Tutorial 2

over R

ctor space.

C over R.

ctr space.

{P_n : degree $\leq n$}

Vector space.

P_n: degree $\geq n$

t a vector space.

(8) $a(\alpha+\beta) = a\alpha + a\beta$

Let $e \in R$

$(a,b),(c,d) \in V$.

$LHS = e((a,b)+(c,d))$

$= e(a+c, b+d)$

$= (e(b+d), e(a+c))$

$RHS = e(a,b) + e(c,d)$

$= (eb, ea) + (ed, ec)$

$= (e(b+d), e(a+c))$

② Given:

$(a,b) + (c,d) = (a+c, b+d)$

$c(a,b) = (cb, ca)$

(7) $(a+b)\alpha = a\alpha + b\alpha$

~~$(c+d)(a+b)$~~

Let $(a,b) \in V$

$c, d \in R$

$\therefore (c+d)(a,b) = ((c+d)b, (c+d)a)$

$RHS = c(a,b) + d(a,b)$

$= (cb, ca) + (db, da)$ // LHS $\neq$ RHS $\therefore$ Not a vector space

(9) $a(b\alpha) = (ab)\alpha$

Let $(a,b) \in V$

$p, q \in R$

$LHS = p(q(a,b))$

$= p(qb, qa)$

$= (pqa, pqb)$

$RHS = (pq)(a,b)$

$= (pqb, pqa)$

~~③ (a+b) + (c+d) = (a+c)~~

③ $(a,b) + (c,d)$

$= (a+c, \cancel{b+d}\,0)$ } equal

$c(a,b) = (\cancel{cb, ca}\ ca, b)$.

(9) Let $(a,b) \in V$

and $c, d \in R$.

$a(b\alpha) = (ab)\alpha$

LHS $= c(d(a,b))$

$= c(da, b)$

$= (cda, b)$

RHS $= (cd)(a,b)$

$= (cda, b)$

(8) $a(\alpha+\beta) = a\alpha + a\beta$

Let $(a,b), (c,d) \in V$

$e \in R$.

LHS $= e((a,b) + (c,d))$

$= e(a+c, 0)$

$= (e(a+c), 0)$

RHS $= e(a,b) + e(c,d)$

$= (ea, b) + (ec, d)$

$= (ea+ec, 0)$

(7) $(a+b)\alpha = a\alpha + b\alpha$

$e, f \in R$

LHS $= (e+f)(a,b)$

$= ((e+f)a, b)$

RHS $= e(a,b) + f(a,b)$

$= (ea, b) + (fa, b)$

$= (ea+fa, 0)$

LHS $\neq$ RHS.

$\therefore$ V is not a vector space

④ Vector space.

⑥ ⓐ $W_1 = \{(a,b,c) \in V \ a \geq 0\}$

~~Let α = a+b~~

Let $\alpha = (a_1, b_1, c_1)$
$\beta = (a_2, b_2, c_2)$ $\Big\} \in W_1$

$p, q \in R$.

$p\alpha + q\beta = (pa_1 + qa_2,$
$pb_1 + qb_2, pc_1 + qc_2)$

Eg: ~~Let p = -2.~~
Let $a = -2 \in R$.
$\alpha = (3, 4, 5) \in V$

$a\alpha = (-6, -8, -10) \notin W_1$

$\Rightarrow W_1$ is not a subspace

(b) $W_2 = ((a, b, c) \in V ; a, b, c \in Q)$
Let $a = \sqrt{2}$.
$\alpha = (1, 2, 3)$

$a\alpha = (\sqrt{2}, 2\sqrt{2}, 3\sqrt{2}) \notin W$

$\Rightarrow W_2$ is not a subspace

(c) $W_3 = \{(a, b, c) \in V;$
$a \le b \le c\}$.

Let $a = -1$
$\alpha = (1, 2, 3)$

$a\alpha = (-1, -2, -3) \notin$
(here $a \ge b \ge c$)

$\therefore W_3$ is not a subspace

(d) $W_4 = \{(a, b, c) \in V;$
$b + 4c = 0\}$

Let $\alpha = (a_1, b_1, c_1)$
$\beta = (a_2, b_2, c_2)$
where $b_1 + 4c_1 = 0$
$b_2 + 4c_2 = 0$

$p\alpha + q\beta$

~~$= pa_1 + qa_2, pa_1 + qb_2$~~

$(pa_1 + qa_2, pb_1 + qb_2,$
$pc_1 + qc_2)$

Let $\alpha = (a_1, b_1, c_1)$
$\beta = (a_2, b_2, c_2)$ $\Big\} \in W_1$

$p, q \in \mathbb{R}$.

