VECTOR SPACES.

V – nonempty collection of elements

Addition

(i) for any $v_1, v_2 \in V$, $v_1 + v_2 \in V$ – closure property

(ii) v_1, v_2, v_3 ; $(v_1 + v_2) + v_3 = v_1 + (v_2 + v_3)$ – associative property

(iii) for any $v \in V$, $v_1 + 0 = 0 + v_1 = v_1$ – additive property

(iv) for any $v_1 \in V$ $-v_1 \in V$, $v_1 + (-v_1) = -v_1 + v_1 = 0$ – inverse

(v) $v_1, v_2 \in V$; $v_1 + v_2 = v_2 + v_1$ – commutative property

Scalar multiplication.

(vi) $c \in R$, $v_1 \in V$; $cv_1 \in V$

(vii) $c_1, c_2 \in R$ $(c_1 + c_2).v_1 = c_1 v_1 + c_2 v_1$

(viii) $c_1 \in R$, $v_1, v_2 \in V$; $c_1(v_1 + v_2) = c_1 v_1 + c_1 v_2$

(ix) $c_1, c_2 \in R$ $c_1(c_2 v_1) = (c_1 c_2) v_1$

(x) $1 \in R$, $1 . v_1 = v_1 . 1$, $v_1 \in V$

Q. Is $V = R^n$ over R a vector space?

Ans: $V_1 = (x_1, \ldots\ldots x_n)$ where $x_i \in R$.

$V_2 = (y_1, \ldots\ldots, y_n)$ where $y_i \in R$.

$V_1 + V_2 = ((x_1+y_1), (x_2+y_2), \ldots\ldots (x_n+y_n))$

closure.

$cV_1 = (cx_1 + \ldots\ldots + cx_n)$.

$V_3 = (z_1, z_2, \ldots\ldots z_n)$.

(ii) $(V_1+V_2)+V_3 = ((x_1+y_1)+z_1 \ldots\ldots (x_n+y_n)+z_n)$

$= (x_1+(y_1+z_1) \ldots\ldots (x_n)+(y_n+z_n))$

$= V_1 + (V_2+V_3)$. associative

(iii) $V_1 + 0 = (x_1, x_2, \ldots\ldots, x_n) + (0, 0, \ldots\ldots 0)$

$= (x_1, x_2, \ldots\ldots x_n)$

$= V_1$ identity

(iv) $-V_1 = (-x_1, -x_2, \ldots\ldots, -x_n)$

$V_1 + (-V_1) = (0, 0, 0, \ldots\ldots) = 0$.

inverse

(v) $V_1 + V_2 = ((x_1+y_1), (x_2, y_2), \ldots\ldots (x_n+y_n))$

$= ((y_1+x_1), (y_2+x_2), \ldots\ldots (y_n+x_n))$

$= (y_1+y_2+\ldots\ldots+y_n) + (x_1+x_2+\ldots\ldots+x_n)$

$= V_2 + V_1$

Commutative

Q. Is $R^2 = V$ over R a vector space? check with conditions.

Ans: (i) (x_1, y_1) $(x_2, y_2) \in R^2$.

$$(x_1, y_1) + (x_2, y_2) = (x_1 + 2x_2, y_1 + 2y_2) \in R^2$$

$$c(x_1, y_1) = (Cx_1, Cy_2) \in R^2.$$

$\therefore$ closed under addition.

(ii) $(x_1, y_1) + [(x_2, y_2) + (x_3, y_3)] = (x_1, y_1) + (x_2 + 2x_3, y_2 + 2y_3)$

$$= (x_1 + 2x_2 + 4x_3, y_1 + 2y_2 + 4y_3$$

$[(x_1, y_1) + (x_2, y_2)] + (x_3, y_3) = (x_1 + 2x_2, y_1 + 2y_2) + (x_3, y_3)$

$$= (x_1 + 2x_2 + 2x_3, \; y_1 + 2y_2 + 2y_3)$$

$\therefore$ Not associative.

(iii) $(x_1, y_1) + (0, 0) = (x_1, y_1)$.

$(0, 0) + (x_1, y_1) = (2x_1, 2y_1)$

$\therefore$ No additive identity

(iv) No inverse

(v) $(x_1, y_1) + (x_2, y_2) \neq (x_2, y_2) + (x_1, y_1)$

$\therefore$ Not commutative.

(vi) closed under scalar multiplication.

(vii) $(c_1+c_2)(x_1,y_1) = ((c_1+c_2)x_1, (c_1+c_2)y_1)$

$$= (c_1 x + c_2 x_1, \; c_1 y_1 + c_2 y_1)$$

$$c_1(x_1,y_1) + c_2(x_2,y_2) = (c_1 x_1 + 2c_2 x_2, \; c_1 y_1 + 2c_2 y_2)$$

$$\therefore (c_1+c_2)(x_1,y_1) \neq c_1(x_1,y_1) + c_2(x_2,y_2)$$

(viii) $(c_1 c_2)(x_1,y_1) = c_1(c_2 x_1, c_2 y_1)$

$$= (c_1 c_2 x_1, \; c_1 c_2 y_1)$$

$$= (c_2 c_1)(x_1,y_1)$$

(ix) $c_1((x_1,y_1)+(x_2,y_2)) = c_1(x_1+2x_2, \; y_1+2y_2)$

$$= (c_1 x_1 + 2c_1 x_2, \; c_1 y_1 + 2c_1 y_2)$$

$$c_1(x_1,y_1) + c_2(x_2,y_2) = c_1 x_1 + 2c_2 x_2, \; c_1 y_1 + 2c_2 y_2$$

$$\therefore c_1((x_1,y_1)+(x_2,y_2)) = c_1(x_1,y_1) + c_2(x_2,y_2)$$

(x) $1.(x_1,y_1) = (x_1,y_1) = (x_1,y_1).1$

Q Let V be a set of all $m \times n$ matrices with real number entries over R. $A, B \in V$, $A = [a_{ij}]$, $B = [b_{ij}]$

$A + B = [a_{ij} + b_{ij}]$

$CA = [c a_{ij}]$

Ans: (i) $A + B \in V$ closed under addition

(ii) $A + (B+C) = [a_{ij}] + [b_{ij} + c_{ij}]$

$= (a_{ij} + b_{ij}) + [c_{ij}]$

$= (A+B) + C$.

∴ associative.

(iii) $A + 0 = [a_{ij} + 0] = [a_{ij}] = A$.

$0 + A = A$ where $\theta = [0]_{m \times n}$.

∴ Hence additive identity exists.

(iv) $A = [a_{ij}]$, $-A = [-a_{ij}]$

$A + (-A) = (-A) + A = 0$.

∴ Inverse exists, i.e, $-A$

(v) $A + B = [a_{ij}] + [b_{ij}] = [a_{ij} + b_{ij}]$

$= [b_{ij} + a_{ij}]$

$= B + A$

∴ It is commutative.

(vi) $CA = [c a_{ij}] \in V$

(vii) $(C_1 + C_2) A = [(C_1 + C_2)[a_{ij}]] = C_1[a_{ij}] + C_2[a_{ij}]$

$= C_1 A + C_2 A$

$C_1 \cdot C_2) A =$

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(viii) $(c_1 c_2) A = c_1 c_2 [a_{ij}]$

$= [c_1 c_2 a_{ij}]$

$= c_1 [c_2 a_{ij}] = c_1 (c_2 \cdot A)$

(ix) $c(A+B) = c[a_{ij} + b_{ij}]$

$= c[a_{ij}] + [b_{ij}] c$

$= cA + cB$.

(x) $1 \cdot A = 1 \cdot [a_{ij}] = [a_{ij}] = A$.

Hence V is a vector space over R.

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* set of all square matrixes with real entries over R is also a vector space.

* set of all square matrix A, such that $|A| = 0$

$\Rightarrow$ $A, B \in V$ $\quad |A| = |B| = 0$

$|A+B| \neq |A| + |B|$.

not closed under addition.

If two matrix have their determinant equal to 0, it is not necessary that their sum will form a matrix whose determinant is 0.

Hence closure property is violated.

$\therefore$ It is not a vector space.

Q. V = set of all continuous real valued function defined on $[a,b]$ over $\mathbb{R}$. $f, g \in V$

$$(f+g)(x) \doteq f(x) + g(x)$$

$$(cf)(x) = c\,f(x)$$

Ans: i) $f+g$ - continuous real valued $\in V$

$\therefore$ It is closed under addition

(ii) $f+(g+h)x = f(x) + (g+h)x$

$= [f(x) + g(x)] + h(x)$

$= (f+g)x + h(x)$.

$\therefore$ associative.

(iii) $0(x) = 0$

$(f+0)x = f(x) + 0(x) = f(x)$.

$\therefore \quad f + 0 = f = 0 + f$.

Additive identity is the 0 function.

(iv) $(-f)(x) = -f(x)$.

$(f+(-f))\,x = f(x) + (-f)x$.

$= \overbrace{f(x) - f(x)}$ $\longrightarrow$ two real numbers.

$= 0 \leftarrow$ real no. zero

$f + (-f) = \underbrace{(-f) + f}_{\text{two function}} = 0 \leftarrow$ zero function

$\therefore$ Additive inverse is $-f$

(v) $(f+g)x = f(x)+g(x)$ $\therefore$ commutative.

$= g(x) + f(x)$

$= (g+f)(x)$

$f+g = g+f$

(vi) $(c_1+c_2)f(x) = c_1 f(x) + c_2 f(x)$

$= (c_1 f + c_2 f)x$

$(c_1+c_2)f = c_1 f + c_2 f$

(vii) $c(f+g)x = c[f(x)+g(x)]$

$= cf(x) + cg(x)$

$= (cf)x + (cg)x$

$\therefore c(f+g) = cf + cg$

(viii) $[(c_1c_2)f]x = c_1c_2 f(x) = c_1(c_2 f(x)) = c_1(c_2 f)x$

If this question is changed to differentiable real valued function

check whether it is closed under addition, scalar multiplication, also whether it has an identity and inverse.

If differentiable real valued function such $f(0)=1$ ×

if $f(0)=1$ condition is not given then ✓

Here when we move from a higher subset to a lower one we should check these conditions.

Linear dependance and independance

⇒ V is a vector space, $v_1, v_2, \ldots, v_n \in V$ are linearly independant if $\alpha_1 v_1 + \ldots + \alpha_n v_n = 0$

$$\Rightarrow \alpha_n = 0 \quad \forall\, i = 1, \ldots n.$$

⇒ If $\exists$ at least one $\alpha_k \neq 0$

$$\alpha_k v_k = -\sum_{i=1}^{n} \alpha_i v_i$$

$$v_k = -\sum_{i=1,}^{n} \frac{\alpha_i}{\alpha_k} v_i$$

v_k is linearly dependant $v_1, \ldots, v_n$

Eg: consider a vector space V in R^3. Let

$$\begin{pmatrix}1\\1\\1\end{pmatrix} \begin{pmatrix}2\\1\\1\end{pmatrix} \begin{pmatrix}1\\2\\1\end{pmatrix}$$ be 3 vectors.

$$\begin{pmatrix}1 & 1 & 1\\2 & 1 & 1\\1 & 2 & 1\end{pmatrix} \sim \begin{pmatrix}1 & 1 & 1\\0 & -1 & -1\\0 & 1 & 0\end{pmatrix} \sim \begin{pmatrix}1 & 1 & 1\\0 & 1 & 1\\0 & 1 & 0\end{pmatrix}$$

$$\sim \begin{pmatrix}1 & 1 & 1\\0 & 1 & 1\\0 & 0 & 1\end{pmatrix}$$

we can either write them as row/column because in the end row rank = column rank

Rank = 3 = no. of vectors

Hence the vectors are linearly independant.

Eg: consider 4 vectors in R^3

$$A = \begin{pmatrix} - \\ - \\ - \\ - \end{pmatrix}_{4\times 3}$$

$R(A) \leq 3 < 4$.

4 vectors are l.d in R^3.

when there exists vectors in R^n greater than n, then they will linearly dependant.

Q. Find the 'a' such that:

$$\begin{pmatrix} 4 \\ 5 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} a \\ 10 \\ 9 \end{pmatrix}$$ are linearly independent.

