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MODULE - 2

Gamma and Beta functions

Gamma function

$$\Gamma n = \int_0^\infty e^{-x} \cdot x^{n-1}\, dx \quad (n>0)$$

called "gamma n"

Applying integration by parts

$$\Gamma n = \left[x^{n-1}(-e^{-x})\right]_0^\infty - \int_0^\infty (n-1)\, x^{n-2} (-e^{-x}).dx.$$

$$= (n-1)\int_0^\infty e^{-x}\, x^{n-2}.dx$$

$$\boxed{\Gamma n = (n-1)\Gamma{n-1}} \rightarrow \text{for any real +ve number}$$

Case 1: If n is a positive integer,

$$\Gamma n = (n-1)(n-2)\ldots\ldots 1\,\Gamma 1$$

$$\Gamma 1 = \int_0^\infty e^{-x}.1.dx = 1.$$

$$\therefore \Gamma n = (n-1)(n-2)\ldots\ldots 1 = (n-1)!$$

$$\therefore \boxed{\Gamma n = (n-1)!} \quad \text{for any +ve integer}$$

Note:-

$$\Gamma\tfrac{1}{2} = \int_0^\infty e^{-x} x^{-1/2}.dx.$$

Put $x = u^2 \Rightarrow dx = 2u.du$

$$= \int_0^\infty e^{-u^2} u^{-1}.(2u).du = \int_0^\infty e^{-u^2}(2).du.$$

$$\Gamma\tfrac{1}{2} = 2\int_0^\infty e^{-u^2}.du \quad \text{——— (1)}$$

$$\Gamma\tfrac{1}{2} = 2\int_0^\infty e^{-v^2}.dv \quad \text{——— (2)}$$

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Multiplying ① and ②

$$\left(\sqrt{\tfrac{1}{2}}\right)^2 = 4\int_0^\infty\int_0^\infty e^{-(u^2+v^2)}\,.du.dv.$$

[This possible only in the case when the limits are constants and the variables are independent]

Let $u = r\cos\theta$, $v = r\sin\theta$

$du = -r\sin\theta.d\theta$ $dv = r\cos\theta.d\theta$

The limits change to $0 \le r \le \infty$ and $0 \le \theta \le \frac{\pi}{2}$.

$$\left(\sqrt{\tfrac{1}{2}}\right)^2 = 4\int_0^{\pi/2}\int_0^\infty e^{-r^2}\, r\,dr.\,d\theta$$

Let $r^2 = t$

$2r = \frac{dt}{dr} \Rightarrow r.dr = \frac{dt}{2}$

$$= 4\int_0^{\pi/2}\left(\int_0^\infty e^{-t}\frac{.dt}{2}\right).d\theta = 2\int_0^{\pi/2}\left[\frac{e^{-t}}{-1}\right]_0^\infty .d\theta$$

$$= 2\int_0^{\pi/2}\left[-e^{-r^2}\right]_0^\infty .d\theta = 2\int_0^{\pi/2}\left[-e^{-\infty}+e^0\right]$$

$$= 2\left(\frac{\pi}{2}\right)(1) = \pi$$

$$\left(\sqrt{\tfrac{1}{2}}\right)^2 = \pi$$

$$\therefore \boxed{\sqrt{\tfrac{1}{2}} = \sqrt{\pi}}$$

$\Rightarrow \sqrt{\tfrac{7}{2}} = \frac{5}{2}\sqrt{\tfrac{5}{2}} = \frac{5}{2}\left(\frac{3}{2}\right)\sqrt{\tfrac{3}{2}} = \frac{5}{2}\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\sqrt{\tfrac{1}{2}} = \frac{15}{8}\sqrt{\pi}$

i.e. $\int_0^\infty e^{-x}x^{5/2}.\,dx = \frac{15}{8}\sqrt{\pi}$

Evaluate

(i) $\int_0^\infty e^{-2x} . x^2 . dx$

Let $2x = t$

$2dx = dt$

$\int_0^\infty e^{-t} \frac{t^2}{4} \frac{dt}{2} = \frac{1}{8}\int_0^\infty e^{-t}.t^2.dt$

$= \frac{1}{8}\int_0^\infty e^{-t} t^{3-1}.dt$

$\Gamma 3 = (3-1)! = 2! = 2$

$\therefore \int_0^\infty e^{-2x}. x^2.dx = \frac{1}{8}(2) = \frac{1}{4}$

(ii) $\int_0^\infty (3x)^3 e^{-x}.dx$

$3(9)\int_0^\infty x^3 e^{-x}.dx = 27\int_0^\infty e^{-x}.x^{4-1}.dx$

$\Gamma n = (n-1)!$

$\Gamma 4 = 3! = 6$

$\therefore \int_0^\infty (3x)^3 e^{-x}.dx = 6(27) = 162$

(iii) $\int_0^\infty (3x)^3 e^{-3x} .dx$

$= 27\int_0^\infty x^3 e^{-3x} .dx$

Let $3x = t$

$dx = \frac{dt}{3}$

$= \int_0^\infty t^{4-1} e^{-t} .\frac{dt}{3} = \int_0^\infty t^{4-1} e^{-t}.\frac{dt}{3}$

Ans:

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$\Gamma 4 = (3!) = 6$

$\therefore \int_0^\infty (3x)^3 e^{-3x} = 3(6) = \underline{18}$

Q. Prove that $\int_0^\infty e^{-m^2x^2} \cdot dx = \frac{\sqrt{\pi}}{2m}$

Ans:

$m^2x^2 = t$

$m^2(2x)\, dx = dt$ $\qquad x = \sqrt{\frac{t}{m^2}} = \frac{t^{1/2}}{m}$

$dx = \frac{dt}{m^2(2x)}$

$$\int_0^\infty \frac{e^{-t}}{2m^2 x}(dt) = \frac{1}{2m^2}\int_0^\infty \frac{e^{-t}}{x}.dt$$

$$= \frac{1}{2m^2}\int_0^\infty e^{-t}\, t^{-1/2}\, m.dt$$

$$= \frac{1}{2m}\int_0^\infty e^{-t}.t^{-1/2}.dt$$

$$= \frac{1}{2m}\int_0^\infty e^{-t}.t^{\frac{1}{2}-1}.dt$$

$$\Gamma\frac{1}{2} = \sqrt{\pi}$$

$$\int_0^\infty e^{-m^2x^2}.dx = \frac{1}{2m}\sqrt{\pi}$$

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Q Prove that : $\beta(m,n) = \int_0^1 \frac{y^{m-1} + y^{n-1}}{(1+y)^{m+n}} \, dy$ $(m>0, n>0)$.

Proof:

$$\beta(m,n) = \int_0^\infty \frac{y^{m-1}}{(1+y)^{m+n}} \, dy$$

$$= \int_0^1 \frac{y^{m-1}}{(1+y)^{m+n}} \, dy + \int_1^\infty \frac{y^{m-1}}{(1+y)^{m+n}} \, dy$$

Put $y = \frac{1}{t}$ As $y \to 1$, $t \to 1$

$dy = -\frac{1}{t^2} \, dt$ $y \to \infty$, $t \to 0$

$$= \int_0^1 \frac{y^{m-1}}{(1+y)^{m+n}} \, dy + \int_1^0 \frac{\left(\frac{1}{t}\right)^{m-1} \cdot t^{m+n}}{(t+1)^{m+n}} \left(\frac{-1}{t^2}\right) dt$$

$$= \int_0^1 \frac{y^{m-1}}{(1+y)^{m+n}} \, dy + \int_0^1 \frac{t^{\cancel{m}+n-\cancel{m}+1-2}}{(t+1)^{m+n}} \, dt$$

$$= \int_0^1 \frac{y^{m-1}}{(1+y)^{m+n}} \, dy + \int_0^1 \frac{t^{n-1}}{(t+1)^{m+n}} \, dt$$

$$\boxed{\beta(m,n) = \int_0^1 \frac{y^{m-1} + y^{n-1}}{(1+y)^{m+n}} \, dy}$$

Q Prove that : $\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2m-1}\theta \, d\theta$.

Proof:

$$\beta(m,n) = \int_0^1 x^{m-1} (1-x)^{n-1} \, dx$$

Let $x = \sin^2\theta = \frac{1-\cos 2\theta}{2} = \frac{1}{2} - \frac{\cos 2\theta}{2}$

$dx = +\frac{1}{2}(\sin 2\theta)(2) \, d\theta$ As $x \to 0$, $\theta \to 0$

$dx = \sin 2\theta \, d\theta$ $x \to 1$ $\theta \to \pi/2$

$$\beta(m,n) = \int_0^1 \sin^{2m-2}\theta \, (\cos^{2n-2}\theta) \cdot \sin 2\theta \, d\theta$$

$$= 2\int_0^{\pi/2} \sin^{2m-2}\theta \cdot \sin\theta \cdot \cos^{2n-2}\cos\theta \cdot d\theta$$

$$\boxed{\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta}$$

Prove that: $\beta(m,n) = \cancel{\frac{\sqrt{m}\cdot\sqrt{n}}{\sqrt{m+n}}} \quad \frac{\Gamma m \, \Gamma n}{\Gamma m+n}$

Proof:

$$\Gamma m = \int_0^\infty e^{-t} t^{m-1} \cdot dt \cdot \int_0^\infty e^{-t} t^{n-1} dt$$

$t = x^2$ $\qquad$ $t = y^2$

$dt = 2x \cdot dx$ $\qquad$ $dt = 2y \cdot dy$

$$\Gamma m = \int_0^\infty e^{-x^2} \cdot x^{2m-2} (2x) \cdot dx \cdot \int_0^\infty e^{-y^2} y^{2n-2} (2y) \cdot dy$$

$$= 2\int_0^\infty \cancel{x}\, e^{-x^2} x^{2m-1} \cdot dx \cdot 2\int_0^\infty e^{-y^2} y^{2n-1} \cdot dy$$

$$= 4\int_0^\infty \int_0^\infty (e^{-x^2} \cdot x^{2m-1} \cdot e^{-y^2} \cdot y^{2n-1}) \, dx \, dy$$

$$= 4\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} \cdot x^{2m-1} \cdot y^{2n-1} \, dx \, dy$$

Let $x = r\cos\theta$, $\quad y = r\sin\theta$

$0 \le \theta \le \frac{\pi}{2}$, $\quad 0 \le r \le \infty$

$$= 4\int_0^{\pi/2} \int_0^\infty e^{-r^2} (r\cos\theta)^{2m-1} (r\sin\theta)^{2n-1} \cdot r \cdot dr \cdot d\theta$$

$$= 4\int_0^{\pi/2} \int_0^\infty e^{-r^2} \cdot r^{2(m+n)-1} \cos^{2m-1}\theta \sin^{2n-1}\theta \cdot dr \, d\theta$$

$$= 2\int_0^{\pi/2} \cos^{2m-1}\theta \, \sin^{2n-1}\theta \cdot d\theta \left(2\int_0^\infty e^{-r^2} r^{2(m+n)-1} \cdot dr \right) \quad \text{—— (1)}$$

Consider $2\int_0^\infty e^{-r^2} r^{2(m+n)-1} \cdot dr$.