$p\alpha + q\beta = (pa_1 + qa_2,\ pb_1 + qb_2,\ pc_1 + qc_2)$.

Eg: ~~Let p = -2~~.
Let $a = -2 \in \mathbb{R}$.
$\alpha = (3, 4, 5) \in V$

$a\alpha = (-6, -8, -10) \notin W_1$

$\Rightarrow W_1$ is not a subspace

(b) $W_2 = ((a, b, c) \in V ;\ a, b, c \in \mathbb{Q})$

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$\alpha = (1, 2, 3)$

$a\alpha = (\sqrt{2}, 2\sqrt{2}, 3\sqrt{2}) \notin W$

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(c) $W_3 = \{(a, b, c) \in V;\ a \le b \le c\}$.

Let $a = -1$
$\alpha = (1, 2, 3)$

$a\alpha = (-1, -2, -3) \notin W_3$
(here $a \ge b \ge c$)

$\therefore W_3$ is not a subspace

(d) $W_4 = \{(a, b, c) \in V;\ b + 4c = 0\}$

Let $\alpha = (a_1, b_1, c_1)$
$\beta = (a_2, b_2, c_2)$
where $b_1 + 4c_1 = 0$
$b_2 + 4c_2 = 0$

$p\alpha + q\beta$

$= \cancel{pa_1 + qa_2,\ pa_1 + qb_2,}$

pa

$(pa_1 + qa_2,\ pb_1 + qb_2,\ pc_1 + qc_2)$.

$(pb_1+qb_2) + 4(pc_1+qc_2)$

$= (pb_1+4pc_1) + (qb_2+4qc_2)$

$= p(b_1+4c_1) + q(b_2+4c_2)$

$= \underline{\underline{0}}$

$\therefore W_4$ is a subspace.

$a\alpha + b\beta \in W$

$\Rightarrow$ vector space.

⑦ Given

$W: \{$set of all diff. eqns : $3\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 7y = 0\}$

Let $\alpha = y_1$, $\beta = y_2 \in W$.

and $a, b \in R$

~~$a\alpha + b\beta$~~

$3\frac{d^2y_1}{dx^2} - 5\frac{dy_1}{dx} + 7y_1 = 0$

and $3\frac{d^2y_2}{dx^2} - 5\frac{dy_2}{dx} + 7y_2 = 0$

⑧ ⓐ $W_1: \left\{\begin{bmatrix} a & b \\ c & d \end{bmatrix} : ad - bc \neq 0\right\}$

Let $\alpha = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}$

$a_1d_1 - b_1c_1 \neq 0$

and $\beta = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}$

$a_2d_2 - b_2c_2 \neq 0$.

~~$p\alpha + q\beta$~~

Let $p, q \in R$

$p\alpha + q\beta$

$= \begin{bmatrix} pa_1+qa_2 & pb_1+qb_2 \\ pc_1+qc_2 & pd_1+qd_2 \end{bmatrix}$

$\Delta = (pa_1+qa_2)(pd_1+qd_2) - (pb_1+qb_2)(pc_1+qc_2)$

$= p^2 a_1 d_1 + \cancel{pq} \; pq a_1 d_2 + \cancel{pq\, a_2 d_1} - p^2 b_1 c_1 - pq b_1 c_2 - \cancel{pq b_2 c_1} - q^2 b_2 c_2$

Let $\alpha = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $|\alpha| = 1$

$\beta = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ $|\beta| = 1$

$a = 1, \quad b = 1$

$a\alpha + b\beta = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$|a\alpha + b\beta| = 0.$

$\therefore W_1$ is not a subspace

ⓑ $W_2 = \left\{ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : A^2 = A \right\}$

$A^2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$= \begin{bmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{bmatrix}$

$\alpha = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

$\beta = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$a = 1, \quad b = 1$

$a\alpha + b\beta = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$

$\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}$

$\neq a\alpha + b\beta.$

$\therefore W_2$ is not a subspace

ⓒ $W_3 = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}, a, b \in \mathbb{R} \right\}$

$\alpha = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$

$\beta = \begin{bmatrix} c & 0 \\ 0 & d \end{bmatrix}$

$p\alpha + q\beta = \begin{bmatrix} pa + qc & 0 \\ 0 & pb + qd \end{bmatrix} \in W_3$

$\therefore W_3$ is a subspace //

(9) (a) Given W_1 and W_2 are two subspaces of a vector space V.