Ans:

$$\begin{pmatrix} 4 & 5 & 1 \\ 3 & 0 & 2 \\ a & 10 & 9 \end{pmatrix} \sim \begin{pmatrix} 1 & 5/4 & 1/4 \\ 3 & 0 & 2 \\ a & 10 & 9 \end{pmatrix}$$

$R_2 \rightarrow R_2 - 3R_1$

$$\begin{pmatrix} 1 & 5/4 & 1/4 \\ 0 & -15/4 & 5/4 \\ 0 & 10-\frac{5a}{4} & 9-\frac{a}{4} \end{pmatrix} \sim \begin{pmatrix} 1 & 5/4 & 1/4 \\ 0 & 1 & -1/3 \\ 0 & 10-\frac{5a}{4} & 9-\frac{a}{4} \end{pmatrix}$$

$\cancel{\frac{5}{4}}R_3 \rightarrow R_3 - \left(10-\frac{5a}{4}\right)R_2$

$$\begin{pmatrix} 1 & 5/4 & 1/4 \\ 0 & 1 & -1/3 \\ 0 & 0 & \frac{37-2a}{3} \end{pmatrix}$$

These vectors are linearly independant if

rank $= 3$

$\therefore \frac{37-2a}{3} \neq 0 \Rightarrow a \neq \frac{37}{2}$

Q Find a, b such that the vectors $\begin{pmatrix}1\\2\\3\end{pmatrix}\begin{pmatrix}a\\0\\0\end{pmatrix}\begin{pmatrix}0\\b\\1\end{pmatrix}$ are l.i

Ans: The vectors are: $\begin{pmatrix}1\\2\\3\end{pmatrix}\begin{pmatrix}a\\0\\0\end{pmatrix}\begin{pmatrix}0\\b\\1\end{pmatrix}$

$$\begin{pmatrix}1 & 2 & 3\\ a & 0 & 0\\ 0 & b & 1\end{pmatrix}$$

$R_2 \to R_2 - aR_1$

$$\begin{pmatrix}1 & 2 & 3\\ 0 & -2a & -3a\\ 0 & b & 1\end{pmatrix} \sim$$

$R_2 \to \frac{R_2}{-2a}$, $a \neq 0$

$$\begin{pmatrix}1 & 2 & 3\\ 0 & 1 & 3/2\\ 0 & b & 1\end{pmatrix}$$

$R_3 \to R_3 - bR_2$

$$\begin{pmatrix}1 & 2 & 3\\ 0 & 1 & 3/2\\ 0 & 0 & 1-\frac{3b}{2}\end{pmatrix}$$

For linear independancy,

$$1 - \frac{3b}{2} \neq 0.$$

$$1 \neq \frac{3b}{2} \Rightarrow b \neq \frac{2}{3}, \quad a \neq 0$$

Q $\{1, x, x^2\}$ check whether they are lineary independant

Ans: $\alpha_1 + \alpha_2 x + \alpha_3 x^2 = 0$

diff w.r.t x

$\alpha_2 + 2\alpha_3 x = 0$

diff. w.r.t x

$2\alpha_3 = 0$

$\Rightarrow \alpha_3 = 0, \alpha_2 = 0, \alpha_1 = 0$

$\therefore$ function is l.i

Q $\{1, \sin t, \cos t\}$ L.I or not?

Ans: $\alpha_1 + \alpha_2 \sin t + \alpha_3 \cos t = 0$

$\alpha_2 \cos t - \alpha_3 \sin t = 0$

$-\alpha_2 \sin t - \alpha_3 \cos t = 0$

$AX = 0$, $|A| = 0$ for a non-trivial solu.

$$\begin{vmatrix} 1 & \sin t & \cos t \\ 0 & \cos t & -\sin t \\ 0 & -\sin t & -\cos t \end{vmatrix} = 1(-\cos^2 t - \sin^2 t) - \sin t(0) + \cos t(0)$$

$$= -1 \neq 0$$

$\therefore$ system has only trivial solution.

$\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$

Q $\{\sin t, \cos t, \sin 2t\}$

$\alpha_1 \sin t + \alpha_2 \cos t + \alpha_3 \sin 2t = 0$

$\alpha_1 \cos t - \alpha_2 \sin t + 2\alpha_3 \cos 2t = 0$

$+\alpha_1 \sin t + \alpha_2 \cos t + 4\alpha_3 (+1) \sin 2t = 0$

$+\alpha_1 \sin t + \alpha_2 \cos t + \alpha_3 \sin 2x = 0$

$$\begin{vmatrix} \sin t & \cos t & \sin 2t \\ \cos t & -\sin t & 2\cos 2t \\ \sin t & \cos t & 4\sin 2t \end{vmatrix} = \sin t(-4\sin t \sin 2t - 2\cos 2t \cos t) - \cos t(4\cos t \sin 2t - 2\sin t \cos 2t) + \sin 2t(\cos^2 t + \sin^2 t)$$

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$= -4\sin^2 t \sin 2t - 2\cos 2t \cos t \sin t$
$\quad - 4\cos^2 t \sin 2t + 2\cos t \sin t \cos 2t$
$\quad + \sin 2t$

$= -4\sin 2t + \sin 2t$

$= -3\sin 2t \neq 0$

<u>Note</u>:- If $V = \mathbb{R}^n$ and vectors are,

$\{0, v_1, \ldots, v_n\}$, i.e, including zero vector

$\alpha_1(0) + \alpha_2(v_1) + \ldots + \alpha_{n+1}(v_n) = 0$

$\therefore \begin{pmatrix} \alpha_1 = k \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ is a non-trivial solution

$\therefore$ linearly dependant.

<u>Note</u>:- $S = \{v_1, \ldots, v_n\}$, linearly independant

$S' = \{\text{only } k \text{ vectors from } S\}$

Any subset of a l.i set is always l.i

$\alpha_1 v_1 + \ldots + \alpha_k v_k = 0$

$\alpha_1 v_1 + \ldots \alpha_k v_k + 0 . v_{k+1} + \ldots 0\, v_n = 0 \Rightarrow \alpha_i = 0$

t)

:2t)

<u>Note:-</u>

Any superset of a linearly dependant set is linearly dependant.

Eg: $2\begin{pmatrix}1\\1\end{pmatrix} - 1\begin{pmatrix}2\\2\end{pmatrix} + 0\begin{pmatrix}1\\2\end{pmatrix} + 0\begin{pmatrix}1\\0\end{pmatrix} + 0\begin{pmatrix}0\\1\end{pmatrix} + \begin{pmatrix}0\\0\end{pmatrix}$

<u>Linear span</u>

$S = \{v_1, \dots v_n\} \in V$

$L(S)$ = linear span $S = \{\alpha_1 v_1 + \dots + \alpha_n v_n \mid \alpha_n \in R\}$
is a vector space

<u>Subspace of a vector space V (over R)</u>

Let $W \subset V$, W is a subspace of V in a vector space.

if i) $w_1, w_2 \in W$ $\quad w_1 + w_2 \in W$
ii) if $\alpha \in R$, $w \in W$ $\quad \alpha w \in W$ } closed under addition and scalar multiplica
↓
This condit can be grouped tog as $\alpha w_1 + \beta$

$-1 \in R$, $w \in W$ $\quad -w \in W$

$w + (-w) = w - w = 0$

Q $V = R^3$ over R

$W = \{(x, y, z) \mid xyz = 0\}$

clearly $W \subset R^3$.

Ans: $w_1 = (x_1, y_1, z_1)$, $w_2 = (x_2, y_2, z_2)$

$w = (x_1 + x_2, y_1 + y_2, z_1 + z_2)$

but $(x_1 + x_2)(y_1 + y_2)(z_1 + z_2)$ need not be zero

Hence it is not closed under addition.

$\alpha W_1 = (\alpha x_1, \alpha y_1, \alpha z_1)$

$\alpha x_1 \alpha y_1 \alpha z_1 \Rightarrow \alpha^3 x_1 y_1 z_1 = 0$

$[\because x_1 y_1 z_1 = 0]$

$\therefore$ It is closed under scalar multiplication.

However it is not a subspace. because it is not closed under addition.

Q $W = \{(2t, 3t, 4t) / t \in R\}$

$W \subset R^3$

$W_1 = (2t_1, 3t_1, 4t_1)$.

$W_2 = (2t_2, 3t_2, 4t_2)$.

$W_1 + W_2 = (2(t_1+t_2), 3(t_1+t_2), 4(t_1+t_2)) \in W$

$\therefore$ It is closed under addition

$\alpha W_1 = (2\alpha t_1, 3\alpha t_1, 4\alpha t_1) \in W$

$\therefore$ It is closed under scalar multiplication.

$\therefore$ W is a subspace

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Q W_1, W_2 subspace of V

Let $u_1, u_2 \in W_1 \cap W_2$.

$u_1 \in W_1$ & W_2.

$u_2 \in W_1$ & W_2

$u_1 + u_2 \in W_1 \cap W_2$

$\alpha u_1 \in W_1 \cap W_2$.

$W_1 \cap W_2$ is a subspace.

$W_1 = \{ t(1,2,3) . t \in R \}$

$W_2 = \{ t'(1,1,1) \ t' \in R \}$

$W_1 \cup W_2$.

Q $W_1 = \{ (x,y,z) \in V : 3z - 5x = 0$ and y is an integer $\}$

$\alpha W_1 = \{ \alpha x, \alpha y, \alpha z \}$ Here αy need not be an integer

Hence it is not closed under multiplication

Therefore not a subspace.

Q

$W_2 = \{(x, y, z) \in V \quad x \leq y + 2z \leq 10\}$

$(x_1, y_1, z_1) \in W_2 \qquad x_1 \leq y_1 + 2z_1 \leq 10$

$(x_2, y_2, z_2) \in W_2 \qquad x_2 \leq y_2 + 2z_2 \leq 10$

$(x_1 + x_2, y_1 + y_2, z_1 + z_2)$

This need $\quad x_1 + x_2 \leq (y_1 + y_2) + 2(z_1 + z_2) \leq 10$
not be true.

~~Eg: (1, 1, 1)~~ $\Rightarrow 1 \leq 1 + 2 \leq 10$

~~$(1, 3, 3) \Rightarrow 1 \leq$~~ Hence not closed under addition

~~$1 \leq$~~ $1 + 2 \leq 10$

$1 \leq$

$\therefore$ It is not a subspace.

Q V be a set of all 2×2 matrices with

and $W_1 = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ $\det A \neq 1$.

$a, b, c \in \mathbb{R}$.

$$\begin{pmatrix} a_1 & b_1 \\ b_1 & c_1 \end{pmatrix} + \begin{pmatrix} a_2 & b_2 \\ b_2 & c_2 \end{pmatrix} = \begin{pmatrix} a_1 + a_2 & b_2 + b_1 \\ b_1 + b_2 & c_1 + c_2 \end{pmatrix}$$

$$\alpha W_1 = \begin{pmatrix} \alpha a & \alpha b \\ \alpha b & \alpha c \end{pmatrix}$$

Now, $a_1 c_1 - b_1^2 \neq 1 \qquad a_2 c_2 - b_2^2 \neq 1$

$\therefore (a_1 + a_2)(c_1 + c_2) - (b_1 + b_2)^2$ need not be 1.

$\therefore$ Since it is not closed under addition, hence it is not a subspace.

Basis of vectorspace

V is a vectorspace such that $S \subset V$, $S = \{v_1, \ldots, v_n\}$ is said to be a basis of V if,

i) S spans (generates) V. ∴ any $v \in V$,

$v = \alpha_1 v_1 + \ldots + \alpha_n v_n$ for some $\alpha_1, \alpha_2, \ldots$

ii) S is a linearly independant set

Eg: R^2, $S = \{(1,0), (0,1)\}$

$(x,y) \in R^2$

then, $(x,y) = x(1,0) + y(0,1)$.

∴ S is a basis of R^2, (standard basis).

or

$S = \{(2,0), (0,3)\}$

$(x,y) \in R^2$, then

$$(x,y) = \frac{x}{2}(2,0) + \frac{y}{3}(0,3).$$

or

$S = \{(1,1), (2,2)\}$ is not a basis because they are linearly dependant.

or

$S = \{(1,1), (2,1)\}$

$$(x,y) = \alpha(1,1) + \beta(2,1)$$

$\alpha + 2\beta = x$
$\alpha + \beta = y$

∴ $\alpha = 2y - x$.
$\beta = x - y$.

Hence, $\therefore$ It is a basis of R^2.

Basis is not unique, however no. of vectors in a basis is unique and it is called the dimension of a vectorspace.

Eg: Consider $V = R^3$.

Standard basis $= \{(1,0,0), (0,1,0), (0,0,1)\}$

$$(x,y,z) = x(1,0,0) + y(0,1,0) + z(0,0,1)$$

or

$$S = \{(2,0,0), (0,\tfrac{1}{2},0), (0,0,3)\}.$$

$$(x,y,z) = \frac{x}{2}(2,0,0) + 2y(0,\tfrac{1}{2},0) + \frac{z}{3}(0,0,3).$$

Note:-

→ Consider R^n with m vectors where $m > n$, then it is linearly independant.

