GAMMA FUNCTIONS

Gamma function is defined as.

$$\Gamma(p) = \int_0^\infty e^{-x} x^{p-1} dx \quad ; p > 0$$

Properties

1) $\Gamma \cancel{p\epsilon} (1) = 1$

$$\Gamma(1) = \int_0^\infty e^{-x} dx = 1$$

2) $\Gamma(p+1) = p\Gamma(p)$ recursance prop

Proof:

$$\int_0^\infty e^{-x} x^p dx = x^p \frac{e^{-x}}{-1}\Big|_0^\infty - \int_0^\infty \frac{e^{-x}}{-1} p x^{p-1} dx$$

$$= p \int_0^\infty e^{-x} x^{p-1} dx$$

$$= p\Gamma(p)$$

$$\lim_{x \to \infty} -x^p e^{-x} = 0$$

e^{-x} will reach 0 faster than $x \to \infty$

$$x \downarrow \infty \qquad e^{-x} \downarrow 0$$

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(3) If $p = n$, a positive integer

$$\Gamma(n+1) = n!$$

*
(4) PT $\Gamma(1/2) = \sqrt{\pi}$
*

$$\Gamma(1/2) = \int_0^\infty e^{-x} x^{1/2-1} \, dx$$

$$= \int_0^\infty e^{-x} x^{-1/2} \, dx$$

Put $x = t^2$; $dx = 2t\,dt$

$$\Gamma(1/2) = \int_0^\infty e^{-t^2} t^{-1} \cdot t\,2\,dt$$

$$= \int_0^\infty 2 e^{-t^2} \, dt$$

$\therefore$ we consider $\Gamma(1/2)\,\Gamma(1/2) = 2\int_0^\infty e^{-x^2} dx \cdot 2\int_0^\infty e^{-y^2} dy$

$$= 4 \int_0^\infty \int_0^\infty e^{-(x^2+y^2)} \, dx\,dy$$

Put $\Rightarrow$ $x = r\cos\theta$, $y = r\sin\theta$

$$\Gamma(1/2)\Gamma(1/2) = 4\int_0^{\pi/2}\int_0^{\infty} e^{-r^2}\, r\, dr\, d\theta$$

$$= \text{[illegible]} = 4 \times \frac{\pi}{2} \times \int_0^{\infty} e^{-r^2}\, r\, dr$$

$$\left[\Gamma(1/2)\right]^2 = 4 \times \frac{\pi}{2} \times \frac{1}{2} = \pi$$

$$\Gamma(1/2) = \sqrt{\pi}$$

BETA FUNCTION

We define $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$

, $m,n>0$

Properties

(1) $\beta(m,n) = \beta(n,m)$

Proof:

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx \quad -(1)$$

Let $y = 1-x \Rightarrow x = 1-y$

$$\beta(n,m) = \int_0^1 (1-y)^{n-1}\,y^{m-1}\,dy \quad -(2)$$

(1) = (2)

*(2) $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$

put $x = \sin^2\theta$

$$\beta(m,n) = \int_0^{\pi/2} \sin^{2(m-1)}\theta\,(1-\sin^2\theta)^{n-1}\,\cancel{dx}\,2\sin\theta\cos\theta\,d\theta$$

$$= \int_0^{\pi/2} \sin^{2(m-1)}\theta\,\cos^{2(n-1)}\theta \cdot 2\sin\theta\cos\theta\,d\theta$$

** $$= 2\int_0^{\pi/2} \sin^{2m-1}\theta\,\cos^{2n-1}\theta\,d\theta$$

Result: (Theorem)

$$B(m,n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$$

Remark

$$\int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta$$

$$= \frac{1}{2} B\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$$

$$= \frac{1}{2} \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\cancel{p+1}\,\frac{p+q+2}{2}\right)}$$

ex: $\int_0^{\pi/2} \sin^p\theta \, d\theta$

$$= \frac{1}{2} \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{p+2}{2}\right)} = \frac{\sqrt{\pi}}{2} \frac{\Gamma\left(\frac{p+1}{2}\right)}{\Gamma\left(\frac{p+2}{2}\right)}$$

To prove: $\beta(m,n) = \dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$

Proof:

$$\Gamma(p) = \int_0^\infty e^{-x} x^{p-1} dx$$

put $x = t^2$

$$\Gamma(p) = \int_0^\infty e^{-t^2} (t^2)^{p-1} \cdot 2t\,dt$$

$$= 2\int_0^\infty e^{-t^2} \cdot t^{2p-1} dt$$

Consider $\Gamma(m)\Gamma(n) = 2\int_0^\infty e^{-x^2} x^{2m-1} dx \cdot 2\int_0^\infty e^{-y^2} y^{2n-1} dy$

$$= 4\int_0^\infty\int_0^\infty e^{-(x^2+y^2)} x^{2m-1} y^{2n-1} dx\,dy$$

put $x = r\cos\theta$, $y = r\sin\theta$

$$\Gamma(m)\Gamma(n) = 4\int_0^{\pi/2}\int_0^\infty e^{-r^2} \cdot (r\cos\theta)^{2m-1} \cdot (r\sin\theta)^{2n-1} r\,dr\,d\theta$$

$$= \left(2\int_0^{\pi/2} \cos^{2m-1}\theta \sin^{2m-1}\theta\, d\theta\right)\left(2\int_0^\infty e^{-r^2} r^{2(m+n)-1} dr\right)$$

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$$= \beta(m,n)\,\Gamma(m+n)$$

$$\Rightarrow \quad \beta(m,n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$$

Q1. Show that $\Gamma(p) = \int_0^1 \left(\log \frac{1}{y}\right)^{p-1} dy \;,\; p>0$

ans:
$$\Gamma(p) = \int_0^\infty e^{-x}\, x^{p-1}\, dx.$$

$$y = e^{-x}$$
$$dy = -e^{-x}\, dx.$$

$$\cancel{\therefore x = \log}\quad e^{x} = \frac{1}{y}$$
$$x = \log \frac{1}{y}$$

$$-\int_1^0 \cancel{y}\left(\log \frac{1}{y}\right)^{p-1} dy = \int_0^1 \left(\log \frac{1}{y}\right)^{p-1} dy$$

Q2. Show that $\beta(m,n) = \int_0^\infty \frac{y^{n-1}}{(1+y)^{m+n}}\, dy$

ans:
$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\, dx$$

$$\cancel{y = \log x \Rightarrow x = e^{y}} \quad x = \frac{1}{1+y}$$

$$dx = \frac{-1}{(1+y)^2}\, dy$$

$$\int_{\infty}^{0} \left(\frac{1}{y+1}\right)^{m-1} \left(1 - \frac{1}{y+1}\right)^{n-1} \frac{-1}{(y+1)^2}\, dy$$

$$= \int_{0}^{\infty} \frac{y^{n-1}}{(y+1)^{m+n}}\, dy$$

Q3. Evaluate

(i) $\Gamma(9/2)$ (ii) $\beta(9/2, 7/2)$

ans:

(1) $\Gamma(9/2) = \frac{7}{2}\Gamma\left(\frac{7}{2}\right)$

$$= \frac{7}{2} \cdot \frac{5}{2}\, \Gamma\left(\frac{5}{2}\right) = \frac{7}{2} \cdot \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2}\, \Gamma(1/2)$$

$$= \frac{105\sqrt{\pi}}{16}$$

(ii) $$\beta(9/2, 7/2) = \frac{\Gamma(9/2)\,\Gamma(7/2)}{\Gamma\left(\frac{9}{2} + \frac{7}{2}\right)}$$

$$= \frac{\frac{105\sqrt{\pi}}{16} \cdot \frac{15\sqrt{\pi}}{8}}{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}$$

$$= \frac{1575\,\pi}{645120}$$

Factorial of fractions :

$$\left(\frac{3}{2}\right)! = \Gamma\left(\frac{3}{2}+1\right) = \Gamma\left(\frac{5}{2}\right)$$

$$\Gamma(p+1) = p\,\Gamma(p)$$

$$\Gamma(p) = \frac{\Gamma(p+1)}{p}$$

$-1 < p < 0 \quad \Rightarrow \quad 0 < p+1 < 1$

$-2 < p < -1 \quad \Rightarrow \quad -1 < p+1 < 0$

$$\frac{1}{p!} = \begin{cases} \dfrac{1}{\Gamma(p+1)}, & p = -1, -2, \ldots \\ 0, & p = -1, -2, \ldots \end{cases}$$

Q4. Express as gamma integral

(i) $\int_0^\infty e^{-x^3}\sqrt{x}\,dx$ (ii) $\int_0^{\pi/2} \sqrt{\cot\theta}\,d\theta$

ans:

(a) $\int_0^\infty e^{-x^3} \sqrt{x}\, dx$

$\Gamma(m) = \int e^{-x} x^{m-1} dx$

$x^3 = t \Rightarrow \cancel{t = t}^{1/3}$

$3x^2 dx = dt$ $\Rightarrow x = t^{1/3}$

$$\int_0^\infty e^{-t} t^{1/6} \cdot \frac{dt}{3t^{2/3}} = \frac{1}{3}\int_0^\infty e^{-t} t^{-1/2} dt$$

$$= \frac{1}{3} \Gamma(1/2) = \frac{\sqrt{\pi}}{3}$$

(b) $\cancel{B}$ $\int_0^{\pi/2} \sqrt{\cot\theta}\, d\theta$

$$\int_0^{\pi/2} \cos^{1/2}\theta \sin^{-1/2}\theta\, d\theta$$

$$= \frac{1}{2} \beta\left(\frac{3}{4}, \frac{1}{4}\right)$$

$$= \frac{1}{2} \frac{\Gamma(\frac{3}{4})\Gamma(\frac{1}{4})}{\Gamma(1)} = \frac{1}{2}\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right)$$

Q5. PT $\int_0^{\pi/2} \sqrt{\sin\theta}\, d\theta \cdot \int_0^{\pi/2} \frac{d\theta}{\sqrt{\sin\theta}} = \pi$

ans:

$$\int_0^{\pi/2} \sin^{1/2}\theta\, d\theta \cdot \int_0^{\pi/2} \sin^{-1/2}\theta\, d\theta .$$

$$\left.\begin{array}{l} \frac{1}{4} \cdot B\left(\frac{3}{4}, 0\right) \cdot \beta\left(\frac{1}{4}, 0\right) \\[2ex] \frac{1}{4}\ \frac{\Gamma\left(\frac{3}{4}\right) \Gamma(0)}{\Gamma\left(\frac{3}{4}\right)} . \end{array}\right\} \text{po}$$

$$\frac{1}{4} B\left(\frac{3}{4}, \frac{1}{2}\right) B\left(\frac{1}{4}, \frac{1}{2}\right)$$

$$= \frac{1}{4} \cdot \frac{\cancel{\Gamma(3/4)} \cdot \Gamma(1/2)}{\Gamma\left(\frac{5}{4}\right)} \cdot \frac{\Gamma(1/4)\, \Gamma(1/2)}{\cancel{\Gamma\left(\frac{3}{4}\right)}}$$

$$= \frac{1}{4} \cdot \frac{[\Gamma(1/2)]^2\, \Gamma(1/4)}{\frac{1}{4} \cancel{\Gamma(1/4)}}$$

$$= \underline{\pi}$$

Fundamental Set / System on Basis of a DE is the set of 2 ind LI soln of eqn

$S = \{y_1, y_2\}$

METHOD OF UNDETERMINED COEFFECIENTS

Q1 $y'' - 3y' + 2y = e^{2x}$ —(1)

ans: $y_g = C_1e^{x} + C_2e^{2x}$

S2. Let $y_p = \cancel{Ae^{2x}}$ be a particular soln

$y_p = Axe^{2x}$

(y roots are same put Ax^2e^{2x})

