TUTORIAL-3
Assignment

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1. Which of the following transformations are linear?

1a) $T(x) = 0$

Clearly, $T(0) = \hat{0}$

Let $\alpha, \beta \in R$ and $a, b \in R$

$\alpha = (x_1)$, $\beta = (x_2)$ $\therefore T(\alpha) = 0$, $T(\beta) = 0$

$a\alpha + b\beta = (ax_1 + bx_2)$

$T(a\alpha + b\beta) = T(ax_1 + bx_2) = 0$

$aT(\alpha) + bT(\beta) = a \times 0 + b \times 0 = 0$

$\therefore T(a\alpha + b\beta) = aT(\alpha) + bT(\beta)$
Hence, $T(x) = 0$ is a linear transformation.

b) $T(x) = x$

Clearly, $T(0) = \hat{0}$
Let $\alpha, \beta \in R$ & $a, b \in R$

$\alpha = (x_1)$ $\beta = (x_2)$ $\therefore T(\alpha) = x_1$ & $T(\beta) = x_2$

$a\alpha + b\beta = ax_1 + bx_2$
$T(a\alpha + b\beta) = T(ax_1 + bx_2) = ax_1 + bx_2$

$aT(\alpha) + bT(\beta) = ax_1 + b2$
$\therefore aT(\alpha) + bT(\beta) = T(a\alpha + b\beta)$ $\therefore$ Hence it is linear transformation

c) $T(x) = x + a$

Clearly, $T(\bar{0}) \neq 0$

Let $\alpha, \beta \in R \quad \forall\ a_1, b_1 \in R$

$\alpha = (x_1)$, $\beta = (x_2)$ $\therefore T(\alpha) = T(x_1) = x_1 + a$

$T(\beta) = T(x_2) = x_2 + a$

$a_1\alpha + b_1\beta = (a_1x_1 + b_1x_2) \therefore T(a_1\alpha + b_1\beta) = a_1x_1 + b_1x_2 + a$

$a_1T(\alpha) + b_1T(\beta) = a_1x_1 + b_1x_2 + a_1\alpha + b_1 a$

$\neq T(a_1\alpha + b_1\beta)$

$\therefore T(x) = x + a$ is not a linear transformation.

d) $T(x) = x^2$

$T(\bar{0}) = \hat{0}$.

Let $\alpha, \beta \in R$ & $a, b \in R$

$\alpha = (x_1)$ $\beta = (x_2)$ $\therefore T(\alpha) = T(x_1) = x_1 2$

$T(\beta) = T(x_2) = x_2^2$

Now, $a\alpha + b\beta = (ax_1 + bx_2)$

$T(a\alpha + b\beta) = (ax_1 + bx_2)^2$

$aT(\alpha) + bT(\beta) = ax_1^2 + bx_2^2 \neq T(a\alpha + b\beta)$

$\therefore T(x) = x^2$ is not a linear transformation.

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$T(x) = ex$

Clearly $T(0) = e^0 \neq 0$ $\therefore$ Not a linear transformation

f) $T(x) = 1$

Clearly $T(0) = 1 \neq 0$ $\therefore$ $T(x) = 1$ is not a linear transformation.

g) $T(x) = \sin x$

Clearly, $T(0) = \hat{0}$

Let $\alpha, \beta \in R$ & $a, b \in R$

$\alpha = (x_1)$ & $\beta = (x_2)$ $\therefore$ $T(\alpha) = T(x_1) = \sin x_1$
$T(\beta) = T(x_2) = \sin x_2$

Now, $a\alpha + b\beta = (ax_1 + bx_2)$

$T(a\alpha + b\beta) = T(ax_1 + bx_2) = \sin(ax_1 + bx_2)$

$aT(\alpha) + bT(\beta) = a\sin x_1 + b\sin x_2 \neq T(a\alpha + b\beta)$

$\therefore$ $T(x) = \sin x$ is not a linear transformation.

2a) $T(x) = (x, x)$

$T(0) = \hat{0}$
Let $\alpha, \beta \in R$ & $a, b \in R$

$\alpha = (x_1)$ & $\beta = (x_2)$ $\therefore$ $T(\alpha) = T(x_1) = (x_1, x_1)$
$T(\beta) = T(x_2) = (x_2, x_2)$

$a\alpha + b\beta = (ax_1 + bx_2)$ $\therefore$ $T(ax_1 + bx_2) = (ax_1 + bx_2, ax_1 + bx_2)$

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$aT(\alpha)+bT(\beta) = a(x_1, x_1) + b(x_2, x_2)$
$= (ax_1+bx_2,\ ax_1+bx_2)$

Clearly $T(a\alpha+b\beta) = aT(\alpha)+bT(\beta)$

Hence, $T(x) = (x, x)$ is a linear transformation.

