}{dx} - 3y + 3 = 0$$

$$\int \frac{dy}{3y-3} = \int \frac{1}{x}\,dx$$

$$\frac{1}{3}\int \frac{dy}{y-1} = \int \frac{1}{x}\,dx \Rightarrow \frac{1}{3}\log_e|y-1| = \log_e|x| + \log_e C$$

$$(y-1)^{1/3} = Cx$$

$$y-1 = Cx^3.$$

If $x = 0$, $y = 0$, the condition is not possible, hence for this initial value there exists no solution.

If $x = 1$, $y = 1$, there is a unique solution, i.e, $y = 1$.

If $x = 0$, $y = 1$, there are infinitely many solutions, i.e, $y = 1 + Cx^3$. (not unique)

Using existence and uniqueness theorem,

$$\frac{dy}{dx} = \frac{3y-3}{x}, \quad \frac{\partial y}{\partial x} = \frac{3}{x}.$$

In the first case, $x = 0$, hence there is not a neighbourhood about 0 that allows $\frac{dy}{dx}$ and $\frac{\partial y}{\partial x}$ to be continuous, hence the solution doesn't exist. and is not unique.

Q Show that $y' = -\frac{x}{y}$ $y(0) = 1$ cannot be extended beyond the interval $(-1, 1)$.

Ans:

$$\frac{dy}{dx} = f(x,y) = -\frac{x}{y}, \qquad \frac{\partial y}{\partial x} = +x\left(\frac{1}{y^2}\right)$$

solution of the given differential equation.

$$\int y\, dy = \int -x.dx$$

$$c - x^2 = y^2 + c. \qquad \text{At} \quad x = 0, \; y = 1$$

$$\therefore \quad c = 1$$

$$x^2 + y^2 = 1.$$

$f(x,y)$, and $\frac{\partial y}{\partial x}$ are not continuous at $y = 0$

hence x^2 should be less than 1. (not less than or equal to because at $x = 1$, $y = 0$)

$$x^2 < 1$$

$$\therefore \quad \underline{\underline{-1 < x < 1}}$$

I Variable Separable Equation

Q1.

$$2\cosh x \cos y . dx = \sinh x \sin y \, dy$$

$$\frac{2\cosh x}{\sinh x} . dx = \frac{\sin y}{\cos y} dy$$

Integrating both sides,

$$\int \frac{2\cosh x}{\sinh x} . dx = \int \frac{\sin y}{\cos y} . dy$$

$$2 \ln \sinh x = -\ln \cos y + \ln C$$

$$\cancel{(\sinh x)^2 = c \cos y}$$

$$\underline{\underline{\sinh^2 x \cos y = C}}$$

Q2 $2xy\,dx + 3x^2\,dy = 0$

$2xy\,dx = -3x^2.dy$

$\int \frac{2x}{-3x^2}\,dx = \int \frac{dy}{y}$

$-\frac{2}{3}\int \frac{1}{x}.dx = \int \frac{dy}{y} \Rightarrow -\frac{2}{3}\ln x = \ln y$

$-\frac{2}{3}\ln x + \ln C = \ln y$

$\ln C = \ln y + \ln x^{2/3} = \ln(y\,x^{2/3})$

$$y\,x^{2/3} = C$$

II Homogeneous Equation

(i) $\frac{dy}{dx} = \frac{a_1 x + b_1 y}{a_2 x + b_2 y}$; Assume $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$.

$y = vx$.

$\frac{dy}{dx} = v + x\frac{dv}{dx}$

Substituting, $\frac{dv}{dx} + v = \frac{a_1 x + b_1 v x}{a_2 x + b_2 v x} \Rightarrow v = \frac{a_1 + b_1 v}{a_2 + b_2 v}$.

Q. $\frac{dy}{dx} = \frac{-(2x+y)}{x+5y}$.

$F(x,y) = \frac{-(2x+y)}{x+5y}$

$F(\lambda x, \lambda y) = \frac{-2\lambda x + \lambda y}{\lambda x + 5\lambda y} = \left(-\frac{2x+y}{x+5y}\right)\lambda^0$

$F(\lambda x, \lambda y) = \lambda^0 F(x,y)$

∴ The given function is homogeneous.

$$y = vx$$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$-\frac{(2x+y)}{x+5y} = v + x\frac{dv}{dx}$$

$$\frac{-(2x+vx)}{x+5vx} = v + x\frac{dv}{dx} \Rightarrow -\frac{(2+v)}{1+5v} = v + x\frac{dv}{dx}$$

$$-x\frac{dv}{dx} = v + \frac{2+v}{1+5v} = \frac{v+5v^2+2+v}{1+5v}$$

$$-x\frac{dv}{dx} = \frac{5v^2+2v+2}{1+5v}$$

$$\int \frac{1+5v}{5v^2+2v+2}\,dv = -\int \frac{1}{x}.dx$$

Let $5v^2+2v+2 = t$

$$(10v+2).dv = dt$$

$$(5v+1).dv = \frac{dt}{2}$$

$$\int \frac{dt}{2t} = -\int \frac{1}{x}.dx$$

$$\frac{1}{2}\log|5v^2+2v+2| = -\log|x| + \log c$$

$$\log x^2(5v^2+2v+2) = \cancel{2}\ \cancel{\log x^2}\ \cancel{+}\ \log c^2.$$

$$x^2\left[\frac{5y^2}{x^2} + \frac{2y}{x} + 2\right] = \cancel{2}.c$$

$$x^2\left[\frac{5y^2+2xy+2x^2}{x^2}\right] = c$$

$$5y^2 + 2xy + 2x^2 = c.$$

Q. Solve: $3x^5 - y(y^2 - x^3)\frac{dy}{dx} = 0$

Put $u = x^3, \; v = y^2$

$$3x^3 . x^2 - (y^3 - x^3 y)\frac{dy}{dx} = 0$$

$$u = x^3$$

$$\frac{du}{dx} = 3x^2 \quad , \quad \frac{dv}{dy} = 2y.$$

$$\frac{dy}{dv} \times \frac{du}{dx} = \frac{1}{2y} \times 3x^2 = \frac{3}{2}\frac{x^2}{y}$$

$$\left(\frac{dy}{dx}\right)\left(\frac{du}{dv}\right) = \frac{3}{2}\frac{x^2}{y} \Rightarrow \frac{dy}{dx} = \left(\frac{3}{2}\frac{x^2}{y}\right)\frac{dv}{du}$$

$$3u . u^{2/3} - (v.v^{1/2} - u.v^{1/2})\left(\frac{3}{2}\frac{u^{2/3}}{v^{1/2}}\right)\left(\frac{dv}{du}\right) = 0$$

$$3u^{5/3} = (v^{3/2} - u.v^{1/2})\left(\frac{3}{2}\frac{u^{2/3}}{v^{1/2}}\right)\frac{dv}{du}$$

$$3u^{5/3} = (v-u)\left(\frac{3}{2}u^{2/3}\right)\frac{dv}{du}.$$

$$\frac{du}{dv} = \frac{(v-u)(\cancel{3}u^{2/3})}{2(\cancel{3})u^{5/3}} = \left(\frac{v-u}{2}\right)u^{-1}.$$

$$\frac{du}{dv} = \frac{v-u}{2u} = \cancel{\frac{v}{2u} - \frac{1}{2}}.$$

$$F(u,v) = \frac{v-u}{2u}$$

$$F(\lambda u, \lambda v) = \lambda^0 F(u,v)$$

Therefore this differential equation is homogeneous.

$$u = mv$$

$$\frac{du}{dv} = m + v\frac{dm}{dv}$$

$$m + v\frac{dm}{dv} = \frac{v - mv}{2mv} = \frac{1-m}{2m}.$$

$$v\frac{dm}{dv} = \frac{1-m}{2m} - m = \frac{1-m-2m^2}{2m} =$$

$$\int \frac{2m}{-2m^2 - m + 1}\,dm = \int \frac{1}{v}\,dv$$

$$= \int \frac{2m}{2m^2 - m + 1}\, dm = \int \frac{1}{V}\, dV$$

$2m^2 + m - 1$

$2m^2 + 2m - m - 1$

$2m(m+1) - 1(m+1)$

$(2m-1)(m+1)$

$$-\int \frac{2m}{2m^2 + m - 1} \cdot dm = \int \frac{1}{V}\, dV$$

$$-\int \frac{2m}{(2m-1)(m+1)}\, dm = \int \frac{1}{V} dV$$

$$\frac{A}{2m-1} + \frac{B}{m+1} = \frac{2m}{(m+1)(2m-1)}$$

$$A(m+1) + B(2m-1) = 2m$$

$$Am + 2Bm + (A - B) = 2m$$

$$m(A + 2B) + (A - B) = 2m + 0$$

$$\therefore A = B$$

$$A + 2B = 2 \Rightarrow 3A = 2$$

$$A = \frac{2}{3} = B.$$

$$-\int \frac{2}{3(2m-1)} + \frac{2}{3(m+1)} \cdot dm = \int \frac{1}{V} \cdot dV$$

$$\frac{2}{3} \int \left(\frac{1}{2m-1} + \frac{1}{m+1} \right) \cdot dm = -\int \frac{1}{V} dV$$

$$\frac{2}{3} \left[\frac{\log |2m-1|}{2} + \log_e |m+1| \right] = -\log_e V + \log_e C$$

$$\log_e (2m-1)^{1/3} + \log_e |m+1|^{2/3} = \log_e \frac{C}{V}$$

$$\log_e (2m-1)^{1/3} (m+1)^{2/3} = \log_e \frac{C}{V}$$

$$\left(2\frac{u}{v} - 1\right)^{1/3} \left(\frac{u}{v} + 1\right)^{2/3} = \frac{C}{V}$$

$$\left(2\frac{x^3}{y^2} - 1\right)^{1/3} \left(\frac{x^3}{y^2} + 1\right)^{2/3} = \frac{C}{y^2}$$

$$\frac{(2x^3 - y^2)^{1/3} (x^3 + y^2)^{2/3}}{y^{2/3} \cdot y^{4/3}} = \frac{C}{y^2} \Rightarrow \underline{\underline{(2x^3 - y^2)^{1/3} (x^3 + y^2)^{2/3} = C}}$$

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(ii) $\frac{dy}{dx} = \frac{ax+by+c}{a'x+b'y+c'}$ $\left(\frac{a}{a'} \neq \frac{b}{b'}\right)$

$x = X+h$, $y = Y+K$

$\frac{dy}{dx} = \frac{dY}{dX}$.

$\frac{dY}{dX} = \frac{a(X+h)+b(Y+k)+c}{a'(X+h)+b'(Y+k)+c'}$

$\frac{dY}{dX} = \frac{aX+bY+(ah+bk+c)}{a'X+b'Y+(a'h+b'k+c')}$.

Choose h, K, such That

$ah+bK+c = 0$

$a'h+b'k+c' = 0$.

$\frac{dY}{dX} = \frac{aX+bY}{a'X+b'Y}$

Then continue with $Y = VX$.

Eg: $\frac{dy}{dx} = \frac{3x-y-5}{-x+3y+7}$.

$x = X+h$, $y = Y+k$

$X = x-h$, $Y = y-K$

$3h - K - 5 = 0$

$-h + 3K + 7 = 0$

$3h - K - 5 = 0$

$-3h + 9k + 21 = 0$

$8k + 16 = 0 \Rightarrow k = -2$, $h = 1$.

$X = x-1$, $Y = y+2$

$\frac{dY}{dX} = \frac{3X-Y+(3(1)-1(-2)-5)}{-X+3Y} = \frac{3X-Y}{-X+3Y}$

Let $Y = VX$

$\frac{dY}{dX} = V + X\frac{dV}{dX}$

$V + X\frac{dV}{dX} = \frac{3X-VX}{-X+3VX} = \frac{3-V}{-1+3V}$

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$$x\frac{dv}{dx} = \frac{3-v}{-1+3v} - v$$

$$= \frac{3-v - v(-1+3v)}{-1+3v}$$

$$x\frac{dv}{dx} = \frac{3-\cancel{v}+\cancel{v}-3v^2}{3v-1} = \frac{3-3v^2}{3v-1}$$

$$\int \frac{3v-1}{3(1-v^2)}\,dv = \int \frac{1}{x}\,dx$$

$$\int \frac{\cancel{3}v}{\cancel{3}(1-v^2)}\,.dv - \int \frac{1}{3(1-v^2)}\,dv = \int \frac{1}{x}\,dx$$

$$1-v^2 = t$$

$$-2v.dv = dt$$

$$vdv = \frac{-dt}{2}$$

$$-\frac{1}{2}\int \frac{dt}{t} - \frac{1}{3}\int \frac{1}{1-v^2}\,.dv = \int \frac{1}{x}\,dx$$

$$-\frac{1}{2}\log_e(1-v^2) - \frac{1}{3}\cdot\frac{1}{2}\log\left|\frac{1+v}{1-v}\right| = \log_e x + \log_e c$$

$$-\log\sqrt{1-v^2} - \log\left(\frac{1+v}{1-v}\right)^{1/6} = \log_e xc$$

$$-\log_e\sqrt{1-\frac{y^2}{x^2}} - \log\left(\frac{1+\frac{y}{x}}{1-\frac{y}{x}}\right)^{1/6} = \log_e xc$$

$$+\log_e\frac{\sqrt{x^2-y^2}}{x} + \log\left(\frac{x+y}{x-y}\right)^{1/6} = -\log_e(xc)$$

$$\frac{\sqrt{x^2-y^2}}{x}\,\frac{(x-y)^{1/6}}{(x+y)^{1/6}} = \frac{1}{xc}.$$

$$\frac{(x-y)^{1/2+1/6}}{\cancel{x}}\,(x+y)^{1/2-1/6} = \frac{1}{c}$$

$$\frac{(x-y)^{2/3}}{\cancel{x}}\,(x+y)^{1/3} = \frac{1}{c}$$

$$\frac{(x-1-y-2)^{2/3}\,(x-1+y+2)^{1/3}}{\cancel{(x-1)^2}} = \frac{1}{c}$$

$$\frac{(x-y-3)^{2/3}\,(x+y+1)^{1/3}}{\cancel{(x-1)^2}} = \frac{1}{c} \neq k$$

$$(x-y-3)^2\,(x+y+1) = \left(\frac{1}{c}\right)^3 = k.$$

Solve:

