15.12.2017.

First order.

Separation of variables | Homogenous equations. | Non-homogenous equations. | Exact differential equation. | Linear differential equation.

19.12.2017.

Separation of variables. (separable differential equations)

$$\frac{dy}{dx} = f(x,y) \qquad f(x,y) = h(x)\, g(y)$$

$$\frac{dy}{dx} = h(x)\, g(y)$$

$$\frac{1}{g(y)}\, dy = h(x)\, dx.$$

1. $$\frac{dy}{dx} = x^2 y^3$$

$$\int \frac{dy}{y^3} = \int x^2\, dx$$

$$\frac{y^{-2}}{-2} = \frac{x^3}{3} + c. \Rightarrow \frac{1}{-2y^2} = \frac{x^3}{3} + c$$

2. $$9y \frac{dy}{dx} + 4x = 0.$$

$$\int 9y\, dy = \int -4x\, dx$$

$$\frac{9y^2}{2} = -\frac{4x^2}{2} + c \Rightarrow \frac{9}{2} y^2 = -2x^2 + c.$$

3. $$\frac{dy}{dx} = (1+y^2)$$

$$\int \frac{dy}{1+y^2} = \int dx \rightarrow \tan^{-1} y = x + c.$$

4. $2xy\dfrac{dy}{dx} = y^2 - x^2$

$y = vx$ $\qquad v = \dfrac{y}{x}$

$\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

$2\dfrac{y}{x}\dfrac{dy}{dx} = \dfrac{y^2}{x^2} - 1$

$2v\dfrac{dy}{dx} = v^2 - 1$

$2v\left(v + x\dfrac{dv}{dx}\right) = v^2 - 1$

$v + x\dfrac{dv}{dx} = \dfrac{v^2-1}{2v}$

$x\dfrac{dv}{dx} = \dfrac{v^2-1}{2v} - v = \dfrac{v^2-1-2v^2}{2v} = \dfrac{-v^2-1}{2v}$

$-\displaystyle\int \frac{2v}{v^2+1}\,dv = \int \frac{dx}{x}$ $\qquad v^2+1 = u$

$2v\,dv = du$

$-\cancel{\log}^{\ln}|(v^2+1)| = \cancel{\log}^{\ln}|x| + c$

$\Rightarrow -\cancel{\log}^{\ln}\left|\left(\frac{y}{x}\right)^2 + 1\right| = \cancel{\log}^{\ln}|x| + c$

$f(x,y) \rightarrow$ homogenous.

$f(\lambda x, \lambda y) = \lambda^n f(x,y)$ some n. $\Rightarrow \dfrac{1}{\left|1+\left(\frac{y}{x}\right)^2\right|} = |x|c$

5. $(2x - 4y + 5)\dfrac{dy}{dx} + (x - 2y + 3) = 0.$

$\dfrac{dy}{dx} = -\left(\dfrac{x - 2y + 3}{2x - 4y + 5}\right)$

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$$\frac{h}{bc'-b'c} = \frac{k}{ca'-c'a} = \frac{1}{ab'-ba'}$$

$a = -1 \quad a' = 2$

$b = 2 \quad b' = -4$

$c = -3 \quad c' = 5$

$$\frac{h}{10+12} = \frac{k}{-6+5} = \frac{1}{4-4}$$

$$\frac{a}{a'} = \frac{b'}{b'} = \frac{-1}{2}.$$

$$\frac{dy}{dx} = \frac{-x+2y+3}{-2(-x+2y)+5}$$

$$-x+2y = t$$

$$-1+2\frac{dy}{dx} = \frac{dt}{dx}$$

$$\frac{1}{2}\left(\frac{dt}{dx}+1\right) = \frac{-x+2y+3}{2x-4y+5} = \frac{t+3}{-2t+5}.$$

$$\frac{dt}{dx} = \frac{2t+6}{-2t+5} - 1 = \frac{2t+6+2t-5}{-2t+5}$$

$$= \frac{4t+1}{-2t+5}.$$

$$4t+1 \overline{\big) -2t+5} \; -1/2$$

$$-2t - 1/2$$

$$\underline{5.5}$$

$$\int \frac{-2t+5}{4t+1}\,dt = \int dx$$

$$\int -\frac{1}{2} + \frac{11}{8t+2} = x+c$$

$$-\frac{t}{2} + \frac{11}{8}\log|8t+2| = x+c.$$

21.12.2017

$$\frac{dy}{dx} = \frac{2x+3y+6}{4x+6y+5} = \frac{2x+3y+6}{2(2x+3y)+5}$$

$2x+3y = t$

$2 + 3\frac{dy}{dx} = \frac{dt}{dx}$

$$\left(\frac{dt}{dx} - 2\right)\frac{1}{3} = \frac{t+6}{2t+5}$$

$$\frac{dt}{dx} - 2 = \frac{3t+18}{2t+5}$$

$$\frac{dt}{dx} = \frac{3t+18+4t+10}{2t+5} = \frac{7t+28}{2t+5}$$

$$\int \frac{2t+5}{7t+28}\,dt = \int dx$$

$$7t+28 \overline{)\,2t+5\,}\;\; \frac{2}{7}$$

$$2t + \cancel{\frac{16}{3}}\frac{8}{7}$$

$$-\frac{29}{7}$$

$$\cancel{\int \frac{2}{7} + \frac{19}{7(7t+8)}}$$

$$\int \frac{2}{7} - \frac{3}{7t+28} = x+c$$

$$\frac{2t}{7} - \int \frac{3}{7t+28} = x+c.$$

$$\frac{2t}{7} - \frac{3}{7}\int \frac{1}{t+4} = x+c$$

$$\frac{2t}{7} - \frac{3}{7}\log|t+4| = x+c.$$

$$\frac{2}{7}(2x+3y) - \frac{3}{7}\log|(2x+3y+4)| = x+c.$$

$$\cancel{\frac{3}{7}(x+y)} - \frac{3}{7}\log|2x+3y+4| = c$$

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$$\frac{dy}{dx} = \frac{2x+3y+6}{4x+y+5} \qquad x = X+h \quad y = Y+k$$

$$= \frac{2(X+h)+3(Y+k)+6}{4(X+h)+(Y+k)+5}$$

$2h+3k+6=0. \Rightarrow 4h+6k+12=0.$

$4h+k+5=0$ $\qquad 5k+7=0 \Rightarrow k = \underline{-7/5}.$

$4h = \frac{7}{5}-5 = \frac{-18}{5} \Rightarrow h = \frac{-18}{4\times5} = \underline{\frac{-9}{10}}.$

$$\frac{dY}{dX} = \frac{dy}{dx} = \frac{2X+3Y}{4X+Y} \qquad \frac{X}{Y} = v$$

$$v + X\frac{dv}{dX} = \frac{2+3v}{4+v}$$

$$X\frac{dv}{dX} = \frac{2+3v-4v-v^2}{4+v} = \frac{-v^2-v+2}{4+v}$$

$$\int \frac{4+v}{-v^2-v+2}\,dv = \int \frac{dx}{x}$$

$$v-1 \overline{)\,(v^2+v-2)}\ \ v+2$$
$$\underline{v^2-v}$$
$$2v-2$$
$$\underline{2v-2}$$
$$\underline{\underline{0}}.$$

$$\frac{4+v}{-(v-1)(v+2)} = \frac{A}{v-1} + \frac{B}{v+2}$$

$$= Av + 2A + Bv - B$$

$+2A-B = 4.$

$A+B = 1.$

~~$+2A-(1-A) = 4 \Rightarrow$~~ ~~$4A+ \ldots A \ldots 5 \Rightarrow A = 5$~~

~~$-5\log|\frac{y}{x}-1| + 4\log|\frac{y}{x}|$~~ $2A-1+A = 4 \Rightarrow 3A-1=4$

$A = 5/3.$ $\quad B = -2/3.$

HW $(x-2y+1)\,dx + (4x-3y-6)\,dy = 0.$

Exact differential equation.

