NATIONAL INSTITUTE OF TECHNOLOGY CALICUT

Department of Mathematics

Winter Semester 2019-20

MA1002D MATHEMATICS II

Tutorial 3

Linear Transformations

1. Which of the following transformations are linear
   1. $T(x) = 0$, $T(x) = x$, $T(x) = x + a$, $T(x) = x^2$, $T(x) = e^x$, $T(x) = 1$, $T(x) = \sin x$
   2. $T(x) = (x, x)$, $T(x) = (x, 0)$, $T(x) = (x^2, x)$, $T(x) = (x, 1)$
   3. $T(x, y) = xy$, $T(x, y) = x + y$, $T(x, y) = 2x + 3y$, $T(x, y) = x^2 + y$
   4. $T(x, y) = (x + y, xy)$, $T(x, y) = (y, x)$, $T(x, y) = (x/y, y/x)$, $T(x) = (|x|, 0)$
2. A linear transformation T on $\mathbb{R}^3$ to itself is defined by $T(e_1) = e_1 + e_2 + e_3$, $T(e_2) = e_2 + e_3$ and $T(e3) = e_2 - e_3$, where $\{e_1, e_2, e_3\}$ is the standard basis of $\mathbb{R}^3$. Determine the image of (2, -1, 3).
3. Find $T(x_1, x_2, x_3)$ where $T: \mathbb{R}^3 \to \mathbb{R}$ defined by $T(1, 1, 1) = 3$, $T(0, 1, -2) = 1$, $T(0, 0, 1) = -2$.
4. Let $T: \mathbb{R}^3 \to \mathbb{R}^2$ be a linear transformation defined by $T(x, y, z) = (2x-3y+z, -2x+5z)$, find the matrix of T relative to the standard basis of $\mathbb{R}^3$.

Kernel, Range and Rank-Nullity Theorem

5. Find the kernel space and range space of linear transformation $T: P[x] \to P[x]$ defined by $T(p(x)) = p'(x)$ (where P[x] is the set of all real polynomials)
6. Verify the Rank- Nullity Theorem for the following function
$$T: M_{2\times2}(\mathbb{R}) \to M_{2\times2}(\mathbb{R}) \text{ by } T(A) = A + A^T$$
(Where $M_{2\times2}(\mathbb{R})$is the set of all real 2×2 matrices)
7. Let T be a linear transformation from U to V. Show that Range space of T is a subspace of V and Kernal of T is a subspace of U.
8. For each of the following linear mappings $T: U \to V$ find a basis and the dimension of its range space and its null space (kernel). Also verify Rank-Nullity Theorem.
   (a) $T(x, y, z) = (y + z, x + y - 2z, x + 2y - 2z)$, $T(x, y, z) = (3x, x - y, 2x + y + z)$
   (b) $T(x_1, x_2, x_3, x_4) = (x_1 - x_2 + x_3 + x_4, x_1 + 2x_3 - x_4, x_1 + x_2 + 3x_3 - 3x_4)$
9. Let V be the vector space of 2 x 2 matrices and let $M = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$. Let $T: V \to V$ be the linear map defined by $T(A) = AM - MA$. Find a basis and the dimension of the null space of T.
10. Let V be the vector space of all polynomials p(x), with real coefficients, whose degree is less than or equal to 6. Compute the basis and dimension of the null space of the linear transformation $T: V \to V$ defined by $T(p(x))=(1/2)(p(x)-p(-x))$, for all $p(x) \in V$.

Invertibility of Linear Transformations

11. Show that $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ defined by $T(x, y, z) = (x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta, z)$ is nonsingular, where θ is any angle.
12. Show that each of the following operators T on $\mathbb{R}^3$ or $\mathbb{R}^2$ is invertible and find T^{-1}.
    a) $T(x, y, z) = (2x, 4x - y, 2x + 3y - z)$
    b) $T(x, y, z) = (x - 3y - 2z, y - 4z, z)$
    c) $T(x, y, z) = (x + z, x - z, y)$
    d) $T(x, y) = (x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta)$

Gram-Schmidt Orthonormalisation

13. Using the Gram-Schmidt process, find an orthonormal basis for the subspace spanned by the following sets of vectors.
    a) $(3, 4), (-1, 1)$.
    b) $(2, 3, 6), (7, 12, 8)$
    c) $(1, 1, 0), (1, 0, 1), (0, 1, 1)$
    d) $(1, 2, 2), (1, 4, 0), (2, 0, 1)$
    e) $(1, 1, 1, 1), (0, 1, 2, 2), (0, 0, 1, 1)$.