b) $T(x) = (x, 0)$

Let $\alpha, \beta \in R$ & $a, b \in R$

$\alpha_1 = (x_1)$ & $\beta = (x_2)$ $\therefore T(\alpha) = (x_1, 0)$
$T(\beta) = (x_2, 0)$

Now, $a\alpha+b\beta = (ax_1+bx_2)$

$T(a\alpha+b\beta) = T(ax_1+bx_2) = (ax_1+bx_2, 0)$

$aT(\alpha)+bT(\beta) = a(x_1, 0)+b(x_2, 0) = (ax_1+bx_2, 0)$

$\rightarrow$ It is a linear transformation.

c) $T(x) = (x^2, x)$

Let $\alpha, \beta \in R$ & $a, b \in R$

$\alpha = (x_1)$ & $\beta = (x_2)$ $\therefore T(\alpha) = T(x_1) = (x_1^2, x_1)$
$T(\beta) = T(x_2) = (x_2^2, x_2)$

$a\alpha+b\beta = (ax_1+bx_2)$ $\therefore T(a\alpha+b\beta) = (ax_1+bx_2)^2$

$aT\alpha+bT(\beta) = ax_1^2+bx_2^2 \neq (ax_1+bx_2)^2$
$\neq T(a\alpha+b\beta)$

$\rightarrow$ Not a linear transformation.

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$T(x) = (x, 1)$

Clearly $(T(0) = (0,1) \neq \hat{0}$ $\therefore$ Not a linear transformation.

3(a) $T(x,y) = xy$

Let $\alpha, \beta \in R^2$ & $a, b \in R$

$\alpha = (x_1, y_1)$ & $\beta = (x_2 y_2)$ $\therefore T(\alpha) = T(x_1, y_1) = x_1 y_1$

$T(\beta) = T(x_2, y_2) = x_2, y_2$

$(a\alpha + b\beta) = (ax_1 + bx_2, \; ay_1 + by_2)$

$T(a\alpha + b\beta) = (ax_1 + bx_2)(ay_1 + by_2)$

$= a^2 x_1 y_1 + ab x_1 y_2 + ab x_2 y_1 + b^2 x_2 y_2$

$aT(\alpha) + bT(\beta) = a x_1 y_1 + b x_2 y_2$

$\neq T(a\alpha + b\beta)$

$\therefore \longrightarrow$ Not a linear transformation

b) $T(x,y) = x + y$

Let $\alpha, \beta \in R^2$ & $a, b \in R$

$\alpha = (x_1, y_1)$ & $\beta = (x_2, y_2)$ $\therefore T(\alpha) = x_1 + y_1$

$T(\beta) = x_2 + y_2$

$a\alpha + b\beta = (ax_1 + bx_2, \; ay_1 + by_2)$ $\therefore T(a\alpha + b\beta) = ax_1 + bx_2 + ay_1 + by_2$

$aT(\alpha) + bT(\beta) = a(x_1 + y_1) + b(x_2 + y_2) = ax_1 + bx_2 + ay + by_2$

$T(a\alpha + b\beta) = aT(\alpha) + bT(\beta) \longrightarrow$ Linear tranformation

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c) $T(x,y) = 2x+3y$

Let $\alpha, \beta \in R^2$ & $a,b \in R$

$\alpha = (x_1, y_1)$ & $\beta = (x_2, y_2)$ $\therefore T(\alpha) = 2x_1 + 3y_1$

$T(\beta) = 2x_2 + 3y_2$

$a\alpha + b\beta = (ax_1 + bx_2, ay_1 + by_2)$

$T(a\alpha + b\beta) = 2ax_1 + 2bx_2 + 3ay_1 + 3by_2$

$aT(\alpha) + bT(\beta) = 2ax_1 + 2bx_2 + 3ay_1 + 3by_2$

$\therefore T(a\alpha + b\beta) = aT(\alpha) + bT(\beta) \rightarrow$ Linear Transformation.

$\lambda T(\alpha) + bT(\beta$

b)

d) $T(x,y) = x^2 + y$

Let $\alpha, \beta \in R^2$ & $a,b \in R$

$\alpha = (x_1, y_1)$ & $\beta = (x_2, y_2)$ $\therefore T(\alpha) = x_1^2 + y_1$

$T(\beta) = x_2^2 + y_2$

$a\alpha + b\beta = (ax_1 + bx_2, ay_1 + by_2)$ $\therefore T(a\alpha + b\beta) = (ax_1 + bx_2)^2 + ay_1 + by_2$

$aT(\alpha) + bT(\beta) = ax_1^2 + ay_1 + bx_2^2 + by_2$

$\neq T(a\alpha + b\beta) \longrightarrow$ Not a linear transform.