$$M(x,y)\,dx + N(x,y)\,dy = 0.$$

$$du = 0.$$

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = 0.$$

$$\frac{\partial u}{\partial x} = M(x,y) \qquad \frac{\partial u}{\partial y} = N(x,y).$$

$$\frac{\partial^2 u}{\partial y\,\partial x} = \frac{\partial M}{\partial y} \qquad \frac{\partial^2 u}{\partial x\,\partial y} = \frac{\partial N}{\partial x}.$$

$$\frac{\partial^2 u}{\partial y\partial x} = \frac{\partial^2 u}{\partial x\partial y}$$

$$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$

$$(3x^2 + 4xy)\,dx + (2x^2 + 2y)\,dy = 0.$$

$$\frac{\partial M}{\partial y} = 4x = \frac{\partial N}{\partial x} = 4x \quad \therefore \text{Exact.}$$

$$\therefore\ x^3 + 2x^2y + 2x^2y + y^2 = c$$

$$x^3 + 4x^2y + y^2 = c$$

$$\frac{\partial u}{\partial x} = 3x^2+4xy$$

$$u = \int \partial u = \int 3x^2+4xy\ \partial x = x^3 + 2x^2y + f(y).$$

$$\frac{\partial u}{\partial y} = 2x^2 + f'(y)$$

$$f'(y) = 2y.$$

$$\frac{d}{dy} f(y) = 2y$$

$$f(y) = \int 2y\, dy = y^2 + c.$$

$$u = x^3 + \underline{2x^2y} + y^2 + c.$$

$$d(x^3 + 2x^2\underline{y} + y^2 + c) = 0.$$

$$x^3 + 2x^2\underline{y} + y^2 = c.$$

$$(2x\cos y + 3x^2y)\,dx + (x^3 - x^2\sin y - y)\,dy = 0.$$

$$x^2\cos y + x^3y + f = u$$

$$\frac{\partial u}{\partial y} = -x^2\sin y + x^3 + f' \qquad \therefore\ f' = -y \Rightarrow f = -\frac{y^2}{2}.$$

$$\therefore\ x^2\cos y + x^3y - \underline{\frac{y^2}{2}} = c$$

$$(2x^2+y)\,dx + (x^2y - x)\,dy = 0.$$

Integrating factor = μ => convert non-exact DE to exact DE.

$$P(x,y)\,dx + Q(x,y)\,dy = 0.$$

$$P_y \neq Q_x \qquad \therefore \text{Not exact}$$

$$\underbrace{\mu(x,y)\,P(x,y)\,dx}_{M} + \underbrace{\mu(x,y)\,Q(x,y)\,dy}_{N} = 0$$

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If $\frac{1}{Q(x,y)}\left[\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right] = f(x)$ function of x only.

$$\mu(x,y) = e^{\int f(x)\,dx}.$$

If $\frac{1}{P(x,y)}\left\{\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right\}$ function of y only. $= f$

$$\mu(x,y) = e^{\int f(y)\,dy}$$

$(2x^2+y)\,dx + (x^2y - x)\,dy = 0.$

$\frac{\partial P}{\partial y} = 1 \neq \frac{\partial Q}{\partial x} = 2xy - 1.$ $\therefore$ Not exact.

$$\frac{1}{(x^2y-x)}\left[1 - 2xy + 1\right] = \frac{1}{(x^2y-x)}\left[2 - 2xy\right]$$

$$= \frac{-2(1-xy)}{x(1-xy)} = \frac{-2}{x}.$$

$$\mu(x,y) = e^{\int -2/x\,dx} = e^{-2\log x} = x^{-2}. \frac{1}{x^2}$$

$\Rightarrow \left(2 + \frac{y}{x^2}\right)dx + \left(y - \frac{1}{x}\right)dy = 0.$ is an exact equation

$u = 2x - \frac{y}{x} + f$ $\quad \frac{\partial u}{\partial y} = -\frac{1}{x} + f' = y - \frac{1}{x}$

$\Rightarrow f' = y \Rightarrow f = \frac{y^2}{2}.$

$\therefore u = 2x - \frac{y}{x} + \frac{y^2}{2} = c.$

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$(2x + y\tan y)\,dx + (x - x^2\tan y)\,dy = 0.$

$\frac{\partial Q}{\partial x} = 1 - 2x\tan y \qquad \frac{\partial P}{\partial y} = \sec^2 y$

$\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} = \sec^2 y - 1 + 2x\tan y = + \tan^2 y + 2x\tan y = \tan y(2x + \tan y)$

$\frac{1}{P}\left[-\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial x}\right] = -\tan y$

ILATE

~~$(-2x\cot y - 1)dx + (-x\cot y + x^2)dy = 0.$~~

$\int \sec x$

$\mu = e^{\int \tan y} = e^{-\log \sec x} = e^{\log \cos x} = \cos x.$

$x\sec x -$

$\int x\sec x \tan x\, d$

$\left(\frac{2x}{\cos x} + \frac{\tan y}{\cos x}\right)dx + \left(\frac{x}{\cos x} - \frac{x^2\tan y}{\cos x}\right)dy = 0.$

$(2x\sec x + \tan y\sec x)\,dx + (x\sec x - x^2\tan y\sec x)\,dy = 0.$

$\int 2x\sec x + \tan y\sec x\;dx$

$= 2\left[x\int\sec x - \int\int\sec x + \tan y\int\sec x\right]$

$(2x\cos x + \tan y\cos x)\,dx + (x\cos x - \tan y\, x^2\cos x)\,dy = 0.$

$2\left[x\sin x - \int \sin x\, dx\right] + \left[\tan y\sin x\right] + f^{\wedge} = c.$

$2x\sin x + 2\cos x + \tan y\sin x + f^{\wedge} = c$

$\frac{\partial u}{\partial y} = \sec^2 y\sin x + f'$

Linear differential equations

$$\frac{dy}{dx} + P(x)\, y = Q(x). \quad \text{— Linear in } y.$$

$$\frac{dx}{dy} + P(y)\, x = Q(y) \quad \text{— Linear in } x.$$

$$dy + (P(x)y - Q(x))\, dx = 0.$$

$$M = P(x)y - Q(x) \qquad N = 1.$$

$$\frac{\partial M}{\partial y} = P(x) \qquad \frac{\partial N}{\partial x} = 0. \qquad \therefore M_y \neq N_x$$

Not exact

$$\frac{1}{1}\left[P(x) - 0\right] = p(x).$$

$$\mu(x,y) = e^{\int P(x)\, dx}.$$

$$e^{\int P(x)\, dx}\left(P(x)y - Q(x)\right)dx + e^{\int P(x)\, dx}\, dy = 0.$$

$\therefore$ this is an exact DE.

Solution $$y \times (IF) = \int Q(x)\cdot(IF)\, dx + c.$$

$$\frac{dy}{dx} + \left(\frac{2x+1}{x}\right)y = e^{-2x}$$

$$P(x) = (2 + 1/x) \qquad Q(x) = e^{-2x}$$

$$IF = e^{\int P(x)dx} = e^{\int 2 + 1/x} = e^{2x + \log x} = xe^{2x}$$

$$y \times xe^{2x} = \int e^{-2x} \times xe^{2x}\, dx$$

$$= \int x = \frac{x^2}{2} + c$$

$(x^2+1)\dfrac{dy}{dx} + 4xy = x$

$\dfrac{dr}{d\theta} + r\tan\theta = \cos\theta.$

$\dfrac{dy}{dx} + \dfrac{4xy}{x^2+1} = \dfrac{x}{x^2+1}$

$e^{\int \frac{4x}{x^2+1}dx} = e^{2\log(x^2+1)} = (x^2+1)^2$

$y(x^2+1)^2 = \int \dfrac{x}{x^2+1}(x^2+1)^2\,dx = \int x^3 + x\,dx = \underline{\dfrac{x^4}{4} + \dfrac{x^2}{2}} + c$

$\dfrac{dr}{d\theta} + r\tan\theta = \cos\theta$

$e^{\int \tan\theta\, d\theta} = e^{\log\sec\theta} = \underline{\sec\theta}.$

$r\sec\theta = \int \cos\theta\sec\theta\, d\theta = \underline{\theta + c}.$

$(x-2y+1)dx + (4x-3y-6)dy = 0.$

$x = X+h$

$y = Y+k$

$\dfrac{dy}{dx} = -\left(\dfrac{x-2y+1}{4x-3y-6}\right)$

$h - 2k + 1 = 0.$

$4h - 3k - 6 = 0.$

$4h - 8k + 4 = 0.$

$\dfrac{dy}{dx} = -\left(\dfrac{X-2Y}{4X-3Y}\right)$

$5k - 10 = 0$

$\boxed{k=2. \quad h=3}$

$v + X\dfrac{dv}{dX} = -\left(\dfrac{1-2v}{4-3v}\right)$

$X\dfrac{dv}{dX} = \dfrac{-1+2v-4v+3v^2}{4-3v} = \dfrac{3v^2-2v-1}{4-3v}$

$\int \dfrac{dX}{X} = \int \dfrac{4-3v}{3v^2-2v-1}\,dv$

$$\log X = 4\int \frac{1}{3v^2-2v-1} - 3\int \frac{v}{3v^2-2v-1}\, dv$$

conversion into exact

$$(2x+\tan y)dx + (x - x^2\tan y)dy = 0.$$

$$\frac{\partial M}{\partial y} = \sec^2 y \qquad \frac{\partial N}{\partial x} = 1 - 2x\tan y.$$

$$\frac{1}{M}\left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right] = \frac{[1-2x\tan y - \sec^2 y]}{2x+\tan y} = \frac{-\tan^2 y - 2x\tan y}{2x+\tan y}$$

$$= -\tan y.$$

$$e^{\int -\tan y\, dy} = e^{-\log \sec y} = \cos y.$$

$$[2x\cos y + \cancel{\ } \sin y]dx + [x\cos y - x^2\sin y]\,dy = 0.$$

$$u = x^2\cos y + x\sin y + f$$

$$\frac{\partial u}{\partial y} = \cancel{\ } \cancel{\ } - x^2\sin y + x\cos y + f'$$

$$f' = 0 \Rightarrow f = c$$

$$x^2\cos y + x\sin y = c$$

26.12.2017.