Let $r^2 = u$

$2r \cdot dr = du \Rightarrow dr = \frac{du}{2\sqrt{u}}$

$$2\int_0^\infty e^{-r^2} r^{2(m+n)-1} \cdot dr = 2\int_0^\infty e^{-u} u^{m+n} \cdot u^{-1/2} \cdot \frac{du}{2\sqrt{u}}$$

$$= \int_0^\infty e^{-u} u^{(m+n)-1} \, du$$

$$= \Gamma(m+n)$$

Hence ① becomes

$$\Gamma m \, \Gamma n = \beta(m,n) \, \Gamma(m+n)$$

$$\boxed{\beta(m,n) = \frac{\Gamma m \, \Gamma n}{\Gamma(m+n)}}$$

$$2\int_0^{\pi/2} \cos^{2m-1}\theta \, \sin^{2n-1}\theta \, d\theta = \beta(m,n)$$

proved from previous question

P.T : $\beta(m,n) = \beta(m+1, n) + \beta(m, n+1)$

$$\beta(m+1,n) + \beta(m,n+1) = \frac{\Gamma(m+1)\,\Gamma n}{\Gamma(m+n+1)} + \frac{\Gamma m \, \Gamma(n+1)}{\Gamma(m+n+1)}$$

$$= \frac{m \, \Gamma m \, \Gamma n}{\Gamma(m+n+1)} + \frac{\Gamma m \; n \, \Gamma n}{\Gamma(m+n+1)}$$

$$= \frac{(m+n) \cdot (\Gamma m \, \Gamma n)}{(m+n)\,\Gamma(m+n)}$$

$$= \frac{\Gamma m \, \Gamma n}{\Gamma(m+n)} = \beta(m,n)$$

$$= \frac{(2n)(2n-1)(2n-2)(2n-3)(2n-4)\cdots}{2.2.2.\cdots(2n)(2n-2)(2n-4)\cdots} \frac{1}{2}\sqrt{\pi}$$

$$\boxed{\Gamma\left(n+\tfrac{1}{2}\right) = \frac{2n!}{2^{2n}\, n!}\sqrt{\pi}}$$

$$* \boxed{\int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta = \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)}$$

Eg: $\int_0^{\pi/2} \sin^4\theta \cos^4\theta \, d\theta = \frac{1}{2}\beta\left(\frac{5}{2}, \frac{5}{2}\right)$

$$= \frac{1}{2} \frac{\Gamma\frac{5}{2}\,\Gamma\frac{5}{2}}{\Gamma 5}$$

$$= \frac{\frac{1}{2}\left[\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\sqrt{\pi}\right]^2}{4\times3\times2\times1}$$

$$= \left(\frac{3}{4}\right)^2 \pi^2 \left(\frac{1}{2}\right)\left(\frac{1}{24}\right)$$

$$= \frac{9\times\pi^2}{4\times8\,(24)} = \underline{\underline{\frac{3\pi^2}{256}}}.$$

Eg: $\int_0^{\pi/2} \sin^p\theta \, d\theta = \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{1}{2}\right)$

$$= \frac{1}{2} \frac{\Gamma\frac{p+1}{2}\,\sqrt{\pi}}{\Gamma\frac{p}{2}+1}$$

$$= \frac{1}{2} \frac{\left(\frac{p}{2}-\frac{1}{2}\right)\left(\frac{p}{2}-\frac{3}{2}\right)\cdots\left(\frac{1}{2}\right)\sqrt{\pi}.\sqrt{\pi}}{\frac{p}{2}\left(\frac{p}{2}-1\right)\left(\frac{p}{2}-2\right)}$$

$$= \frac{1}{2} \frac{\left(\frac{p-1}{2}\right)\left(\frac{p-3}{2}\right)\cdots\left(\frac{1}{2}\right)\pi}{\left(\frac{p}{2}\right)\left(\frac{p-2}{2}\right)\left(\frac{p-4}{2}\right)\cdots}$$

$$\int_0^{\pi/2} \sin^p\theta \, d\theta = \int_0^{\pi/2} \cos^p\theta \, d\theta = \begin{cases} \frac{p-1}{p} \times \frac{p-3}{p-2} \times \cdots \frac{1}{2} \times \frac{\pi}{2} & \text{if } p \text{ is even} \\ \frac{p-1}{p} \times \frac{p-3}{p-2} \times \cdots \frac{2}{3} \times 1 & \text{if } p \text{ is odd.} \end{cases}$$

Eg:

$$\int_0^{\pi/2} \sin^7\theta . d\theta = \frac{7-1}{7} \times \frac{7-3}{7-2} \times \frac{7-5}{7-4} \times \cancel{1/4}$$

$$= \frac{\cancel{6}^{2}}{7} \times \frac{4}{5} \times \frac{2}{\cancel{3}} \times 1.$$

$$= \frac{16}{35}$$

Eg:

$$\int_0^{\pi/2} \cos^{10}\theta . d\theta = \frac{10-1}{10} \times \frac{10-3}{10-2} \times \frac{10-5}{10-4} \times \frac{10-7}{10-6} \times \frac{10-9}{10-8} \times \pi$$

$$= \frac{9}{10_2}\left(\frac{7}{8}\right)\left(\frac{\cancel{5}}{\cancel{6}_2}\right)\left(\frac{\cancel{3}}{4}\right)\left(\frac{1}{2}\right) \pi$$

$$= \frac{63}{256} \pi$$

Q. Given $\int_0^\infty \frac{x^{p-1}}{1+x} dx = \frac{\pi}{\sin \pi p}$ p is a non-integer.

Prove that $\Gamma x \, \Gamma 1-x = \frac{\pi}{\sin \pi x}$. ($x$ is a non-integer).

Proof: $\Gamma x \, \Gamma 1-x = \beta(x, 1-x) \, \Gamma 1$

$$= \int_0^\infty \frac{y^{x-1}}{(1+y)} . dy = \frac{\pi}{\sin(\pi x)}$$

Q. Prove that $\boxed{\Gamma \tfrac{1}{2}+x \; \Gamma \tfrac{1}{2}-x = \frac{\pi}{\cos \pi x}}$, x is a non-integer.

$\frac{1}{2} + x = y$ where y is a non-integer

$1 - y = \frac{1}{2} - x$

$$\Gamma y \, \Gamma 1-y = \frac{\pi}{\sin \pi y} = \frac{\pi}{\sin \pi\left(\frac{1}{2}+x\right)}$$

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$$\Gamma y\,\Gamma 1-y = \frac{\pi}{\cos(\pi x)} \Rightarrow \Gamma\tfrac{1}{2}+x\,\Gamma\tfrac{1}{2}-x = \frac{\pi}{\cos(\pi x)}$$

Q. Prove that $\Gamma\frac{1}{9}\Gamma\frac{2}{9}\Gamma\frac{3}{9}\ldots\ldots\Gamma\frac{8}{9} = \frac{16}{3}\pi^4$

1.11.S

$$= \left(\Gamma\tfrac{1}{9}\,\Gamma 1-\tfrac{1}{9}\right)\left(\Gamma\tfrac{2}{9}\,\Gamma\tfrac{7}{9}\right)\left(\Gamma\tfrac{3}{9}\,\Gamma\tfrac{6}{9}\right)\left(\Gamma\tfrac{4}{9}\,\Gamma\tfrac{5}{9}\right)$$

$$= \left(\frac{\pi}{\sin\frac{\pi}{9}}\right)\left(\frac{\pi}{\sin\frac{2\pi}{9}}\right)\left(\frac{\pi}{\sin\frac{3\pi}{9}}\right)\left(\frac{\pi}{\sin\frac{4\pi}{9}}\right)$$

$$= \frac{\pi^4}{\sin 20^\circ \sin 40^\circ \sin 60^\circ \sin 80^\circ}$$

$$= \frac{-2\pi^4}{(\cancel{\cos}\,30^\circ - \cos 10^\circ)\frac{\sqrt{3}}{2}\sin 80^\circ} = \frac{-4\pi^4}{\sqrt{3}\left(\frac{\sqrt{3}}{2} - \cos 10^\circ\right)\sin 80^\circ}$$

$$= -\frac{4}{\sqrt{3}}\,\frac{\pi^4}{\frac{\sqrt{3}}{2}\sin 80^\circ - \cos 10^\circ \sin 80^\circ}$$

$$= \frac{16}{3}\pi^4$$

Q Prove that $\int_0^1 \frac{1}{(1-x^6)^{1/6}}\,.dx = \frac{\pi}{3}$.

Ans:

$$x^3 = \sin\theta$$

$$3x^2.dx = \cos\theta.\,d\theta$$

$$dx = \frac{\cos\theta}{3\sin^{2/3}\theta}.\,d\theta$$

$$= \int_0^1 \frac{1}{(1-\sin^2\theta)^{1/6}}\,\frac{\cos\theta}{3\sin^{2/3}\theta}.\,d\theta$$

$$= \int_0^1 \frac{\cos\theta\,\cancel{\sin}}{3\cos^{1/3}\theta\,\sin^{2/3}\theta}.\,d\theta = \frac{1}{3}\int_0^{\pi/2} \cos^{2/3}\theta\,\sin^{-2/3}\theta\,.d\theta.$$

$$= \left(\frac{1}{3}\right)\left(\frac{1}{2}\right)\beta\left(\frac{5}{6},\frac{1}{6}\right) = \frac{1}{6}\,\frac{\Gamma\frac{5}{6}\,\Gamma\frac{1}{6}}{\Gamma 1} = \frac{1}{6}\,\Gamma 1-\tfrac{1}{6}\,\Gamma\tfrac{1}{6} = \frac{1}{6}\,\frac{\pi}{\sin\frac{\pi}{6}}$$

$$= \frac{1}{6}\,\frac{\pi(2)}{1} = \frac{\pi}{3}$$

Q. Evaluate $\int_0^a x^6 (a^6 - x^6)^{1/3} . dx$

As $x \to 0, \quad y \to 0$
$x \to a \quad y \to 1$

Ans: Put $x^6 = a^6 y$

$$= \int_0^a a^6 y \, (a^6 - a^6 y)^{1/3} . dx$$

$$6x^5 dx = a^6 dy \Rightarrow dx = \frac{a^6}{6x^5} . dy$$

$$dx = \frac{a^6 dy}{6 . a\, y^{5/6}} = \frac{a^5}{6y^{5/6}} dy$$

$$= \int_0^a a^6 y \, (a^6 - a^6 y)^{1/3} \frac{a^5}{6y^{5/6}} . dy$$

$$= \int_0^a \frac{a^{11}}{6} y^{1/6} a^{6/3} (1-y)^{1/3} . dy$$

$$= \int_0^a \frac{a^9}{6} y^{1/6} (1-y)^{1/3} . dy = \int_0^1 \frac{a^9}{6} y^{1-5/6} (1-y)^{1-\frac{2}{3}} . dy$$

$$= \frac{a^9}{6} \beta\left(\frac{7}{6}, \frac{4}{3}\right)$$

$$= \frac{a^9}{6} \frac{\Gamma\frac{7}{6} \Gamma\frac{4}{3}}{\Gamma\left(\frac{7}{6} + \frac{4}{3}\right)} = \frac{a^9}{6} \frac{\frac{1}{6}\Gamma\frac{1}{6} \; \frac{1}{3}\Gamma\frac{1}{3}}{\Gamma\frac{5}{2}}$$

$$= \frac{a^9}{6}\left(\frac{1}{18}\right) \frac{\Gamma\frac{1}{6} \Gamma\frac{1}{3}}{\frac{3}{2}\Gamma\frac{3}{2}} = \frac{a^9}{6} \times \frac{1}{18} \times \frac{2}{3} . \frac{\Gamma\frac{1}{6} \Gamma\frac{1}{3}}{\frac{1}{2}\sqrt{\pi}}$$

$$= \frac{a^9}{\cancel{6}_3} \times \frac{1}{\cancel{18}_9} \times \frac{\cancel{2}}{3} \times \frac{\cancel{2}}{\sqrt{\pi}} \Gamma\frac{1}{6} . \Gamma\frac{1}{3}$$

$$= \frac{a^9}{81\sqrt{\pi}} \Gamma\frac{1}{6} \Gamma\frac{1}{3}$$

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$\int_0^{\pi/2} \sqrt{\tan x}\,.dx$