→ Hence a basis of R^n cannot have more than n variables.

→ Dimension of V = max. no. of linearly independant vectors in V.

→ Any 'n' linearly independant vectors in R^n is a basis of R^n.

Proof:- To prove $S = \{v_1, \ldots, v_n\}$ generates V → linear span of V

Let us assume that $\exists$ a $v_{n+1} \notin L(S)$

$$\alpha_1 v_1 + \cdots + \alpha_n v_n + \alpha_{n+1} v_{n+1} = 0$$

$\therefore \quad v_{n+1} = \Sigma \quad \Rightarrow \alpha_{n+1} = 0$

$\Rightarrow \alpha_1 v_1 + \dots + \alpha_n v_n = 0 \Rightarrow \alpha_i = 0, \quad i = 1, \dots n.$

$\alpha_i = 0, \quad i = 1, \dots, n+1 \Rightarrow v_1, \dots v_n, v_{n+1}$ is linearly independent

This is a contradiction to the assumption that dimension of $R^n = n$.

$v_{n+1} \in L(S)$

$\therefore$ S generates V.

Extension of a basis.

Let $V = R^n$, $v_1, \dots v_k$ 'k' linearly independant vectors of R, $k < n$

$S = \{v_1, \dots v_k\}$.

Choose v_{k+1}, $v_{k+1} \notin L(S)$

$S_1 = \{v_1, \dots v_k, v_{k+1}\}$ is linearly independant.

if $k+1 = n$, S_1 is a basis of R^n

if not $v_{k+2} \notin L(v_1, \dots v_k, v_{k+1})$ $S_1 \cup \{v_{k+2}\}$ is linearly indep

if $k+2 = n$, then $S_2 = S_1 \cup \{v_{k+1}\}$ is a basis

$\vdots$

$k + r = n$

To form a basis of R^4,

$$\begin{pmatrix} 1 & -2 & 5 & -3 \\ 0 & 1 & -9/7 & 2/7 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Find the dimension and a basis of the subspace W of $P_3(t)$ spanned by $w_1 = t^3+2t^2-3t+4$

$w_2 = 2t^3+5t^2-4t+7.$

$w_3 = t^3-4t^2+t+2.$ $\quad L(S) = L\{w_1, w_2, w_3\} \neq P_3(t)$

Soln:

$$\begin{pmatrix} 1 & 2 & -3 & 4 \\ 2 & 5 & -4 & 7 \\ 1 & -4 & 1 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & -3 & 4 \\ 0 & 1 & 2 & -1 \\ 0 & -6 & 4 & -2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & -7 & 6 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 16 & -8 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -7 & 6 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 1 & -1/2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 5/2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1/2 \end{pmatrix}$$

$$\begin{array}{cc} 1 & 2 \\ 2 & 5 \\ -3 & -4 \\ 4 & 7 \end{array}$$

$r(A) =$ no. of unknown $= 3.$

Hence unique solution exists.

i.e., a trivial solution

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Q. Suppose in R^3 we are given two vectors $(1,1,1)$ $(1,2,1)$. Find a basis.

Soln:

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

We must add a 3^{rd} row such that the rank of the matrix changes to 3. We should choose such that it isn't $(0,K,0)$ or (m,m,m) as it can easily simplified and hence the 3^{rd} row would become $(0,0,0)$.

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

2 2 2

0 3 0

Q. Consider $U_{2\times2}$ = set of all 2×2 matrix with real entries. Find a basis.

Ans:

$$U_{2\times2} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$S = \left\{ a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + d\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

Q. Consider $V = \left\{ \begin{pmatrix} a & -a \\ b & c \end{pmatrix} \right\}$ a,b,c real $V \subset U_{2\times2}$.

Ans:

$$S = \left\{ \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$ is a basis and $\dim V = 3$.

Q $P_3(x)$ = set of all polynomials of degree atmost 3

$a_0 + a_1x + a_2x^2 + a_3x^3$

Ans: $S = \{1, x, x^2, x^3\}$ is a basis $P_3(x)$

a) $S = \{2, 3x, 5x^2, x^3\}$ $a_0 + a_1x + a_2x^2 + a_3x^3$

$$\frac{a_0}{2}(2) + \frac{a_1}{3}(3x) + \frac{a_2}{5}(5x^2) + a_3x^3.$$

Q) $S = \{1+x, 1+x^2, 1+x^3\}$ → prove is not a basis of $P_3(x)$

it cannot span or generate

$(a_0 + a_1 + a_2) + a_0x + a_1x^2 + a_2x^3$ $a_0(1+x) + a_1(1+x^2) + a_2(1+x^3)$

Q Let W be the subspace of $\mathbb{R}^2$ spanned by the vectors

$w_1 = (1, -2, 5, -3)$

$w_2 = (2, 3, 1, -4)$

$w_3 = (3, 8, -3, -5)$

Find the dim W and basis. Also extend this to a basis of R^4.

Soln:

$$\begin{pmatrix} 1 & -2 & 5 & -3 \\ 2 & 3 & 1 & -4 \\ 3 & 8 & -3 & -5 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & 5 & -3 \\ 0 & 7 & -9 & 2 \\ 0 & 14 & -18 & 4 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -2 & 5 & -3 \\ 0 & 1 & -9/7 & 2/7 \\ 0 & 14 & -18 & 4 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & 5 & -3 \\ 0 & 1 & -9/7 & 2/7 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The w_1, w_2, w_3 are linearly dependant.

Bases = $\{(1, -2, 5, -3), (2, 3, 1, -4)\}$

dim W = 2

To form a basis of R^4,

$$\begin{pmatrix} 1 & -2 & 5 & -3 \\ 0 & 1 & -9/7 & 2/7 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Q. Find the dimension and a basis of the subspace W of $P_3(t)$ spanned by

$$W_1 = t^3 + 2t^2 - 3t + 4$$
$$W_2 = 2t^3 + 5t^2 - 4t + 7.$$
$$W_3 = t^3 - 4t^2 + t + 2.$$

$L(S) = L\{w_1, w_2, w_3\} \neq P_3(t)$

Soln:

$$\begin{pmatrix} 1 & 2 & -3 & 4 \\ 2 & 5 & -4 & 7 \\ 1 & -4 & 1 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & -3 & 4 \\ 0 & 1 & 2 & -1 \\ 0 & -6 & 4 & -2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & -7 & 6 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 16 & -8 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -7 & 6 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 1 & -1/2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 5/2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1/2 \end{pmatrix}$$

$r(A) =$ no. of unknown $= 3$.

Hence unique solution exists.

i.e, a trivial solution

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Hence W_1, W_2, W_3 are linearly independant.

The given 3 are the basis of W.

Linear Transformation

Let V, W be vectorspace.

A mapping, $T = V \to W$ is a linear map if

This is the addition occuring in V

This addition occurs over W.

i) $T(v_1 + v_2) = T(v_1) + T(v_2)$

linear operator

ii) $T(\alpha v_1) = \alpha T(v_1)$.

V, W over R.

Eg:(1) $T : R^2 \to R^2$

$T(x, y) = (x+y, x-y)$. Is it a linear transformation?

Soln:- Let $v_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$, $v_2 = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$

$T(v_1 + v_2) = T(x_1 + x_2, y_1 + y_2)$

$= (x_1 + x_2 + y_1 + y_2, x_1 + x_2 - y_1 - y_2)$

$= ((x_1 + y_1) + (x_2 + y_2), (x_1 - y_1) + (x_2 - y_2))$

$= (x_1 + y_1, x_1 - y_1) + (x_2 + y_2, x_2 - y_2)$

$= T(v_1) + T(v_2)$

$T(\alpha v_1) = T(\alpha x_1, \alpha y_1)$

$= T(\alpha x_1 + \alpha y_1, \alpha x_1 - \alpha y_1) = \alpha(x_1 + y_1, x_1 - y_1)$

(ii) $T : R^2 \longrightarrow R^2$

$T(x, y) = (x^2, y^2)$

Soln: Let $V_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$, $V_2 = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$

$T(V_1 + V_2) = T(x_1 + x_2, \; y_1 + y_2)$.

$= \cancel{x^2 + x_2} \; ((x_1 + x_2)^2, (y_1 + y_2)^2)$

$T(V_1) + T(V_2) = (x_1^2, y_1^2) + (x_2^2, y_2^2)$

$= (x_1^2 + x_2^2, \; y_1^2 + y_2^2)$.

$\therefore \; T(V_1 + V_2) \neq T(V_1) + T(V_2)$.

(iii) $T : R^2 \longrightarrow R^2$

$T(x, y) = (xy)$

$T(V_1 + V_2) = (x_1 + x_2)(y_1 + y_2)$

$T(V_1) + T(V_2) = x_1 y_1 + x_2 y_2$.

$T(V_1 + V_2) \neq T(V_1) + T(V_2)$

(iv) $T : R^3 \longrightarrow R^2$.

$\rightarrow T(x, y, z) = (x, y)$ ← linear transformation ✓

$\rightarrow T(x, y, z) = (x + y, \; y + z)$ ✓

$\rightarrow T(x, y, z) = (x + 1, y + 1, z + 1)$ ×

since $T(V_1 + V_2) = (x_1 + x_2 + 1, \; y_1 + y_2 + 1, \; z_1 + z_2 + 1)$

$T(V_1) + T(V_2) = (x_1 + x_2 + 2, \; y_1 + y_2 + 2, \; z_1$

(4) V = set of all continuous real valued functions on R.

1. $T : V \to V$ defined as $T(f) = f'$

$$[T(f)]x = f'(x).$$

$$f, g \in V$$

$$T(f+g)(x) = (f+g)' = f' + g'$$

$$T(\alpha f)(x) = (\alpha f)' = \alpha T(f).$$

$\therefore$ It is a linear transformation

2. V = set of all $n \times n$ matrices and M be a fixed matrix in V

$T: V \to V$, $T(A) = MA$

$$T(A+B) = M(A+B).$$

$$= MA + MB$$

$$= T(A) + T(B)$$

$$T(\alpha A) = M(\alpha A)$$

$$= \alpha(MA)$$

$$= \alpha T(A).$$

$\therefore$ It is a linear transformation

3. $T: V \to V$

$$T(A) = M + A$$

$$T(A+B) = M + A + B.$$

$$T(A) + T(B) = M + A + M + B.$$

$$\therefore T(A+B) \neq T(A) + T(B)$$

$$T(\alpha A) = M + \alpha A \neq \alpha(M+A) = \alpha T(A)$$

$\therefore$ Not a linear transformation.

$T : V \to V$.

*

Let $v \in V$

$$v + (-v) = 0 \in V$$

$$T(0) = T(v + (-v))$$

$$= T(v) + T(-v).$$

$$= T(v) + (-1)\,T(v)$$

$$= T(v) - T(v)$$

$$= 0 \quad \in W$$

* for all linear transformation on the zero element of one vector space is mapped to the zero element of the other vector space.

* But there can be other elements from one vector space can be mapped to zero element apart from from the zero element.

* Image of the linear combination is the linear combination of the images.

$$v_1 \ldots\ldots v_n \in V$$

~~$T\,(T(\alpha_1 v_1) + T(\alpha_2$~~

$$T(\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n)$$

$$= T(\alpha_1 v_1) + T(\alpha_2 v_2 + \ldots + \alpha_n v_n)$$

$$= T(\alpha_1 v_1) + T(\alpha_2 v_2) + \ldots + T(\alpha_n v_n)$$

$$= \alpha_1 T(v_1) + \alpha_2 T(v_2) + \ldots + \alpha_n T(v_n)$$

T (linear combination) = linear combination (Images)

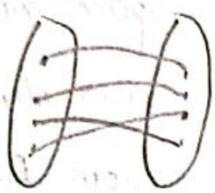
One-One Linear Transformation (injective)

if $v_1 \neq v_2$ $\quad T(v_1) \neq T(v_2)$

or if $T(v_1) = T(v_2) \Rightarrow v_1 = v_2$

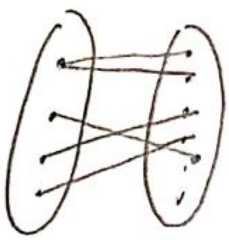
Onto mapping (surjective)

for every $w \in W$ $\exists$ $v \in V$, $T(v) = w$

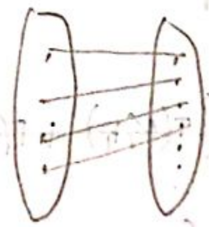


Bijective

If T is both one and onto.