$y_p' = A(x2e^{2x} + e^{2x})$

$y''_p = A(x4e^{2x} + 2e^{2x} + 2e^{2x})$

(1) => $\cancel{y_p'' \; A(2e^{2x}}$

$A(4e^{2x} + 4xe^{2x}) - 3A(2xe^{2x} + e^{2x})$

$+ 2Ae^{2x}x = e^{2x}$

$Ae^{2x} = e^{2x}$

=> $A = 1$ $\quad y_p = xe^{2x}$

to find y_p, $y_p = A\,R(x)$

is $R(x)$ is there in y_g put $Ax\,R(x)$

$$y_p = \frac{1}{D^2-3D+2} e^{2x} = \frac{x\cancel{2}e^{2x}}{2D-3} = \frac{\cancel{x2}e^{2x}}{F}$$

$$= xe^{2x}$$

th

LAPLACE TRANSFORMS

Let $f(t)$ be a fn defined for $t > 0$ We define the Laplace transform of $f(t)$ as

$$L[f(t)] = F(s) = \int_0^{\infty} e^{-st} f(t)\, dt$$

We define the inverse Laplace transform as

$$L^{-1}[F(s)] = f(t) \Leftrightarrow L[f(t)] = F(s)$$

Q1: $f(t) = 1$ for $t > 0$

$$L(1) = \int_0^{\infty} e^{-st}\, 1\, dt = \frac{1}{s}, \quad s > 0$$

$$\therefore \quad L^{-1}[1/s] = 1$$

Q2: $f(t) = t \quad ; \quad t > 0$

ILATE

$$L[t] = \int_0^{\infty} e^{-st}\, t\, dt$$

$$= t\, \frac{1}{s} - \frac{1}{s}\int_0^{\infty} e^{-st}\, dt = \frac{1}{s^2} \quad s > 0$$

3. $f(t) = \begin{cases} t & 0 < t < 1 \\ 1 & 1 \le t < 2 \\ 0 & 2 \le t \end{cases}$

ans:

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_0^1 + \int_1^2 + \int_2^\infty$$

$$= \int_0^1 e^{-st} t\,dt + \int_1^2 e^{-st}\,dt + \int_2^\infty e^{-st} \cdot 0\,dt$$

$$= -\frac{t}{s}\left(e^{-st}\right)\Big|_0^1 + \frac{1}{s}\int_0^1 e^{-st} + -\frac{1}{s} e^{-st}\Big|_1^2$$

$$= -\frac{\cancel{t}1}{s} e^{-s} - \frac{1}{s^2} e^{-s} - \frac{1}{s}\left(e^{-2s} - e^{-s}\right) + 1$$

$$= -\frac{e^{-2s}}{s} - \frac{e^{-s}}{s^2} + 1$$

Pdt rule:

$$\int uv = uv_1 - u'v_2 + u''v_3 + \ldots$$

Sufficient Condn for the existence of Laplace Transform

Laplace transform $f(t)$ exists if

a) $f(t)$ is cont or atleast piecewise cont in $(0, \infty)$

b) There exists 2 nos $M > 0$, $\alpha > 0$ such that $|f(t)| \leq M e^{\alpha t}$ for all t

Some fns violate these 2 condn but still has Laplace T

PROPERTIES

#1 : Linearity

$$L[af(t) + bg(t)] = a L[f(t)] + b L[g(t)]$$

$a, b = \text{const.}$

Proof: $L[af + bg]$

$$= \int_0^\infty e^{-st} (af + bg)\, dt$$

$$= a \int_0^\infty e^{-st} f dt + b \int_0^\infty e^{-st} g dt$$

$$= a L[f(t)] + b L[g(t)]$$

1. Find LT of $f(t) = e^{at}$

ans:
$$L[e^{at}] = \int_0^\infty e^{-st} e^{at} dt$$

$$= \int_0^\infty e^{(a-s)t} dt = \frac{1}{a-s} e^{(a-s)t}\Big|_0^\infty$$

$$= \frac{e^{-(s-a)t}}{-(s-a)} = \frac{1}{s-a} \quad s>a$$

2. Find LT of sin at and cos at

ans:
$$L[\sin at] = \int_0^\infty e^{-st} \sin at \, dt$$

Consider $f(t) = e^{iat}$

$$L[e^{iat}] = \frac{1}{s-ia}$$

$$L[\cos at + i\sin at] = \frac{s+ia}{s^2+a^2}$$

$$L[\cos at] = \frac{s}{s^2+a^2} \quad \text{and}$$

$$L[\sin at] = \frac{a}{s^2+a^2}$$

3. Find LT of sinh at and cosh at

ans: $L[\sinh at] = L\left[\frac{e^{at} - e^{-at}}{2}\right]$

$= \frac{1}{2}\left(L[e^{at}] - L[e^{-at}]\right)$

$= \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{s+a}\right] = \frac{a}{s^2-a^2}$

$L[\cosh at] = L\left[\frac{e^{at} + e^{-at}}{2}\right]$

$= \frac{s}{s^2-a^2}$

First Shifting Theorem

If $L[f(t)] = F(s)$ then $L[e^{at} f(t)] = F(s-a)$

$L[e^{at} f(t)] = \int_0^\infty e^{-st} \cdot e^{at} f(t)\, dt$

$= \int_0^\infty e^{-(s-a)t} f(t)\, dt$

$= F(s-a)$

1. Find LT of $f(t) = t^p$; $p > -1$

ans: $L[t^p] = \int_0^\infty e^{-st} t^p \, dt$

Put $st = u$

$t = \frac{u}{s}$ $\quad dt = \frac{du}{s}$

$$L[t^p] = \int_0^\infty e^{-u} \left(\frac{u}{s}\right)^p \frac{du}{s}$$

$$= \frac{1}{s^{p+1}} \int_0^\infty e^{-u} u^p \, du$$

$$\Gamma(p) = \int_0^\infty e^{-x} x^{p-1} dx$$

$$L[t^p] = \frac{\Gamma(p+1)}{s^{p+1}}$$

If $p = n$, a +ve integer.

$$L[t^n] = \frac{n!}{s^{n+1}}$$

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$f(t)$	$F(s)$	
1	$1/s$	$s>0$
t	$1/s^2$	
t^p	$\Gamma(p+1)/s^{p+1}$	
e^{at}	$\frac{1}{s-a}$	
$\sin at$	$\frac{a}{s^2+a^2}$	
$\cos at$	$\frac{s}{s^2+a^2}$	
$\sinh at$	$\frac{a}{s^2-a^2}$	
$\cosh at$	$\frac{s}{s^2-a^2}$	
$e^{at}t^p$	$\frac{\Gamma(p+1)}{(s-a)^{p+1}}$	{by 1st Shifting Th.
$e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2}$	
$e^{at}\cos bt$	$\frac{(s-a)}{(s-a)^2+b^2}$	
$e^{at}\sinh bt$	$\frac{b}{(s-a)^2-b^2}$	
$e^{at}\cosh bt$	$\frac{(s-a)}{(s-a)^2-b^2}$	

$F(s)$	$f(t)$
$\frac{1}{s}$	1
$\frac{1}{s^p}$	$\frac{t^{p-1}}{\Gamma(p)}$
$\frac{1}{s-a}$	e^{at}
$\frac{1}{s^2+a^2}$	$\frac{1}{a}\sin at$

Laplace Transform of Derivative

If $L[f(t)] = F(s)$ then,

$$L[f'(t)] = sF(s) - f(0)$$

<u>Proof:</u>

$$L[f'(t)] = \int_0^\infty e^{-st} f'(t)\,dt$$

$$= e^{-st} f(t)\Big|_0^\infty - \int_0^\infty (-s) e^{-st} f(t)\,dt$$

$$= 0 - f(0) + sF(s)$$

$$= sF(s) - f(0)$$ provided $e^{-st} f(t) \to 0$ as $t \to \infty$

Note: $L[f''(t)] = s^2 F(s) - s f(0) - f'(0)$

$$L[f''] = s L[f'] - f(0)$$

$$= s^2 F(s) - s f(0) - f'(0)$$

$$L[f'''(t)] = s^3 F(s) - s^2 f(0) - s f'(0) - \cancel{s} f''(0)$$

$$L[f^{(n)}(t)] = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) \cdots - f^{n-1}(0)$$

Laplace Transform of Integral

If $L[f(t)] = F(s)$, then

$$L\left[\int_0^t f(u)\,du\right] = \frac{F(s)}{s}$$

Proof:

Take $g(t) = \int_0^t f(u)\,du$

$g'(t) = f(t)$, $g(0) = 0$

$L[g'(u)] = s\,L[g(t)] - g(0)$ • by previous result

$L[f(t)] = s\,L\left[\int_0^t f(u)\,du\right] - 0$

$$L\left[\int_0^t f(u)\,du\right] = \frac{F(s)}{s}$$

Q1 Find LT

i) $f(t) = (t+2)e^t$

ii) $f(t) = \cos^2 t$

iii) $f(t) = e^{-t}\cos^2 t$

iv) $f(t) = \sinh at \cdot \sin bt$

ans:

$L[t+2] = L[t] + L[2]$

$= \frac{1}{s^2} + \frac{2}{s}$

$L[e^t(t+2)] = \frac{1}{(s-1)^2} + \frac{2}{(s-1)}$

(ii) $f(t) = \cos^2 t$

$$\cos 2t = 2\cos^2 t - 1$$

$$\cos^2 = \frac{\cos 2t + 1}{2}$$

$$L\left[\frac{\cos 2t + 1}{2}\right] = \frac{1}{2} L[\cos 2t] + \frac{1}{2} L[1]$$

$$= \frac{1}{2} \cdot \frac{s}{s^2+4} + \frac{1}{2s}$$

(iii) $f(t) = e^{-t} \cos^2 t$

$$L[\cos^2 t] = \frac{1}{2}\left[\frac{s}{s^2+4} + \frac{1}{s}\right]$$

$$L[e^{-t} \cos^2 t] = \frac{1}{2}\left[\frac{s+1}{(s+1)^2+4} + \frac{1}{s+1}\right]$$

