Day 1

MODULE III

Kinematics of a Particle – Simple Relative Motion

Kinematics – study of the motion of particles & rigid bodies without consideration of cause of motion

1. Dynamics of a single particle
2. Dynamics of a system of particles
3. Dynamics of a rigid body
4. Dynamics of a system of rigid bodies
5. Dynamics of a continuous deformable medium

If all points of a body have the same velocity at any time t (translatory motion) – only motion of mass center reqd.

Trajectory motion of planet

Differentiation of a Vector with Respect to Time

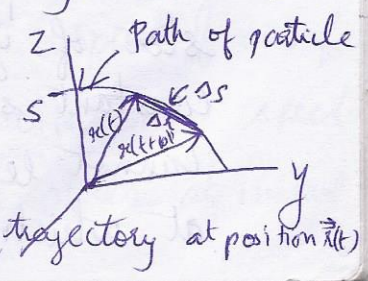
Scalar definition of time derivative

$$\frac{d f(t)}{dt} = \lim_{\Delta t \to 0} \left[\frac{f(t+\Delta t) - f(t)}{\Delta t} \right]$$

$$\frac{d \vec{F}}{dt} = \lim_{\Delta t \to 0} \left[\frac{\vec{F}(t+\Delta t) - \vec{F}(t)}{\Delta t} \right]$$

$$\frac{d\vec{r}}{dt} = \lim_{\Delta t \to 0} \left[\frac{\vec{r}(t+\Delta t) - \vec{r}(t)}{\Delta t} \right] = \text{velocity vector } \vec{V}$$

$$\frac{d\vec{r}}{dt} = \lim_{\Delta t \to 0} \left(\frac{\Delta \vec{r}}{\Delta t} \right) = \lim_{\Delta t \to 0} \left(\frac{\Delta \vec{r}}{\Delta s} \frac{\Delta s}{\Delta t} \right)$$



$\Delta t \to 0 \Rightarrow \Delta r$ approaches tangency to trajectory at position $\vec{r}(t)$
$= \Delta s$ in magnitude

$$\frac{\Delta \vec{r}}{\Delta s} \rightarrow \frac{\Delta s \vec{\epsilon}_t}{\Delta s} = \vec{\epsilon}_t \quad \therefore \quad \frac{d\vec{r}}{ds} = \vec{\epsilon}_t$$

$$\frac{d\vec{r}}{dt} = \vec{V} = \lim_{\Delta t \to 0} \left[\left(\frac{\Delta s}{\Delta t}\right)\left(\frac{\Delta \vec{r}}{\Delta s}\right) \right] = \frac{ds}{dt}\vec{\epsilon}_t$$

$\frac{d\vec{r}}{dt}$ leads to a vector having a magnitude to the speed of the particle & a direction tang the trajectory.

$$\vec{a} = \frac{d\vec{V}}{dt} = \frac{d^2\vec{r}}{dt^2}$$

Velocity & Acceleration Calculations

Rectangular Components

$$\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$$

$x(t)$, $y(t)$, $z(t)$ – scalar functions of time

$$\frac{d\vec{r}}{dt} = \vec{V}(t) = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}k$$

$$= \dot{x}(t)\hat{i} + \dot{y}(t)\hat{j} + \dot{z}(t)\hat{k}$$

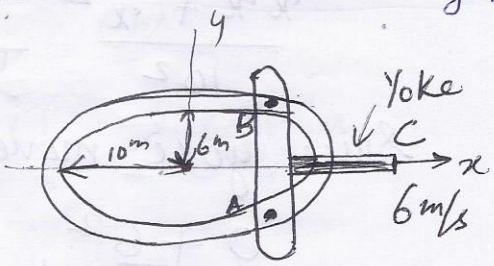
$$\frac{d^2\vec{r}}{dt^2} = \vec{a} = \ddot{x}(t)\hat{i} + \ddot{y}(t)\hat{j} + \ddot{z}(t)\hat{k}$$

$$V_x(t) = \int \ddot{x}(t)\,dt + C_1$$

$$x(t) = \int \left[\int \ddot{x}(t)\,dt\right] dt + C_1 t + C_2$$

1. Pins A & B must always remain in the vert slot of yoke C, which moves to the right at constant speed of 6 m/s in Fig. Furthermore, the p cannot leave the elliptic slot (a) What is the at which the pins approach each other when

yoke slot is at $x = 5m$? (b) What is the rate of change of speed toward each other when the yoke slot is at $x = 5m$?



Sol: Eqn of elliptic path in which pins must move

$$\frac{x^2}{10^2} + \frac{y^2}{6^2} = 1 \qquad (a)$$

If coordinates (x, y) are coordinates of pin B, they must be time functions such that for any time t, the values $x(t)$ & $y(t)$ satisfy Eqn (a).

Also $\dot{x}(t)$ & $\dot{y}(t)$ must be such that pin B moves at all times on the elliptic path.

$$\frac{2x\dot{x}}{10^2} + \frac{2y\dot{y}}{6^2} = 0 \qquad (b)$$

Pin B must have a velocity $\dot{x} = 6$ m/s bcoz of yoke

When $x = 5m$, (a) $\Rightarrow$ $\frac{5^2}{10^2} + \frac{y^2}{6^2} = 1$

$$y = 5.20 \text{ m}$$

(b) $\Rightarrow$ $\frac{2 \times 5 \times 6}{10^2} + \frac{2 \times 5.2\,\dot{y}}{6^2} = 0$

$$\dot{y} = -2.08 \text{ m/s}$$

Thus pin B moves downward with a speed of 2.08 m/s. Clearly pin A must move upward with same speed of 2.08 m/s. The pins approach each other at instant of interest at speed of 4.16 m/s.

To get acceleration $\ddot{y}$ of pin B, differentiate (6) w

$$\frac{x\ddot{x} + \dot{x}^2}{10^2} + \frac{y\ddot{y} + \dot{y}^2}{6^2} = 0$$

Since yoke moves at constant speed, $\ddot{x} = 0$

$$\frac{0 + 6^2}{10^2} + \frac{5.2\ddot{y} + 2.08^2}{6^2} = 0$$

$$\ddot{y} = \underline{-3.32 \text{ m/s}^2}$$

Pin B must be accelerating downward at a r of 3.32 m/s² while pin A accelerates upward the same rate. The pins accelerate toward each then, at a rate of 6.64 m/s².

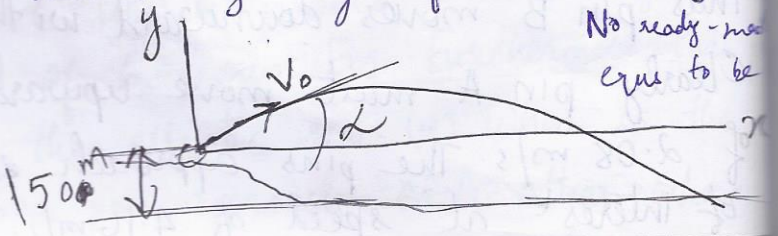
Day 2

Ballistic Problems

$\ddot{y} = -g \quad \ddot{x}(t) = \ddot{z}(t) = 0$

1. A shell is fired from a hill 150 m above plain. The angle α of firing is 15° above horizontal, & the muzzle velocity V_0 is ~~30~~ 90 At what hor. distance, d, will the shell h plain if we neglect ~~friction~~ of the air? Wh the max ht of the shell above the plain? Final determine the trajectory of the shell (y = f

No ready-mad eqns to be



Sol: $\ddot{y}(t) = \frac{dV_y}{dt} = -9.81 \ m/s^2$ (a)

$\ddot{x}(t) = \frac{dV_x}{dt} = 0$ (b)

We do not consider $\ddot{z}(t) \rightarrow$ coplanar

$$dV_y = -9.81 \, dt$$

$$dV_x = 0 \, dt$$

$$V_y(t) = -9.81t + C_1$$

$$V_x(t) = C_2$$

$t=0$ at instant canon is fired.