Non-linear problems

$$\boxed{\frac{dy}{dx} + P(x)y = Q(x)\,y^n \quad n>0.} \rightarrow \text{Bernoulli equations.}$$

$$\Rightarrow y^{-n}\frac{dy}{dx} + P(x)y^{1-n} = Q(x)$$

put $v = y^{1-n}$

$$\frac{dv}{dx} = (1-n)y^{-n}\frac{dy}{dx}.$$

$$y^{-n}\frac{dy}{dx} = \frac{1}{(1-n)}\frac{dv}{dx}$$

$$\therefore\ y^{-n}\frac{dy}{dx} + P(x)y^{1-n} = Q(x) \Rightarrow \frac{1}{(1-n)}\frac{dv}{dx} + P(x)v = Q(x).$$

$$\frac{dv}{dx} + (1-n)P(x)v = (1-n)Q(x) \Rightarrow \text{linear in } v.$$

Ex. $\frac{dy}{dx} + \cancel{x}y = xy^3$ $\qquad y^{-2} = v$

$$\frac{1}{y^3}\frac{dy}{dx} + \frac{\cancel{x}1}{y^2} = x. \qquad \frac{-2}{y^3}\frac{dy}{dx} = \frac{dv}{dx}$$

$$-\frac{1}{2}\frac{dv}{dx} + \cancel{x}v = x$$

$$+\frac{dv}{dx} - 2\cancel{x}v = -2x.$$

IF $= e^{\int -2\cancel{x}\,dx} = e^{-2x\cancel{x}}$

$$\cancel{ve^{-x^2} = \int -2xe^{-x^2} = \int e^{u}\,du = e^{-x^2}+c}$$

$$\cancel{\frac{e^{-x^2}}{y^2}} = \cancel{e^{-x^2}+c}.$$

$$ve^{-2x} = \int -2xe^{-2x} \qquad +2x = u.$$

$$+2dx = du.$$

$$= \int -ue^{-u}\frac{du}{+2}$$

$$= \left(+u \times e^{-u} - \int (-1)e^{-u}\right)\frac{1}{+2}$$

$$= \left(ue^{-u} + e^{-u} + c\right)\frac{1}{2}.$$

$$\frac{+2e^{-2x}}{y^2} = +2xe^{-2x} + e^{-2x} + c$$

HW. $\frac{dy}{dx} + \frac{y}{2x} = \frac{x}{y^3}$, $y(1) = 2$.

$P(x) = \frac{1}{2x}$ $Q(x) = x$.

$y^3 \frac{dy}{dx} + \frac{y^4}{2x} = x$

$y^4 = u$

$\frac{1}{4}\frac{du}{dx} + \frac{u}{2x} = x$.

$4y^3 \frac{dy}{dx} = \frac{du}{dx}$

$\frac{du}{dx} + \frac{2u}{x} = 4x$.

$\text{IF} = e^{\int 2/x \, dx} = e^{2\log x} = x^2$

$ux^2 = \int x^2 \times 4x \, dx$

$= \int 4x^3 dx = x^4 + c$

$y^4 x^2 = x^4 + c$.

$(2)^4 = 1 + c \Rightarrow c = 16 - 1 = 15$.

$\therefore y^4 x^2 = x^4 + 15$.

APPLICATIONS.

$F(x,y,c) = 0$ —①.

c: Parameter.

To find the orthogonal & trajectory of ①.

1. Write the DE for ①.

$$\frac{dy}{dx} = g(x,y) \text{ —— ②}.$$

2. Write the DE of orthogonal trajectory of ①.

$x^2 + y^2 = a^2$

$2x + 2y \frac{dy}{dx} = 0$

$\frac{dy}{dx} = \frac{-x}{y}$. for $\perp r$ curves $m_1 \cdot m_2 = -1$

$\therefore m_1 = \frac{dy}{dx} = \frac{-x}{y}$ $\quad m_2 = \frac{-1}{dy/dx} = \frac{-1}{-x/y} = \frac{y}{x}$.

(3) — $\frac{dy}{dx} = \frac{y}{x}$. $\Rightarrow$ DE of orthogonal trajectory.

3. Solve ~~the~~ (3) and get solution.

$\int \frac{dy}{y} = \int \frac{dx}{x} \Rightarrow \log y = \log x + \log c$

$\Rightarrow y = cx$.

Find the orthogonal trajectories of the family of parabolas $y = cx^2$.

Ans:- $\frac{dy}{dx} = 2cx \Rightarrow c = \frac{1}{2x} \frac{dy}{dx}$.

$\rightarrow y = \frac{1}{2x} \frac{dy}{dx} x^2 = \frac{x}{2} \frac{dy}{dx} \rightarrow \frac{dy}{dx} = \frac{2y}{x}$. — (1).

$m_2 = \frac{-x}{2y} = \frac{dy}{dx}$ — (2).

$\int -x\,dx = \int 2y\,dy \Rightarrow c - \frac{x^2}{2} = y^2$

$c = y^2 + \frac{x^2}{2}$

$\Rightarrow c = 2y^2 + x^2$.

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HW. Find the orthogonal trajectories

1. $y = cx^3$.

2. $cx^2 + y^2 = 1$.

3. $y = x - 1 + ce^{-x}$.

4. $x = \frac{y^2}{4} + \frac{c}{y^2}$

Ans:- 1. $\frac{dy}{dx} = 3cx^2$

$c = \frac{1}{3x^2}\frac{dy}{dx}$

$y = \frac{x}{3}\frac{dy}{dx}$

$\frac{3y}{x} = \frac{dy}{dx} \Rightarrow \frac{dy'}{dx'} = \frac{-x}{3y}$

$\frac{3y^2}{2} = \frac{-x^2}{2} + c$

$3y^2 + x^2 = c$

2. $2cx + 2y\frac{dy}{dx} = 0$.

$2cx = -2y\frac{dy}{dx} \rightarrow c = \frac{-y}{x}\frac{dy}{dx}$

$-xy\frac{dy}{dx} + y^2 = 1$

$$\frac{dy}{dx} = \frac{y^2-1}{xy}$$

$$\frac{dy'}{dx'} = \frac{xy}{1-y^2}$$

$$\int \frac{1-y^2}{y}\,dy = \int x\,dx$$

$$\log y - \frac{y^2}{2} = \frac{x^2}{2} + c$$

3. $\frac{dy}{dx} = 1 - ce^{-x}$

$$\left(\frac{dy}{dx} - 1\right)\frac{-1}{e^{-x}} = c.$$

$$y = x - 1 + 1 - \frac{dy}{dx} = x - \frac{dy}{dx}$$

$$\frac{dy}{dx} = x - y.$$

$$\frac{dy}{dx} = \frac{-1}{x-y}$$

$\cancel{v + x\frac{dv}{dx} = x + vx = x(1+v)}$

$\cancel{x\frac{dv}{dx} = x + vx - v.}$

$\cancel{\frac{dy}{dx} + y = x}$ $\cancel{IF = e^{\int dx} = e^x}$.