4(a) $T(x,y) = (x+y, xy)$

Let $\alpha, \beta \in R^2$ & $a,b \in R$

$\alpha = (x_1, y_1)$ & $\beta = (x_2, y_2)$ $\therefore T(\alpha) = (x_1 + y_1, x_1 y_1)$

$T(\beta) = (x_2 + y_2, x_2 y_2)$

$T(a\alpha + b\beta) = (ax_1 + bx_2 + ay_1 + by_2, (ax_1 + bx_2)(ay_1 + by_2))$

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$$aT(\alpha)+bT(\beta) = (a(x_1+y_1, x_1y_1) + b(x_2+y_2, x_2y_2)$$

$$= (ax_1+ay_1+bx_2+by_2,\ ax_1y_1+bx_2y_2)$$

$$\neq T(a\alpha+b\beta) \rightarrow \underline{\text{Not a L.T.}}$$

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b) $T(x,y) = (y,x)$

Let $\alpha, \beta \in R^2$ & $a, b \in R$

$\alpha = (x_1, y_1)$ & $\beta = (x_2, y_2)$ $\therefore T(\alpha) = (y_1, x_1)$

$T(\beta) = (y_2, x_2)$

$$T(a\alpha+b\beta) = (ay_1+by_2,\ ax_1+bx_2)$$

$$aT(\alpha)+bT(\beta) = (a(y_1,x_1)+b(y_2,x_2))$$

$$= (ay_1+by_2,\ ax_1+bx_2)$$

$\therefore T(a\alpha+b\beta) = aT(\alpha)+bT(\beta) \rightarrow$ <u>Linear Transformation.</u>

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c) $T(x) = (|x|, 0)$

Let $\alpha, \beta \in R^2$ & $a, b \in R$

$\alpha = (x_1, y_1)$ $\beta = (x_2, y_2)$ $\therefore T(\alpha) = (|x_1|, 0)$

$T(\beta) = (|x_2|, 0)$

$$T(a\alpha+b\beta) = T(ax_1+bx_2,\ ay_1+by_2) = (|ax_1+bx_2|, 0)$$

$$aT(\alpha)+bT(\beta) = a|x_1|+b|x_2| \neq T(a\alpha+b\beta)$$

$\rightarrow$ Not a Linear Transformation

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2. A linear transformation T on R^3 to itself is def. by $T(e_1) = e_1+e_2+e_3$, $T(e_2) = e_2+e_3$ and $T(e_3) = e_2$ where $\{e_1, e_2, e_3\}$ is standard basis of R^3. Determine image of $(2,-1,3)$.

→ we know that image of $(2,-1,3)$ is $T(2,-1,3)$

$$T(e_1,e_2,e_3) = \{(e_1+e_2+e_3), (e_2+e_3), (e_2-e_3)\}$$

let $e_1 = 2$, $e_2 = -1$ $e_3 = 3$

$$T(2,-1,3) = (2+-1+3, -1+3, -1-3)$$

$$= (4, 2, -4)$$

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3. Find $T(x_1, x_2, x_3)$ where $T: R^3 \to R$ defined by $T(1,1,1) = 3$, $T(0,1,-2) = 1$, $T(0,0,1) = -2$.

→ Let $(x,y,z) \in R^3$ be such that

$$(x,y,z) = a(1,1,1) + b(0,1,-2) + c(0,0,1)$$

$\therefore$ $a = x$
$a+b = y$
$a-2b+c = z$

$\therefore$ $a = x$
$b = y-x$
$c = z-x+2y$

$\therefore$ $(x,y,z) = x(1,1,1) + (y-x)(0,1,-2) + (z-x+2y)(0,0,1)$

$$T(x,y,z) = x\,T(1,1,1) + (y-x)\,T(0,1,-2) + (z-x+2y)\,T(0,0,1)$$

$$= x \times 3 + (y-x)1 + (z-x+2y) \times -2$$

$$= 8x - 3y - 2z$$

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Let $T : R^3 \to R^2$ be a linear transformation defined by $T(x,y,z) = (2x-3y+z, -2x+5z)$, find the matrix of T relative to standard basis of R^3.

We have, $T : R^3 \to R^2$ given by

$$T(x,y,z) = (2x-3y+z, -2x+5z)$$

Let A be matrix representation of T w.r.t. stand basis

$$\therefore A\bar{x} = T(x)$$

where $\bar{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

Let, $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$

Using $A\bar{x} = T(x)$

$$\therefore \quad a_{11}x + a_{12}y + a_{13}z = 2x - 3y + z$$
$$a_{21}x + a_{22}y + a_{23}z = -2x + 5z$$

$\therefore a_{11} = 2, \; a_{12} = -3, \; a_{13} = 1, \; a_{21} = -2 \quad a_{22} = 0 \quad a_{23} = 5$

$\therefore A = \begin{bmatrix} 2 & -3 & 1 \\ -2 & 0 & 5 \end{bmatrix}$ is required matrix representation of T relative to standard basis of R^3

5. Find the kernel space of linear transformation $T: P[x] \longrightarrow P[x]$ defined by $T(p(x)) = p'(x)$ (where Let is the set of all real polynomials).

