Special Functions :-

1) Gamma functions . $\Gamma x = \int_0^{\infty} e^{-t} t^{x-1} dt \quad x > 0$

2) β function

3) Bessel function $\Big\}$ diff. eq.

4) legendre's fn

Gamma functions :-

$$\Gamma x = \int_0^{\infty} e^{-t} t^{x-1} dt$$

$e^{-t} \rightarrow$ is alway positive

(i) $\Gamma x \geq 0$

(ii) $\Gamma 1 = 1$

n is the positive integer

(iii)

$$\Gamma n = \int_0^{\infty} e^{-t} t^{n-1} . dt$$

$$= \underbrace{\left[t^{n-1}(-e^{-t})\right]_0^{\infty}}_{0} + \int_0^{\infty} (n-1) t^{n-2} e^{-t} . dt$$

$$= 0 + (n-1)\Gamma n-1$$

$\left.\begin{array}{l} t^{n-2} \rightarrow \infty \\ e^{-t} \rightarrow 0 \end{array}\right\}$ at upper limit

$$= (n-1)\Gamma n-1$$

But the rate at which $e^{-t} \rightarrow 0$ is more than the other $\therefore$ It had become zero.

$$= (n-1)(n-2) \text{----} 2 \times \Gamma 1$$

$$\Gamma n = (n-1)!$$

(iv) $\Gamma\frac{1}{2} = \int_0^\infty e^{-t} t^{-1/2} . dt$

put $\sqrt{t} = y$

$t = y^2$

$dt = 2y\,dy$

$$\int_0^\infty e^{-t} t^{-1/2} dt = \int_0^\infty e^{-y^2} . y^{-1} \times 2y\,dy$$

$$= \int_0^\infty 2e^{-y^2} . dy$$

$$\left(\Gamma\tfrac{1}{2}\right)^2 = 2\int_0^\infty e^{-u^2} du \,.\, 2\int_0^\infty e^{-v^2} . dv$$

u and v both are independent Var.

$$= 4\int_0^\infty e^{-(u^2+v^2)} . du\,dv$$

When $u = r\cos\theta$, $v = r\sin\theta$

~~du = r d\theta~~ , ~~dv = r d\theta~~

~~du = r(-\sin\theta) d\theta + \cos\theta\, dr~~

Applying the Jacobian

$$= 4\int_{\theta=0}^{\pi/2}\int_{r=0}^{\infty} e^{-r^2} . r\,dr . d\theta$$

put $r^2 = z$

$2r.dr = dz$

$r\,dr = dz/2$

$$= 4\int_{\theta=0}^{\pi/2}\int_0^\infty e^{-z} . \frac{dz}{2} . d\theta$$

$$= 2\int_{\theta=0}^{\pi/2}\int_0^\infty e^{-z} . dz\,d\theta$$

$$= 2\int_0^{\pi/2} \left(-e^{-z}\right)_0^\infty . d\theta$$

$$= 2\int_0^{\pi/2} 1 . d\theta$$

$$= \cancel{2} \times \frac{\pi}{\cancel{2}}$$

$$= \pi$$

$$\left(\Gamma\tfrac{1}{2}\right)^2 = \pi$$

$$\boxed{\Gamma\tfrac{1}{2} = \sqrt{\pi}}$$

$$\Gamma\tfrac{3}{2} = \tfrac{1}{2}\Gamma\tfrac{1}{2},$$

$$\boxed{\Gamma\tfrac{3}{2} = \tfrac{1}{2}\sqrt{\pi}}$$

If 'n' is +ve integer then prove

$$\Gamma\left(n+\tfrac{1}{2}\right) = \frac{2n!}{2^{2n} . n!}\sqrt{\pi}$$

$\therefore$

$$\Gamma\left(n+\tfrac{1}{2}\right) = (n-\tfrac{1}{2})(n-\tfrac{3}{2})(n-\tfrac{5}{2}) \ldots \tfrac{1}{2}\sqrt{\pi}$$

$$= \frac{(2n-1)(2n-3)(2n-5)\ldots 1\sqrt{\pi}}{2^n}$$

$$= \frac{(2n)!\ \sqrt{\pi}}{2^n \left(\overset{(2n)}{2n-2}\right)(2n-4)\ldots 2)},$$

$$= \frac{(2n)!\ \sqrt{\pi}}{2^n \times 2^n\ (n)(n-1)(n-2)\ldots 1}$$

$$= \frac{(2n)!\ \sqrt{\pi}}{(2^{2n})\ (n!)}$$

$$\boxed{\Gamma\left(n+\tfrac{1}{2}\right) = \frac{(2n)!}{2^{2n}\ (n!)}\sqrt{\pi}}$$

* $\underline{\beta}$ - Functions:-

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1} dx$$

Here there is no quaries, it is clearly bounded one.

$$B(n,m) = \int_0^1 y^{n-1}(1-y)^{m-1} dy$$

$$B(m,n) = B(n,m)$$

Prove that $B(m,n) = \dfrac{\Gamma m\,\Gamma n}{\Gamma m+n}$

$$\Gamma m = \int_0^\infty e^{-t} t^{m-1} . dt$$

put $t = x^2$

$dt = 2x\,dx$

$$\Gamma m = \int_0^\infty e^{-x^2} x^{2m-1} \times 2dx$$

$$= 2\int_0^\infty e^{-x^2} x^{2m-1} dx$$

$$\Gamma n = 2\int_0^\infty e^{-y^2} y^{2n-1} . dy$$

$$\Gamma m\,\Gamma n = 4\int_0^\infty e^{-x^2} x^{2m-1} . e^{-y^2} . y^{2n-1} . dx\,dy$$

$$= 4\int_0^\infty e^{-(x^2+y^2)} x^{2m-1} y^{2n-1} dx\,dy$$

Put $x = r\cos\theta$, $y = r\sin\theta$

$$\Gamma m\,\Gamma n = 4\int_0^{\pi/2}\int_0^{\infty} e^{-r^2}\cdot r^{2(m+n)-2}\cdot\cos^{2m-1}\theta\,\sin^{2n-1}\theta\, r\,dr\,d\theta$$

~~Put~~ $r^2 = z$ $\quad 2r\,dr = dz$

$$\Gamma m\,\Gamma n = 4\int_0^{\pi/2}\int_0^{\infty} e^{-z}$$

$$= 2\int_0^{\pi/2}\left(2\int_0^{\infty} e^{-r^2}\cdot r^{2(m+n)-1}\,dr\right)\cos^{2m-1}\theta\,\sin^{2n-1}\theta\,d\theta$$

$$= 2\int_0^{\pi/2}\Gamma{m+n}\cdot\cos^{2m-1}\theta\,\sin^{2n-1}\theta\,d\theta$$

$$= 2\Gamma{m+n}\int_0^{\pi/2}\cos^{2m-1}\theta\cdot\sin^{2n-1}\theta\,d\theta$$

$$\beta(m,n) = \int_0^1 x^{m-1}\cdot(1-x)^{n-1}$$

Put $x = \cos^2\theta$

$dx = -2\cos\theta\sin\theta\,d\theta$

$$\beta(m,n) = \int_{\pi/2}^{0}\cos^{2m-2}\theta\cdot\sin^{2n-2}\theta\cdot(-2)\cos\theta\sin\theta\,d\theta$$

$$\boxed{\beta(m,n) = 2\int_0^{\pi/2}\cos^{2m-1}\theta\cdot\sin^{2n-1}\theta\,d\theta}$$

$\therefore\ \Gamma m\,\Gamma n = \Gamma{m+n}\,\beta(m,n)$

Hence proved.

$$\Gamma x = \int_0^{\infty} e^{-t}\,t^{x-1}\,dt$$

$\Gamma x = (x-1)\Gamma{x-1}$; $\Gamma\tfrac{1}{2} = \sqrt{\pi}$

$\Gamma n = (n-1)!$ if n is an integer

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx = \beta(n,m)$$

$$\beta(m,n) = \frac{\Gamma m\,\Gamma n}{\Gamma{m+n}}$$

Prove that $\displaystyle\int_0^1 \frac{1}{\sqrt{-\ln x}}\,dx = \sqrt{\pi}$

put $-\ln x = y$

$x = e^{-y}$

$dx = -e^{-y}dy$

then $\displaystyle\int_{\infty}^{0}\frac{(-e^{-y})\,dy}{\sqrt{y}}$

$$= \int_0^{\infty}\frac{e^{-y}}{\sqrt{y}}\,dy$$

$$= \int_0^{\infty} e^{-y}\,y^{-1/2}\,dy$$

$= \Gamma\tfrac{1}{2}$

$= \sqrt{\pi}$

Prove that $\int_0^\infty e^{-m^2x^2}\,dx = \frac{\sqrt{\pi}}{2m}$

Sol:- put $m^2x^2 = t$ then $2m^2x\,dx = dt$

$$\int_0^\infty e^{-t}\left(\frac{dt}{2m^2x}\right) = \frac{1}{2m^2}\int_0^\infty e^{-t}x^{-1}\,dt = \frac{1}{2m^2}\int_0^\infty \frac{e^{-t}}{\sqrt{\frac{t}{m^2}}}\,dt$$

$$= \frac{m}{2m^2}\int_0^\infty e^{-t}t^{-1/2}\,dt$$

$$= \frac{1}{2m}\int_0^\infty e^{-t}t^{-1/2}\,dt = \frac{1}{2m}\Gamma\tfrac{1}{2} = \frac{\sqrt{\pi}}{2m}$$

Prove that $\Gamma x\,\Gamma 1-x = \frac{\pi}{\sin\pi x}$ (x is not an integer)

Assume $\int_0^\infty \frac{x^{p-1}}{(1+x)}\,dx = \frac{\pi}{\sin\pi p}$

$$B(m,n) = \frac{\Gamma m\,\Gamma n}{\Gamma m+n}$$

then $\Gamma m\,\Gamma n = B(m,n)\,\Gamma m+n$

Let us prove

$$B(m,n) = \int_0^\infty \frac{y^{m-1}}{(1+y)^{m+n}}\,dy$$

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

put $x = \frac{1}{1+y}$

then $dx = \frac{-1}{(1+y)^2}\,dy$

$$B(m,n) = \int_\infty^0 \left(\frac{1}{1+y}\right)^{m-1}\left(1-\frac{1}{1+y}\right)^{n-1}\left(\frac{-1}{(1+y)^2}\right)dy$$

$$= \int_0^\infty \left(\frac{1}{1+y}\right)^{m-1}\left(\frac{y}{1+y}\right)^{n-1}\left(\frac{+1}{(1+y)^2}\right)dy$$

$$= \int_0^\infty \frac{y^{n-1}}{(y+1)^{m-1+n-1+2}}\,dy = \int_0^\infty \frac{y^{n-1}}{(1+y)^{m+n}}\,dy = \int_0^\infty \frac{y^{m-}}{(1+y)}$$