$\int_0^{\pi/2} \sin^{1/2} x \cos^{-1/2} x\,.dx$ $= \frac{1}{2}\beta\left(\frac{3}{4}, \frac{1}{4}\right)$

$= \frac{1}{2}\left[\frac{\Gamma\frac{3}{4}\Gamma\frac{1}{4}}{\Gamma 1}\right] = \frac{1}{2}\left[\Gamma\left(1-\frac{1}{4}\right)\Gamma\frac{1}{4}\right]$

$= \frac{1}{2} \cdot \frac{\pi}{\sin\frac{\pi}{4}} = \frac{\pi}{\sqrt{2}}$

LAPLACE transformation

Let $f(t)$ be a function that is piece-wise continuous on every finite interval for $t \geq 0$ and satisfies $|f(t)| \leq Me^{kt}$ for bounded some M ($M>0$) and k ($k>0$). Then the Laplace transform $L\{f(t)\}$ exists for all $s > k$. $L\{f(t)\} = \int_0^\infty e^{-st} f(t).dt = F(s)$

⇒ Is t^n exponentially bounded?

$$t^n \leq e^t$$

$$|t^n| \leq e^t \quad , \text{ Let } M=1,\ k=1$$

Hence it is exponential bounded.

⇒ Is e^{t^2} expontentially bounded?

$e^{t^2} \not\leq Me^{kt}$, so it is not exponentially bounded

These two conditions are not necessary, but if they do satisfy both then they will have laplace transformation. One condition may be violated and even then laplace transformation may exist.

Q1. $L\{1\} = \int_0^\infty e^{-st} f(1).dt = \left[\frac{e^{-st}}{-s}\right]_0^\infty = -\frac{1}{s}\ [0-1] = \frac{1}{s}$

Q2. If n is a +ve integer

$$L\{t^n\} = \int_0^\infty e^{-st}\, t^n.dt$$

$$= \frac{t^n e^{-st}}{-s} - \left[-\frac{1}{s}\int e^{-st}.\, t^{n-1}(n).dt\right]$$

$$= \left[\frac{t^n e^{-st}}{-s}\right]_0^\infty + \frac{1}{s}\int_0^\infty e^{-st}.\, n\, t^{n-1}.dt.$$

$$= 0 + \frac{n}{s}\int_0^\infty e^{-st}.\, t^{n-1}.dt$$

$$= \frac{n}{s}\, L\{t^{n-1}\}$$

$$\therefore\ L\{t^n\} = \frac{n}{s}\, L\{t^{n-1}\}$$

$$= \frac{n}{s}.\frac{n-1}{s}.\dots\dots\frac{1}{s}\, L\{1\}$$

$$\boxed{L\{t^n\} = \frac{n!}{s^{n+1}}}$$

Q3. $L\{\sin at\} = ?$, $L\{\cos(at)\} = ?$

$$L\{\sin(at)\} = \int_0^\infty e^{-st}\sin at\ .dt$$

$$I = \left[\frac{\sin at\ e^{-st}}{-s}\right]_0^\infty + \frac{1}{s}\int e^{-st}\cos at\,(a).dt$$

$$= -\left[\frac{\sin at\ e^{-st}}{s}\right]_0^\infty + \frac{a}{s}\left[\frac{-\cos at\ e^{-st}}{s} - \int \frac{e^{-st}}{s}\sin at\,(a)\,dt\right]$$

$$L\{\sin(at)\} = \left[\frac{e^{-st}}{s^2+a^2}(-s\sin at - a\cos at)\right]_0^\infty$$

$$\boxed{L\{\sin(at)\} = \frac{a^2}{s^2+a^2}}$$

Formula.

$$\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)$$

$$\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)$$

$$L\{\cos at\} = \int_0^\infty e^{-st}\cos(at).dt$$

$$= \left[\frac{e^{-st}}{s^2+a^2}(-s\cos at + a\sin at)\right]_0^\infty$$

$$\boxed{L\{\cos(at)\} = \frac{s}{s^2+a^2}}$$

$L\{e^{at}\} = ?$

$$L\{e^{at}\} = \int_0^\infty e^{-st}.e^{at}.dt$$

$$= \int_0^\infty e^{(-s+a)t}.dt$$

$$= \left[\frac{e^{(-s+a)t}}{(-s+a)}\right]_0^\infty = \left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^\infty$$

$$\boxed{L\{e^{at}\} = \frac{1}{s-a}}$$

$$\boxed{L\{e^{-at}\} = \frac{1}{s+a}}$$

For two

Linearity Property.

For two functions, $f(t)$ and $g(t)$ whose Laplace transformation exists, and for constants a, b.

$$\boxed{L\{af(t) + bg(t)\} = a\,L\{f(t)\} + b.L\{g(t)\}}$$

Q Find $L\{t^2+2t+5\}$

$$L\{t^2+2t+5\} = L\{t^2\} + L\{2t\} + L\{5\}$$

$$= \int_0^\infty e^{-st}.t^2.dt + 2\int_0^\infty e^{-st}.t.dt + 5\int_0^\infty e^{-st}.dt$$

$$= \frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s}(5)$$

$$= \frac{2+2s+5s^2}{s^3}$$

$$= \frac{(5s^2+2s+2)}{s^3}.$$

Q Find $L\{\cos^2 at\}$

$$L\{\cos^2 at\} = L\left\{\frac{1}{2} + \frac{\cos 2at}{2}\right\}$$

$$= \frac{1}{2}L\{1\} + \frac{1}{2}L\{\cos 2at\}$$

$$= \frac{1}{2}\left(\frac{1}{s}\right) + \frac{1}{2}.\frac{s}{s^2+4a^2}$$

$$= \frac{1}{2s} + \frac{s}{2(s^2+4a^2)}$$

$$= \frac{(s^2+4a^2)+s^2}{2s^2(s^2+4a^2)}$$

$$= \frac{\cancel{2}(s^2+2a^2)}{\cancel{2}s^2(s^2+4a^2)} = \frac{s^2+2a^2}{s^2(s^2+4a^2)}.$$

Q $L\{\sin 2t\cos 2t\} = ?$

$$L\{\sin 2t\cos 2t\} = L\left\{\frac{1}{2}\sin 4t\right\} = \frac{1}{2}\frac{s^2}{s^2+16}$$

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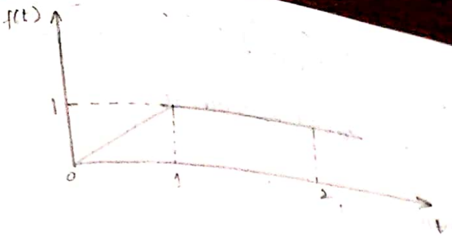
For the given graph, determine the laplace transform.



Ans:

$$f(t) = \begin{cases} t & , 0 < t < 1 \\ 1 & , 1 < t < 2 \\ 0 & , t > 2 \end{cases}$$

$$L\{f(t)\} = \int_0^1 e^{-st} \cdot t \, dt + \int_1^2 e^{-st} \, dt$$

$$= t \frac{e^{-st}}{-s} + \int \frac{e^{-st}}{s} \cdot dt + \left[\frac{e^{-st}}{-s}\right]_1^2$$

$$= \left[-\frac{t}{s} e^{-st} + \frac{1}{s}\left(-\frac{1}{s}\right) e^{-st}\right]_0^1 + \left[\frac{e^{-st}}{-s}\right]_1^2$$

$$= \left[-\frac{t}{s} e^{-st} - \frac{1}{s^2} e^{-st}\right]_0^1 - \frac{1}{s}\left[e^{-st}\right]_1^2$$

$$= \left(-\frac{1}{s} e^{-s} - \frac{1}{s^2} e^{-s} + 0 + \frac{1}{s^2}\right) - \frac{1}{s}\left[e^{-2s} - e^{-s}\right]$$

$$= \cancel{\frac{-e^{-s}}{s}} - \frac{e^{-s}}{s^2} + \frac{1}{s^2} - \frac{1}{s} e^{-2s} + \cancel{\frac{1}{s} e^{-s}}$$

$$= \frac{-e^{-s} + 1}{s^2} - \frac{1}{s} e^{-2s}.$$

Inverse transform

$$L^{-1}\{F(s)\} = f(t)$$

Q Find inverse transform of $L^{-1}\left\{\frac{s-10}{s^2+25}\right\} = ?$

Note:- $L\{a f(t) + b y(t)\} = a F(s) + b G(s)$

$L^{-1}\{a F(s) + b G(s)\} = a f(t) + b g(t) = a L^{-1}\{F(s)\} + b L^{-1}\{G(s)\}$

$\int_0^\infty e^{-st} \cdot dt$

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$$L^{-1}\left\{\frac{s-10}{s^2+25}\right\} = L^{-1}\left\{\frac{s}{s^2+25}\right\} - 2L^{-1}\left\{\frac{5}{s^2+25}\right\}$$

$$= \cos 5t - 2\sin 5t.$$

Find the inverse transforms of :

(i) $\dfrac{s-2}{s^2+1}$ (ii) $\dfrac{s^2+2s+4}{s^5}$ (iii) $\dfrac{s^2+5s+4}{(s+2)^3}$.