(iv) $f(t) = \sinh at \cdot \sin bt$

ans: $\sinh at \cdot \sin bt = \left(\frac{e^{at} - e^{-at}}{2}\right) \sin bt$

$$= \frac{e^{at} \sin bt}{2} - \frac{e^{-at} \sin bt}{2}$$

$$L[\sin bt] = \frac{b}{s^2+b^2}$$

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$$L[\sinh at \cdot \sin bt] = \frac{1}{2}\left[\frac{b}{(s\pm a)^2+b^2} - \frac{b}{(s+a)^2+b^2}\right]$$

$$f(t) = \sin at \sin bt$$

Q2

ans: $\sin at \, \cancel{\cos} \sin bt = \frac{1}{2}(\cos (a-b)t - \cos (a+b))$

$$L[f(t)] = \frac{1}{2} L[\cos (a-b)t] - \frac{1}{2} L[\cos (a+b)t]$$

$$\cancel{= \frac{1}{2}\left(\frac{s}{s^2+(s-a)^2}\right.}$$

$$= \frac{1}{2}\left(\frac{s}{s^2+(a-b)^2} - \frac{s}{s^2+(a+b)^2}\right)$$

Q3. Find L^{-1} of $\frac{1}{s^2+2s+3}$

ans: $L^{-1}\left[\frac{1}{(s+1)^2+(\sqrt{2})^2}\right] = \frac{e^{-t}}{\sqrt{2}} \sin \sqrt{2}t$

Q4. L^{-1} of $\frac{1}{(s^2+2s+3)(s+3)}$

Apply method of partial fractions

ans:

$$\frac{1}{(s+3)(s^2+2s+3)} = \frac{A}{s+3} + \frac{Bs+C}{s^2+2s+3}$$

$$A+B=0$$
$$2A+3B+C=0$$
$$3A+3C=1$$

$$-A+\frac{1-3A}{2}=0$$
$$-5A+1=0$$

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$$A = \frac{1}{5}$$

$$B = \frac{-1}{5}$$

$$C = \frac{1 - 3/5}{2} = \frac{1}{5}$$

$$\frac{1}{(s+3)(s^2+2s+3)} = \frac{1}{5(s+3)} + \frac{1}{5(s^2+2s+3)} - \frac{1}{5}\frac{s}{s^2+2s+3}$$

$$L^{-1}\left[\frac{1}{5}\cdot\frac{1}{s+3}\right] = \frac{1}{5}e^{-3t}$$

$$L^{-1}\left[\frac{1}{5}\cdot\frac{1}{(s+1)^2+(\sqrt{2})^2}\right] = \frac{1}{5}\frac{e^{-t}}{\sqrt{2}}\sin\sqrt{2}t$$

$$L^{-1}\left[-\frac{1}{5}\cdot\frac{s}{s^2+2s+3}\right] = \frac{-1}{5}\frac{e^{-t}}{\sqrt{2}}\cos\sqrt{2}t$$

$$L^{-1}\left[\frac{1}{(s^2+2s+3)(s+3)}\right] = \frac{1}{5}\left(e^{-3t} + \frac{e^{-t}}{\sqrt{2}}\sin\sqrt{2}t - \frac{e^{-t}}{\sqrt{2}}\cos\sqrt{2}t\right)$$

Partial Fractions Decomposition

1. $\frac{px+q}{(x-a)(x-b)}$	$\frac{A}{x-a} + \frac{B}{x-b}$
2. $\frac{px+q}{(x-a)^2}$	$\frac{A}{x-a} + \frac{B}{(x-a)^2}$
3. $\frac{px^2+qx+r}{(x-a)(x-b)(x-c)}$	$\frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c}$

4. $\dfrac{px^2+qx+r}{(x-a)^2(x-b)}$ | $\dfrac{A}{x-a}+\dfrac{B}{(x-a)^2}+\dfrac{C}{x-b}$

5. $\dfrac{px^2+qx+r}{(x-a)(x^2+bx+c)}$ | $\dfrac{A}{(x-a)}+\dfrac{Bx+C}{x^2+bx+c}$

Changing Scale

If $L[f(t)] = F(s)$, then $L[f(at)] = \frac{1}{a}F(s/a)$
$a > 0$

Ex: $L[\sin t] = \dfrac{1}{s^2+1}$

$$L[\sin at] = \frac{1}{a} \cdot \frac{1}{(s/a)^2+1} = \frac{a}{s^2+a^2}$$

Multiplication by t

If $L[f(t)] = F(s)$, then $L[tf(t)] = -\dfrac{dF}{ds}$

Proof:

$$F(s) = \int_0^\infty e^{-st} f(t)\,dt$$

Differentiate both sides partially wrt to s

$$\frac{dF}{ds} = \int_0^\infty -t\,e^{-st} f(t)\,dt$$

$$-\frac{dF}{ds} = \int_0^\infty t\,e^{-st} f(t)\,dt = \int_0^\infty e^{-st}\, tf(t)\,dt$$

$$\Rightarrow L[t f(t)] = -\frac{dF}{ds}$$

Note: $L[f(t)] = F(s)$ then

$$L[t^n f(t)] = (-1)^n \frac{d^n F}{ds^n} \quad ; \; n = \text{+ve int}$$

Division By t

If $L[f(t)] = F(s)$ then $L\left[\frac{f(t)}{t}\right] = \int_s^\infty F(u)\,du$

Proof:

$$F(s) = \int_0^\infty e^{-st} f(t)\,dt$$

$$\int_s^\infty F(s)\,ds = \int_0^\infty \left[\frac{e^{-st}}{-t}\right]_s^\infty f(t)\,dt$$

$$= \int_0^\infty e^{-st} \frac{f(t)}{t}\,dt$$

$$= L\left[\frac{f(t)}{t}\right]$$

$$L\left[\frac{f(t)}{t^n}\right] = \underbrace{\int_s^\infty \int_s^\infty \cdots \int}_{n \text{ times}} F(u)\,du$$

8) Find LT

a) $f(t) = t \sin at$

ans: $L[\sin at] = \dfrac{a}{s^2+a^2}$

$L[t \sin at] = -\dfrac{d}{ds}\left(\dfrac{a}{s^2+a^2}\right)$

$= \dfrac{-a \times -2sa}{(s^2+a^2)^2} = \dfrac{2a^2 s}{(s^2+a^2)^2}$

(b) $f(t) = t \cos at$

ans $L[\cos at] = \dfrac{s}{s^2+a^2}$

$L[t \cos at] = -\dfrac{d}{ds}\left(\dfrac{s}{s^2+a^2}\right)$

$= -\dfrac{(s^2+a^2 - s \cdot 2s)}{(s^2+a^2)^2}$

$= \dfrac{s^2-a^2}{(s^2+a^2)^2}$

(c) $f(t) = \dfrac{\sin t}{t}$

ans: $L[\sin t] = \dfrac{1}{s^2+1}$

$$L\left[\frac{\sin t}{t}\right] = \int_s^\infty \frac{1}{s^2+1}\,ds$$

$$= \tan^{-1} s \Big|_s^\infty$$

$$= \frac{\pi}{2} - \tan^{-1} s$$

(4) $f(t) = \dfrac{1-\cos t}{t}$

ans: $L[1-\cos t] = L[1] - L[\cos t]$

$$= \frac{1}{s} - \frac{s}{s^2+a^2}$$

#

$$L\left[\frac{1-\cos t}{t}\right] = \int_s^\infty \frac{1}{s}\,ds - \int_s^\infty \frac{s}{s^2+a^2}\,ds$$

$$= \cancel{\ln s \Big|_s^\infty - \frac{1}{2}\ln(s^2+a^2)\Big|_s^\infty}$$

$$= \ln \frac{s}{\sqrt{s^2+a^2}}\Big|_0^\infty = -\ln s + \ln\sqrt{s^2+1}$$

$$= \ln\left(\frac{\sqrt{s^2+1}}{s}\right)$$

$$\ln 1 - \ln\frac{s}{\sqrt{s^2+1}} = \ln\frac{\sqrt{s^2+1}}{s}$$

(5) $f(t) = \dfrac{\cos at - \cos bt}{t}$

ans: $L[\cos at - \cos bt] = \dfrac{s}{s^2+a^2} - \dfrac{s}{s^2+b^2}$

$L\left[\dfrac{\cos at - \cos bt}{t}\right] = \ln(\sqrt{s^2+a^2}) - \ln(\sqrt{s^2+b^2}) \Big|_{s}^{\infty}$

$= \ln \dfrac{\sqrt{s^2+a^2}}{\sqrt{s^2+b^2}} \Bigg|_{s}^{\infty}$

$= \ln \dfrac{\sqrt{s^2+b^2}}{\sqrt{s^2+a^2}}$

$\left.\begin{aligned} &= \ln 1 - \ln \frac{a}{b} \\ &= \ln \frac{b}{a} \end{aligned}\right\}$ PO

2. Find inv LT

i) $F(s) = \dfrac{s-1}{s^4}$ $\qquad L[t^n] = \dfrac{n!}{s^{n+1}}$

$= \dfrac{1}{s^{\cancel{4}3}} - \dfrac{1}{s^{\cancel{3}4}}$

$= L\left[\dfrac{t^2}{2}\right] - L\left[\dfrac{t^3}{6}\right]$

$= L\left[\dfrac{t^2}{2} - \dfrac{t^3}{\cancel{3}6}\right]$

$f(t) = \cancel{t} \; \dfrac{3t^2 - \cancel{2}t^3}{6}$

(2) $F(s) = \dfrac{1}{s^2+3s+2}$

ans : $F(s) = \dfrac{1}{\left(s+\left(\frac{3}{2}\right)\right)^2}$

$$F(s) = \frac{1}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2}$$

$$A + B = 0$$
$$2A + B = 1$$
$$A = 1 \qquad \underline{\underline{B = -1}}$$

$$F(s) = \frac{1}{s+1} - \frac{1}{s+2}$$

$$L^{-1}\left[\frac{1}{s^2+3s+2}\right] = e^{-t} - e^{-2t}$$

<u>M-2</u>

$$\frac{1}{(s+3/2)^2 - (1/2)^2} = 2e^{-3/2 t} \sinh 1/2 t$$

(3) $F(s) = \dfrac{1}{s^4-1}$

ans: $F(s) = \dfrac{1}{(s^2+1)(s-1)(s+1)}$

$$= \frac{As+B}{s^2+1} + \frac{C}{s-1} + \frac{D}{s+1}$$

$$= A$$

~~so~~ also correct but long

$$F(s) = \frac{1}{(s^2+1)(s^2-1)}$$

$$= \frac{-1}{2}\left(\frac{1}{s^2+1} - \frac{1}{s^2-1}\right)$$

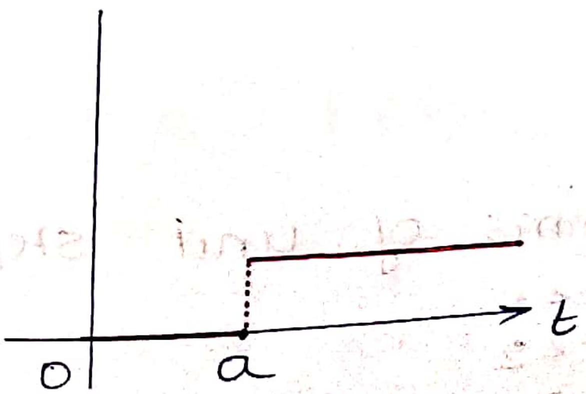
$$L^{-1}[F(s)] = \frac{-1}{2}\left(\sin t - \sinh t\right)$$

$$= \frac{1}{2}\left(\sinh t - \sin t\right)$$

Heaviside's Unit Step Function

For $a > 0$, we define the unit step function as

$$u(t-a) = \begin{cases} 0 & ; \quad t \le a \\ 1 & ; \quad t > a \end{cases}$$



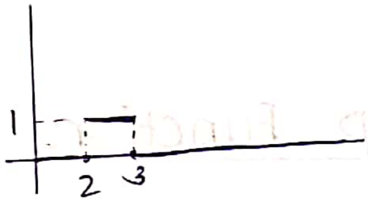
Q1. Find $L[u(t-a)]$

ans. $$L[u(t-a)] = \int_0^\infty e^{-st}\, u(t-a)\, dt$$

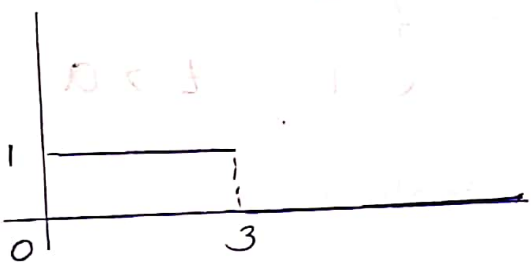
$$= \int_a^\infty e^{-st}\, 1\, dt = \frac{1}{s} e^{-as}$$