Now $\Gamma m\,\Gamma n = B(m,n)\,\Gamma m+n$

$\Gamma x\,\Gamma 1-x = B(x,1-x)\,\Gamma \cancel{x}+1-\cancel{x}$

$$\left(\int_0^\infty \frac{(k)^{x-1}}{(k)^{x+1-x}}\,dx\right)\Gamma 1 = \int_0^\infty \frac{k^{x-1}}{(k+1)}\,dx = \frac{\pi}{\sin}$$

$$\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}x\,dx \cdot \cos^{2n-1}x\,dx$$

$2m-1=0$

$$\int_0^{\pi/2} \sin^4 x \cdot \cos^4 x \cdot dx = \frac{1}{2}\beta\left(\frac{5}{2}, \frac{5}{2}\right)$$

$$= \frac{1}{2}\frac{\Gamma\frac{5}{2}\Gamma\frac{5}{2}}{\Gamma\left(\frac{5}{2}+\frac{5}{2}\right)} = \frac{1}{2}\frac{\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)^2}{\Gamma 5}$$

$$= \frac{1}{2}\frac{\left(\frac{3}{2}\left(\frac{1}{2}\right)\sqrt{\pi}\right)^2}{4!}$$

$$= \frac{1}{2}\frac{(9\pi)}{16\,(24)} = \frac{3\pi}{256}$$

$2m = 3/2$

$m = 3/4$

Show that $\int_0^{\pi/2}\sqrt{\tan x}\,.dx = \frac{\pi}{\sqrt{2}}$

$$= \int_0^{\pi/2} \sin^{1/2}x \cdot \cos^{-1/2}x\,dx = \frac{1}{2}\beta\left(\frac{3}{4}, \frac{1}{4}\right)$$

$3/4 - 1$

$-1/4$

$$= \frac{1}{2}\frac{\Gamma\frac{3}{4}\Gamma\frac{1}{4}}{\Gamma 1} = \frac{1}{2}\left(\frac{\pi}{\sin \pi/4}\right)$$

$1-x$

$1-1/4$

$$= \frac{1}{2}\left(\frac{\pi}{1/\sqrt{2}}\right)$$

$$= \frac{\pi}{\sqrt{2}}$$

$$\int_0^{\pi/2} \sin^m x \cdot dx = \frac{1}{2}\beta\left(\frac{m+1}{2}, \frac{1}{2}\right) = \frac{1}{2}\beta\left(\frac{1}{2}, \frac{m+1}{2}\right) = \int_0^{\pi/2}\cos^m x\,dx$$

$m=4$

$$= \frac{1}{2}\frac{\Gamma\frac{m+1}{2}\Gamma\frac{1}{2}}{\Gamma\left(\frac{m}{2}+1\right)} = 1/2$$

$$\frac{1}{2}\times\frac{\left(\frac{m-1}{2}\right)\left(\frac{m-3}{2}\right)\left(\frac{m-5}{2}\right)\ldots\ldots\sqrt{\pi}}{\frac{m}{2}\times\frac{m-2}{2}\times\frac{m-4}{2}\times\frac{m-6}{2}\ldots}$$

if m is even then

$$\int_0^{\pi/2}\sin^m x\,dx = \frac{\frac{m-1}{2}\cdot\frac{m-3}{2}\cdot\frac{m-5}{2}\ldots\frac{1}{2}\Gamma\frac{1}{2}\sqrt{\pi}}{2\,\frac{m}{2}\cdot\frac{m-2}{2}\cdot\frac{m-4}{2}\ldots\frac{2}{2}}$$

$$= \frac{1}{2\times 2}\frac{(m-1)(m-3)(m-5)\ldots(1)(\sqrt{\pi})(\sqrt{\pi})}{(m)(m-2)(m-4)\ldots(1)}$$

('2' will be cancelled one two will here in deno Extra..

$$= \frac{1}{4}\frac{(m-1)(m-3)(m-5)\ldots(1)\,\pi}{(m)(m-2)(m-4)\ldots(1)}$$

if m is odd then

$$\int_0^{\pi/2}\sin^m x\cdot dx = \frac{1}{2}\times\frac{\frac{m-1}{2}\cdot\frac{m-3}{2}\cdot\frac{m-5}{2}\ldots\frac{2}{2}\sqrt{\pi}}{\frac{m}{2}\cdot\frac{m-2}{2}\cdot\frac{m-4}{2}\ldots\frac{1}{2}\sqrt{\pi}} =$$

$$= \frac{m-1}{m}\cdot\frac{m-3}{m-2}\ldots\ldots\frac{2}{3}\times 1$$

$$\int_0^{\pi/2} \sin^8 x\,dx = \frac{7}{8}\times\frac{5}{6}\times\frac{3}{4}\times\frac{1}{2}\times\frac{\pi}{2}$$

$$= \frac{35\pi}{256}$$

$$\int_0^{\pi/2} \cos^9 x\,dx = \left(\frac{8}{9}\right)\left(\frac{6}{7}\right)\left(\frac{4}{5}\right)\left(\frac{2}{3}\right)$$

$$= \frac{128}{315}$$

• Prove that $\int_0^1 \frac{1}{(1-x^6)^{1/6}}\,dx = \pi/3$

Put $x^2 = \sin\theta$

then $3x^2\,dx = \cos\theta\,d\theta$

$$\int_0^{\pi/2} \frac{1}{(1-\sin^2\theta)^{1/6}} \cdot \frac{\cos\theta\,d\theta}{3\sin^{2/3}}$$

$2m-1 = 2/3$

$2m = 2/3 + 1$

$2m = 5/3$

$$= \int_0^{\pi/2} \frac{\cos\theta\,d\theta}{3\cos^{2\times 1/6}\theta\,.\sin^{2/3}} = \int_0^{\pi/2} \frac{1}{3}\frac{\cos\theta\,d\theta}{\cos^{1/3}.\sin^{2/3}}$$

$$= \frac{1}{3}\int_0^{\pi/2} \cos^{2/3}\theta\,.\sin^{-2/3}.d\theta\,.$$

$$= \frac{1}{6}\,B\left(\frac{1}{6},\frac{5}{6}\right) = \frac{1}{6}\,\frac{\Gamma\frac{1}{6}\,\Gamma\frac{5}{6}}{\Gamma 1} = \frac{1}{6}\,\Gamma\tfrac{1}{6}\,\Gamma\tfrac{5}{6}$$

$$= \frac{1}{6}\,\frac{\pi}{\sin\frac{\pi}{6}} = \frac{\pi\times 2}{6_3} = \frac{\pi}{3}$$

$\frac{35}{4\times 7}$ / $\frac{}{4\ 5}$

Evaluate $\int_0^a x^9\sqrt[3]{a^6-x^6}\,.dx\,. \Rightarrow \frac{a^{12}\pi}{9\sqrt{3}}$

put $\cancel{x = a\sin\theta}$ $\quad x^6 = a^6 y$ $\qquad \sin(\pi - \pi/3)$

$\cancel{dx = a\cos\theta\,d\theta}$ $\quad 6x^5.dx = a^6 dy$ $\qquad -\sin\pi/3 = -$

$\int_0^{\pi/2} a^9\sin^9\theta\,($ $\qquad +\sqrt{3}$

$$\int_0^a x^9\sqrt[3]{a^6-x^6}\,dx = \int_0^1 \sqrt[3]{a^6-a^6y}\,.\frac{a^6.dy}{6\,a^5.y^{5/6}} \qquad a^9.y^{9/6}$$

$$= \int_0^1 \frac{1}{6}(a^6)^{1/3}(1-y)^{1/3}.a^{10}.y^{9/6-5/6}\,dy$$

$$= \frac{1}{6}\int_0^1 a^{12}.y^{4/6}.(1-y)^{1/3}.dy$$

$m = 2/3 +$

$2m = 2/3 + 1$

$5/6$

$$= \frac{a^{12}}{6}\int_0^1 y^{2/3}.(1-y)^{1/3}dy\,.$$

$1/3 + 1$

$$= \cancel{\frac{a^{12}}{6}\,\frac{\Gamma\frac{5}{3}\Gamma\frac{4}{3}}{\Gamma 1}} = \cancel{\frac{a^{12}\pi}{6\sin\pi/3}} = \cancel{\frac{a^{12}}{6}\,\frac{\pi\times 2}{\sqrt{3}\times\sqrt{3}}} = \cancel{\frac{a^{12}\pi\sqrt{3}}{9}}$$

$$= \frac{a^{12}}{6}\,\frac{\Gamma\frac{5}{3}\,\Gamma\frac{4}{3}}{\Gamma 3} = \frac{a^{12}}{6\times 2}\,\cancel{\frac{\pi}{\ldots}}\ \Gamma\tfrac{2}{3}\,\Gamma\tfrac{1}{3} = \frac{a^{12}}{12}\,\frac{\pi}{\sin\pi/3}$$

$$= \frac{a^{12}}{6\times 2} \qquad = \frac{a^{12}}{\frac{12}{6}}\,\frac{\pi\times 2}{\sqrt{3}}$$

$$\beta(m,n)=\int_0^\infty \frac{y^{m-1}}{(1+y)^{m+n}}\,dy=\int_0^\infty \frac{y^{n-1}}{(1+y)^{m+n}}\,dy$$

Prove that $\beta(m,n)=\int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}\,dx$

Sol:- We have $\beta(m,n)=\int_0^\infty \frac{y^{m-1}}{(1+y)^{m+n}}\,dy$

$$=\int_0^1 \frac{y^{m-1}}{(1+y)^{m+n}}\,dy+\int_1^\infty \frac{y^{m-1}}{(1+y)^{m+n}}\,dy$$

put $y=1/t$ in the second integral

then $dy=-1/t^2\,dt$

$$\int_1^\infty \frac{y^{m-1}}{(1+y)^{m+n}}\,dy=\int_1^0 \frac{(1/t)^{m-1}}{(1+1/t)^{m+n}}\left(\frac{-1}{t^2}\right)dt$$

$$=+\int_0^1 \frac{t^{m+n}}{(t+1)^{m+n}\cdot t^{m-1+2}}\,dt$$

$$=\int_0^1 \frac{t^{m+n-m-1}}{(t+1)^{m+n}}\,dt$$

$$=\int_0^1 \frac{t^{n-1}}{(t+1)^{m+n}}\,dt \qquad \Gamma n+1=n\Gamma n$$

$$\therefore\ \beta(m,n)=\int_0^1 \frac{y^{m-1}}{(1+y)^{m+n}}\,dy+\int_0^1 \frac{y^{n-1}}{(1+y)^{m+n}}\,dy$$

$$=\int_0^1 \frac{y^{m-1}+y^{n-1}}{(1+y)^{m+n}}\,dy$$

Prove then $\frac{\beta(m,n+1)}{n}=\frac{\beta(m+1,n)}{m}=\frac{\beta(m,n)}{m+n}$

$$\beta(m,n+1)=\frac{\Gamma m\,\Gamma n+1}{\Gamma m+n+1}=\frac{\Gamma m\,\Gamma n\,(n)}{\Gamma m+n\ (m+n)}=\frac{n}{m+n}\,\beta(m,n)$$

$$\frac{\beta(m,n+1)}{n}=\frac{\beta(m,n)}{m+n}$$

In the same method the other one.