$\cancel{ye^x =}$ $\frac{dx}{dy} = y - x.$

$$\frac{dx}{dy} + x = y \qquad IF = e^{\int dy} = e^y$$

$$xe^y = \int ye^y\,dy = ye^y - e^y + c$$

4. $x = \frac{y^2}{4} + \frac{c}{y^2}$

$1 = \left(\frac{2y}{4} - \frac{2c}{y^3}\right)\frac{dy}{dx}$

$\cancel{\frac{dx}{dy}\left(1-\frac{y}{2}\right)\frac{y^3}{-2} = c = \left(\frac{(y-2)y^3}{4}\right)\frac{dx}{dy}}$

$\cancel{x = \frac{y^2}{4} +}$

$\left(\frac{dx}{dy} - \frac{y}{2}\right)\frac{y^3}{-2} = \left(\frac{y}{2} - \frac{dx}{dy}\right)\frac{y^3}{2} = c.$

$x = \frac{y^4}{4} + \frac{y}{2}\left(\frac{y}{2} - \frac{dx}{dy}\right)$

$= \frac{y^4}{4} + \frac{y^2}{4} - \frac{y}{2}\frac{dx}{dy}.$

$\frac{y}{2}\frac{dx}{dy} + x = \frac{y^4+y^2}{4}$

$\frac{dx}{dy} + \frac{2x}{y} = \frac{y^3+y}{2}.$

$IF = e^{\int 2/y\,dy} = y^2$

$xy^2 = \int y^2 \times \left(\frac{y^3+y}{2}\right) dy = \frac{1}{2}\left[\frac{y^6}{6} + \frac{y^4}{4}\right] + c.$

Ans $24xy^2 = 2y^6 + 3y^4 + c$

2. $cx^2+y^2=1$

$$2cx+2y\frac{dy}{dx}=0.$$

$$2cx=-2y\frac{dy}{dx}$$

$$c=\frac{-y}{x}\frac{dy}{dx}$$

$$\Rightarrow -xy\frac{dy}{dx}+y^2=1.$$

$$\frac{dy}{dx}=\frac{1-y^2}{-xy}=\frac{y^2-1}{xy}.$$

$$\frac{-1}{\frac{dy}{dx}}=\frac{-1(xy)}{y^2-1}=\frac{xy}{1-y^2}=\frac{dy}{dx}$$

$$\int x\,dx=\int\frac{1-y^2}{y}dy$$

$$\frac{x^2}{2}=\log y-\frac{y^2}{2}+c.$$

$$x^2+y^2=2\log y+c$$

28.12.2017.

ORTHOGONAL TRAJECTORIES IN POLAR FORM.

$$F(r,\theta,c)=0.$$

*. slope $= r\frac{dr}{d\theta}$.

$\boxed{P\,dr+Q\,d\theta=0}\xrightarrow{O.T}\boxed{Qr^2dr-P\,d\theta=0}$

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28. ii. $r = a\cos\theta$.

$\frac{dr}{d\theta} = -a\sin\theta \Rightarrow a = -\frac{1}{\sin\theta}\frac{dr}{d\theta}$

$\cancel{r\frac{dr}{d\theta} = ar\sin\theta}$

$dr + a\sin\theta\, d\theta = 0$

$P = 1 \quad Q = a\sin\theta$

$= r\tan\theta$

$\cancel{r\frac{dr}{d\theta} = \frac{1}{ar\sin\theta}}$

$\cancel{\int ar^2 dr = \int \operatorname{cosec}\theta\, d\theta = \int\frac{d\theta}{\sin\theta}}$

$\cancel{\frac{ar^3}{3} = \log|\operatorname{cosec}x - \cot x| + c}$

$r^3 + \tan\theta\, dr - d\theta = 0$

$r^3 + \tan\theta\, dr = d\theta$

$r^3 dr = \cot\theta\, d\theta$

$\frac{r^4}{4} = \log|\sin\theta|$

$r = -\cot\theta\frac{dr}{d\theta}$

$-\tan\theta\, d\theta = \frac{dr}{r} \Rightarrow \log|\cos\theta| = \log|r| + c$

$= \cos\theta = rc$

NEWTON'S COOLING AND HEATING PROBLEM.

$\frac{dT}{dt} \propto (T-M)$

$\frac{dT}{dt} = k(T-M)$.

Eg:. A steak is removed from a freezer and put into the refrigerator to thaw. The freezer is kept at $-10°c$ and the fridge is kept at $40°c$. After 4 hours the temperature of steak was $-6°c$. When will the steak be thawed to $2^{\circ}c$.

Ans :-
$$\frac{dT}{dt} \propto (T-M)$$

$$\frac{dT}{dt} = k(T-M)$$

$$\frac{dT}{T-M} = k\,dt$$

$$\frac{dT}{\cancel{(T+10+4)}\,(T-M)} = +k\,dt$$

~~log T+4 = kt~~ $\pm c$.

$$\log(T-M) = +kt + c.$$

$$\log T = \log m + kt + c.$$

$$T = M + ce^{kt}$$

$M = 4$. When $t=0$, $T = -10^{\circ}c$.

$t = 4$ hrs, $T = -6^{\circ}c$.

then $-10 = 4 + C$

$\Rightarrow$ $c = -14$

$$-6 = 4 - 14e^{4k}.$$

$$\frac{10}{14} = e^{4k} \Rightarrow \log\left(\frac{5}{7}\right) = 4k$$

$$k = \frac{1}{4}\log\frac{5}{7} \Rightarrow$$

Then

$$2 = 4 + -14\left[e^{\log(5/7)^{1/4} \cdot T}\right]$$

$k = -0.08.$

$$-2 = -14\left[\left(\frac{5}{7}\right)^{1/4}\right]^{T}$$

$$\log\frac{1}{7} = T\log\left(\frac{5}{7}\right)^{1/4}$$

$$T = \frac{\log(1/7)}{\log(5/7)^{1/4}} = \underline{\underline{23.13\,^{\circ}c}}.$$

$$= \underline{\underline{\frac{4\log(1/7)}{\log(5/7)}}}.$$

<u>1.1.2018</u>.

Growth <u>and decay problems.</u>

$$\frac{dy}{dt} \propto y$$

$$\frac{dy}{dt} = ky$$

$$\int\frac{dy}{y} = \int k\,dt$$

$$\ln|y| = kt + \ln c.$$

$$\boxed{y = ce^{kt}}$$

1. Suppose it is known that the cells of a given bacterial culture divide every 3.5 hours (on average). If there are 500 cells in a dish to begin with, how many will there be after 12 hours?

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Ans:

$$y = ce^{kt}$$

$$500 = c$$

$$1000 = 500e^{3.5k}$$

$$2 = e^{3.5k}$$

$$\ln 2 = 3.5k$$

$$\Rightarrow \quad k = \frac{\ln 2}{3.5}$$

$$y = 500e^{\frac{\ln 2}{3.5} \times 12}$$

$$= 500\left(e^{\ln 2}\right)^{12/3.5}$$

$$= 500\,(2)^{12/3.5}$$

$$= 5383.60 \text{ is } 5383 \text{ cells.}$$

Second order differential equation (linear).

$$f(y, y', y'') = 0.$$

eg: $y'' + xy' + y = 4x.$

$$\boxed{y'' + p(x)y' + q(x)y = r(x)} \Rightarrow \text{Linear 2nd order DE}$$

If $r(x) = 0$, then it is called homogenous.

If $r(x) \neq 0$, then it is called non-homogenous.

Homogenous → If y_1 and y_2 are solutions, then $c_1y_1 + c_2y_2$ is also a solution of this equation.

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$y'' - y = 0.$

$y_1 = e^{x} \qquad y_2 = e^{-x}$

$y = c_1 e^{x} + c_2 e^{-x}.$

constant coefficients. (real).

$$y'' + ay' + by = 0$$

$$y = e^{\lambda x}$$

$$y' = \lambda e^{\lambda x}, \quad y'' = \lambda^2 e^{\lambda x}$$

$$(\lambda^2 + a\lambda + b) e^{\lambda x} = 0.$$

$\boxed{\Rightarrow \lambda^2 + a\lambda + b = 0} \Rightarrow$ auxiliary equation

characteristic equation.

$$\lambda = \frac{-a \pm \sqrt{a^2 - 4b}}{2}$$

If. $\lambda_1 = \dfrac{-a + \sqrt{a^2-4b}}{2}.$

$\lambda_2 = \dfrac{-a - \sqrt{a^2-4b}}{2}.$

Case - I :- λ_1 & λ_2 are real and $\lambda_1 \neq \lambda_2$.

the solution $\boxed{y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}}$

eg:- $y'' - y = 0.$

$\lambda^2 - 1 = 0.$

$\lambda = \pm 1$

$\therefore y = c_1 e^{x} + c_2 e^{-x}$

$y'' + 2y' + y = 0.$

$\lambda^2 + 2\lambda + 1 = 0.$ $\qquad y = ce^{-x}$

$(\lambda+1)^2 = 0$

$\Rightarrow \lambda = -1$

$y = xe^{-x}$

~~$x^3e^{-x} - 2x^2e^{-x} + xe^{-x} = 0.$~~

~~$xe^{-x}(x^2 - 2x + 1) = 0$~~

$y' = -xe^{-x} + e^{-x}$

$y'' = -2xe^{-x} + x^3e^{-x} - e^{-x}$

$-2xe^{-x} + x^3e^{-x} - e^{-x} + -2x^2e^{-x} + 2e^{-x} + xe^{-x}$

$= -xe^{-x} - 2x^2e^{-x} + e^{-x} + x^3e^{-x}.$

$y'' = -e^{-x} + xe^{-x} \qquad -e^{-x}$

$\rightarrow -\cancel{e^{-x}} + \cancel{xe^{-x}} - \cancel{e^{-x}} - \cancel{2x}e^{-x} + \cancel{2}e^{-x} + xe^{-x}.$

$= 0$

Case B. λ_1 & λ_2 are real and $\lambda_1 = \lambda_2 = \lambda = -\frac{a}{2}$.