Q2. $u(t-2) - u(t-3)$

~~t-3~~



Q3. $1 - u(t-3)$ Graph



Q4. Express in terms of unit step fn

$$f(t) = \begin{cases} 1 & 0 < t < 1 \\ t & 1 \le t \le 2 \\ t^2 & t > 2 \end{cases}$$

ans:

$$f(t) = 1[1 - u(t-1)] + t[u(t-1) - u(t-2)] + t^2 u(t-2)$$

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Second Shifting Theorem

If $L[f(t)] = F(s)$ then the theorem says $L[f(t-a)\,u(t-a)] = e^{-as}F(s)$; $a > 0$

Proof:

$$L[f(t-a)\,u(t-a)] = \int_0^\infty e^{-st} f(t-a)u(t-a)\,dt$$

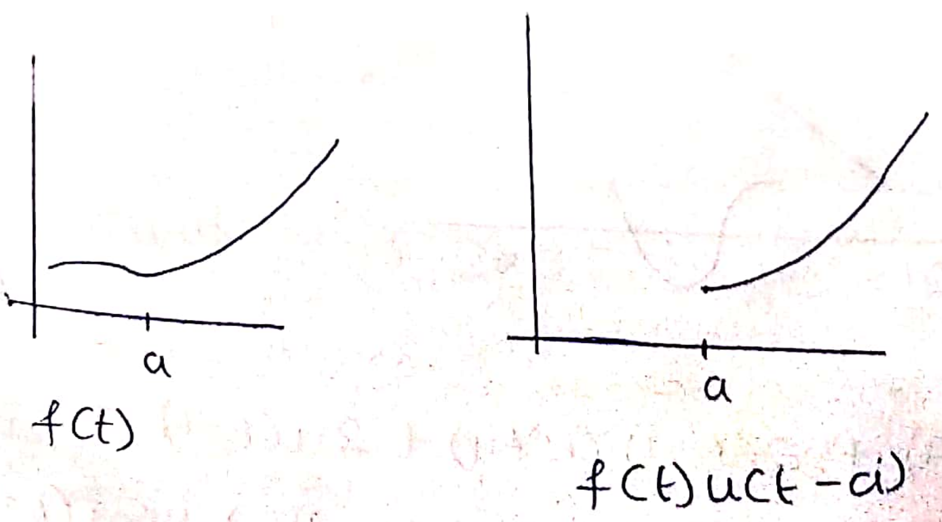
$$= \int_a^\infty e^{-st} f(t-a)\,dt$$

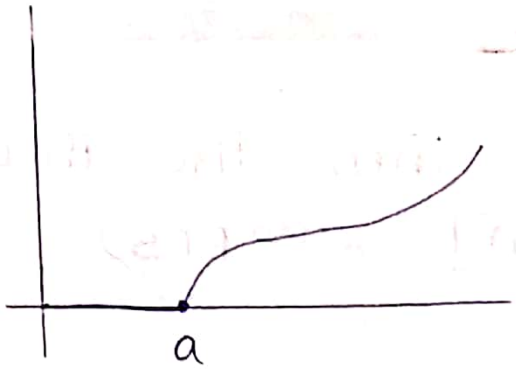
Put $t-a = u$

$$= \int_0^\infty e^{-(u+a)s} f(u)\,du$$

$$= e^{-as}\int_0^\infty e^{-us} f(u)\,du$$

$$= e^{-as} F(s)$$





$f(t-a)\,u(t-a)$

$f(t-a)$ mean graph is shifted to a

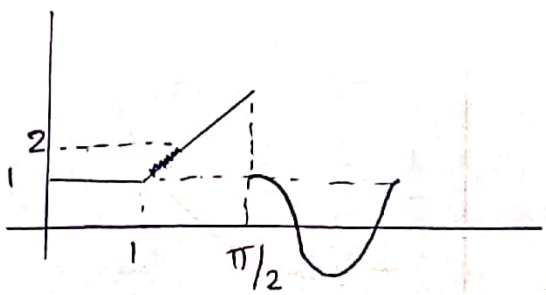
Q1. using the unit step function evaluate the Laplace transf of

(1) $L[$

$$f(t) = \begin{cases} 1 & 0<t<1 \\ 2t & 1<t<\pi/2 \\ \sin t & t>\pi/2 \end{cases}$$

ans:

$$f(t) = 1\,(1 - u(t-1)) + 2t\,(u(t-1) - u(t-\pi/2))$$
$$+ \sin t\,(u(t-\pi/2))$$



$$= 1 - u(t-1) + 2(t-1)\,u(t-1) + 2u(t-1)$$
$$- 2(t-\pi/2)\,u(t-\pi/2) - \pi u(t-\pi/2) + \cos(t-\pi/2)\,u(t-\pi/2)$$

By Second Shifting th.

$$L[f(t)] = \frac{t}{s} + \frac{e^{-as}}{s} - \frac{\pi e^{-\frac{\pi}{2}s}}{s} + \frac{2e^{-s}}{s^2}$$

$$-2e^{-\pi/2 s} \cdot \frac{1}{s^2} + e^{-\frac{\pi}{2}s} \frac{s}{s^2+a^2}$$

32.

$$f(t) = \begin{cases} 1 & 0 < t < 1 \\ t & 1 < t < 2 \\ t^2 & t > 2 \end{cases}$$

ans: $f(t) = \cancel{t}(u(1 - u(t-1)) + t(u(t-1) - u(t-2))$

$+ t^2(u(t-2))$

$= 1 - u(t-1) + (t-1)u(t-1) + u(t-1)$

$- (t-2)u(t-2) - 2u(t-2)$

$+ [(t-2)^2 + 4(t-2) + 4]u(t-2)$

$=$ ~~1+2u(t~~ $1 + 2u(t-2) + 3(t-2)u(t-2)$

$+ (t-1)u(t-1) + (t-2)^2 u(t-2)$

$$= \frac{1}{s} + \frac{2e^{-2s}}{s} + \frac{3e^{-2s}}{s^2}$$

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PERIODIC FUNCTIONS

we say that $f(t)$ is a periodic with period T if T is the smallest +ve no such that

$$f(t+T) = f(t) \text{ for all } t$$

Then

$$f(t+2T) = f(t)$$
$$f(t+3T) = f(t)$$
$$\Rightarrow \quad f(t+nT) = f(t) \; ; \; n = 1, 2 \ldots$$

Result:

If $f(t)$ is periodic with period T

Then

$$L[f(t)] = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\,dt$$

Proof:

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_0^T e^{-st} f(t)\,dt + \int_T^{2T} e^{-st} f(t)\,dt + \int_{2T}^{3T} e^{-st} f(t)\,dt \ldots$$

in 2nd Integral, $t = u+T$

In 3rd Integral, $t = u+2T$

etc

$$= \int_0^T e^{-st} f(t)\,dt + \int_0^T e^{-s(u+T)} f(u+T)\,du$$

$$+ \int_0^T e^{-s(u+2T)} f(u+2T)\,du$$

$f(u+T) = f(u+2T)$
$= \cdots \quad f(u)$

$$= \int_0^T e^{-st} f(t)\,dt \left[1 + e^{-sT} + e^{-2sT} + \cdots \right]$$

$$= \int_0^T e^{-st} f(t)\,dt \cdot \frac{1}{1-e^{-sT}}$$

$$= \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\,dt$$

Q1. Find Laplace Transform of the square wave fn given by $f(t) = \begin{cases} 1 & 0 < t < a \\ -1 & a < t < 2a \end{cases}$

and $f(t+2a) = f(t)$ for all t

ans:

Here $T = 2a$

$$L[f(t)] = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-2as}} \int_0^{2a} e^{-st} f(t)\, dt$$

$$= \frac{1}{1-e^{-2as}} \left[\int_0^{a} e^{-st}\, dt - \int_a^{2a} e^{-st}\, dt \right]$$

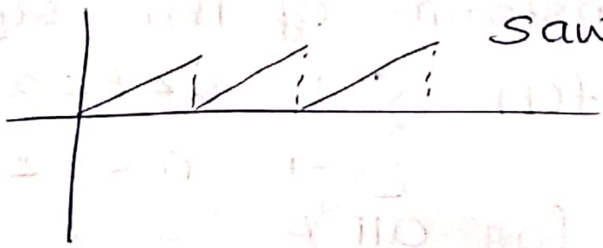
$$= \frac{1}{1-e^{-2as}} \left[\frac{e^{-st}}{-s} \Big|_0^{a} + \frac{e^{-st}}{\cancel{+}s} \Big|_a^{2a} \right]$$

$$= \frac{1}{1-e^{-2as}} \left[\frac{1}{s} + \frac{e^{-2as}}{s} - \frac{2e^{-sa}}{s} \right]$$

Q 32. $f(t) = t \qquad 0 < t < a$

$f(t+a) = f(t)$

ans:



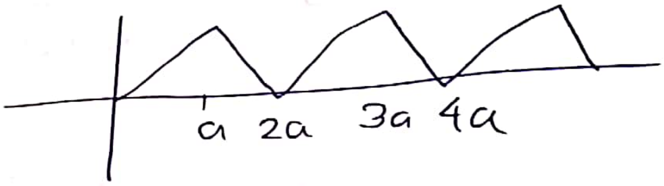
$T = a$

$$L[f(t)] = \frac{1}{1-e^{-as}} \int_0^{a} e^{-st}\, t\, dt$$

$$= \frac{1}{1-e^{-as}} \left[t\frac{e^{-st}}{-s} \Big|_0^{a} - \frac{e^{-st}}{s^2} \Big|_0^{a} \right]$$