$$\frac{\beta(m+1,n)}{m}=\frac{\Gamma m+1\,\Gamma n}{m\,\Gamma m+n+1}=\frac{\cancel{m}\,\Gamma m\,\Gamma n}{\cancel{m}(m+n)\,\Gamma m+n}=\frac{\beta(m,n)}{m+n}$$

What is $\beta(m+1,n) + \beta(m,n+1) = \frac{\beta(m,n)\times m}{m+n} + \frac{n\times\beta(m,n)}{m+n}$

$= \frac{\beta(m,n)}{\cancel{m+n}}\,(\cancel{m+n})$

$= \beta(m,n)$

Evaluate $\int_0^\infty x^8 . e^{-2x} . dx$

put $2x = y$

then $2dx = dy$

$\int_0^\infty \left(\frac{y}{2}\right)^8 e^{-y} . \frac{dy}{2} = \int_0^\infty \frac{1}{2^9}\, y^8 . e^{-y} dy$

$= \frac{1}{2^9}\,\Gamma 9 = \frac{1}{2^9}(8!) = \frac{\cancel{8}\times 7\times \cancel{6}\times 5\times \cancel{4}\times 3\times \cancel{2}}{\cancel{2}\times \cancel{2}\times \cancel{2}\times \cancel{2}\times \cancel{2}\times 2^4} = \frac{4\times 7\times 9\times 5}{2\times 4}$

$= \frac{35\times 9}{4} = \frac{315}{4}$

Evaluate $\int_0^\infty x\sqrt{x}\, e^{-x^2} dx$

$x = y^{1/2}$

$x^2 = y$

$2x\,dx = dy$

$\int_0^\infty \frac{dy}{2} (y^{1/2})^{1/2} e^{-y}$

$= \frac{1}{2}\int_0^\infty e^{-y} y^{1/4} = \frac{1}{2}\Gamma\frac{5}{4} = \frac{1}{2}\cdot\frac{1}{4}\Gamma\frac{1}{4} = \frac{1}{8}\Gamma\frac{1}{4}$

Laplace Transforms

$f(t)$

$L\{f(t)\} = \int_0^\infty e^{-st} f(t).dt = F(s)$ $\boxed{s > k}$

(i) $f(t)$ should be piecewise continous

(a,b) $x_0\ x_1\ x_2$? . . . x_n

then $(a\ x_0)(x_0\ x_1)$ - - - - (x_n, b)

It should be continous in these intervals

(ii) $f(t)$ is of exponential order

$|f(t)| < Me^{-kt}$

$\boxed{M,k>0}$

If this two conditions are satisfied then the Laplace transform exist.

Theorem:

Let $f(t)$ be a function ie piecewise continous on every finite interval in the range $t \geq 0$ and satisfies $|f(t)| < Me^{-kt}$ for some k and k positive then $L\{f(t)\}$ exists for all $s > k$

Example

$f(t) = 1$

$L\{f(t)\} = \int_0^\infty e^{-st} . 1 . dt = -\frac{1}{s}\left(e^{-st}\right)_0^\infty = \frac{1}{s}$

2) $f(t)=e^{at}$

$$L\{f(t)\} = \int_0^\infty e^{-st}.e^{at}dt = \int_0^\infty e^{-(s-a)t}.dt = \frac{1}{s-a} \quad \boxed{s>a}$$

3) $f(t)=e^{-at}$

$$L\{f(t)\} = \frac{1}{s+a}$$

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4) $$L\{\sin at\} = \int_0^\infty e^{-st}\sin at\,dt$$

$$= -\sin at\left(\frac{e^{-st}}{s}\right)_0^\infty + \int_0^\infty a\cos at.\frac{e^{-st}}{s}.dt$$

$$= \cancel{\sin at}(1)\Big|_0^\infty + \frac{a}{s}\left((\cos at)\left(-\frac{e^{-st}}{s}\right)\right)_0^\infty - \frac{a}{s}\int_0^\infty -a\sin at\left(-\frac{e^{-st}}{s}\right)$$

$$I = \frac{\cancel{\sin at}}{s} + \frac{a}{s^2}\overset{\cos 0}{\cancel{\cos at}} - \frac{a^2}{s^2}\int_0^\infty e^{-st}\sin at$$

$$I\left(1+\frac{a^2}{s^2}\right) = \frac{\cancel{\sin at}}{s} + \frac{a}{s^2}\overset{\cos 0}{\cancel{\cos at}}$$

$$I\frac{(s^2+a^2)}{\cancel{s^2}} = \frac{\cancel{s\sin at} + a\cos at}{\cancel{s^2}}$$

$$I(s^2+a^2) = a$$

$$I = \frac{a\cos at + s\sin at}{s^2+a^2} \qquad \boxed{I = \frac{a}{a^2+s^2}}$$

$$L\{\sin at\} = \frac{a}{s^2+a^2}$$

$$L\{\cos at\} = \frac{s}{s^2+a^2}$$

5) $L\{t^n\}$ if n is +ve integer.

$$= \int_0^\infty e^{-st}t^n.dt \qquad \text{put } st=u,\quad t=u/s,\quad dt=\tfrac{1}{s}du$$

$$= \int_0^\infty e^{-u}(u/s)^n.\tfrac{1}{s}du$$

$$= \frac{1}{s^{n+1}}\int_0^\infty e^{-u}u^n.du$$

$$= \frac{1}{s^{n+1}}\Gamma{n+1} = \frac{n!}{s^{n+1}}$$

Linearity Property:-

$$L\{af(t)+bg(t)\} = aL\{f(t)\} + bL\{g(t)\}$$

$$L\{af(t)+bg(t)\} = \int_0^\infty (af(t)+bg(t)).e^{-st}dt$$

$$= a\int_0^\infty e^{-st}f(t).dt + b\int_0^\infty e^{-st}g(t).dt$$

$$= aL\{f(t)\} + bL\{g(t)\}$$

Example:- $L\{t^2+5t+4\} = L\{t^2\} + 5L\{t\} + 4L\{1\}$

$$= 2/s^3 + 5/s^2 + 4/s$$

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2) $L\{\cos t \cos 2t\} = L\{\frac{1}{2}(\cos 3t + \cos t)\}$

$= \frac{1}{2} L\{\cos 3t\} + \frac{1}{2} L\{\cos t\}$

$= \frac{1}{2}\frac{s}{s^2+9} + \frac{1}{2}\frac{s}{s^2+1} = \frac{s}{2}\left(\frac{1}{s^2+9} + \frac{1}{s^2+1}\right)$

$\cos 2t = 2c$

3) $L\{\cos^2 \omega t\} = L\left\{\frac{1+\cos 2\omega t}{2}\right\}$

$= \frac{1}{2} L\{1\} + \frac{1}{2} L\{\cos 2\omega t\} = \frac{1}{2}(1/s) + \frac{1}{2}\left(\frac{s}{s^2+4\omega^2}\right)$

$= \frac{1}{2}\left(\frac{s}{s^2+4\omega^2} + \frac{1}{s}\right)$

Shifting Theorem:-

$L\{f(t)\} = F(s) \qquad (s>k)$

$L\{e^{at} f(t)\} = F(s-a) \qquad (s-a>k)$

Proof:- $L\{f(t)e^{at}\} = \int_0^\infty e^{at} \cdot e^{-st} \cdot f(t) \cdot dt$

$= \int_0^\infty e^{-(s-a)t} f(t) \cdot dt$

$= F(s-a)$

Example:- 1) $L\{t^2\} = \frac{2}{s^3}$ $\quad L\{e^{at} t^2\} = \frac{2}{(s-a)^3}$

2) $L\{\cosh at\} = L\left\{\frac{e^{at}+e^{-at}}{2}\right\} = \frac{1}{2} L\{e^{at}\} + \frac{1}{2} L\{e^{-at}\}$

$= \frac{1}{2}\frac{1}{s-a} + \frac{1}{2}\frac{1}{s+a} = \frac{1}{2}\left(\frac{s+\cancel{a}+s-\cancel{a}}{s^2-a^2}\right)$

$= \frac{s}{s^2-a^2}$

3) $L\{\sinh at\} = \frac{a}{s^2-a^2}$

Inverse transforms:-

$L\{af(t) + bg(t)\} = aL\{f(t)\} + bL\{g(t)\}$

$= aF(s) + bG(s)$

$L^{-1}\{aF(s) + bG(s)\} = af(t) + bg(t)$

$= aL^{-1}\{f(t)\} + bL^{-1}\{g(t)\}$

Example: 1) $L^{-1}\left\{\frac{s+2}{s^2+1}\right\}$ $\qquad$ 2) $L^{-1}\left\{\frac{s^2+2s+4}{s^5}\right\}$

$= L^{-1}\left\{\frac{s}{s^2+1} + \frac{2}{s^2+1}\right\}$

$= L^{-1}\left\{\frac{s}{s^2+1}\right\} + 2L^{-1}\left(\frac{1}{s^2+1}\right)$

$= \cos t + 2\sin t$

$= L^{-1}\left(\frac{1}{s^3}\right) + 2L^{-1}\left(\frac{1}{s^4}\right) + 4L^{-1}\left\{\frac{1}{s^5}\right\}$

$= \frac{t^2}{2!} + \frac{2}{3!}t^3 + \frac{4}{4!}t^4 = \frac{t^2}{2}\left(1 + \frac{2}{3}t + \frac{t^2}{3}\right)$