Ans: (i) $L^{-1}\left(\dfrac{s-2}{s^2+1}\right) = L^{-1}\left(\dfrac{s}{s^2+1}\right) - 2L^{-1}\left(\dfrac{1}{s^2+1}\right)$

$$= \cos t - 2\sin t$$

(ii) $L^{-1}\left(\dfrac{s^2+2s+4}{s^5}\right) = L^{-1}(s^{-3}) + 2L^{-1}(s^{-4}) + 4L^{-1}(s^{-5})$

$$= \frac{L^{-1}}{2}\left(\frac{2}{s^3}\right) + \frac{2}{6}L^{-1}\left(\frac{6}{s^4}\right) + \frac{4}{24}L^{-1}\left(\frac{24}{s^5}\right)$$

$$= \frac{t^2}{2} + \frac{1}{3}t^3 + \frac{1}{6}t^4$$

$$= \frac{1}{6}\left(3t^2 + 2t^3 + t^4\right)$$

(iii) $\dfrac{s^2+5s+4}{(s+2)^3}$

$$= \frac{s^2+4s+s+4}{(s+2)^3}$$

$$= \frac{s(s+4)+1(s+4)}{(s+2)^3}$$

$$= \frac{(s+1)(s+4)}{(s+2)^3}$$

$$\frac{A}{(s+2)} + \frac{Bs+C}{(s+2)^2} + \frac{Ds^2+Es+F}{(s+2)^3} = \frac{s^2+5s+4}{(s+2)^3}$$

$$A(s+2)^2 + (B(s)+C)(s+2) + Ds^2 + Es + F = s^2+5s+4$$

$$A(s^2+4+4s) + (Bs+C)(s+2) + Ds^2 + Es + F$$

$$As^2 + 4A + 4As + Bs^2 + 2Bs + Cs + 2C + Ds^2 + Es + F$$

$$s^2(A+B+D) + s(4A+2B+C+E) + (4A+2C+F) =$$

$\cancel{A+B+D=1}$

$\cancel{4A+2B+C+E=5}$

$\cancel{4A+2C+F=4}.$

$$\frac{s^2+5s+4}{(s+2)^3} = \frac{(s^2+4s+4)+s}{(s+2)^3}$$

$$= \frac{(s+2)^2+s}{(s+2)^3} = \frac{1}{(s+2)} + \frac{s}{(s+2)^3}$$

$$L^{-1}\left\{\frac{1}{(s+2)} + \frac{s}{(s+2)^3}\right\} = L^{-1}\left\{\frac{1}{s+2}\right\} + L^{-1}\left\{\frac{s}{(s+2)^3}\right\}$$

$$= e^{-2t} + L^{-1}\left\{\frac{s+2-2}{(s+2)^3}\right\}$$

$$= e^{-2t} + L^{-1}\left\{\frac{1}{(s+2)^2} - \frac{2}{(s+2)^3}\right\}$$

$$= e^{-2t} + L^{-1}\left\{\frac{1}{(s+2)^2}\right\} - 2L^{-1}\left\{\frac{1}{(s+2)^3}\right\}$$

<u>Note:-</u> <u>First shifting theorem</u> (shift in frequency domain)

Given $L\{f(t)\} = F(s)$

$$L\{e^{at}f(t)\} = \int_0^\infty e^{-st}e^{at}f(t).dt$$

$$= \int_0^\infty e^{-(s-a)t}f(t).dt \qquad s-a>0$$

$$\boxed{L\{e^{at}f(t)\} = F(s-a)}$$

$$L\{e^{-at}f(t)\} = F(s+a)$$

$$L^{-1}\{F(s-a)\} = e^{+at}f(t)$$

$$= e^{+at}L^{-1}\{F(s)\}$$

$$L^{-1}\{F(s+a) = e^{-at}L^{-1}\{F(s)\}$$

Continuing prev. question,

$$= e^{-2t} + e^{-2t}L^{-1}\left\{\frac{1}{s^2}\right\} - 2e^{-2t}L^{-1}\left\{\frac{1}{s^3}\right\}$$

$$= e^{-2t} + e^{-2t}t. - 2e^{-2t}.\frac{t^2}{2!}$$

$$= e^{-2t} + e$$

Q $L^{-1}\left\{\frac{s^4+6s-18}{s^5-3s^4}\right\}$

Ans: $\frac{s^4+\cancel{18}(s-3)}{s^4(s-3)} = \frac{\cancel{s^4}}{\cancel{s^4}(s-3)} + \frac{6}{s^4}$

$L^{-1}\left\{\frac{1}{s-3}\right\} + L^{-1}\left\{\frac{6}{s^4}\right\} = e^{3t} + t^3$

Q

$L\{\sinh(at)\}$

$= L\left\{\frac{e^{at}-e^{-at}}{2}\right\} = \frac{1}{2}L\{e^{at}\} - \frac{1}{2}L\{e^{-at}\}$

$= \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{s+a}\right]$

$= \frac{1}{2}\left[\frac{\cancel{s}+a-\cancel{s}+a}{s^2-a^2}\right]$

$= \frac{1}{2}\left[\frac{2a}{s^2-a^2}\right] = \frac{a}{s^2-a^2}.$

$$\therefore \boxed{\begin{aligned} L\{\sinh(at)\} &= \frac{a}{s^2-a^2} \\ L\{\cosh(at)\} &= \frac{s}{s^2-a^2} \end{aligned}}$$

Q Find, $L^{-1}\left\{\frac{2s^3}{s^4-1}\right\}$

$\frac{2s^3}{(s^2-1)(s^2+1)} = \frac{As+B}{(s^2+1)} + \frac{Cs+D}{s^2-1}$

$2s^3 = (As+B)(s^2-1) + (Cs+D)(s^2+1)$

$2s^3 = As^3 + Bs^2 - As - B + Cs^3 + Cs + Ds^2 + D$

$2s^3 = (A+C)s^3 + (B+D)s^2 + s(-A+C) + D-B$

$A + C = 2$

$2A = 2$ $\qquad$ $B + D = 0$

$A = 1$ $\qquad$ $-A + C = 0$

$C = 1$ $\qquad$ $D - B = 0 \Rightarrow$ ~~$\Rightarrow$~~ $A = C.$

$B = 0$ $\qquad \Rightarrow D = B$

$D = 0$

$$L^{-1}\left\{\frac{s}{s^2+1} + \frac{1}{s^2-1}\right\}$$

$$= L^{-1}\left\{\frac{s}{s^2+1}\right\} + L^{-1}\left\{\frac{1}{s^2-1}\right\}$$

$$= \cos(t) + \cosh(t)$$

$\Rightarrow$ If the transform of $f(t)$ is $F(s)$, i.e,

$L\{f(t)\} = F(s)$, prove that $L\{f(at)\} =$

$$L\{f(at)\} = \int_0^\infty e^{-st} f(at).\,dt = \int_0^\infty e^{(\frac{u}{a})s} f(u)\,\frac{1}{a}.\,du.$$

Let $at = u.$

$t = \frac{u}{a}$

$$= \frac{1}{a}\int_0^\infty e^{(\frac{s}{a})u} f(u).\,du.$$

$$\boxed{L\{f(at)\} = \frac{1}{a}F\left(\frac{s}{a}\right)}$$

Laplace transform of derivatives.

suppose $f(t)$ is continuous for all $t > 0$ and exponentially bounded and has a derivative $f'(t)$ which is piecewise continuous on every finite interval for $t \geq 0$, then the laplace transform of $f'(t)$ exists and is equal to,

$$\boxed{L\{f'(t)\} = sL\{f(t)\} - f(0)}$$

Q $L^{-1}\{$

Ans: $\dfrac{s^4 + 3\ldots}{s^4(\ldots}$

$= -\int_0^\infty e^{-st} (-s) f(t).dt$

$= \int_0^\infty e^{-st} f(t).dt$

$= f(0) + sF(s)$

$= f(0) + s L\{f(t)\}.$ —— ①

$= \int_0^\infty e^{-st} f''(t).dt$

from ①

$L\{f''(t)\} = f'(0) + sL\{f'(t)\}$

$= s[sL\{f(t)\} - f(0)] - f'(0)$

$= s^2 L\{f(t)\} - s f(0) - f'(0)$

Therefore generally,

$$\boxed{L\{f^{n}(t)\} = s^n L\{f(t)\} - s^{n-1} f(0) - s^{n-2} f'(0) \ldots\ldots\ldots - f^{n-1}(0)}$$

Eg: $L\{\sin^2 t\}$

$f(t) = \sin^2 t \qquad , f(0) = 0$

$f'(t) = 2\sin t\cos t = \sin 2t$

$L\{f'(t)\} = sL\{f(t)\} - f(0)$

$L\{\sin 2t\} = s\, L\{\sin^2 t\}$

$L\{\sin^2 t\} = \frac{1}{s} L\{\sin 2t\} = \frac{1}{s} \frac{2}{(s^2+4)}$

(ii) $L\{t\ldots$

$f(t)$

$f'(t)$

$f''(\ldots$

(iii)

Proof:
$$L\{f'(t)\} = \int_0^\infty e^{-st} f'(t).dt$$

$$= \left[e^{-st} f(t)\right]_0^\infty - \int_0^\infty e^{-st} (-s) f(t).dt.$$

$$= f(0) + s\int_0^\infty e^{-st} f(t).dt$$

$$= f(0) + s F(s)$$

$$= f(0) + s L\{f(t)\}. \quad \text{—(1)}$$

$$L\{f''(t)\} = \int_0^\infty e^{-st} f''(t).dt$$

From (1)

$$L\{f''(t)\} = f'(0) + sL\{f'(t)\}$$

$$= s[sL\{f(t)\} - f(0)] - f'(0)$$

$$= s^2 L\{f(t)\} - sf(0) - f'(0)$$

Therefore generally,

$$\boxed{L\{f^{n}(t)\} = s^n L\{f(t)\} - s^{n-1}f(0) - s^{n-2}f'(0) \cdots\cdots + f^{n-1}(0)}$$

Eg: $L\{\sin^2 t\}$

$f(t) = \sin^2 t$, $f(0) = 0$

$f'(t) = 2\sin t\cos t = \sin 2t$

$L\{f'(t)\} = sL\{f(t)\} - f(0)$

$L\{\sin 2t\} = s\, L\{\sin^2 t\}$

$$L\{\sin^2 t\} = \frac{1}{s} L\{\sin 2t\} = \frac{1}{s}\frac{2}{(s^2+4)}$$

(ii) $L\{t\sin wt\}$

$f(t) = t\sin wt$

$f'(t) = t\cos wt \cdot w + \sin wt$, $f(0) = 0$

$f''(t) = -t w^2 \sin wt + w\cos wt + \cos wt \cdot w$, $f'(0) = 0$

$= -t w^2 \sin wt + 2w\cos wt.$ $\rightarrow f''(0) = 2w$

$L\{f''(t)\} = s^2 L\{f(t)\} - s f(0) - f'(0)$

$= s^2 L\{t\sin wt\} - s(0) - 0$

$= s^2 L\{t\sin wt\}$

$L\{t\sin wt\} = -\frac{w^2}{s^2} L\{+t\sin wt\} + \frac{1}{s^2}(2w) L\{\cos wt\}$

$L\{t\sin wt\}\left(1 + \frac{w^2}{s^2}\right) = \frac{2w}{s^2} L\{\cos wt\}$

$L\{\sin wt\} = \frac{2w}{s^2+w^2} \cdot \frac{s}{s^2+w^2}$

$$\boxed{L\{\sin wt\} = \frac{2ws}{(s^2+w^2)^2}}$$

(iii) $L\{t\cos wt\}$

$f(t) = t\cos wt$, $f(0) = 1.$

$f'(t) = -t\sin wt \cdot w + \cos wt$, $f'(0) = 1.$

$f''(t) = -\sin wt \cdot w + w^2 t \cos wt - w\sin wt.$

$= -w^2 t\cos wt - 2w\sin wt$

$L\{f''(t)\} = s^2 L\{f(t)\} - s f(0) - f'(0) = s^2 L\{f(t)\}$

$-w^2 L\{t\cos wt\} - 2w\, L\{\sin wt\} = s^2 L\{t\cos wt\}$

$-2w\, L\{\sin wt\} = (s^2+w^2)\, L\{t\cos wt\}$

$$\boxed{L\{t\cos wt\} = \frac{-2w}{(s^2+w^2)^2}}$$

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Q Solve:-

(i) $y' + 3y = 10\sin t$, $y(0) = 0$

Let $y = f(t)$

$L\{y' + 3y\} = L\{10\sin t\}$

$L\{y'\} + 3L\{y\} = \frac{10}{s^2+1}$.