Q3. $f(t) = \begin{cases} t & ; 0 < t < a \\ \cancel{2t-a} \; t-2a & ; a < t < 2a \end{cases}$

ans: Triangular W/F



Q1. Find Laplace Transform of

(i) $\int_0^t \frac{\sin u}{u} \, du$ (ii) $\int_0^t \frac{e^{-au} - e^{-bu}}{u} \, du$

ans. $L[\sin t] = \frac{1}{s^2+1}$

$$L\left[\frac{\sin t}{t}\right] = \int_s^\infty F(s) \, ds$$

$$= \int_s^\infty \frac{1}{u^2+1} \, du = \tan^{-1} u \Big|_s^\infty$$

$$= \frac{\pi}{2} - \tan^{-1} s$$

$$\therefore \; L\left[\int_0^t \frac{\sin u}{u}\, du\right] = \frac{F(s)}{s}$$

$$= \frac{\pi}{2s} - \frac{\tan^{-1} s}{s}$$

(ii) $f(u) = e^{-au} - e^{-bu}$

$$L[f(u)] = \frac{1}{s+a} - \frac{1}{s+b}$$

$$L\left[\frac{e^{-au} - \cancel{t}e^{-bu}}{u}\right] = \int_s^\infty \frac{1}{s+a}\, ds - \int_s^\infty \frac{1}{s+b}\, ds$$

$$= \ln \frac{s+a}{s+b}\Big|_s^\infty$$

$$= \ln \frac{s+b}{s+a}$$

$$L\left[\int_0^t \frac{e^{-au} - e^{-bu}}{u}\right] = \frac{1}{s} \ln\left(\frac{s+b}{s+a}\right)$$

Q2. Evaluate $\int_0^\infty \frac{\sin t}{t} dt$ or $\int_0^\infty e^{-2t} \frac{\sin t}{t} dt$

ans. $L\left[\frac{\sin t}{t}\right] = \frac{\pi}{2} - \tan^{-1} s$

$$\int_0^\infty e^{-st} \frac{\sin t}{t} dt = \frac{\pi}{2} - \tan^{-1} s$$

Take $s = 0$

$$\int_0^\infty \frac{\sin t}{t} = \frac{\pi}{2}$$

put $s = 2$

$$\int_0^\infty e^{-2t} \frac{\sin t}{t} dt = \frac{\pi}{2} - \tan^{-1} 2$$

Q3. Find Inv Laplace transform

(i) $F(s) = \tan^{-1} s$ (ii) $F(s) = \log \frac{s+b}{s+a}$

(iii) $F(s) = \log \frac{s^2+a^2}{s^2+b^2}$

ans. Let $L^{-1}[\tan^{-1} s] = f(t)$; to find $f(t)$

$$L[t f(t)] = \frac{-d}{ds}[\tan^{-1} s]$$

$$= \frac{-1}{1+s^2}$$

$$\therefore \quad t f(t) = -\sin t$$

$$\Rightarrow \quad f(t) = -\frac{\sin t}{t}$$

(ii) $L^{-1}[F(s)] = f(t)$ to find $f(t)$

$$L[t f(t)] = \frac{-d}{ds}\left[\log(s+b) - \log(s+a)\right]$$

$$= +\frac{1}{s+a} - \frac{1}{s+b}$$

$$t f(t) = +e^{-at} - e^{-bt}$$

$$f(t) = \frac{+e^{-at} - e^{-bt}}{t}$$

(iii) $L^{-1}[F(s)] = f(t)$; to find $f(t)$

$$L[t f(t)] = -\frac{d}{ds}\left[\log s^2+a^2 - \log s^2+b^2\right]$$

$$= \frac{-2s}{s^2+a^2} + \frac{2s}{s^2+b^2}$$

$$f(t) = \frac{\cancel{2} -2\cos at + 2\cos bt}{t}$$

CONVULATION

The convulation of 2 functions $f(t)$ and $g(t)$ defined for $t>0$ is

$$f*g\,(t) = \int_0^t f(u)\,g(t-u)\,du$$

Properties

1. $f*g\,(t) = g*f(t)$

2. $f*1 \neq f$

ex: $f(t) = t$

$$t*1 = \int_0^t u\cdot(1)\,du = \frac{t^2}{2}$$

Convulation Theorem

If $L[f(t)] = F(s)$, $L[g(t)] = G(s)$

then $L[f(t) * g(t)] = F(s)\,G(s)$

i.e $L^{-1}[F(s)\,G(s)] = f*g\,(t)$

Proof:

$$L[f*g(t)] = \int_0^\infty e^{-st} f*g\,(t)\,dt$$

$$= \int_0^\infty \int_0^t e^{-st} f(u)\,g(t-u)\,du\,dt$$

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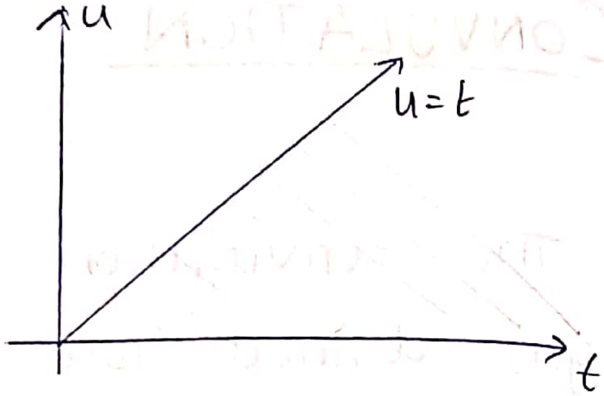
$$= \int_0^\infty \int_u^\infty e^{-st} f(u)\, g(t-u)\, dt\, du$$



Put $t - u = v$.

$$= \int_0^\infty f(u) \int_0^\infty e^{-s(u+v)} g(v)\, dv\, du$$

$$= \int_0^\infty e^{-su} f(u)\, du \cdot \cancel{\int_0^\infty e^{-sv} g(v)\, dv}\ G(s).$$

$$= F(s) \cdot G(s)$$

Q1. Using convulation th. Find L^{-1} of $\frac{1}{s^2(s+1)}$

ans: $F(s) \cdot G(s) = \frac{1}{s^2} \cdot \frac{1}{s+1}$

$L^{-1}[F(s)] = t = f(t)$

$L^{-1}[G(s)] = e^{-t} = g(t)$

using convulation th.

$$L^{-1}\left[\frac{1}{s^2(s+1)}\right] = f * g(t)$$

$$= \cancel{\int_0^t t \cdot e^{-t+u}\, du}$$

$$= \int_0^t u \cdot e^{-t+u}\, du$$

$$= e^{-t} \int_0^t u e^{u} du$$

$$= e^{-t} \left[u e^{u} \Big|_0^t - e^{u} \Big|_0^t \right]$$

$$= e^{-t} \left[t e^{t} - e^{t} + 1 \right]$$

$$= t - 1 + e^{-t} = t - 1 + e^{-t}$$

Q2. L^{-1} of $\dfrac{1}{(s^2+1)(s^2+4)}$

ans: $L^{-1}\left[\dfrac{1}{s^2+1}\right] = \sin t = f(t)$

$L^{-1}\left[\dfrac{1}{s^2+4}\right] = \dfrac{1}{2}\sin 2t = g(t)$

using convulation th.

$$L^{-1}\left[\frac{1}{(s^2+1)(s^2+4)}\right] = \int_0^t \sin u \cdot \frac{1}{2}\sin(2t-2u)\, du$$

$$= \int_0^t \cancel{\sin u \cdot \sin(t-u)\cos(t-u)\, du}$$

$$= \frac{1}{2\times 2}\int_0^t \cos(u-t+u) - \cos(u+t-u)\, du$$

$$= \frac{1}{4}\int_0^t \cos(2u-t) \cdot \cancel{\sin}^{\cos} t \, du$$

$$= \frac{\cancel{\sin}^{\cos} t}{4} \; \frac{\cancel{\cos} \cdot \sin(2u-t)}{-1} \Big|_0^t$$

$$= -\frac{\cancel{\sin}^{\cos} t}{4}\left(\sin(\cancel{m}\,t) - \sin(-t)\right)$$

$$= -\frac{\sin^{\cancel{2}} t \cos t}{2} \cos t$$

Solution of DE using Laplace Transforms

Let $y = y(t)$

$Y = Y(s) = L[y]$

$$L[y'] = sY - y(0)$$

$$L[y''] = s^2Y - sy(0) - y'(0)$$

Q1. Solve $y'' + y = 0$; $y(0) = 1$ and $y'(0) = 0$

ans: <u>M-1</u>

$$m^2 + 1 = 0$$
$$m = \pm i$$

$$y = c_1 \cos t + c_2 \sin t$$

$y(0) = 1$ $\Rightarrow$ $1 = c_1$

$y'(0) = 0$ $\quad 0 = +c_2$

soln: $y = \cos t$

<u>M-2</u> - Laplace Transform

$$y'' + y = 0 \quad \text{—(1)}$$

Take L of (1)

$$L[y''] + L[y] = L[0]$$

$$s^2Y - sy(0) - y'(0) + sY - y(0) = 0$$

$$s^2Y + sY - s = 0$$

Solve for Y

$$Y = \frac{s}{s^2+1}$$

$\therefore \; y = L^{-1}[Y] = \cos t$

Q1. Solve $y'' + y = \sin t$, $y(0) = 0$, $y'(0) = 0$

ans:

$$s^2Y - sy(0) - y'(0) + \cancel{s}Y - y(0) = \cancel{s}\frac{1}{s^2+1}$$

$$s^2Y + \cancel{s}Y = \frac{1}{s^2+1}$$

$$Y = \frac{1}{(s^2+1)(s^2+\cancel{1})}$$ PO.

$$\frac{1-\cos 2t}{2}$$

$$= \frac{1}{s(s+1)(s}$$

$$s^2Y + Y = \frac{1}{s^2+1}$$

$$Y = \frac{1}{(s^2+1)(s^2+1)}$$

$y = L^{-1}(Y)$

$\cancel{y =}$ $f(t) = \sin t$
$g(t) = \sin t$.

$$y = \int_0^t \sin u \sin(t-u)\, d\cancel{t}u.$$

$$= \int_0^t \sin u\,(\sin t\cos u - \cos t\sin u)\,du$$

$$= \sin t \cdot \frac{\sin^2 u}{2}\Big|_0^t - \frac{\cos t}{2}\left(u - \frac{\sin 2t}{2}\right)\Big|_0^t$$

$$= \frac{\sin^3 t}{2} - \frac{\cos t}{2}\left(t - \frac{\sin 2t}{2}\right)$$

Q2. Solve $y'' + 2y' + 5y = e^{-t}\sin t$

$y(0) = 0$, $y'(0) = 1$

ans:

$$s^2Y - \cancel{sy(0)}^{\,0} - y'(0) + 2sY - \cancel{2y(0)}^{\,0} + 5Y = \frac{1}{(s+1)^2+1}$$

$$s^2Y + 2sY + 5Y - 1 = \frac{1}{(s+1)^2+1}$$

$$\cancel{\frac{(s+1)^2+2}{((s+1)^2+1)(s^2-1)^2}}$$

$$(s^2+2s+5)\,Y = \frac{(s+1)^2+2}{(s+1)^2+1}$$

$$Y = \frac{(s+1)^2+2}{(s^2+2s+5)((s+1)^2+1)}$$

$$= \int_0^t \sin u \, (\sin t \cos u - \cos t \sin u) \, du$$

$$= \sin t \cdot \frac{\sin^2 u}{2} \Big|_0^t - \frac{\cos t}{2}\left(u - \frac{\sin 2u}{2}\right)\Big|_0^t$$

$$\therefore \frac{\sin^3 t}{2} - \frac{\cos t}{2}\left(t - \frac{\sin 2t}{2}\right)$$