Not onto or one



one-one not onto

Th: Let $T : V \to W$ be a linearly transformation

$v_1, \ldots v_n \in V$ and are linearly independant.

Is $T(v_1) \ldots T(v_n)$ linearly independant ?

Note:- Any set that includes the zero vector is linearly dependant.

Soln: It is not necessary to be so.

$T(x,y) = (x,0)$

consider $(1,0)$ & $(0,1)$ are l.i

If we start this step shown in the method of this then we will begin like this

$T(1,0) = (1,0)$

$T(0,1) = (0,0)$

$\alpha_1 T(v_1) + \alpha_2 T(v_2) + \dots + \alpha_n T(v_n)$

$T(\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n) =$

$(0,0)$ and $(1,0)$ are linearly dependant.

They never told it 0.

Th: Let $T: V \to W$ be a linear transformation

Let $\{w_1, \dots w_n\}$ a l.i in W $\exists$ $v_i \in V$ s.t

$T(v_i) = w_i$ Then $\{v_1, \dots, v_n\}$ l.i

soln: consider $\alpha_1 v_1 + \dots + \alpha_n v_n = 0$

$$T(\alpha_1 v_1 + \dots + \alpha_n v_n) = T(0) = \theta$$

$$\alpha_1 T(v_1) + \dots + \alpha_n T(v_n) = 0$$

$$\alpha_1 w_1 + \dots + \alpha_n w_n = 0$$

$$\Rightarrow \alpha_i = 0 \quad i = 1, \dots, n.$$

Hence they are linearly independant.

Kernal of a Linear Transformation. (Nullspace)

Th: $\ker$ or T or $N(T) = \{v \in V / T(v) = 0\}$ is a subspace of V.

Proof: let $v_1, v_2 \in V$

$T(v_1) = 0$, $T(v_2) = 0$

$T(v_1+v_2) = T(v_1)+T(v_2) = 0+0 = 0$

$T(\alpha v) = \alpha T(v_1) = \alpha . 0 = 0$.

Range of Linear Transform, T(R) & Image of LT

Th: $Im(T) = R(T) = \{w \in W / w = T(v), \text{ for some } v \in V\}$ is a subspace of W

Proof: $w_1, w_2 \in Im(T)$ $\exists$ $v_1, v_2 \in V$ $T(v_1) = w_1$. $T(v_2) = w_2$

$w_1 + w_2 = T(v_1) + T(v_2) = T(v_1+v_2) \Rightarrow w_1+w_2 \in Im(T)$

$\alpha w_1 = \alpha(T(v_1)) = T(\alpha v_1) \Rightarrow \alpha w_1 \in Im\, T$

Th: Let $T: V \rightarrow W$ T is one-one iff ~~kerT~~ $\ker T = \{0\}$

Proof: Part I

Suppose T is one-one. ~~Let $v_1, v_2 \in V$ such that~~

~~$T(v_1) =$~~

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If possible let there be $v \neq 0$ such that $T(v) = 0$,

But $T(0) = 0$

$$T(v) = T(0) \Rightarrow v = 0$$

$$\Rightarrow \ker T = \{0\}$$

Part II

Let $\ker T = \{0\}$

To prove T is one-one. Let us consider that

$v_1, v_2 \in V$ at $T(v_1) = T(v_2)$

$$T(v_1) - T(v_2) = 0 \in W$$

Since T is a L.T

$$T(v_1 - v_2) = 0 \in W$$

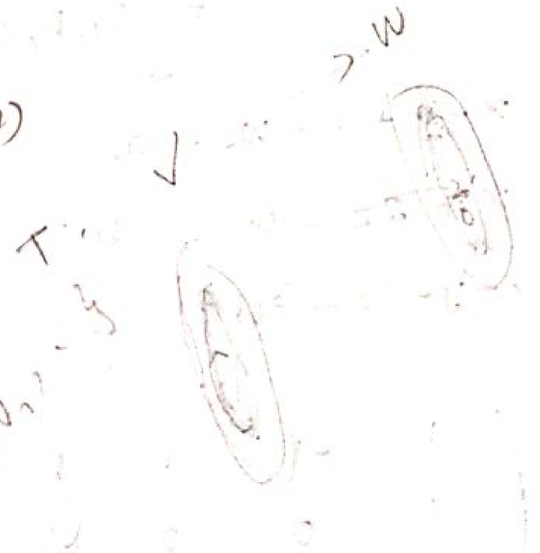
$v_1 - v_2 = 0 \in V$ [as zero element is mapped to zero element].

$\therefore v_1 = v_2$

If $T: V \to W$ T is one-one or $\ker(T) = \{0\}$ then $T(v_1), \ldots, T(v_n)$ is l.i whenever $v_1, v_2, \ldots, v_n$ is l.i

$T(v_1) = T(v_2)$

$v_1 = v_2$



Rank - Nullity Theorem

Rank of a linear transform $=\dim \mathrm{Im}(T)$

Nullity of a linear transform $=\dim \ker(T)$

Th: Let $T: V \to W$ then

$$\dim V = \dim(\mathrm{Im}\,T) + \dim(\ker T)$$

$$= \mathrm{Rank}(T) + \mathrm{Nullity}(T)$$

$$n = r + s.$$

Q Let $T: R^4 \to R^3$ be a linear transformation

$$T(x,y,z,w) = (x - y + z + w,\ x + 2z - w,\ x+y+3z-3$$

Find $\mathrm{Im}(T)$ and $\ker(T)$. Find rank and nullity of the linear transformation T.

Ans:

$$T(x,y,z,w) = (x - y + z + w,\ x + 2z - w,\ x + y + 3z - 3w)$$

$$= (a,b,c)$$

$$x - y + z + w = a$$

$$x + 2z - w = b.$$

$$x + y + 3z - 3w = c.$$

$$\left(\begin{array}{cccc:c} 1 & -1 & 1 & 1 & a \\ 1 & 0 & 2 & -1 & b \\ 1 & 1 & 3 & -3 & c \end{array}\right)$$

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$$\sim \left(\begin{array}{cccc|c} 1 & -1 & 1 & 1 & a \\ 0 & 1 & 1 & -2 & b-a \\ 0 & 2 & 2 & -4 & c-a \end{array}\right)$$

$$\sim \left(\begin{array}{cccc|c} 1 & -1 & 1 & 1 & a \\ 0 & 1 & 1 & -2 & b-a \\ 0 & 0 & 0 & 0 & a-2b+c \end{array}\right)$$

* For a solution to be present, i.e, system is consistent if

$$a-2b+c=0$$

let $c = k_1$, $b = k_2$.

$$a = 2k_2 - k_1$$

$$Im(T) = \left\{ \begin{pmatrix} 2k_2-k_1 \\ k_2 \\ k_1 \end{pmatrix} \right\} = k_1 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + k_2 \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$$

Hence all images can be represented by these 2 vectors

$\therefore$ Basis of $Im(T) = \{(-1,0,1), (2,1,0)\}$

$$\dim(Im(T)) = 2.$$

Kernel of T, i.e, $T(x,y,z,w) = (0,0,0)$

$$x-y+z+w=0$$
$$x+2z-w=0$$
$$x+y+3z-3w=0$$

$$\begin{pmatrix} 1 & -1 & 1 & 1 \\ 1 & 0 & 2 & -1 \\ 1 & 1 & 3 & -3 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & -1 & 1 & 1 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$x - y + z + w = 0$$

$$y + z - 2w = 0$$

$$w = k_1, \quad z = k_2.$$

$$x = y - z - w = y - k_1 - k_2.$$

$$y = -z + 2w = -k_2 + 2k_1$$

$$\therefore \quad x = -k_2 + 2k_1 - k_1 - k_2$$

$$x = k_1 - 2k_2.$$

$$\begin{pmatrix} k_1 - 2k_2 \\ -k_2 + 2k_1 \\ k_2 \\ k_1 \end{pmatrix} = k_1 \begin{pmatrix} 1 \\ +2 \\ 0 \\ 1 \end{pmatrix} + k_2 \begin{pmatrix} -2 \\ -1 \\ 1 \\ 0 \end{pmatrix}$$

Basis of $\ker(T) = \{(1,2,0,1), (-2,-1,1,0)\}$

$\dim \ker(T) = 2$

Rank of l.t = ~~dim~~ $\dim \operatorname{Im}(T) = 2$

Nullity of l.t = $\dim \ker(T) = 2.$

Q. Let $M_{2\times2}$ set of all 2×2 matrices.

$V = W = M_{2\times2}$.

$T: V \to W$, $T\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a+b & b+c \\ c+d & d+a \end{bmatrix}$

verify Rank-Nullity Theorem.

Ans: $T\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$

$$a+b = x$$
$$b+c = y$$
$$c+d = z$$
$$d+a = w$$

$$\left[\begin{array}{cccc:c} 1 & 1 & 0 & 0 & x \\ 0 & 1 & 1 & 0 & y \\ 0 & 0 & 1 & 1 & z \\ 1 & 0 & 0 & 1 & w \end{array}\right]$$

$$\sim \left[\begin{array}{cccc:c} 1 & 1 & 0 & 0 & x \\ 0 & 1 & 1 & 0 & y \\ 0 & 0 & 1 & 1 & z \\ 0 & -1 & 0 & 1 & w-x \end{array}\right]$$

$$\sim \left[\begin{array}{cccc:c} 1 & 1 & 0 & 0 & x \\ 0 & 1 & 1 & 0 & y \\ 0 & 0 & 1 & 1 & z \\ 0 & 0 & 1 & 1 & w-x+y \end{array}\right]$$

$$\sim \left[\begin{array}{cccc:c} 1 & 1 & 0 & 0 & x \\ 0 & 1 & 1 & 0 & y \\ 0 & 0 & 1 & 1 & z \\ 0 & 0 & 0 & 0 & w-x+y-z \end{array}\right]$$

for the system to be consistent if

$$w - x + y - z = 0$$

let $x = k_1,\ y = k_2,\ z = k_3$

$$w = x - y + z = k_1 - k_2 + k_3$$

$$\left\{ \begin{pmatrix} k_1 \\ k_2 \\ k_3 \\ k_1 - k_2 + k_3 \end{pmatrix} \right\} = k_1 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} + k_2 \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} + k_3 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$$

· Basis of $Im(T) = \{(1,0,0,1), (0,1,0,-1), (0,0,$

dimension = 3.

To find $ker(T)$, $T\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$a + b = 0$
$b + c = 0$
$c + d = 0$
$d + a = 0$

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$x + y = 0$ $\qquad z = k_1$, ~~$y = c_2$~~
$y + z = 0$
$z + w = 0$

$x = +c_1$
$y = -c_1$
$w = -c_1$

$$\cancel{\begin{pmatrix} c_1 \\ -c_1 \\ c_1 \\ -c_1 \end{pmatrix}}$$

$$\left\{\begin{pmatrix} k_1 & k_2 \\ k_3 & k_1-k_2+k_3 \end{pmatrix}\right\} = k_1\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + k_2\begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} + k_3\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$$

$$\ker(T) = \begin{pmatrix} c_1 & -c_1 \\ c_1 & -c_1 \end{pmatrix} = c_1\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$$

$\dim \ker(T) = 1$

rank of a l.t $= 3$

nullity of a l.t $= 1$.

$\therefore \dim U_{2\times 2} = 3+1 = 4$.

Inverse of a linear transformation

* If T is one-one and onto (bijective), T^{-1} exists.

Th: $T: V \to W$ is a linear transformation, which is bijective then $T^{-1}: W \to V$ is also linear transformation.

then, $T^{-1}(w_1 + w_2) = T^{-1}(w_1) + T^{-1}(w_2)$

$T^{-1}(\alpha w_1) = \alpha T^{-1}(w_1)$

Proof: Let $w_1, w_2 \in W$ $\exists$ v_1, v_2 such That $T(v_1) = w_1$, $T(v_2) = w_2$.

$$\left.\begin{aligned} v_1 &= T^{-1}(w_1) \\ v_2 &= T^{-1}(w_2) \end{aligned}\right\} \text{— ①}$$

$$w_1 + w_2 = T(v_1) + T(v_2)$$
$$= T(v_1 + v_2)$$

$$v_1 + v_2 = T^{-1}(w_1 + w_2)$$

$$T^{-1}(w_1 + w_2) = T^{-1}(w_1) + T^{-1}(w_2) \quad [\text{from } ①]$$

Hence first condition is satisfied

consider $\alpha w_1 = \alpha T(v_1)$
$$= T(\alpha v_1).$$

$$T^{-1}(\alpha w_1) = \alpha v_1 = \alpha T^{-1}(w_1)$$

Hence second condition is satisfied.