We know that Range of $T = R(T) = \{T(\alpha) \in P[x] ; \alpha \in P(x)\}$

kernel of T = $Ker(T) = \{\alpha \in P(x) \therefore T(\alpha) = \hat{0}\}$

Let $\alpha = P(x)$

$T(\alpha) = T(p(x)) = 0$

$\therefore p'(x) = 0 \quad \therefore p(x) = K$ such that $T(p(x)) = 0$

$\therefore$ ~~Ker T~~ $= \{$ ~~K(constant) $\in$ P(x) $\in$ T(P(x)) = $\hat{0}$~~ $\}$

~~Range T = R(T) = $\{$ p'(x) $\ni$ p(x) $\in$ set of all real polynomials $\}$~~

$Ker(T) = \{ K(\text{constant}) \in P(x) \ni T(p(x)) = \hat{0} \}$

Range $T = R(T) = \{ p'(x) \ni p(x) \in$ set of all real polynomials $\}$

6. Verify the rank-Nullity theorem for the following function:

$T: M_{2\times2}(R) \longrightarrow M_{2\times2}(R)$ by $T(A) = A + A^T$

(where $M_{2\times2}$ is set of all real 2×2 matrices)

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Let $M_{2\times2} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \forall\ a,b,c,d \in R$

Linear transformation T is given by

$$T(A) \Rightarrow A + A^T$$

i) Rank of T

Let $S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

Such that $L[S] = M$

Now, $S_1 = \left\{ T\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, T\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, T\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, T\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

$$= \left\{ \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} \right\}$$

$\therefore$ S_1 generates $R[T] \Rightarrow L[S_1] = R[T]$

$$S_1 \equiv \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \equiv \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ S_1 is L.D $\Rightarrow$ No basis

S_2 is L.I where $S_2 = \{(2,0,0,0), (0,1,1,0), (0,0,0,2)\}$

Clearly S_2 is basis of $R[T]$ ($\because L[S_1] = L[S_2] = R(T)$)

$$\dim R[T] = \rho(T) = 3$$

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i) Nullity of T

$Ker(T) = \{\alpha \in R^4 : T(\alpha) = \hat{0}\}$

$= \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Let $(a,b,c,d) \in Ker(T)$

$\therefore T(a,b,c,d) = 0$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} a & c \\ b & d \end{bmatrix} = 0$$

$$\begin{bmatrix} 2a & b+c \\ c+b & 2d \end{bmatrix} = 0 \Rightarrow a=0, d=0$$
$$b+c=0$$

Let $b=k$, then $c=-k$

$$\therefore \alpha = \begin{bmatrix} 0 \\ k \\ -k \\ 0 \end{bmatrix} = k\begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} \in R^4$$

$\therefore \alpha = L[S_3]$ where $S_3 = \{(0,1,-1,0)\}$

$\rightarrow$ S_3 is Linearly independent & $L[S_3] = Ker(T)$

$\therefore$ S_3 is basis of $ker(T)$

$\dim ker(T) = \nu(T) = 1$

$\Rightarrow$ $\dim(M) = \dim R^4 = 4$

$\dim(R(T)) + \dim(ker(T)) = 3+1 = 4$

Clearly, $\dim(M) = \rho(T) + \nu(T)$ $\therefore$ Hence, Nullity Rank theorem verified

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Let T be a linear transformation $T: U \to V$.
~~basis~~ Show that Range space of T is a subspace of V and Kernal of T is a subspace of U.

$\Rightarrow$ We know, $R(T) \leq V$ & $Ker(T) \leq U$

Let $\alpha_1, \alpha_2 \in U \Longrightarrow T(\alpha_1) = \beta_1, \; T(\alpha_2 = \beta_2)$

$\Rightarrow \beta_1, \beta_2 \in R(T)$

Let $a, b \in F$ & $\beta_1, \beta_2 \in R(T)$

$\because \alpha_1, \alpha_2 \in U \Rightarrow a\alpha_1 + b\alpha_2 \in U$

$T(a\alpha_1 + b\alpha_2) \in R(T)$
$aT\alpha_1 + bT\alpha_2 \in R(T)$

$\therefore a\beta_1 + a\beta_2 \in R(T)$

Now, $\beta_1, \beta_2 \in R(T) \Rightarrow a\beta_1 + b\beta_2 \in R(T)$

$\therefore R(T)$ is subspace of V

$\Rightarrow$ Now, $Ker(T) \leq U$

$\alpha_1, \alpha_2 \in Ker(T)$ & $a, b \in F$

$T(\alpha_1) = \hat{0}, \; T(\alpha_2) = \hat{0}$

$T(a\alpha_1 + b\alpha_2) = aT(\alpha_1) + bT(\alpha_2) = 0$

$\therefore a\alpha_1 + b\alpha_2 \in Ker(T) \; \forall \; \alpha_1, \alpha_2 \in Ker(T)$
$\therefore Ker\,T$ is subspace of U