$3x^5 - y$ $\quad (2x^2+3y^2-7)\,x - (3x^2+2y^2-8)\,y y' = 0$

$$x^2 = u, \quad y^2 = v$$

$$2x\,dx = du, \quad 2y\,dy = dv.$$

$$dx = \frac{du}{2x}, \quad dy = \frac{dv}{2y}$$

$$\frac{dy}{dx} = \frac{dv}{2y}\frac{(2x)}{du} = \frac{x}{y}\frac{dv}{du}$$

$$(2u+3v-7)\sqrt{u} - (3u+2v-8)\sqrt{v}\,\frac{dy}{dx} = 0$$

$$\frac{2u^{3/2} + 3v\sqrt{u} - 7\sqrt{u}}{3u\sqrt{v} + 2v^{3/2} - 8\sqrt{v}} = \frac{dy}{dx} = \frac{\sqrt{u}}{\sqrt{v}}\frac{dv}{du}.$$

$$\frac{\cancel{\sqrt{u}}\,(2u+3v-7)}{\cancel{\sqrt{v}}\,(2v+3u-8)} = \frac{\cancel{\sqrt{u}}}{\cancel{\sqrt{v}}}\frac{dv}{du}$$

$$\frac{dv}{du} = \frac{2u+3v-7}{2v+3u-8}$$

$$u = U+h, \quad v = V+k.$$

$$2h+3k-7 = 0 \times 3 \Rightarrow 6h+9k-21 = 0$$

$$3h+2k-8 = 0 \times 2 \Rightarrow -6h \mp 4k \mp 16 = 0$$

$$5k - 5 = 0$$

$$k = 1$$

$$h = \frac{7-3}{2} = 2.$$

$$\frac{dU}{dV} = \frac{2(U+2)+3(V+1)-7}{2(V+1)+3(U+2)-8} = \frac{2U+3V}{3U+2V}$$

let $\quad u = tV$

$$\frac{du}{dV} = t + V\frac{dt}{dV}$$

$$t + V\frac{dt}{dV} = \frac{2tV + 3V}{3tV + 2V} = \frac{2t+3}{3t+2}$$

$$V\frac{dt}{dV} = \frac{2t+3}{3t+2} - t = \frac{\cancel{2t}+3-3t^2-\cancel{2t}}{3t+2}$$

$$V\frac{dt}{dV} = \frac{3(1-t^2)}{3t+2}$$

$$\int \frac{3t+2}{3(1-t^2)}\,dt = \int \frac{1}{V}\,dV$$

$$\int \frac{t}{1-t^2}\,dt + \frac{2}{3}\int \frac{1}{1-t^2}\,dt = \log_e V + \log_e C$$

$1-t^2 = m$

$-2t\,dt = dm \Rightarrow t\,dt = -\frac{1}{2}dm$

$$-\frac{1}{2}\int \frac{dm}{m} + \frac{\cancel{2}}{3}\frac{1}{\cancel{2}}\log\left|\frac{1+t}{1-t}\right| = \log_e VC$$

$$\log_e(1-t^2)^{-1/2} + \log_e\left(\frac{1+t}{1-t}\right)^{1/3} = \log_e VC.$$

$$\log_e \frac{(1+t)^{1/3}}{(1-t)^{1/2}(1+t)^{1/2}(1-t)^{1/3}} = \log_e VC$$

$$\frac{1}{(1+t)^{1/6}(1-t)^{5/6}} = VC.$$

$$\frac{1}{\left(1+\frac{u}{V}\right)^{1/6}\left(1-\frac{u}{V}\right)^{5/6}} = VC$$

$$\frac{1}{(V+u)^{1/6}(V-u)^{5/6}} = C$$

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$U = u - 2, \quad V = v - 1$

$$\frac{1}{(V-1+U-2)^{1/6}\,(V-1-U+2)^{5/6}} = C_1$$

$$(V+U-3)^{1/6}\,(V-U+1)^{5/6} = \frac{1}{C_1} = C_2$$

$(V+U-3)$

Ans: $(x^2 \cdot y^2 - 1)^5 = (x^2 + y^2 - 3)\,c$

(iii) $\dfrac{dy}{dx} = \dfrac{ax+by+c}{a'x+b'y+c'}$, where $\dfrac{a}{a'} = \dfrac{b}{b'}$

Eg: $\dfrac{dy}{dx} = \dfrac{x+y+1}{2x+2y+3}$.

Put $x+y=t$.

$$1+\frac{dy}{dx} = \frac{dt}{dx} \Rightarrow \frac{dy}{dx} = \frac{dt}{dx} - 1$$

$$\frac{dt}{dx} - 1 = \frac{t+1}{2t+3}.$$

$$\frac{dt}{dx} = \frac{t+1+2t+3}{2t+3} = \frac{3t+4}{2t+3}.$$

$$\int \frac{\cancel{\varnothing}\, t+3}{3(-1)+(4)}\,dt = \int dx$$

$$\frac{\cancel{3}}{2}\int \frac{t}{t+2}\,dt + \frac{3}{2}\int \frac{1}{t+2}\,dt = \int dx$$

$$\frac{1}{2}\int \frac{t+2}{t+2}\,dt - \frac{2}{2}\int \frac{1}{t+2}\,dt + \frac{3}{2}\int \frac{1}{t+2}\,dt = \int dx$$

$$\frac{1}{2}t - \log_e(t+2) + \frac{3}{2}\log_e(t+2) = x + C$$

$$\frac{t}{2} + \frac{1}{2}\log_e(t+2) = x + C$$

$$\frac{t}{2} + \log_e\sqrt{t+2} = x + C.$$

$$\frac{x+y}{2} + \log_e\sqrt{x+y+2} = x + C.$$

$$\log_e\sqrt{x+y+2} = \frac{x}{2} - \frac{y}{2} + C_1.$$

$$2\log_e\sqrt{x+y+2} = x - y + C_2$$

$$x+y+2 = e^{x-y}\,C_3.$$

$$x+y+2 = C_3\,e^{x-y}$$

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$$\int \frac{2t+3}{3t+4}\,dt = \int dx$$

$$\int \frac{2t}{3t+4}\,dt + 3\int \frac{1}{3t+4}\,dt = \int dx$$

$$\frac{2}{3}\int \frac{3t+4-4}{3t+4}\,dt + 3\int \frac{1}{3t+4}\,dt = \int dx$$

$$\frac{2}{3}\int \frac{3t+4}{3t+4}\,dt - \frac{4}{3}\int \frac{1}{3t+4}\,dt + 3\int \frac{1}{3t+4}\,dt = \int dx$$

$$\frac{2}{3}t + \left(3 - \frac{4}{3}\right)\int \frac{1}{3t+4}\,dt = x + C$$

$$\frac{2t}{3} + \frac{5}{3}\,\frac{\log(3t+4)}{3} = x + C.$$

$$\frac{2t}{3} + \frac{5}{9}\log(3t+4) = x + C$$

$$\frac{2(x+y)}{3} + \frac{5}{9}\log_e(3x+3y+4) = x + C$$

$$\frac{5}{9}\log_e(3x+3y+4) = \frac{x}{3} - \frac{2y}{3} + C.$$

$$\frac{5}{3}\log_e(3x+3y+4) = x - 2y + C.$$

$$\log_e(3x+3y+4)^5 = 3x - 6y + C$$

$$\underline{(3x+3y+4)^5 = Ce^{3x-6y}}$$

$$\therefore\ 3x+3y+4 = Ce^{3x-6y}$$

$$3x+3y+4 = Ae^{3x-6y}$$

Exact differential equation.

$u = f(x,y) \qquad x = f(t) \;,\; y = g(t)$

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}$$

called (total derivative)

$u(x,y) = c$

$du = 0$

$M dx + N dy = 0.$

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy \quad \text{—①}$$

$$= M(x,y)\,dx + N(x,y)\,dy \quad \text{—②}$$

Comparing ① and ②

$$\frac{\partial u}{\partial x} = M \quad , \quad N = \frac{\partial u}{\partial y}$$

$$\frac{\partial M}{\partial y} = \frac{\partial^2 u}{\partial x \partial y} \quad , \quad \frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial y \partial x}.$$

$$\therefore \text{since } \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

Hence if an equation is exact, then the condition is $M_y = N_x$. (and vice versa).

Solution: $\int M dx + \int (\text{terms independant of } x \text{ in } N).dy = c$

Eg: $y^2 dx + 2xy\, dy = 0$

$M(x,y) = y^2 \quad ; \quad \frac{\partial M}{\partial y} = 2y$

$N(x,y) = 2xy \quad ; \quad \frac{\partial N}{\partial x} = 2y$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$\therefore$ It is an exact differential equation.

$\psi_x = M$

$\psi = \int M.dx = \int y^2.dx = xy^2 + h(y)$

$\psi_y = x(2y) + h'(y)$

$2xy = 2xy + h'(y)$

$h'(y) = 0$

$h(y) = 0 + C$

$\psi = xy^2 + C_1$

The solution is $\underline{xy^2 = C_2}$

2. $(2x\sin y + y^3e^x)dx + (x^2\cos y + 3y^2e^x)dy = 0$

$M = 2x\sin y + y^3e^x$; $M_y = 2x\cos y + e^x(3y^2)$

$N = x^2\cos y + 3y^2e^x$; $N_x = 2x\cos y + 3y^2e^x$

$$M_y = N_x$$

$\therefore$ It is an exact differential equation.

$\psi_x = M \Rightarrow \psi = \int M\,dx$

$= \int 2x\sin y + y^3e^x \cdot dx$

$= 2\sin y \cdot \frac{x^2}{2} + y^3e^x + h(y)$

$\psi_y = x^2\cos y + 3y^2e^x + h'(y)$

$\psi_y = N = x^2\cos y + 3y^2e^x$

$\cancel{x^2\cos y} + \cancel{3y^2e^x} = \cancel{x^2\cos y} + \cancel{3y^2e^x} + h'(y)$

$h'(y) = 0$

$h(y) = C_1$

$\psi = x^2\sin y + y^3e^x + C_1$

The required solution,

$$\underline{x^2\sin y + y^3e^x = c.}$$

<u>Note</u>:- Those factors which when multiplied to a differential equation to make it exact are called integrating factor.

Eg: $y\,dx + 2x\,dy = 0$ $\rightarrow$ not exact

y is the integrating factor as after its multiplication it makes the differential equation exact.

$y^2dx + 2xy\,dy = 0$ $\rightarrow$ exact

I.F ⇒ If the differential equation $Mdx + Ndy = 0$ is not exact and

(i) $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = g(x)$, is a function of x alone, then

$$I.F = e^{\int g(x)\,dx}$$

(ii) $\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = h(y)$, is a function of y alone, then

$$I.F = e^{\int h(y).dy}$$

Q. Solve :

$$y + (3xy + y^2 - 1)\frac{dy}{dx} = 0$$

$M = y \quad ; \quad M_y = 1$

$N = 3xy + y^2 - 1 \quad ; \quad N_x = 3y$

$\therefore M_y \neq N_x$; hence its not exact.

$$\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = \frac{1}{y}(3y - 1) = h(y)$$

$$I.F = e^{\int h(y).dy} = e^{\int \frac{3y-1}{y}.dy}$$

$$= \cancel{e^{\left(\frac{3y^2}{2} - y\right)}} \quad e^{\int 3 - \frac{1}{y}.dy} = e^{3y - \log_e y}$$

$$I.F = \frac{e^{3y}}{y}$$

$$y\,dx + (3xy + y^2 - 1)dy = 0$$

$$e^{3y}.dx + \left(3xe^{3y} + ye^{3y} - \frac{e^{3y}}{y}\right)dy = 0$$

$M_y = e^{3y}(3) \qquad N_x = 3e^{3y}$

$$\psi_x = M \Rightarrow \psi = \int e^{3y}.dx = xe^{3y} + h(y)$$

$$\psi_y = xe^{3y}(3) + h'(y)$$

$$\cancel{3xe^{3y}} + ye^{3y} - \frac{e^{3y}}{y} = \cancel{3xe^{3y}} + h'(y)$$

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$$h(y) = \int y e^{3y} - \frac{e^{3y}}{y} \cdot dy$$

$$h(y) = \left[y e^{3y}(3) - 3\int e^{3y} \cdot dy\right] - \int \frac{e^{3y}}{y} \cdot dy$$

$$= \left[3y e^{3y} - 9e^{3y}\right] - \int \frac{e^{3y}}{y} \cdot dy$$

$$= 3y e^{3y} - 9e^{3y} - \left[\frac{1}{y} \cancel{e^{3y}(3) - \int e^{3y}(3)}\right.$$

Linear differential equations.

$$\left(\frac{dy}{dx}\right) + P(x)y = Q(x)$$

$\frac{dy}{dx}$ and y are both of degree 1.

$$[P(x)y - Q(x)].dx + dy = 0$$

Let $M = P(x)y - Q(x)$, $N = 1$

$$\frac{\partial M}{\partial y} = P(x) \quad , \quad \frac{\partial N}{\partial x} = 0$$