$$V_y(0) = \cancel{3000}\ 900 \sin 15 = -9.81 \times 0 + C_1$$

$$C_1 = 232.94 \ m/s$$

$$V_x(0) = 900 \cos 15 = 869.33 \ m/s = C_2$$

$$V_y(t) = -9.81t + 232.94 \quad (c)$$

$$V_x(t) = 869.33 \quad (d)$$

$$y(t) = -9.81 \frac{t^2}{2} + 233t + C_3$$

$$x(t) = 869t + C_4$$

$t=0 \Rightarrow x=0, y=0 \Rightarrow C_3 = C_4 = 0$

$$y(t) = -4.905t^2 + 233t \quad (e)$$

$$x(t) = 869t \quad (f)$$

To determine distance d, find t for impa
shell on plain

In (e), $y = -150$

$$-150 = -4.905t^2 + 233t$$

$$t = 48.14\ s$$

In(f),

$$\boxed{d = 869t = 41833.66\ m}$$

Max. ht y_{max}

Find t when $V_y = 0$

(c) $$0 = -9.81t + 233$$

$$t = 23.75\ s$$

$$y_{max} = -4.905 \times 23.75^2 + 233 \times 23.75$$

$$\boxed{y_{max} = 2767.02\ m}$$

Trajectory of shell

Solve for t in (f) & substitute in (e)

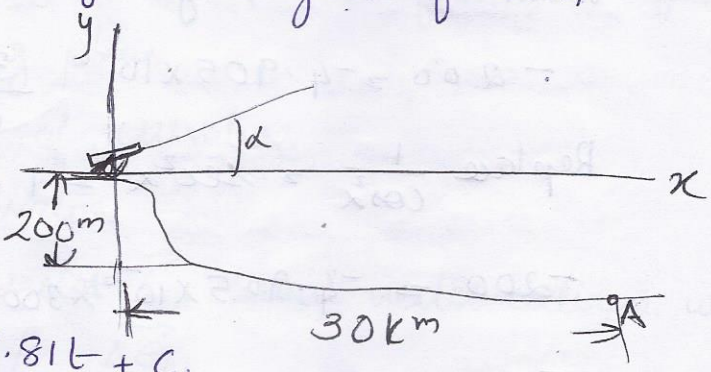
$$y = -4.905\left(\frac{x}{869}\right)^2 + 233\left(\frac{x}{869}\right)$$

$$\boxed{y = -6.495 \times 10^{-6}\, x^2 + 0.268\, x}$$

Clearly, the trajectory is a parabola.

2. A gun emplacement (military installation) is shown on a cliff in Fig
The muzzle velocity of the gun is 1000 m/s. At w
angle α must the gun point in order to hit ta

A shown in the diagram? Neglect friction



Sol: Newton's law:

$\ddot{y}(t) = -9.81$

$\ddot{x}(t) = 0$

Integrating

$\dot{y}(t) = V_y(t) = -9.81t + C_1$

$\dot{x}(t) = V_x(t) = C_2$

$t=0 \Rightarrow \dot{y} = 1000\sin\alpha$ & $\dot{x} = 1000\cos\alpha$

$1000\sin\alpha = C_1$

$C_2 = 1000\cos\alpha$

$\dot{y}(t) = -9.81t + 1000\sin\alpha$

$\dot{x}(t) = 1000\cos\alpha$

Integrating

$y(t) = -9.81\frac{t^2}{2} + 1000\sin\alpha\, t + C_3$

$x(t) = 1000\cos\alpha\, t + C_4$

$t=0 \Rightarrow x=0, y=0 \Rightarrow C_3 = C_4 = 0$

$y(t) = -4.905t^2 + 1000\sin\alpha\, t$

$x(t) = 1000\cos\alpha\, t$

To get trajectory,

$$y = -4.905\frac{x^2}{(1000\cos\alpha)^2} + 1000\sin\alpha \times \frac{x}{1000\cos\alpha}$$

$$= -4.905\times10^{-6}\frac{x^2}{\cos^2\alpha} + x\tan\alpha$$

When $x = 30\,km$, $y = -200\,m$

$$-200 = -4.905\times10^{-6}\,\frac{30000^2}{\cos^2\alpha} + 30000\tan\alpha$$

Replace $\frac{1}{\cos^2\alpha} = \sec^2\alpha = 1 + \tan^2\alpha$

$$-200 = -4.905\times10^{-6}\times30000^2(1+\tan^2\alpha) + 30000$$

$$-200 = -4414.5 - 4414.5\tan^2\alpha + 30000\,ta$$

$$4414.5\tan^2\alpha - 30000\tan\alpha + 4214.5 = 0$$

$$\tan^2\alpha - 6.796\tan\alpha + 0.955 = 0$$

$$\tan\alpha_1 = 6.65 \quad \text{or} \quad \tan\alpha_2 = 0.1435$$

$$\alpha_1 = \underline{\underline{81.45^\circ}} \qquad \alpha_2 = \underline{\underline{8.17^\circ}}$$

Day 3 Velocity & Acceleration in terms of Path Variables

$$\vec{V} = \frac{ds}{dt}\,\epsilon_t$$

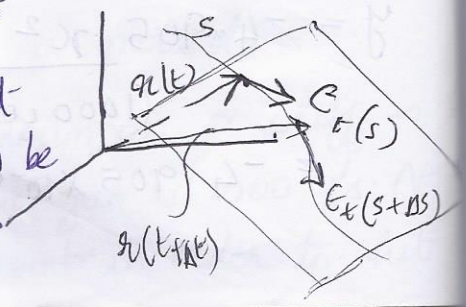
$\frac{ds}{dt} \rightarrow$ speed along the path

$\epsilon_t = \frac{d\vec{r}}{ds}$ – unit vector tangent to the path

$$\frac{d\vec{V}}{dt} = \vec{a} = \frac{d^2s}{dt^2}\epsilon_t + \frac{ds}{dt}\frac{d\epsilon_t}{dt}$$

$$\vec{a} = \frac{d^2s}{dt^2}\epsilon_t + \left(\frac{ds}{dt}\right)^2\frac{d\epsilon_t}{ds}$$

If Δs is small enough, unit vectors $\epsilon_t(s)$ & $\epsilon_t(s+\Delta s)$ can be considered to intersect & thus to form a plane.

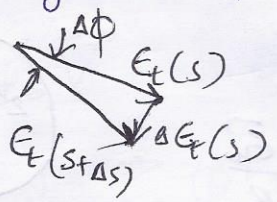


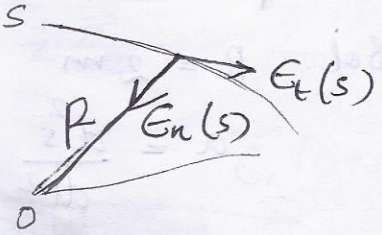
If $\Delta s \to 0$, these unit vectors then form a limiting plane – osculating plane

Point O – center of curvature

R – radius of curvature

Principal normal vector $\vec{\epsilon}_n$ – unit vector collinear with the limiting vector for $\Delta\vec{\epsilon}_t$.





$$|\Delta\epsilon_t| = |\epsilon_t|\,\Delta\phi = \Delta\phi$$

$$\Delta\phi = \frac{\Delta s}{R}$$

$$\Delta\vec{\epsilon}_t \approx \frac{\Delta s}{R}\,\vec{\epsilon}_n$$

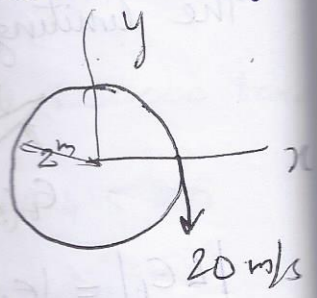
$$\frac{d\vec{\epsilon}_t}{ds} = \lim_{\Delta s\to 0}\left(\frac{\Delta\vec{\epsilon}_t}{\Delta s}\right) = \lim_{\Delta s\to 0}\left[\frac{(\Delta s/R)\,\vec{\epsilon}_n}{\Delta s}\right] = \frac{\vec{\epsilon}_n}{R}$$

$$\vec{a} = \frac{d^2 s}{dt^2}\,\vec{\epsilon}_t + \left(\frac{ds}{dt}\right)^2 \frac{\vec{\epsilon}_n}{R}$$

one component in a direction tangent to the path & one component in the osculating plane at right angles to the path & pointing toward the center of curvature.

$$R = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$$

1. A particle is moving along a circular path in the xy plane. When the particle crosses the x axis it has an acceleration along the path of 5 m/s² and is moving with a speed of 20 m/s in the -ve y direction. What is the total acceleration of the particle?



Sol: $R = 2\,m$

$$a = \frac{d^2s}{dt^2}\vec{\epsilon_t} + \left(\frac{ds}{dt}\right)^2 \frac{\vec{\epsilon_n}}{R}$$

$$= 5\,\vec{\epsilon_t} + \frac{20^2}{2}\vec{\epsilon_n}$$

$$= -5\hat{j} - 200\,\hat{i}\ m/s^2$$

Cylindrical Coordinates

Specified by cylindrical coordinates θ, r & z.