To prove: $W_1 \cap W_2$ is also a subspace of V.

~~Proof~~: Let $\alpha \in W_1 \cap W_2$

and $\beta \in W_1 \cap W_2$.

<u>To show</u>: $\alpha+\beta \in W_1 \cap W_2$.

and $a\alpha \in W_1 \cap W_2$.

Proof: We know that

$W_1 \cap W_2 \subset W_1$

and $W_1 \cap W_2 \subset W_2$.

W_1 is a subspace $\Rightarrow \alpha+\beta \in W_1$ and $a\alpha \in W_1$

~~$\Rightarrow \alpha+\beta$~~

$\Rightarrow \alpha+\beta \in W_1 \cap W_2 \subset W_1$

Illy W_2 is a subspace

$\Rightarrow \alpha+\beta \in W_2$.

$\Rightarrow \alpha$~~$+\beta \in W_1 \cap W_2 \subset W_2$~~,

and $a\alpha \in W_2$

$\Rightarrow \alpha+\beta \in W_1 \cap W_2$ and also $a\alpha \in W_1 \cap W_2 \Rightarrow W_1 \cap W_2$ is a subspace of V.

<u>Now</u>, ~~$a\alpha \in W_1$~~

and ~~$a\beta \in W_2$~~

$\Rightarrow$ ~~$a\alpha \in W_1 \cap W_2$~~. $\in W_1$

~~(b) Eg: Let $V = R^2$.~~

~~Assume $U \cup V$ is a subspace of~~ R^n.

(b) Consider two subspaces W_1 and W_2.

Let there be ~~a~~ an element $\alpha \in W_1$ but $\alpha \notin W_2$.

Assume that $W_1 \cup W_2$ is a subspace.

We know that,

$\alpha, \beta \in W_1 \cup W_2$.

$\Rightarrow \alpha+\beta \in W_1 \cup W_2$

$\Rightarrow \alpha+\beta \in W_1$ or $\alpha+\beta \in W_2$

If $\alpha+\beta \in W_1$, then there exists $\alpha' \in W_1$ such that

$\alpha+\beta = \alpha'$

$\because \alpha \in W_1$, $\alpha' \in W_1$, then $\alpha' - \alpha = \beta$ should also belong to W_1. But $\beta \in W_2$.

This is a contradiction.

$\therefore \alpha+\beta \in W_2$

$\Rightarrow \alpha+\beta = \beta'$

$\because \beta \in W_2, \beta' \in W_2,$

$\alpha = \beta'-\beta$ should also belong to W_2. But $\alpha \in W_1$.

$\Rightarrow$ This is a contradiction.

$\therefore W_1 \cup W_2$ need not be a subspace.

(10) (a) V: {set of all continuous and diff. real func defined on R}

W_1: {set of all func $f: R \to R$ such that $f(2)=0$ and $f(0)=2$}

~~$f(0)=2$~~

f ~~$0 = f(2)$~~

~~$f(0)=2$~~

Let clearly $W_1 \subset V$.

(11) V: {set of all real polynomials}

W: {polynomials of degree = 7}

Clearly $W \subset V$.

Let $\alpha = x^7 - x^2$

$\beta = -x^7 - x^2$

$\alpha \in W$, $\beta \in W$.

$\alpha+\beta = -2x^2 \notin W$.

$\therefore W$ is not a subspace.

(12) Let α be in the L.C of x_1, x_2, x_3.

$\alpha = ax_1 + bx_2 + cx_3$

$(1,-5,3) = a(1,-3,2) + b(2,-4,-1) + c(1,5,7)$.