$sL\{y\} - y(0) + 3L\{y\} = \frac{10}{s^2+1}$

$(s+3)\, L\{y\} = \frac{10}{s^2+1}$.

$L\{y\} = \frac{10}{(s+3)(s^2+1)}$

$y = L^{-1}\left\{\frac{10}{(s+3)(s^2+1)}\right\}$

$\frac{10}{(s+3)(s^2+1)} = \frac{As+B}{s^2+1} + \frac{C}{s+3}$

$10 = (As+B)(s+3) + C(s^2+1)$

$10 = As^2 + 3As + Bs + 3B + Cs^2 + C$

$10 = s^2(A + C) + (3A + B)s + (3B + C)$

$A + C = 0$ $\quad\quad$ $3A + B = 0$

$3B + C = 10$

$C = 10 - 3B$

$A - 3B = -10$

$9A + 3B = 0$

$10A = -10 \Rightarrow \boxed{A = -1}$

$\boxed{B = 3}$

$\boxed{C = 1}$

Q (ii)

$$y = L^{-1}\left\{ \frac{-s+3}{s^2+1} + \frac{1}{s+3} \right\}$$

$$y = L^{-1}\left\{ \frac{-s+3}{s^2+1} \right\} + L^{-1}\left\{ \frac{1}{s+3} \right\}$$

$$= L^{-1}\left\{ \frac{-s}{s^2+1} \right\} + 3\left\{ \frac{1}{s^2+1} \right\} + L^{-1}\left\{ \frac{1}{s+3} \right\}$$

$$y = -\cos t + 3\sin t + e^{-3t}$$

Q.

(ii) $y'' + y = 2\cos t$, $y(0) = 3$, $y'(0) = 4$

Let $y = f(t)$.

$$L\{y'' + y\} = L\{2\cos t\}$$

$$L\{y''\} + L\{y\} = L\{2\cos t\}$$

$$s^2 L\{y\} - s f(0) - f'(0) + L\{y\} = L\{2\cos t\}$$

$$(s^2+1) L\{y\} - s(3) - 4 = 2L\{\cos t\}$$

$$L\{y\} = \left(\frac{2s}{s^2+1} + 3s + 4 \right) \frac{1}{s^2+1}$$

$$L\{y\} = \left(\frac{2s + (3s+4)(s^2+1)}{(s^2+1)^2} \right)$$

$$= \frac{2s + 3s^3 + 4s^2 + 3s + 4}{(s^2+1)^2}$$

$$L\{y\} = \frac{3s^3 + 4s^2 + 5s + 4}{(s^2+1)^2}$$

$$\cancel{L\{y\} =}$$

$$y = L^{-1}\left\{ \frac{2s}{(s^2+1)^2} + \frac{3s}{s^2+1} + \frac{4}{s^2+1} \right\}$$

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$$y = L^{-1}\left\{ \frac{2s}{(s^2+1)^2} \right\} + 3\cos t + 4\sin t$$

$$\cancel{\frac{2s}{(s^2+1)^2} = \frac{As+B}{(s^2+1)^2} + \frac{C}{s^2+1}}$$

$$y = t\sin t + 3\cos t + 4\sin t$$

H.W

$$y'' + 2y' - 3y = 6e^{-2t}, \quad y(0) = 3, \; y'(0) = 4.$$

$\left(\text{ans: } y = -2e^{-2t} + \frac{11}{2}e^{-3t} - \frac{3}{2}e^{t} \right)$

$$L\{y'' + 2y' - 3y\} = L\{6e^{-2t}\}$$

$$L\{y''\} + 2L\{y'\} - 3\{y\} = 6L\{e^{-2t}\}$$

$$s^2 L\{y\} - sy(0) - y'(0) + 2sL\{y\} - y(0) - 3L\{y\} = \frac{6}{s+2}$$

$$(s^2 + 2s - 3)L\{y\} - 3s - 4 - 3 = \frac{6}{s+2}$$

$$(s^2 + 2s - 3)L\{y\} - 3s - 7 = \frac{6}{s+2}$$

$$(s^2 + 2s - 3)L\{y\} = \frac{6}{s+2} + 3s + 7$$

$$L\{y\} = \frac{6}{(s+2)(s^2+2s-3)} + \frac{3s}{(s^2+2s-3)} + \frac{7}{s^2+2s-3}$$

$s^2 + 3s - s - 3$ $\quad s(s+3) - 1(s+3)$

$$\frac{6}{(s+2)(s+3)(s-1)} + \frac{3s}{(s+3)(s-1)} + \frac{7}{(s+3)(s-1)}$$

$$6 = A(s+3)(s-1) + B(s+2)(s-1) + C(s+2)(s+3)$$

$$= A(s^2+2s-3) + B(s^2+s-2) + C(s^2+5s+6)$$

$$= s^2(A+B+C) + s(2A+B+5C) + (-3A-2B+6C)$$

$A+B+C = 0, \quad 2A+B+5C = 0$

$-3A - 2B + 6C = 1$

$$\begin{aligned} A+B+C &= 0 \\ -2A-B-5C &= 0 \\ \hline -A \quad -4C &= 0 \end{aligned}$$

$A + 4C = 0$

$A + 16C = 1$

$$\begin{aligned} -A + 4C &= 0 \\ A + 16C &= 1 \\ \hline 20C &= 1 \end{aligned}$$

$C = 0$ $\quad C = \frac{1}{12}$

$A = 1 - 16$

$$\begin{aligned} 3A + 4B + 20C &= 0 \\ -6A - 4B + 12C &= 2 \\ \hline 2A + 32C &= 2 \end{aligned}$$

$2A = -30$

$A = -15$

$A = -\frac{1}{3}$

$-\frac{1}{3} + B + \frac{1}{12} = 0$

$-\frac{4+1}{12} + B = 0$

$B = \frac{3}{12}$

$B = \frac{1}{4}$

Q solve: $y'' + y = 2t$, $y(\pi/4) = \pi/2$, $y'(\pi/4) = 2-\sqrt{2}$

Let $\tilde{t} = t - \frac{\pi}{4}$

$$\tilde{y}'' + \tilde{y} = 2(\tilde{t} + \pi/4)$$

$y(t) \to \tilde{y}(t - \frac{\pi}{4})$
$y' \to \tilde{y}'$
$y'' \to \tilde{y}''$

$\tilde{y}'(0) = y'(\pi/4)$
$\tilde{y}(0) = y(\pi/4)$

$$L\{\tilde{y}'' + \tilde{y}\} = 2\, L\{\tilde{t} + \tfrac{\pi}{4}\}.$$

$$L\{\tilde{y}''\} + L\{\tilde{y}\} = 2\, L\{\tilde{t} + \tfrac{\pi}{4}\}$$

$$s^2 L\{\tilde{y}\} + s\,\tilde{y}(0) + \tilde{y}'(0) + L\{\tilde{y}\} = 2\, L\{\tilde{t} + \tfrac{\pi}{4}\}$$

$$(s^2+1)\, L\{\tilde{y}\} + s\, y(\pi/4) + y'(\tfrac{\pi}{4}) = 2\, L\{\tilde{t} + \tfrac{\pi}{4}\}$$

$$(s^2+1)\, L\{\tilde{y}\} + s(\tfrac{\pi}{2}) + (2-\sqrt{2}) = 2\, L\{\tilde{t} + \tfrac{\pi}{4}\}$$

$$(s^2+1)\, L\{\tilde{y}\} = 2\, L\{\tilde{t} + \tfrac{\pi}{4}\} - \tfrac{\pi}{2} s - 2 + \sqrt{2}$$

$$L\{\tilde{y}\} = \underbrace{2\, L\{\tilde{t} + \tfrac{\pi}{4}\}}_{\frac{2}{s^2} + \frac{\pi/2}{s}} - \frac{\pi}{2}\frac{s}{(s^2+1)} + \frac{(2+\sqrt{2})}{(s^2+1)}$$

$$L\{\tilde{y}\} = \frac{2}{s^2(s^2+1)} + \frac{\pi/2}{s(s^2+1)} + \frac{\pi/2\, s}{s^2+1} + \frac{-2+\sqrt{2}}{s^2+1}$$

$$\tilde{y} = L^{-1}\left\{\frac{2}{s^2(s^2+1)}\right\} + \frac{\pi}{2} L^{-1}\left\{\frac{1}{s(s^2+1)}\right\} + \frac{\pi}{2} L^{-1}\left\{\frac{s}{s^2+1}\right\}$$
$$(-2+\sqrt{2}) L^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$$\tilde{y} = 2L^{-1}\left\{\frac{1}{s^2} + \frac{-1}{s^2+1}\right\} + \frac{\pi}{2} L^{-1}\left\{\frac{1}{s} - \frac{s}{s^2+1}\right\}$$
$$+ \frac{\pi}{2}\frac{s}{s^2+1} + (2-\sqrt{2})\frac{-1}{s^2+1}$$

$$\frac{1}{s^2(s^2+1)} = \frac{A}{s^2} + \frac{B}{s^2+1}$$

$$1 = As^2 + A + Bs^2$$

$$1 = (A+B)s^2 + A$$

$$A = 1, \quad B = -1$$

$$\tilde{y} = 2\tilde{t} - 2\sin\tilde{t} + \frac{\pi}{2}(1 - \cos\tilde{t})$$
$$+ \frac{\pi}{2}\cos\tilde{t} + (2-\sqrt{2})\sin\tilde{t}$$
$$= 2\tilde{t} + \frac{\pi}{2} - \sqrt{2}\sin\tilde{t}$$
$$y = 2(t - \pi/4) + \frac{\pi}{2} - \sqrt{2}\sin(t - \pi/4)$$

$$y = 2t - \sqrt{2}\sin(t - \pi/4)$$

Suppose $f(t)$ is continuous and of exponential order and if $L\{f(t)\} = F(s)$ then

$$\boxed{L\left\{\int_0^t f(t).dt\right\} = \frac{1}{s}F(s)}$$

$$L^{-1}\left\{\frac{1}{s}F(s)\right\} = \int_0^t f(t).dt = \int_0^t L^{-1}\{F(s)\}.dt.$$