The solution $\boxed{y = c_1e^{\lambda x} + c_2xe^{\lambda x}}$

$= e^{\lambda x}(c_1 + c_2x)$

$y_1 = e^{-\frac{a}{2}x}$

$y_1' = \frac{-a}{2}e^{-\frac{a}{2}x}$

$y_2 = vy_1 = ve^{-a/2\,x}$

2.1.2018. Case II :- λ_1 & and λ_2 are complex, $\lambda_1 = \alpha + i\beta$ & λ_2.

of case II. contd....

$y_2' = v'y_1 + vy_1'$ $\quad y_2'' = v''y_1 + 2v'y_1' + vy_1''$

$(v''y_1 + 2v'y_1' + vy_1'') + a(v'y_1 + vy_1') + b(vy_1) = 0$

$v\underbrace{(y_1'' + ay_1' + by_1)}_{=0} + v'\underbrace{(2y_1' + ay_1)}_{=0} + v''y_1' = 0.$

$\Rightarrow v''y_1' = 0. \Rightarrow v'' = 0 \Rightarrow v = cx + d.$

$\Rightarrow y_2 = xy_1$

1. Find the solution of.

$y'' - 6y' + 9y = 0.$

Ans:- $\lambda^2 - 6\lambda + 9 = 0.$

$\lambda = \frac{6 \pm \sqrt{36-36}}{2} = \underline{3, 3}$

$\therefore \; y = c_1 e^{3x} + c_2 x e^{3x}$

$y' = 3e^{3x} + 3xe^{3x} + e^{3x} = 4e^{3x} + 3xe^{3x}$

$y'' = 12e^{3x} + 9xe^{3x} + 3e^{3x}$

$y'' - 6y' + 9y = \cancel{12e^{3x}} + \cancel{9xe^{3x}} + \cancel{3e^{3x}}$
$- \cancel{24e^{3x}} - \cancel{18xe^{3x}} + \cancel{9e^{3x}} + \cancel{9xe^{3x}} = \underline{0}$

Case III.

$a^2 < 4b$

$\lambda = \frac{-a \pm \sqrt{a^2 - 4b}}{2a} \rightarrow \frac{-a \pm i\sqrt{4b - a^2}}{2} = \alpha \pm i\beta$

$y'' + y = 0$

$y = \sin x, \cos x$

$y = e^{\alpha x}(A\cos\beta x + B\sin\beta x)$.

$y = c_1 e^{\alpha_1 x} + c_2 e^{\alpha_2 x}$

$= c_1\left(e^{(\alpha+i\beta)x}\right) + c_2\left(e^{(\alpha-i\beta)x}\right)$

$= c_1 e^{\alpha x}\cdot e^{i\beta x} + c_2 e^{\alpha x}\cdot e^{-i\beta x}$

$= e^{\alpha x}\left(c_1 e^{i\beta x} + c_2 e^{-i\beta x}\right)$

$= e^{\alpha x}\left(c_1(\cos\beta x + i\sin\beta x) + c_2(\cos\beta x - i\sin\beta x)\right)$

$= e^{\alpha x}(A\cos\beta x + B\sin\beta x)$.

1. $y'' - 6y' + 25y = 0$.

$\alpha = \frac{6 \pm \sqrt{36-100}}{2} = 3 \pm \overset{4}{\cancel{8}}i$

$y = c_1 e^{(3+\overset{4}{8}i)x} + c_2 e^{(3-\overset{4}{8}i)x}$

$= e^{3x}(A\cos\overset{4}{8}x + B\sin\overset{4}{8}x)$

$2\,|\,256$
$2\,|\,128$
64

HW.

2. $3y'' \mp 14y' - 5y = 0$. $\quad 3\alpha^2 \mp 14\alpha - 5 = 0$.

$\alpha = \frac{+14 \pm \sqrt{196+60}}{6} = \frac{14 \mp \sqrt{256}}{6} = \frac{14 \pm 16}{6} = -\frac{1}{3}, 5$.

$y = c_1 e^{-x/3} + c_2 e^{5x}$

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Non-homogenous equation

$$y'' + ay' + by = r(x), \quad r(x) \neq 0.$$

Part I :- Consider the homogenous part.

$$y'' + ay' + by = 0$$

Suppose y_h is the solution of homogenous part.

Part II :- Find any particular solution, say y_p

Final solution is $y = y_h + y_p$.

4.8.2018.

Methods to find y_p.

1. Method of undetermined coefficients.
2. Method of variation of parameters.

1. Method of undetermined coefficients

$$y'' + ay' + by = (\text{exponential})(\text{polynomial})\underset{(\sin/\cos)}{(\text{trigonometric})}$$

$$y'' - 2y' - 3y = 2e^{4x}$$

$$y'' - 2y - 3y = 0.$$

$$\lambda^2 - 2\lambda - 3 = 0$$

$$\lambda = \frac{2 \pm \sqrt{4+12}}{2} = \frac{2 \pm 4}{2} = 3, -1$$

$$y_h = c_1 e^{3x} + c_2 e^{-x}.$$

$y_p = Ae^{4x}$ could be a solution.

$$y_p' = 4Ae^{4x}$$

$$y_p'' = 16Ae^{4x}$$

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$16A - 2(4A) - 3(A) = 2$

$16A - 8A - 3A = 2 \Rightarrow 5A = 2 \Rightarrow A = \frac{2}{5}$

$y_p = \frac{2}{5} e^{4x}$

$y = y_h + y_p = c_1 e^{3x} + c_2 e^{-x} + \frac{2}{5} e^{4x}$

$r(x)$	
$e^{\alpha x}$	$A e^{\alpha x}$
$x^2 + x + 1$	$Ax^2 + Bx + C$
5	A
$\sin 3x$	$A \sin 3x + B \cos 3x$
$e^{\alpha x} \sin \beta x$	$e^{\alpha x}(A \sin \beta x + B \cos \beta x)$
$(x^2+1) \cos 5x$	$(Ax^2 + Bx + C)(\cos 5x + \sin 5x)$
$(x^2 + 5x + 1) e^{4x}$	$(Ax^2 + Bx + C) e^{4x}$

2. $y'' - 2y' - 3y = 2e^{3x}$

$y_h = c_1 e^{3x} + c_2 e^{-x}$

$y_p = A e^{3x}$

$y_p' = 3A e^{3x}$

$y_p'' = 9A e^{3x}$

$9A - 2(3A) - 3(A) = 2$

$9A - 6A - 3A = 2 \Rightarrow 0 = 2 \Rightarrow$ not possible

So take $y_p = Axe^{3x}$

$y_p' = 3Axe^{3x} + Ae^{3x}$

$y_p'' = 3Ae^{3x} + 9Axe^{3x} + 3Ae^{3x}$

$= 6Ae^{3x} + 9Axe^{3x}$

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$6Ae^{3x} + 9Axe^{3x} - 2(3Axe^{3x} + Ae^{3x}) - 3Axe^{3x} = 2e^{3x}$

$6A - 2A = 2 \rightarrow 4A = 2 \rightarrow A = \underline{\underline{1/2}}$.

$9A - 6A - 3A = 0$.

$y_p = \frac{1}{2}xe^{3x}$.

$y = y_h + y_p = \underline{c_1e^{3x} + c_2e^{-x}} + \frac{1}{2}xe^{3x}$

3. $y'' - 3y' + 2y = x^2e^x$.

4. $y'' - 2y' - 3y = 2e^x - 10\sin x$.

5. $y'' - 3y' + 2y = 2x^2 + e^x + 2xe^x + 4e^{3x}$

~~6. $y'' - 2y' - 3y = 2e^x - 10\sin x$.~~

7. $y'' + 2y' + 5y = 6\sin 2x + 7\cos 2x$.

8. $y'' - 3y' + 2y = \frac{\sin x}{x} \Rightarrow$ cannot apply method 1.

3. $\lambda^2 - 3\lambda + 2 = 0$.

$\frac{3 \pm \sqrt{9-8}}{2} = \frac{3 \pm 1}{2} = \underline{2, 1}$.

$y_h = \underline{c_1e^{2x} + c_2e^{x}}$

$y_p = (Ax^2 + Bx + C)e^x$

$y_p' = e^x(Ax^2 + Bx + C) + e^x(2Ax + B)$

$y_p'' = e^x(Ax^2 + Bx + C) + e^x(2Ax + B) + e^x(2Ax + B) + e^x(2A)$.