$(a+2b+c, -3a-4b+5c, 2a-b+7c)$

$\Rightarrow a+2b+c = 2$

$-3a-4b+5c = -5$

$2a-b+7c = 3$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ -3 & -4 & 5 & -5 \\ 2 & -1 & 7 & 3 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 2 & 8 & 1 \\ 0 & -3 & 5 & -1 \end{array}\right]$$

(13) $x = (1,-2,5)$

$x_1 = (1,1,1)$

$x_2 = (1,2,13)$

$x_3 = (2,-1,1)$

$x = a x_1 + b x_2 + c x_3$

$(1,-2,5) = a(1,1,1) + b(1,2,13) + c(2,-1,1)$

$= (a+b+2c,\ a+2b-c,\ a+3b+c)$

$\Rightarrow a+b+2c = 1$

$a+2b-c = -2$

$a+3b+c = 5$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 1 & 2 & -1 & -2 \\ 1 & 3 & 1 & 5 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & 1 & -3 & -3 \\ 0 & 2 & -1 & 4 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & 1 & -3 & -3 \\ 0 & 0 & 5 & 10 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 2 & 8 & 1 \\ 0 & 0 & -14 & -5 \end{array}\right]$$

$\Rightarrow -14c = -5$

$c = \frac{5}{14}$

$2b + 8c = 1$

$2b = 1 - \frac{40}{14}$

$2b = \frac{-26}{14}$

$b = \frac{-13}{14}$

$a + 2b + c = 2$

$a = 2 + \frac{26}{14} - \frac{5}{14}$

$a = \frac{49}{14}$

$x = \frac{49}{14} x_1 - \frac{13}{14} x_2 + \frac{5}{14} x_3$ //

$\therefore x$ can be expressed as a l.c of x_1, x_2, x_3.

$\Rightarrow 5c = 10$

$c = 2$

$b - 3c = -3$

$b - 6 = -3$

$b = 3$

$a + b + 2c = 1$

$a + 3 + 4 = 1$

$a = -6$

$\therefore x = -6x_1 + 3x_2 + 2x_3$

(14) $x = \begin{bmatrix} 3 & -1 \\ 1 & -2 \end{bmatrix}$

$x_1 = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$

$x_2 = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$

$x_3 = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$

$x = ax_1 + bx_2 + cx_3$

$\begin{bmatrix} 3 & -1 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} a & a \\ 0 & -a \end{bmatrix} + \begin{bmatrix} b & b \\ -b & 0 \end{bmatrix} + \begin{bmatrix} c & -c \\ 0 & 0 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 3 & -1 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} a+b+c & a+b-c \\ -b & -a \end{bmatrix}$

$\Rightarrow a = 2 \qquad b = -1$

$a + b + c = 3 \Rightarrow c = 2$

$\therefore x = 2x_1 - x_2 + 2x_3$

(15) Given

$S = \{(1,2,3), (0,1,2), (0,0,1)\}$

We know that $L[S] \subseteq R^3$ — (1)

Let $(x,y,z) \in R^3$ and

let $(x,y,z) \in L[S]$.

$\Rightarrow (x,y,z) = a(1,2,3) + b(0,1,2) + c(0,0,1)$

for some $a,b,c \in R$

$\Rightarrow x = a$

$y = 2a + b$

$z = 3a + 2b + c$

$\Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & x \\ 2 & 1 & 0 & y \\ 3 & 2 & 1 & z \end{array}\right]$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & x \\ 0 & 1 & 0 & y-2x \\ 0 & 2 & 1 & z-3x \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & x \\ 0 & 1 & 0 & y-2x \\ 0 & 0 & 1 & x-2y+z \end{array}\right]$$

$\Rightarrow$ $a = x$

$b = y - 2x$

$c = x - 2y + z$

$\therefore$ Solution exists.

$\Rightarrow (x, y, z) \in L[S]$

and $(x, y, z) \in R^3$

$\Rightarrow R^3 \subseteq L[S]$ — (2)

From (1) & (2),

$R^3 = L[S]$

or x_1, x_2, x_3 generate R^3.

(16) $u = (1, 2, 3)$

$v = (3, 1, 5)$

$w = (3, -4, 7)$

Linear span of S:

$L[S] = \{au + bv, \forall\ a, b \in R\}$

$L[T] = \{cu + dv + ew\ \forall\ c, d, e \in R\}$

If w can be expressed as l.c. of u and v, then $L[S] = L[T]$.