Q $L^{-1}\left\{\frac{1}{s(s^2+w^2)}\right\} = ?$

$$L^{-1}\{F(s)\} = F^{-1}\left\{\frac{1}{s^2+w^2}\right\} = \frac{1}{w}\sin wt$$

$$L^{-1}\left\{\frac{1}{s(s^2+w^2)}\right\} = \int_0^t \frac{1}{w}\sin wt.dt$$

$$= \frac{1}{w^2}(-\cos wt)_0^t$$

$$= \frac{1}{w^2}(-\cos wt + \cos 0)$$

$$= \frac{1}{w^2}(-\cos wt + 1)$$

Q $L^{-1}\left\{\frac{1}{s^2(s-1)}\right\} = ?$

$$L^{-1}\left\{\frac{1}{s(s-1)}\right\} = \int_0^t e^t.dt = [e^t]_0^t = e^t - 1$$

$$L^{-1}\left\{\frac{1}{s[s(s-1)]}\right\} = \int_0^t (e^t - 1).dt = [e^t - t]_0^t$$

$$L^{-1}\left\{\frac{1}{s^2(s-1)}\right\} = e^t - t - 1$$

$$L^{-1}\left\{\frac{1}{s^2}\left(\frac{s-1}{s+1}\right)\right\} = ?$$

$$L^{-1}\left\{\frac{s}{s+1}\right\} - L^{-1}\left\{\frac{1}{s+1}\right\} = L^{-1}\left\{\frac{s}{s+1}\right\} - e^{-t}$$

$$L^{-1}\left\{\frac{s-1}{s+1}\right\} = L^{-1}\left\{\frac{s+1}{s+1} - \frac{1}{s+1}\right\} - e^{-t}$$

$$= L^{-1}\left\{1 - \frac{1}{s+1}\right\} - e^{-t}$$

$$= L^{-1}\{1\} - L^{-1}\left\{\frac{1}{s+1}\right\} - e^{-t}$$

$$= L^{-1}\{1\} - e^{-t} - e^{-t}$$

$$= \frac{1}{s} - 2e^{-t}$$

$$L^{-1}\left\{\frac{1}{s^2}\left(\frac{s+1-2}{s+1}\right)\right\} = L^{-1}\left\{\frac{1}{s^2}\left(1 - \frac{2}{s+1}\right)\right\}$$

$$= L^{-1}\left\{\frac{1}{s^2} - \frac{2}{s^2(s+1)}\right\}$$

$$= L^{-1}\left\{\frac{1}{s^2}\right\} - 2L^{-1}\left\{\frac{1}{(s+1)s^2}\right\}$$

$$= t - 2L^{-1}\left\{\frac{1}{(s+1)s^2}\right\}$$

$$L^{-1}\left\{\frac{1}{s(s+1)}\right\} = \int_0^t e^{-t}\,dt = \left[-e^{-t}\right]_0^t = -e^{-t} + 1$$

$$L^{-1}\left\{\frac{1}{s^2(s+1)}\right\} = \int_0^t (e^{-t}+1)\,dt = \left[e^{-t}+t\right]_0^t = e^{-t} + t - 1$$

$$L^{-1}\left\{\frac{1}{s^2}\frac{(s-1)}{(s+1)}\right\} = t - 2e^{-t} + 2t + 2$$

$$= -3t - 2e^{-t} + 2.$$

Q $L^{-1}\left\{\frac{9}{s^2}\left(\frac{s+1}{s^2+9}\right)\right\}$

$$L^{-1}\left\{\frac{s+1}{s^2+9}\right\} = L^{-1}\left\{\frac{s}{s^2+9}\right\} + L^{-1}\left\{\frac{1}{s^2+9}\right\}$$

$$= \cos 3t + \frac{\sin 3t}{3}$$

$$L^{-1}\left\{\frac{1}{s}\left(\frac{s+1}{s^2+9}\right)\right\} = \int_0^t \left(\cos 3t + \frac{\sin 3t}{3}\right).dt$$

$$= \left[\frac{\sin 3t}{3} + \frac{\cos 3t\,(-1)}{3(3)}\right]_0^t$$

$$= \frac{1}{3}\left(\sin 3t - \frac{\cos 3t}{3} - 0 + \frac{1}{3}\right)$$

$$= \frac{1}{3}\left(\sin 3t - \frac{\cos 3t}{3} + \frac{1}{3}\right)$$

$$L^{-1}\left\{\frac{1}{s^2}\left(\frac{s+1}{s^2+9}\right)\right\} = \frac{1}{3}\int_0^t \left(\sin 3t - \frac{\cos 3t}{3} + \frac{1}{3}\right).dt$$

$$= \frac{1}{3}\left[-\frac{\cos 3t}{3} - \frac{\sin 3t}{3(3)} + \frac{t}{3}\right]_0^t$$

$$= \frac{1}{3}\left[-\frac{\cos 3t}{3} - \frac{\sin 3t}{3(3)} + \frac{t}{3} + \frac{1}{3} + 0 - 0\right]$$

$$= \frac{1}{3}\left[-\frac{\cos 3t}{3} - \frac{\sin 3t}{3(3)} + \frac{t}{3} + \frac{1}{3}\right]$$

$$L^{-1}\left\{\frac{9}{s^2}\left(\frac{s+1}{s^2+9}\right)\right\} = \frac{\cancel{9}}{\cancel{9}}\left(-\cos 3t - \frac{\sin 3t}{3} + 3t + 1\right)$$

Derivative of Laplace Transform

$$L\{f(t)\} = F(s)$$

$$-F'(s) = L\{t f(s)\}$$

Proof:

derivative of transform

$$\boxed{F'(s) = \frac{d}{ds}\int_0^\infty e^{-st} f(t).dt}$$

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$$= \int_0^\infty \frac{d}{ds}\left(e^{-st} f(t).dt\right)$$ ← This step can be done because differentiation is done w.r.t to t and integration w.r.t t and both are independant variables

$$= \int_0^\infty e^{-st}(-t) f(t).dt$$

$$= -\int_0^\infty e^{-st}\left[t f(t)\right].dt = -L\{t f(t)\}$$

$$L^{-1}\{-F'(s)\} = t f(t)$$

$$= t\, L^{-1}\{F(s)\}$$

or

$$\boxed{L^{-1}\{F(s)\} = -\frac{L^{-1}\{F'(s)\}}{t}}$$

Q $L\{te^{-t}\} = ?$

$$L\{te^{-t}\} = -\frac{d}{ds} L\{e^{-t}\}$$

$$= -\frac{d}{ds}\left(\frac{1}{s+1}\right) = \frac{1}{(s+1)^2}$$

Q $L\{t \sin \beta t\} = ?$

$$L\{t \sin \beta t\} = -\frac{d}{ds} L\{\sin \beta t\}$$

$$= -\frac{d}{ds}\left(\frac{\beta}{\beta^2+s^2}\right)$$

$$= -\beta\left(\frac{(\beta^2+s^2)(0) - \cancel{\beta^2+s^2}\, 1\,(2s)}{(\beta^2+s^2)^2}\right)$$

$$= -\frac{\beta(-2s)}{(\beta^2+s^2)^2} = \frac{2\beta s}{(\beta^2+s^2)^2}$$

Q Find $L^{-1}\left\{\log_e \frac{(s^2+1)}{(s-1)^2}\right\}$

$$F(s) = \log_e \frac{(s^2+1)}{(s-1)^2}$$

$$F'(s) = \frac{(s-1)^2}{s^2+1}\left[\frac{(s-1)^2(2s) - (s^2+1)(2(s-1))}{(s-1)^4}\right]$$

$$F'(s) = \frac{\cancel{(s-1)^2}}{(s^2+1)} \frac{[2s(s-1)^2 - 2(s-1)(s^2+1)]}{(s-1)^4}$$

$$= \frac{2(s-1)}{(s^2+1)(s-1)^2}[2s(s-1) - s^2 - 1]$$

$$= \frac{2}{(s^2+1)(s-1)}[\cancel{s^2} - s - \cancel{s^2} - 1]$$

$$= \frac{-2s-2}{(s^2+1)(s-1)}$$

$$\frac{-2s-2}{(s^2+1)(s-1)} = \frac{As+B}{s^2+1} + \frac{C}{s-1}$$

$$= (As+B)(s-1) + C(s^2+1)$$

$$= As^2 - As + Bs - B + Cs^2 + C$$

$$-2s-2 \quad = s^2(A+C) + s(B-A) - B + C$$

$C = -2$

$A+C = 0 \Rightarrow A = 2$

$B-A = -2 \Rightarrow B = 0$

$-2 = B$

$B - A = -2$

$-A + C = -2$

$A + C = 0$

$2C = -2 \Rightarrow C = -1, \quad A = 1$

$B - 1 = -2$

$B = -1$

$-A + C = -2 \Rightarrow$

$A + C = 0$

$2C = -2 \Rightarrow C = -1, \quad A = 1.$

$B = -1$

$$F'(s) = \frac{s-2}{s^2+1} + \frac{-1}{s-1}.$$

$$= \frac{s}{s^2+1} - \frac{2}{s^2+1} - \frac{1}{s-1}$$

$$L^{-1}\{F'(s)\} = \cos t - 2\sin t - e^t$$

$$F'(s) = \frac{s-1}{s^2+1} + \frac{(-1)}{s-1}$$

$$F'(s) = \frac{2s}{s^2+1} + \frac{(-2)}{s-1}$$

$$= 2\cos t - 2e^t$$

$$L^{-1}\{F(s)\} = -\frac{L^{-1}\{F'(s)\}}{t} = -\frac{2\cos t}{t} - \frac{2e^t}{t}$$

$$-\frac{2}{t}\cos t + \frac{2e^t}{t}$$

Integration of Laplace Transform

$L\{f(t)\} = F(s)$

$\int_s^\infty F(s).\,ds = L\{\frac{f(t)}{t}\}$

Integral of transform

the other two results they are transform of derivative and integral

Proof: $\int_s^\infty F(s).ds = \int_s^\infty \int_0^\infty e^{-st} f(t).dt$

$= \int_0^\infty \int_s^\infty e^{-st} f(t).ds\,.dt$

$= \int_0^\infty \left(\frac{e^{-st}}{-t}\right)_s^\infty f(t).dt = \int_0^\infty e^{-st} \frac{f(t)}{t}\,dt$

$= L\{\frac{f(t)}{t}\}$

Unit step function (Heaviside's function).

$$u(t-a) = \begin{cases} 0, & t<a \\ 1, & t>a \end{cases}$$

$L\{u(t-a)\} = \int_0^\infty e^{-st} u(t-a).dt$

$= \int_a^\infty e^{-st}.dt = \left[\frac{e^{-st}}{-s}\right]_a^\infty = \frac{e^{-as}}{s}$

Second shifting theorem (shift in the time domain)

$L\{f(t)\} = F(s)$

then $\boxed{L\{f(t-a)\,u(t-a)\} = e^{-as}.F(s)}$

Proof: $L\{f(t-a)\,u(t-a)\} = \int_a^\infty e^{-st} f(t-a).dt$

$= \int_0^\infty e^{-s(u+a)} f(u).du.$

Let $t-a=u$

when $t\to a,\ u\to 0$

$t\to\infty,\ u\to\infty$

$= e^{-as}\int_0^\infty e^{-su} f(u).du$

$= e^{-as}\,F(s)$

2

$\Rightarrow A=2$

$= -2 \Rightarrow B=0$

$A=1$

$-1=-2$

$\cancel{B=-1}$

$\frac{s-1}{s^2+1} + \frac{(-1)}{s-1}.$

$\frac{2s}{s^2+1} + \frac{(-2)}{s-1}$

$2\cos t - 2e^t$

$+\frac{2e^t}{t}$

Q $L\{(t-1)\,u(t-1)\} = ?$

Ans: Using second shifting theorem, ie,

$L\{f(t-a)\,u(t-a)\} = e^{-as}F(s)$

$L\{(t-1)\,u(t-1)\} = e^{-s}L\{t\} = \frac{e^{-s}}{s^2}$.