Transformation eqns b/n Cartesian & cylindrical coordinates:

$x = \bar{r}\cos\theta$ $\qquad \bar{r} = \sqrt{x^2+y^2}$

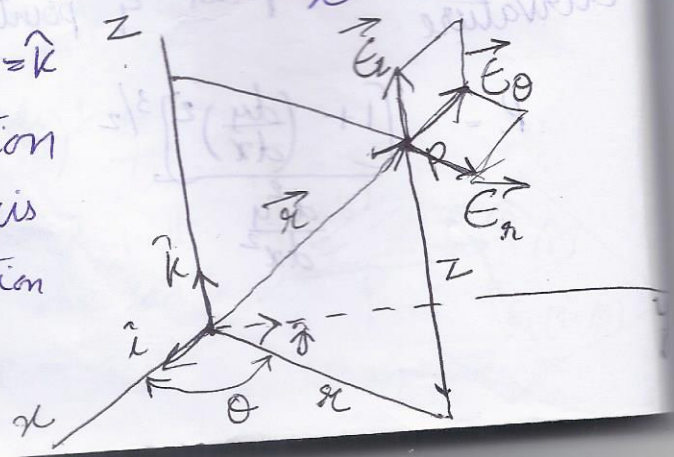
$y = \bar{r}\sin\theta$ $\qquad \theta = \tan^{-1}\frac{y}{x}$

$\vec{\epsilon_z} \rightarrow \parallel^{el}$ to z axis $= \hat{k}$

axial direction

$\vec{\epsilon_r} \rightarrow$ normal to z axis

radial direction



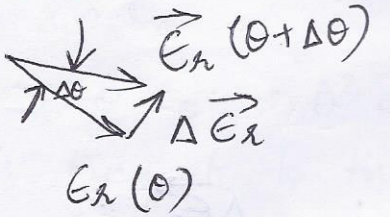
ϵ_θ – normal to plane formed by $\vec{\epsilon}_z$ & $\vec{\epsilon}_r$
– transverse direction

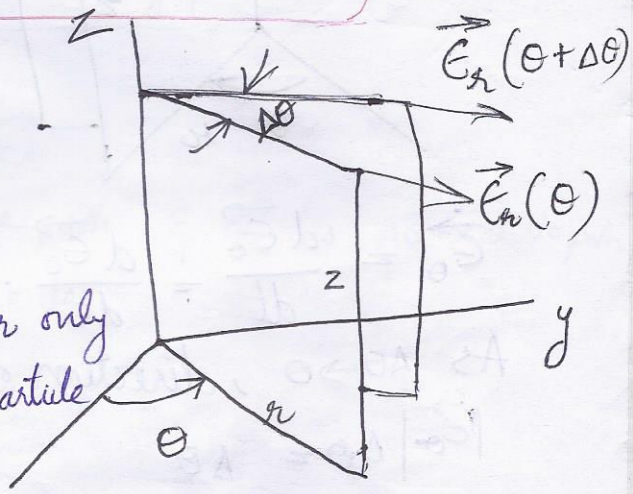
$\vec{\epsilon}_r$ & $\vec{\epsilon}_\theta$ will change direction as the particle moves relative to xyz reference – fns of time

$\vec{\epsilon}_z$ – const. vector

$$\vec{r} = r\vec{\epsilon}_r + z\vec{\epsilon}_z$$

$$\boxed{\frac{d\vec{r}}{dt} = \vec{V} = r\dot{\vec{\epsilon}}_r + \dot{r}\vec{\epsilon}_r + \dot{z}\vec{\epsilon}_z}$$





change in $\vec{\epsilon}_r$ occur only when θ coordinate of particle changes.

$$\dot{\vec{\epsilon}}_r = \frac{d\vec{\epsilon}_r}{dt} = \frac{d\vec{\epsilon}_r}{d\theta}\frac{d\theta}{dt} = \frac{d\vec{\epsilon}_r}{d\theta}\cdot\dot{\theta}$$

$$|\Delta\vec{\epsilon}_r| = |\vec{\epsilon}_r|\Delta\theta = \Delta\theta$$

As $\theta \to 0$, direction of $\Delta\vec{\epsilon}_r$ approaches that of ϵ_θ.

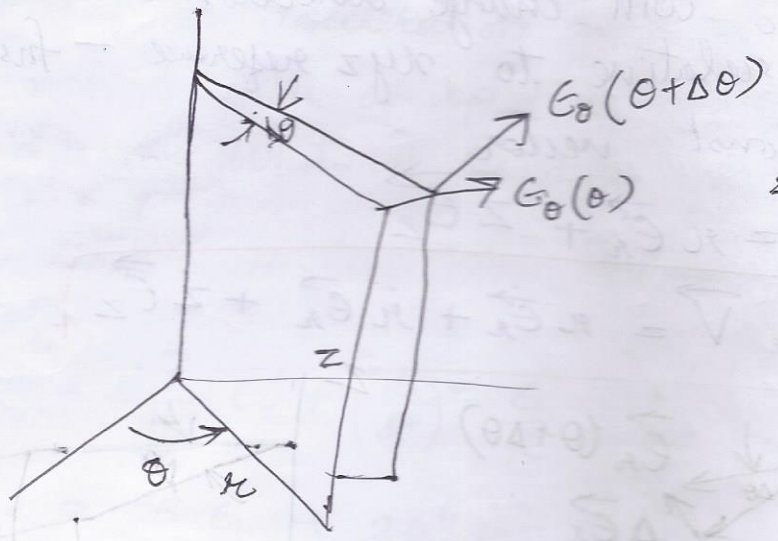
$$\Delta\vec{\epsilon}_r = |\Delta\vec{\epsilon}_r|\,\vec{\epsilon}_\theta \approx \Delta\theta\,\vec{\epsilon}_\theta$$

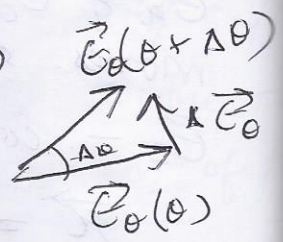
$$\frac{d\vec{\epsilon}_r}{dt} = \frac{d\vec{\epsilon}_r}{d\theta}\cdot\dot{\theta} \approx \frac{\Delta\vec{\epsilon}_r}{\Delta\theta}\cdot\dot{\theta} = \frac{\Delta\theta\,\vec{\epsilon}_\theta}{\Delta\theta}\cdot\dot{\theta} = \dot{\theta}\vec{\epsilon}_\theta$$

$$\boxed{\vec{V} = \dot{r}\vec{\epsilon}_r + r\dot{\theta}\vec{\epsilon}_\theta + \dot{z}\vec{\epsilon}_z}$$

$$\vec{a} = \frac{d\vec{V}}{dt} = \ddot{r}\vec{\epsilon}_r + \dot{r}\dot{\vec{\epsilon}}_r + \dot{r}\dot{\theta}\vec{\epsilon}_\theta + r\ddot{\theta}\vec{\epsilon}_\theta + r\dot{\theta}\dot{\vec{\epsilon}}_\theta + \ddot{z}\vec{\epsilon}_z$$

$\vec{e}_\theta$ can vary only when a change in coordi
θ causes a change in direction of this vecto





$$\dot{\vec{e}}_\theta = \frac{d\vec{e}_\theta}{dt} = \frac{d\vec{e}_\theta}{d\theta}\cdot\dot{\theta} = \frac{\Delta\vec{e}_\theta}{\Delta\theta}\dot{\theta}$$

As $\Delta\theta \to 0$, direction of $\Delta\vec{e}_\theta = -\vec{e}_r$ & magnitude

$|\vec{e}_\theta|\Delta\theta = \Delta\theta$.