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = 1\,(P(x) - 0) = P(x)$$

$$I.F = e^{\int P(x)\,dx}.$$

$$e^{\int P(x).dx}\frac{dy}{dx} + e^{\int P(x).dx}P(x)\,y = Q(x)e^{\int P(x).dx}.$$

$$\frac{d}{dx}\left(y\,e^{\int P(x).dx}\right) = Q(x)\,e^{\int P(x).dx}.$$

$$d\left(y\,e^{\int P(x).dx}\right) = \left[Q(x)\,e^{\int P(x).dx}\right].dx$$

Integrating,

$$\boxed{y\,e^{\int P(x).dx} = \int Q(x)\,e^{\int P(x).dx}.dx}$$

Eg ①
$$\frac{dy}{dx} + \left(\frac{2x+1}{x}\right)y = e^{-2x}$$

This of the form $\frac{dy}{dx} + Py = Q$

$$P = \frac{2x+1}{x}, \quad Q = e^{-2x}$$

$$I.F = e^{\int \frac{2x+1}{x}dx}$$

$$I_1 = \int \frac{2x+1}{x}dx = \int\left(2 + \frac{1}{x}\right)dx = 2x + \log_e x$$

$$I.F = e^{2x}.x = xe^{2x}$$

$$y x e^{2x} = \int e^{-2x} \cdot x e^{2x} . dx$$

$$y x e^{2x} = \int x . dx$$

$$\Rightarrow \quad y x e^{2x} = \frac{x^2}{2} + C.$$

(2) Solve the following IVP,

$$(x^2+1)\frac{dy}{dx} + 4xy = x \quad ; \quad y(2) = 1$$

$$\frac{dy}{dx} + \left(\frac{4x}{x^2+1}\right) y = \frac{x}{x^2+1}$$

$$P = \frac{4x}{x^2+1} \quad , \quad Q = \frac{x}{x^2+1}$$

$$I.F = e^{\int P . dx} = e^{\int \frac{4x}{x^2+1} . dx}$$

$$I_1 = \int \frac{4x}{x^2+1} . dx$$

Let $x^2+1 = t$

$2x \, dx = dt$

$$I_1 = \int \frac{2 \, dt}{t} = 2\log_e t = \log_e t^2$$

$$I.F = e^{\log_e t^2} = t^2 = (x^2+1)^2$$

$$(x^2+1)^2 y = \int \frac{(x^2+1)^{\cancel{2}}(x)}{\cancel{x^2+1}} dx$$

$$= \int x^3 + x \; dx$$

$$(x^2+1)^2 y = \frac{x^4}{4} + \frac{x^2}{2} + C$$

At $y = 1, \quad x = 2$

$$5^2 = \frac{\cancel{16}^{4}}{\cancel{4}} + \frac{\cancel{4}^{2}}{\cancel{2}} + C \Rightarrow 25 - 6 = C.$$

$C = 19$

$$\therefore (x^2+1)^2 y = \frac{x^4}{4} + \frac{x^2}{2} + 19$$

Bernoulli's Equation

$$\frac{dy}{dx} + P(x)y = Q(x)\,y^n$$

$$\frac{1}{y^n}\frac{dy}{dx} + \frac{P(x)}{y^{n-1}} = Q(x)$$

Put $\frac{1}{y^{n-1}} = z$

$$(1-n)\,y^{1-n-1}\frac{dy}{dx} = \frac{dz}{dx}.$$

$$\frac{1}{y^n}\frac{dy}{dx} = \frac{1}{1-n}\frac{dz}{dx}$$

$$\left(\frac{1}{1-n}\right)\frac{dz}{dx} + P(x)z = Q(x)$$

$$\frac{dz}{dx} + (1-n)P(x)\,z = Q(x)\,(1-n)$$

eg ① $\frac{dy}{dx} + y = xy^3$

Let $z = \frac{1}{y^2}$

$$\frac{dz}{dx} = \frac{-2}{y^3}\frac{dy}{dx}.$$

Now, $\frac{dz}{dx} + (1-n)P(x)z = Q(x)(1-n)$

$$\cancel{-\frac{2}{y^3}\frac{dy}{dx} + \frac{(-2)(1)}{y^2} = x(-2)}$$

$$\cancel{\frac{1}{y^3}\frac{dy}{dx} + \frac{1}{y^2} = x.}$$

$$\frac{dz}{dx} + (-2)(1)z = -2x$$

$$I.F = e^{\int -2x\,dx} = e^{-2\int x\,dx} = e^{-\frac{2x^2}{2}} = e^{-2x}$$

$$\cancel{e^{-x^2}x = \int -2x.e^{-x^2}dx}$$

$$e^{-2x}.x = \int -2x.e^{-2x}dx$$

$$e^{-2x}.x = -2\int xe^{-2x}.dx$$

$$xe^{-2x} = -2\left[\frac{xe^{-2x}}{-2} + \int\frac{e^{-2x}}{2}dx\right] = -2\cdot\left[\frac{xe^{-2x}}{-2} - \frac{1}{4}e^{-2x}\right] + C$$

$$x e^{-2x} = x e^{-2x} + \frac{1}{2} e^{-2x} + C$$

$$\frac{e^{-2x}}{y^3} = x e^{-2x} + \frac{1}{2} e^{-2x} + C$$

$$\frac{1}{y^2} = x + \frac{1}{2} + Ce^{2x}$$

(2)

$$\frac{dy}{dx} + \frac{y}{2x} = \frac{x}{y^3}, \quad y(1) = 2$$

$$\frac{dy}{dx} + \left(\frac{1}{2x}\right) y = x y^{-3}$$

$$P = \frac{1}{2x}, \quad Q = \cancel{y}\, x$$

Put $\frac{1}{y^{-3-1}} = z \Rightarrow z = y^4.$

$$\frac{dz}{dx} + P(x)(1-n) z = Q(x)(1-n)$$

$$\frac{dz}{dx} + \frac{1}{2x}(1+3) z = x(1+3)$$

$$\frac{dz}{dx} + \frac{4z}{2x} = 4x \Rightarrow \frac{dz}{dx} + \frac{2z}{x} = 4x$$

$$I.F = e^{\int \frac{2}{x} . dx} = e^{2\int \frac{1}{x} dx} = e^{2 \log_e x}$$

$$= x^2$$

$$x^2 z = \int x^2 (4x) \, dx$$

$$x^2 z = \frac{\cancel{4} x^4}{\cancel{4}} + C$$

$$x^2 y^4 = x^4 + C.$$

At $y = 2, \; x = 1$

$$1(16) = 1 + C \Rightarrow C = 15$$

$$x^2 y^4 = x^4 + 15$$

(3) $\frac{dy}{dx} + y = f(x) = \begin{cases} e^{-x} & 0 < x < 2 \\ e^{-2} & x \geq 2 \end{cases}$ $y(0) = 1$.

For case $0 < x < 2$

$\frac{dy}{dx} + y = e^{-x}$

$P = 1, \quad Q = e^{-x}$

$I.F = e^{\int P dx} = e^{\int dx} = e^{x}$

$e^{x}.y = \int e^{-x} \cdot \cancel{y} e^{x} . dx$

$ye^{x} = \int dx = x + C$

when $x = 0, \; y = 1$

$1 e^{0} = 0 + C \Rightarrow C = 1$

$ye^{x} = x + 1$

For case $x \geq 2$

$\frac{dy}{dx} + y = e^{-2}$

$P = 1, \quad Q = e^{-2}$.

$I.F = e^{\int P dx} = e^{\int dx} = e^{x}$

$e^{x}.y = \int e^{x}.e^{-2} dx$

$ye^{x} = e^{-2}.e^{x} + C \Rightarrow ye^{x} = e^{x-2} + C$.

As $x \to 2$

consider $y = e^{-2}$

$\lim_{x \to 2^-} e^{-2} = e^{-2}$.

$ye^{2} = 3$

$y = \frac{3}{e^2}$

$\frac{3}{e^2} \cdot e^{x} = e^{x-2} + C$

$\frac{3}{\cancel{e^2}} \cancel{e^{2}} = e^{2-2} + C \Rightarrow C = 2$

$ye^{x} = e^{x-2} + 2$

$$y = \begin{cases} e^{-x}(x+1) & ; 0 \le x < 2 \\ 2e^{-x} + e^{-2} & x \geq 2. \end{cases}$$

Orthogonal Trajectories

$f(x, y, c) = 0$, c is a parameter.

Step 1. Form differential equation of the given family.

Step 2. Replace $\frac{dy}{dx}$ by $-1/\frac{dy}{dx}$, differential equation of O.T

Step 3. Solve these differential equations to get equations of O.T

Q1. Find the orthogonal trajectory of $x^2+y^2=c^2$

Ans:

$$2x + 2y\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = -\frac{x}{y}$$

Replace $\frac{dy}{dx}$ by $-1/\frac{dy}{dx}$

$$-\frac{1}{\frac{dy}{dx}} = -\frac{x}{y} \Rightarrow \frac{1}{\frac{dy}{dx}} = \frac{x}{y}$$

$$\frac{y}{x} = \frac{dy}{dx} \rightarrow \int \frac{1}{x}\,dx = \int \frac{1}{y}\,dy$$

$$\log_e x = \log_e y + \log_e c$$

$$x = cy$$

Q2. Experiments show that the electric lines of force of two opposite charges of the same strength at $(-1,0)$ and $(1,0)$ are circles through $(-1,0)$ and $(1,0)$. Show that these circles can be represented by $x^2+(y-c)^2 = 1+c^2$. Also prove that the equipotential lines or orthogonal trajectories are the circles $(x+c')^2+y^2 = c'^2-1$

Ans:

$x^2+y^2+2gx+2fy+c=0$ —— ①

At $(1,0)$

$1+0+2g+0+c=0$

$\Rightarrow 1+2g+c=0$

At $(-1,0)$

$1+0-2g+0+c=0$

$\Rightarrow 1-2g+c=0$

$$\begin{array}{r} 2g+c=-1 \\ -2g+c=-1 \\ \hline 2c=-2 \\ c=-1 \\ g=0. \end{array}$$

Substituting g and c in ①

$x^2+y^2+2fy-1=0$

$x^2+(y^2+2fy+f^2)-1-f^2=0$

$x^2+(y+f)^2=1+f^2$

$f=-c.$

$x^2+(y-c)^2=1+c^2$

$x^2+y^2+c^2-2cy=1+c^2 \Rightarrow x^2+y^2-2cy=1.$

$\cancel{x^2+y^2+1+2y-1+1}$

$\cancel{x^2+y^2}$

$x^2+y^2-2cy-1=0$ —— ②

$2x+2yy'-2cy'=0$

$2x+(2y-2c)y'=0$

$x+(y-c)y'=0$

$x+yy'-cy'=0$

$\dfrac{x+yy'}{y'}=c.$ —— ③

Substitute ③ in ②

$x^2+y^2-2\left(\dfrac{x+yy'}{y'}\right)y-1=0$

$x^2+y^2-\dfrac{2xy}{y'}-\dfrac{2yy'y}{y'}-1=0$

$x^2+y^2-\dfrac{2xy}{y'}-2y^2-1=0$

$x^2-y^2-1=\dfrac{2xy}{y'}$

$y'=\dfrac{2xy}{x^2-y^2-1}$

Replace y' to $-\dfrac{1}{y'}$

$-\dfrac{1}{y'}=\dfrac{2xy}{x^2-y^2-1}$

$\dfrac{-x^2+y^2+1}{2xy}=\dfrac{dy}{dx}$

$2xy\,dy + (y^2 + x^2 - 1)\,dx = 0$

$2xy\,dy - (-x^2 + y^2 + 1).dx = 0$

$2xy\,dy + (x^2 - y^2 - 1)\,dx = 0$

$M = 2xy, \quad N = x^2 - y^2 - 1$

$\frac{\partial M}{\partial y} = 2x \qquad \frac{\partial N}{\partial x} = 2x.$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$\psi_x = M \Rightarrow \psi = \int M.dx$

$\psi = \int 2xy\,dx = \cancel{2}y \frac{x^2}{\cancel{2}} + h(y)$

$\psi = x^2 y + h(y)$

$\psi_y = x^2 + h'(y)$

$\psi = N = x^2 - y^2 - 1$

$h'(y) = -y^2 - 1$

$h(y) = -\frac{y^3}{3} - y + c.$

$\psi = x^2 y - \frac{y^3}{3} - y + c_1$

The solution

$x^2 y - \frac{y^3}{3} - y = c.$

Q. show that $y^2 = 4a(x+a)$ where a is a parameter is self orthogonal.

Ans:

$$y^2 = 4a(x+a) \quad \text{—— ①}$$

$$2yy' = 4a$$

$$yy' = 2a \rightarrow y' \neq 2a$$

$$\frac{yy'}{2} = a \quad \text{—— ②}$$

Substitute ② in ①

$$y^2 = \frac{4yy'}{2}\left(x + \frac{yy'}{2}\right) = \frac{4xyy'}{2} + \frac{2y^2y'^2}{2} \quad \text{—— ②}$$

$$\cancel{y^2 = 2xy + 2y^2 \quad (2y^2)m^2 + 2xym - y^2 = 0}$$

$$\cancel{m = \frac{-2xy \pm \sqrt{4x^2y^2 - 4(2y^2)(-y^2)}}{2(2)y^2}}$$

$$\cancel{= \frac{-2xy \pm \sqrt{4x^2y^2 + 8y^4}}{4y^2}}$$

Substitute $y' = -\frac{1}{y'}$

$$y^2 = 4y\left(\frac{-1}{y'}\right)\left(x + \frac{y(-1)}{2y'}\right)$$

$$y^2 = \left(\frac{-2y}{y'}\right)\left(x - \frac{y}{2y'}\right)$$

$$y^2 = -\frac{2xy}{y'} + \frac{2y^2}{2y'^2}$$

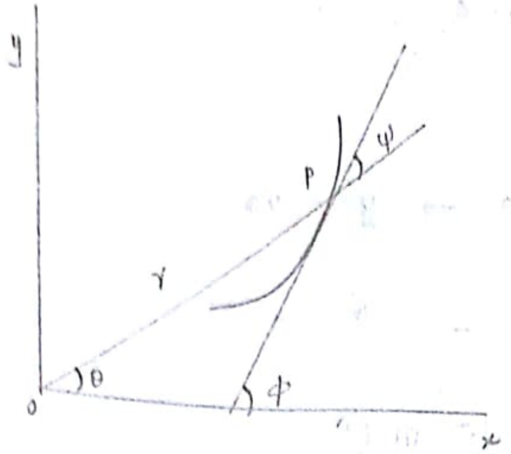
$$y^2 = -\frac{2xy}{y'} + \frac{y^2}{y'^2}$$

$$y^2y'^2 = -2xyy' + y^2$$

$$y^2 - 2xyy' - y^2y'^2 = 0 \quad \text{—— ③}$$

② and ③ are equivalent, hence it is self orthogonal.