$= \cancel{\frac{t^3}{3!}\left(1 + \frac{2t}{4} + \frac{t^2}{5}\right)}$

Laplace Transform of Derivatives:-

Theorem:- $f(t)$ is continuous : $t \geq 0$; exponential order for some k, M

$f'(t)$ is piecewise cont. $t \geq 0$

then the laplace transform of $f'(t)$ exists i.e,

$L\{f'(t)\}$ exists, $s > k$

$$L\{f'(t)\} = sL\{f(t)\} - f(0)$$

Proof:- $f'(t)$ is piecewise and exponentional order

when $f(t)$ is exponential order then $f(t)$ also

$$L\{f'(t)\} = \int_0^\infty e^{-st} f'(t)\,dt$$

$$= \left(e^{-st} f(t)\right)_0^\infty + \int_0^\infty s e^{-st} f(t)\,dt$$

$$= 0 - f(0) + s\int_0^\infty e^{-st} f(t)\,dt$$

$$= sL\{f(t)\} - f(0)$$

If all the conditions are satisfied then $L\{f''(t)\} = sL\{f'(t)\} - f'(0)$

$$= s\{sL\{f(t)\} - f(0)\} - f'(0)$$

$$L\{f''(t)\} = s^2L\{f(t)\} - sf(0) - f'(0)$$

$$\therefore \boxed{L\{f^n(t)\} = s^nL\{f(t)\} - s^{n-1}f(0) - s^{n-2}f'(0) \cdots\cdots f^{n-1}(0)}$$

Example:-

$L\{t\sin\omega t\} = L\{f(t)\}$

Let $f(t) = t\sin\omega t$

$f'(t) = t\omega\cos\omega t + \sin\omega t$

$f''(t) = -t\omega^2\sin\omega t + \omega\cos\omega t + \omega\cos\omega t$

$= -\omega^2 t\sin\omega t + 2\omega\cos\omega t$

$L\{f''(t)\} = L\{-\omega^2 t\sin\omega t + 2\omega\cos\omega t\} = \cancel{s} -\omega^2 L\{t\sin\omega t\} + 2\omega$

$L\{f''(t)\} = s^2L\{f(t)\} - sf(0) - f'(0)$

$= s^2L\{t\sin\omega t\} - s(0) - 0$

$-\omega^2L\{t\sin\omega t\} + 2\omega L\{\cos\omega t\} = s^2L\{t\sin\omega t\}$

$2\omega \frac{s}{s^2+\omega^2} = (s^2+\omega^2)L\{t\sin\omega t\}$

$$\boxed{L\{t\sin\omega t\} = \frac{2\omega s}{(s^2+\omega^2)^2}}$$

1) solve $y'+3y=10\sin t$ initial condition $y(0)=0$

Sol:- Take transformer on both sides

$$L\{y'+3y\} = L\{10\sin t\}$$

$$L\{y'\}+3L\{y\} = L\{\sin t\}\times 10$$

$$sL\{y\} - f(0) + 3L\{y\} = 10\times\frac{1}{s^2+1}$$

$$(s+3)L\{y\} = \frac{10}{s^2+1}$$

$$L\{y\} = \frac{10}{(s^2+1)(s+3)}$$

Now get the inverse transform to find the solution of y

$$y = L^{-1}\left\{\frac{10}{(s+3)(s^2+1)}\right\}$$

$$\frac{10}{(s+3)(s^2+1)} = \frac{As+B}{s^2+1} + \frac{C}{s+3}$$

$$= \frac{(As+B)(s+3)+C(s^2+1)}{(s^2+1)(s+3)}$$

$$As^2+3As+Bs+3B+Cs^2+C = 10$$

$As^2+Cs^2=0$ $\quad A+C=0$ $\quad \boxed{A=-C}$

$3A+B=0$ $\quad 3A=-B$ $\quad 3(-C)=-B$ $\quad \boxed{B=3C}$

$\boxed{3B+C=10}$

$9C+C=10$

$10C=10$

$\boxed{C=1}$

$\boxed{B=3}$

$\boxed{A=-1}$

$$y = L^{-1}\left\{\frac{-s+3}{s^2+1} + \frac{1}{s+3}\right\}$$

$$y = L^{-1}\left\{\frac{-s+3}{s^2+1}\right\} + L^{-1}\left\{\frac{1}{s+3}\right\}$$

$$= L^{-1}\left\{\frac{-s}{s^2+1}\right\} + L^{-1}\left\{\frac{3}{s^2+1}\right\} + L^{-1}\left\{\frac{1}{s+3}\right\}$$

$$\boxed{y = -\cos t + 3\sin t + e^{-3t}}$$

2) solve $y''+y=2\cos t$ initial conditions $y(0)=3$, $y'(0)=4$

$$L\{y''\} + L\{y\} = 2L\{\cos t\}$$

$$s^2L\{y\} - sy(0) - y'(0) + L\{y\} = 2\frac{s}{s^2+1}$$

$$(s^2+1)L\{y\} - s(3) - 4 = \frac{2s}{s^2+1}$$

$$(s^2+1)L\{y\} = \frac{2s}{s^2+1} + (4+3s)$$

$$L\{y\} = \frac{2s}{(s^2+1)^2} + \frac{3s+4}{s^2+1}$$

$$(s^2+1)L\{y\} = \frac{2s+(4+3s)(s^2+1)}{(s^2+1)}$$

$$L\{y\} = \frac{2s+4s^2+4+3s^3+3s}{(s^2+1)^2} = \frac{3s^3+4s^2+5s+4}{(s^2+1)^2}$$

$$\frac{3s^3+4s^2+5s+4}{(s^2+1)^2} = \frac{Ax+B}{(s^2+1)} + \frac{Cx^3+Dx^2+ex+f}{(s^2+1)^2}$$

$$\frac{(As+B)(s^2+1) + cs^3+Ds^2+es+f}{(s^2+1)^2}$$

$$\frac{As^3+As+Bs^2+B+cs^3+Ds^2+es+f}{(s^2+1)^2}$$

$$\frac{(A+c)s^3+(B+D)s^2+(A+e)s+B+f}{(s^2+1)^2}$$

Now equating the coefficients $A+c=3$ $\boxed{D=f}$

$B+D=4$

$A+e=5$

$B+f=4$

$$y = L^{-1}\left\{\frac{2s}{(s^2+1)^2} + \frac{3s+4}{s^2+1}\right\}$$

$$= L^{-1}\left\{\frac{2s}{(s^2+1)^2}\right\} + 3L^{-1}\left\{\frac{s}{s^2+1}\right\} + 4L^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$$= t\sin\omega t + 3\cos t + 4\sin t = t\sin t + 3\cos t + 4\sin t$$

$\omega=1$

3) Solve $y''+2y'-3y = 6e^{-2t}$ $y(0)=2,\ y'(0)=-14$

$$L\{y''\}+2L\{y'\}-3L\{y\} = 6L\{e^{-2t}\}$$

$$s^2L\{y\}-sy(0)-y'(0)+2\{sL\{y\}-y(0)\}-3L\{y\} = 6\frac{1}{s+2}$$

$$s^2L\{y\}-2s+14+2sL\{y\}-2(2)-3L\{y\} = \frac{6}{s+2}$$

$$(s^2+2s-3)L\{y\}-2s+14-4 = \frac{6}{s+2}$$

$$(s^2+2s-3)L\{y\}-2s+10 = \frac{6}{s+2}$$

$$(s^2+2s-3)L\{y\} = \frac{6}{s+2} + 2s-10$$

$$(s-1)(s+3)L\{y\} = \frac{6}{s+2} + (2s-10)$$

$$L\{y\} = \frac{6}{(s+2)(s+3)(s-1)} + \frac{2s-10}{(s-1)(s+3)}$$

$$\frac{6}{(s+2)(s+3)(s-1)} = \frac{A}{s-1} + \frac{B}{s+2} + \frac{C}{s+3} = \frac{1}{2(s-1)} + \frac{(-2)}{s+2} + \cdot$$

$$= \frac{A(s+2)(s+3)+B(s-1)(s+3)+C(s-1)(s+2)}{(s-1)(s+2)(s+3)}$$

$$(s^2+5s+6)A + (s^2+2s-3)B + (s^2+s-2)C = 6$$

$A+B+C=0$ — ①

$5A+2B+C=0$ — ②

$6A-3B-2C=6$ — ③

$4A+B=0$

$\boxed{4A=-B}$

$A+(-4A)=-C$

$-3A=C$

$\boxed{C=3A}$

$6A-3(-4A)-2(3A$

$\cancel{6A}+12A-\cancel{6A}=6$

$\boxed{A=\frac{1}{2}}$

$\boxed{B=-2}$

$\boxed{C=}$

s^2+3s

$s(s+$

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$$\frac{2s-10}{(s-1)(s+3)} = \frac{A}{s-1} + \frac{B}{s+3} = \frac{-2}{s-1} + \frac{4}{s+3}$$

$$2s-10 = A(s+3) + B(s-1)$$

$$As + Bs = 2s \qquad A + B = 2$$

$$3A - B = -10 \qquad \boxed{A = 2-B}$$

$$3(2-B) - B = -10 \qquad A = 2-4$$

$$6-3B-B = -10 \qquad \boxed{A=-2}$$

$$6-4B = -10$$

$$16 = 4B$$

$$\boxed{B=4}$$

$$L\{y\} = \frac{1}{2(s-1)} + \frac{(-2)}{s+2} + \frac{3}{2(s+3)} + \frac{(-2)}{s-1} + \frac{4}{s+3}$$

$$= \frac{1}{s-1}\left\{\tfrac{1}{2}-2\right\} + \frac{1}{s+3}\left\{\tfrac{3}{2}+4\right\} + \left\{\frac{-2}{s+2}\right\}$$

$$= \frac{-3}{2(s-1)} + \frac{11}{2(s+3)} + \frac{(-2)}{s+2}$$

$$y = \frac{-3}{2}L^{-1}\left\{\frac{1}{s-1}\right\} + \frac{11}{2}L^{-1}\left\{\frac{1}{s+3}\right\} - 2L^{-1}\left\{\frac{1}{s+2}\right\} = \frac{-3}{2}e^{t} + \frac{11}{2}e^{-3t} - 2e^{-2t}$$

Transform of integrals :-

Theorem:- Let $F(s)$ be the transform of $f(t)$, if $f(t)$ is piecewise continous and of exponential order then $\quad L\{f(t)\} = F(s)$

$$L\left\{\int_0^t f(u).du\right\} = \frac{1}{s}F(s)$$

Proof:-

Let $g(t) = \int_0^t f(u).du$

This $g(t)$ is piecewise continous since $f(t)$ is.

$$|g(t)| \le \int_0^t |f(u)|.du \le \int_0^t Me^{ku}du = \frac{M}{k}\left(e^{ku}\right)_0^t = \frac{M}{k}\left(e^{kt}-1\right)$$

$$|g(t)| \le \frac{M}{k}e^{kt}$$

$\therefore g(t)$ is of exponential order.