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8) For each of the following mapping $T: U \to V$ find and the dimensions of its range space & its null sp. Also verify Rank-Nullity theorem

a) $T(x,y,z) = (y+z, x+y-2z, x+2y-2z)$

Let $S = [(1,0,0),(0,1,0),(0,0,1)]$

such that $L[S] = U$

& $S_1 = [T(1,0,0), T(0,1,0), T(0,0,1)]$

$= \{(0,1,1),(1,1,2),(1,-2,-2)\}$

$\therefore$ S_1 generates $R(T) \Rightarrow L[S_1] = R[T]$

$$S_1 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & -2 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 1 & -2 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & -3 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

$\therefore$ S_1 is L. Independent & $L[S_1] = R[T]$

$\therefore$ S_1 is basis of $R(T)$

$\dim R(T) = \rho(T) = 3$

we have, $\ker(T) = \{\alpha \in R3, T(\alpha) = \hat{0}\}$

Let $(x,y,z) \in \ker(T)$

$\therefore$ $y+z=0$

$x+y-2z=0$

$x+2y-2z=0$

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$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & -2 & 0 \\ 1 & 2 & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

$\therefore$ Using back substitution, $x=0, y=0, z=0$

$\therefore$ $S_2 = \{0\} \rightarrow$ Linearly dependent

$\dim S_2 = \rho(t) = 0$

We know, $U = R^3$ is vector space

$\dim U = \dim R^3 = 3 =$

$\therefore$ $\dim R(T) + \dim \ker(T) = \dim U$

$\dim(U) = \rho(T) + \nu(T)$

$\therefore$ Hence rank-nullity theorem verified.

(ii) $T(x,y,z) = (3x, x-y, 2x+y+z)$

Let $S = \{(1,0,0), (0,1,0), (0,0,1)\}$

and let $S_1 = \{T(1,0,0), T(0,1,0), T(0,0,1)\}$

$= \{(3,1,2), (0,-1,1), (0,0,1)\}$

$\therefore$ S_1 generates $R(T)$ & $L[S_1] = R(T)$

$S_1 = \begin{bmatrix} 3 & 1 & 2 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow$ S_1 is linearly independent

$\therefore$ S is basis of $R[T]$

$\dim R(T) = \rho(T) = 3$

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Let $Ker\ T = \{ \alpha \in R^3,\ T\alpha = \hat{0} \}$

$(x, y, z) \in Ker\ T$

$\therefore$ $3x = 0$ $\Rightarrow$ $x = 0$, $y = 0$ & $z = 0$
$x - y = 0$
$2x + y + z = 0$

$\therefore S_2 = \{0\} \rightarrow$ Linearly dependent ~~dim~~

$\dim K(T) = 0 = \nu(T)$

Now $U = R^3$ is vector space

$\dim U = \dim R^3 = 3 = \dim K(T) + \dim R(T)$

$\therefore \dim U = \rho(T) + \nu(T)$

Hence, rank-nullity theorem is verified.

b) $T(x_1, x_2, x_3, x_4) = (x_1 - x_2 + x_3 + x_4,\ x_1 + 2x_3 - x_4,\ x_1 + x_2 + 3x_3 - 3x_4)$

Let $S = \{ (1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1) \}$

& Let $S_1 = \{ T(1,0,0,0), T(0,1,0,0), T(0,0,1,0), T(0,0,0,1) \}$

$\therefore S_1 = \{ (1,1,1), (-1,0,1), (1,2,3), (1,-1,-3) \}$

$\therefore L[S_1] = R[T]$

$$S_1 = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & 2 & 3 \\ 1 & -1 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ -1 & 0 & 1 \\ 1 & -1 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \\ 1 & -1 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

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$\therefore S_1$ is linearly dependent

$S_2 = \{(1,1,1), (0,1,2)\}$ forms a basis on $R(T)$

$L[S_1] = L[S_2] = R(T)$

$\therefore \dim R(T) = \rho(T) = 2$

Now, let $\ker(T) = \{\alpha \in R^4, T(\alpha = \hat{0})\}$
$= (x_1, x_2, x_3, x_4) \in R^4$

$\therefore x_1 - x_2 + x_3 + x_4 = 0$
$x_1 + 2x_3 - x_4 = 0$
$x_1 + x_2 + 3x_3 - 3x_4 = 0$

$$\Rightarrow \begin{bmatrix} 1 & -1 & 1 & 1 & 0 \\ 1 & 0 & 2 & -1 & 0 \\ 1 & 1 & 3 & -3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 1 & 1 & 0 \\ 0 & 2 & 2 & -4 & 0 \\ 1 & 0 & 2 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 1 & 1 & 0 \\ 0 & 2 & 2 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Using Back subst.