Q Consider a linear mapping, $T: R^3 \to R^3$

$$T(x, y, z) = (2x, 4x - y, 2x + 3y - z)$$

show T is invertible and find T^{-1}

Ans: (i) To show T is 1-1

$$T(x_1, y_1, z_1) = (2x_1, 4x_1 - y_1, 2x_1 + 3y_1 - z_1)$$

$$T(x_2, y_2, z_2) = (2x_2, 4x_2 - y_2, 2x_2 + 3y_2 - z_2)$$

$$2x_1 = 2x_2 \Rightarrow x_1 = x_2$$

$$4x_1 - y_1 = 4x_2 - y_2 \Rightarrow y_1 = y_2 \quad [\because x_1 = x_2]$$

$$2x_1 + 3y_1 - z_1 = 2x_2 + 3y_2 - z_2$$

$$\therefore z_1 = z_2 \qquad [\because x_1 = x_2 \;\&\; y_1 = y_2]$$

$\therefore$ The L.T is 1-1.

(ii) To show T is onto,

$$T(x, y, z) = (r, s, t) \quad \text{where } (r, s, t) \in R^3$$

$$r = 2x \Rightarrow x = \frac{r}{2}$$

$$4x - y = s \Rightarrow y = \frac{4r}{2} - s = 2r - s$$

$$2x + 3y - z = t \Rightarrow z = \frac{2r}{2} + 3(2r - s) - t =$$

$$= r + 6r - 3s - t$$

$$= 7r - 3s - t$$

$$T\left(\frac{r}{2}, 2r - s, 7r - 3s - t\right) = (r, s, t).$$

Hence T is onto.

Also,

$$T^{-1}(r, s, t) = \left(\frac{r}{2}, 2r - s, 7r - 3s - t\right)$$

Q Find a linear transformation whose kernel is spanned by $(1, 2)$ $\quad T : R^2 \to R^2$

soln: [Note :-

$T : V \to W$

Basis of $V = \{v_1, \ldots v_n\}$

specify images of basis, $T(v_1), T(v_2), \ldots, T(v_n)$]

If we can specify images of basis then we can find the L.T because the image of basis is unique.

Standard Basis $= \{(1,0),(0,1)\}$ for R^2.

$\ker T = \{k(1,2) / k \in R\} \subset R^2$

$T(1,2) = (0,0)$

choose $B = \{(1,2),(0,1)\}$

choose the image of $(0,1)$ as $T(0,1) = (2,1)$ randomly chosen, easy

Now, $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \qquad T(x,y)$.

$(x,y) = \alpha(1,2) + \beta(0,1)$

$= x(1,2) + (y-2x)(0,1)$

$T(x,y) = T[x(1,2)] + T[(y-2x)(0,1)]$

$= x\,T(1,2) + (y-2x)\,T(0,1)$

$= (y-2x)(2,1)$.

$= (2y-4x, \; y-2x)$

This l.t is not unique. This may change according to what we put here.

Q. Find a l.t from $R^4 \to R^3$ whose kernel is spanned by $(1,2,3,4)$ and $(0,1,1,-1)$.

Soln:- $\ker T = \{k(1,2,3,4), (0,1,1,-1) / k \in R\}$

$T(1,2,3,4) = T(0,1,1,-1) = (0,0,0)$

Let Basis $= \{(1,2,3,4), (0,1,1,-1), (0,0,1,0), (0,0,0,1)\}$ extend basis

Let $T(0,0,1,0) = (1,0,2)$

and $T(0,0,0,1) = (2,0,3)$

Now, $T: R^4 \longrightarrow R^3$

~~$(x, y) =$~~

$$(x, y, z, w) = x(1,2,3,4) + (y-2x)(0,1,1,-1) + (z-y+x)(0,0,1,0) + (w+y-6x)(0,0,0,1).$$

$$T(x,y,z,w) = T[x(1,2,3,4)] + (y-2x)\,T(0,1,1,-1) + (z-y+x)\,T(0,0,1,0) + (w+y-6x)\,T(0,0,0,1)$$

$$= x(0,0,0) + (y-2x)(0,0,0) + (z-y+x)(1,0,2) + (w+y-6x)(2,0,3)$$

$$= (z-y+x,\ 0,\ 2z-2y+x) + (2w+2y-12x,\ 0,\ 3w+3y-18x)$$

$$= (2w+2y-12x+z-y+x,\ 0,\ 2z-2y+x+3w+3y-18x)$$

$$T(x,y,z,w) = (-11x+y+z+2w,\ 0,\ -17x+y+2z+3w)$$

Q. Find a l.t whose image is spanned by (1,1,1)
$T: R^2 \to R^3$

Soln. Basis of $R^2 = \{(1,0), (0,1)\}$

$T(1,0) = (1,1,1)$

$T(0,1) = (1,2,1)$

$(x,y) = x(1,0) + y(0,1)$

$T(x,y) = x\,T(1,0) + y\,T(0,1)$

$= x(1,1,1) + y(1,2,1)$

$T(x,y) = (x+y,\ x+2y,\ x+y)$

Note ... not ... If we take a basis ... (1,1) and ... then we ...

$T(1,2) = ($ ...

$T(1,1) = ($ ...

$(x,y) = x($ ...

Q. Find a l.t : $T: R^3 \to R^3$ whose image is spanned (1,2,3), (4,5,6).

Basis $= \{(1,0,0), (0,0,1), (0,1,0)\}$

$T(1,0,0) = (1,2,3)$

$T(0,1,0) = (4,5,6)$

$T(0,0,1) = (1,2,3)$ This ... be ... comb... and (...

$(x,y) = x(1,0,0) + y(0,1,0) + z(0,0,1)$

$T(x,y) = x\,T(1,0,0) + y\,T(0,1,0) + z\,T(0$ ...

$= x(1,2,3) + y(4,5,6) + z(1,2,$ ...

$= (x + 4y + z,\ 2x + 5y + 2z,\ 3x +$ ...

Q. Find a l.t whose image is spanned by (1,1,1) and (1,2,1)
$T: R^2 \to R^3$

Note- dimension of zero subspace is 0 — not related, just a random note

Soln: Basis of $R^2 = \{(1,0), (0,1)\}$

If we take a different basis suppose (1,1) and (1,2) then our l.t

$T(1,0) = (1,1,1)$

$T(0,1) = (1,2,1)$

$(x,y) = x(1,0) + y(0,1)$

$T(x,y) = x\,T(1,0) + y\,T(0,1)$

$= x(1,1,1) + y(1,2,1)$

$T(1,2) = (1,1,1)$
$T(1,1) = (1,2,1)$

$\underline{T(x,y) = (x+y,\ x+2y,\ x+y)}$

$(x,y) = x(1,2) + \ldots (1,1)$

Q. Find a l.t $: T: \to R^3 \to R^3$ whose image is spanned by (1,2,3), (4,5,6).

Basis $= \{(1,0,0), (0,0,1), (0,1,0)\}$

$T(1,0,0) = (1,2,3)$

$T(0,1,0) = (4,5,6)$

$T(0,0,1) = (1,2,3)$

This vector should be a linear combination of (1,2,3) and (4,5,6) and should not be l.i because if it is l.i then the image will be spanned by 3 vectors in the Q whereas they say only the given two vectors.

$(x,y) = x(1,0,0) + y(0,1,0) + z(0,0,1)$

$T(x,y) = x\,T(1,0,0) + y\,T(0,1,0) + z\,T(0,0,1)$

$= x(1,2,3) + y(4,5,6) + z(1,2,3)$

$= \underline{(x+4y+z,\ 2x+5y+2z,\ 3x+6y+3z)}$

Eg:

Matrix of a linear transformation

Eg: $T : R^3 \longrightarrow R^3$

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x + y + z \\ 2x + y + z \\ x + 2y + 3z \end{pmatrix}$$

$$\underset{X}{\begin{pmatrix} x \\ y \\ z \end{pmatrix}} \underset{A}{\begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix}}$$

Basis standard $= \{(0,0,1), (0,1,0), (1,0,0)\}$

$$T\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

$$T\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

$$T\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$$

Generally,

$T : R^n \longrightarrow R^m$

$A X \in R^m$ — A: $n \times m$, X: $n \times 1$.

$T(X) = A\,\underline{x}$

$$T\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = AX = \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix}$$

ld

is 1·1 image spanned as whereas they say the given vectors.

Inner Product

Let V be a vector space over R, $v_1, v_2 \in V$ $\quad V \times V \to R$

$\langle v_1, v_2 \rangle$ – inner product of v_1 and v_2 real number.

Satisfying the following property

i) $\langle v, v \rangle \geq 0$ (positivity)

$\langle v, v \rangle = 0$ if and only if $v = 0$

ii) $\langle V_1, V_2 \rangle = \langle V_2, V_1 \rangle$; i.e., there is symmetry

iii) $\langle V_1 + V_2, W \rangle = \langle V_1, W \rangle + \langle V_2 + W \rangle$ (additivity)

iv) $\langle \alpha V_1, V_2 \rangle = \alpha \langle V_1, V_2 \rangle$ (homogenity)

A vectorspace that has an inner product defined over it is called an inner product space.

Note:-

Q Let R^n be the vector space

(a) $\langle u, v \rangle = u^T v$ $\quad u = \begin{pmatrix} u \\ \vdots \\ u_n \end{pmatrix}, \; v = \begin{pmatrix} v \\ \vdots \\ v_n \end{pmatrix}$

$= u_1 v_1 + u_2 v_2 + \dots + u_n v_n$,

soln:- (a) $\langle v, v \rangle = v_1^2 + \dots + v_n^2 \geq 0$

$= 0 \Rightarrow v_i = 0 \Rightarrow v = 0$

(b) $\langle u, v \rangle = u_1 v_1 + \dots + u_n v_n$

$= v_1 u_1 + \dots + v_n u_n = \langle v, u \rangle$

(c) $\langle u + v, w \rangle = (u_1 + v_1) w_1 + \dots (u_n + v_n) w_n$

$= (u_1 w_1 + u_2 w_2 + \dots u_n w_n)$

$+ (v_1 w_1 + \dots + v_n w_n)$

$= \langle u, w \rangle + \langle v, w \rangle$

(d) $\langle \alpha u, v \rangle = \alpha u_1 v_1 + \dots + \alpha u_n v_n$

$= \alpha (u_1 v_1 + \dots + u_n v_n)$

$= \alpha . \langle u, v \rangle$

$\therefore \langle u, v \rangle$ is an inner product

8. R^2 be a vector space

$<u, v> = u_1 v_1 - u_2 v_1 - u_1 v_2 + u_2 v_2$

where $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$

soln: (a) $<v, v> = v_1^2 - v_1 v_2 - v_1 v_2 + v_2^2$

$= (v_1 - v_2)^2 \geq 0$

and $<v, v> = 0 \Rightarrow v_1 = v_2$

Hence since it violates the condition that $<v, v> = 0$ only if $v = 0$

(b) $< u, v > = u_1 v_1 - u_2 v_1 - u_1 v_2 + u_2 v_2$

$< v, u > = v_1 u_1 - v_2 u_1 - v_1 u_2 + v_2 u_2$.

