TUTORIAL – I

① (a)

$$x + y - z = 9$$
$$8y + 6z = -6$$
$$-2x + 4y - 6z = 40$$

$$\tilde{A} = \left[\begin{array}{ccc|c} 1 & 1 & -1 & 9 \\ 0 & 8 & 6 & -6 \\ -2 & 4 & -6 & 40 \end{array}\right]$$

$R_3 \to R_3 + 2R_1$ $\sim \left[\begin{array}{ccc|c} 1 & 1 & -1 & 9 \\ 0 & 8 & 6 & -6 \\ 0 & 6 & -8 & 58 \end{array}\right]$

$R_3 \to 4R_3 - 3R_2$ $\sim \left[\begin{array}{ccc|c} 1 & 1 & -1 & 9 \\ 0 & 8 & 6 & -6 \\ 0 & 0 & -50 & 250 \end{array}\right]$

-32
18

232
18

$\rho(A) = 3$

$\rho(\tilde{A}) = 3$

$n = 3$

Solution exists.

$-50z = 250$

$\underline{\underline{z = -5}}$

$8y + 6z = -6 \Rightarrow 8y = 24$
$y = 3$

$x + y - z = 9$

$\underline{\underline{x = 1}}$

(1) $4y + 3z = 8$

$2x - z = 2$

$3x + 2y = 5$

$$\tilde{A} = \left[\begin{array}{ccc|c} 0 & 4 & 3 & 8 \\ 2 & 0 & -1 & 2 \\ 3 & 2 & 0 & 5 \end{array}\right]$$

$$R_2 \leftrightarrow R_1 \sim \left[\begin{array}{ccc|c} 2 & 0 & -1 & 2 \\ 0 & 4 & 3 & 8 \\ 3 & 2 & 0 & 5 \end{array}\right]$$

$$R_3 \to 2R_3 - 3R_1 \sim \left[\begin{array}{ccc|c} 2 & 0 & -1 & 2 \\ 0 & 4 & 3 & 8 \\ 0 & 4 & 3 & 4 \end{array}\right]$$

$$R_3 \to R_3 - R_2 \sim \left[\begin{array}{ccc|c} 2 & 0 & -1 & 2 \\ 0 & 4 & 3 & 8 \\ 0 & 0 & 0 & -4 \end{array}\right]$$

$$\left.\begin{array}{l} \rho(A) = 2 \\ \rho(\tilde{A}) = 3 \end{array}\right\} \text{inconsistent}$$

③ $10x + 4y - 2z = -4$

$-3w - 17x + y + 2z = 2$

$w + x + y = 6$

$8w - 34x + 16y - 10z = 4$

$$\tilde{A} = \begin{array}{cccc|c} x & y & z & w & \\ \end{array}$$

$$\tilde{A} = \left[\begin{array}{cccc|c} 10 & 4 & -2 & 0 & -4 \\ -17 & 1 & 2 & -3 & 2 \\ 1 & 1 & 0 & 1 & 6 \\ -34 & 16 & -10 & 8 & 4 \end{array}\right]$$

$R_2 \to 10R_2 + 17R_1$

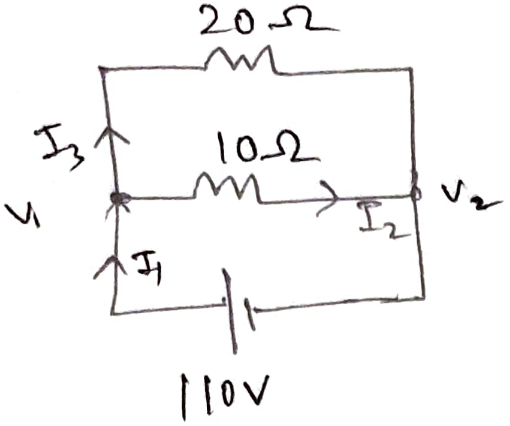
$R_3 \to 10R_3 - R_1$

$R_4 \to 5R_4 + 17R_1$

$$\sim \left[\begin{array}{cccc|c} 10 & 4 & -2 & 0 & -4 \\ 0 & 78 & -14 & -30 & -48 \\ 0 & 6 & 2 & 10 & 64 \\ 0 & 148 & -84 & 40 & -48 \end{array}\right]$$

