NATIONAL INSTITUTE OF TECHNOLOGY CALICUT

Department of Mathematics, Winter Semester 2019-20

MA1002D MATHEMATICS II

Tutorial sheet- 2

Vector Spaces and Subspaces

1. Prove /disprove that the following are examples of vector spaces under usual (natural) addition and scalar multiplication.
   (a) $\mathbb{R}^n$ over $\mathbb{R}$, (b) $\mathbb{C}$ over $\mathbb{R}$
   (c) P = {Polynomials in x with real coefficients}, $P_n = \{p(x) \in P : \deg p(x) \leq n\}$
   (d) P = {Polynomials in x with real coefficients}, $Pn = \{p(x) \in P : \deg p(x) = n\}$.

2. Let V be the set of all pairs (a,b) of real numbers and R be the field of real numbers. With the operation $(a,b)+(c,d)=(a+c,b+d)$; $c(a,b)=(cb,ca)$ prove or disprove that V(R) is a vector space.

3. Let V be the set of all pairs (a,b) of real numbers and R be the field of real numbers. Show that with the operation $(a,b)+(c,d)=(a+c,0)$; $c(a,b)=(ca,b)$, V(R) is not a vector space.

4. Let V be the set of all 2x2 real matrices and R be the field of real numbers. Show that V(R) is a vector space.

5. Identify (geometrically) all subspaces of $\mathbb{R}^3$.

6. Let V be the vector space $\mathbb{R}^3$. Examine whether the following are subspaces or not.
   (a) $W1 = \{ (a, b, c) \in V ;\ a \geq 0 \}$
   (b) $W2= \{ (a, b, c) \in V ;\ a, b, c \in Q \}$
   (c) $W3 = \{(a, b, c) \in V;\ a \leq b \leq c\}$
   (d) $W4= \{(a, b, c) \in V ;\ b + 4c = 0\}$

7. If V is the vector space of real valued continuous functions, then show that the set W of all solutions of the differential equation $3\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 7y = 0$ is a subspace of V.

8. Let V be the vector space of all 2 x 2 matrices with real entries. Determine whether Wi (i=1,2,3) is a subspace of V or not, where
   (a) W1 consists of all matrices with non-zero determinant.
   (b) W2 consists of all matrices A such that $A^2 = A$.
   (c) W3 consists of all diagonal matrices.

9. Let W1 and W2 are two nontrivial subspaces of a vector space V($\mathbb{R}$). Prove or disprove
   (a) Intersection of W1 and W2 is also a subspace of V
   (b) Union of W1 and W2 is also a subspace of V.

10. Let V be the set of all continuous and differentiable real valued functions defined on R. Verify whether the following subsets of V are subspaces of V or not? Justify your answers.
    (a) W1 is the set of functions $f : R \to R$ such that $f(2) = 0$ and $f(0) = 2$.
    (b) W2 is the set of functions $f : R \to R$ such that $f(1) = 0$ and $f'(2)=0$

11. Let V be the vector space of set of all real polynomials over the field of real numbers R. Let W be the subset of V of all real polynomials of degree 7. Is W, a subspace of V?

Linear Combination and Span of Vectors

12 Is the vector (2, -5, 3), a linear combination of vectors $x_1 = (1, -3, 2)$, $x_2 = (2, -4, -1)$ and $x_3 = (1, 5, 7)$?

13 Write the vector $x = (1, -2, 5)$ as a linear combination of the vectors $x_1 = (1, 1, 1)$, $x_2 = (1, 2, 3)$ and $x_3 = (2, -1, 1)$ in the vector space $\mathbb{R}^3$.

14 Write the vector $x = \begin{bmatrix} 3 & -1 \\ 1 & -2 \end{bmatrix}$ in the vector space of all 2 x 2 matrices with real entries, as a linear combination of

$$x_1 = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}, \quad x_2 = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} \text{ and } x_3 = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

15 Show that the vectors $x_1 = (1, 2, 3)$, $x_2 = (0, 1, 2)$ and $x_3 = (0, 0, 1)$ generate $\mathbb{R}^3$.

16 In the vector space $\mathbb{R}^3$, let $u = (1, 2, 3)$, $v = (3, 1, 5)$, $w = (3, -4, 7)$. Prove that the subspace S spanned by u and v and the subspace T spanned by u, v and w are the same.

17 Is the vector (3, -1, 0, -1) an element in the subspace of $\mathbb{R}^4$ spanned by the vectors (2, -1, 3, 2), (-1, 1, 1, -3) and (1, 1, 9, -5).

18 Prove that the polynomials $1, 2 - x, 3 + x^2, 4 - x^3$ span the vector space P_3 of Ex 1 (c)

Linear Independence of Vectors

19 Determine whether the following vectors are linearly independent or not.

(a) (1, 2, -3), (1, -3, 2), (2, 1, -5)

(b) (0, 2, -4), (1, -2, -1), (1, -4, 3)

(c) (0, 1, -2), (1, -1, 1), (1, 2, 1)

20 If x, y, z are linearly independent vectors in a vector space V then prove that $x + y$, $y + z$, $z + x$ are also linearly independent.

21 Under what condition on a, the vectors $(1 + a, 1 - a)$ and $(1 - a, 1 + a)$ in $\mathbb{R}^2$ are linearly independent?

22 Find 'a' if the vectors (1, -1, 3), (1, 2, -3) and (a, 0, 1) are linearly dependent.

Basis and Dimension

23 Examine whether the following set of vectors form a basis for $\mathbb{R}^3$.

(a) (1, 1, 2), (1, 2, 5), (5, 3, 4) (b) (0, -3, 2), (1, 2, 1), (1, 0, -1)

24 Show that the set $S = \{1, x, x^2, \ldots, x^n\}$ is a basis for the vector space P_n of Ex 1 (c).

25 Can you give any example of 3 linearly independent vectors in $\mathbb{R}^2$?

26 Let V be the vector space of all 2 x 2 matrices and let W be the set of matrices of the form $\begin{bmatrix} x & -x \\ y & z \end{bmatrix}$.

Verify that W is a Subspace of V and hence find dim W.

27 Find a basis and dimension of the subspace W of $\mathbb{R}^4$ generated by (1, -4, -2, 1), (1, -3, -1, 2), (3, -8, -2, 7). Also extend the basis of W to a basis of the whole space $\mathbb{R}^4$.

28 V_1 and V_2 are subspaces of $\mathbb{R}^4$ given by $V_1 = \{(a, b, c, d); b - 2c + d = 0\}$, $V_2 = \{(a, b, c, d); a = d, b = 2c\}$. Find a basis and dimension of V_1, V_2 and $V_1 \cap V_2$.

29 Find a basis and dimension of the solution space W of the following system of equations.

a)
$$\begin{aligned} x + 2y - 4z + 3s - t &= 0 \\ x + 2y - 2z + 2s + t &= 0 \\ 2x + 4y - 2z + 3s + 4t &= 0 \end{aligned}$$

(b)
$$\begin{aligned} x + 2y - 2z + 2r + s &= 0 \\ 2x + 4y - 6z + 5r &= 0 \\ 2x + 4y - 2z + 3r + 4s &= 0 \\ 3x + 6y - 8z + 7r + s &= 0 \end{aligned}$$

(c)
$$\begin{aligned} x + y + z &= 0 \\ 2x + y - 2z &= 0 \\ y - z &= 0 \end{aligned}$$