$$g(t) = \int_0^t f(u).du$$

$$g'(t) = f(t)$$

$$L\{g'(t)\} = sL\{g(t)\} - g(0)$$

$$L\{g(t)\} = \frac{L\{g'(t)\} + g(0)}{s} = \frac{L\{f(t)\}}{s} = \frac{F(s)}{s}$$

$$L\left\{\int_0^t f(u).du\right\} = \frac{1}{s}F(s)$$

$$L\left\{\int_0^t f(u).du\right\} = \frac{1}{s}F(s)$$

$$L^{-1}\left\{\frac{1}{s}F(s)\right\} = \int_0^t f(u).du$$

$$= \int_0^t L^{-1}\{F(s)\}.du$$

Find $L^{-1}\left\{\frac{1}{s(s^2+\omega^2)}\right\}$

Let $F(S) = \frac{1}{S^2+\omega^2}$ then $L^{-1}\{F(S)\} = \frac{1}{\omega}\sin\omega t$

$$L^{-1}\left\{\frac{1}{S}\cdot F(S)\right\} = \int_0^t \frac{1}{\omega}\sin\omega u.du$$

$$= \frac{1}{\omega^2}(-\cos\omega u)_0^t$$

$$= \frac{1}{\omega^2} - \cos\omega t + \frac{1}{\omega^2}$$

$$L^{-1}\left\{\frac{-1}{s(s^2+\omega^2)}\right\} = \frac{-\cos\omega t + 1}{\omega^2}$$

$$L\left\{\underbrace{\int_0^t\int_0^t \cdots\cdots \int_0^t}_{n\ \text{times}} f(u).du \cdots\cdots\right\} = \frac{1}{s^n}F(n)$$

Find $L^{-1}\left\{\frac{1}{s^3-s^2}\right\}$

then $L^{-1}\left\{\frac{1}{s^2(s-1)}\right\}$

$$\cancel{\int_0^t (e^t-1)du}$$

$$\cancel{e^t(u)_0^t - (u)_0^t}$$

$$\cancel{e^t}$$

$f(s) = \frac{1}{s-1}$ then $L^{-1}\{F(s)\} = e^t$

$$L^{-1}\left\{\frac{1}{s^2}F(s)\right\} = \int_0^t\int_0^t L^{-1}(F(s))\,du = \int_0^t\int_0^t e^t\,dt\,du = \int_0^t (e^u)_0^t\,dt$$

$$= \int_0^t (e^t-1)dt$$

$$= (e^t - t)_0^t = e^t - 1 - t$$

$$= \underline{e^t - t - 1}$$

Find $L^{-1}\left\{\frac{9}{s^2}\,\frac{(s+1)}{s^2+9}\right\}$

$F(s) = \frac{9(s+1)}{s^2+9}$ $\quad L^{-1}\{F(s)\} = 9L^{-1}\left\{\frac{s}{s^2+9} + \frac{1}{s^2+9}\right\}$

$$= 9(\cos 3t + \tfrac{1}{3}\sin 3t) = 9\cos 3t + 3\sin 3t$$

$$\int_0^t\int_0^t (9\cos 3t + 3\sin 3t)dt\,dt = \int_0^t\left(\frac{9}{3}\sin 3t + \frac{3}{3}(-\cos 3t)\right)_0^t dt$$

$$= \int_0^t (3\sin 3t - \cos 3t)_0^t\,dt$$

$$\int_0^t 3\sin 3t - \cos 3t + 1\,dt$$

$= \cancel{3\cos}$

$$= \left(\frac{-3}{3}\cos 3t - \tfrac{1}{3}\sin 3t + t\right)_0^t = -\cos 3t - \tfrac{1}{3}\sin 3t + t + 1$$

$y'' + y = 2t \qquad y(\pi/4) = \pi/2\,, \quad y'(\pi/4) = 2-\sqrt{2} \qquad \underline{y(0)\quad y'(0)}$

$\tilde{t} = t - \pi/4 \qquad \tilde{y}(0) = \pi/2$

$t = \tilde{t} + \pi/4 \qquad \tilde{y}'(0) = 2-\sqrt{2}$

$y(t) = y(\tilde{t} - \pi/4)$

$= \tilde{y}$

$\tilde{y}'' + \tilde{y} = 2(\tilde{t} + \pi/4) \qquad L(\tilde{y}'') + L(\tilde{y}) = 2\left(L(\tilde{t}) + L(\pi/4)\right)$

$\tilde{y}'' + \tilde{y} =$

$s^2 L\{\tilde{y}\} - s\tilde{y}(0) - \tilde{y}'(0) + L\{\tilde{y}\} = \dfrac{2}{s^2} + \dfrac{2\pi}{4}(1/s)$

$(s^2+1) L\{\tilde{y}\} = \dfrac{2}{s^2} + \dfrac{\pi}{2s} + s(\pi/2) + 2-\sqrt{2}$

$\cancel{= \dfrac{2+\pi}{2s^2}}$

$L\{\tilde{y}\} = \dfrac{2}{(s^2+1)s^2} + \dfrac{\pi}{2s(s^2+1)} + \dfrac{s\pi}{2(s^2+1)} + \dfrac{2-\sqrt{2}}{s^2+1}$

$= \dfrac{2}{s^2(s^2+1)} + \dfrac{2-\sqrt{2}}{(s^2+1)} + \dfrac{\pi}{2(s^2+1)}\left(\dfrac{1}{s} + s\right) = \dfrac{2}{s^2(s^2+1)} + \dfrac{\pi}{2s} + \dfrac{2-\sqrt{2}}{s^2+1}$

$= \dfrac{As+B}{s^2+1} + \dfrac{Cs+D}{s^2} + \dfrac{2}{s^2+1} - \dfrac{\sqrt{2}}{s^2+1} + \dfrac{\pi}{2s}$

$\tilde{y} = L^{-1}\left\{\dfrac{2}{s^2(s^2+1)}\right\} + \dfrac{\pi}{2} L^{-1}\{1/s\} + 2-\sqrt{2}\, L^{-1}\left\{\dfrac{1}{s^2+1}\right\}$

$F(s) = \dfrac{2}{s^2+1} \qquad L^{-1}(F(s)) = 2L^{-1}\left(\dfrac{1}{s^2+1}\right) = 2\sin\tilde{t}$

$L^{-1}\left\{\dfrac{2}{s^2(s^2+1)}\right\} = \int_0^{\tilde{t}} \int_0^{\tilde{t}} 2\sin\tilde{t}\, dt = 2\int_0^{\tilde{t}} \left(-\cos\tilde{t}\right)_0^{t} dt = 2\int_0^{\tilde{t}} -\cos\tilde{t} + 1 \, dt$

$= 2(-\sin\tilde{t})_0^{\tilde{t}} + 2(t)_0^{\tilde{t}}$

$= 2(-\sin\tilde{t}) + 2\tilde{t}.$

$\tilde{y} = -2\sin\tilde{t} + 2\tilde{t} + \dfrac{\pi}{2}(1) + 2-\sqrt{2}\,\sin\tilde{t}$

$= -\sqrt{2}\sin\tilde{t} + 2\tilde{t} + \pi/2$

$y = -\sqrt{2}\sin(t - \pi/4) + 2(t - \pi/4) + \pi/2$

$= -\sqrt{2}\left(\sin t . \cos\pi/4 - \cos t \sin\pi/4\right) + 2t - \cancel{\pi/2} + \cancel{\pi/2}$

$= -\dfrac{\cancel{\sqrt{2}}\sin t}{\cancel{\sqrt{2}}} + \dfrac{\sqrt{2}}{\sqrt{2}}\cos t + 2t = 2t + \cos t - \sin t$

Differentiation of Transform:-

$-F'(s) = L\{f(t).t\}$, $L\{f(t)\} = F(s)$

$\boxed{f(t) = L^{-1}(F(s))}$

Proof:-

$$\frac{d}{ds}F(s) = \frac{d}{ds}\int_0^\infty e^{-st}f(t).dt$$

$$= \int_0^\infty \frac{d}{ds}e^{-st}f(t).dt = \int_0^\infty -te^{-st}f(t).dt = -\int_0^\infty e^{-st}(tf(t))dt$$

$$= -L\{tf(t)\}$$

Extending it to finite no. of times.

$$\boxed{L\{t^n f(t)\} = (-1)^n F^n(s)}$$

$$L^{-1}\{F'(s)\} = -t.L^{-1}\{F(s)\}$$

$$L^{-1}\{F(s)\} = \frac{-L^{-1}\{F'(s)\}}{t}$$

Question $L\{t\cos at\}$

We know $L\{\cos at\} = \frac{s}{s^2+a^2} = F(s)$

$$F'(s) = \frac{(s^2+a^2)(1)-(s)(2s)}{(s^2+a^2)^2} = \frac{s^2+a^2-2s^2}{(s^2+a^2)^2} = \frac{a^2-s^2}{(s^2+a^2)^2}$$

$$-F'(s) = \frac{s^2-a^2}{(s^2+a^2)^2}$$

$$L\{te^{-t}\cos at\} = \frac{(s+1)^2-a^2}{((s+1)^2+a^2)^2}$$

Find $L^{-1}\left\{\ln\left(\frac{s+3}{s+2}\right)\right\}$

Let $F(s) = \ln\left(\frac{s+3}{s+2}\right) = \ln(s+3) - \ln(s+2)$

$$F'(s) = \frac{1}{s+3} - \frac{1}{s+2}$$

$$L^{-1}\{F'(s)\} = L^{-1}\left\{\frac{1}{s+3}\right\} - L^{-1}\left\{\frac{1}{s+2}\right\}$$

$$= e^{-3t} - e^{-2t}$$

$$L^{-1}\{F(s)\} = \frac{-L^{-1}\{F'(s)\}}{t}$$

$$L^{-1}\left\{\ln\left(\frac{s+3}{s+2}\right)\right\} = \frac{-(e^{-3t}-e^{-2t})}{t} = \frac{e^{-2t}-e^{-3t}}{t}$$

Find $L^{-1}\left\{\ln\left(1+\frac{1}{s^2}\right)\right\}$

$$F(s) = \ln\left(\frac{s^2+1}{s^2}\right) = \ln(s^2+1) - \ln(s^2)$$

$$F'(s) = \frac{2s}{s^2+1} - \frac{2s}{s^2} = \frac{2s}{s^2+1} - \frac{2}{s}$$

$$L^{-1}(F'(s)) = 2L^{-1}\left(\frac{s}{s^2+1}\right) - 2L^{-1}(1/s) = 2\cos t - 2$$

$$L^{-1}\left\{\ln(1+1/s^2)\right\} = \frac{2-2\cos t}{t}$$

Integration of Laplace Transform:-

If $L\{f(t)\} = F(s)$ then the $\int_s^\infty F(s)ds = L\left\{\frac{f(t)}{t}\right\}$