Q2. Solve $y'' + 2y' + 5y = e^{-t} \sin t$

$y(0) = 0, \quad y'(0) = 1$

ans:

$$s^2 Y - \cancel{s y(0)}^{\,0} - y'(0) + 2sY - \cancel{2y(0)}^{\,0} + 5Y = \frac{1}{(s+1)^2 + 1}$$

$$s^2 Y + 2sY + 5Y - 1 = \frac{1}{(s+1)^2 + 1}$$

$$\cancel{\frac{(s+1)^2 + 2}{((s+1)^2 + 1)(s^2 - 1)^2}}$$

$$(s^2 + 2s + 5) Y = \frac{(s+1)^2 + 2}{(s+1)^2 + 1}$$

$$Y = \frac{(s+1)^2 + 2}{(s^2 + 2s + 5)((s+1)^2 + 1)}$$

$$Y = \frac{s^2 + 2s + 3}{(s^2+2s+2)(s^2+2s+5)} \quad -(2)$$

$$y = L^{-1}[Y]$$

$$Y = \frac{As + B}{s^2 + 2s + 2} + \frac{Cs + D}{s^2 + 2s + 5}.$$

$$A + C = 0$$

$$2A + B + 2C + D = 1$$

$$\Rightarrow \cancel{B + D} = 0 \qquad B + D = 1$$

$$2B + 5A + 2D + 2C = 2.$$

$$\begin{array}{r} 5A + 2C = 0 \\ 2A + 2C = 0 \\ \hline 3A = 0 \end{array}$$

$$\underline{A = 0}$$

$$\underline{C = 0}$$

$$5B + 2D = 3$$

$$3B + 2 = 3$$

$$B = 1/3$$

$$D = 2/3$$

$$Y = \frac{1}{3}\frac{1}{s^2+2s+2} + \frac{2}{3}\frac{1}{s^2+2s+5}$$

Solution of Integral Equations

$$L\left[\int_0^t f(t)\,dt\right] = \frac{F(s)}{s}$$

Q1. Solve $y(t) - \int_0^t y(u)\sin(t-u)\,du = \cos t$

ans: $y - y * \sin t = \cos t$

Taking Laplace Transform

$$Y - Y \cdot \frac{1}{s^2+1} = \frac{s}{s^2+1}$$

$$Y\left(\frac{s^2}{s^2+1}\right) = \frac{s}{s^2+1}$$

$$Y = 1/s$$

$$y = L^{-1}[Y]$$

$$y = \cancel{t}\,1$$

Q2. Solve $y(t) + \int_0^t (t-u)\,y(u)\,du = 1$

ans:

$$y + y * t = 1.$$

$$Y + Y \cdot \frac{1}{s^2} = \frac{1}{s}$$

$$Y\left(\frac{s^2+1}{s^2}\right) = \frac{1}{s}$$

$$Y = \frac{s}{s^2+1}$$

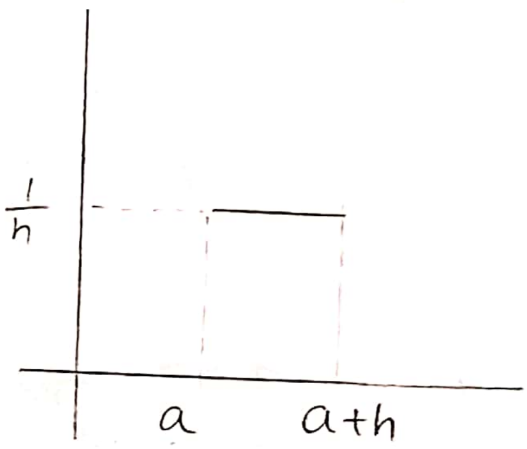
$$y = \cos t$$

Dirac delta function [unit - impulse fn]

delta fn $\delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$

~~physical~~ ex:

$$\delta_h(t-a) = \begin{cases} \frac{1}{h} & ; a \leq t \leq a+h \\ 0 & ; \text{o/w} \end{cases}$$



area $= \frac{1}{h} \times h = 1$

notation :- $\delta(t-a) = \lim_{h \to 0} \delta_h(t-a)$

$$L[\delta(t-a)] = e^{-as}$$

MATRICES

Matrix : rectangular arrangement of real / complex nos

$m \times n$: m = no of rows
n = no of columns

n-row vector , n - column vector

Consider $n \times 1$ matrix $\begin{bmatrix} a_{11} \\ \vdots \\ a_{n1} \end{bmatrix}$ n - column vector

$1 \times n$ matrix $[a_{11} \ a_{12} \ \cdots \ a_{1n}]$ n row vector

Equality of matrix

$A = B$

→ order of both matrices should be same
→ corresponding (all) elements are equal

Operations on Matrices

① Addition $A + B = C$
* both should have the same order

C is the matrix obtained by adding the corresponding elements

It is commutative $A+B = B+C$

associative $(A+B)+C = A+(B+C)$

(2) Scalar multiplication

Each and every element of matrix is multiplied by the scalar multiplier

(3) Product of Matrices

$$A_{m \times \underline{n}} \times B_{\underline{n} \times p} = C_{m \times p}$$

$$A = [a_{ij}]_{m \times n} \quad B = [b_{ij}]_{n \times p} \; ; \; C = [c_{ij}]_{m \times p}$$

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

$$A = \begin{bmatrix} a_{i1} & a_{i2} & \cdots & a_{in} \end{bmatrix} \quad B = \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{bmatrix}$$

* multiplication is not always commutative

* " is associative $A(BC) = (AB)C$

Diagonal Matrix, upper triangular, lower triangular matrix

* all these are defined only for square matrix

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & \vdots \\ \vdots & & \ddots & \vdots \\ a_{m1} & \cdots & \cdots & a_{mn} \end{bmatrix}$$

→ upper triangular (include diag)

→ main diagonal

lower triangular

diagonal matrix : non zero elements only on the main diagonal

i.e $a_{ij} = 0$; $i \neq j$

upper Δ matrix, U = except the upper Δ all elements should be 0

$$U \cdot U = U$$

$$L \cdot L = L$$

Simultaneous System of Linear Equations

Consider,

$$a_{11}x_1 + a_{12}x_2 + \cdots a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \cdots a_{2n}x_n = b_2$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \cdots a_{mn}x_n = b_m$$

* m equations with n unknowns

→ a_{ij}'s and b_i's are known scalars and x_i's are the unknowns.

The above system has m equations and there are n unknowns

The system is $AX = B$ — (1) where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

A = matrix of coefficients

Solution : a set of values $x_1, x_2 \cdots x_n$ is called a soln of the system if it satisfies the system.

If atleast one soln exists, we say that the system is consistent.

If no soln is possible, system is called inconsistent.

ex :(1) $x + y = 1$ $2x + 2y = 4$ (2) $x + y = 1$ $x - y = 0$ (3) $x + y = 1$ $2x + 2y = 2$

inconsistent no soln — one soln — ∞ soln

$$A_{m \times n} \quad X_{n \times 1} = B_{m \times 1}$$

Augmented Matrix

For the system $AX = B$, the matrix obtained from A by adding an extra coloumn as B is called the augmented matrix

$$[A, B] \text{ or } [A\ B]$$

Elementary Row operations of [A, B]

The following are the elementary row operation

(1) Interchanging any 2 rows. If ith and jth rows are interchanged we shall write $R_i \longleftrightarrow R_j$

(2) Multiply any row by a non zero const. If ith row is multiplied by k, we shall write $R_i \longrightarrow kR_i$, $k \neq 0$

(3) Adding a const multiple of another row with a row. If row i is added with k times row j, we write

$R_i \longrightarrow R_i + kR_j$

Row Reduced Form of a Matrix

We say that a matrix is in the row reduced form if the following are true:

(1) 0 rows appear only after non zero rows

(2) In a non zero row the 1st non zero element is 1 and all elements directly below that are 0's

(3) The 1st non zero element in a row appears only after atleast one column after the non zero element in the prev row

Note: Applying row elimination we change the system but will not change the solns

[A, B] after row oper [A', B']

soln of both r same

GAUSS ELIMINATION

To solve the system $AX = B$, we consider the augmented matrix $[A, B]$ using elementary row operations, conv it into a row reduced form. Then we write the soln by the method of back substitution.

using gauss elimination solve the system

1) $x + 2y + z = 3$

$2x + 3y - z = -6$

$3x - 2y - 4z = -2$

Ans:

The augmented matrix:

$$[A, B] = \begin{bmatrix} 1 & 2 & 1 & 3 \\ 2 & 3 & -1 & -6 \\ 3 & -2 & -4 & -2 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1 \qquad R_3 \to R_3 - 3R_1$

$$= \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & -1 & -3 & -12 \\ 0 & -8 & -7 & -11 \end{bmatrix}$$

$R_2 \to -1R_2 \qquad R_3 \to R_3 + 8R_2$

$$= \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 1 & 3 & 12 \\ 0 & 0 & 17 & 85 \end{bmatrix}$$

$R_3 \to \frac{1}{17} R_3$

$$= \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 1 & 3 & 12 \\ 0 & 0 & 1 & 5 \end{bmatrix}$$

Hence system is

$$x + 2y + z = 3$$
$$y + 3z = 12$$
$$z = 5$$

By back substitution we get,

$\therefore \; z = 5 \; , \; y = -3 \; , \; x = 4$

Q. By method of Gauss elimination solve

$x + 3y = 4$

$2x - y = 1$

$3x + 2y = 5$

$5x + 15y = 20$

Ans: The augmented matrix

$$[A, B] = \begin{bmatrix} 1 & 3 & 4 \\ 2 & -1 & 1 \\ 3 & 2 & 5 \\ 5 & 15 & 20 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$ $\quad R_3 \to R_3 - 3R_1$ $\quad R_4 \to R_4 - 5R_1$

$$= \begin{bmatrix} 1 & 3 & 4 \\ 0 & -7 & -7 \\ 0 & -7 & -7 \\ 0 & 0 & 0 \end{bmatrix}$$

$R_2 \to -\frac{1}{7} R_2$ $\quad R_3 \to R_3 + 7R_1$

$$\begin{bmatrix} 1 & 3 & 4 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

System is

$$x + 3y = 4$$
$$y = 1$$

By back substitution

$$y = 1 \, , \; x = 1$$

Q3. By Gauss elimination solve

$$2c + 3d = 4$$
$$a + 3c + d = 2$$
$$a + b + 2c = 0$$

ans. The augmented matrix

$$[A, B] = \begin{bmatrix} 0 & 0 & 2 & 3 & 4 \\ 1 & 0 & 3 & 1 & 2 \\ 1 & 1 & 2 & 0 & 0 \end{bmatrix}$$