$= e^x(Ax^2 + (4A + B)x + (2A + 2B + C))$

$y_p' = e^x(Ax^2 + (2A+B)x + B+C)$

$e^x(Ax^2 + (4A+B)x + (2A+2B+C)) -$

$3(Ax^2 + (2A+B)x + (B+C))e^x$

$+ 2e^x(Ax^2 + Bx + C)$

$= A - 3A + 2A$

$\lambda^2 - 2\lambda - 3 = 0$

$\lambda = \frac{2 \pm \sqrt{4+12}}{2} = \frac{2 \pm 4}{2} = 3, -1$

$y_h = C_1e^{3x} + C_2e^{-x}$

$y_P = Ae^x - B\sin x - C\cos x$

$y_P' = Ae^x - B\cos x + C\sin x$

$y_P'' = Ae^x + B\sin x + C\cos x$

$A - 2A - 3A = 2$

$-4A = 2 \Rightarrow A = -1/2$

$-B - 2C - 3B = -10$

$-2C - 4B = -10 \Rightarrow -2C - 8C = -10$

$\Rightarrow C = 1$

$-C + 2B - 3C = 0$

$2B = 4C$ $\quad$ $B = 2$

5. $y'' - 3y' + 2y = 2x^2 + e^x + 2xe^x + 4e^{3x}$

$\lambda^2 - 3\lambda + 2 = 0.$

$\lambda = \frac{3 \pm \sqrt{9-8}}{2} = \frac{3 \pm 1}{2} = 2, -1.$

$y_h = c_1 e^{2x} + c_1 e^{-x}$

$y_p = (Ax^2 + Bx + C) + e^x(Dx + E) + Fe^{3x}.$

$y_p' = (2Ax + B) + e^x(Dx+E) + e^x(D) + 3Fe^{3x}$

$y_p'' = 2A + e^x(Dx+E) + e^x(D) + De^x + 9Fe^{3x}$

~~$E + D + D - 3(E+D) + 2(E) = 1.$~~

~~$E + 2D - 3E - 3D + 2E = 1$~~

~~$D = 1 \Rightarrow D = 1$~~

$A = 2.$

~~Equations~~ $D + D + E - 3(E+D) + 2(E)$

$\Rightarrow D = -1$

$F + 3F + 9F = 4$

$13F = 4$

$F = 4/13$

$D - 3D + 2D = 2.$

7.

3.

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7. $y'' + 2y' + 5y = 6\sin 2x + 7\cos 2x$

$\lambda^2 + 2\lambda + 5$ ———— AE.

$\lambda = \frac{-2 \pm \sqrt{4-20}}{2} = \frac{-2 \pm i4}{2}$

$\alpha = -1 \qquad \beta = 2.$

$y_h = e^{-x}(A\cos 2x + B\sin 2x)$.

$y_p = A\sin 2x + B\cos 2x.$

$y_p' = 2A\cos 2x - 2B\sin 2x.$

$y_p'' = -4A\sin 2x - 4B\cos 2x.$

$-4A + 2(-2B) + 5A = 6.$

$\Rightarrow \underline{A - 4B = 6}$

$-4B + 2(2A) + 5B = 7.$

$\underline{B + 4A} = 7$

$B + (6+4B)4 = 7.$

$B + 24 + 16B = 7$

$17B = 7 - 24 = -17$

$\Rightarrow \underline{B = -1}. \qquad \underline{A = 2}.$

3.

3. $y'' - 3y' + 2y = x e^x$.

$y_p = (Ax^2 + Bx + C)e^x = \underline{Ax^2 e^x} + Bxe^x + Ce^x$

$y_p' = \underline{Ax^2 e^x} + 2Axe^x + Be^x + Bxe^x + Ce^x$.

$y_p'' = 2Axe^x + \underline{Ax^2 e^x} + 2Ae^x + 2Axe^x + Be^x + Bxe^x + Be^x$

$A - 3A + 2A = 1$.

$y_p = Ax^3 e^x + \underline{Bx^2 e^x} + \underline{Cxe^x}$.

$y_p' = \underline{3Ax^2 e^x} + Ax^3 e^x + \underline{2Bxe^x} + \underline{Bx^2 e^x} + \underline{Ce^x} + \underline{Cxe^x}$.

$y_p'' = \underline{6Axe^x} + \underline{3Ax^2 e^x} + \underline{3Ax^2 e^x} + Ax^3 e^x$
$+ \underline{2Be^x} + \underline{2Bxe^x} + \underline{2Bxe^x} + \underline{Bx^2 e^x} + \underline{Ce^x} + \underline{Ce^x} + Cx$

B $(6A + B) - 3(3A + B) + 2(B) = 1$.

$6A - 9A + \cancel{B} - \cancel{3B} + \cancel{2B} = 1$.

$A = \underline{-1/3}$.

$4B + C + 6A - 3(2B + C) + 2(C) = 0$.

$2B + 2C - 3C = 0$.

B $\underline{2B = C}$.

$4B + 2B - 2 - 3(2B + 2B) + 2(2B) = 0$.

$\cancel{6A} - 2 - 6B - 6B + 4B = 0$.

5. $y'' - 3y' + 2y = 2x^2 + e^x + 2xe^x + 4e^{3x}$

$y_P = Ax^2 + Bx + C + xe^x(Dx + E) + Fe^{3x}$

$y_P' = 2Ax + B + e^x(Dx+E) + Dxe^x + 3Fe^{3x}$

$y_P'' = 2A + e^x(Dx+E) + De^x + De^x + Dxe^x + 9Fe^{3x}$

$9F - 9F + 2F = 4$ $\qquad A = 1$

$F = 2$

$2D + E - 3E + 2 = 1$

$2D - 2E = -1$

$D + D - 3D - 3D + 2E = 2$

$-4D + 2E = 2$

$-4D + 2D + 1 = 2$

$-2D = -1 \Rightarrow D = 1/2$

$E = 1$

$2A - 3B + 2C = 0$

$2 = 3B - 2C$ $\qquad 9 - 2 = 7 = 2C$

$-6A + 2B = 0$ $\qquad C = 7/2$

5.1.2018 $\qquad B = 3$

Variation of parameters

Let $y'' + ay' + by = r(x)$

Wronskian $W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1y_2' - y_2y_1' \neq 0$

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Let $y_p = u(x)y_1 + v(x)y_2$ be a particular solution

$y_p' = u(x)y_1' + v(x)y_2' + (u'(x)y_1 + v'(x)y_2)$

Choose $u(x)$ and $v(x)$ in such a way that

$u'(x)y_1 + v'(x)y_2 = 0.$

$\Rightarrow \quad y_p' = u(x)y_1' + v(x)y_2'$

$y_p'' = u(x)y_1'' + v(x)y_2'' + (u'(x)y_1' + v'(x)y_2')$

$(u(x)y_1'' + v(x)y_2'' + u'(x)y_1' + v'(x)y_2') + a(u(x)y_1' + v(x)y_2')$

$+ b(u(x)y_1 + v(x)y_2) = r(x).$

$u(x)(y_1'' + ay_1' + y_1) + v(x)(y_2'' + ay_2' + by_2)$

$+ u'(x)y_1' + v'(x)y_2' = r(x).$

$\Rightarrow \quad u'(x)y_1' + v'(x)y_2' = r(x).$

$u'(x) = \dfrac{-v'(x)y_2}{y_1}$

$-v'(x)\dfrac{y_2y_1'}{y_1} + v'(x)y_2' = r(x).$

$v'(x)\left(\dfrac{y_2y_1' + y_2'y_1}{y_1}\right) = r(x)$

$v'(x) = \dfrac{r(x)y_1}{y_2y_1' + y_2'y_1} = \dfrac{r(x)y_1}{W(y_1,y_2)}$

$u'(x) = \dfrac{-r(x)y_2}{y_2y_1' + y_2'y_1} = -\dfrac{r(x)y_2}{W(y_1,y_2)}$

$$u(x) = -\int \frac{r(x)\, y_2}{W(y_1, y_2)}\, dx$$

$$v(x) = \int \frac{r(x)\, y_1}{W(y_1, y_2)}\, dx.$$

$$\boxed{y_P = \left(-\int \frac{r(x) y_2}{W(y_1,y_2)}\, dx\right) y_1 + \left(\int \frac{r(x) y_1}{W(y_1,y_2)}\, dx\right) y_2}$$

1. Solve $y'' + y = \tan x$.

$\lambda^2 + \cancel{\lambda}1 = 0$.

$\lambda = \cancel{0}, \pm i$

$\cancel{y_1 = e^0 = 1 \qquad y_2 = e^{-x}}$

$\cancel{r(x) = \tan x.}$

$\cancel{W(y_1\, y_2) = \begin{vmatrix} 1 & e^{-x} \\ 0 & -e^{-x} \end{vmatrix} = -e^{-x}}$

$y_h = A\cos x + B\sin x$.