$\therefore <u, v> = <v, u>$

(c) $< u + v, w > = u$

8. $V = R^2$, $u, v \in R^2$ $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$, $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$

$< u, v > = u_1 v_1 - u_2 v_1 + 4 u_2 v_2 - u_2 v_1$

soln:

(a) $< v, v > = v_1^2 - 2v_1 v_2 + 4 v_2^2$

$= (v_1 - v_2)^2 + 3v_2^2 \geq 0$

$<v, v> = 0 \Rightarrow v_2 = 0$

$v_1 = v_2 = 0 \Rightarrow v_1 = 0$

(b) $\langle u, v \rangle = u_1 v_1 - u_1 v_2 - u_2 v_1 + 4 u_2 v_2$

$\langle v, u \rangle = v_1 u_1 - v_1 u_2 - v_2 u_1 + 4 v_2 u_2$

$= \langle u, v \rangle$

(c) $\langle u + v, w \rangle = (u_1 + v_1) w_1 - 2(u_1 + v_1) w_2 - (u_2 + v_2) w_1 + 4(u_2 + v_2) w_2$

$\langle v, w$ $\cancel{= u_1 w_1 + v_1 w_1 - 2u_1 w_2 - 2v_1 w_2 + 4u_2 w_2 + 4v_2 w_2}$

$\cancel{= (u_1 w_1 - 2u_1 w_2 + 4u_2 w_2) + (v_1 w_1 - 2v_1 w_2 + 4v_2 w_2)}$

$= u_1 w_1 + v_1 w_1 - (u_1 w_2) + v_1 w_2 - u_2 w_1 - v_2 w_1 + 4u_2 w_2 + 4v_2 w_2$

$= (u_1 w_1 - u_1 w_2 - u_2 w_1 + 4u_2 w_2) + (v_1 w_1 - v_1 w_2 - v_2 w_1 + 4v_2 w_2)$

$= \langle u + w \rangle + \langle v, w \rangle$

(d) $\langle \alpha u, v \rangle = \alpha u_1 v_1 - \alpha u_1 v_2 - \alpha u_2 v_1 + 4 \alpha u_2 v_2$

$= \alpha (u_1 v_1 - u_1 v_2 - u_2 v_1 + 4 u_2 v_2)$

$= \alpha \langle u, v \rangle$

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Q. V = set of all square matrices of order n

$$\langle A, B\rangle = \text{Trace}(A^T B)$$

$$= \text{sum of diagonal elements of } (A^T B)$$

soln: (a) $\langle A, A\rangle = \text{Trace}(A^T . A)$

Eg: $\overset{A^T}{\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}} \overset{A}{\begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix}}$

$$= \begin{bmatrix} a_{11}^2 + a_{12}^2 & a_{11}a_{21} + a_{12}a_{22} \\ a_{21}a_{11} + a_{22}a_{12} & a_{21}^2 + a_{22}^2 \end{bmatrix}$$

$$\langle A, A\rangle = \sum_{j=1}^{n} \sum_{i=1}^{n} a_{ij}^2 \geq 0.$$

$$\langle A, A\rangle = 0 \text{ if } a_{ij} = 0$$

$$\Rightarrow A = 0$$

(b) $\langle A, B\rangle = \text{Trace}(A^T B)$

$$= \text{Trace}(B^T \cdot A) = \langle B, A\rangle$$

Eg: $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$

$$= \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix}$$

(b

(c) $\langle A+B, C \rangle = \langle (A+B)^T C \rangle$

$= \langle (A^T + B^T) C \rangle$

$= \langle A^T C + B^T C \rangle$

$= \langle A, C \rangle + \langle B, C \rangle$

(d) $\langle \alpha A, B \rangle = \langle (\alpha A)^T . B \rangle$

$= \alpha \langle A^T B \rangle$

$= \alpha \langle A, B \rangle$.

Hence proved.

Q

V = set of all continuous real valued functions defined on $[0,1]$, $f, g \in V$

$$\langle f, g \rangle = \int_0^1 f(t) \cdot g(t) \cdot dt$$

(a) $\langle f, f \rangle = \int_0^1 [f(t)]^2 \cdot dt \geq 0$

$\langle f, f \rangle = 0$ unless $f(t) = 0$

(b) $\langle f, g \rangle = \int_0^1 f(t)\, g(t) \cdot dt$

$= \int_0^1 g(t)\, f(t) \cdot dt = g(t) \cdot dt$

(c) $\langle f+g, h \rangle = \int_0^1 (f+g)(t) \cdot h(t) \cdot dt$

$= \int_0^1 [f(t)\, h(t) + g(t)\, h(t)]\, dt$

$= \langle f, h \rangle + \langle g, h \rangle$

(d) $\langle \alpha f, g \rangle = \alpha \int_0^1 f(t)\, g(t)\, dt = \alpha \langle f, g \rangle$

Length of a vector v,

$$\boxed{\|v\|^2 = \langle v, v\rangle} \quad \text{(norm of } v\text{)}.$$

$$|v|^2 = \langle v, v\rangle = v^T v = v \cdot v$$

$$\|v\| = 0 \text{ iff } v = 0$$

Properties associated with norm

(i) $\|v\| = 0$ iff $v = 0$

(ii) $\|\alpha v\| = |\alpha| \, \|v\|$

Proof: $\|\alpha v\|^2 = \langle \alpha v, \alpha v\rangle$

$= \alpha \langle v, \alpha v\rangle$

$= \alpha \cdot \alpha \langle v, v\rangle$

$= \alpha^2 \langle v, v\rangle$

$= \alpha^2 \|v\|$

$\|\alpha v\| = |\alpha| \, \|v\|$

(iii) $\|v_1 + v_2\| \leq \|v_1\| + \|v_2\|$ — triangle inequality

Proof: Let u, v be two vectors in the vectorspace and t is a scalar

$\|tu + v\|^2 = \langle tu + v, tu + v\rangle$

$\cancel{t^2 \langle u, u\rangle} = t^2 \langle u, u\rangle + 2t \langle u, v\rangle + \langle v, v\rangle \geq 0$

Quadratic expression in t

Note:- If $ax^2+bx+c \geq 0$

then, $D = b^2-4ac \leq 0$

$$4 <u,v>^2 - 4<u,u><v,v> \leq 0$$

$$<u,v>^2 \leq \|u\|^2\|v\|^2 \quad \text{——— (1)}$$

$$<u,v> \leq \|u\|\|v\| \quad \text{or} \quad <v_1,v_2> \leq \|v_1\|\|v_2\|$$

$$\|v_1+v_2\|^2 = <v_1+v_2,\ v_1+v_2>$$

$$= <v_1,v_1> + 2<v_1,v_2> + <v_2,v_2>$$

$$= \|v_1\|^2 + 2\|v_1\|\|v_2\| + \|v_2\|^2$$

$$= (\|v_1\| + \|v_2\|)^2$$

$$\|v_1+v_2\| \leq \|v_1\| + \|v_2\|.$$

(iv) $\|v_1+v_2\|^2 = \|v_1\|^2 + \|v_2\|^2$

if $<v_1, v_2> = 0$

Note:-

v_1, v_2 are said to orthogonal when $<v_1,v_2> = 0$.

Eg: R^2 $\begin{pmatrix}1\\0\end{pmatrix} \begin{pmatrix}0\\1\end{pmatrix}$ are orthogonal. $\quad <u,v> = u^Tv$

$\begin{pmatrix}1\\1\end{pmatrix} \begin{pmatrix}1\\-1\end{pmatrix}$ are orthogonal $\}$ l.i

A set of orthogonal vectors which does not include 0 element is l.i

$\begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}-2\\1\end{pmatrix}$ orthogonal

$\left\{\begin{pmatrix}1\\3\end{pmatrix}, \begin{pmatrix}-3\\1\end{pmatrix}, \begin{pmatrix}0\\0\end{pmatrix}\right\}$ mutually orthogonal

Any set which includes the zero element cannot be l.d ~~l.i~~

orthogonal set

Hence this set is l.d set

let V be a v.s

let $\{v_1, \ldots v_n\}$ be a set of nonzero orthogonal vectors

Then $\{v_1, \ldots, v_n\}$ is a l.i set

Proof: $\alpha_1 v_1 \ldots + \alpha_n v_n = 0.$

$< \alpha_1 v_1 + \ldots \alpha_n v_n , v_i > = <0, v_i> = 0$

$\alpha_1 <v_1, v_i> + \ldots \alpha_i <v_i, v_i> + \ldots \alpha_n <v_n, v_n>$

$\alpha_i <v_i, v_i> = 0$

$\alpha_i = 0$

$\alpha_n = 0, \quad i = 1, \ldots, n.$

Gram - Schmidt orthogonalization process

Given $u_1, \ldots, u_n$ in an inner product space.

To construct n orthogonal vectors

Let $v_1 = u_1$

$$v_2 = u_2 - \frac{<u_2, v_1>}{<v_1, v_1>} v_1$$

$$\Big[<v_2, v_1> = <u_2 - kv_1, v_1>$$

$$= <u_2, v_1> - k <v_1, v_1>$$

$$= <u_2, v_1> - <\frac{u_2, v_1}{v_1, v_1}> <v_1, v_1> = 0 \Big]$$

$$V_3 = u_3 - \frac{\langle u_3, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1 - \frac{\langle u_3, v_2 \rangle}{\langle v_2, v_2 \rangle} v_2$$

$\vdots$

$$v_n = u_n - \frac{\langle u_n v_1 \rangle}{\langle v_1, v_1 \rangle} v_1 - \frac{\langle u_n v_2 \rangle}{\langle v_2 v_2 \rangle} \cdot v_2 - \dots - \frac{\langle u_n, v_{n-1} \rangle}{\langle v_{n-1}, v_{n-1} \rangle} v_{n-1}$$

$\{v_1, v_2, \dots v_n\}$ is a set of orthogonal vectors.

$$\langle v_i, v_j \rangle = 0 \quad , i \neq j$$

$w_i = \frac{v_i}{\|v_i\|}$ $\quad \{w_1, \dots w_n\}$ is an orthonormal vectors.

Eg of orthogonal set

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} -1 \\ -1 \end{pmatrix} ; \quad \begin{pmatrix} 2 \\ 1 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

Eg: of orthonormal set

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$\begin{pmatrix} 2 \\ 1 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \end{pmatrix}$ can be made orthonormal by

$$\begin{pmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{pmatrix} \begin{pmatrix} -1/\sqrt{5} \\ 2/\sqrt{5} \end{pmatrix}$$

Q: Construct an orthonormal set of vectors from $u_1 = (1,1,1)$
$u_2 = (-1, 1, 0)$, $u_3 = (1, 2, 1)$

Ans:

$$v_1 = u_1 = (1, 1, 1)$$

$$v_2 = u_2 - \frac{\langle u_2, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1$$

$$= (-1, 1, 0) - 0\, v_1$$

$$= (-1, 1, 0)$$

$$V_3 = U_3 - \frac{\langle U_3, V_1\rangle}{\langle V_1, V_1\rangle} V_1 - \frac{\langle U_3, V_2\rangle}{\langle V_2, V_2\rangle} V_2$$

$\langle U_3, V_1\rangle = 4$,

$\langle V_1, V_1\rangle = 3$

$\langle U_3, V_2\rangle = 1$.

$\langle V_2, V_2\rangle = 2$

$$V_3 = (1,2,1) - \frac{4}{3}(1,1,1) - \frac{1}{2}(-1,1,0)$$

$$= (1,2,1) - \left(\frac{4}{3}, \frac{4}{3}, \frac{4}{3}\right) - \left(-\frac{1}{2}, \frac{1}{2}, 0\right)$$

$$= \left(1 - \frac{4}{3} + \frac{1}{2},\ 2 - \frac{4}{3} - \frac{1}{2},\ 1 - \frac{4}{3}\right)$$

$$= \left(\frac{6-8+3}{6},\ \frac{12-8-3}{6},\ -\frac{1}{3}\right)$$

$$V_3 = \left(\frac{1}{6}, \frac{1}{6}, -\frac{1}{3}\right)$$

$$W_1 = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$

$$\frac{1}{36} + \frac{1}{36} + \frac{1}{9}$$

$$W_2 = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$$

$$\frac{2}{36} + \frac{1}{9}$$

$$\frac{1}{18} + \frac{1}{9}$$

$$\cancel{W_3 = \left(\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)}$$

$$\frac{1+2}{18}$$

$$W_3 = \left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{\sqrt{2}}{\sqrt{3}}\right)$$

Q Find an orthonormal basis for the subspace that

$$W = \{(x, y, z) / x + 2y - z = 0\}$$

Ans:

Let $z = k_1$

$y = k_2$

$x = -2k_2 + k_1$

$$\begin{pmatrix} -2k_2 + k_1 \\ k_2 \\ k_1 \end{pmatrix} = k_1 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + k_2 \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$$

$$\therefore \quad \text{Basis} = \{ \underset{u_1}{(1,0,1)}, \underset{u_2}{(-2,1,0)} \}$$

To get orthogonal basis,

$$v_1 = u_1 = (1,0,1)$$

$$v_2 = u_2 - \frac{\langle u_2, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1$$

$$= (-2,1,0) - (-1)(1,0,1)$$

$$= (-2,1,0) - (-1,0,-1) = (-1,1,1)$$

Orthonormal basis,

$$\left\{ \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right), \left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right) \right\}$$