Orthogonal Trajectories with polar coordinates



$$\boxed{\tan\psi = \frac{r}{dr/d\theta} = \frac{r}{r'}}$$

Proof:

$\psi = \tan(\phi - \theta)$

$= \dfrac{\tan\phi - \tan\theta}{1+\tan\phi\tan\theta}$

$\dfrac{\frac{dy}{dx} - \frac{y}{x}}{1+\left(\frac{y}{x}\right)\left(\frac{dy}{dx}\right)} = \dfrac{x\,dy - y\,dx}{x\,dx + y\,dy}$

$= \dfrac{x\frac{dy}{d\theta} - y\frac{dx}{d\theta}}{x\frac{dx}{d\theta} + y\frac{dy}{d\theta}}$

$\psi = \dfrac{x\frac{dy}{d\theta} - y\frac{dx}{d\theta}}{1+\left(\frac{dy}{dx}\right)\left(\frac{y}{x}\right)} = \dfrac{x\frac{dy}{d\theta} - y\frac{dx}{d\theta}}{x\frac{dx}{d\theta} + y\frac{dy}{d\theta}}$

$\psi = \dfrac{r^2}{r r'}$

$r = f(\theta)$

$x = r\cos\theta$

$y = r\sin\theta$

Q

$r = a(1+\cos\theta)$

$\dfrac{dr}{d\theta} = a(-\sin\theta) = -a\sin\theta$

$a = \dfrac{dr/d\theta}{-\sin\theta}$

$r = -\dfrac{dr}{d\theta}\left(\dfrac{1+\cos\theta}{\sin\theta}\right)$

$r = -\dfrac{dr}{d\theta}\cot\theta/2$

$\int \tan\theta/2\, d\theta = \int -\frac{1}{r}\,dr$

$\cancel{-2\log|\cos\theta/2| = -\log_e r + \log_e C}$

$\cancel{\log\cos^2\theta/2 = \log_e \frac{r}{c}}$

$\cancel{\cos^2\theta/2 = \frac{r}{c}}$

$\cancel{1+\cos\theta = \frac{r}{c}}$

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$$r = \frac{\cot\theta/2}{dr/d\theta}$$

$$\frac{r\,dr}{d\theta} = \cot\theta/2 \implies$$

$$\frac{r}{r'} = \tan\theta/2$$

$$r = \tan\theta/2 \frac{dr}{d\theta}$$

$$\int \cot\theta/2 \, d\theta = \int \frac{1}{r} dr$$

$$\log_e r + \log_e C = 2\log_e \sin\theta/2$$

$$\log_e rC = \log_e \sin^2\theta/2$$

$$Cr = 1 - \cos\theta$$

$$r = A(1-\cos\theta)$$

Q. $\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} = 1$, show it is self-orthogonal.

Application of Ordinary Differential Equations.

(i) <u>Growth</u> : $\frac{dP}{dt} = KP$

(ii) <u>Decay</u> : $\frac{dx}{dt} = -Kx$

(iii) <u>Interest</u> : $\frac{dP}{dt} = \frac{Pr}{100}$

(iv) <u>Electric circuit</u> : • $E_L = L\frac{dI}{dt}$ (voltage drop across inductor)

• $E_R = RI$ (voltage drop across resistance)

R. L circuit

• $E_L + E_R = E(t)$

• $L\frac{dI}{dt} + RI = E(t)$

• $\frac{dI}{dt} + \frac{R}{L} I = \frac{E(t)}{L}$

• $I e^{R/L t} = \int \frac{E(t)}{L} e^{\frac{R}{L}t} . dt$

dividing throughout by $e^{\frac{R}{L}t}$

$I(t) = e^{-\frac{R}{L}t} \int (\quad) + C e^{-\frac{R}{L}t}$

(v) <u>Psychology</u> : y = learners skill level at t.

p - individual learner

n - nature of tasks

$$\frac{dy}{dt} = \frac{2p}{\sqrt{n}} y^{3/2}(1 - y^{3/2})$$

(vi) <u>Newton's law of cooling</u> : $\frac{dT}{dt} \propto (T - T_0)$

$$\frac{dT}{dt} = -K(T - T_0)$$

Q1. A cup of hot coffee initially at 95°C cools to 80°C in 5 minutes where the room temperature is 21°C. Using Newton's law of cooling determine when the temperature is 50°C

Ans:

$$\frac{dT}{dt} = -k(T_1 - T_0)$$

$T_0 = 21°C$, $T_1 = 95°C$, $T_2 = 80°C$.

$$\cancel{\frac{dT}{dt} = -k(95-21)}$$

$$\cancel{\frac{dT}{dt} =} \qquad \frac{dT}{dt} = -k(T_1 - T_0)$$

$$\int_{T_1}^{T_2} \frac{dT}{T_1 - T_0} = -\int_{t_1}^{t_2} k\,dt$$

$$\left[\log_e (T - T_0)\right]_{T_1}^{T_2} = -k[t]_{t_1}^{t_2}$$

$$\log_e \frac{(T_2 - T_0)}{T_1 - T_0} = -k(t_2 - t_1)$$

substituting the values:

$$\log_e \left(\frac{80-21}{95-21}\right) = -k(5)$$

$$\log_e \left(\frac{59}{74}\right) = -5k$$

$$-0.226527 = -5k$$

$$k = 0.0453$$

$$\int_{T_1}^{T_2} \frac{dT}{T_1 - T_0} = -\int_{t_1}^{t_2} k\,dt$$

$$\log_e \frac{(T_2 - T_0)}{(T_1 - T_0)} = -0.0453(t_2 - t_1)$$

$$\log_e \frac{(50-21)}{(95-21)} = -0.0453(\Delta t)$$

$$\log_e (0.392) = -0.0453\ \Delta t$$

$$\Delta t = 20.673 \text{ minutes}$$

∴ 15.68 minutes after changing to 80°C

Q. A object of mass m and is given an initial ~~velocity~~ v downward velocity v. Assuming that gravitational force is constant and force due to air resistance proportional to the velocity of the object, Determine the equation of motion.

Ans:

$$F = F_1 + F_2 \qquad F_1 = mg, \quad F_2 \propto v$$

$$F_2 = -bv$$

$$F_t = mg - bv$$

$$m\frac{dv}{dt} = mg - bv$$

$$\frac{dv}{dt} = g - \frac{b}{m}v$$

$$\int_{v_0}^{v} \frac{dv}{g - \frac{b}{m}v} = \int_0^t dt \Rightarrow \frac{\left[\log_e\left(g - \frac{b}{m}v\right)\right]_{v_0}^{v}}{-b/m} = t \cancel{+c}$$

$$-\frac{m}{b}\left[\ln\left(g - \frac{b}{m}v\right)\right]_{v_0}^{v} = t \cancel{+c}$$

$$\left[\ln\left(g - \frac{b}{m}v\right)\right]_{v_0}^{v} = -\frac{b}{m}t \cancel{-\frac{b}{m}c}$$

$$\frac{g - \frac{b}{m}v}{g - \frac{b}{m}v_0} = k e^{-\frac{b}{m}t}$$

$$\frac{g - k e^{-\frac{b}{m}t}}{g - \frac{b}{m}v_0} = \frac{b}{m}v$$

$$\frac{mg}{b} - \frac{mk}{b}e^{-\frac{b}{m}t} = v$$

$$\frac{dx}{dt} = \frac{mg}{b} - \frac{mk}{b}e^{-\frac{b}{m}t}$$

$$\int dx = \frac{mgt}{b} - \frac{mk}{b}\frac{e^{-\frac{b}{m}t}}{-b/m} \cancel{\left(-\frac{b}{m}\right)}$$

$$x(t) = \frac{mg\,t}{b} -$$

$$\left(g - \frac{b}{m} v\right) = \left(g - \frac{b}{m} v_0\right) \cancel{K} e^{-\frac{b}{m} t}$$

$$g - \left(g - \frac{b}{m} v_0\right) e^{-\frac{b}{m} t} = \frac{b}{m} v$$

$$g - g e^{-\frac{b}{m} t} + \frac{b}{m} v_0 e^{-\frac{b}{m} t} = \frac{b}{m} v$$

Second order linear differential equations

$$y'' + p(x)y' + q(x)y = r(x)$$

let $r(x) = 0$

$y'' + p(x)y' + q(x)y = 0$ homogeneous equation

$y'' + ay' + by = 0$ with constant coefficients.

Let $y = e^{\lambda x}$

$y' = e^{\lambda x}(\lambda)$

$y'' = e^{\lambda x}(\lambda^2)$

$$\lambda^2 e^{\lambda x} + a\lambda e^{\lambda x} + b e^{\lambda x} = 0 \Rightarrow e^{\lambda x}(\lambda^2 + a\lambda + b) = 0$$

since $e^{\lambda x} \neq 0$; $\lambda^2 + a\lambda + b = 0$

Case i) : roots are real and distinct

$$\lambda_1 \neq \lambda_2$$

$y_1 = e^{\lambda_1 x}$, $y_2 = e^{\lambda_2 x}$

General solution: $y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$

Case ii) : Roots are real and repeated.

$$\lambda_1 = \lambda_2$$

We can say linearly dependant for any no of terms if

$f(x) = kg(x)$

$f(x) - kg(x) = 0$

$c_1 f(x) + c_2 g(x) = 0$ (l.d)

if $c_1 = c_2 = 0$ (l.i)

Note:-

If we divide the functions we get a constant $\leftarrow$ x $\quad 2x^2$ (linearly dependent)

If we divide the function we get a function of x $\leftarrow$ x $\quad x^2$ (linearly independent)

x $\quad x+1$ (linearly independant)

x $\quad |x|$, $x > 0$ (l.d)

x $\quad |x|$, $x < 0$ (l.d)

x $\quad |x|$, $-1 \le x < 1$ (l.i)

linearly dependant functions must have the same constant throughout the defined interval

here the sign changes hence constant changes

$\sin x \cos x$ $\quad \sin 2x$ (l.d.)

Let $y = c(x)\, e^{\lambda_1 x}$

$$y' = c'(x)\, e^{\lambda_1 x} + c(x)\, \lambda_1 e^{\lambda_1 x}$$

$$y'' = c''(x)\, e^{\lambda_1 x} + c(x)\, \lambda_1^2 e^{\lambda_1 x} + 2\lambda_1 e^{\lambda_1 x} c'(x)$$

Substituting in (1)

$$c''(x)\, e^{\lambda_1 x} + c(x)\, \lambda_1^2 e^{\lambda_1 x} + 2\lambda_1 c'(x) e^{\lambda x}$$
$$+ a c'(x) e^{\lambda_1 x} + a c(x) \lambda_1 e^{\lambda_1 x} + b\, c(x) e^{\lambda_1 x}$$

$$c''(x)\, e^{\lambda_1 x} + c'(x)\left[2\lambda_1 e^{\lambda_1 x} + a e^{\lambda_1 x}\right] + c(x)\left[a\lambda_1 e^{\lambda_1 x} + b e^{\lambda_1 x} + e^{\lambda_1 x} \lambda_1^2\right] =$$

$$c''(x)\, e^{\lambda_1 x} + c'(x)\, e^{\lambda_1 x}\underbrace{\left[2\lambda_1 + a\right]}_{0} + c(x) e^{\lambda_1 x}\underbrace{\left[a\lambda_1 + \lambda_1^2 + b\right]}_{0} = 0$$

$$c''(x)\, e^{\lambda_1 x} = 0 \Rightarrow c''(x) = 0$$

$$c(x) = c_1 x + c_2$$

General solution : $\boxed{y = (c_1 x + c_2)\, e^{\lambda_1 x}}$

Case iii) : Roots are complex. , $\lambda = p \pm iq$

$$y_1 = e^{(p+iq)x} \qquad , \qquad y_2 = e^{(p+iq)x}$$

$$y_1 = e^{px} \cdot e^{iqx} \qquad y_2 = e^{px} \cdot e^{iqx}$$

$$y_1 = e^{px}(\cos qx + i \sin qx) \qquad y_2 = e^{px}(\cos qx + i \sin qx)$$

$$y = c_1 y_1 + c_2 y_2$$

General solution $\boxed{y = e^{px}\left[\underbrace{(c_1 + c_2)}_{A} \cos qx + \underbrace{i(c_1 + c_2)}_{B} \sin qx\right]}$

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Q1 $y'' - y = 0$

$$\lambda^2 - 1 = 0$$

Ans: $\lambda = \pm 1$

∴ since the roots are real and distinct then,

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} = c_1 e^{x} + c_2 e^{-x}$$

Q2. $y'' + y = 0$

Ans:

$$\lambda^2 + 1 = 0 \Rightarrow \lambda^2 = -1$$

$$\lambda = \pm i$$

The general solution is,

$$y = e^{px}\left[(c_1 + c_2)\cos qx + i(c_1 + c_2)\sin qx\right]$$

$$\lambda = p \pm iq$$

$$p = 0, \quad q = 1.$$

$$y = e^{0}\left[(c_1 + c_2)\cos x + i(c_1 - c_2)\sin x\right]$$

$$y = 1\left[A\cos x + B\sin x\right] \qquad \left[\begin{array}{l} c_1 + c_2 = A \\ i(c_1 - c_2) = B \end{array}\right]$$