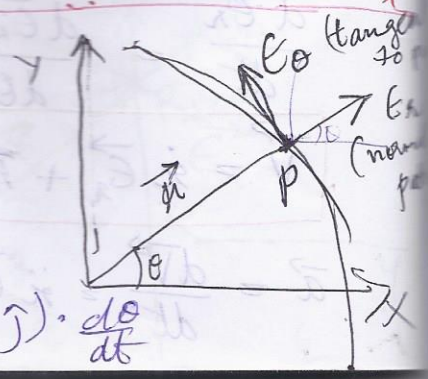
$$\Delta\vec{e}_\theta = -\Delta\theta\,\vec{e}_r$$

$$\dot{\vec{e}}_\theta = -\dot{\theta}\,\vec{e}_r$$

$$\vec{a} = \ddot{r}\vec{e}_r + \dot{r}\dot{\theta}\vec{e}_\theta + \dot{r}\dot{\theta}\vec{e}_\theta + r\ddot{\theta}\vec{e}_\theta - r\dot{\theta}^2\vec{e}_r + \ddot{z}\vec{e}$$

$$\boxed{\vec{a} = (\ddot{r} - r\dot{\theta}^2)\vec{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\vec{e}_\theta + \ddot{z}\vec{e}_z}$$

To evaluate $\dot{\vec{e}}_r$



$$\vec{e}_r = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}$$

$$\vec{e}_\theta = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j}$$

$$\dot{\vec{e}}_r = \frac{d\vec{e}_r}{dt} = (-\sin\theta\,\hat{i} + \cos\,\hat{j})\cdot\frac{d\theta}{dt}$$

$= \vec{\epsilon}_\theta \cdot \dot{\theta}$

$$\boxed{\vec{\epsilon}_r = \dot{\theta}\, \vec{\epsilon}_\theta}$$

$$\dot{\vec{\epsilon}}_\theta = \frac{d\vec{\epsilon}_\theta}{dt} = \left(-\cos\theta\,\hat{i} + -\sin\theta\,\hat{j}\right)\frac{d\theta}{dt} = -\vec{\epsilon}_r \cdot \dot{\theta}$$

$$\boxed{\dot{\vec{\epsilon}}_\theta = -\dot{\theta}\,\vec{\epsilon}_r}$$

For motion in a circle in xy plane, $\dot{r} = \dot{z} = 0$, $r = r$

$$\vec{V} = r\dot{\theta}\,\vec{\epsilon}_\theta$$

$$\vec{a} = r\ddot{\theta}\,\vec{\epsilon}_\theta - r\dot{\theta}^2\,\vec{\epsilon}_r$$

$\vec{\epsilon}_\theta$ is tangent to the path & $\vec{\epsilon}_r$ is normal to path & points away from center

$$r\ddot{\theta} = \frac{d^2s}{dt^2}$$

$$r\dot{\theta}^2 = \frac{V^2}{r}$$

Linear motion	Angular motion	$V = r\omega$
u	ω_0	
V	ω	
a	α	
s	θ	
$V = \frac{ds}{dt}$	$\omega = \frac{d\theta}{dt}$	
$a = \frac{dv}{dt}$	$\alpha = \frac{d\omega}{dt}$	

HW 11.50 $s = 3t^2 + \frac{t^3}{6}$ ft $\quad r = 40$ ft

$V = 6t + \frac{3t^2}{6}$

$V = r\dot{\theta}$

$\dot{\theta} = \frac{6t}{40} + \frac{3t^2}{6 \times 40} = 1.0625$ rad/s

$\ddot{\theta} = \frac{6}{40} + \frac{6t}{6 \times 40} = 0.225$ rad/s

1. A grain of plutonium is being tracked in a turbulent atmosphere. Relative to reference xyz, the displacement components are $x = 6t$ m, $y = 10t$ m, $z = t^3 + 10$ m

11.60

Express the position, velocity & acceleration vector of the particle using cylindrical coordinates with components in the axial, transverse & radial dir

Sol: $\vec{r} = r\vec{e}_r + z\vec{e}_z$

$r = \sqrt{x^2+y^2} = \sqrt{36t^2+100t^2} = 2\sqrt{34}\,t$

$\dot{r} = 2\sqrt{34}$ $\quad \ddot{r} = 0$

$\theta = \tan^{-1}\frac{y}{x} = \tan^{-1}\frac{10t}{6t} = 59°$

$\vec{e}_r = \cos\theta\,\hat{i} + \sin\theta\,\hat{j} = 0.514\,\hat{i} + 0.857\,\hat{j}$

$\vec{e}_\theta = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j} = -0.857\,\hat{i} + 0.514\,\hat{j}$

$\vec{r} = \underset{11.66}{2\sqrt{34}}\,t\,\vec{e}_r + (t^3+10)\,\vec{e}_z$ m

$\vec{V} = \dot{r}\vec{e}_r + r\dot{\theta}\vec{e}_\theta + \dot{z}\vec{e}_z$ $\quad \pi - 180°$, $\alpha - 59°$

$= 2\sqrt{34}\,\vec{e}_r + (2\sqrt{34}t).\overset{0}{\cancel{\tan 29}}\,\vec{e}_\theta + 3t^2\,\vec{e}_z$

$\vec{V} = 11.66\,\vec{e}_r + \cancel{12t\,\vec{e}_\theta} + 3t^2\,\vec{e}_z$ m/s

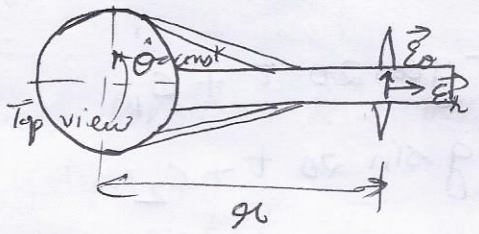
$\vec{a} = (\ddot{r} - r\dot{\theta}^2)\vec{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\vec{e}_\theta + \ddot{z}\,\vec{e}_z$

$= (-2\sqrt{34}\,t \times 0)\vec{e}_r + 0 + 6t\,\vec{e}_z$

$\vec{a} = 6t\,\vec{e}_z$ m/s²

2. Consider a scaled model of a towing tank which is being moved out radially so that in one revolution of the main beam it has gone, at const. speed relative to the main beam, from position $r = 3.3m$ to $r = 4m$. The angular speed of the beam is 3 rpm. What is the acceleration of the model relative to the water when $r = 4^m$?

Sol:



Time $t = \dfrac{1}{\frac{3}{60}} = 20\,s$

$\dot{r} = \dfrac{4 - 3.3}{20} = 0.035\ m/s$

$r = 4m$

$\dot{r} = 0.035\ m/s$

$\ddot{r} = 0$

$\dot{\theta} = 3 \times \dfrac{2\pi}{60} = 0.314\ rad/s$

$\ddot{\theta} = 0$

$\ddot{z} = 0$

$\vec{a} = (\ddot{r} - r\dot{\theta}^2)\vec{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\vec{e}_\theta + \ddot{z}\vec{e}_z$

$= (0 - 4 \times 0.314^2)\vec{e}_r + (4 \times 0 + 2 \times 0.035 \times 0.314)\vec{e}_\theta + 0$

$= -0.394\,\vec{e}_r + 0.022\,\vec{e}_\theta\ m/s^2$

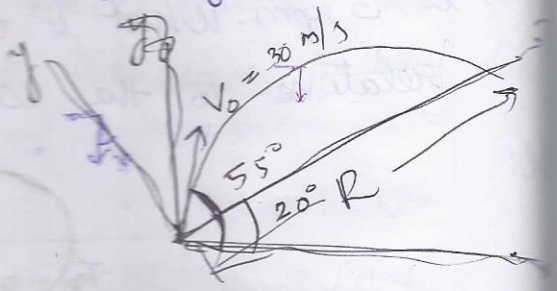
$|\vec{a}| = 0.395\ m/s^2$

3. A particle is projected from a point, on an inclined plane, with a velocity of 30 m/s. The angle of projection & the angle of plane are 55 & 20° to the horizontal respectively. Find the range & the time of flight of the particle.