$y_1 = \cos x \qquad y_2 = \sin x$

$W(y_1, y_2) = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1.$

$r(x) = \tan x$.

$$y_P = \left(-\int \tan x \sin x\, dx\right) \cos x + \left(\int \tan x \cos x\, dx\right) \sin x$$

$$= \cos x \left(-\int \frac{\sin^2 x}{\cos x} dx\right) + -\cos x \sin x$$

$=$

$$-\int \frac{\sin^2 x \, dx}{\cos x} = -\int \frac{1-\cos^2 x}{2\cos x}\, dx$$

$$= -\frac{1}{2}\int \sec x + \frac{1}{2}\int \frac{\cos 2x}{\cos x}\, dx.$$

$$\int \frac{\tan^2 x}{\sec x}\, dx = -\frac{1}{2}\int \sec x\, dx + \frac{1}{2}\int \cos x \, dx$$

$$\left(-\frac{1}{2}\log|\sec x + \tan x| + \frac{1}{2}\sin x\right)\cos x.$$

$$y_p = -\cos x \ln|\sec x + \tan x|.$$

$$y = y_h + y_p = A\cos x + B\sin x - \cos x \ln|\sec x + \tan x|$$

4. $y_1 = e^{3x}$

$y_2 = e^{-x}$

$y_1' = 3e^{3x}$

$y_2' = -e^{-x}.$

$$W = \begin{vmatrix} e^{3x} & e^{-x} \\ 3e^{3x} & -e^{-x} \end{vmatrix} = -e^{2x} - 3e^{2x} = -4e^{+2x}.$$

$$y_p = -\int \frac{(2e^x - 10\sin x)e^{-x}}{-4e^{+2x}} + \int \frac{(2e^x - 10\sin x)e^{3x}}{-4e^{2x}}.$$

$$\int \sin x \, e^{3x} = 3\sin x \, e^{3x} - \int \cos x \, 3e^{3x}$$

$$\int \cos x \, e^{3x} = 3\cos x \, e^{3x} + \int \sin x \, 3e^{3x}$$

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$\int \sin x e^{3x} = 3\sin x e^{3x} - 3\cos x e^{3x} - 9\int \sin x e^{3x}$.

$\int \sin x e^{3x} = \dfrac{3e^{3x}(\sin x - \cos x)}{10}$.

3. $y_1 = e^{2x}$

$y_2 = e^{-x}$

$y_1' = 2e^{2x}$

$y_2' = -e^{-x}$

$W = \begin{vmatrix} e^{2x} & e^{-x} \\ 2e^{2x} & -e^{-x} \end{vmatrix} = -e^{x} - 2e^{x} = -3e^{x}$.

$y_p = -\int \dfrac{x^2e^{x} \times e^{-x}}{-3e^{x}} dx + \int \dfrac{x^2e^{x} \times e^{2x}}{-3e^{+x}} dx$

$= \dfrac{1}{3}\int x^2 e^{-x} dx + -\dfrac{1}{3}\int x^2 e^{2x} dx$.

$\int x^2 e^{2x} = \dfrac{x^2e^{2x}}{2} - \int \dfrac{2xe^{2x}}{2} dx$

$\int xe^{2x} = \dfrac{xe^{2x}}{2} - \int e^{2x} dx = \dfrac{xe^{2x}}{2} - \dfrac{e^{2x}}{2}$.

1. $y_1 = e^{-x}\cos 2x$

$y_2 = e^{-x}\sin 2x$.

$y_1' = -e^{-x}\cos 2x - 2e^{-x}\sin 2x$

$y_2' = -e^{-x}\sin 2x + 2e^{-x}\cos 2x$.

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$$W = \begin{vmatrix} e^{-x}\cos 2x & e^{-x}\sin 2x \\ -e^{-x}\cos 2x - 2e^{-x}\sin 2x & -e^{-x}\sin 2x + 2e^{-x}\cos 2x \end{vmatrix}$$

$$= -\cancel{e^{-2x}\frac{\sin 4x}{2}} + 2e^{-2x}\cos^2 2x.$$

$$+ \cancel{e^{-2x}\frac{\sin 4x}{4}} + 2e^{-2x}\sin^2 2x$$

$$= \underline{2e^{-2x}}.$$

$$y_p = -\int \frac{6\sin 2x + 7\cos 2x}{2e^{-2x}} \left[e^{-x}\sin 2x\right] dx.$$

$$+ \int \frac{6\sin 2x + 7\cos 2x}{2e^{-2x}} \left[e^{-x}\cos 2x\right] dx.$$

8. 1.2018.

$y'' - 2y' + y = e^x \log x.$

$y'' - 2y' + y = 0.$

$\lambda^2 - 2\lambda + 1 = 0.$

$\lambda = \frac{2 \pm \sqrt{4-4}}{2} = \underline{1}.$ $\qquad y_h = c_1 e^x + c_2 x e^x.$

$y_1 = e^x.$

$y_2 = xe^x.$

$y_1' = e^x.$

$y_2' = e^x + xe^x.$

$$W = \begin{vmatrix} e^x & xe^x \\ e^x & e^x + xe^x \end{vmatrix} = e^{2x} + \cancel{xe^{2x}} - \cancel{xe^{2x}} = e^{2x}$$

$$y_P = \left(-\int \frac{e^x \log x \times xe^x}{e^{2x}}\,dx\right)e^x + \left(\int \frac{e^x \log x \, e^x}{e^{2x}}\,dx\right)xe^x$$

$$= \left(-\int x \log x\,dx\right)e^x + \left(\int \log x\,dx\right)xe^x$$

$$= -\left[\frac{x^2 \log x}{2} - \int \frac{x}{2}\,dx\right]e^x + \left[x\log x - \int dx\right]xe^x$$

$$= \left(-\frac{x^2 \log x}{2} + \frac{x^2}{4}\right)e^x + \left(x\log x - x\right)xe^x + c.$$

$$y_P = u_1(x)e^x + v(x)xe^x.$$

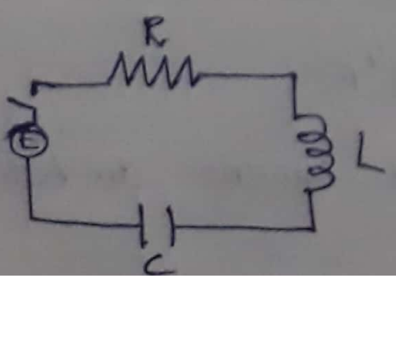
$$= -\frac{x^2e^x \log x}{2} + \frac{x^2e^x}{4} + x^2e^x \log x - x^2e^x + c.$$

$$= \frac{-3x^2e^x \log x}{4} - \frac{x^2e^x}{2} + c$$

$$y = c_1e^x + c_2xe^x + -\frac{3x^2e^x \log x}{4} - \frac{x^2e^x}{2} + c$$

Applications of Diff. eqns

I. Electrical circuits



E: Electromotive force.

R: Resistance.

L: Inductance.

C: capacitance.

At time t, charge in capacitor be $Q(t)$,

current $I = \frac{dQ}{dt}$.

Voltage drop across $R = IR$

Voltage drop across $L = L\frac{dI}{dt}$.

Voltage drop across $C = \frac{Q}{C}$.

According to Kirchoff's law;

Sum of voltage drops = supplied voltage.

$$IR + L\frac{dI}{dt} + \frac{Q}{C} = E(t).$$ Putting $I = \frac{dQ}{dt}$.

$$\boxed{L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1Q}{C} = E(t)} \longrightarrow (1).$$

At time $t=0$, Then initial conditions,

$\boxed{Q(0) = Q_0;\ Q'(0) = I(0) = I_0}$

$$I = \frac{dQ}{dt}$$

Differentiate (1) w.r.t t & substitute $\frac{dQ}{dt} = I$;

$$L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{1}{C} I = \frac{dE(t)}{dt} = E'(t)$$

1. Find the charge and current at time t in a series LCR circuit if $R = 40\Omega$, $L = 1H$, $C = 16\times10^{-4}F$. $E(t) = 100\cos 10t$. Initial charge and current are both zero

Ans:- $\frac{d^2Q}{dt^2} + 40\frac{dQ}{dt} + \frac{1}{16\times10^{-4}} Q = 100\cos 10t$