Q3. $y'' - 2y' + y = 0$

Ans:

$$\lambda^2 - 2\lambda + 1 = 0$$

$$(\lambda - 1)^2 = 0 \Rightarrow \lambda = 1, 1$$

$$y = (c_1 x + c_2)e^{x} \qquad [\because \lambda = 1]$$

Q4. $25y'' + 40y' + 16y = 0$

Ans:

$$25\lambda^2 + 40\lambda + 16 = 0$$

$$\lambda^2 + \frac{40}{25}\lambda + \frac{16}{25} = 0$$

$$25\lambda^2 + 20\lambda + 20\lambda + 16 = 0$$

$$5\lambda(5\lambda + 4) + 4(5\lambda + 4) = 0 \Rightarrow \lambda = -\frac{4}{5}, -\frac{4}{5}$$

The required general solution: $y = (c_1 x + c_2) e^{\lambda_1 x}$

$$y = (c_1 x + c_2) e^{-1/5 x}$$

Q5. $y'' - y' - 2y = 0 \qquad y(0) = -4, \; y'(0) = -17$ (Initial value problem)

$\lambda^2 - \lambda - 2 = 0$

$\lambda^2 - 2\lambda + \lambda - 2 = 0$

$\lambda(\lambda - 2) + 1(\lambda - 2) = 0$

$(\lambda + 1)(\lambda - 2) = 0$

$\therefore \lambda = -1, 2$

The general solution,

$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$

$\therefore y = c_1 e^{-x} + c_2 e^{2x}$

If $y = -4, \; x = 0$

$-4 = c_1 e^{+0} + c_2 e^{0} \Rightarrow c_1 + c_2 = -4. \quad —(1)$

If $y' = -17, \; x = 0$

$y' = c_1 e^{-x}(-1) + c_2 e^{2x}(2)$

$-17 = -c_1 + 2c_2. \quad —(2)$

From (1) and (2)

$$\begin{aligned} \cancel{c_1} + c_2 &= -4 \\ -\cancel{c_1} + 2c_2 &= -17 \\ \hline 3c_2 &= -21 \\ c_2 &= -7 \end{aligned}$$

$c_1 = 2c_2 + 17 = 2(-7) + 17 = 3.$

$\therefore \; y = 3e^{-x} - 7e^{2x}.$

Q6. $y'' + 2y' + 2y = 0 \qquad y(0) = 1, \; y(\pi/2) = 0$

$\lambda^2 + 2\lambda + 2 = 0$

$$\lambda = \frac{-2 \pm \sqrt{4 - 4(1)(2)}}{2} = \frac{-2 \pm \sqrt{4 - 8}}{2}$$

$$\lambda = \frac{-2 \pm \sqrt{-4}}{2} = \frac{-2 \pm \sqrt{4}\, i}{2}$$

$$\lambda = \frac{-2 \pm 2i}{2} = -1 \pm i \qquad [\lambda = p \pm iq]$$

$p = -1, \quad q = 1.$

$y = e^{px}\left[(C_1+C_2)\cos qx + i(C_1-C_2)\sin qx\right]$

$y = e^{-x}\left[(C_1+C_2)\cos x + i(C_1-C_2)\sin x\right]$

At $y = 1,\ x = 0$

$1 = 1\left[(C_1+C_2)\cos 0 + i(C_1-C_2)\sin 0\right]$

$1 = A\cos 0 + 0 \Rightarrow \underline{\underline{A = 1}}.$

At $y = 0,\ x = \frac{\pi}{2}$

$0 = e^{-\pi/2}\left[A\cos\frac{\pi}{2} + B\sin\frac{\pi}{2}\right]$

$\cancel{0 = e^{-\pi/3}\left[\frac{1}{2} + B\frac{\sqrt{3}}{2}\right]}$

$\cancel{e^{\pi/3} = \frac{1+\sqrt{3}B}{2}}$

$\cancel{\frac{2e^{\pi/3}-1}{\sqrt{3}}} = B.$

$0 = e^{-\pi/2}\left[A(0) + B\right]$

$B = 0.$

$\therefore\ y = e^{-x}\left[A\cos x + B\sin x\right]$

$y = e^{-x}\left[\cos x\right]$

<u>Non-homogeneous equation</u> (with constant coefficient)

$y'' + ay' + by = r(x)$ —①

$y'' + ay' + by = 0$ —② y_h (general solution of ②)

Let y_p - (particular integral of (1))

$y = y_h + y_p$ (solution to ①)

Proof: $y' = y_h' + y_p'$

$y'' = y_h'' + y_p''$

$$L.H.S = y_h'' + y_p'' + a(y_h' + y_p') + b(y_h + y_p)$$

$$= (y_h'' + ay_h' + by_h) + (y_p'' + ay_p' + by_p)$$

$$= 0 + r(x)$$

$$= R.H.S$$

Cases to obtain y_p - (Method of undetermined coefficients.)

$\Rightarrow$ If $r(x) =$ exponential function, ~~whose~~ choice of y_p = exp fn.

$\Rightarrow$ If $r(x) = \sin x$ or $\cos x$, ~~whose~~ choose y_p as $= A\cos kx + B\sin kx$.

$\Rightarrow$ If $r(x) =$ a polynomial choose for y_p = a polynomial.

$\Rightarrow$ If $r(x) = e^{\alpha x}\cos\omega x$ choose for $y_p = e^{\alpha x}(A\cos\omega x + B\sin\omega x)$

(I) Method of undetermined coefficients.

$r(x)$	choice of y_p
(i) $Ke^{\lambda x}$	$Ce^{\lambda x}$
(ii) a polynomial of degree n.	$K_n x^n + \dots\dots + K_1 x + K_0$
(iii) $K\cos\omega x$, $K\sin\omega x$	$K_1\cos\omega x + K_2\sin\omega x$
(iv) $Ke^{\alpha x}\cos\omega x$, $Ke^{\alpha x}\sin\omega x$	$e^{\alpha x}(K_1\cos\omega x + K_2\sin\omega x)$

Basic rule.

If $r(x)$ is one of the functions in the 1^{st} column, choose the corresponding choice of y_p from the second column and determine the undetermined coefficient while substituting y_p and its derivative in

③. Modification rule

If a term in your choice of y_p happens to be a solution of the homogeneous equation ② then multiply your choice of y_p by x or x^2 if this function corresponds to a double root of the characteristic equation of the homogeneous equation.

$$y'' + y = \sin x \quad \text{---} (1)$$

$$\lambda = \pm i$$

$$y_h = c_1 \cos x + c_2 \sin x$$

$$y_p = x(k_1 \cos x + k_2 \sin x)$$

$$y_p' = x(-k_1 \sin x + k_2 \cos x) + (k_1 \cos x + k_2 \sin x)$$

$$y_p'' = x(-k_1 \cos x - k_2 \sin x) + k_2 \cos x - k_1 \sin x + k_1 \sin x + k_2 \cos x$$

$$y_p'' + y_p = -k_1 x \cos x - k_2 x \sin x + k_2 \cos x - k_1 \sin x - k_1 \sin x + k_2 \cos x + x k_1 \cos x + k_2 \sin x \cdot x$$

$$y_p'' + y_p = 2k_2 \cos x - 2k_1 \sin x$$

Comparing with (1)

$$k_2 = 0, \quad k_1 = -\frac{1}{2}$$

$$y_p = x\left(-\frac{\cos x}{2} + 0\right) = -\frac{x \cos x}{2}$$

The general solution of (1)

$$y = c_1 \cos x + c_2 \sin x - \frac{x \cos x}{2}$$

Eg: (2) $y'' - 2y' + y = 2e^x$ —— (1)

Homogeneous ; $y'' - 2y' + y = 0$

$$\lambda^2 - 2\lambda + 1 = 0$$

$$(\lambda - 1)^2 = 0 \Rightarrow \lambda = \pm 1. \quad \text{(double root)}$$

$$y_h = (c_1 x + c_2) e^x \Rightarrow y_h = c_1 x e^x + c_2 e^x$$

$$y_p = x^2 \cdot c e^{\lambda x}$$

$$y_p' = x^2 \cdot c e^{\lambda x} (\lambda) + c e^{\lambda x} (2x)$$

$$y_p'' = c e^{\lambda x} (\lambda)(2x) + x^2 c e^{\lambda x} \lambda^2 + 2c[e^{\lambda x} + x e^{\lambda x} \cdot \lambda]$$

$$= 2c\lambda x e^{\lambda x} + c\lambda^2 x^2 e^{\lambda x} + 2c e^{\lambda x} + 2c\lambda x e^{\lambda x}$$

Substituting in (1) L.H.S

$$2c\lambda x e^{\lambda x} + c\lambda^2 x^2 e^{\lambda x} + 2c e^{\lambda x} + 2c\lambda x e^{\lambda x} - 2x^2 \lambda c e^{\lambda x} - 4x c e^{\lambda x} + x^2 c e^{\lambda x}$$

$y = y_h + y_p$

$y = c_1 x e^x + c_2 e^x + c e^x$

or

$c_1 x e^x + (c_2 + c) e^x$

$c = 1 \quad ; \quad y_p = x^2 e^x$

The general solution : $y = y_h + y_p$

$y = (c_1 x + c_2) e^{x} + x^2 e^x$

$y_p' = c[x^2 e^{\lambda x}\lambda + e^{\lambda x}(2x)] = c[\lambda x^2 e^{\lambda x} + 2x e^{\lambda x}]$

$y_p'' = c\{\lambda[x^2 e^{\lambda x}\lambda + e^{\lambda x}(2x)] + 2[x e^{\lambda x}\lambda + e^{\lambda x}]\}$

$y_p'' = c\{\lambda^2 x^2 e^{\lambda x} + 2\lambda x e^{\lambda x} + 2\lambda x e^{\lambda x} + 2e^{\lambda x}\}$

$= c\{\lambda^2 x^2 e^{\lambda x} + 4\lambda x e^{\lambda x} + 2e^{\lambda x}\}$

Substituting ① in L.H.S

$= c\lambda^2 x^2 e^{\lambda x} + 4\lambda c x e^{\lambda x} + 2ce^{\lambda x} - 2c\lambda x^2 e^{\lambda x} - 2cxe^{\lambda x} + x^2 c e^{\lambda x}$

$= \cancel{cx^2 e^x} + \cancel{4cxe^x} + 2ce^x - \cancel{2cx^2e^x} - \cancel{4cxe^x} + \cancel{x^2ce^x}$

$= 2ce^x \qquad \therefore c = 1$

Q. $y'' + 2y' + 10y = 25x^2 + 3.$

$y'' + 2y' + 10y = 0.$

$\lambda^2 + 2\lambda + 10 = 0$

$\lambda = \dfrac{-2 \pm \sqrt{4 - 4(1)(10)}}{2} = \dfrac{-2 \pm \sqrt{4-40}}{2}$

$\lambda = \dfrac{-2 \pm \sqrt{-36}}{2} = \dfrac{-2 \pm \sqrt{36}\, i}{2}$

$\lambda = -\dfrac{2 \pm 6i}{2} = -\cancel{2} \pm 3i$

$\lambda = -1 \pm 3i \quad ; \quad p = -1, \quad q = 3$

$y_h = e^{-x}[(c_1 + c_2)\cos 3x + i(c_1 + c_2)\sin 3x]$

$y_h = e^{-x}[A\cos 3x + B\sin 3x]$

No need to use modification Because the y_h is not similar to the corresponding y_p value in the 2nd column hence we can proceed as such

choose $y_p = K_2 x^2 + K_1 x + K_0$

$y_p' = 2K_2 x + K_1$

$y_p'' = 2K_2$

$= 2K_2 + 2(2K_2 x + K_1) + 10(K_2 x^2 + K_1 x + K_0)$

$= 2K_2 + 4K_2 x + 2K_1 + 10K_2 x^2 + 10K_1 x + 10K_0$

$= K_2(x^2 10K_2) + x(4K_2 + 10K_1) + 2K_1 + 2K_2 + 10K_0$

$10K_2 = 25 \Rightarrow K_2 = 5/2 \qquad 4K_2 + 10K_1 = 0 \Rightarrow K_1 = -\dfrac{4K_2}{10}$

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$$k_1 = -\frac{1}{10}\left(\frac{5}{2}\right) = -\frac{1 \times 5}{2 \times 10} = -\frac{1}{2}$$

$$2k_1 + 2k_2 + 10k_0 = 2(-1) + 2\left(\frac{5}{2}\right) + 10k_0 = 3$$

$$-2 + 5 + 10k_0 = 3$$

$$k_0 = 0$$

$$\therefore k_2 = \frac{5}{2}, \quad k_1 = -1, \quad k_0 = 0$$

$$y_p = \frac{5}{2}x^2 - x.$$

The complete solution is $y = e^{-x}(A\cos 3x + B\sin 3x) + \frac{5}{2}x^2$

a

$$y'' + 10y' + 25y = e^{-5x} \quad \text{——} (1)$$

$$y'' + 10y' + 25y = 0$$

$$\lambda^2 + 10\lambda + 25 = 0$$

$$\lambda^2 + 5\lambda + 5\lambda + 25 = 0 \Rightarrow \lambda(\lambda+5) + 5(\lambda+5) = 0$$

$$(\lambda+5)^2 = 0 \Rightarrow \lambda = -5, -5$$

$$y_h = (c_1 x + c_2)e^{-5x}$$

$$y_p = \cancel{x^2 e^{5x}(c_1 x + c_2)} \quad x^2 C e^{\lambda x}$$

$$y_p' = c\left[x^2 e^{\lambda x}(\lambda) + e^{\lambda x}(2x)\right]$$

$$y_p'' = c\left\{\lambda\left[2xe^{\lambda x} + x^2 e^{\lambda x}(\lambda)\right] + 2\left[xe^{\lambda x}\lambda + e^{\lambda x}\right]\right\}$$

Substituting (1) in L.H.S.

$$= c\left\{-5\left[2xe^{-5x} + x^2(-5)e^{-5x}\right] + 2\left[5xe^{-5x} + e^{-5x}\right]\right\}$$

$$= c\left\{-10xe^{-5x} + 25x^2e^{-5x} - 10xe^{-5x} + 2e^{-5x}\right\}$$

$$+ 10c\left[-5x^2e^{-5x} + 2xe^{-5x}\right] + 25x^2Ce^{-5x}$$

$$= -10Cxe^{-5x} + 25Cx^2e^{-5x} - 10Cxe^{-5x} + 2Ce^{-5x} - 50Cx^2e^{-5x}$$

$+ 20Cxe^{-5x} + 25x^2Ce^{-5x}$

$= -20Cxe^{-5x} - 25Ce^{-5x}.x^2 + 20Cxe^{-5x} + 25x^2Ce^{-5x} + 2Ce^{-5x}$

$= 2Ce^{-5x}$

comparing,

$2C = 1 \Rightarrow C = \frac{1}{2}$

The solution is $y = y_p + y_h = \frac{x^2}{2}e^{-5x} + (C_1x + C_2)e^{-5x}$

Q Solve: $y'' + 4y = \sin 2x$.

