NATIONAL INSTITUTE OF TECHNOLOGY CALICUT

Department of Mathematics, Winter Semester 2019-2020

MA1002D MATHEMATICS II, Tutorial-4

1. Find the spectrum and eigenvectors of the following matrices

a) $\begin{bmatrix} 10 & -4 \\ 18 & -12 \end{bmatrix}$ b) $\begin{bmatrix} 3 & 0 & 0 \\ 0 & -8 & 0 \\ 0 & 0 & 4 \end{bmatrix}$ c) $\begin{bmatrix} 3 & 5 & 3 \\ 0 & 4 & 6 \\ 0 & 0 & 1 \end{bmatrix}$

2. Are the following matrices symmetric, skew symmetric or orthogonal? Find their eigenvalues

a) $\begin{bmatrix} 0.96 & -0.28 \\ 0.28 & 0.96 \end{bmatrix}$ b) $\begin{bmatrix} 1 & 4 \\ -4 & 1 \end{bmatrix}$ c) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$ d) $\begin{bmatrix} 0 & 9 & -12 \\ -9 & 0 & 20 \\ 12 & -20 & 0 \end{bmatrix}$

3. Prove the following for a square matrix A

   (a) $\lambda = 0$ is an eigenvalue of A if and only if A is nonsingular

   (b) If λ is an eigenvalue of A, then $a_0\lambda^2 + a_1\lambda + a_2$ is the eigenvalue of $B = a_0A^2 + a_1A + a_2I$

   (c) Both A and A^T have the same eigenvalues

   (d) If A is nonsingular and if λ is an eigenvalue of A then $\frac{1}{\lambda}$ is an eigenvalues of A^{-1}

   (e) If λ is an eigenvalue of A then λ^k is an eigenvalue of A^k for any positive integer k

   (f) If λ is an eigenvalue of A then $\lambda + k$ is an eigenvalue of $A + kI$ where k is a scalar.

   (g) Let λ_1 & λ_2 be the distinct eigenvalues of A and x_1, x_2 be the corresponding eigenvectors. Prove that $x_1 + x_2$ is not an eigenvector of A.

   (h) Eigenvalues of an idempotent matrix A are either zero or one.

   (i) For a real symmetric matrix, show that the eigenvectors corresponding to two distinct eigenvalues are orthogonal.

   (j) If λ is an eigenvalue of an orthogonal matrix A, then $1/\lambda$ is also an eigenvalue of A

4. Prove that

   (a) If is any square matrix of order, then the matrix is always symmetric matrix.

   (b) If is any square matrix of order, then the matrix is always skew symmetric matrix.

   (c) Any square matrix can be expressed as sum of symmetric matrix and skew symmetric matrix.

5. Without (actually) finding all eigenvalues, find the sum and the product of the eigenvalues of each of the following matrices A, B and C

6. A matrix B is said to be similar to A if $B = P^{-1}AP$, for an invertible matrix P. Prove that similar matrices have same eigenvalues. Verify this for the following. Also show that $X = PY$ if X is an eigenvector of A and Y an eigenvector of $P^{-1}AP$.

$$A = \begin{bmatrix} 10 & -3 & 5 \\ 0 & 1 & 0 \\ -15 & 9 & -10 \end{bmatrix}, \quad P = \begin{bmatrix} 2 & 0 & 3 \\ 0 & 1 & 0 \\ 3 & 0 & 5 \end{bmatrix}$$

7. Prove that

(a) If A and B are symmetric matrices, then AB is symmetric if and only if AB = BA.
(b) If a matrix A can be diagonalized using an orthogonal matrix, then A is symmetric.
(c) If A and B are symmetric matrices, then AB and BA have same eigenvalues but different eigenvectors.
(d) If A and B are symmetric matrices, then $A^{-1}B$ and BA^{-1} have same eigenvalues but different eigenvectors.

8. If possible, diagonalize the matrix and compute

9. If A , then compute , where .

10. The eigenvectors of a 3 x 3 matrix A corresponding to the eigenvalues 1, 1, 3 are $[1\ 0\ -1]^T$, $[0\ 1\ -1]^T$, $[1\ 1\ 0]^T$ respectively. Find the matrix A.

11. Find the defect of each of the eigenvalues of the following matrices.
If possible, find a basis of eigenvectors and hence diagonalize the following matrices.

a) $\begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}$ b) $\begin{bmatrix} 16 & 0 & 0 \\ 48 & -8 & 0 \\ 84 & -24 & 4 \end{bmatrix}$ c) $\begin{bmatrix} 5 & 10 & -10 \\ 10 & 5 & -20 \\ 5 & -5 & -10 \end{bmatrix}$

12. Use Cayley Hamilton theorem to find the inverse of $\begin{bmatrix} 1 & 2 & 4 \\ -1 & 0 & 3 \\ 3 & 1 & -2 \end{bmatrix}$.

13. If $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$, verify Cayley Hamilton theorem and hence find the matrix represented by $A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 - 8A^2 - 2A + 1$.

14. Find the symmetric coefficient matrix C of the quadratic form $Q = X^TCX$ given by

a) $4x_1^2 - 8x_1x_2 + 5x_2^2$ b) $-2x_1^2 + 2x_1x_3 + 4x_2x_3 - 9x_3^2$ c) $0.5x_1^2 + 0.8x_1x_2 - 1.4x_2x_3$

d) $(x_1 - 2x_2 + 3x_3 - x_4)^2$e) $(x_1 - x_2)^2 - 4x_3^2$

15. Classify each of the following quadratic forms:
(a) $3x_1^2 + 3x_2^2 + 6x_3^2 - 2x_1x_2 - 4x_1x_2$
(b) $x_1^2 + 4x_2^2 + 4x_3^2 + 4x_1x_2 + 4x_1x_3 + 6x_2x_3$
(c) $x_1^2 + 2x_2^3 + 2x_3^2 + 4x_4^2 - 2x_1x_2 + 2x_2x_3 + 6x_3x_4$

16. Find out the conic section represented by the given quadratic form. Transform it into principal axes
a) $7x_1^2 + 6x_1x_2 + 7x_2^2 = 200$ b) $9x_1^2 - 6x_1x_2 + x_2^2 = 40$ c) $32x_1^2 - 60x_1x_2 + 7x_2^2 = -52$

17. Reduce the quadratic forms to sum of squares form/Canonocal form/ Principal axes form, and find the corresponding linear transformation. Also find the index and signature
a) $6x_1^2 + 3x_2^2 + 3x_3^2 - 4x_1x_2 - 2x_2x_3 + 4x_3x_1$
b) $10x^2 + 2y^2 + 5z^2 + 6yz - 10zx - 4xz$
c) $4x^2 + 3y^2 + z^2 - 8xy - 6yz + 4yz$