$R_3 \to 13R_3 - R_2$

R_4



$$R_{eq} = \frac{20\times10}{20+10} = \frac{200}{30} = \frac{20}{3}$$

$$I_1 = \frac{110}{20/3} = \frac{11\cancel{0}\times3}{2\cancel{0}} = \frac{33}{2}$$

$$I_2 + I_3 = I_1 = \frac{33}{2}$$

$$10I_2 = 110$$

$$I_2 = 10A$$

$$20I_3 = 110$$

$$I_3 = \frac{11}{2}$$

$$I_2 + I_3 = \frac{11}{2} + 10 =$$

$\cancel{V_1 = 110}$

$$\frac{V_1 - V_2}{10} + \frac{V_1 - V_2}{20} = 0$$

$$V_1 - 110 = V_2$$

$$V_1 - V_2 = 110$$

$$-20I_3 + 10I_2 = 0$$

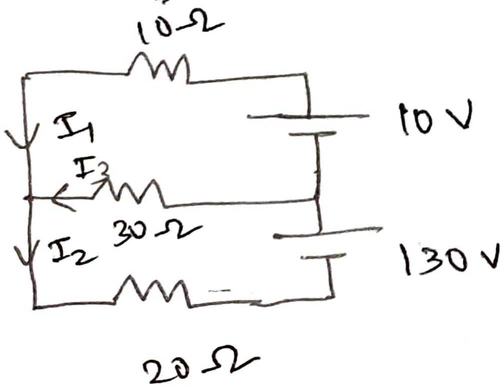
$$110 \cancel{+}\,{-}10I_2 = 0$$

$$I_1 \cancel{+} = I_2 + I_3$$

$$I_2 = \underline{\underline{11}}\,A$$

$$I_3 = \frac{-110}{-2\cancel{0}} = \underline{\underline{5.5A}}$$

$$I_1 = \underline{\underline{16.5A}}$$



$$I_2 = I_1 + I_3 \quad -③$$

$$10I_1 - 10 - 30I_3 = 0 \quad -①$$

$$30I_3 - 130 + 20I_2 = 0 \quad -②$$

$$\begin{pmatrix} 10 & 0 & -30 \\ 0 & 20 & 30 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} I_1 \\ I_2 \\ I_3 \end{pmatrix} = \begin{pmatrix} 10 \\ 130 \\ 0 \end{pmatrix}$$

$$\tilde{A} = \left[\begin{array}{ccc|c} 10 & 0 & -30 & 10 \\ 0 & 20 & 30 & 130 \\ 1 & -1 & 1 & 0 \end{array}\right]$$

$$R_3 \to 10R_3 - R_1 \sim \begin{bmatrix} 10 & 0 & -30 & 10 \\ 0 & 20 & 30 & 130 \\ 0 & -10 & 40 & -10 \end{bmatrix}$$

$$R_3 \to 2R_3 + R_2 \sim \left[\begin{array}{ccc|c} 10 & 0 & -30 & 10 \\ 0 & 20 & 30 & 130 \\ 0 & 0 & 110 & 110 \end{array}\right]$$

$\rho(A) = 3 \qquad \rho(\tilde{A}) = 3 \qquad n = 3.$

$110 I_3 = 110$

$\underline{\underline{I_3 = 1A}}$

$20 I_2 + 30 I_3 = 130$

$\underline{\underline{I_2 = 5A}}$

$10 I_1 - 30 I_3 = 10$

$\underline{\underline{I_1 = 4A}}$

(3) $-400 - 600 + x_1 + x_4 = 0$

$-x_2 - x_1 + 800 + 800 = 0$

$-x_4 + 1000 - x_3 + 600 = 0$

$x_2 + x_3 - 1200 - 1000 = 0$

$$\tilde{A} = \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & \cancel{1200}\ 1000 \\ 1 & 1 & 0 & 0 & 1600 \\ 0 & 0 & 1 & 1 & 1600 \\ 0 & 1 & 1 & 0 & 2200 \end{array}\right]$$

$R_2 \to R_2 - R_1$ $\sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 1000 \\ 0 & 1 & 0 & -1 & 600 \\ 0 & 0 & 1 & 1 & 1600 \\ 0 & 1 & 1 & 0 & 2200 \end{array}\right]$

$R_4 \to R_4 - R_2$ $\sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 1000 \\ 0 & 1 & 0 & -1 & 600 \\ 0 & 0 & 1 & 1 & 1600 \\ 0 & 0 & 1 & 1 & 1600 \end{array}\right]$

$R_4 \to R_4 - R_3$ $\sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 1000 \\ 0 & 1 & 0 & -1 & 600 \\ 0 & 0 & 1 & 1 & 1600 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

$\rho(\tilde{A}) = 3$ $\quad \rho(A) = 3$ $\quad n = 4$

$\therefore$ Solution is not unique.