For the homogeneous equation,

$y'' + 4y = 0$. $\qquad \lambda = \pm 2i$

$y_h = c_1 \cos 2x + C_2 \sin 2x$.

$y_p = x(A\cos 2x + B\sin 2x) = Ax\cos 2x + Bx\sin 2x$

$y_p' = -Ax\sin 2x(2) + A\cos 2x + B\sin 2x + Bx\cos 2x(2)$

$= -2Ax\sin 2x + A\cos 2x + B\sin 2x + 2Bx\cos 2x$

$y_p'' = -2A[\sin 2x + 2x\cos 2x] + A(-\sin 2x(2)) + B\cos 2x(2)$
$+ 2B[\cos 2x - 2x\sin 2x]$

$= -2A\sin 2x - 4Ax\cos 2x - 2A\sin 2x + 2B\cos 2x$
$+ 2B\cos 2x - 4Bx\sin 2x$

$= 4B\cos 2x - 4A\sin 2x - 4Ax\cos 2x - 4Bx\sin 2x$

$4B\cos 2x - 4A\sin 2x - \cancel{4Ax\cos 2x} - \cancel{4Bx\sin 2x} + 4(\cancel{Ax\cos 2x}) + \cancel{4Bx\sin 2x}$

$=$ on comparing, $B = 0$, $A = -\frac{1}{4}$

$y = C_1\cos 2x + C_2\sin 2x - \frac{1}{4}x\cos 2x$.

$y'' + 2y' + y = e^x + x^2$ —— ①

The homogeneous equation, $y'' + 2y' + y = 0$

$\lambda^2 + 2\lambda + 1 = 0 \Rightarrow (\lambda+1)^2 = 0$

$\Rightarrow \lambda = -1, -1$

$y_h = (c_1 x + c_2) e^{-x}$

$y_p = Ke^{+x} + (k_2 x^2 + k_1 x + k_0)$

$y_p' = +Ke^{+x} + k_2(2x) + k_1$

$y_p'' = +Ke^{+x} + 2k_2$

Substituting in ①

$2k_2 + Ke^{+x} + 2(+Ke^{+x}) + 4xk_2 + 2k_1 + Ke^{+x} + k_2 x^2 + k_1 x + k_0$

$= 2k_2 + Ke^{+x} + 2Ke^{+x} + 4xk_2 + 2k_1 + Ke^{+x} + k_2 x^2 + k_1 x + k_0$

$= e^x (K + 2K + K) + k_2 x^2 + x(k_1 + 4k_2) + k_0 + 2k_2 + 2k_1$

on comparing,

$4K = 1$ $\qquad k_2 = 1$

$K = \frac{1}{4}$ $\qquad k_1 + 4k_2 = 0$ $\qquad 2k_1 + k_0 + 2k_2 = 0$

$k_1 = -4$ $\qquad k_0 = 8 - 2 = 6$

$y_p = \frac{1}{4} e^x + (x^2 + (-4x) + 6)$

The general solution ; $y = (c_1 x + c_2) e^{-x} + \frac{e^x}{4} + x^2 - 4x + 6.$

Lab CSE I

(II) Method of variation of parameters

$$y'' + ay' + by = r(x) \quad \text{--- (1)}$$

solve homogeneous equation, i.e, $y'' + ay' + by = 0$ --- (2)

y_1, y_2 linearly independant solutions of (2)

$$y = c_1 y_1 + c_2 y_2.$$

let $y = u(x) y_1 + v(x) y_2$ be a solution of (1)

$$y' = u y_1' + v y_2' + v' y_2 + u' y_1$$

Impose the condition $u' y_1 + v' y_2 = 0$ --- (3)

$$y' = u y_1' + v y_2'$$

$$y'' = u' y_1' + u y_1'' + v y_2'' + v' y_2'$$

Substituting in (1)

$$u' y_1' + u y_1'' + v y_2'' + v' y_2' + a(u y_1' + v y_2') + b\underbrace{y}_{b(u y_1 + v y_2)} = r(x).$$

$$u(y_1'' + a y_1' + b y_1) + v(y_2'' + a y_2' + b y_2) + u' y_1' + v' y_2' = r(x).$$

$$u(0) + v(0) + u' y_1' + v' y_2' = r(x) \quad [\text{from } (2)]$$

$$u' y_1' + v' y_2' = r(x) \quad \text{--- (4)}$$

$$u' = \frac{-y_2 r(x)}{\begin{vmatrix} y_1 & y_2 \\ y_3 & y_4 \end{vmatrix}} = \frac{-y_2 r(x)}{W}$$

W — wronskian.

$$v' = \frac{y_1 r(x)}{W}$$

Eg: $y'' - 4y' + 4y = \dfrac{e^{2x}}{x}$

Homogeneous equation, $y'' - 4y' + 4y = 0$

$$\lambda^2 - 4\lambda + 4 = 0$$

$$(\lambda - 2)^2 = 0 \Rightarrow \lambda = 2, 2.$$

$$y_1 = x e^{2x}, \quad y_2 = e^{2x}$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$W = \begin{vmatrix} xe^{2x} & e^{2x} \\ xe^{2x}(2)+e^{2x} & 2e^{2x} \end{vmatrix}$$

$$W = 2x.e^{4x} - e^{2x}(2xe^{2x}+e^{2x})$$

$$= 2xe^{4x} - e^{4x}(2x+1) \Rightarrow -e^{4x}. = W$$

$$u' = \frac{-e^{2x}.e^{2x}}{x.e^{4x}} = +\frac{1}{x}.$$

$$u = \ln x + C_1$$

$$v' = \frac{xe^{2x}.e^{2x}}{-x(e^{4x})} = -1$$

$$v = -x + C_2.$$

$$y = u(x)y_1 + v(x)y_2$$

$$y = (\ln x + C_1)\,xe^{2x} + (-x + C_2)e^{2x}.$$ is the general solution of ①

Q $y'' + y = \tan x.$

$y'' + y = 0$

$\lambda^2 + 1 = 0 \Rightarrow \lambda = \pm i$

$y_h = A\cos x + B\sin x$

$\therefore y_1 = \cos x, \quad y_2 = \sin x.$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1.$$

$$u' = -\frac{y_2\, r(x)}{W} = -\frac{\sin x\,(\tan x)}{1}$$

$$\Rightarrow \quad u = \int -\frac{\sin^2 x}{\cos x}\,dx = \int \left(\frac{\cos^2 x - 1}{\cos x}\right) dx$$

~~let $\cos x = t$~~

$$u = \int (\cos x - \sec x)\,dx = \sin x - \log|\sec x + \tan x| + c_1$$

$$v' = +\frac{y_1\, r(x)}{W} = \cos x \tan x$$

$$v = \int \cos x \tan x\,.dx = \int \sin x\,.dx = -\cos x + C_2.$$

$$y = u y_1 + v y_2$$

The general solution is,

$$y = (\sin x - \log|\sec x + \tan x| + c_1)\cos x + (-\cos x + C_2)\sin x$$

$$y_p = (\sin x - \log|\sec x + \tan x|)\cos x + (-\cos x)\sin x.$$

$$y_p = -\log|\sec x + \tan x|\cdot\cos x.$$

Euler - cauchy equation

$$x^2 y'' + a x y' + b y = r(x) \quad \text{--- (1)}$$

Consider homogeneous equation.

$$x^2 y'' + a x y' + b y = 0 \quad \text{--- (2)}$$

$$y = x^m, \quad y' = m x^{m-1}, \quad y'' = m x^{m-2}(m-1)$$

Substituting in (2)

$$x^2\, m . x^{m-2}(m-1) + a x\, m\, x^{m-1} + b x^m = 0$$

$$m(m-1)x^m + a m x^m + b x^m = 0$$

$$x^m\,[\,m(m-1) + am + b\,] = 0$$

$m^2 + m(a-1) + b = 0$ (corresponding auxiliary equation)

Case I: when m_1 and m_2 are real and distinct.

$y = c_1 x^{m_1} + c_2 x^{m_2}$. (solution to homogeneous equation)

Case II: $m_1 = m_2$ (real and equal).

$y = (c_1 \ln x + c_2) x^{m_1}$

Case III: let $\lambda \pm i\mu$ (roots are complex).

$y = x^{\lambda} (c_1 \cos \mu \ln x + c_2 \sin \mu \ln x)$.

Method II $x = e^t$

$\frac{dx}{dt} = e^t = x$.

$y' = \frac{dy/dt}{dx/dt} = \frac{1}{x}\frac{dy}{dt} = \frac{1}{x} \cdot \theta y$. (here θ is the operator $\frac{dy}{dt}$ and not a variable)

$xy' = \theta y$ — ③

$D = \frac{d}{dx}$.

$y'' = \frac{d}{dt}\left(\frac{dy/dt}{dx/dt}\right)\frac{dt}{dx}$.

$= \frac{\left[\left(\frac{d^2y}{dt^2}\right)\left(\frac{dx}{dt}\right) - \left(\frac{dy}{dt}\right)\left(\frac{d^2x}{dt^2}\right)\right]}{\left(\frac{dx}{dt}\right)^3} = \frac{1}{x^2}\left(\theta^2 y - \theta y\right)$

$x^2 y'' = \theta(\theta - 1)y$. — ④

substituting ③ and ④ in ②

$\theta(\theta-1)y + a\theta y + by = 0$

$\theta^2 y + (a-1)\theta y + by = 0$

i) $y = c_1 e^{m_1 t} + c_2 e^{m_2 t}$

ii) $y = (c_1 t + c_2) e^{m_1 t} = (c_1 \ln x + c_2) x^{m_1}$

iii) $y = e^{\lambda t}(c_1 \cos \mu t + c_2 \sin \mu t)$.

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Solve: i) $x^2y'' - 4xy' + 6y = 0$

Ans: $x^2y'' - 4xy' + 6y = 0$

$m^2 + (a-1)m + b = 0$; $a = -4,\ b = 6.$

$m^2 + (-5)m + 6 = 0$

$m^2 - 5m + 6 = 0 \Rightarrow m^2 - 3m - 2m + 6 = 0$

$m(m-3) - 2(m-3) = 0$

$m = 3, 2.$

$y = c_1 x^3 + c_2 x^2.$ is the general solution.

(ii) $x^2y'' - xy' + 2y = 0$

Ans: $m^2 + (a-1)m + b = 0$; $a = -1,\ b = 2$

$m^2 + (-2m) + 2 = 0$

$m^2 - 2m + 2 = 0$

$m = \frac{2 \pm \sqrt{4 - 4(1)(2)}}{2} = \frac{2 \pm \sqrt{-4}}{2} = \frac{2 \pm 2i}{2}$

$m = 1 \pm i$; $\lambda = 1,\ \mu = 1$

$y = x(c_1 \cos \ln x + c_2 \sin \ln x)$

(iii) $x^2y'' + 7xy' + 9y = 0$

Ans: $m^2 + (a-1)m + b = 0$; $a = 7,\ b = 9$

$m^2 + 6m + 9 = 0$

$m^2 + 3m + 3m + 9 = 0$

$m(m+3) + 3(m+3) = 0$

$(m+3)^2 = 0 \Rightarrow m = -3, -3$

$y = (c_1 \ln x + c_2) x^{-3}.$

Q. $x^2y'' - 4xy' + 6y = 21x^{-4}$

Ans: $x = e^t$, $xy' = \theta y$

$x^2y' = \theta(\theta-1)y$

$\theta(\theta-1)y - 4\theta y + 6y = 21e^{-4t}$

$\Rightarrow \theta^2 y + y(-\theta - 4\theta) + 6y = 21e^{-4t}$

$\Rightarrow \theta^2 y + \cancel{y}(-5\theta)y + 6y = 21e^{-4t}$ —— ①

Consider homogeneous equation,

$m^2 - 5m + 6 = 0$

$m_1 = 3, \; m_2 = 2.$

$y = c_1 e^{3t} + c_2 e^{2t}$

$y_h = c_1 x^3 + c_2 x^2$

Let $y_p = ke^{-4t}$.

$y_p' = -4ke^{-4t}$

$y_p'' = 16ke^{-4t}$

Substituting in ①

$16ke^{-4t} - 5(-4ke^{-4t}) + 6ke^{-4t} = 21e^{-4t}$

$(16 + 20 + 6)\, ke^{-4t} = 21e^{-4t}$

$42(k)e^{-4t} = 21e^{-4t}$

$\therefore k = \frac{1}{2}.$

$e^{-4\log_e x}$

$y_p = \frac{1}{2}e^{-4t} = \frac{1}{2}x^{-4}$

The general solution: $y = c_1x^3 + c_2x^2 + \frac{1}{2}x^{-4}$

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Solve: i) $x^2y'' - 4xy' + 6y = 0$

Ans:

$x^2y'' - 4xy' + 6y = 0$

$m^2 + (a-1)m + b = 0$; $a = -4, \quad b = 6.$

$m^2 + (-5)m + 6 = 0$

$m^2 - 5m + 6 = 0 \Rightarrow m^2 - 3m - 2m + 6 = 0$

$m(m-3) - 2(m-3) = 0$

$m = 3, 2.$

$y = c_1 x^3 + c_2 x^2.$ is the genural solution.