Q $L\{t^2 u(t-1)\} = ?$ $t^2u(t-1) = \begin{cases} 0, & t<1 \\ t^2, & t>1 \end{cases}$

Ans: $L\{t^2(u(t-1)\} = \cancel{L}$

$t^2 = (t-1)^2+2(t-1)+1$.   to use second shifting theorem we require t^2 in the form of $t-1$

$L\{((t-1)^2+2(t-1)+1)u(t-1)\}$

$= L\{(t-1)^2u(t-1)+2(t-1)u(t-1)+u(t-1)\}$

$= L\{(t-1)^2u(t-1)\}+2L\{(t-1)u(t-1)\}+L\{u(t-1)\}$

$= \frac{2e^{-s}}{s^3} + \cancel{2}\frac{e^{-s}}{s^2} + \frac{e^{-s}}{s}$

?? Q Let $f(t) = \begin{cases} 2 & 0<t<\pi \\ 0 & \pi<t<2\pi \\ \sin t & t>2\pi \end{cases}$ Use shifting theorem to find laplace transform

Ans: $f(t) = 2u(t) - 2u(t-\pi) + \sin(t-2\pi)\,u(t-2\pi)$

$L\{f(t)\} = 2\{u(t)\} - 2L\{u(t-\pi)\} + L\{\sin(t-2\pi)\,u(t-2\pi)\}$

$= \frac{2}{s} - 2\frac{e^{-\pi s}}{s} + \frac{e^{-2\pi s}}{s^2+1}$

?? Q $f(t) = \begin{cases} 1 & 0<t<1 \\ t & 1<t<2 \\ t^2 & t>2 \end{cases}$

$f(t) = u(t) + (t-1)u(t-1) + [(t-2)^2+3(t-2)+2]\,u(t-2)$

$L\{f(t)\} = \cancel{u(t)} + \frac{e^{0}}{s} + \frac{e^{-2s}}{s^2} + \frac{2e^{-2s}}{s^3} + \frac{3e^{-2s}}{s^2} + \frac{2e^{-2s}}{s}$.

HW Q. $f(t) = \begin{cases} 1 & 0 < t < 1 \\ t & 1 < t < 2 \\ 4 - t & 2 < t < 3 \\ 0 & t > 3 \end{cases}$

Finding inverse transforms.

$$L^{-1}\{e^{-as}F(s)\} = f(t-a)\,u(t-a)$$

Q. $L^{-1}\left\{ \frac{2}{s^2} - \frac{2e^{-s}}{s^2} - \frac{4e^{-2s}}{s^2} \right\}$

Ans: $= 2t\,u(t) - 2(t-1)\,u(t-1) - 4(t-2)\,u(t-2).$

Q. $L^{-1}\left\{ \frac{se^{-\pi s}}{s^2+1} \right\}$

Ans:- $\cos(t-\pi)\,u(t-\pi)$

Q. $L^{-1}\left\{ \frac{(s+1)e^{-\pi s}}{s^2+s+1} \right\} = f(t-\pi)\,u(t-\pi) = e^{-\frac{1}{2}(t-\pi)} \cos\frac{\sqrt{3}}{2}(t-\pi) + \frac{1}{\sqrt{3}} e^{-\frac{1}{2}(t-\pi)} \sin\frac{\sqrt{3}}{2}(t-\pi)$

$$\frac{s+1}{s^2+s+1} = \frac{(s+\frac{1}{2}) + \frac{1}{2}}{\left[(s+\frac{1}{2})^2 + \left(\frac{\sqrt{3}}{2}\right)^2\right]}.$$

$$= \frac{s+\frac{1}{2}}{(s+\frac{1}{2})^2 + \left(\frac{\sqrt{3}}{2}\right)^2} + \frac{\frac{1}{2}}{(s+\frac{1}{2})^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

L^{-1} $f(t) = e^{-t/2} \cos\frac{\sqrt{3}}{2}t + \frac{1}{2} \times \frac{2}{\sqrt{3}} e^{-t/2} \sin\sqrt{\frac{3}{2}}\,t$

Transform of Periodic Function

Let $f(t)$ with $t > 0$ be a periodic function with period

$$f(t) = \{$$

$$f(t+T) = f(t)$$

$$f(t+2T) = f(t+T) = f(t)$$

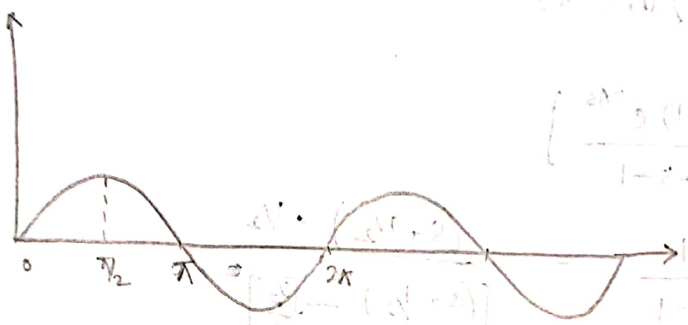
$$f(t+3T) = f(t+T) = f(t)$$

$\vdots$

$$f(t+nT) = f(t).$$

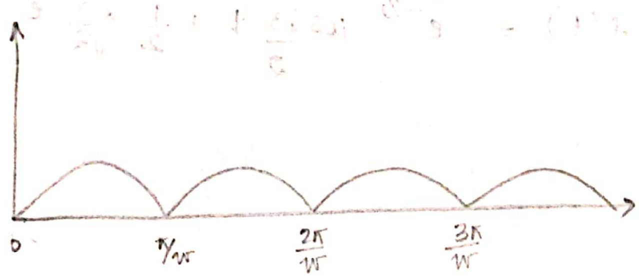
Eg: ① $f(t) = \sin t \qquad 0 < t < 2\pi$

$T = 2\pi$



② $f(t) = |\sin wt| \qquad 0 < t < \frac{\pi}{w}$

$T = \frac{\pi}{w}$



$$\sin w\left(\frac{3\pi}{2w}\right) = \sin w\left(\frac{\pi}{w} + \frac{\pi}{2w}\right) = \sin w\left(\frac{\pi}{2w}\right) = \underline{\underline{1}}$$

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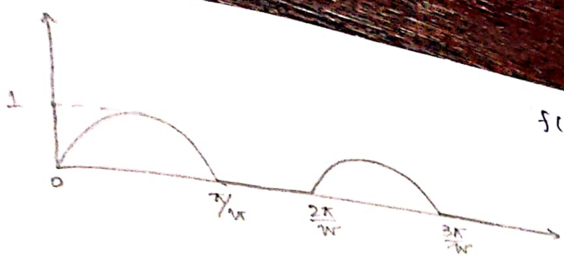
(3)



$f(t) = \begin{cases} \sin wt & 0 < t < \pi/w \\ 0 & \frac{\pi}{w} < t < \frac{2\pi}{w} \end{cases}$

$$L\{f(t)\} = \int_0^{\infty} e^{-st} f(t).dt$$

$$= \int_0^{T} e^{-st} f(t).dt + \int_T^{2T} e^{-st} f(t).dt + \int_{2T}^{3T} \cdots$$

Put $t = u + T$. Put $v + 2T = t$

$dt = du$. $dv = dt$

when $t = T$, $u = 2T$ when $t =$

when $t = 2T$, $u = T$

$$L\{f(t)\} = \int_0^{T} e^{-st} f(t).dt + \int_0^{T} e^{-s(u+T)} f(u+T).du$$

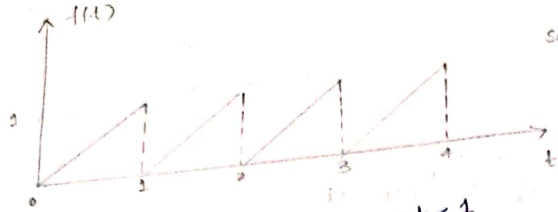
$$+ \int_0^{T} e^{-s(v+2T)} f(v+2T).dv + \cdots$$

$$= \int_0^{T} e^{-st} f(t).dt + e^{-sT}\int_0^{T} e^{-su} f(u).du + e^{-2Ts}\int_0^{T} e^{-sv} f(v).dv + \cdots$$

$$= \int_0^{T} e^{-st} f(t).dt \left[1 + e^{-sT} + e^{-2sT} + \cdots\right]$$

$$\boxed{L\{f(t)\} = \frac{1}{1-e^{-sT}} \int_0^{T} e^{-st} f(t).dt}$$ → for periodic functions

Q Find the L.T of



saw toothed wave

Ans: $T = 1$, $f(t) = t$ $\forall\ 0 < t < 1$

$$L\{f(t)\} = L\{t\} = \frac{1}{1-e^{-s}} \int_0^1 e^{-st} . t . dt$$

$$= \frac{1}{1-e^{-s}} \left\{ \left[\frac{te^{-st}}{-s} \right]_0^1 + \int_0^1 \frac{e^{-st}}{-s} . dt \right\}$$

$$= \frac{1}{1-e^{-s}} \left\{ \left[\frac{e^{-s}}{-s} \right] + \left(-\frac{1}{s}\right) \left[\frac{e^{-st}}{-s} \right]_0^1 \right\}$$

$$= \frac{1}{1-e^{-s}} \left\{ -\frac{e^{-s}}{s} + \frac{1}{s^2} \left[e^{-s} - e^{0} \right] \right\}$$

$$= \frac{1}{1-e^{-s}} \left\{ -\frac{e^{-s}}{s} + \frac{(e^{-s}-1)}{s^2} \right\}$$

$$= \cancel{\frac{e^{-s}}{1-e^{-s}} \left\{ \frac{1}{s^2} - \frac{1}{s} \right\}}$$

Q

$$f(t) = \begin{cases} t & , 0 < t < 1 \\ 2-t & , 1 < t < 2 \end{cases}$$

$T = 2$.

At $0 < t < 1$,

$$L\{f(t)\} = \frac{1}{1-e^{-2s}} \int_0^2 e^{-st} (t)$$

$$= \frac{1}{1-e^{-2s}} \left[\frac{te^{-st}}{-s} + \frac{1}{s} \int e^{-st} . dt \right]_0^2$$

$$= \frac{1}{1-e^{-2s}} \left[-\frac{te^{-st}}{s} + \frac{1}{s} \frac{e^{-st}}{(-s)} \right]_0^2$$

$$= \frac{1}{1-e^{-2s}} \left[-\frac{\cancel{2} e^{-2s}}{\cancel{2}} + \left(-\frac{1}{s^2}\right) e^{-2s} + 0 + \frac{1}{s^2} \right]$$

$$= \frac{1}{1-e^{-2s}} \left[-e^{-2s} - \frac{e^{-2s}}{s^2} + \frac{1}{s^2} \right]$$

$$= \frac{1}{1-e^{-2s}} \left[-e^{-2s} + \frac{(1-e^{-2s})}{s^2} \right]$$

At $1 < t < 2$.

$$L\{f(t)\} = \frac{1}{1-e^{-2s}} \int_0^2 e^{-st}(2-t).\,dt$$

$$= \frac{1}{1-e^{-2s}} \int_0^2 (2e^{-st} - t e^{-st}).dt = \frac{1}{1-e^{-2s}} \left\{ \left[\frac{2e^{-st}}{-s} \right]_0^2 - \int_0^2 e^{-st}.dt \right\}$$

$$= \frac{1}{1-e^{-2s}} \left\{ \left[\frac{2e^{-2s}}{-s} + \frac{2}{s} e^{-0} \right] - \left(-e^{-2s} + \frac{(1-e^{-2s})}{s^2} \right) \right\}$$

$$= \frac{1}{(1-e^{-2s})} \left\{ \frac{-2e^{-2s}}{s} + \frac{2}{s} + e^{-2s} - \frac{(1-e^{-2s})}{s^2} \right\}$$

Consider $L\{f(t)\} = F(s)$ and $L\{g(t)\} = G(s)$.