Suppose $w = au + bv$, for $a, b \in R$

$(3, -4, 7) = a(1, 2, 3) + b(3, 1, 5)$

$(3, -4, 7) = (a+3b, 2a+b, 3a+5b)$

$a + 3b = 3$

$2a + b = -4$

$3a + 5b = 7$

$$\left[\begin{array}{cc|c} 1 & 3 & 3 \\ 2 & 1 & -4 \\ 3 & 5 & 7 \end{array}\right]$$

$$\sim \begin{bmatrix} 1 & 3 & 3 \\ 0 & -5 & -10 \\ 0 & -4 & -2 \end{bmatrix}$$

$$\sim \left[\begin{array}{cc|c} 1 & 3 & 3 \\ 0 & -5 & -10 \\ 0 & 0 & +30 \end{array}\right]$$

(17) $(3,-1,0,-1) = a(2,-1,3,2) + b(-1,1,1,-3) + c(1,1,9,-5)$

$= (2a-b+c,\ -a+b+c,\ 3a+b+9c,\ 2a-3b-5c)$

$2a-b+c = 3$

$-a+b+c = -1$

$3a+b+9c = 0$

$2a-3b-5c = -1$

$$\left[\begin{array}{ccc|c} 2 & -1 & 1 & 3 \\ -1 & 1 & 1 & -1 \\ 3 & 1 & 9 & 0 \\ 2 & -3 & -5 & -1 \end{array}\right]$$

$\sim$

$$\left[\begin{array}{ccc|c} 2 & -1 & 1 & 3 \\ 0 & 1 & 3 & 1 \\ 0 & 5 & 15 & -9 \\ 0 & -2 & -6 & -4 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 2 & -1 & 1 & 3 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & -14 \\ 0 & 0 & 0 & -2 \end{array}\right]$$

$\rho(A) = 2$

$\rho(\bar{A}) = 4$

$\Rightarrow \rho(A) < \rho(\bar{A})$

$\Rightarrow$ infinitely many solutions $\Rightarrow$ does not span

~~Take~~

(18) 1

$2-x$

$3+x^2$

$4-x^3$

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 2 \\ 0 & 1 & 0 & 3 \\ -1 & 0 & 0 & 4 \end{bmatrix}$$

$$\sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & -1 & 0 & 0 \\ 3 & 0 & 1 & 0 \\ 4 & 0 & 0 & -1 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$\Rightarrow$ L.I

Let $ax^3 + bx^2 + cx + d \in P_3$

$ax^3 + bx^2 + cx + d$

$= \cancel{p(1,0,0,0)} +$

$\cancel{q(2,-1,0,0)} + \cancel{r(3,0,1,0)} +$

$\cancel{s(4,0,0,-1)}$

$= p(1) +$

$q(2-x) + r(3+x^2) +$

$s(4-x^3)$.

p, q, r, s

$= (p + 2q + 3r + 4s) +$

$-qx + rx^2 - sx^3$.

Comparing,

$\Rightarrow$

$a = -s$

$b = r$

$c = -q$

$d = p + 2q + 3r + 4s$

$\Rightarrow s = -a$

$r = b$

$q = -c$

& $p = d + 2c - 3b + 4a$

$\Rightarrow (a, b, c, d) \in P_3$

$\Rightarrow (p, q, r, s) \in P_3$.

$\therefore$ The polynomials span the vector space P_3

19 (a) $(1,2,-3), (1,-3,2), (2,1,-5)$

Let $a(1,2,-3) + b(1,-3,2) + c(2,1,-5) = \bar{0} = (0,0,0)$.

$\Rightarrow (a+b+2c,\ 2a-3b+c,\ -3a+2b-5c) = (0,0,0)$.

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 2 & -3 & 1 & 0 \\ -3 & 2 & -5 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & -5 & -3 & 0 \\ 0 & 5 & 1 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & -5 & -3 & 0 \\ 0 & 0 & -2 & 0 \end{array}\right]$$

$\rho(A) = \rho(\bar{A}) = 3$

$\Rightarrow a = b = c = 0$.

$\Rightarrow$ unique solution

$\Rightarrow$ L.I.//

(b) $(0,2,-4), (1,-2,-1), (1,-4,3)$.

Let $a(0,2,-4) + b(1,-2,-1) + c(1,-4,3) = (0,0,0)$

$(b+c,\ 2a-2b-4c,\ -4a-b+3c) = (0,0,0)$.

$$\left[\begin{array}{ccc|c} 0 & 1 & 1 & 0 \\ 2 & -2 & -4 & 0 \\ -4 & -1 & +3 & 0 \end{array}\right]$$

$$\sim \begin{array}{c} \begin{array}{ccc} c & b & a. \end{array} \\ \left[\begin{array}{cccc} 1 & 1 & 0 & 0 \\ -4 & -2 & 2 & 0 \\ 3 & -1 & -4 & 0 \end{array}\right] \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & -4 & -4 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\Rightarrow$ infinite sol. $\Rightarrow$ a, b, c can be non-zero $\Rightarrow$ L.D

(c) $(0,1,-2),(1,-1,1),(1,2,1)$

Let $a(0,1,-2)+b(1,-1,1)+c(1,2,1)=(0,0,0)$

$b+c=0$

$a-b+2c=0$

$-2a+b+c=0$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 2 & -1 & 1 & 0 \\ 1 & 1 & -2 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & -3 & 1 & 0 \\ 0 & 0 & -2 & 0 \end{array}\right]$$

$\sim$ ~~1 1 0 0~~

~~0 -3 1~~ 0

$\Rightarrow$ L.I

(20) Given x,y,z are L.I.