Sol:

$$\ddot{y} = -g\cos 20$$

$$\ddot{x} = -g\sin 20$$



$$\dot{y} = -g\cos 20\, t + C_1$$

$$\dot{x} = -g\sin 20\, t + C_2$$

At $t=0$; $\dot{y}(0) = 30\sin 35$

$\dot{x}(0) = 30\cos 35$

$$30\sin 35 = -9.81\cos 20 \times 0 + C_1$$

$$C_1 = 17.21$$

$$C_2 = 30\cos 35 = 24.57$$

$$\dot{y} = -g\cos 20\, t + 17.21$$

$$\dot{x} = -g\sin 20\, t + 24.57$$

$$y = -g\cos 20\,\frac{t^2}{2} + 17.21t + C_3$$

$$x = -g\sin 20\,\frac{t^2}{2} + 24.57t + C_4$$

At $t=0$, $y(0) = x(0) = 0$

$C_3 = C_4 = 0$

$y = 0$, $x = d$?

$$0 = -g\cos 20\,\frac{t^2}{2} + 17.21\,t$$

$t \neq 0$, $$t = \frac{17.21 \times 2}{g\cos 20} = 3.73\,s$$

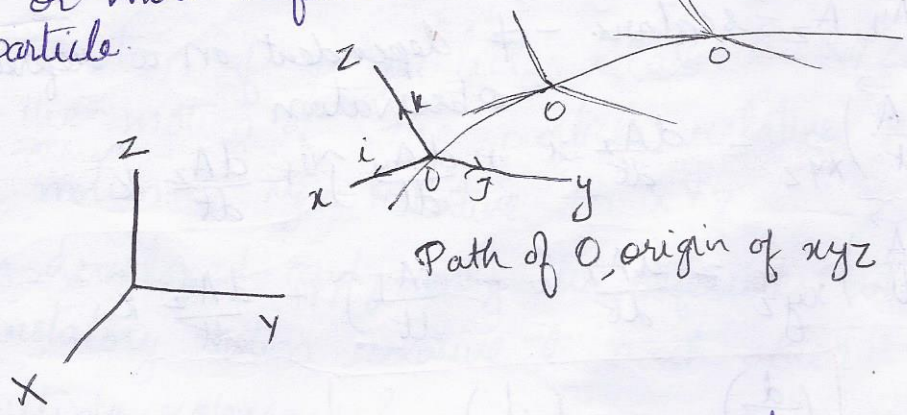
$$x = -g\sin 20\,\frac{3.73^2}{2} + 24.57 \times 3.73$$

$$= 68.35\,m$$

Day 4 Simple Kinematical Relations & Applications

Simple Relative Motion

2 or more references used to describe the motion of the particle.



direction of the axes of xyz always retain the same orientation relative to XYZ. Such a motion of xyz relative to XYZ is called translation.

Suppose we have a vector $\vec{A}(t)$ which varies with time. In general case, the time variation of $\vec{A}$ will depend on the reference from which we are observing the time variation.

$$\left(\frac{d\vec{A}}{dt}\right)_{XYZ} \quad \& \quad \left(\frac{d\vec{A}}{dt}\right)_{xyz}$$

Decompose $\vec{A}$ into components $\parallel^{el}$ to the xyz axes,

$$\left(\frac{d\vec{A}}{dt}\right)_{XYZ} = \left[\frac{d}{dt}\left(A_x\hat{i} + A_y\hat{j} + A_z\hat{k}\right)\right]_{XYZ}$$

A_x, A_y, A_z - scalar components of $\vec{A}$ along the xyz axes

Bcoz xyz translates relative to XYZ, the unit vectors of xyz, denoted as $\hat{i}, \hat{j}, \hat{k}$ are const vectors as seen from XYZ.

They may change their lines of action, but they do not change magnitude & direction as seen from XYZ & are thus const vectors.

$$\left(\frac{d\vec{A}}{dt}\right)_{XYZ} = \left(\frac{dA_x}{dt}\right)_{XYZ}\hat{i} + \left(\frac{dA_y}{dt}\right)_{XYZ}\hat{j} + \left(\frac{dA_z}{dt}\right)_{XYZ}\hat{k}$$

A_x, A_y, A_z - scalars - ≠ dependent on a reference of observation

$$\left(\frac{d\vec{A}}{dt}\right)_{XYZ} = \frac{dA_x}{dt}\hat{i} + \frac{dA_y}{dt}\hat{j} + \frac{dA_z}{dt}\hat{k}$$

$$\left(\frac{d\vec{A}}{dt}\right)_{xyz} = \frac{dA_x}{dt}\hat{i} + \frac{dA_y}{dt}\hat{j} + \frac{dA_z}{dt}\hat{k}$$

$$\boxed{\left(\frac{d}{dt}\right)_{XYZ} = \left(\frac{d}{dt}\right)_{xyz}}$$

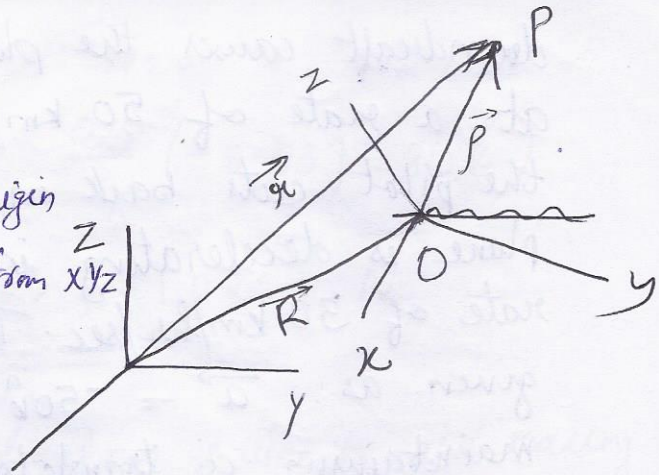
Time derivative of vector is same for all reference axes that are translating relative to each other

Motion of a Particle Relative to a Pair of Translating Axes

The velocity vector of any particle P depends on reference from which the motion is observed.

$$\vec{V}_{XYZ} = \left(\frac{d\vec{r}}{dt}\right)_{XYZ}$$

$$\vec{V}_{xyz} = \left(\frac{d\vec{\rho}}{dt}\right)_{xyz}$$



$\left(\frac{d\vec{R}}{dt}\right)_{XYZ}$ – velocity of the origin of xyz reference from XYZ

$$\vec{r} = \vec{R} + \vec{\rho}$$

$$\left(\frac{d\vec{r}}{dt}\right)_{XYZ} = \left(\frac{d\vec{R}}{dt}\right)_{XYZ} + \left(\frac{d\vec{\rho}}{dt}\right)_{XYZ}$$

$$\boxed{\vec{V}_{XYZ} = \dot{\vec{R}} + \vec{V}_{xyz}}$$

$$\boxed{\vec{a}_{XYZ} = \ddot{\vec{R}} + \vec{a}_{xyz}}$$

The motion of a particle relative to XYZ is the sum of the motion of the particle relative to xyz + the motion of xyz relative to XYZ

Eqn developed apply only to references which have a translatory motion relative to each other.

Multiple references?

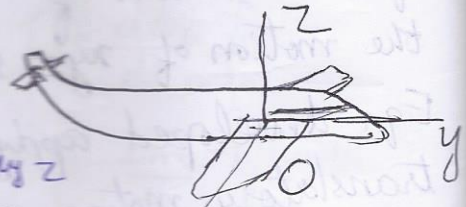
Motion of particle is known relative to a given rigid body & the motion of this body is known relative to a ~~given rigid body~~ ground or other convenient reference.

1. A jet airliner is shown in Fig flying at a speed of 600 km/hr in a translatory manner relative to the ground reference XYZ. At the instant of interest, a

downdraft causes the plane to accelerate downward at a rate of 50 km/hr/sec. While this is happening the pilot cuts back on the throttle (valve that regulates supply of fuel to engine) so that the plane is decelerating in the Y direction at the rate of 30 km/hr/sec. Thus, the plane has an acceleration given as: $\vec{a} = -50\hat{k} - 30\hat{j}$ km/hr/sec, while maintaining a translatory attitude. While this is happening, a solenoid is operated to close a valve gate that weighs 0.5 N. What is the force on the valve gate from the plane at the instant when the valve gate is moving downward relative to the airplane at a speed of 10 m/s & accelerating downward relative to the plane at a rate of 16.1 m/s