4. (a)
$$x + 2y + 4z = 3$$
$$3x + 8y + 14z = 13$$
$$2x + 6y + 13z = 4$$

Let $A = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 14 \\ 2 & 6 & 13 \end{bmatrix} = LU$

$= \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$

$= \begin{matrix} l_{11} & l_{11}u_{12} & l_{11}u_{13} \\ l_{21} & l_{21}u_{12} + l_{22} & l_{21}u_{13} + l_{22}u_{23} \\ l_{31} & l_{31}u_{12} + l_{32} & l_{31}u_{13} + l_{32}u_{23} + l_{33} \end{matrix}$

$l_{11} = 1$

$l_{21} = 3$

$l_{31} = 2$

$u_{12} = 2$

$u_{13} = 4$

$6 + l_{22} = 8$

$\underline{\underline{l_{22} = 2}}$

$12 + 2u_{23} = 14$

$\underline{\underline{u_{23} = 1}}$

$6 + l_{32} = 6$

$\underline{\underline{l_{32} = 0}}$

$8 + 0 + l_{33} = 13$

$\underline{\underline{l_{33} = 5}}$

$\therefore L = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 2 & 0 \\ 2 & 0 & 5 \end{bmatrix}$

$U = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

$A\bar{x} = \bar{b} \Rightarrow LU\bar{x} = \bar{b}$

put $U\bar{x} = \bar{y}$

$L\bar{y} = \bar{b}$

$\bar{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_2 \end{bmatrix}$.

$$L\bar{y} = \begin{bmatrix} y_1 \\ 3y_1+2y_2 \\ 2y_1+5y_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 13 \\ 4 \end{bmatrix} \Rightarrow \underline{\underline{y_1=3}}$$

$\underline{\underline{y_2 = 2}}$

$\underline{\underline{y_3 = -2/5}}$

$U\bar{x} = \bar{y}$

$$\begin{bmatrix} x+2y+4z \\ y+z \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ -2/5 \end{bmatrix} \Rightarrow z = -2/5$$

$y = 2 + 2/5 = \underline{\underline{12/5}}$

$x = 3 - \frac{24}{5} + \frac{8}{5} = \underline{\underline{-\frac{1}{5}}}$

1 (b) $\bar{b} = \begin{bmatrix} 17 \\ 61 \\ 53 \end{bmatrix} = \begin{bmatrix} y_1 \\ 3y_1+2y_2 \\ 2y_1+5y_3 \end{bmatrix} \Rightarrow \underline{\underline{y_1 = 17}}$

$y_2 = \underline{\underline{5}}$

$y_3 = \underline{\underline{\frac{9}{5}}}$

$U\bar{x} = \bar{y}$

$$\therefore \begin{bmatrix} x+2y+4z \\ y+z \\ z \end{bmatrix} = \begin{bmatrix} 17 \\ 5 \\ 9/5 \end{bmatrix} \Rightarrow \begin{aligned} z &= \underline{\underline{9/5}} \\ y &= 5 - 9/5 = \underline{\underline{16/5}} \\ x &= \frac{85}{5} - \frac{32}{5} - \frac{36}{5} \\ &= \underline{\underline{\frac{17}{5}}} \end{aligned}$$

5) a)

$$A = \begin{bmatrix} 3 & 2 \\ 0 & 1 \end{bmatrix} \qquad \begin{aligned} [A_1] &= 3 & |A_1| &= 3 \\ [A_2] &= A & |A_2| &= 3 \end{aligned} \Bigg\} \text{ LU decomposition}$$

b)

$$A = \begin{bmatrix} 1 & -3 & 7 \\ -2 & 6 & 1 \\ 0 & 3 & -2 \end{bmatrix} \qquad A_1 = [1]$$

$$A_2 = \begin{bmatrix} 1 & -3 \\ -2 & 6 \end{bmatrix} \Rightarrow |A_2| = 0$$

$\Rightarrow$ No LU decomposition.

c)

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 1 & 3 & 4 \end{bmatrix} \qquad A_1 = [1]$$

$$A_2 = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \qquad |A_2| = 0$$

$\Rightarrow$ No LU decomposition.