Proof:-

Consider $\int_s^\infty F(s)ds = \int_s^\infty \int_0^\infty e^{-st} f(t).dt.ds$

$$= \int_0^\infty \int_s^\infty e^{-st} f(t)\, ds.dt$$

$$= -\int_0^\infty \frac{f(t).(e^{-st})_s^\infty}{t}.dt$$

$$= -\int_0^\infty \frac{f(t)}{t}\left[0 - e^{-st}\right] dt$$

$$= \int_0^\infty \frac{f(t)e^{-st}}{t} dt$$

$$\boxed{\int_s^\infty F(s)\, ds = L\left\{\frac{f(t)}{t}\right\}}$$

$$\boxed{t\, L^{-1}\left\{\int_s^\infty F(s)ds\right\} = L^{-1}(F(s))}$$

$L\left\{\frac{\sin t}{t}\right\} = \int_s^\infty \frac{1}{s^2+1}.ds = \left(\tan^{-1}(s)\right)_s^\infty = \tan^{-1}\infty - \tan^{-1}s$

$= \pi/2 - \tan^{-1}s$

$L\left\{\frac{\cosh t}{t}\right\} = \frac{1}{2}\int_s^\infty \frac{2s}{s^2-1} ds = \frac{1}{2}\left(\ln|s^2-1|\right)_s^\infty = \frac{1}{2}\left(\ln\infty - \ln(s^2-1)\right)$

$= \frac{1}{2}\ln\left(\frac{\infty}{s^2-1}\right) = \frac{1}{2}\ln\alpha$

$L^{-1}\left\{\frac{s}{(s^2+4)^2}\right\}$

Let $F(s) = \frac{s}{(s^2+4)^2}$ then $\int_s^\infty F(s).ds$

$$= \int_s^\infty \frac{s}{(s^2+4)^2} ds = \int_{s^2+4}^\infty \frac{1}{2u^2} du$$

Let $s^2+4 = u$

then $\boxed{2sds = du}$

$$= \frac{1}{2}\int_{s^2+4}^\infty u^{-2} du = \frac{1}{2}\left[\frac{u^{-1}}{-1}\right]$$

$$= -\frac{1}{2}\left[\frac{1}{u}\right]_{s^2+4}^\infty$$

$$= -\frac{1}{2}\left[\frac{1}{\infty} - \frac{1}{s^2+4}\right]$$

$$= \frac{1}{2(s^2+4)}$$

$L^{-1}\left(\int_s^\infty F(s).ds\right) = L^{-1}\left(\frac{1}{2}\frac{1}{s^2+4}\right) = \frac{1}{4}\sin 2t$

$L^{-1}\left\{\frac{s}{(s^2+4)^2}\right\} = \frac{t}{4}\sin 2t$

$$L^{-1}\left(\frac{s^2-\pi^2}{(s^2+\pi^2)^2}\right)$$

Let $F(s) = \dfrac{s^2-\pi^2}{(s^2+\pi^2)^2}$

$$\int_s^\infty \frac{s^2-\pi^2}{(s^2+\pi^2)^2}\,ds = \int_s^\infty \frac{(s+\pi)(s-\pi)}{(s^2+\pi^2)^2}\,ds$$

$$= \int_s^\infty \frac{(s^2+\pi^2)-2\pi^2}{(s^2+\pi^2)^2}\,ds = \int_s^\infty \frac{s^2+\pi^2}{(s^2+\pi^2)^2}\,ds - \frac{2\pi^2}{(s^2+\pi^2)^2}\,ds$$

$$= \int_s^\infty \frac{1}{s^2+\pi^2}\,ds - 2\int_s^\infty \frac{\pi^2}{(s^2+\pi^2)^2}\,ds$$

$$= \frac{1}{\pi}\left(\tan^{-1}\left(\frac{s}{\pi}\right)\right)_s^\infty - 2\pi^2\int_s^\infty \frac{1}{(s^2+\pi^2)^2}\,ds$$

$$= \frac{1}{\pi}\left(\frac{\pi}{2} - \tan^{-1}\frac{s}{\pi}\right) - 2\pi^2\int_s^\infty \frac{1}{(s^2+\pi^2)^2}\,ds$$

$$= \frac{1}{2} - \frac{1}{\pi}\tan^{-1}\frac{s}{\pi} - 2\pi^2\int_s^\infty \frac{1}{(s^2+\pi^2)^2}\,ds$$

$$L^{-1}\{F(s)\} = \frac{-L^{-1}\{F'(s)\}}{t}$$

Let $F(s) = \dfrac{s}{s^2+\pi^2}$ then $F'(s) = \dfrac{(s^2+\pi^2)(1)-(s)(2s)}{(s^2+\pi^2)^2} = \dfrac{s^2+\pi^2-2s^2}{(s^2+\pi^2)^2}$

$$= \frac{-s^2+\pi^2}{(s^2+\pi^2)^2}$$

$$L^{-1}\{F(s)\} = \frac{-L^{-1}\{F'(s)\}}{t}$$

$$t\,L^{-1}\left\{\frac{s}{s^2+\pi^2}\right\} = -L^{-1}\left\{\frac{-s^2+\pi^2}{(s^2+\pi^2)^2}\right\}$$

$$t\cos\pi t = +1\,L^{-1}\left(\frac{-\pi^2+s^2}{(s^2+\pi^2)^2}\right)$$

$$\boxed{\therefore L^{-1}\left\{\frac{s^2-\pi^2}{(s^2+\pi^2)^2}\right\} = t\cos\pi t}$$

Unit step function:- (Heav side function)

$$u(t-a) = \begin{cases} 0 & t<a \\ 1 & t>a \end{cases} \qquad f(t-a)\,u(t-a) = \begin{cases} 0 & t<a \\ f(t-a) & t>a \end{cases}$$

$$f(t).u(t-a) = \begin{cases} 0 & t<a \\ f(t) & t>a \end{cases}$$

Laplace transform of the unit step function:-

This is the second shifting theorem

$L\{f(t)\} = F(s)$

then the transform of $L\{f(t-a).u(t-a)\} = e^{-as}F(s)$

Proof:- $L\{f(t-a).u(t-a)\} = \int_0^\infty e^{-st} f(t-a).u(t-a)\,dt = \int_a^\infty e^{-st} f(t-a).dt$

$t-a = v$ $\quad \boxed{t = a+v}$

$dt = dv$

$$= \int_0^\infty e^{-s(u+a)} f(u)\,du$$

$$= e^{-sa} \int_0^\infty \left(e^{-su} \cdot f(u)\,du\right)$$

$$= e^{-sa} F(s)$$

$$\boxed{L\{f(t-a) \cdot u(t-a)\} = e^{-sa} F(s)}$$

From this $L\{u(t-a)\} = e^{-sa} L\{f(t)=1\} = e^{-sa} L\{1\} = \frac{e^{-sa}}{s}$

What is the transform of $(t-1)\,u(t-1)$

1) $L\{(t-1) \cdot u(t-1)\} = e^{-s} L\{t\} = \frac{e^{-s}}{s^2}$

2) $L\{t^2 \cdot u(t-1)\} = e^{-s} L\{(t+1)^2\} = e^{-s} \left(\frac{2!}{s^3} + \frac{2}{s^2} + \frac{1}{s}\right)$

$f(t-1) = t^2$

$= (t-1)^2 - 1 + 2t$ $\qquad = e^{-s}\left(\frac{2 + 2s + s^2}{s^3}\right)$

$= (t-1)^2 + 2(t-1) + 1$

$f(t) = t^2 + 2t + 1$ $\qquad = \frac{e^{-s}(s^2 + 2s + 2)}{s^3}$

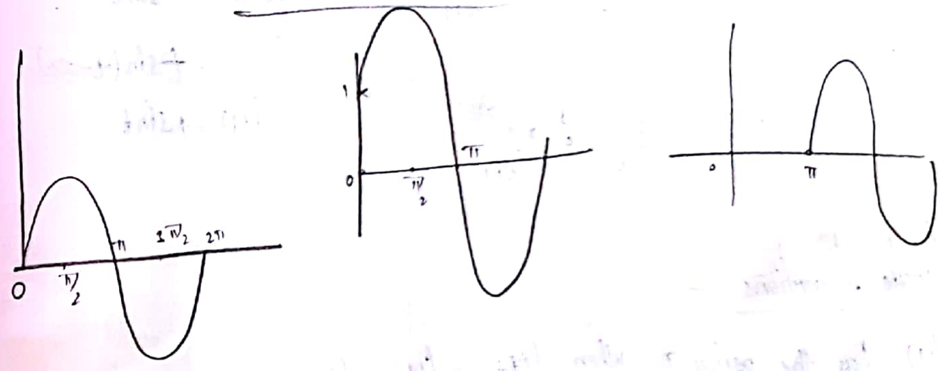
3) $L\{\cos t \cdot u(t-\pi)\} = e^{-\pi s} L\{-\cos t\} = -e^{-\pi s} \frac{s}{s^2+1}$

Sol: $f(t-\pi) = \cos t$

$f(t-\pi) = -\cos(t-\pi)$

then $f(t) = -\cos t$

All are sine curves



$\rightarrow L\{f(t-a)\,u(t-a)\} = e^{-as} F(s)$

$L^{-1}\{e^{-as} F(s)\} = f(t-a)\,u(t-a)$

4) $L^{-1}\left\{\frac{2}{s^2} - \frac{2e^{-s}}{s^2} - 4\frac{e^{-2s}}{s}\right\} = L^{-1}\{2/s^2\} - 2L^{-1}\left\{\frac{e^{-s}}{s^2}\right\} - 4L^{-1}\left\{\frac{e^{-2s}}{s}\right\}$

$= 2t - 2(t-1)\,u(t-1) - 4u(t-2)$

$= 2t - 2(t-1)\,u(t-1) - 4u(t-2)$

5) $L^{-1}\left\{\frac{s}{s^2+4} e^{-\pi s}\right\} = \cos 2(t-\pi)\,u(t-\pi)$

6) $f(t) = \begin{cases} 2 & 0 < t < \pi \\ 0 & \pi < t < 2\pi \\ \sin t & t > 2\pi \end{cases}$

Converting it into unit step function form

$$f(t) = 2u(t) - 2u(t-\pi) + \sin t \, u(t-2\pi)$$

$$L\{f(t)\} = \frac{2e^{-0}}{s} - 2\frac{e^{-\pi s}}{s} + \frac{e^{-2\pi s}}{s^2+1}$$