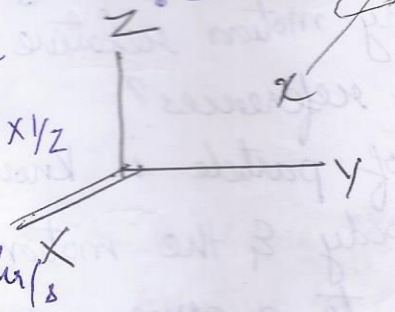
Sol: Fix xyz to the plane

XYZ to the ground



Motion of valve gate relative to xyz

$\vec{a}_{xyz} = -16.1\hat{k}$ m/s²



accln of O, relative to XYZ

$\ddot{\vec{R}} = -50\hat{k} - 30\hat{j}$ km/hr/s

$$\vec{a}_{XYZ} = \ddot{\vec{R}} + \vec{a}_{xyz}$$

$$= (-50\hat{k} - 30\hat{j}) \times \frac{1000}{60 \times 60} + -16.1\hat{k}$$

$$= -8.33\hat{j} - 30\hat{k} \text{ m/s}^2$$

Newton's Law, $\vec{F} = m\vec{a}_{xyz}$

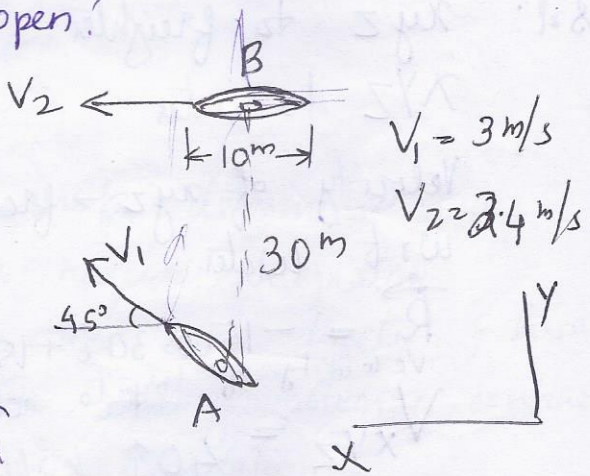
$$\vec{F}_{plane} - 0.5\hat{k} = \frac{0.5}{g}\left(-8.33\hat{j} - 30\hat{k}\right)$$

$$\vec{F}_{plane} = -0.425\hat{j} - 1.029\hat{k}\ N$$

HW

11.80 Two 10-m international-class sailboats are racing. They are on different tracks. If there is no change in course, will A hit B? If so, where from the center of B does this happen?

Sol:

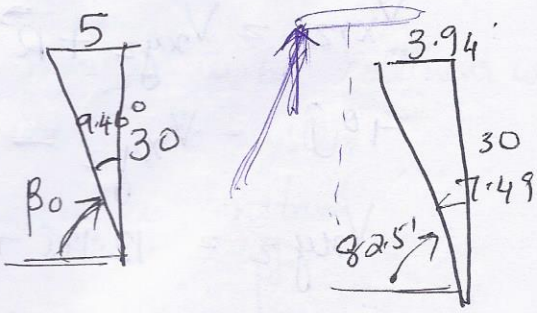


$$V_{XYZ} = V_{xyz} + \dot{R}$$

$$+ 3\cos 45\,\hat{i} + 3\sin 45\,\hat{j}$$

$$= V_{xyz} + 2.4\,\hat{i}$$

$$V_{xyz} = -0.278\hat{i} + 2.12\hat{j}$$



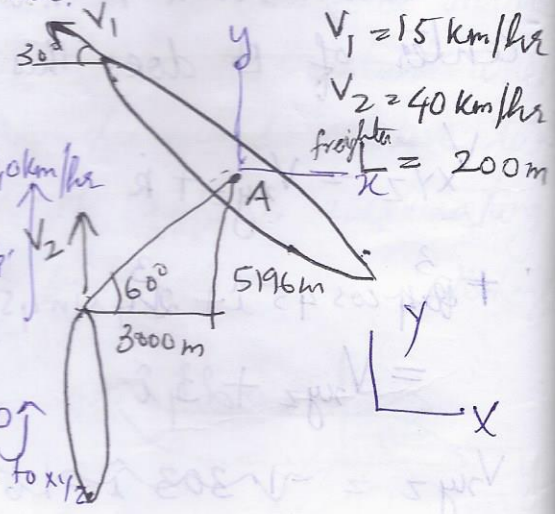
$$\beta_0 = 90 - 9.46° = 80.54°$$

$$\beta_{out} = \tan^{-1}\frac{2.12}{0.278}$$

$$= 82.51°$$

So A hit B at a distance 3.94 m from center of B.

2. The freighter in Fig is moving at a steady speed V_1 of 15 km/hr relative to the water. The freighter is 200 m long at the waterline with point A at midship. A stalking submerged submarine fires a torpedo when the submarine & freighter are at the positions shown in the diagram. The torpedo maintains a steady speed V_2 of 40 km/hr relative to the water. Will the torpedo hit the freighter?



Sol: xyz to freighter

XYZ to water

Velocity of xyz → freighter wrt water

$\vec{R} = -15\cos 30\,\hat{i} + 15\sin 30\,\hat{j}$

Velocity of torpedo relative to xyz

$V_{XYZ} = 40\hat{j}$ km/hr

$\vec{V}_{XYZ} = \vec{V}_{xyz} + \vec{R}$

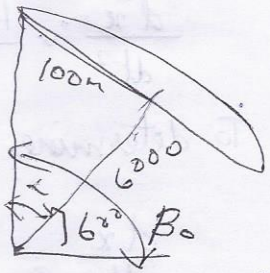
$40\hat{j} = V_{xyz} - 15\cos 30\,\hat{i} + 15\sin 30\,\hat{j}$

$V_{xyz} = 12.99\hat{i} + 32.5\hat{j}$ km/hr

To just miss the freighter, V_{xyz} must have a course such that the vector forms an angle β_0 with hor. axis.

$\beta_0 = \alpha + 60 = \tan^{-1}\frac{100}{6000} + 60 = 60.95°$

$$\beta_{ACT} = \tan^{-1}\frac{(V_{xyz})_y}{(V_{xyz})_x} = \tan^{-1}\frac{32.5}{12.99} = 2.5$$



$$= 68.21^\circ$$

$\beta_{ACT} > \beta_0 \Rightarrow$ torpedo just misses the freighter

Day 5 Particle Dynamics

Newton's Law for Rectangular Coordinates

$$F_x = ma_x = m\frac{dV_x}{dt} = m\frac{d^2x}{dt^2}$$

$$F_y = ma_y = m\frac{dV_y}{dt} = m\frac{d^2y}{dt^2}$$

$$F_z = ma_z = m\frac{dV_z}{dt} = m\frac{d^2z}{dt^2}$$

If motion is known, force can be determined using algebraic eqns. But inverse requires integration.
Resultant force on particle has same direction & line of action at all times. The resulting motion is confined to a straight line – rectilinear translation.

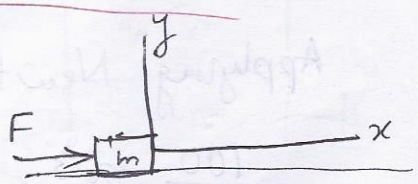
Rectilinear Translation

Consider line of action of motion to be collinear with one axis of a rectilinear coordinate system

1 Force is a function of time or a constant

No friction

W = Normal force



$\therefore F_R = F(t)$

$$F(t) = m\frac{d^2x}{dt^2}$$

$$\frac{d^2x}{dt^2} = \frac{F(t)}{m}$$

To determine motion, $\frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{F(t)}{m}$

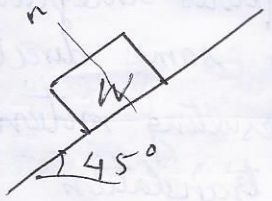
$$\frac{dx}{dt} = \int \frac{F(t)}{m}\,dt + C_1$$

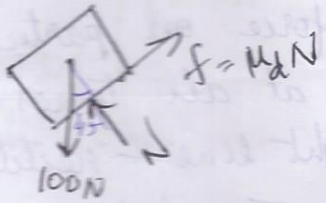
$$x = \int\left[\int \frac{F(t)}{m}\,dt\right]dt + C_1 t + C_2$$

Initial conditions - time $t=0 \Rightarrow V = V_0$ & $x = x_0$

1. A 100-N body is initially stationary on a 45° incl as shown in Fig. The coefficient of dynamic friction μ_d b/n the block & incline is 0.5. What distance along the incline must the weight slide before it reaches a speed of 40 m/s?

Sol:





Since acceleration is zero in the direction normal to the incline, $100\cos 45 - N = 0$

$$N = 70.7\ N$$

Applying Newton's law in a direction along incline

$$\frac{100}{9.81}\frac{d^2s}{dt^2} = 100\sin 45 - \mu_d N$$

$$\frac{d^2s}{dt^2} = (70.7 - 0.5\times 70.7)\times\frac{9.81}{100} = 3.46$$

$$d\left(\frac{ds}{dt}\right) = 3.46\, dt$$

Integrating, $\frac{ds}{dt} = 3.46\, t + c_1$

$$s = 3.46 \frac{t^2}{2} + c_1 t + c_2$$

$t=0$, $s = \frac{ds}{dt} = 0 \rightarrow c_1 = c_2 = 0$

$\frac{ds}{dt} = 40$ m/s. $\quad 40 = 3.46t$

$$t = 11.53\ s$$

Distance travelled $s = \frac{3.46 \times 11.53^2}{2} = 230.83\ m$

2. A charged particle is shown in Fig at time $t=0$ between large $\parallel^{el}$ condenser plates separated by a distance d in a vacuum. A time-varying voltage V given as $V = 6 \sin \omega t$ is applied to the plates. What is the motion of the particle if it has a charge q Coulombs & if we do not consider gravity?