Let $a,b,c \in R$

$ax+by+cz=\bar{0}$

$\Rightarrow a=b=c=0.$

Consider

$a(x+y)+b(y+z)+c(z+x)=\bar{0}$

$(a+c)x+(a+b)y+(b+c)z=\bar{0}.$

$a+c=0$

$a+b=0$

$b+c=0 \Rightarrow a=b=c=0$

$\Rightarrow \{x+y,\ y+z,\ z+x\}$ is L.I.

(21) $(1+a,1-a)\ (1-a,1+a)$

Let $p,q \in R$

Let $p(1+a,1-a)+q(1-a,1+a)=\bar{0}=(0,0)$

$\Rightarrow (p(1+a)+q(1-a),\ p(1-a)+q(1+a))=(0,0)$

~~p(1+a) + q(1-a)~~

$p(1+a) + q(1-a) = 0$

$p(1-a) + q(1+a) = 0$

$p + ap + q - aq = 0$

$p - ap + q + aq = 0$

$\Rightarrow (p+q) + (p-q)a = 0$ —(1)

$(p+q) + (q-p)a = 0$ —(2)

(1) − (2)

$\Rightarrow (p-q-q+p)\,a = 0$

$\Rightarrow \cancel{2p-2q}\,(p-q)\,a = 0$

$$\begin{bmatrix} 1+a & 1-a \\ 1-a & 1+a \end{bmatrix}\begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \left[\begin{array}{cc|c} 1+a & 1-a & 0 \\ 1-a & 1+a & 0 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{cc|c} 1+a & 1-a & 0 \\ 0 & (1+a)^2 & 0 \end{array}\right]$$

$\cancel{\Rightarrow (1+a)^2 q = 0.}$

$\cancel{\Rightarrow (1+a)^2 \neq 0 \Rightarrow q = 0.}$

If $(1+a)^2 = 0$, system is L.D.

$\therefore$ Set is L.I iff $(1+a)^2 \neq 0$.

$\Rightarrow a \neq -1$

Q2 $(1,-1,3)$ $(1,2,-3)$ $(a,0,1)$ are L.D.

Let $p(1,-1,3) + q(1,2,-3) + r(a,0,1) = (0,0,0)$

$(p+q+ar,\ -p+2q,\ 3p-3q+r) = (0,0,0)$.

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & a & 0 \\ -1 & 2 & 0 & 0 \\ 3 & -3 & 1 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & a & 0 \\ 0 & 3 & a & 0 \\ 0 & -6 & 1-3a & 0 \end{array}\right]$$

$$\sim \begin{bmatrix} 1 & 1 & a & 0 \\ 0 & 3 & a & 0 \\ 0 & 0 & 1-a & 0 \end{bmatrix}$$

It is L.D $\Rightarrow 1-a=0$

$$\underline{\underline{a=1}}$$

(23) (a)
$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 5 \\ 5 & 3 & 4 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 3 \\ 0 & -2 & -6 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow$ <u>L.D</u> $\Rightarrow$ Not a basis.

(b) $\cancel{0 \quad -3 \quad 2}$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -3 & 2 \\ 1 & 0 & -1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & -3 & 2 \\ 0 & -2 & -2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & -3 & 2 \\ 0 & 0 & -10 \end{bmatrix}$$

$\Rightarrow$ L.I.