6 a

$$A = \begin{bmatrix} 3 & 1 & 4 \\ 0 & 5 & 8 \\ -3 & 4 & 4 \end{bmatrix}$$

$$R_3 \to R_3 + R_1 \sim \begin{bmatrix} 3 & 1 & 4 \\ 0 & 5 & 8 \\ 0 & 5 & 8 \end{bmatrix}$$

$$R_3 \to R_3 - R_2 \sim \begin{bmatrix} 3 & 1 & 4 \\ 0 & 5 & 8 \\ 0 & 0 & 0 \end{bmatrix} \qquad \rho(A) = 2$$

b

$$\begin{bmatrix} 0 & 8 & -1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \\ 0 & 4 & 5 \end{bmatrix}$$

$$R_2 \leftrightarrow R_1 \sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & 8 & -1 \\ 0 & 0 & 3 \\ 0 & 4 & 5 \end{bmatrix}$$

$$R_4 \to 2R_4 - R_2 \quad \begin{bmatrix} 1 & 2 & 0 \\ 0 & 8 & -1 \\ 0 & 0 & 3 \\ 0 & 0 & \text{[illegible]} \end{bmatrix}$$

$R_4 \to 3R_4 - 11R_3$

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 8 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \rho(A) = 3$$

c) $$A = \begin{bmatrix} 9 & 3 & 1 & 0 \\ 3 & 0 & 1 & -6 \\ 1 & 1 & 1 & 1 \\ 0 & -6 & 1 & 9 \end{bmatrix}$$

$R_2 \to 3R_2 - R_1$

$R_3 \to 9R_3 - R_1$

$$\begin{bmatrix} 9 & 3 & 1 & 0 \\ 0 & -3 & 2 & -18 \\ 0 & 6 & 8 & 9 \\ 0 & -6 & 1 & 9 \end{bmatrix} \quad 9 + 36$$

$R_3 \to R_3 + 2R_2$

$R_4 \to R_4 - 2R_2$

$$\begin{bmatrix} 9 & 3 & 1 & 0 \\ 0 & -3 & 2 & -18 \\ 0 & 0 & 12 & -27 \\ 0 & 0 & -3 & 45 \end{bmatrix} \quad \begin{array}{r} 180 \\ -27 \\ \hline 153 \end{array}$$

$R_4 \to 4R_4 + R_3$

$$\begin{bmatrix} 9 & 3 & 1 & 0 \\ 0 & -3 & 2 & -18 \\ 0 & 0 & 12 & -27 \\ 0 & 0 & 0 & 153 \end{bmatrix} \quad \rho(A) = 4$$

(d) $A = \begin{bmatrix} 1 & 1 & 2 \\ 4 & 5 & 5 \\ 5 & 8 & 1 \\ -1 & -2 & 2 \end{bmatrix}$

$R_2 \to R_2 - 4R_1$, $R_3 \to R_3 - 5R_1$, $R_4 \to R_4 + R_1$
$\sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -3 \\ 0 & 3 & -9 \\ 0 & -1 & 4 \end{bmatrix}$

$R_3 \to R_3 - 3R_2$, $R_4 \to R_4 + R_2$
$\sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 6 \end{bmatrix}$

$R_4 \leftrightarrow R_3$
$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 6 \\ 0 & 0 & 0 \end{bmatrix}$

$\rho(A) = 3$

(e) $\begin{bmatrix} 1 & 2 & -3 & -2 & -3 \\ 1 & 3 & -2 & 0 & -1 \\ 3 & 8 & -7 & -2 & -11 \\ 2 & 10 & 9 & -10 & -3 \end{bmatrix}$

$R_2 \to R_2 - R_1$
$R_3 \to R_3 - 3R_1$
$R_4 \to R_4 - 2R_2$

$\begin{bmatrix} 1 & 2 & -3 & -2 & -3 \\ 0 & 1 & 1 & 2 & -1 \\ 0 & 2 & 2 & 4 & -2 \\ 0 & 6 & 15 & -6 & 3 \end{bmatrix}$

$R_3 \to R_3 - 2R_2$
$R_4 \to R_4 - 6R_2$

$\begin{bmatrix} 1 & 2 & -3 & -2 & -3 \\ 0 & 1 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 9 & -18 & 9 \end{bmatrix}$

$R_4 \leftrightarrow R_3$
$\begin{bmatrix} 1 & 2 & -3 & -2 & -3 \\ 0 & 1 & 1 & 2 & -1 \\ 0 & 0 & 9 & -18 & 9 \end{bmatrix}$

$\rho(A) = 3$

f) $A = \begin{bmatrix} m & n & p \\ n & m & p \end{bmatrix}$

$R_2 \to mR_2 - nR_1$

$\sim \begin{bmatrix} m & n & p \\ 0 & m^2-n^2 & mp-np \end{bmatrix}$

$\rho(A) = 2$

(7) $A = \begin{bmatrix} 4 & 4 & -3 & 1 \\ 1 & 1 & -1 & 0 \\ k & 2 & 2 & 2 \\ 9 & 9 & k & 3 \end{bmatrix}$