$x_2 + x_3 = 2x_4$
$x_1 - x_2 + x_3 + x_4 = 0$

Let $x_2 = k_1$ & $x_3 = k_2$

$\therefore x_4 = \dfrac{k_1 + k_2}{2}$ $\quad x_1 = \dfrac{k_1 - 3k_2}{2}$

$$\therefore \text{Sol}^n \text{ vector } \bar{x} = \begin{bmatrix} \frac{k_1 - 3k_2}{2} \\ k_1 \\ k_2 \\ \frac{k_1 + k_2}{2} \end{bmatrix} = \frac{k_1}{2}\begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} + \frac{k_2}{2}\begin{bmatrix} -3 \\ 0 \\ 2 \\ 1 \end{bmatrix}$$

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$\therefore S_2 = \{(1,2,0,1), (-3,0,2,1)\}$

$\because L[S_2] = \ker(T)$ & S_2 is linearly independent set.

$\therefore \dim \ker(T) = \nu(T) = 2$

$\dim U = \dim \mathbb{R}^4 = 4 = \dim R(T) + \dim K(T)$

$\therefore \dim U = \rho(T) + K(T)$

$\therefore$ Hence rank nullity theorem verified.

9. Let V be the vector space of a 2×2 matrices

Let $M = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$. Let $T: V \to V$, be linear map, defined by $T(A) = AM - MA$. Find a basis and dimension of null space of T.

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \ker(T)$

$\therefore T(A) = 0,$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} a & 2a+3b \\ c & 2c+3d \end{bmatrix} - \begin{bmatrix} a+2c & b+2d \\ 3c & 3d \end{bmatrix} = 0$$

$$\begin{bmatrix} -c & 2a+2b-d \\ -c & c \end{bmatrix} = 0$$

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$\therefore C=0, \; 2a+2b-2d=0$

Let $a=k_1, \; b=k_2, \; d=k_1+k_2$

$$\therefore \bar{X}=\begin{bmatrix} k_1 \\ k_2 \\ 0 \\ k_1+k_2 \end{bmatrix} = k_1\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} + k_2\begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \in R_4 = L[S]$$

$\therefore S=\{(1,0,0,1),(0,1,0,1)\} \rightarrow$ Linearly independent

$Ker(T)=L[S]$

$\therefore$ S is a basis of $Ker(T)$

$\dim Ker(T)=\nu(T)=2$

) $\therefore$ Let V be the vector space of all polynomials $p(x)$, with real coefficients, degree less than or equal to 6. Compute the basis and dimension of null space of the linear transformation $T: V \rightarrow V$ defined by

$T(p(x))=\frac{1}{2}(p(x)-p(-x))$ for all $p(x) \in V$

Let $p(x)=\alpha \in V$

We have $Ker(T)=\{\alpha \in V ; T(p(x))=0)$

$\therefore T(p(x))=0$

ie $p(x)=p(-x)$

$\Rightarrow p(x)$ is an even function of degree 6

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$\therefore p(x) = ax^2 + bx^4 + cx^6$

Let $S = \{x^2, x^4, x^6\}$

$L[S] = ax^2 + bx^4 + cx^6 = 0 \Rightarrow a = b = c = 0$

Hence S is linearly independent & $L[S] = \ker(T)$

$\therefore \dim K(T) = \nu(T) = 3$

Show that $T: R^3 \to R^3$ defined by $T(x, y, z) = (x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta, z)$ is nonsingular where θ is any angle.

Let $\alpha \in R^3$

$\alpha = (x, y, z)$

$\therefore T(\alpha) = \hat{0} \Rightarrow$ $x\cos\theta = y\sin\theta$
$x\sin\theta + y\cos\theta = 0$
and $z = 0$

$\therefore$ $x\cos\theta - y\sin\theta = 0$
$x\sin\theta + y\cos\theta = 0$

Solving we get $x = 0,\ y = 0,\ z = 0$

$\therefore$ Linear Transformation $T(\alpha) = \hat{0}$ only when $\alpha = \bar{0}$

Hence, $T: R^3 \to R^3$ defined by

$T(x, y, z) = (x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta, z)$ is nonsingular

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Show that each of following operators T on R^3 or R^2 is invertible and find T^{-1}

a) $T(x, y, z) = (2x, 4x-y, 2x+3y-z)$

Let $\alpha \in R^3$, $\alpha = (x, y, z)$

$T(\alpha) = \hat{0} \Rightarrow$ $2x = 0$
$4x - y = 0$
$2x + 3y - z$

$\therefore$ Solving we get $x = 0, y = 0, z = 0$

Clearly $T(\alpha) = \hat{0}$ only when $\alpha = \bar{0}$

Hence, $T(x, y, z)$ is non singular Linear Transformation.