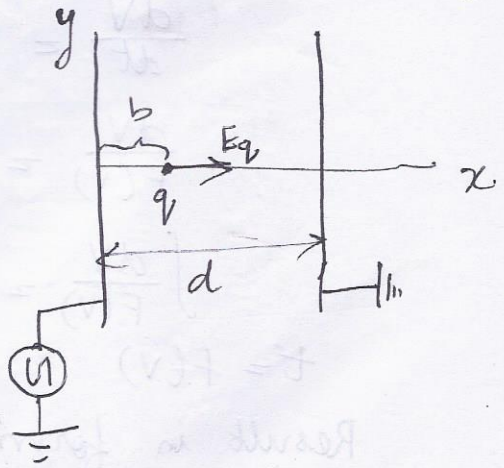
Sol: Electric field $E = \frac{V}{d}$

Force on the particle is qE

Using Newton's law,

$$\frac{d^2x}{dt^2} = q\,\frac{6 \sin \omega t}{md}$$

$$d\left(\frac{dx}{dt}\right) = q\,\frac{6 \sin \omega t}{md}\, dt$$



Integrating, $\frac{dx}{dt} = -\frac{6q}{m\omega d}\cos\omega t + C_1$

$$x = -\frac{6q}{m\omega^2 d}\sin\omega t + C_1 t + C_2$$

Applying initial conditions $x = b$ & $\frac{dx}{dt} = 0$ at t

$$b = C_2$$

$$C_1 = \frac{6q}{m\omega d}$$

$$\therefore x = \frac{-6q}{m\omega^2 d}\sin\omega t + \frac{6q}{m\omega d}t + b$$

The motion of the charged particle will be of sinusoidal oscillation in which center of oscillation drifts from left to right.

Day6

2 Force is a Function of speed

Eg:- aerodynamic drag force on an airplane or m

Newton's law

$$\frac{dV}{dt} = \frac{F(V)}{m}$$

$$\frac{dV}{F(V)} = \frac{1}{m}dt$$

$$\int\frac{dV}{F(V)} = \frac{1}{m}t + C_1$$

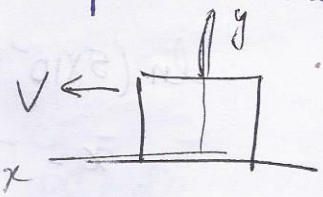
$$t = f(V)$$

Result in form of $V = H(t, C_1)$

Replace V by $\frac{dx}{dt}$

$$x = \int H(t, C_1)\,dt + C_2$$

1. A high-speed land racer is moving at a speed of 100 m/s. The resistance to motion of the vehicle is primarily due to aerodynamic drag, which for this speed can be approximated as $0.2V^2$ N with V in m/s. If the vehicle has a mass of 4000 kg, what distance will it coast before its speed is reduced to 70 m/s?



Sol: Newton's law

$$\frac{dV}{dt} = \frac{-0.2V^2}{4000} = -5\times10^{-5}V^2$$

$$\frac{dV}{V^2} = -5\times10^{-5}\,dt$$

Integrating, $-\frac{1}{V} = -5\times10^{-5}t + C_1$

At $t=0$, $V=100 \Rightarrow -\frac{1}{100} = C_1$

Replacing V by $\frac{dx}{dt}$,

$$\frac{1}{V} = \frac{dt}{dx} = 5\times10^{-5}t + \frac{1}{100} \quad -(1)$$

$$\frac{dt}{5\times10^{-5}t + \frac{1}{100}} = dx$$

Let $\eta = 5\times10^{-5}t + \frac{1}{100}$

$$d\eta = 5\times10^{-5}\,dt$$

$$\frac{d\eta}{5\times10^{-5}\eta} = dx$$

$$\ln \eta = 5\times10^{-5} x + C_2$$

$$\ln\left(5\times10^{-5} t + \frac{1}{100}\right) = 5\times10^{-5} x + C_2$$

When $t=0$, $x=0$ & $C_2 = \ln\left(\frac{1}{100}\right)$

$$\ln\left(5\times10^{-3} t + 1\right) = 5\times10^{-5} x \quad -(2)$$

When $V=70$, $t = 85.7\ s$

$$\ln\left(5\times10^{-3}\times 85.7 + 1\right) = 5\times10^{-5} x$$

$$x = 7132.5\ m \simeq \underline{\underline{7.13\ km}}$$

2. A block slides on a film of oil. The resistance 12B4 motion ~~of~~ the block is proportional to the spe of the block relative to the incline at the rat of 7.5 N/m/sec. If the block is released from rest, what is the terminal speed? What is th distance moved after 10 sec? $mg - 7.5V = 0$ $22.4\ m/s$

Sol:

$$F \propto V$$

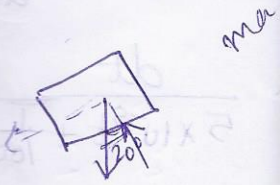
$$F(V) = 7.5V$$

$$\frac{dV}{dt} = \frac{F(V)}{m}$$

$$\frac{dV}{dt} = \frac{7.5V}{50}$$

$$\frac{dV}{V} = 0.15\, dt$$

$$\ln V = 0.15t + C_1$$



$50\times9.81\cos 20 =$

~~At t=0, V=0 ⇒~~

$F(V) = 50 \times 9.81 \sin 20 = 167.76\ N$

$V = \frac{167.76}{7.5} = 22.4\ m/s$

$V =$

$S = ut + \frac{1}{2}at^2$

$= \frac{1}{2} \times 3.352 \times$

Resistance $\propto V$

$ma = (\cdot \quad \cdot)$

$a =$

$F = mg\sin\alpha - kV$

$\alpha = 20°$

$k = 7.5\ N/m/s$

$m = 50\ kg$

$m\frac{dV}{dt} = mg\sin\alpha - kV$

$\int_0^V \frac{dV}{mg\sin\alpha - kV} = \int_0^t \frac{dt}{m}$

$-\frac{1}{k}\ln(mg\sin\alpha - kV) = \frac{1}{m}t + C$

$t=0, V=0,$ $\ln(mg\sin\alpha - kV) = \frac{-kt}{m} + C_1$

$C_1 = \ln mg\sin\alpha$

$1 - \frac{kV}{mg\sin\alpha} = e^{-\frac{kt}{m}}$

$V = \frac{mg\sin\alpha}{k}\left(1 - e^{-\frac{kt}{m}}\right) = 22.368(1 - e^{-0.15t})$

$\int_0^x dx = \int_0^t \frac{mg\sin\alpha}{k}\left(1 - e^{-\frac{kt}{m}}\right) dt$

$x = \frac{mg}{k}\sin\alpha\left[t + \frac{m}{k}\left(e^{-\frac{kt}{m}} - 1\right)\right] = 22.368\left[t + \frac{20}{3}\left(e^{-0.15t} - 1\right)\right]$

Terminal speed → $a=0$ ⇒ $F=0$ ⇒ $mg\sin\alpha - kV = 0$

$V_T = 22.368\ m/s$

After 10s, $x = 107.83\ m$

HW

12.12 $F = -27.5V^2$ $\qquad m = 8\,Mg = 8000\,kg$

$$m\frac{dV}{dt} = -27.5V^2$$

$$\frac{dV}{V^2} = \frac{-27.5}{8000}dt$$

$$\frac{-1}{V} = -3.4375\times10^{-3}t + C_1 \quad - (1)$$

$t=0$, $V = 350$ km/hr

$C_1 = -0.0103$

$$\frac{-dt}{dx} = -3.4375\times10^{-3}t - 0.0103$$

$$dx = \frac{dt}{3.4375\times10^{-3}t + 0.0103}$$

$$\frac{dx}{x} = 3.4375\times10^{-3}dx$$

$$\ln(3.4375\times10^{-3}t + 0.0103) = 3.4375\times10^{-3}x + C$$

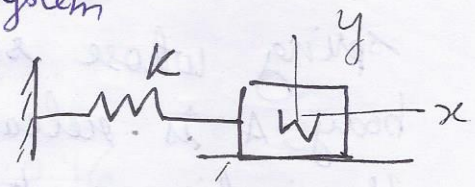
$t=0$, $x=0 \Rightarrow C_2 = -4.575$

In (1), $V = 200$ km/hr, $t = 2.24$ s

(2) $\Rightarrow$ $x = 162.8$ m

3 Force is a function of Position

Frictionless mass-spring system



Force $= f(x)$

$$m\frac{dV}{dt} = F(x)$$

$$m\frac{dV}{dt} = m\frac{dV}{dx}\frac{dx}{dt} = mV\frac{dV}{dx}$$

$$mV\,dV = F(x)\,dx$$

Integrating, $\frac{mV^2}{2} = \int F(x)\,dx + C_1$

$$\frac{dx}{dt} = \sqrt{\frac{2}{m}\int F(x)\,dx + C_1}$$

$$t = \int \frac{dx}{\sqrt{\frac{2}{m}\int F(x)\,dx + C_1}} + C_2$$

Linear restoring force

$\boxed{F = kx}$ k – spring const

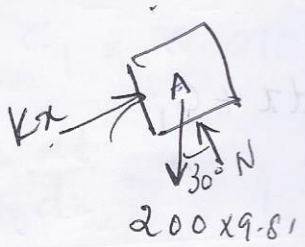
$x \to +ve$, spring force points in $-ve$ direction