$R_2 \to 4R_2 - R_1$

$R_3 \to 4R_3 - kR_1$

$R_4 \to 4R_4 - 9R_1$

$\sim \begin{bmatrix} 4 & 4 & -3 & 1 \\ 0 & 0 & -1 & -1 \\ 0 & 8-4k & 8+3k & 8-k \\ 0 & 0 & 4k+27 & 3 \end{bmatrix}$

$R_3 \leftrightarrow R_2$

$\begin{bmatrix} 4 & 4 & -3 & 1 \\ 0 & 8-4k & 8+3k & 8-k \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 4k+27 & 3 \end{bmatrix}$

$R_4 \to R_4 + (4k+27) R_1$

$\sim \begin{pmatrix} 4 & 4 & -3 & 1 \\ 0 & 8-4k & 8+3k & 8-k \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & -4k-24 \end{pmatrix}$

$3 \neq -4k-27$

For $\rho(A)$ to be 3,

~~For A to~~

$-4k-24=0$

$\Rightarrow k = -6$

$k=2$

(9) $x - 3y + 2z = 1$

$2x - 2y = k^2$

$3x - 5y + z = 0$

$-2x + 8y + 4z = 49$

$101 - 14k^2 - 14$

87

$$\bar{A} = \begin{bmatrix} 1 & -3 & 2 & 1 \\ 2 & -2 & 0 & k^2 \\ 3 & -5 & 1 & 0 \\ -2 & 8 & 4 & 49 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$

$R_3 \to R_3 - 3R_1$

$R_4 \to R_4 + 2R_1$

$-2 - (-6)$

$$\sim \begin{bmatrix} 1 & -3 & 2 & 1 \\ 0 & 4 & -4 & k^2-2 \\ 0 & 4 & -5 & -3 \\ 0 & 2 & 8 & 51 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$R_4 \to 2R_4 - R_2$

$-3 - k^2 + 2$

$$\sim \begin{bmatrix} 1 & -3 & 2 & 1 \\ 0 & 4 & -4 & k^2-2 \\ 0 & 0 & -1 & -k^2-1 \\ 0 & 0 & 14 & 101 \end{bmatrix}$$

$R_4 \to R_4 + 14R_3$

$$\sim \left[\begin{array}{ccc|c} 1 & -3 & 2 & 1 \\ 0 & 4 & -4 & k^2 \\ 0 & 0 & -1 & -k^2 \\ 0 & 0 & 0 & -14k^2 \end{array}\right]$$

$\rho(A) = 3$

$\rho(\bar{A}) = 4$

in consistent.

(10) $$\bar{A} = \begin{bmatrix} 3 & 4 & 5 & a \\ 4 & 5 & 6 & b \\ 5 & 6 & 7 & c \end{bmatrix}$$

$R_2 \to 3R_2 - 4R_1$

$R_3 \to 3R_3 - 5R_1$

$$\sim \left[\begin{array}{ccc|c} 3 & 4 & 5 & a \\ 0 & -1 & -2 & 3b-4a \\ 0 & -2 & -4 & 3c-5a \end{array}\right]$$

$R_3 \to R_3 - 2R_2$

$$\sim \left[\begin{array}{ccc|c} 3 & 4 & 5 & a \\ 0 & -1 & -2 & 3b-4a \\ 0 & 0 & 0 & \cancel{3c-5a-6b+8a} \\ & & & 3a-6b+3c \end{array}\right]$$

$\rho(A) = 2$

$\rho(\bar{A}) = 2$ iff $3a-6b+3c=0$

$\Rightarrow a+c=2b$

(a) $\cancel{\rho(A)} =$ no solution.

If $a-8=0$ & $b-15 \neq 0$

$\rho(A) = 2$ $\rho(\bar{A}) = 3$

$\Rightarrow$ no solution

(b) $a-8 \neq 0$ $\cancel{b-15 \neq 0}$

$\rho(A) = \rho(\bar{A}) = n = 3$

unique solution

(11)
$$\tilde{A} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 1 & 3 & 5 & 9 \\ 2 & 5 & a & b \end{array}\right]$$

$R_2 \to R_2 - R_1$

$R_3 \to R_3 - 2R_1$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & a-6 & b-12 \end{array}\right]$$

$R_3 \to R_3 - R_2$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & a-8 & b-15 \end{array}\right]$$

(c) $b-15=0$, $a-8=0$

$\rho(A) = \rho(\bar{A}) = 2 < 3 = n$.

$\Rightarrow$ infinite solutions.