$R_1 \leftrightarrow R_2$, $R_3 \to R_3 - R_1$

$$\begin{bmatrix} 1 & 0 & 3 & 1 & 2 \\ 0 & 0 & 2 & 3 & 4 \\ 0 & 0 & -1 & -1 & -2 \end{bmatrix}$$

Row reduced form

$$\begin{bmatrix} 1 & 0 & 3 & 1 & 2 \\ 0 & 1 & -1 & -1 & -2 \\ 0 & 0 & 1 & 3/2 & 2 \end{bmatrix}$$

System is :

$$a + 3c + d = 2$$
$$b - c - d = -2$$
$$c + 3/2\, d = 2$$

Let $d = d$, arbitary value

$$c = 2 - \frac{3}{2}d$$

$$b = -\frac{1}{2}d \qquad \infty \text{ soln}$$

$$a = -4 + \frac{7}{2}d$$

Q4. By Gauss elimination solve.

$$2x + 4y + 3z = 8$$
$$3x - 4y - 4z = 3$$
$$\cancel{4}\ 5x - z = 12$$

ans: The augmented matrix

$$[A, B] = \begin{bmatrix} 2 & 4 & 3 & 8 \\ 3 & -4 & -4 & 3 \\ 5 & 0 & -1 & 12 \end{bmatrix}$$

$R_2 \to R_2 - 3R_1$ $\quad$ $R_3 \to R_3 - 5R_1$

$R_1 \to \frac{1}{2} R_1$

$$= \begin{bmatrix} 1 & 2 & 3/\cancel{2}4 & 4 \\ 0 & \cancel{10}\,1 & 0.85 & 0.9 \\ 0 & 0 & 0 & 1 \end{bmatrix} \leftarrow \text{Row reduced form}$$

system

$$x + 2y + \frac{3}{\cancel{4}2} z = 4$$

$$y + 0.85 z = 0.9$$

$$0 = 1$$

No soln

LU DECOMPOSITION [Crout's method]

We have to solve a system of n equations in n unknowns. Consider the system $AX = B$ —(1)

$A_{n \times n}$, $X_{n \times 1}$, $B_{n \times 1}$

Find a lower triangular matrix L and an upper triang matrix U such that

$A = LU$

Then $(1) \Rightarrow (LU)X = B$

$$L(UX) = B \quad -(2)$$

Take $UX = Y$ — (3)

$(2) \Rightarrow \quad L \cdot Y = B$

We solve this system for Y using forward substitution

Then $(3) \Rightarrow$ ~~$LY = B$~~ $UX = Y$

This equation is solved for X by backward substitution.

To Find L and U

Suppose A is 3X3

Take L as a 3x3 matrix

$$L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}_{3\times 3}$$

$$U = \begin{bmatrix} \cancel{u_{11}}\,1 & u_{12} & u_{13} \\ 0 & 1\,\cancel{u_{22}} & u_{23} \\ 0 & 0 & 1\,\cancel{u_{33}} \end{bmatrix}$$

We find L_{ij} and U_{ij} by equating the elements from $A = LU$

Q1) Using LU decomposition solve

$$x + y = 2$$
$$2x + 3y = 5$$

ans: Here $A = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$

Take

$$L = \begin{bmatrix} l_{11} & 0 \\ l_{21} & l_{22} \end{bmatrix} \qquad u = \begin{bmatrix} 1 & u_{12} \\ 0 & 1 \end{bmatrix}$$

If $A = LU$

$$\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} l_{11} & 0 \\ l_{21} & l_{22} \end{bmatrix} \begin{bmatrix} 1 & u_{12} \\ 0 & 1 \end{bmatrix}$$

$l_{11} = 1$

$l_{11} \cdot u_{12} + \cancel{1} = 1$

$\cancel{k}\ u_{12} = \cancel{0}1$

$l_{21} = 2$

$l_{21} \cdot u_{12} + l_{22} = 3$

$l_{22} = \cancel{2}1$

<u>Note</u>:

* Arrange the system such that the main diagonal of A is non zero

$$L = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \qquad u = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$X = \begin{bmatrix} x \\ y \end{bmatrix} \qquad B = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$$

Take

$$(LU)x = B$$

$$L(ux) = B$$

$$ux = y \quad -(3)$$

$$LY = B$$

$$\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$$

$$\alpha = 2$$

$$2\alpha + \beta = 5 \Rightarrow \beta = \cancel{3}1$$

$$Y = \begin{bmatrix} 2 \\ \cancel{3}1 \end{bmatrix}$$

$$(1) \rightarrow UX = Y$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$x + y = 2$$

$$y = 1$$

$$\Rightarrow x = 1 \qquad X = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Q2. By LU decomposition, solve

$$x + y + z = 9$$

$$2x - 3y + 4z = 13$$

$$3x + 4y + 5z = 40$$

ans:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -3 & 4 \\ 3 & 4 & 5 \end{bmatrix}$$

$$L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \qquad U = \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

$A = LU$

$l_{11} = 1$

$l_{11} u_{12} = 1 \Rightarrow u_{12} = 1.$

$l_{11} u_{13} = 1 \Rightarrow u_{13} = 1.$

$l_{21} + \cancel{l_{22}} = 2$

$l_{21} u_{12} + l_{22} = -3$

$l_{22} = -5$

$l_{21} u_{13} + l_{22} u_{23} = \cancel{3}\, 4.$

$u_{23} = 2/-5$

$l_{31} = 3$

$l_{31} u_{12} + l_{32} = 4$

$l_{32} = 1$

$l_{31} u_{13} + l_{32} u_{23} + l_{33} = 5$

$3 - \frac{2}{5} + l_{33} = 5$

$l_{33} = 12/5$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & -5 & 0 \\ 3 & 1 & 12/5 \end{bmatrix}$$

$$U = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -2/5 \\ 0 & 0 & 1 \end{bmatrix}$$

$$LY = B$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & -5 & 0 \\ 3 & 1 & 12/5 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 9 \\ 13 \\ 40 \end{bmatrix}$$

$$\alpha = 9$$

$$2\alpha - 5\beta = 13 \Rightarrow \beta = 1$$

$$3\alpha + \beta + \frac{12}{5}\gamma = 40 \Rightarrow \gamma = 5$$

$$Y = \begin{bmatrix} 9 \\ 1 \\ 5 \end{bmatrix}$$

$UX = Y$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -2/5 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 1 \\ 5 \end{bmatrix}$$

$x + y + z = 9$

$y - 2/5\, z = 1$

$\underline{z = 5}$

$\underline{y = 3}$ $\underline{x = 1}$

Q3 Using LU decomposition, solve:

$2y + z = -1$

$x + y + 3z = 8$

$2x - y - z = 1$

DETERMINANTS

Suppose A is a square matrix. Corresponding to matrix A. we define a scalar denoted as det A or |A| as follows.

If $A = [a_{11}]$

we define

$$|A| = a_{11}$$

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$$

If $A = [a_{ij}]_{n \times n}$

M_{ij} as the determinant of the subsquare matrix obt by deleting the ith row and jth column

M_{ij} is called the minor of element a_{ij}

we define $A_{ij} = (-1)^{i+j} M_{ij}$, A_{ij} is called the i,j th cofactor

$|A| = a_{i1}A_{i1} + a_{i2}A_{i2} + \dots + a_{in}A_{in}$; $i = 1, 2$

OR

$|A| = a_{1j}A_{1j} + a_{2j}A_{2j} + \dots \; a_{nj}A_{nj}$; $j = 1, 2, 3$

Properties

(1) $|A| = |A^T|$

(2) If we interchange any 2 rows/columns the value of the determinant changes only in sign

(3) In matrix A, if any 2 rows/columns are identical then $|A| = 0$

(4) If any rows or coloumn is added with a const multiple of another row/coloumn, the det does not change.

(5) If a matrix is either lower/upper triangular, then value of determinants it is the pdt of elements ~~of~~ on diagonal

Rank of a Matrix

Suppose A is a $m \times n$ matrix, we say that the rank of A is r if ~~the~~:

(i) the matrix has a non zero minor of order r

(2) Any minor of A having order gr than r is zero

Linear Independance

We consider a set of vectors $v_1, v_2 \dots v_m$. we say that these vectors are linearly independent if $\alpha_1 v_1 + \alpha_2 v_2 + \dots \alpha_m v_m = 0 \Leftrightarrow \alpha_1 = \alpha_2 = \dots \alpha_m = 0$

We say that the vectors $v_1, v_2, \dots v_m$ are linearly dependent if there exists scalars $\alpha_1, \alpha_2, \dots \alpha_m$ with atleat one $\alpha_i \neq 0$ such that $\alpha_1 v_1 + \alpha_2 v_2 + \dots \alpha_m v_m = 0$

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Q1. Check for Linear independance

(1) $v_1 = [1 \quad 2]$ $\quad v_2 = [3 \quad 4]$

ans: Suppose $\alpha_1 v_1 + \alpha_2 v_2 = 0$ for some scalar α_1, α_2

$$\alpha_1 [1 \quad 2] + \alpha_2 [3 \quad 4] = 0$$

$$[\alpha_1 + 3\alpha_2 \quad 2\alpha_1 + 4\alpha_2] = 0$$

$$\alpha_1 + 3\alpha_2 = 0$$
$$2\alpha_1 + 4\alpha_2 = 0$$

$$\Rightarrow \alpha_2 = 0, \alpha_1 = 0$$

$\therefore$ v_1 and v_2 are LI

(2) $v_1 = [1 \quad 2 \quad 3]$ $\quad v_2 = [2 \quad 4 \quad 0]$

$v_3 = [0 \quad 0 \quad 1]$

ans:

$$\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = 0$$

$$\alpha_1 [1 \quad 2 \quad 3] + \alpha_2 [2 \quad 4 \quad 0] + \alpha_3 [0 \quad 0 \quad 1] = 0$$

$\alpha_1 + 2\alpha_2 = 0$ $\qquad \alpha_1 = -2$

$2\alpha_1 + 4\alpha_2 = 0$ $\qquad \alpha_2 = 1$

$3\alpha_1 + \alpha_3 = 0$ $\qquad \alpha_3 = 6$

LD

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(3) $v_1 = 0$

$v_2 = [2 \quad 1 \quad 3]$

$v_3 = [4, 8, 9]$

always LD

If $\alpha_1, \alpha_2 \dots \alpha_n$ are scalars and $v_1, v_2 \dots v_n$ are vectors then the expression $\alpha_1 v_1 + \alpha_2 v_2 + \dots \alpha_n v_n$ is called a linear combination of the vectors $v_1, v_2 \dots v_n$

Result

1. A set of vectors $\{v_1, v_2 \dots v_n\}$ ~~are~~ is LD if and only if one of them is a linear combination of other

Proof:

If $v_1, v_2 \dots v_n$ are linearly dependent then $\alpha_1 v_1 + \alpha_2 v_2 + \dots \alpha_n v_n = 0$ {has some non zero soln} for atleast one $\alpha_i \neq 0$

Suppose $\alpha_1 \neq 0 \Rightarrow v_1 = \dfrac{-\alpha_2 v_2 - \alpha_3 v_3 \dots - \alpha_n v_n}{\alpha_1}$

2. If the set of vectors $\{v_1, v_2 \ldots v_n\}$ contains the zero vector, then the set is LD

3. If $\{v_1, v_2 \ldots v_n\}$ is LI, then any subset of it is also linearly independent. If the above set is LD then any superset of it also linearly dependent.