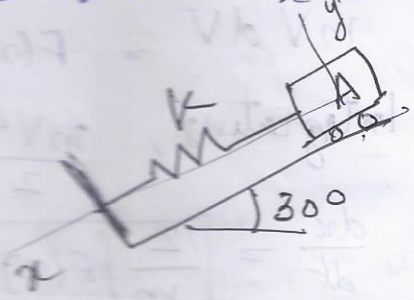
$x \to -ve$, " $+ve$ "

So it always point towards $x=0$ for which spring is undeformed.

1. A cart A having a mass of 200 kg is held on an incline so as to just touch an undeformed spring whose spring constant k is 50 N/mm. If body A is released very slowly, what distance down the incline must A move to reach an eqlbm configuration? If body A is released suddenly what is its speed when it reaches the aforementioned eqlbm configuration for a slow release?

Sol:





$$k = \frac{F}{\delta} \qquad \delta - \text{compression of the spring}$$

$$\delta = \frac{F}{k} = \frac{200 \times 9.81 \sin 30}{50} = 19.62 \text{ mm}$$

Quick release: Newton's law

$k = 50 \times 1000$
x - meters

$$200\ddot{x} = 200 \times 9.81 \sin 30 - 50 \times 1000\, x$$

$$\ddot{x} = 4.905 - 250\, x$$

$a = \frac{dV}{dt}$

$$V \cdot \frac{dV}{dx} = 4.905 - 250x$$

$$V\, dV = (4.905 - 250x)\, dx$$

$$\frac{V^2}{2} = 4.905x - 250\frac{x^2}{2} + C_1$$

$x=0$ when $V=0$, $C_1 = 0$

$x = 0.01962\ m \Rightarrow V^2 = 0.096$

$$V = 0.31\ m/s$$

HW 12.126

$M = 1$ slug $\quad k = 10\ lb/in \quad x = 1\ in \quad F = 5\ lb$

$$\ddot{x} = 5 - 10x$$

$$V\frac{dV}{dx} = 5 - 10x$$

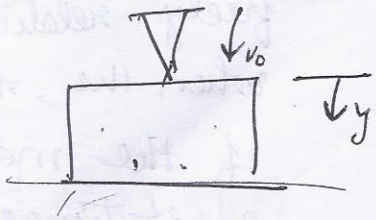
$$\frac{V^2}{2} = 5x - \frac{10x^2}{2} + C_1$$

$x=0,\ V=0 \Rightarrow C_1 = 0$

$x = 1,\ V = 0$

2. A The cone falling with a speed V_0 strikes & penetrates the block of packing material. The acceln of the cone after impact is $a = g - cy^2$, where c is a positive const & y is the penetration distance. If the max. penetration depth is observed to be y_m, determine const c.

Sol: $a = g - cy^2$



$\ddot{y} = g - cy^2$

$$V\frac{dV}{dy} = g - cy^2$$

$$\frac{V^2}{2} = gy - \frac{cy^3}{3} + C_1$$

$$y=0, \quad V = V_0 \Rightarrow \quad C_1 = \frac{V_0^2}{2}$$

$$y = y_{max}, \; V = 0 \Rightarrow \quad g y_{max} - \frac{c y_{max}^3}{3} + \frac{V_0^2}{2} = 0$$

$$\boxed{c = \frac{6 g y_{max} + 3 V_0^2}{2 y_{max}^3}}$$

Day 7 Newton's Law for Cylindrical Coordinates

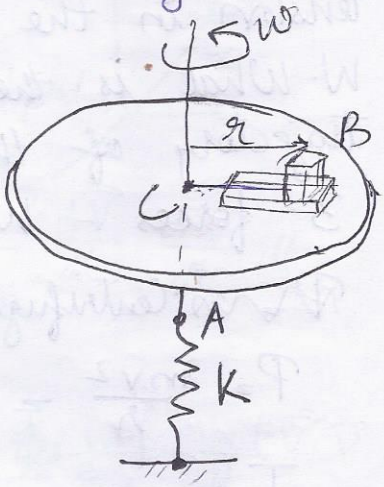
$$\left.\begin{aligned} F_r &= m(\ddot{r} - r\dot{\theta}^2) \\ F_\theta &= m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) \end{aligned}\right\} \text{non-linear eqns}$$

$$F_z = m\ddot{z}$$

1. A platform shown in Fig has a const. angular velocity ω equal to 5 rad/s. A mass B of 2 kg slides in a frictionless chute attached to the platform. The mass is connected via a light inextensible cable to a linear spring having a spring const K of 20 N/m. A swivel connector at A allows the cable to turn freely relative to the spring. The spring is unstretched when the mass B is at the center C of the platform. If the mass B is released at $r = 200$ mm from a stationary position relative to the platform,

what is its speed relative to the platform when it has moved to position $r = 400$ mm? What is the transverse force on the body B at this position?

Sol: Coplanar motion



$$F_r = m(\ddot{r} - r\dot{\theta}^2)$$

$$-\overset{Kr}{20}r = 2(\ddot{r} - 25r)$$

$$\ddot{r} = 15r \quad (a)$$

$$\ddot{r} = \frac{dV_r}{dt} = \frac{dV_r}{dr}\frac{dr}{dt} = V_r\frac{dV_r}{dr} = 15r$$

$$V_r\, dV_r = 15r\, dr$$

Integrating, $\frac{V_r^2}{2} = \frac{15r^2}{2} + C_1 \quad -(b)$

$r = 200$ mm, $V_r = 0 \Rightarrow C_1 = -\frac{15}{2} \times 0.2^2$

$= -0.3$

$$(b) \Rightarrow \frac{V_r^2}{2} = \frac{15r^2}{2} - 0.3$$

$$V_r^2 = 15r^2 - 0.6 \quad (c)$$

When $r = 400$ mm, $V_r = \sqrt{15 \times 0.4^2 - 0.6} = 1.342$ m/s

To get transverse force F_θ.

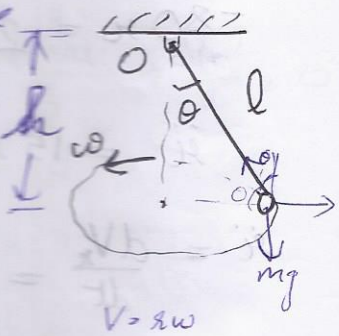
$$F_\theta = m(r\ddot{\theta} + 2\dot{r}\dot{\theta})$$

$$= 2[0.4 \times 0 + 2 \times 1.342 \times 5] = 26.84\text{N}$$

12.56 HW

2. A conical pendulum of length l is shown. The pendulum is made to rotate at a const. angular speed of ω about the vertical axis. Compute the tension in the cord if the pendulum bob has wt W. What is the distance of the plane of the trajectory of the bob from the support at O?

Sol: 3 forces – W, T, centrifugal force



~~P = m~~ Centrifugal accln, $a = \frac{v^2}{r}$

$$P = \frac{mv^2}{r} = m\omega^2 r$$

$$T\sin\theta = m\omega^2 r$$

$$T = \frac{m\omega^2 r}{\sin\theta} = \frac{m\omega^2 r}{r/l} =$$

$$\boxed{T = m\omega^2 l}$$

$$\Sigma M_O = 0 \Rightarrow m\omega^2 r h = mgr$$

$$\boxed{h = \frac{g}{\omega^2}}$$