$\lambda^2 + 40\lambda + 625 = 0.$

$625 \times 4 = 2500$

$\lambda = \frac{-40 \pm \sqrt{1600-2500}}{2}$

$= \frac{-40 \pm 30i}{2} = -20 \pm 15i$

$y_h = e^{-20x}(A\cos 15x + B\sin 15x).$

$y_1 = e^{-20x}\cos 15x.$

$y_2 = e^{-20x}\sin 15x.$

$\frac{d^2Q}{dt^2} + 40\frac{dQ}{dt} + 625Q = 100\cos 10t.$

$Q_p = A\sin 10t + B\cos 10t.$

$Q'_p = 10A\cos 10t - 10B\sin 10t.$

$Q''_p = -10^2 A\sin 10t - 10^2 B\sin 10t.$

$-10^2(A\sin 10t + B\cos 10t) + 40(10A\cos 10t - 10B\sin 10t)$

$+ 625(A\sin 10t + B\cos 10t) = 100\cos 10t.$

$-100A - 400B + 625A = 0 \Rightarrow 525A - 400B = 0.$

$-100B + 400A + 625B = 100 \rightarrow 400A + 525B = 100.$

$A = \frac{\cancel{400}^{16}}{\cancel{525}_{21}} B$

$400\left(\frac{16B}{21}\right) + 525B = 100.$

$$\frac{256B}{21} + 21B = \frac{\cancel{100}\,4}{\cancel{25}}$$

$$256B + 441B = 0.4 \times 21$$

$$697B - \cancel{185B} = 4\times 21 = \underline{84}$$

$$\cancel{B = \frac{4}{185}}$$

$$A = \frac{84}{697} \qquad A = \frac{64}{691}$$

441
256
185

21×
21
21
42
441.

$A = \frac{84}{697}$ $B = \frac{64}{697}$

$Q(0) = 0.$

$$0 = C_1 + \frac{84}{691} \Rightarrow C_1 = \frac{-84}{697}.$$

$I(0) = 0.$

$$I(t) = \frac{dQ}{dt} = e^{-20t}(-15c_1 \sin 15t + 15c_2 \cos 15t)$$

$$- 20e^{-20t}(c_1 \cos 15t + c_2 \sin 15t).$$

$$+ \frac{1}{691}(-840 \sin 10t + 640 \cos 10t). = 0.$$

$$15c_2 - 20c_1 + \frac{640}{691} = 0.$$

$$15C_2 = -\frac{640}{691} + \frac{20 \times -84}{697}$$

$$= \frac{-640 - 1680}{697}.$$

$$= -\frac{2320}{697} \qquad C_2 = \frac{-464}{\cancel{208}\ 2091}$$

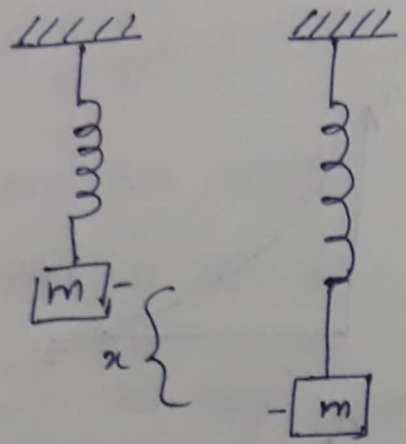
-1680
+640
1040

208
~~1040~~
15 3

9
691×
3
2091.

9.1.2018.

II. vibrating spring. Simple harmonic motion.



By Hooke's law;

Restoring force $= -kx$ where k is the Spring constant $(k>0)$;

By newton's second law,

$$m\frac{d^2x}{dt^2} = -kx$$

$$\rightarrow \boxed{m\frac{d^2x}{dt^2} + kx = 0}$$

Auxiliary eqn, :- $m\alpha^2 + k = 0$.

$\alpha = \pm i\sqrt{\frac{k}{m}}$. $\sqrt{\frac{k}{m}} = \omega$.

Soln $x(t) = c_1 \cos \omega t + c_2 \sin \omega t$.

Damped vibrations.

* Motion of spring subject to a frictional force / damping force.

Assume damping force $\propto$ velocity of mass & acts in direction opposite to the motion.

damping force $= -c\frac{dx}{dt}$

where $c>0$ is damping constant.

$$m\frac{d^2x}{dt^2} = -kx - c\frac{dx}{dt}$$

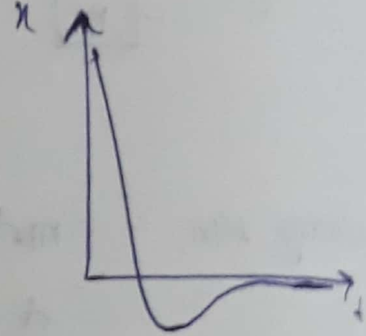
$$\boxed{m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = 0.}$$

Auxilliary equation : $m\lambda^2 + c\lambda + k = 0$.

$$\lambda = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m}$$

Case I :- $c^2 - 4mk > 0$. (over damping)

* λ_1 and λ_2 are real and distinct.
* $x = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$.
* Since $m > 0$, $k > 0$, $\sqrt{c^2 - 4mk} < c$.
* $\therefore \lambda_1$ and, $\lambda_2 < 0$.
* As $t \to \infty$, $x \to 0$.



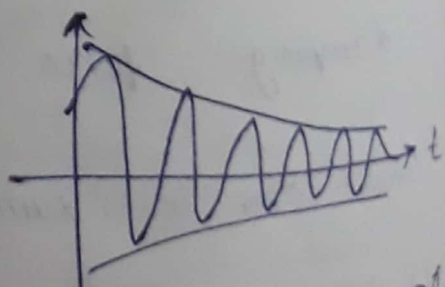
Case II : $c^2 - 4mk = 0$. (critical damping.)

* Damping force is just sufficient to suppress vibrations.
* Real and equal roots, $\lambda_1 = \lambda_2 = \frac{-c}{2m}$.
* Graph same as I
* $x = c_1 e^{\left(\frac{-c}{2m}\right)t} + c_2 t e^{-(c/2m)t}$.
* As $t \to \infty$, $x \to 0$.

Case III : $c^2 - 4mk < 0$. (under damping).

* Roots are imaginary.
* $\lambda_1 = \frac{-c}{2m} + wi$; $\lambda_2 = \frac{-c}{2m} - wi$.

where $w = \frac{\sqrt{4mk - c^2}}{2m}$.



As $t \to \infty$; $x \to 0$

* Soln $x(t) = e^{-c/2m}(c_1 \cos wt + c_2 \sin wt)$.
* Oscillations are damped by $e^{-c/2m}$.

eg:- 1. A spring with a mass of 2 kg has natural length 0.5 m. A force of 25.6 N is required to maintain it stretched to a length of 0.7 m. If the spring is stretched to a length of 0.7 m, and then released with an initial velocity 0, find the position of the mass at time t.

Given :- $m = 2kg$; $x = 0.2m$.

$x(0) = 0.2m.$

$F = -kx$

$x'(0) = \frac{dx}{dt}(0) = 0$ m/s.

$k = \frac{-25.6}{0.2} = \frac{-256}{2} = \underline{-128 \text{ N/m}}.$

$2\frac{d^2x}{dt^2} + 128x = 0. \Rightarrow \frac{d^2x}{dt^2} + 64x = 0.$

$d^2 + 64 = 0 \Rightarrow d = \pm 8i.$

$x = c_1 \cos 8t + c_2 \sin 8t.$

$0.2 = c_1$

$\frac{dx}{dt} = -8c_1 \sin 8t + 8c_2 \cos 8t.$

$0 = 8c_2 \Rightarrow c_2 = 0.$

$\therefore \underline{x(t) = 0.2 \cos 8t}.$

2. If the spring in eg.1, is immersed in a fluid with damping constant $c = 40$, find the position of the mass at time t, if it starts from equilibrium position and is given a push

to start it with initial velocity 0.6m/s.

Ans: $2\frac{d^2x}{dt^2} + 40\frac{dx}{dt} + 128x = 0.$

$$\frac{d^2x}{dt^2} + 20\frac{dx}{dt} + 64x = 0.$$

$$\lambda^2 + 20\lambda + 64 = 0.$$

$$\lambda = \frac{-20 \pm \sqrt{400-256}}{2} = \frac{-20 \pm \sqrt{144}}{2}$$

$$= \frac{-20 \pm 12}{2} = \frac{-32}{2}, \frac{-8}{2} \Rightarrow -16, -4.$$

$$x = c_1 e^{-16t} + c_2 e^{-4t}.$$

$$x(0) = 0 = c_1 + c_2.$$

$$x'(0) = -16c_1 e^{-16(0)} + -4c_2 e^{-4(0)}$$

$$0.6 = -16c_1 - 4c_2.$$

$$= 16c_2 - 4c_2 = 12c_2$$

$$c_2 = \frac{0.6}{12} = \frac{0.1}{2} = 0.05 \qquad c_1 = -c_2 = -0.05$$

$$\therefore x(t) = -0.05e^{-16t} + 0.05e^{-4t}.$$