$f(t-2\pi) = \sin t$

$= +\sin(t-2\pi)$

$f(t) = +\sin t$

$$= \frac{2}{s} - \frac{2e^{-\pi s}}{s} + \frac{e^{-2\pi s}}{s^2+1}$$

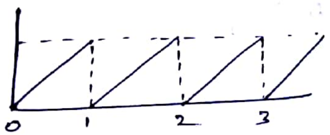
Periodic Functions :-

$f(t)$ has the period T when $f(t+T) = f(t)$ $t \geq 0$

when $f : (0, \infty) \to R$

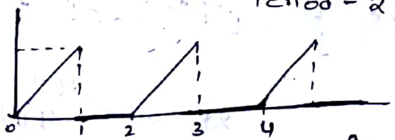
Saw tool head Wave :-

$f(t) = t$ $0 < t < 1$



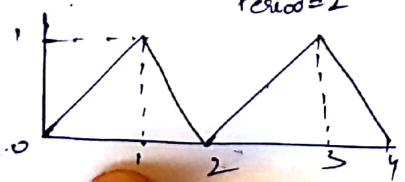
period is 1

Period - 2



$f(t) = \begin{cases} t & 0 < t < 1 \\ 0 & 1 < t < 2 \end{cases}$

Period = 2



$f(t) = \begin{cases} t & 0 < t < 1 \\ 2-t & 1 < t < 2 \end{cases}$

Transform of periodic functions :-

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t).dt$$

$f(t+T) = f(t)$

$f(t+2T) = f(t+T+T)$

$= f(t+T)$

$= f(t)$

$$= \int_0^T e^{-st} f(t).dt + \int_T^{2T} e^{-st} f(t).dt + \int_{2T}^{3T} \ldots$$

put $t = t' - T$

$$= \int_0^T e^{-st} f(t).dt + \int_0^T e^{-s(t'+T)} f(t'+T).dt' + \ldots$$

$$= \int_0^T e^{-st} f(t)dt + \int_0^T e^{-sT}.e^{-st'} f(t').dt' + \ldots$$

$$= \int_0^T e^{-st} f(t)\,dt + e^{-sT}\int_0^T e^{-st'} f(t')dt' + e^{-2sT}\int_0^T e^{-st''} f(t'')dt'' + \ldots$$

$$= \int_0^T e^{-st} f(t) \left[1 + e^{-sT} + e^{-2sT} + e^{-3sT} + \ldots\right]$$

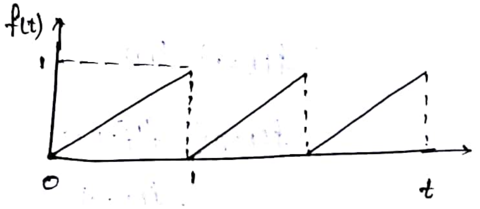
e^{-s}

s should be +ve

$1 - e^{-sT} < 1$

$$= \int_0^T e^{-st} f(t) \cdot \left(\frac{1}{1-e^{-sT}}\right) = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\,dt$$

Find the Laplace transform of $f(t)$.



Here $T=1$ $\quad f(t)=t, \quad 0<t<1$

then $L\{f(t)\} = \dfrac{\int_0^T e^{-st} f(t)\,dt}{1-e^{-sT}}$

$$L\{f(t)\} = \frac{1}{1-e^{-s}} \int_0^1 \underset{II}{e^{-st}}\,\underset{I}{t}\,dt \qquad \text{ILATE}$$

$$= \frac{1}{1-e^{-s}}\left[\left(t\,\frac{e^{-st}}{-s}\right)_0^1 - \int_0^1 \frac{e^{-st}}{-s}\right]$$

$$= \frac{1}{1-e^{-s}}\left[\frac{e^{-s}}{-s} + \frac{1}{s}\left(\frac{e^{-st}}{-s}\right)_0^1\right]$$

$$= \frac{1}{1-e^{-s}}\left[\frac{e^{-s}}{-s} - \frac{1}{s^2}\left(e^{-s}-e^{0}\right)\right]$$

$$= \frac{1}{1-e^{-s}}\left[\frac{-e^{-s}}{s} - \frac{e^{-s}}{s^2} + \frac{1}{s^2}\right]$$

$$= \frac{+1}{1-e^{-s}}\left[\frac{1-e^{-s}-se^{-s}}{s^2}\right] = \frac{1}{s^2} - \frac{e^{-s}}{s(1-e^{-s})}$$

Find the Laplace transform of periodic function $f(t)=t \quad 0<t<1 \quad T=2$

$\qquad\qquad\qquad = 2-t \quad 1<t<2$

Sol: $$L\{f(t)\} = \frac{1}{1-e^{-2s}}\left(\int_0^1 e^{-st}\cdot t\,dt + \int_1^2 e^{-st}(2-t)\,dt\right)$$

$$= \frac{1}{1-e^{-2s}}\left[\frac{1-e^{-s}-se^{-s}}{s^2} + 2\left(\frac{e^{-st}}{-s}\right)_1^2 - \int_1^2 t e^{-st}\,dt\right]$$

$$= \frac{1}{1-e^{-2s}}\left[\frac{1-e^{-s}-se^{-s}}{s^2} \oplus \frac{2}{s}\left(e^{-2s}-e^{-s}\right) - \left(\frac{t e^{-st}}{-s}\right)_1^2 + \int_1^2 \frac{e^{-st}}{-s}\cdot dt\right]$$

$$= \frac{1}{1-e^{-2s}}\left[\frac{1-e^{-s}-se^{-s}}{s^2} \oplus \frac{2}{s}\left(e^{-2s}-e^{-s}\right) + \left(\frac{2e^{-2s}}{s} - \frac{e^{-s}}{s}\right) + \frac{1}{s^2}\left(e^{-st}\right)_1^2\right]$$

$$= \frac{1}{1-e^{-2s}}\left[\frac{1-e^{-s}-se^{-s}}{s^2} + \frac{se^{-s}}{s} + \frac{1}{s^2}\left(e^{-2s}-e^{-s}\right)\right]$$

$$= \frac{1}{1-e^{-2s}}\left[\frac{1-e^{-s}-\cancel{se^{-s}}+\cancel{\ldots}+\cancel{se^{-s}}+e^{-2s}-e^{-s}}{s^2}\right]$$

$$= \frac{1}{s^2(1-e^{-2s})}\left(\cancel{\ldots} - \cancel{\ldots} + e^{-2s} - 2e^{-s} + 1\right) = \frac{1+e^{-2s}-2e^{-s}}{s^2(1-e^{-2s})}$$

Convolution theorem:-

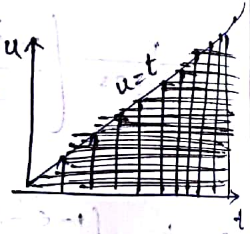
Let $f(t)$ and $g(t)$ satisfy conditions of existence theorem and $L\{f(t)\} = F(s)$ & $L\{g(t)\} = G(s)$ then $F(s).G(s) = L\left\{\int_0^t f(u).g(t-u).du\right\}$

$\int_0^t f(u).g(t-u).du$ → this is called convolution of f & g

Proof:-

$$L\left\{\int_0^t f(u).g(t-u).du\right\} = \int_0^\infty e^{-st}\left(\int_0^t f(u).g(t-u).du\right).dt$$

$$= \int_{t=0}^{t=\infty}\int_{u=0}^{u=t} f(u)\,g(t-u).e^{-st}.du.\,dt$$



$0 \le u \le t$ $\quad u \le t \le \infty$

$0 \le t \le \infty$ $\quad 0 \le u \le \infty$

$$= \int_{u=0}^{u=\infty}\int_{t=u}^{t=\infty} f(u).g(t-u).e^{-st}.dt.du$$

$$= \int_{u=0}^{u=\infty} f(u)\left(\int_{t=u}^{t=\infty} g(t-u)e^{-st}dt\right)du$$

Put $t-u=v$, $dt=dv$

$$= \int_{u=0}^{u=\infty}\int_{v=0}^{v=\infty} f(u).g(v).e^{-s(u+v)}.dv.du$$

$$= \int_{u=0}^{u=\infty} f(u)e^{-su}\left(\int_{v=0}^{v=\infty} e^{-sv}.g(v).dv\right).du$$

$$= \int_{u=0}^{u=\infty} e^{-su} f(u).G(s)\,du$$

$$= \int_{u=0}^{u=\infty} f(u).e^{-us}.G(s).du = G(s)\int_{u=0}^{u=\infty} f(u).e^{-us}.du$$

$= F(s).G(s)$

$= G(s).F(s)$

$$\boxed{L\left\{\int_0^t f(u).g(t-u).du\right\} = F(s).G(s)}$$

$$L\left\{\int_0^t f(u).g(t-u).du\right\} = F(s)\,G(s)$$

$$L^{-1}\{F(s)\,G(s)\} = \int_0^t f(u).g(t-u).du$$

Let $F(s) = \frac{1}{s}$ and $G(s) = \frac{1}{s+3}$ } But our choise of choosing $f(t)$, $g(t)$ is $g(t-u)$ should be simple so inter-change them.

then $f(t)=1$ $\quad g(t) = e^{-3t}$

$G(s) = 1/s$ $\quad F(s) = \frac{1}{s+3}$

$g(t)=1$ & $f(t) = e^{-3t}$

$g(t-u)=1$

$$L^{-1}\left\{\frac{1}{(s+3)s}\right\} = \int_0^t e^{-3u}.du = \left(\frac{e^{-3u}}{-3}\right)_0^t = \frac{-1}{3}\left[e^{-3t}-1\right] = \frac{1-e^{-3t}}{3}$$

2) $L^{-1}\left\{\frac{1}{(s+1)(s^2+1)}\right\}$

Let $G(s) = \frac{1}{s+1}$ then $g(t) = e^{-t}$ $\quad$ -then!