Let $T^{-1}(x, y, z) = (a, b, c)$

$\therefore$ $T(a, b, c) = (x, y, z)$

$\Rightarrow$ $2a = x$, $a = x/2$
$4a - b = y$; $b = 2x - y$

$2a + 3b - c = z$; $c = 7x - 3y - z$

$\therefore$ $T^{-1}(x, y, z) = (x/2, 2x-y, 7x-3y-z)$

---

b) $T(x, y, z) = (x-3y-2z, y-4z, z)$

Let $\alpha \in R^3$, $\alpha = (x, y, z)$

$T(\alpha) = \hat{0} \Rightarrow$ $x - 3y - 2z = 0$
$y - 4z = 0$
$z = 0$

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$\therefore$ Solving we get $\alpha = \bar{0}$

Hence, given Transformation is non singular

Let $T^{-1}(x,y,z) = (a,b,c)$

$\therefore T(a,b,c) = (x,y,z)$

$(a-3b-2c,\ b-4c,\ c) = (x,y,z)$

$$\therefore \begin{aligned} c &= z \\ b &= y+4z \\ a &= 3y+14z \end{aligned}$$

$\therefore T^{-1}(x,y,z) = (3y+14z+x,\ y+4z,\ z)$

---

c) $T(x,y) = (x\cos\theta - y\sin\theta,\ y\cos\theta + x\sin\theta)$

Let $\alpha \in R^3$ such that $\alpha = (x,y)$

$T(\alpha) = \hat{0} \Rightarrow \begin{aligned} x\cos\theta - y\sin\theta &= 0 \\ y\cos\theta + x\sin\theta &= 0 \end{aligned}$

$\Rightarrow x = 0 \ \& \ y = 0$

$\therefore T(\alpha) = \hat{0}$ only when $\alpha = \bar{0}$

$\therefore T$ is a non singular Linear Transformation.

Let $T^{-1}(x,y,z) = (a,b)$

$\therefore T(a,b) = x,y \Rightarrow \begin{aligned} a\cos\theta - b\sin\theta &= x \\ a\sin\theta + b\cos\theta &= y \end{aligned}$

$\therefore$ Solving we get $\begin{aligned} b &= y\cos\theta - x\sin\theta \\ a &= x\cos\theta + y\sin\theta \end{aligned}$

$\therefore T^{-1}(x,y) = (x\cos\theta + y\sin\theta,\ y\cos\theta - x\sin\theta)$

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Using the Gram-Schmidt process, find an ~~orthonormal~~ orthonormal basis of following set of vectors:

a) $(3,4), (-1,1)$

Here, $\beta_1 = (3,4)$, $\beta_2 = (-1,1)$

$\gamma_1 = \beta_1 = (3,4)$

$$\gamma_2 = \beta_2 - \frac{\langle \beta_2, \gamma_1 \rangle \gamma_1}{|\gamma_1|^2}$$

$\langle \beta_2, \gamma_1 \rangle = 1$

$\langle \gamma_1 \rangle = (3,4) \quad |\gamma_1|^2 = 25, \|\gamma_1\| = 5$

$$\therefore \gamma_2 = (-1,1) - \frac{(3,4)}{25} = \left(\frac{-28}{25}, \frac{21}{25}\right)$$

$$\|\gamma_2\| = \sqrt{\left(\frac{-28}{25}\right)^2 + \left(\frac{21}{25}\right)^2} = \frac{35}{25}$$

$$\therefore S = \left\{ \frac{\gamma_1}{\|\gamma_1\|}, \frac{\gamma_2}{\|\gamma_2\|} \right\}$$ is required orthonormal basis

$$\therefore S = \left\{ \left(\frac{3}{5}, \frac{4}{5}\right), \left(-\frac{4}{5}, \frac{3}{5}\right) \right\}$$ is required orthonormal basis

(b) $(2,3,6), (7,12,18)$

Let $\beta_1 = (2,3,6)$ & $\beta_2 = (7,12,8)$

$\gamma_1 = \beta_1 = (2,3,6)$

$$\gamma_2 = \beta_2 - \frac{\langle \beta_2, \gamma_1 \rangle \gamma_1}{\|\gamma_1|^2}$$

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$\langle \beta_2, \gamma_1 \rangle = 14 + 36 + 48 = 98$

$\|\gamma_1\|^2 = 49 \quad \therefore \|\gamma_1\| = 7$

$\therefore \gamma_2 = (7, 12, 8) - \frac{98(2,3,6)}{49}$

$= (3, 6, -4)$

$|\gamma_2| = \sqrt{9+36+16} = \sqrt{61}$

$\therefore S = \left\{ \frac{\gamma_1}{|\gamma_1|}, \frac{\gamma_2}{|\gamma_2|} \right\}$ is req. orthonormal basis