Eigen Values and Vectors

$$T : R^n \longrightarrow R^n$$

$$T(x) = \lambda x$$

$$x \in R^n \quad T\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \longrightarrow \begin{pmatrix} \lambda x_1 \\ \vdots \\ \lambda x_n \end{pmatrix}$$

$$T(X) = AX = \lambda X$$

$$AX = \lambda X$$

$$\begin{pmatrix} a_{11}x_1 + \dots + a_{1n}x_n \\ \vdots \\ a_{n1}x_1 \dots \dots a_{nn}x_n \end{pmatrix} = \begin{pmatrix} \lambda x_1 \\ \vdots \\ \lambda x_n \end{pmatrix}$$

$$AX - \lambda X = 0$$

$$(A - \lambda I)X = 0$$

For this system to have a non-trivial solution

$$|A - \lambda I| = 0$$

→ polynomial of degree n. This polynomial is called characteristic polynomial

The roots of this polynomial is called latent roots / Eigen values

Eg: consider $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$

then $A - \lambda I = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix}$$

$$= (1-\lambda)^2 - 4$$

$$= 1 + \lambda^2 - 4 - 2\lambda$$

$$= \lambda^2 - 2\lambda - 3. \quad \leftarrow \text{characteristic polynomial}$$

so this a 2 degree polynomial

$$\lambda^2 - 2\lambda - 3 = 0$$

$$\lambda^2 - 3\lambda + \lambda - 3 = 0$$

$$\lambda(\lambda - 3) + 1(\lambda - 3) = 0$$

$$\therefore \lambda = -1, 3 \quad \text{eigen values}$$

vectors vectors

For $\lambda = 3$

$$A - 3I = \begin{pmatrix} 1-3 & 2 \\ 2 & 1-3 \end{pmatrix} = \begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix}$$

$$(A - 3I)X = 0$$

$$\begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -2x_1 + 2x_2 \\ 2x_1 - 2x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$x_1 = x_2 = K$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} K \\ k \end{pmatrix} \rightarrow K\underbrace{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}_{\text{Eigen vector}}$$

For $\lambda = -1$

$$A + I = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$$

$$(A + I)X = 0$$

$$\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$$\begin{pmatrix} 2x_1 + 2x_2 \\ 2x_1 + 2x_2 \end{pmatrix} = 0 \qquad x_1 = -x_2$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} k \\ -k \end{pmatrix} = k\overbrace{\begin{pmatrix} 1 \\ -1 \end{pmatrix}}^{\text{Eigen vector}}$$

① ~~distinct~~ eigen values and linearly independant eigen vectors.

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \qquad \lambda = -1,\ 3 \text{ (distinct)}$$

(previous example)

$$\begin{pmatrix} 1 \\ -1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{l.i}$$

② Repeated eigen values l.i eigen vectors.

$$X \neq 0 \qquad A = \begin{pmatrix} 3 & -2 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$

$$AX = \lambda X$$

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 3-\lambda & -2 & 0 \\ -2 & 3-\lambda & 0 \\ 0 & 0 & 5-\lambda \end{vmatrix} = 0$$

$$(3-\lambda)\left[(5-\lambda)(3-\lambda)\right] + 2\left[-2(5-\lambda)\right] = 0$$

$$(3-\lambda)\left[15 - 8\lambda + \lambda^2\right] - 4\left[(5-\lambda)\right] = 0$$

$$(3-\lambda)\left[15 - 8\lambda + \lambda^2\right] - 20 - 4\lambda = 0$$

$$\cancel{(\lambda^2 - 12\lambda - 5)(3-\lambda) = 0}$$

$$-\lambda^3 + 11\lambda^2 - 35\lambda + 25 = 0$$

By inspection, $\lambda = 1$.

$$\lambda \cdot 1 \quad (\lambda - 1)(-\lambda^2 + 10\lambda - 25) = 0$$

$$(\lambda - 1)(\lambda^2 - 10\lambda + 25) = 0$$

$$(\lambda - 1)(\lambda - 5)^2 = 0$$

$\lambda = 1, 5, 5$

(only 2 distinct eigen values).

$$\begin{array}{r|l} & -\lambda^2 + 10\lambda - 25 \\ \hline \lambda - 1 & -\lambda^3 + 11\lambda^2 - 35\lambda + 25 \\ & (-)\ \lambda^3 \ (+)\ \lambda^2 \\ \hline & 10\lambda^2 - 35\lambda \\ & (-)\ 10\lambda^2 \ (+)\ 10\lambda \\ \hline & -25\lambda + 25 \\ & -25\lambda \end{array}$$

Eigen vector corresponding to $\lambda = 1$,

$$\begin{pmatrix} 2 & -2 & 0 \\ -2 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$2x_1 - 2x_2 = 0 \Rightarrow x_1 - x_2 = 0$

$4x_3 = 0 \Rightarrow x_3 = 0$

$\therefore$ vector, $V = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

Eigen vector corresponding to $\lambda = 5$

$$\begin{pmatrix} -2 & -2 & 0 \\ -2 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\sim \begin{pmatrix} -1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$\therefore -x_1 - x_2 = 0$

$x_1 = -x_2.$

Let $x_3 = K_1$

$x_2 = K$

$x_1 = -K_2$

$\therefore$ Eigen vectors

$$\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

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$\therefore$ The vectors are, $\begin{pmatrix}1\\1\\0\end{pmatrix} \begin{pmatrix}-1\\1\\0\end{pmatrix} \begin{pmatrix}0\\0\\1\end{pmatrix}$

These vectors are linearly independant

$\Rightarrow$ Algebraic multiplicity of eigen value (λ_a)

= No. of times eigen value occurs.

For eg :- in the above problem, the algebraic multiplicity of $\lambda = 5$ is 2.

* Eigen space is the space spanned by the vectors corresponding to a particular eigen value.

$\Rightarrow$ Geometric multiplicity (λ_g) = dimension of the eigen space

For $\lambda = 5$, eigen space is spanned by

$\{(1,1,0), (0,0,1)\}$

$\therefore$ dimension = 2.

③ <u>Repeated Eigen values but insufficient eigen vectors.</u>

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

$$|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 2-\lambda & 1 & 0 \\ 0 & 2-\lambda & 1 \\ 0 & 0 & 2-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)^3 - 1(0) = 0$$

$$(2-\lambda)^3 = 0 \quad \Rightarrow \quad \lambda = 2, 2, 2$$

For $\lambda = 2$

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$x_2 = x_3 = 0$ Let $x_1 = k$

Eigen vector

$$\begin{pmatrix} k \\ 0 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \lambda_g = 1.$$

Characteristic equation

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$$

$$(1-\lambda)^2 - 4 = 0$$

$$\lambda^2 - 2\lambda - 3 = 0$$

Note:- Cayley - Hamilton Theorem.

Every square matrix satisfies its characteristic equation.

Eq: $A^2 - 2A - 3I = 0$, $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

Note:-

characteristic equation. $P_0 \lambda^n + P_1 \lambda^{n-1} + \dots + P_n = 0$

sum of eigen value $= -P_1/P_0$

Product of eigen value $= P_n/P_0 = |A|$

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RESULTS :

(i) If λ is an eigen value of A, $\bar{\lambda}$ is an eigen value of $\bar{A}$

(ii) A and A^T have the same eigen values

(iii) If λ is an eigen value of A, then $k\lambda$ is an eigen value of kA.

(iv) If λ is an eigen value of A, then $\lambda - k$ is an eigen value of $(A - kI)$ $(k \in \mathbb{R})$.

(v) If λ, $\lambda \neq 0$ is a non-singular matrix A, then $\frac{|A|}{\lambda}$ is eigen vector of adj A

(vi) If A and B are square matrixes, and A is invertible, i.e, $|A| \neq 0$, then matrix B and $A^{-1}BA$ have the same eigen vectors.

(vii) Eigen values of a Hermitian matrix are real.

(viii) Eigen values of a real symmetric matrix are real.

(ix) Eigen values of a skew symmetric matrix are purely imaginary.

(x) Eigen values of a unitary matrix are of absolute value 1. $(A^{-1} = \bar{A}^{T})$

(xi) Eigen values of an orthogonal matrix are of ab. value 1

$(A^{-1} = A^T)$

(xii) If A is triangular matrix, eigen value of A are principal diagonal elements of A.

PROOF :

(i) $|A - \lambda I| = 0$

Now, $|\bar{A} - \bar{\lambda} I| = |\overline{(A - \lambda I)}|$

$\therefore \; |\overline{(A - \lambda I)}| = 0$

$|\bar{A} - \overline{\lambda I}| = 0$

$|\bar{A} - \bar{\lambda} I| = 0$

$\Rightarrow \; \bar{\lambda}$ is an eigen value of $\bar{A}$.

(ii) Given $|A - \lambda I| = 0$

consider $|A^T - \lambda I|$

$= |(A - \lambda I)^T|$

$= |A - \lambda I| \qquad [\because |A| = |A^T|]$

$= 0$

$\therefore \; \lambda$ is an eigen value of A^T

(iii) Given $|A - \lambda I| = 0$

consider $|KA - K\lambda I|$

$= |K(A - \lambda I)|$

$= K|A - \lambda I| = 0$

$\therefore \; K\lambda$ is an eigen value of KA.

(iv) $\left|\left((A-KI)-(\lambda-K)I\right)\right|$ Given $|A-\lambda I|=0$

$= |A-\lambda I|$

$= 0$

(v) If $\lambda \neq 0$ is an eigen value of A, $\frac{|A|}{\lambda}$ is an eigen value of adj A.

Proof: Since $\lambda \neq 0$ is an eigen value of A.

$|A-\lambda I| = 0$

$|\text{adj} A|\,|A-\lambda I| = 0$

Note:-
$|A|\,|B| = |AB|$
$|A| + |B| \neq |A+B|$

$|(\text{adj} A)(A-\lambda I)| = 0$

$|(\text{adj} A).A - (\text{adj} A)\lambda I| = 0$

$\left||A| I - (\text{adj} A)\lambda I\right| = 0$

$\left|\frac{|A|}{\lambda} I - (\text{adj} A)\right| = 0$

$\left|\text{adj} A - \frac{|A|}{\lambda} I\right| = 0$

$\therefore \frac{|A|}{\lambda}$ is an eigen value of adj A.

Or $AX = \lambda X$

$(\text{adj} A).AX = (\text{adj} A)\lambda X$

$|A| X = \lambda (\text{adj} A) X$

2\. Let $A = \begin{pmatrix} -2 & 4 & 3 \\ 0 & 0 & 0 \\ -1 & 5 & 2 \end{pmatrix}$. compute $A^{593} - 2A^{15}$

Ans:- characteristic equation, $|A - \lambda I| = 0$

$$\begin{vmatrix} -2-\lambda & 4 & 3 \\ 0 & -\lambda & 0 \\ -1 & 5 & 2-\lambda \end{vmatrix} = 0$$

$$(-2-\lambda)[-\lambda(2-\lambda)] - 4[0] + 3[-\lambda] = 0$$

$$-\lambda(-2-\lambda)(2-\lambda) - 3\lambda = 0$$

$$+\lambda(2+\lambda)(2-\lambda) - 3\lambda = 0$$

$$\lambda(4-\lambda^2) - 3\lambda = 0$$

$$4\lambda - \lambda^3 - 3\lambda = 0$$

$$-\lambda^3 + \lambda = 0 \quad [\because -A^3 + A = 0]$$

$$\lambda(-\lambda^2+1) = 0$$

$\lambda = 0$, $\lambda = \pm 1$ are the eigen values.

consider $\lambda^{593} - 2\lambda^{15} = (-\lambda^3 + \lambda)q(\lambda) + r(\lambda)$

$$r(\lambda) = a_0 + a_1\lambda + a_2\lambda^2$$

$$\therefore \lambda^{593} - 2\lambda^{15} = (-\lambda^3 + \lambda)q(\lambda) + (a_0 + a_1\lambda + a_2\lambda^2)$$

Put $\lambda = 0$

$$0 = 0 + a_0$$

$$\underline{a_0 = 0}$$

Put $\lambda = 1$

$$-1 = 0 + a_0 + a_1 + a_2$$

$$-1 = a_1 + a_2 \quad \text{—— (1)}$$

Put $\lambda = -1$

$$1 = 0 + a_0 - a_1 + a_2$$

$$1 = -a_1 + a_2. \quad \text{—— (2)}$$

solving (1) and (2)

$$\cancel{a_1} + a_2 = -1$$
$$-\cancel{a_1} + a_2 = 1$$

$$2a_2 = 0 \Rightarrow \underline{\underline{a_2 = 0}}$$

$$a_1 = \underline{\underline{-1}}$$

$$r(\lambda) = -\lambda$$

Now, $A^{593} - 2A^{15} = (-A^3 + A)\,q(A) + r(A)$

$$= 0 + (-A)$$

$$= -A$$

$$= \begin{pmatrix} 2 & -4 & -3 \\ 0 & 0 & 0 \\ 1 & -5 & -2 \end{pmatrix}$$

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Q: Compute A^{735} if $A = \begin{bmatrix} 2 & 5 \\ 1 & -2 \end{bmatrix}$

Ans:- characteristic equation. $|A - \lambda I| = 0$

$$\begin{vmatrix} 2-\lambda & 5 \\ 1 & -2-\lambda \end{vmatrix} = 0$$

$$-(2-\lambda)(2+\lambda) - 5 = 0$$

$$4 - \lambda^2 + 5 = 0$$

$\lambda = +3, -3.$ are the eigen values.

consider $\lambda^{735} = (\lambda^2 - 9)\, q(\lambda) + r(\lambda)$

$$r(\lambda) = a_0 + a_1 x.$$

$$\lambda^{735} = (\lambda^2 - 9)\, q(\lambda) + (a_0 + a_1 \lambda)$$

Let $\lambda = 3$

$$3^{735} = a_0 + 3a_1$$

Let $\lambda = -3.$

$$-3^{735} = (0) + a_0 - 3a_1.$$

$$a_0 + 3a_1 = 3^{735}$$

$$a_0 - 3a_1 = -3^{735}$$

$$2a_0 = 0$$

$$a_0 = 0$$

$$a_1 = 3^{734}.$$

$$\lambda^{735} = 3^{734} \cdot \lambda$$

Now,

$$A^{735} = 3^{734} A.$$

Note:-

* Sum of eigen values = Trace (A)
* Product of eigen values = $|A|$.