(ii) $x^2y'' - xy' + 2y = 0$

Ans:

$m^2 + (a-1)m + b = 0$; $a = -1, \quad b = 2$

$m^2 + (-2m) + 2 = 0$

$m^2 - 2m + 2 = 0$

$m = \frac{2 \pm \sqrt{4 - 4(1)(2)}}{2} = \frac{2 \pm \sqrt{-4}}{2} = \frac{2 \pm 2i}{2}$

$m = 1 \pm i$; $\lambda = 1, \quad \mu = 1$

$y = x(c_1 \cos \ln x + c_2 \sin (\ln x))$

(iii) $x^2y'' + 7xy' + 9y = 0$

Ans:

$m^2 + (a-1)m + b = 0$; $a = 7, \quad b = 9$

$m^2 + 6m + 9 = 0$

$m^2 + 3m + 3m + 9 = 0$

$m(m+3) + 3(m+3) = 0$

$(m+3)^2 = 0 \Rightarrow m = -3, -3$

$y = (c_1 \ln x + c_2) x^{-3}.$

Q. $x^2y'' - 4xy' + 6y = 21x^{-4}$

Ans: $x = e^t$, $xy' = \theta y$

$x^2y'' = \theta(\theta-1)y$

$\theta(\theta-1)y - 4\theta y + 6y = 21e^{-4t}$

$\theta(\theta-1)y + y(-\theta-4\theta) + 6y = 21e^{-4t}$

$\Rightarrow \theta^2 y + y(-\theta - 4\theta) + 6y = 21e^{-4t}$ —— ①

$\Rightarrow \theta^2 y + \cancel{y}(-5\theta)y + 6y = 21e^{-4t}$

Consider homogeneous equation,

$m^2 - 5m + 6 = 0$

$m_1 = 3, \ m_2 = 2.$ $y_h = c_1 e^{3t} + c_2 e^{2t}$

$y_h = c_1 x^3 + C_2 x^2$

Let $y_p = ke^{-4t}.$

$y_p' = -4ke^{-4t}$

$y_p'' = 16ke^{-4t}$

Substituting in ①

$16ke^{-4t} - 5(-4ke^{-4t}) + 6ke^{-4t} = 21e^{-4t}$

$(16 + 20 + 6)ke^{-4t} = 21e^{-4t}$

$42(k)e^{-4t} = 21e^{-4t}$

$\therefore k = \frac{1}{2}.$

$e^{-4\log_e x}$

$e^{\log x}$

$y_p = \frac{1}{2}e^{-4t} = \frac{1}{2}x^{-4}$

The general solution: $y = C_1x^3 + C_2x^2 + \frac{1}{2}x^{-4}$

Q

$x^2y'' - 2xy' + 2y = x^3\cos x$

Let $x^2y'' = \theta(\theta-1)y$, $xy' = \theta y$

Let $x = e^t$

$\theta(\theta-1)y - 2\theta y + 2y = e^{3t}\cos(e^t)$

$\theta^2 y - \theta y - 2\theta y + 2y = e^{3t}\cos(e^t)$

$\theta^2 y - 3\theta y + 2y = e^{3t}\cos(e^t)$

Consider homogeneous equation,

$\theta^2 y - 3\theta y + 2y = 0 \Rightarrow m^2 - 3m + 2 = 0$

$m^2 - 2m - m + 2 = 0$

$m(m-2) - 1(m-2) = 0$

$(m-1)(m-2) = 0$; $m_1 = 1$, $m_2 = 2$.

$W = \begin{vmatrix} e^t & e^{2t} \\ e^t & 2e^{2t} \end{vmatrix} = e^{3t}$

$y = c_1 e^t + c_2 e^{2t}$; $y_1 = e^t$, $y_2 = e^{2t}$.

$u' = -\dfrac{y_2\, r(t)}{W} = -\dfrac{e^{2t}\cancel{e^{3t}}\cos(e^t)}{\cancel{e^{3t}}}$

$u' = -e^{2t}\cos(e^t)$ Let $e^t = b$.

$e^t dt = db$.

$u = -\int e^{2t}\cos e^t\, dt$

$= -\int b \cos b.\, db.$ $= -\left[b^2\sin b - \int 2b\sin b.db\right]$

$= -\left[b^2\sin b - 2\left[-b\cos b + \int \cos b.db\right]\right]$

$-\left[b\sin b - \int \sin b.db\right]$

$= -\left[b^2\sin b + 2b\cos b - 2\sin b\right] + c_1$. $-\left[b\sin b + \cos b.db\right]$

$= -\left[e^{2t}\sin(e^t) + 2e^t\cos(e^t) - 2\sin(e^t)\right] + c_1$

$-b\sin b - \cos b$

$= -e^t\sin(e^t) - \cos(e^t)$

$v' = \dfrac{y_1\, r(t)}{W} = \dfrac{e^t(\cancel{e^{3t}}\cos e^t)}{\cancel{e^{3t}}} = e^t\cos e^t$

1. $v = \int e^t \cos e^t \, dt$

Let $e^t = a \Rightarrow e^t . dt = da$

$v = \int \cos a . da = \sin a$

$v = \sin(e^t) + C_2$

$y = uy_1 + vy_2 = [-e^t \sin(e^t) - \cos(e^t) + C_1] e^t + e^{2t} . [\sin(e^t) ...$

$$y = [-x\sin x - \cos x + C_1] x + x^2 [\sin x + C_2]$$

~~$y_p = (-x\sin x - \cos x) . x + x^2 \sin x.$~~

Q $x^2y'' - xy' + 2y = x\ln x$

Ans: $m^2 + m(-2) + 2 = 0$

$m = \frac{2 \pm \sqrt{4 - 4(1)(2)}}{2} = \frac{2 \pm 2i}{2} = 1 \pm i$

~~For homogeneous equation,~~

~~$x^2y'' - xy' + 2y = 0$~~

~~$y_h = x(C_1 \ln x + C_2)$~~

$y_h = x(C_1 \cos \ln x + C_2 \sin \ln x)$

$y_1 = x \cos \ln x \quad , y_2 = x \sin \ln x$

$y = uy_1 + vy_2$

$$W = \begin{vmatrix} x\cos \ln x & x\sin \ln x \\ -\frac{x}{x} \sin \ln x + \cos \ln x & \frac{x\cos \ln x}{x} + \sin \ln x \end{vmatrix}$$

$u' = -\frac{y_2 \, r(x)}{W}$

$u' = -\frac{(\cancel{x}\sin \ln x)(\cancel{x} \ln x)}{\cancel{x} \quad x^{\cancel{2}}}$

$W = x\cos \ln x (\cos \ln x + \sin \ln x)$

$- (x \sin \ln x)(-\sin \ln x + \cos \ln x)$

$W = x\cos^2 \ln x + \cancel{x \cos \ln x \sin \ln x}$

$+ x \sin^2 \ln x - \cancel{x \sin \ln x \cos \ln x}$

$W = x$

$$u' = \frac{-x \ln x \sin \ln x}{x}$$

$$\frac{du}{dx} = \frac{-\ln x \sin \ln x}{x} \Rightarrow u = -\int \frac{\ln x \sin \ln x}{x}.dx$$

Let $\ln x = t$

$\frac{1}{x} dx = dt$

$$u = -\int t \sin t . dt = \left[-t\cos t + \int \cos t . dt\right]$$

$$u = +t\cos t - \sin t + c_1$$

$$v' = \frac{y_1 \, r(x)}{W} = \frac{(x \cos \ln x)(x \ln x)}{x^2 . x}$$

$$v' = \frac{\ln x \cos \ln x}{x}$$

$$v = \int \frac{\ln x \cos \ln x}{x} dx \qquad \text{Let } \ln x = t, \quad \frac{1}{x} dx = dt$$

$$v = \int t \cos t . dt = \left[t \sin t - \int \sin t . dt \right] = t \sin t + \cos t + C_2.$$

$$y = u y_1 + v y_2$$

$$y = (+\ln x \cos(\ln x) - \sin(\ln x) + c_1)\, x \cos(\ln x)$$
$$+ (t \sin t + \cos t + C_2)\, x \sin(\ln x).$$

$$y = x \cos^2(\ln x) \ln x - x \cos(\cancel{\ln x}) \sin(\ln x) + c_1 x \cos(\ln x)$$
$$+ \ln x \sin^2(\ln x). x + x \sin(\cancel{\ln x}) \cos(\ln x)$$
$$+ C_2 x \sin(\ln x)$$

$$\underline{y = x \ln x + c_1 x \cos(\ln x) + C_2 x \sin(\ln x).}$$

$\sin(et) + c_2]$

$x \sin \ln x$

$\frac{x \cos \ln x}{x} + \sin \ln x$

$x + \cos(\ln x)$

$\cos \ln x$

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Another form of Euler- Cauchy equation

$$(ax+b)^2 y'' + k_1(ax+b)y' + k_2 y = 0 \quad \text{—— (1)}$$

Put $(ax+b) = e^t$

$$a\frac{dx}{dt} = e^t = (ax+b)$$

$$y' = \frac{dy}{dx} = \frac{(dy/dt)}{(dx/dt)} = \frac{a}{ax+b}\left(\frac{dy}{dt}\right)$$

$$(ax+b)y' = a\frac{dy}{dt} = a\left(\frac{d}{dt}\right)(y) = a\theta y.$$

similarly, $(ax+b)^2 y'' = a^2\theta(\theta-1)y$

substituting in (1)

$$a^2\theta(\theta-1)y \cdot \cancel{y} + k_1 a\theta y + k_2 y = 0$$

$$a^2\theta^2 y - a^2\theta y + k_1 a\theta y + k_2 y = 0$$

$$\theta^2 y a^2 - (a^2 + k_1 a)\theta y + k_2 y = 0 \quad \text{(Equation with constant coefficient)}$$

Q Solve: $(1+x)^2 y'' + (1+x)y' + y = \cos\ln(1+x)$

Let $(1+x)^2 y'' = (1)\,\theta(\theta-1)y$, $(1+x) = e^t$

$$(1+x)y' = (1)\theta y.$$

$$\theta(\theta-1)y + \theta y + y = \cos\ln e^t$$

$$\theta^2 y + y = \cos\ln e^t = \cos t \quad \text{—— (1)}$$

Homogeneous equation,

$$\theta^2 y + y = 0$$

$$m^2 + 1 = 0 \Rightarrow m = \pm i$$

$$y_h = (c_1\cos \cancel{\ln x}\,t + c_2\sin \cancel{\ln x}\,t)$$

~~$y_1 = \cos\ln x$, $y_2 = \sin\ln x$~~

$u' = -\frac{y_2 r(t)}{W}$, $v' = \frac{y_1 r(t)}{W}$

$W = \begin{vmatrix} \cos(\ln x) & \sin(\ln x) \\ -\frac{\sin \ln x}{x} & \frac{\cos(\ln x)}{x} \end{vmatrix}$

$W = \frac{\cos^2(\ln x)}{x} + \frac{\sin^2(\ln x)}{x} = \frac{1}{x}$

$u' = -\underline{\sin(\ln x)\cos \ln e^t}$

$y_p = t(A\cos t + B\sin t)$.

$y_p' = t(-A\sin t + B\cos t) + (B\sin t + A\cos t)$

$= -At\sin t + Bt\cos t + B\sin t + A\cos t$

$y_p'' = -A[t\cos t + \sin t] + B[-t\sin t + \cos t] + B\cos t - A\sin t$

Substitute in 1

~~$-At\cos t - A\sin t - Bt\sin t + B\cos t + B\sin t + A\cos t$~~

~~$\cos t (A \cos t (-At +$ $-A$~~

$-\cancel{At\cos t} - A\sin t + B\cos t - \cancel{Bt\sin t} + B\sin t + A\cos t$
$+ \cancel{At\cos t} + \cancel{Bt\sin t} = \cos t$

$(B-A)\sin t + (A+B)\cos t = \cos t$

$A + B = 1$

$B - A = 0 \Rightarrow B = A$.

$2A = 1 \Rightarrow A = \frac{1}{2}$.

$-\cancel{At\cos t} - 2A\sin t - \cancel{Bt\sin t} + 2B\cos t + \cancel{At\cos t} + \cancel{Bt\sin t}$

$= \cos t$

$2B = 1 \Rightarrow B = \frac{1}{2}$.