Let A be a matrix. The max no of LI rows of A is known as row rank of A

The max no of LI columns of A is known as the column rank of A

Result:

For a matrix A, row rank is equal to its column rank and that is known as the rank of matrix.

Elementary Row operations does not change the rank of a matrix

Then the rank of A is the no of non zero rows of the row reduced matrix

$A \simeq B$ (B is equivalent to A)

B is obt from A using row operation.

Q1. Find the rank of the matrix

$$A = \begin{bmatrix} 1 & 3 & 4 \\ 2 & -1 & 1 \\ 3 & 2 & 5 \\ 5 & 15 & 20 \end{bmatrix}$$

ans:

After row operation, we get

$$\cancel{A} \begin{bmatrix} 1 & 3 & 4 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$r(A) = 2$ (rank)

Q2. Find the rank of

$$A = \begin{bmatrix} 1 & 2 & 1 & 3 \\ 2 & 3 & -1 & -6 \\ 3 & -2 & -4 & -2 \end{bmatrix}$$

ans: Row reduced form of A

$$A \simeq \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 1 & 3 & 12 \\ 0 & 0 & 1 & 5 \end{bmatrix}$$

$r(A) = 3$ as three non-zero rows are there

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Q3. Check for Linear Independence

$$\begin{bmatrix} 2 \\ 6 \\ 2 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} \begin{bmatrix} 8 \\ 16 \\ -3 \end{bmatrix}$$

ans: Consider $A = \begin{bmatrix} 2 & 3 & 8 \\ 6 & 1 & 16 \\ 2 & 2 & -3 \end{bmatrix}$

Row reduced form $\quad R_1 \to \frac{R_1}{2}$ R

$$A = \begin{bmatrix} 1 & 3/2 & 4 \\ 6 & 1 & 16 \\ 2 & 2 & -3 \end{bmatrix}$$

final ~~row~~ ~~reduced~~ 'form'

$R_2 \to R_2 - 6R_1$
$R_3 \to R_3 - 2R_1$

$$\sim \quad A = \begin{bmatrix} 1 & 3/2 & 4 \\ 0 & -8 & -8 \\ 0 & -1 & -11 \end{bmatrix}$$

$R_2 \to \frac{R_2}{-8} \qquad R_3 \to R_3 + R_1$

$$\begin{bmatrix} 1 & 3/2 & 4 \\ 0 & 1 & 1 \\ 0 & 0 & -10 \end{bmatrix}$$

$r(A) = 2$

* If A is a matrix of rank r then by elementary row operations and column operations we can bring to the form

$$A \sim \left[\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right] \qquad I_r = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & & \ddots \end{bmatrix}$$

where I_r is the identity matrix of order r. This form is known as the normal form of the matrix

Consistency of System of Equation (Imp)

Consider the system $AX = B$ where A is a $m \times n$, X is $n \times 1$, and B is $m \times 1$.

Result:

The system $AX = B$ is consistent iff $r(A) = r([AB])$ { $[AB]$ = aug matrix }

Case 1: If $r(A) < r(AB)$

Then system is inconsistent or no soln.

Case 2: If $r(A) = r(AB) = r = n$

n is the no of unknowns ($A_{m\times n}$)

then unique soln

Case 3 If $r(A) = r(AB) = r < n$

Infinitely many soln.

Here $n-r$ variables can be chosen arbitarly ~~values~~.

Q1. Check if the system is consistent.

$x + y - z = 0$

$2x - y + z = 3$

$4x + 2y - 2z = 2$

Ans: $n = 3$

$$[A\ B] = \begin{bmatrix} 1 & 1 & -1 & 0 \\ 2 & -1 & 1 & 3 \\ 4 & 2 & -2 & 2 \end{bmatrix}$$

After row elementary operations

$$\begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Then rank of A = 2 {don't consider the last colo of [A B]}

$r(A) = 2$

$r([A\ B]) = 2$

$r(A) = r([A\ B]) < 3 \Rightarrow$ system is consistent

$r < n \quad \therefore 2 < 3$

$\therefore \Rightarrow \infty$ many soln

$(n-r)$ can be taken arbitary

System is

$$x + y - z = 0$$
$$y - z = a - 1$$
$$0 = 0$$

Take z arbitary, $z = a$

$$y = -1 + a$$
$$x = 1$$

Q2. Find values of a and b for which the system
$$x + 2y + 3z = 6$$
$$x + 3y + 5z = 9$$
$$2x + 5y + az = b$$
has

(i) no soln
(ii) unique soln
(iii) ∞ soln

Solve in 2nd and 3rd case

ans: $n = 3$

$$[A \ B] = \begin{bmatrix} 1 & 2 & 3 & 6 \\ 1 & 3 & 5 & 9 \\ 2 & 5 & a & b \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 3 & 6 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & a-6 & b-12 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 3 & 6 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & a-8 & b-15 \end{bmatrix}$$

(i) If $a=8$; $b \neq 15$

$r(A) = 2$

$r([A\ B]) = 3$

$r(A) < r([A\ B])$ $\therefore$ no soln

(ii) If $a \neq 8$, ~~$b \neq 15$~~ b = any value

$r(A) = 3$

$r([A\ B]) = 3$

unique soln

System is

$$x + 2y + 3z = 6$$
$$y + 2z = 3$$
$$(a-8)z = b-15$$

$z = \dfrac{b-15}{a-8}$ $\quad y = \dfrac{3a-2b+6}{a-8}$

$x = \dfrac{b-15}{a-8}$

(iii) If $a=8$, $b=15$

$r(A) = r([A\ B]) = 2 < 3$

System is:

$$x + 2y + 3z = 6$$
$$y + 2z = 3$$

∞ many soln

Let z be p

$$y = 3 - 2p$$

$$x + 6 - p = 6$$

$$x = 6 + p$$

Q3. Find values of a, b for which the foll sys has

(i) no soln (ii) unique soln (iii) ∞ soln

$$x + y + z = 6$$
$$x + 2y + 3z = 3$$
$$x + 2y + az = b$$

Homogeneous and non-Homogeneous

Consider the system $AX = B$, if $B = 0$ the system is called homogeneous and the system will always be consistent

If $r(A) = 3$ (Then $|A| \neq 0$) then system has unique soln. $X = 0$ is ~~always~~ only a soln

If $r(A) < 3$ ($|A| = 0$) then ∞ soln

Non homogeneous system

$$AX = B \quad -(1)$$

corresponding HE, $AX = 0 \quad -(2)$

If X_1 is a soln of (1) and X_2 is any soln of (2) then $X = X_1 + X_2$ is a soln of (1)

$AX_1 = B$

$AX_2 = 0$

$$AX = A(X_1 + X_2)$$
$$= AX_1 + AX_2$$
$$= B$$

X_g is the general soln of (2)
X_p is any soln of (1)

$$X = X_g + X_p$$

Inverse of a Matrix

$$A_{m \times n} \cdot B = I_n = B \cdot A$$

B is called A^{-1}

$|A| = 0 \Rightarrow$ singular

$$A^{-1} = \frac{1}{|A|} \operatorname{adj}(A)$$

$AX = B$

$\cancel{X = (BA^{T})}$ if A^{-1} exists

$X = A^{-1} B$

VECTOR SPACE

Definition

Let V be a non-empty set, elements of V are called as vectors

1. In V there is defined an addition of vectors denoted by $+$ with the following properties

   a) If v_1 and $v_2 \in V$, then $v_1 + v_2 \in V$

   (b) $v_1 + v_2 = v_2 + v_1$ for all $v_1, v_2 \in V$

   (c) ~~There~~ $v_1 + (v_2 + v_3) = (v_1 + v_2) + v_3$ for any $v_1, v_2, v_3 \in V$

   (d) There exists $0 \in V$ such that $v + 0 = v$ for all $v \in V$

   (e) For $v \in V$, there exists $-v \in V$
   $v + (-v) = 0$

2) Let R be the real ~~AO~~ line. Elements of R are called as scalars. Between any element in R and any element in V there is defined a scalar multiplication with the following properties.

(a) If $\alpha \in R$, $v \in V$, then $\alpha v \in V$
(V is closed under scalar multi)

(b) $1 \in R$, then $1v = v$ for all $v \in V$

(c) $\alpha, \beta \in R$, $v \in V$ then $\alpha(\beta v) = (\alpha\beta)v$

(d) $\alpha, \beta \in R$, $v \in V$ $(\alpha+\beta)v = \alpha v + \beta v$
(Distributive)

(e) If $\alpha \in R$; $v_1, v_2 \in V$ then $\alpha(v_1 + v_2)$
$= \alpha v_1 + \alpha v_2$
(Distributive over vector space)

The set V is called a vector space over R if all the above prop are satisfied

Ex: $V = R^2 = \{(x,y), x,y \in R\}$
If $(x_1, y_1), (x_2, y_2) \in V$
$(x_1, y_1) + (x_2, y_2) = \{x_1 + x_2, y_1 + y_2\}$

If $\alpha \in R$, $(x,y) \in V$ then $\alpha(x,y) = (\alpha x, \alpha y)$

Under this def of addition and scalar multiplication R^2 is a vector space over R

(2) $V = C = \{a+ib; (a,b) \in R\}$

If $z_1 = x_1 + iy_1$
$z_2 = x_2 + iy_2$
$z_1 + z_2 = x_1 + x_2 + i(y_1 + y_2)$

$\alpha z_1 = \alpha x_1 + i \alpha y_1$

(3) $V = \{ (x, 0), x \in R \}$

ans: ~~use~~ $(x_1, 0) + (x_2, 0) = (x_1 + x_2, 0)$

$\alpha(x_1 + x_2) + 0$ ✓

V is a vector space over R.

(4) $V = \{ (x, 1) \mid x \in R \}$

Not a vector space

Reason (1): 0 is not an element i.e (0,0)

(2): addn not possible

Note: If a line is passing through origin then it is a vector space

(5) ⇒ $V = R^n = \{ (x_1, x_2 \ldots x_n) \mid x_1, x_2 \ldots x_n \in R \}$

n dimensional Euclidean space

R^n is a vector space over R

(6) V = set of all m×n matrices

V is a vector space as addition of 2 m×n matrices is a m×n matrix

Similarly scalar pdt.

V = set of all square matrices

(7) V is not a vector space.

as 2×2, 3×3 all are there

{but if it is square matrice of order n}

{ V is a vector space

(8) Set V = set of all cont fn from $R \to R$

V is a vector space

If $f(x)$, $g(x)$ are 2 fns

$(f+g)(x) = f(x) + g(x)$

$\alpha[f(x)] = \alpha f(x)$

V is a vector space

(9) V = set of all discont fn

V is not a vector space.

(10) V = Set of all polynomial of degree $\leq n$ with real coeff.

V is a vector space

degree $= n$; not a vector space

(11) V = set of all soln of $y'' + y = 0$

V is a vector space