$\therefore S = \left\{ \left(\frac{2}{7}, \frac{3}{7}, \frac{6}{7} \right) \left(\frac{3}{\sqrt{61}}, \frac{6}{\sqrt{61}}, \frac{-4}{\sqrt{61}} \right) \right\}$ is req. orthonormal basis

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c) $(1,1,0), (1,0,1), (0,1,+1)$

$\gamma_1 = \beta_1 = (1,1,0)$

$\gamma_2 = \beta_2 - \frac{\langle \beta_2 \gamma_1 \rangle}{\|\gamma_1\|^2} \cdot \gamma_1$

$\therefore \langle \beta_2, \gamma_1 \rangle = 1 \qquad |\gamma_1|^2 = 2$

$\therefore \gamma_2 = (1,0,1) - \frac{1}{2}(1,1,0) = (1/2, -1/2, 1), |\gamma_2|^2 = 3/2$

$\gamma_3 = \beta_3 - \frac{\langle \beta_3, \gamma_1 \rangle}{|\gamma_1|^2} \gamma_1 - \frac{\langle \beta_3, \gamma_2 \rangle}{|\gamma_2|^2} \gamma_2$,

$\beta_3 = (0,1,+1) \qquad \langle \beta_3, \gamma_2 \rangle = 1/2$

$\langle \beta_3, \gamma_1 \rangle = 1$

$$\therefore r_3 = (0,1,1) - \frac{1}{2}(1,1,0) - \frac{1\cdot\cancel{2}}{\cancel{2}\cdot 3}\left(\frac{1}{2}, -\frac{1}{2}, 1\right)$$

$$= \left(-\frac{2}{3}, \frac{2}{3}, \frac{2}{3}\right)$$

$$|r_3| = \frac{2}{\sqrt{3}}$$

$\therefore S = \left\{ \frac{x_1}{|x_1|}, \frac{x_2}{|x_2|}, \frac{r_3}{|r_3|} \right\}$ is req. orthonormal set

$\therefore S = \left\{ \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right), \left(\frac{1}{\sqrt{6}}, \frac{-1}{\sqrt{6}}, \frac{2}{\sqrt{6}}\right)\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) \right\}$, is req. orthonormal set

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Q) $(1,1,1,1), (0,1,2,2), (0,0,1,1)$

$$r_1 = B_1 = (1,1,1,1)$$

$$r_2 = B_2 - \frac{\langle B_2, r_1 \rangle}{|x_1|^2} r_1$$

$$\langle B_2, r_1 \rangle = 5 \quad |r_1|^2 = 4$$

$$\therefore r_2 = (0,1,2,2) - \frac{5}{4}(1,1,1,1)$$

$$= \left(-\frac{5}{4}, -\frac{1}{4}, \frac{3}{4}, \frac{3}{4}\right)$$

$$|r_2|^2 = \frac{44}{16} \qquad \therefore r_2 = \frac{\sqrt{11}}{2}$$

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$$\gamma_3 = \beta_3 - \frac{\langle \beta_3, \gamma_1 \rangle}{|\gamma_1|^2}\gamma_1 - \frac{\langle \beta_3, \gamma_2 \rangle}{|\gamma_2|^2}\gamma_2$$

$$\langle \beta_3, \gamma_1 \rangle = 2 \qquad \langle \beta_3, \gamma_2 \rangle = \frac{3}{2}$$

$$\therefore \gamma_3 = (0,0,1,1) - \frac{2}{42}(1,1,1,1) - \frac{3 \times 16}{2 \cdot 44}\left(\frac{-5}{4}, \frac{-1}{4}, \frac{3}{4}, \frac{3}{4}\right)$$

$$= \left(-\frac{1}{2} + \frac{15}{22}, -\frac{1}{2} + \frac{3}{22}, \frac{1}{2} - \frac{9}{22}, \frac{1}{2} - \frac{9}{22}\right)$$

$$= \left(\frac{4}{22}, \frac{-8}{22}, \frac{2}{22}, \frac{2}{22}\right)$$

$$= \left(\frac{2}{11}, \frac{-4}{11}, \frac{1}{11}, \frac{1}{11}\right)$$

$$|\gamma_3| = \frac{1}{\sqrt{22}}$$

$\therefore S = \left\{ \frac{\gamma_1}{|\gamma_1|}, \frac{\gamma_2}{|\gamma_2|}, \frac{\gamma_3}{|\gamma_3|} \right\}$ is req. orthonormal basis

$$S = \left\{ \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right), \left(\frac{-5}{2\sqrt{11}}, \frac{-1}{2\sqrt{11}}, \frac{3}{2\sqrt{11}}, \frac{3}{2\sqrt{11}}\right) \left(\frac{2\sqrt{2}}{\sqrt{11}}, \frac{-4\sqrt{2}}{\sqrt{11}}, \frac{\sqrt{2}}{11}, \frac{\sqrt{2}}{11}\right) \right\}$$

is req. orthonormal basis