Q Find an eigen vector of the matrix $\begin{bmatrix} \frac{1}{3} & \frac{1}{2} \\ \frac{2}{3} & \frac{1}{2} \end{bmatrix}$ corresponding to $\lambda = 1$.

Ans: If λ is an eigen value of A then,

$$AX = \lambda X$$

$$(A - \lambda I)X = 0$$

$$\begin{pmatrix} \frac{1}{3} - 1 & \frac{1}{2} \\ \frac{2}{3} & \frac{1}{2} - 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -\frac{2}{3} & \frac{1}{2} \\ \frac{2}{3} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -\frac{2}{3} & \frac{1}{2} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$-\frac{2}{3} x_1 + \frac{x_2}{2} = 0$$

$$-4x_1 + 3x_2 = 0 \Rightarrow \frac{3}{4} x_2 = x_1$$

If $x_2 = k$.

$x_1 = \frac{3}{4}k$.

then the vector, $\begin{pmatrix} \frac{3}{4}k \\ k \end{pmatrix}$

Eg: $\begin{pmatrix} 3/4 \\ 1 \end{pmatrix}, \begin{pmatrix} 3/2 \\ 2 \end{pmatrix}$

dimension $= 1$. (because only one vector is required to form other vectors, i.e, $\begin{pmatrix} 3/4 \\ 1 \end{pmatrix}$)

Note:- To get other eigen value

$$\lambda_1 + \lambda_2 = \frac{5}{6}$$

$$1 + \lambda_2 = \frac{5}{6}$$

$$\lambda_2 = -\frac{1}{6}$$

Q. If A^2 is a $\overset{\leftarrow \text{matrix}}{0}$ matrix, then prove that $\overset{\leftarrow \text{real no.}}{0}$ is the only eigen value of A.

Ans: Let λ be an eigen value of A,

$$AX = \lambda X$$

$$A(AX) = A(\lambda X)$$

$$A^2X = \lambda(AX) = \lambda(\lambda X)$$

$$A^2X = \lambda^2 X$$

If $A^2 = 0$, $A^2X = 0$

$$\Rightarrow \lambda^2 X = 0$$

$$\Rightarrow \lambda^2 = 0 \quad [\because X \neq 0]$$

$$\lambda = 0$$

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Q | If $A^2 = A$ 0 and 1 are the only eigen values

Ans:- Let λ be the eigen value of A

$$AX = \lambda X$$

$$A(AX) = A(\lambda X)$$

$$A^2X = \lambda(AX)$$

$$AX = \lambda^2 X$$

$$\lambda X = \lambda^2 X$$

$$X(\lambda - \lambda^2) = 0$$

$$X\lambda(1 - \lambda) = 0$$

$$\therefore \lambda = 0 \text{ or } \lambda = 1 \quad [\because X \neq 0]$$

Diagonalization,

Eg:

$$A = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 4 \\ 3 & 2-\lambda \end{vmatrix}$$

$$= (1-\lambda)(2-\lambda) - 12$$

$$= 2 - \lambda - 2\lambda + \lambda^2 - 12$$

$$= \lambda^2 - 3\lambda - 10$$

$$= \lambda^2 - 5\lambda + 2\lambda - 10$$

$$= \lambda(\lambda - 5) + 2(\lambda - 5)$$

$$= (\lambda + 2)(\lambda - 5)$$

$$\lambda = -2, 5$$

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Eigen vector $\lambda = 5$

$$\begin{pmatrix} -4 & 4 \\ 3 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$-x_1 + x_2 = 0 \Rightarrow x_1 = x_2 \qquad \begin{pmatrix} k \\ k \end{pmatrix} \downarrow \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Eigen vector $\lambda = -2$

$$\begin{pmatrix} 3 & 4 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 4 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$3x_1 + 4x_2 = 0$$

$$x_1 = -\frac{4x_2}{3}$$

$$x_2 = k, \quad x_1 = -\frac{4k}{3}.$$

$$\begin{pmatrix} -\frac{4k}{3} \\ k \end{pmatrix} \rightarrow \begin{pmatrix} -4/3 \\ 1 \end{pmatrix}.$$

Let $P = \begin{pmatrix} 1 & -4 \\ 1 & 3 \end{pmatrix}$

$$P^{-1} = \frac{1}{7} \begin{pmatrix} 3 & 4 \\ -1 & 1 \end{pmatrix}$$

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$$P^{-1}AP = \frac{1}{7}\begin{pmatrix}3 & 4\\ -1 & 1\end{pmatrix}\begin{pmatrix}1 & 4\\ 3 & 2\end{pmatrix}\begin{pmatrix}1 & -4\\ 1 & 3\end{pmatrix}$$

$$= \frac{1}{7}\begin{pmatrix}15 & 20\\ 2 & -2\end{pmatrix}\begin{pmatrix}1 & -4\\ 1 & 3\end{pmatrix}$$

$$= \frac{1}{7}\begin{pmatrix}35 & 0\\ 0 & -14\end{pmatrix}$$

$$= \begin{pmatrix}5 & 0\\ 0 & -2\end{pmatrix}$$

orthogonalise the vectors

$u_1 = \begin{pmatrix}1\\1\end{pmatrix}$, $u_2\begin{pmatrix}-4\\3\end{pmatrix}$

$k = \langle v_2, v_1 \rangle = $

$v_1 = \begin{pmatrix}1\\1\end{pmatrix}$

$v_2 = \begin{pmatrix}-4\\3\end{pmatrix} + \frac{1}{2}\begin{pmatrix}1\\1\end{pmatrix}$

$$P = \begin{pmatrix}\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{pmatrix}$$

$$P^{-1} = P^{T}$$

$$P^{-1}AP = P^{T}AP$$

Q. Let $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ diagonalization.

Soln:
$$|A - \lambda I| = \begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)^2 - 1 = 0$$

$$(2-\lambda)^2 = 1$$

$$\lambda^2 + 4 - 4\lambda - 1 = 0$$

$$\lambda^2 - 4\lambda + 3 = 0$$

Eigen values. $\lambda = 1, 3.$

Eigen vectors, for $\lambda = 1$

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$x_1 + x_2 = 0$$

$$x_1 = -x_2.$$

$$\begin{pmatrix} k \\ -k \end{pmatrix} \rightarrow \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Eigen vector, $\lambda = 3$

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$-x_1 + x_2 = 0 \Rightarrow x_1 = x_2 \qquad \begin{pmatrix} k \\ k \end{pmatrix} \rightarrow \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$u_1 = (-1, 1)$, $u_2 = (1, 1)$

$P = ($

$v_1 = (-1, 1)$,

$$v_2 = (1,1) - \frac{\langle u_2, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1$$

$$= (1, 1)$$

orthonormal vector,

$$w_1 = \frac{v_1}{\| v_1 \|} = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right)$$

$$w_2 = \frac{v_2}{\| v_2 \|} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right)$$

$$P = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$P^{-1} = P^{T} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \quad \text{since } P.P^{T} = I$$

$$P^{-1}AP = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{2}{\sqrt{2}} + \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} + \frac{2}{\sqrt{2}} \\ \frac{2}{\sqrt{2}} + \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} + \frac{2}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{3}{\sqrt{2}} & \frac{3}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$$

Some results.

(i) If λ is some eigen value of A, λ^2 is an eigen value of A^2

Proof: $X \neq 0$, $AX = \lambda X$

$$A.(AX) = A(\lambda X)$$
$$= \lambda(AX)$$
$$A^2 X = \lambda^2 X$$

$\therefore \lambda^2$ is an eigen value of A^2.

(ii) $\lambda = 0$ is an eigen value of A iff A is singular or $|A| = 0$

Proof: λ is an eigen value of A iff $|A - \lambda I| = 0$

$\lambda = 0$ is an eigen value of A iff $|A| = 0$.

(iii) If λ is an eigen value of A, $\lambda \neq 0$, $\frac{1}{\lambda}$ is an eigen value of A^{-1}.

Proof: $AX = \lambda X$ $\quad \lambda \neq 0$

$A^{-1}.AX = A^{-1}.\lambda X$ ($\because A^{-1}$ exists as A is not singular) as $\lambda \neq 0$

$$IX = A^{-1}\lambda X$$

$$\frac{1}{\lambda}X = A^{-1}X$$

$\therefore \frac{1}{\lambda}$ is an eigen value of A^{-1}

(iv) If A and B are two square matrices, and A is invertible, i.e, $|A| \neq 0$ then B and $A^{-1}BA$ have the same eigen value.

Proof: λ is an eigen value of B.

$$|B - \lambda I| = 0$$

Consider $|A^{-1}BA - \lambda I| = |A^{-1}(B - \lambda I)A|$

$$= |A^{-1}|\,|B - \lambda I|\,|A|$$

$$= 0$$

$\therefore$ B and $A^{-1}BA$ have the same eigen value.

(v) Eigen values of an orthogonal matrix are of absolute value 1.

Proof: Note: A is orthogonal if $A^T = A^{-1}$ or $AA^T = A^TA = I$.

$$AX = \lambda X$$

$$A^T.AX = A^T\lambda X$$

$$A^TAX = \lambda A^TX$$

$$IX = \lambda.\lambda X$$ [since A^T and A have the same eigen value] (already proved)

$$X = \lambda^2 X$$

$$\Rightarrow \lambda^2 = 1$$

$$\lambda = \pm 1.$$

(vi) If A is triangular matrix, eigen value of A are principal diagonal elements of A.

Proof: A is triangular

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ 0 & 0 & a_{33} \cdots & a_{3n} \\ \vdots & & & \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ 0 & a_{22}-\lambda & \cdots & a_{2n} \\ \vdots & & & a_{nn}-\lambda \\ 0 & 0 & \cdots & \end{vmatrix}$$

$$= (a_{11}-\lambda)(a_{22}-\lambda)\cdots\cdots(a_{nn}-\lambda)$$

$$= 0$$

$$\therefore \lambda = a_{11}, \quad \lambda = a_{22}, \quad \cdots\cdots \quad \lambda = a_{nn}$$

(vii) Eigen values of a matrix A, where $A^2 = 0$

Proof:

$$AX = \lambda X$$

$$A \cdot AX = A\lambda X$$

$$A^2 X = \lambda(AX) = \lambda^2 X$$

$A^2 = 0$

$\therefore A^2 X = 0 = \lambda^2 X$

$X \neq 0, \quad \lambda^2 X = 0.$

$\lambda^2 = 0 \Rightarrow \lambda = 0.$

if

$A^2 = A$

idempotent.

$A^2 X = \lambda^2 X$

$AX = \lambda^2 X$

$\lambda X = \lambda^2 X$

$\lambda = 0$ or $\lambda = 1$.

Q If 1, 3 are eigen values of A. Find eigen value of: i) $(A^{-1})^2$ (ii) $(A^{-1} - 2I)$.

Soln: (i) Eigen value of $A \longrightarrow 1, 3$

$\therefore$ e.v of $A^{-1} \longrightarrow 1, \frac{1}{3}$

e.v of $(A^{-1})^2 \longrightarrow 1, \frac{1}{9}$.

(ii) E.V of $A \longrightarrow 1, 3$

E.V of $A^{-1} \longrightarrow 1, \frac{1}{3}$

E.V of $A^{-1} - 2I \longrightarrow 1 - 2, \frac{1}{3} - 2$

$-1, -\frac{5}{3}$