$L\{$convolution of $f(t)$ and $g(t)\} = F(s).G(s)$

Convolution of $f(t)$ and $g(t)$,

$$(f*g)(t) = \int_0^t f(u)\, g(t-u).du.$$

Properties:

(i) $g*f = f*g$

(ii) $(f*g)*h = f*(g*h)$

(iii) $f*(c_1 g + c_2 h) = c_1 f*g + c_2 f*h$

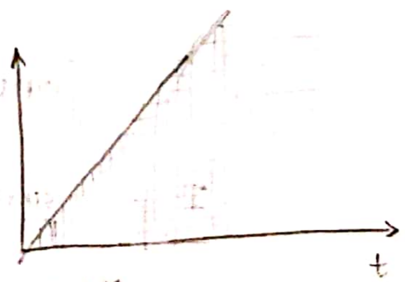
$$\boxed{L\left\{ \int_0^t f(u)\, g(t-u).du \right\} = F(s)\, G(s)}$$

Proof:

$$L\left\{ \int_0^t f(u)\, g(t-u).du \right.$$

$$= \int_{t=0}^{\infty} e^{-st} \left(\int_{u=0}^{t} f(u)\, g(t-u).du \right).dt$$

$$= \int_{t=0}^{\infty} \int_{u=0}^{t} e^{-st} f(u)\, g(t-u).du.dt$$



$0 < t \leq \infty$ vertical strips for $t \longrightarrow 0 \leq u < t$

for u, $u \leq t \leq \infty$

$0 \leq u \leq \infty$

$u < t < \infty \longleftarrow$ Horizontal strips for u

$$= \int_{u=0}^{\infty}\int_{t=u}^{\infty} e^{-st} f(u)\, g(t-u) . dt . du .$$

$t - u = v .$

~~When $t = u$, $t = v + u$.~~

~~When $v = \infty$, $t = \infty$~~

$$= \int_{u=0}^{\infty}\int_{v=0}^{\infty} e^{-s(u+v)} f(u)\, g(v) . dv . du .$$

When $t = u$, $v = 0$

When $t = \infty$, $v = \infty$

$$= \int_{u=0}^{\infty} e^{-su} f(u) \left(\int_0^{\infty} e^{-sv} g(v) \right) du \}$$

$$= F(s) G(s)$$

Q Find $L^{-1}\left\{ \frac{1}{(s+2)(s^2+1)} \right\}$.

$\Big[F(s) G(s) = L \{ \int_0^t f(u)\, g(t-u) . du \}$.

$L^{-1}\{ F(s)\, G(s) \} = \int_0^t f(u)\, g(t-u) . du .$

$f(t) = L^{-1}\{F(s)\}$, $g(t) = L^{-1}\{G(s)\} . \Big]$

Ans: $F(s) = \frac{1}{s^2+1}$, $G(s) = \frac{1}{s+2}$.

$f(t) = \sin t$ $\quad g(t) = e^{-2t}$.

$$\int_0^t \sin u \; e^{-2(t-u)} . du .$$

$$= e^{-2t} \int_0^t \sin u . e^{2u} . du .$$

$= \cancel{e^{-2t} \Big[si}$

$$I = \int \sin u . e^{2u} = \frac{\sin u . e^{2u}}{2} - \int \frac{\cos u . e^{2u}}{2} du$$

$$I = \frac{\sin u . e^{2u}}{2} - \frac{1}{2}\left[\frac{\cos u . e^{2u}}{2} + \int \frac{\sin u . e^{2u}}{2} du \right]$$

$$I = \frac{\sin u . e^{2u}}{2} - \frac{1}{4} \cos u . e^{2u} - \frac{1}{4} I$$

$$\frac{5}{4} I = \left(\sin u . e^{2u} - \frac{1}{2} \cos u . e^{2u} \right) \frac{1}{2}$$

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$$I = \frac{4}{5}\left(\frac{1}{2}\right)\left(\sin u \cdot e^{2u} - \frac{\cos u}{2} \cdot e^{2u}\right)$$

$$I = \frac{2}{5} e^{2u}\left(\sin u - \frac{\cos u}{2}\right).$$

$$\int_0^t \sin u \cdot e^{-2(t-u)} \, du = e^{-2t} \frac{(2)}{5}\left[e^{2u}\left(\sin u - \frac{\cos u}{2}\right)\right]_0^t$$

$$= e^{-2t} \frac{2}{5}\left[e^{2t}\left(\sin t - \frac{\cos t}{2}\right) - 1\left(-\frac{1}{2}\right)\right]$$

$$= \frac{2}{5} e^{-2t}\left[e^{2t}\left(\sin t - \frac{\cos t}{2}\right) + \frac{1}{2}\right]$$

$$= \frac{2}{5}\left[\sin t - \frac{\cos t}{2} + \frac{e^{-2t}}{2}\right] = \frac{2}{5}\sin t - \frac{\cos t}{5} + \frac{e^{-2t}}{5}.$$

Q. Solve for $y(t)$

$$y(t) - \int_0^t \underbrace{e^{t-u} \, y(u)}_{g(t-u)\, f(u)} \, du = t.$$

Taking transform on both sides,

$$L\{y(t)\} - L\left\{\int_0^t e^{t-u} y(u) \, du\right\} = L\{t\}.$$

$$F(s) - F(s)\, L\{e^t\} = \frac{1}{s^2}.$$

$$F(s)\left(1 - L\{e^t\}\right) = \frac{1}{s^2}$$

$$F(s)\left(1 - \frac{1}{s-1}\right) = \frac{1}{s^2}.$$

$$F(s) = \frac{(s-1)}{(s-2)s^2}.$$

$$y(t) = L^{-1}\left\{\frac{s-1}{s^2(s-2)}\right\}.$$

$$\frac{s-1}{s^2(s-2)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{(s-2)}$$

$$(s-1) = As(s-2) + B(s-2) + Cs^2$$

$$s-1 = A(s^2 - 2s) + Bs - 2B + Cs^2.$$

$$s-1 = s(-2A + B) + s^2(A + C) - 2B.$$

$-2A + B = 1.$

$-2B = -1 \Rightarrow \boxed{B = \frac{1}{2}}$

$B - 1 = 2A$

$\frac{1}{2} - 1 = 2A$

$\boxed{A = -\frac{1}{4}}$

$A + C = 0 \Rightarrow C = -A$

$\boxed{C = \frac{1}{4}}$

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$$\frac{s-1}{s^2(s-2)} = \frac{-1/4}{s} + \frac{1/2}{s^2} + \frac{1/4}{s-2}.$$

$$y(t) = -\frac{1}{4}L\{\frac{1}{s}\} + \frac{1}{2}\{\frac{1}{s^2}\} + \frac{1}{4}\{\frac{1}{s-2}\}$$

$$= -\frac{1}{4}\cdot + \frac{1}{2}t + \frac{1}{4}e^{2t}.$$

Q. Solve

$$y' - y = \int_0^t (t-u)e^u\, du. \quad , y(0) = 1.$$

Ans:

$$y' - y = \int_0^t (t-u)e^u\, du$$

$$g(t-u)\ f(u)$$

Applying L.T to both sides,

$$L\{y' - y\} = L\{\int_0^t (t-u)e^u\, du\}$$

$$L\{y'\} - L\{y\} = \left(\frac{1}{s^2}\right)\left(\frac{1}{s-1}\right)$$

$$sL\{y\} - y(0) - L\{y\} = \frac{1}{s^2(s-1)}$$

$$(s-1)\ L\{y\} - 1 = \frac{1}{s^2(s-1)}$$

$$(s-1)\ L\{y\} = \frac{1}{s^2(s-1)} + 1 = \frac{1 + s^2(s-1)}{s^2(s-1)}$$

$$L\{y\} = \frac{1+s^2(s-1)}{s^2(s-1)^2}.$$

$$y(t) = L^{-1}\left\{\frac{1+s^2(s-1)}{s^2(s-1)^2}\right\}$$

$$= L^{-1}\left\{\frac{1}{s^2(s-1)^2}\right\} + L^{-1}\left\{\frac{\cancel{s^2}(s-1)}{\cancel{s^2}(s-1)^2}\right\}$$

$$= L^{-1}\left\{\frac{1}{s^2(s-1)^2}\right\} + L^{-1}\left\{\frac{1}{s-1}\right\}$$

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$$\frac{1}{s^2(s-1)^2} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{(s-1)} + \frac{D}{(s-1)^2}$$

$$1 = As(s-1)^2 + B(s-1)^2 + C(s-1)s^2 + Ds^2$$

$$1 = A\,s(s^2+1-2s) + B(s^2-2s+1) + C(s^3-s^2) + Ds^2$$

$$= A\,s^3 + As - 2As^2 + Bs^2 - 2Bs + B + Cs^3 - Cs^2 + Ds^2$$

$$= s^3(A+C) + s^2(-2A+B-C+D) + s(A-2B) + B$$

$\underline{B=1}$

$A - 2B = 0 \Rightarrow A = 2B \rightarrow \underline{A=2}$

$A + C = 0$

$\underline{C = -2}$

$-2A + B - C + D = 0.$

$-4 + 1 + 2 + D = 0$

$\underline{D=1}.$

$$L^{-1}\left\{\frac{1}{s^2(s-1)^2}\right\} = 2L^{-1}\left\{\frac{1}{s}\right\} + L^{-1}\left\{\frac{1}{s^2}\right\} + (-2)L^{-1}\left\{\frac{1}{s-1}\right\} + L^{-1}\left\{\frac{1}{(s-1)^2}\right\}$$

$$y = 2 + t - 3e^t + te^t = 2\cancel{t} + t - 2e^t + L^{-1}\left\{\frac{1}{(s-1)^2}\right\}$$

$$y(t) = 2 + t - 2e^t.$$

LINEAR Equations

$a_{11}x_1 + \cdots\cdots a_{1n}x_n = b_1$

$\vdots$

$a_{m1}x_1 + \cdots\cdots a_{mn}x_n = b_m$

$$A = \begin{pmatrix} a_{11} & \cdots\cdots & a_{1n} \\ \vdots & & \\ a_{m1} & \cdots\cdots & a_{mn} \end{pmatrix} \quad x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \quad b = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$$

$$AX = b$$

Model of Questions :

(i) Gypotography

(ii) Traffic problem

(iii) Transportation problem

(iv) Electric circuit

(v) Economics - Market models.