ILATE

$F(s) = \frac{1}{s^2+1}$ then $f(t) = \sin t$

$$L^{-1}\left\{\frac{1}{(s^2+1)(s+1)}\right\} = \int_0^t \sin u.e^{-(t-u)}.du \quad e^{-t}\int_0^t (e^u.\sin u.du).$$

$$= e^{-t}\left[(\sin u.e^u)_0^t - \int_0^t \cos u.e^u\right]$$

$$= e^{-t}\left[e^t\sin t - ((\cos u)e^u)_0^t + \int_0^t -\sin u\,e^u\right]$$

$$= e^{-t}\left(\frac{e^t\sin t - e^t\cos t + 1}{2}\right) = \frac{e^{-t} + \sin t - \cos t}{2}$$

H.W find $L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\}$ $\quad \frac{s}{(s^2+4)^2}$ $a^2=4$ $\boxed{a=2}$ $\quad \cos 2\theta = 1-2\sin^2\theta$

$F(s) = \frac{1}{s^2+a^2}$ then $f(t) = \frac{\sin at}{a}$ $\quad \frac{t}{2}\sin 2t$

$G(s) = \frac{s}{s^2+a^2}$ then $g(t) = \cos at$

$$L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\} = \frac{1}{a}\int_0^t \sin au \cdot \cos a(t-u) \cdot du$$

$$= \frac{1}{a}\int_0^t \sin au \left(\cos at \cdot \cos au + \sin at \sin au\right) \cdot du$$

$$= \frac{1}{a}\left(\cos at \int_0^t \sin au \cdot \cos au \cdot du + \sin at \int_0^t \sin^2 au \cdot du\right)$$

$$= \frac{1}{a}\left(\frac{\cos at}{2}\int_0^t \sin 2au \cdot du + \frac{\sin at}{2}\int_0^t (1-\cos 2au)\, du\right)$$

$$= \left(\frac{\cos at}{2}\left[\frac{-\cos 2au}{2a}\right]_0^t + \frac{\sin at}{2}[t] - \frac{\sin at}{2}\left(\left[\frac{\sin 2au}{2a}\right]_0^t\right)\right)\frac{1}{a}$$

$$= \left(\frac{\cos at}{4a}\left[-\cos 2at + \cos 0\right] + \frac{t \sin at}{2} - \frac{\sin at}{4a}\left(\sin 2at - \sin 0\right)\right)\frac{1}{a}$$

$$= \left(+\frac{\cos at}{4a}\left[1-\cos 2at\right] + \frac{t\sin at}{2} - \frac{\sin at \sin 2at}{4a}\right)\frac{1}{a}$$

$$= \frac{-\sin at \sin 2at + 2at \sin at - \cos at \cos 2at + \cos at}{4a^2}$$

$$= \frac{2at \sin at + \cos at - \cancel{a}\cos(2at - at)}{4a^2}$$

$$= \frac{2at \sin at + \cancel{\cos at} - \cancel{\cos at}}{4a^2}$$

$$= \frac{\cancel{2}\cancel{a}t \sin at}{\cancel{4}_2 \cancel{a^2}}$$

$$= \frac{t}{2a}\sin at$$

$$L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\} = \frac{t}{2a}\sin at$$

Laplace transformer method for solving system of differential equation

Simultaneous differential eq^n.

t is the independent variable

x, y are the dependent variables.

Solve:- $\frac{dx}{dt} - \frac{dy}{dt} - 2x - y = 6e^{3t}$ $\qquad 2\frac{dx}{dt} + \frac{dy}{dt} - 3x - 3y = 6e^{2t}$

initial conditions $x(0)=3$, $y(0)=0$

Applying laplace transform to both the eq^n.

$$sL\{x\} - x(0) - sL\{y\} + y(0) - 2L\{x\} - L\{y\} = \frac{6}{s-3}$$

$$(s-2)L\{x\} - (s+1)L\{y\} = \frac{6}{s-3} + 3 \quad \text{—①}$$

$2\left[sL\{x\}-x(0)\right] + sL\{y\} - y(0) - 3L\{x\} - 3L\{y\} = \frac{6}{s-3}$

$(2s-3)L\{x\} + (s-3)L\{y\} = \frac{6}{s-3} + 6$ — ②

Multiply — ① × (s−3) + ② × (s+1)

$(2s-3)(s+1)L\{x\} + \cancel{(s-3)(s+1)L\{y\}} = \frac{6(s+1)}{s-3} + 6(s+1)$

$(s-2)(s-3)L\{x\} - \cancel{(s+1)(s-3)L\{y\}} = \frac{6}{s-3}(s-3) + 3(s-3)$

$\left[(2s-3)(s+1) + (s-2)(s-3)\right]L\{x\} = \frac{6(s+1)}{s-3} + 6s + 6 + 6 + 3s - 9$

$\left[\cancel{2s^2} + 2s - 3s - 3 + \cancel{s^2} - 5s + 6\right]L\{x\} = 9s + 3 + \frac{6(s+1)}{s-3}$

$(3s^2 - 6s + 3)L\{x\} = 9s + 3 + \frac{6(s+1)}{s-3}$

$3(s^2 - 2s + 1)L\{x\} = 9s + 3 + \frac{6(s+1)}{s-3}$

$(s-1)^2 L\{x\} = 3s + 1 + \frac{2(s+1)}{s-3}$

$L\{x\} = \frac{3s}{(s-1)^2} + \frac{1}{(s-1)^2} + \frac{2(s+1)}{(s-1)^2(s-3)} =$

$\frac{A}{(s-1)} + \frac{Bs+C}{(s-1)^2}$

$A(s-1) + Bs + C$

$\cancel{As} - A + \cancel{Bs} + C$

$A+B = -2$

$C - A = 0$

$A = C$

$C = A$

$\frac{A\cancel{s}+B}{(s-1)^2} + \frac{C}{s-3}$

$\cancel{As^2} - 3As + Bs - 3B + \cancel{Cs^2} - 2Cs + C$

$A + C = 0$

$-3A - 2C + B = 2$

$C - 3B = 2$

$3C - 2C + B = 2$

$C + B = 2$

$C - 3B = 2$

$B = 0$

$C = 2$

$A = -2$

$x = 3L^{-1}\left(\frac{s}{(s-1)^2}\right) + L^{-1}\left(\frac{1}{(s-1)^2}\right) + L^{-1}\left\{\frac{-2s}{(s-1)^2}\right\} + L^{-1}\left\{\frac{2}{(s-3)}\right\}$

$x = L^{-1}\left\{\frac{s}{(s-1)^2}\right\} + L^{-1}\left\{\frac{1}{(s-1)^2}\right\} + 2L^{-1}\left\{\frac{1}{s-3}\right\}$

$= L^{-1}\left\{\frac{(s-1)+1}{(s-1)^2}\right\} + L^{-1}\left\{\frac{1}{(s-1)^2}\right\} + 2L^{-1}\left\{\frac{1}{s-3}\right\}$

$= L^{-1}\left\{\frac{1}{(s-1)}\right\} + 2L^{-1}\left\{\frac{1}{(s-1)^2}\right\} + 2L^{-1}\left\{\frac{1}{s-3}\right\}$

$\int u e^{u} e^{2t} e^{-3u}\,du$

$x = e^{t} + 2e^{t}.t + 2e^{3t}$

$e^{2t}\int u e^{-2u}\,du$

$\boxed{x = (2t+1)e^{t} + 2e^{3t}}$ $\quad \cancel{y = e^{t} - te^{t} - e^{2t}}$

$e^{2t}\left[\left(\frac{ue^{-2u}}{-2}\right)_0^t + \int_0^t \frac{e^{-2u}}{2}\right]$

$-\frac{t}{2}e^{-2t} + \frac{1}{2}\left(\frac{-e^{-2u}}{2}\right)_0^t$

$\left(-\frac{t}{2}e^{-2t} + \frac{1}{4} - e^{-2t} + \frac{1}{4}\right)e^{2t}$

- Solving for y.

Substitute $L\{x\} = \frac{s}{(s-1)^2} + \frac{1}{(s-1)^2} + \frac{2}{s-3}$ in eq①

$(s-2)\left[\frac{s}{(s-1)^2} + \frac{1}{(s-1)^2} + \frac{2}{s-3}\right] - \frac{6}{s-3} - 3 = (s+1)L\{y\}$

$\frac{s(s-2)}{(s-1)^2} + \frac{s}{(s-1)^2} - \frac{2}{(s-1)^2} + \frac{2s}{s-3} - \frac{4}{s-3} - \frac{6}{s-3} - 3 = (s+1)L\{y\}$

$L\{y\} = \frac{-3}{s+1} + \frac{10}{(s-3)(s+1)} + \frac{2s}{(s-3)(s+1)} + \frac{s^2-2s+s-2}{(s-1)^2(s+1)}$

$L\{y\} = \frac{-3}{s+1} + \frac{2s-10}{(s-3)(s+1)} + \frac{s^2-s-2}{(s-1)^2(s+1)}$

$$L\{y\} = \frac{-3}{s+1} + \frac{2s-10}{(s-3)(s+1)} + \frac{s^2-s-2}{(s-1)^2(s+1)}$$

$$\frac{A}{s-3} + \frac{B}{s+1}$$

$AS + A + BS - 3B = 2s - 10$

$\cancel{A} + B = 2 \quad A = -1$

$\cancel{A} - 3B = -10$

$- \quad + \quad +$

$4B = 12$

$B = 2$

$$\frac{A}{s+1} + \frac{B}{s-1} + \frac{Cs+D}{(s-1)^2}$$

$= A(s-1)^2 + B(s-1)(s+1) + (Cs+D)(s+1)$

$= As^2 - 2As + A + Bs^2 - B + Cs^2 + Cs + Ds + D$

$= (A+B+C)s^2 + (C+D-2A)s + A - B + D$

$A - B + D = -2$

$A + C = 1 \quad C + D - 2A = -1$

$A = 1 - C \quad \cancel{C+D} \quad 1 - A - 2 - A - 2A = -1$

$C = 1 - A \quad -1 - 4A = -1$

$C = 1 \quad A = 0$

$D = -2$

$$\frac{s-2}{(s-1)^2}$$

Solve Eq ① & eq ② as above for x and get y.

$$\frac{d^2x}{dt^2} + \frac{dy}{dt} + 3x = 15e^{-t}$$

$$\frac{d^2y}{dt^2} - 4\frac{dx}{dt} + 3y = 15\sin 2t$$

initial conditions $x(0) = 35$, $x'(0) = -48$, $y(0) = 27$, $y'(0) = -55$

Ans

$\to x = 30\cos t - 15\sin 3t + 3e^{-t} + 2\cos 2t$

$y = 30\cos t - 60\sin t - 3e^{-t} + \sin 2t$.

i.e. $x \; y$

Sol:- $s^2L\{x\} - sx(0) - x'(0) + sL\{y\} - y(0) + 3L\{x\} = \frac{15}{s+1}$

$(s^2+3)L\{x\} + sL\{y\} - 35s + 48 - 27 = \frac{15}{s+1}$

$(s^2+3)L\{x\} + sL\{y\} = \frac{15}{s+1} + 35s + 21$

$s^2L\{y\} - sy(0) - y'(0) - 4sL(x) + 4x(0) + 3L(y) = \frac{15 \cdot 2}{s^2+4}$

$s^2L\{y\} - 27s + 55 - 4sL\{x\} + 35 \times 4 + 3L(y) = \frac{30}{s^2+4}$

$-4sL\{x\} + (s^2+3)L\{y\} = \frac{30}{s^2+4} + 27s - 195$

$35 \times 4 = 140$, $140 + 55 = 195$

~~$-4s$~~ $+ ① \times s^2+3 + ② \times s$

$(s^2+3)^2L(x) + \cancel{s(s^2+3)L(y)} = \frac{15(s^2+3)}{s+1} + 35s(s^2+3) + 21(s^2+3)$

$-4s^2L(x) + \cancel{s(s^2+3)L(y)} = \frac{30s}{s^2+4} + 27s^2 - 195s$

$+ \quad - \quad - \quad - \quad +$

$$((s^2+3)^2 + 4s^2)L(x) = \frac{15(s^2+3)}{s+1} + 35s^3 + 35 \times 3s + 21s^2 + 21 \times 3 - \frac{30s}{s^2+4} - 27s^2 +$$