(24) ~~$S = \{1, x, x^2, \ldots x^n\}$~~

(25) Not possible.

only 2 L.I vectors will be possible in R^2

(26)
$$W = \begin{bmatrix} x & -x \\ y & z \end{bmatrix}.$$

Let $\alpha = \begin{bmatrix} x_1 & -x_1 \\ y_1 & z_1 \end{bmatrix}$

and $\beta = \begin{bmatrix} x_2 & -x_2 \\ y_2 & z_2 \end{bmatrix} \in W.$

$a, b \in R.$

$$a\alpha + b\beta = \begin{bmatrix} ax_1 + bx_2 & -ax_1 - bx_2 \\ ay_1 + by_2 & az_1 + bz_2 \end{bmatrix}$$

$\in W$

$\Rightarrow$ W is a <u>subspace</u>

Let

$$x\begin{bmatrix}1 & -1\\ 0 & 0\end{bmatrix} + y\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix} + z\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix} = \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$$

$\Rightarrow x = y = z = 0.$

$\Rightarrow$ L.I

$\Rightarrow$ Dimension $= 3$.

(27) $S_1 = \begin{bmatrix}1 & -4 & -2 & 1\\ 1 & -3 & -1 & 2\\ 3 & -8 & -2 & 7\end{bmatrix}$

$\sim \begin{bmatrix}1 & -4 & -2 & 1\\ 0 & 1 & 1 & 1\\ 0 & 4 & 4 & 4\end{bmatrix}$

$\sim \begin{bmatrix}1 & -4 & -2 & 1\\ 0 & 1 & 1 & 1\\ 0 & 0 & 0 & 0\end{bmatrix}$

S_1 is L.D.

Let $S_2 = \{(1,-4,-2,1), (0,1,1,1)\}$

S_2 is L.I.

Also $L[S_1] = W_1$

$\therefore L[S_2] = L[S_1] = W_1$.

$\therefore S_2$ is a basis of W_1

dim $= 2$.

$\{(1,-4,-2,1), (0,1,1,1), (0,0,1,0), (0,0,0,1)\}$ is a basis of R^4.

(28) solved qstn.

(29) (a) $x + 2y - 4z + 3s - t =$

$x + 2y - 2z + 2s + t =$

$2x + 4y - 2z + 3s + 4$

$\sim \begin{bmatrix}1 & 2 & -4 & 3 & -1\\ 1 & 2 & -2 & 2 & 1\\ 2 & 4 & -2 & 3 & 4\end{bmatrix}$

$\sim \begin{bmatrix}1 & 2 & -4 & 3 & -\\ 0 & 0 & 2 & -1 & 2\\ 0 & 0 & 6 & -3 & 6\end{bmatrix}$

$\sim \begin{bmatrix}1 & 2 & -4 & 3 & \\ 0 & 0 & 2 & -1 & 2\\ 0 & 0 & 0 & 0 & 0\end{bmatrix}$

$\therefore S = \{(1, 2, -4, 3, -1),$
$(0, 0, 2, -1, 2)\}.$

$L[S] = W.$

It is a basis with

$\dim = 2.$

(b) $x + 2y - 2z + 2r + s = 0$

$2x + 4y - 6z + 5r = 0$

$2x + 4y - 2z + 3r + 4s = 0$

$3x + 6y - 8z + 7r + s = 0$

$x \quad y \quad z \quad r \quad s$

~~1 2 -2 2 1~~

~~2 4 -6 5~~

$$\begin{bmatrix} 1 & 2 & -2 & 2 & 1 \\ 2 & 4 & -6 & 5 & 0 \\ 2 & 4 & -2 & 3 & 4 \\ 3 & 6 & -8 & 7 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & -2 & 2 & 1 \\ 0 & 0 & -2 & 1 & -2 \\ 0 & 0 & 2 & -1 & 2 \\ 0 & 0 & -2 & 1 & -2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 2 & -2 & & 2 & 1 \\ 0 & 0 & -2 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$S = \{(1, 2, -2, 2, 1)$
$(0, 0, -2, 1, -2)\}$

$\dim = 2$

(c)
$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & -2 \\ 0 & 1 & -1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -4 \\ 0 & 1 & -1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -4 \\ 0 & 0 & -5 \end{bmatrix}$$

$\dim = 3.$

29 a

$$\begin{bmatrix} 1 & 2 & -4 & 3 & -1 \\ 1 & 2 & -2 & 2 & 1 \\ 2 & 4 & -2 & 3 & 4 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & -4 & 3 & -1 \\ 0 & 0 & 2 & -1 & 2 \\ 0 & 0 & 6 & -3 & 6 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & -4 & 3 & -1 \\ 0 & 0 & 2 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

~~here x and y~~

Let $z = k_1$ $s = k_2$ $t = k_3$

$x + 2y - 4z + 3s - t = 0$

$2z - s + 2t = 0$