$-2A = 0 \Rightarrow A = 0$,

$V = \int f(x).\, t\, dt$

Higher order D.E

Q1. $y''' + 6y'' + 11y' + 6y = 0$

$\lambda^3 + 6\lambda^2 + 11\lambda + 6 = 0$

$\lambda = -1$ is a root

$$
\begin{array}{r|l}
 & \lambda^2 + 5\lambda + 6 \\
\lambda + 1 & \lambda^3 + 6\lambda^2 + 11\lambda + 6 \\
(-) & -\lambda^3 \pm \lambda^2 \\
 & 5\lambda^2 + 11\lambda \\
(-) & -5\lambda^2 \pm 5\lambda \\
 & 6\lambda + 6 \\
 & -6\lambda \pm 6 \\
 & 0
\end{array}
$$

$\therefore (\lambda+1)(\lambda^2+5\lambda+6) = 0$

$(\lambda+1)(\lambda^2+3\lambda+2\lambda+6) = 0$

$(\lambda+1)[\lambda(\lambda+3)+2(\lambda+3)] = 0$

$(\lambda+1)(\lambda+2)(\lambda+3) = 0$

$\therefore \lambda = -1, -2, -3.$

$\therefore y_h = c_1 e^{-x} + c_2 e^{-2x} + c_3 e^{-3x}.$

Q $y''' + 3y'' + 3y' + y = 0$

$\lambda^3 + 3\lambda^2 + 3\lambda + 1 = 0$

$\lambda = -1$ is a root

$$
\begin{array}{r|l}
 & \lambda^2 + 2\lambda + 1 \\
\lambda + 1 & \lambda^3 + 3\lambda^2 + 3\lambda + 1 \\
 & -\lambda^3 \pm \lambda^2 \\
 & 2\lambda^2 + 3\lambda \\
 & -2\lambda^2 \pm 2\lambda \\
 & \lambda + 1
\end{array}
$$

$(\lambda+1)(\lambda+1)^2 = 0$

$\lambda = 1, 1, 1$

$y = (C_1 x^2 + C_2 x + C_3) e^{-x}$

Q $y'''' + 8y'' + 16y = 0$

$\lambda^4 + 8\lambda^2 + 16 = 0$

$(\lambda^2 + 4)^2 = 0$

$\lambda^2 = -4$

$\lambda = \pm 2i, \pm 2i$

$y = \cancel{x(C_1 \cos 2x + C_2}$

$(C_1 x + C_2)\cos 2x + (C_3 x + C_4)\sin 2x$

System of

Q1 $\frac{d(y_1)}{dt}$

$\frac{d(y_2}{dt}$

$\Rightarrow$

Sub

System of Linear Equations.

Q1

$$\frac{d(y_1)}{dt} = y_1' = y_1 + 2y_2 \quad \text{--- (i)}$$

$$\frac{d(y_2)}{dt} = y_2' = 2y_1 + y_2 \quad \text{--- (2)}$$ y_1 and y_2 are free of t.

$$\Rightarrow y_1 = \frac{1}{2}(y_2' - y_2)$$

$$y_1' = \frac{1}{2}(y_2'' - y_2')$$

substitute in (1)

$$\frac{1}{2}(y_2'' - y_2') = \frac{1}{2}(y_2' - y_2) + 2y_2$$

$$\frac{y_2''}{2} - \frac{y_2'}{2} = \frac{y_2'}{2} - \frac{y_2}{2} + 2y_2$$

$$\frac{y_2''}{2} - y_2' = \frac{-y_2 + 4y_2}{2}$$

$$\frac{y_2''}{2} - y_2' = \frac{3y_2}{2}$$

$$y_2'' - 2y_2' - 3y_2 = 0$$

Homework:

$y_1' = 3y_1 + y_2$

$y_2' = y_1 + 2y_2$.

on solving,

$$\lambda^2 - 2\lambda - 3 = 0$$

$$\lambda^2 - 3\lambda + \lambda - 3 = 0$$

$$\lambda(\lambda - 3) + 1(\lambda - 3) = 0 \Rightarrow \lambda = 3, -1.$$

$$y_2 = c_1 e^{3t} + c_2 e^{-t}$$

$$y_2' = 3c_1 e^{3t} + c_2 e^{-t}(-1)$$

substitute in (2)

$$3c_1 e^{3t} - c_2 e^{-t} = 2y_1 + c_1 e^{3t} + c_2 e^{-t}$$

$$2c_1 e^{3t} - 2c_2 e^{-t} = 2y_1$$

$$c_1 e^{3t} - c_2 e^{-t} = y_1$$

Solve: $y_1' = 3y_1 + y_2$ ⑪

$y_2' = y_1 + 3y_2$

Ans:

$$y_1' = 3y_1 + y_2 \quad \text{—①}$$

$$y_2' = y_1 + 3y_2 \quad \text{—②}$$

$$y_1 = y_2' - 3y_2$$

$$y_1' = y_2'' - 3y_2'$$

Substituting in ①

$$y_2'' - 3y_2' = 3(y_2' - 3y_2) + y_2 = 3y_2' - 9y_2 + y_2$$

$$y_2'' - 6y_2' + 8y_2 = 0$$

Solving,

$$\lambda^2 - 6\lambda + 8 = 0 \Rightarrow \lambda^2 - 4\lambda - 2\lambda + 8 = 0$$

$$\lambda(\lambda - 4) - 2(\lambda - 4) = 0$$

$$\lambda = 4, 2$$

$$\underline{y_2 = C_1 e^{4t} + C_2 e^{2t}}$$

$$y_2' = 4C_1 e^{4t} + 2C_2 e^{2t}$$

Substituting in ②

$$4C_1 e^{4t} + 2C_2 e^{2t} = y_1 + 12C_1 e^{4t} + 6C_2 e^{2t}$$

$$\underline{y_1 = -8C_1 e^{4t} - 4C_2 e^{2t}}$$

Second order linear differential equation.

$$y'' = f(x, y, y')$$

Put $y = y_1$

$y' = y_2$

$y'' = y_3$

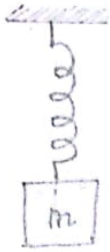
$$y_1' = y' = y_2$$

$$y_2' = y'' = y_3$$

$$y_1' = y_2 \quad \text{and} \quad y_2' = y_3 = f(y_1, y_2)$$

Applications of second order differential equations.

i) **Mass-spring system**



k - spring constant
$y(t)$ - displacement
b - damping constant
t - time

$$F = -by' - ky.$$

$$F = my''$$

$$my'' + by' + ky = 0$$

Case I: When there is no damping.

$$my'' + ky = 0$$

$$y'' + \frac{k}{m} y = 0.$$

$$\lambda^2 + \frac{k}{m} = 0$$

$$\lambda = \pm \sqrt{\frac{k}{m}}\, i$$

Let $w_0 = \sqrt{\frac{k}{m}}$, $\lambda = \pm w_0 i$

$$y(t) = A\cos w_0 t + B\sin w_0 t$$

If $A = C\cos\delta$, $B = C\sin\delta$

$$y(t) = C\cos(w_0 t - \delta)$$

Period $= \frac{2\pi}{w_0}$, Frequency $= \frac{w_0}{2\pi}$ cycles/second.

Case II: When there is damping.

$$my'' + by' + ky = 0$$

$$m\lambda^2 + b\lambda + k = 0$$

$$\lambda^2 + \frac{b}{m}\lambda + \frac{k}{m} = 0$$

ii) Forced oscillations

$$my'' + by' + ky = r(t)$$
(input)

$y(t) = ?$ output
response of the system.

$r(t) = F_0 \cos wt$, y_h .....

$$y_p = A\cos wt + B\sin wt$$

$\left[w_0 = \sqrt{\frac{k}{m}}\right]$; $A = \dfrac{F_0(w_0^2 - w^2)}{m(w_0^2-w^2)^2 + b^2w^2}$, $B = \dfrac{F_0 wb}{m^2(w_0^2-w^2)^2 + w^2b^2}$

case i) undamped system , $b = 0$

$b = 0$; $y_h = c\cos(w_0 t - \delta)$

$B = 0$, $A = \dfrac{F_0}{m(w_0^2 - w^2)}$

$$y = c\cos(w_0 t - \delta) + \frac{F_0}{m(w_0^2-w^2)}\cos wt$$

(natural frequency) $\frac{w_0}{2\pi}$; $\frac{w}{2\pi}$ (frequency of input source).

Maximum amplitude $= \dfrac{F_0}{m(w_0^2-w^2)} = \dfrac{F_0}{mw_0^2\left[1-\left(\frac{w}{w_0}\right)^2\right]}$

As $w \to w_0$, resonance occurs.

Case ii) Damped forced oscillation

$y = y_h + y_p$ (transient state)

$y_h \to 0$. as $t \to \infty$, then $y \to y_p$ (steady state).

(practical case

simple harmonic oscillator then frequency is same as that of input

Q) A spring is stretched 5 cm by a 2 kg mass. It is set in motion from its equilibrium motion with an upwards velocity of 2 m/s. The damping force constant, b = 4. Find the equation of mass at time t.

Ans: $b = 4, \quad m = 2kg, \quad x = 5cm, \quad v = 2m/s$

$$mg = kx.$$

$$\frac{9.8 \times 2}{0.05} = k \Rightarrow k = \frac{9.8 \times 200}{5}$$

$$k = 9.8 \times 40 = 98 \times 4 = 392 \ N/m$$

The equation,

$$my'' + by' + ky = 0$$

$$2y'' + 4y' + 392y = 0. \quad , y(0) = 0$$

$$y'(0) = -2.$$

$$2\lambda^2 + 4\lambda + 392 = 0$$

$$\lambda^2 + 2\lambda + 196 = 0$$

$$\lambda = \frac{-2 \pm \sqrt{4 - 4(1)(196)}}{2}$$

$$\lambda = \frac{-2 \pm \sqrt{4(1-196)}}{2}$$

$$\lambda = -1 \pm \sqrt{195}\, i$$

$$y(t) = \frac{-2}{\sqrt{195}} e^{-t} \sin\sqrt{195}t$$

$$y = e^{-t}\left[A\cos\sqrt{195}\,t + B\sin\sqrt{195}\,t\right]$$

$$\lambda = \frac{-\frac{b}{m} \pm \sqrt{\frac{b^2}{m^2} - 4(1)\left(\frac{k}{m}\right)}}{2(1)}$$

$$\lambda = \frac{-\frac{b}{m} \pm \sqrt{\frac{b^2}{m^2} - \frac{4k}{m}}}{2}$$

$$\lambda = -\frac{b}{2m} \pm \frac{\sqrt{b^2 - 4km}}{2m} = -\alpha \pm \beta$$

(i) If $b^2 > 4ka$, over damping.

$$y(t) = c\, e^{+(-\alpha+\beta)t} + c_2 e^{-(\alpha+\beta)t}$$

as $t \to \infty$, $y(t) \to 0$

(ii) If $b^2 = 4km$; critical damping.

$$y(t) = (c_1 t + c_2)\, e^{-\alpha t}$$

As $t \to \infty$, $y(t) \to 0$ at most for a 't' value

(iii) If $b^2 - 4km < 0$ under damping.

$$y(t) = e^{-\frac{b}{2m}t}\left[A \cos \overset{w^*}{\cancel{\beta}} t + B \sin \overset{w^*}{\cancel{\beta}} t\right]$$

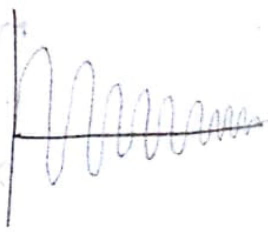
$$= e^{-\alpha t}\left[A \cos \overset{w^*}{\cancel{\beta}} t + B \sin \overset{w^*}{\cancel{\beta}} t\right]$$

$A = C \cos\delta$, $B = C \sin\delta$

$$y(t) = e^{-\alpha t}\, C \cos(w^* t - \delta)$$

$$y(t) = C e^{-\alpha t} \cos(w^* t - \delta)$$

As $t \uparrow$ $y(t) \to 0$



Q. A mass of 0.2 kg stretches a string by 10cm. The damping constant $b = 0.4$. External vibrations create a force of $F(t) = 0.2 \sin 4t$ N. Find the equation of position of mass m at time t.

(Ans. $y = \left(\frac{2}{1697} e^{-t} \cos\sqrt{97}\, t - \frac{80}{1697\sqrt{1697}} e^{-t} \sin\sqrt{97}\, t\right) + \frac{81}{3394} \sin 4t - \frac{2}{1697} \cos 4t$

Ans: $m = 0.2$, $x = 10$ cm, $b = 0.4$

$F(t) = 0.2 \sin 4t$

$$(0.2)(9.8) = k(0.01)$$

$$k = (98)(0.2) = 19.6 \text{ N/m}$$

$$0.2y'' + 0.4y' + 19.6y = 0.2 \sin 4t$$

$$y'' + 2y' + 98y = \sin 4t \qquad y(0) = 0, \quad y'(0) = 0$$

Considering homogeneous equations,

$$y'' + 2y' + 98y = 0$$

$$\lambda^2 + 2\lambda + 98 = 0$$

$$\lambda = \frac{-2 \pm \sqrt{4 - 4(98)}}{2}$$

$$\lambda = -1 \pm \sqrt{97}\, i$$

(iii) Electric circuits.

→ R.L.C circuit

$E_R = IR$

$E_L = L\frac{dI}{dt}$

$E_C = \frac{1}{C}\int I(t)\,dt$

$L\frac{dI}{dt} + RI + \frac{1}{C}\int I(t).dt = E_0 \sin wt.$

differentiating,

$L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{1}{C} I = E_0 w \cos wt$

Q. Given $R = 8\,\Omega$, $L = 2H$, $C = 0.1\,F$ and $E = 160 \cos 5t$.

$2\frac{d^2I}{dt^2} + 8\frac{dI}{dt} + \frac{1}{0.1} I = -160 (5) \sin 5t$

$2\frac{d^2I}{dt^2} + 8\frac{dI}{dt} + 10 I = \cancel{160 \cos 5t} \; -160(5)\sin wt$

$\frac{d^2I}{dt^2} + 4\frac{dI}{dt} + 5I = -\frac{800}{2}\sin 5t = -400 \sin 5t$

Consider homogeneous equation,

$\lambda^2 + 4\lambda + 5 = 0$

$\lambda = \frac{-4 \pm \sqrt{16-4(5)}}{2} = \frac{-4 \pm \sqrt{4(4-5)}}{2}$

$\lambda = \frac{-4 \pm 2\sqrt{-1}}{2} = \frac{-4 \pm 2i}{2}$

$\lambda = -2 \pm i$

$I\, y_h = e^{-2t}(A\cos t + B\sin t)$

$I\, y_p = K_1 \cos 5t + K_2 \sin 5t$

$I_p' = -5K_1 \sin 5t + 5K_2 \cos 5t$

$I_p'' = -25K_1 \cos 5t - 25K_2 \sin 5t$