MAE 510 – NUMERICAL METHODS FOR MOVING INTERFACES

SPRING 2016 – HW #1

03/07/2016

SUBMITTED BY

SIDDHANT S APHALE

PERSON # 50164327

1. Mean Curvature Expression:

Consider an interface with a parametric expression of $r(s) = (x(s), y(s))$, where s is the parameter. Curvature, κ(s) is defined as the magnitude of the derivative of the unit tangent vector with respect to arc length.

The arc-length parameter is defined as

$$s(t) = \int \|\dot{r}\| dt = \int \sqrt{\dot{x}^2 + \dot{y}^2}\, dt \quad (1)$$

where $\dot{r}$ represents dr/dt.

Thus, we have

$$\frac{ds}{dt} = \sqrt{\dot{x}^2 + \dot{y}^2} \quad (2)$$

Using the chain rule we can write,

$$\frac{dr}{dt} = \frac{dr}{ds}\frac{ds}{dt} = \frac{ds}{dt}T \quad (3)$$

$$\frac{d^2r}{dt^2} = \frac{dr}{ds}\frac{d^2s}{dt^2} + \left(\frac{ds}{dt}\right)^2 \frac{d^2r}{ds^2} = \frac{d^2s}{dt^2}T + \left(\frac{ds}{dt}\right)^2 \kappa N \quad (4)$$

Now we take scalar product of both sides of this expression with N. T and N are both unit lengths and are perpendicular. Thus we have, $\langle T, T\rangle = \langle N, N\rangle = 1$ and $\langle T, N\rangle = 0$. Thus we can write,

$$\langle \frac{d^2r}{dt^2}, N\rangle = \kappa \left(\frac{ds}{dt}\right)^2 \quad (5)$$

No we know that $\frac{ds}{dt} = (x\dot{^2} + y\dot{^2})^{1/2}$ and $(ds/dt)^2 = x\dot{^2} + y\dot{^2}$.

Using this we now can write:

$$\kappa\left(x\dot{^2} + \dot{y}^2\right) = \langle \frac{d^2r}{dt^2}, N\rangle \quad (6)$$

$$\kappa = \frac{1}{x\dot{^2} + \dot{y}^2} \langle (\ddot{x}, \ddot{y}), \frac{(-\dot{y}, \dot{x})}{\sqrt{\dot{x}^2 + \dot{y}^2}} \rangle \quad (7)$$

$$\kappa(s) = \frac{1}{x\dot{^2} + \dot{y}^2}\left(\frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2}}\right) \quad (8)$$

$$\kappa(s) = \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{(\dot{x}^2 + \dot{y}^2)^{3/2}} \quad (9)$$

Equation 9 is the mean curvature expression for an interface with a parametric representation of $r(s) = (x(s), y(s))$.

To verify this expression, consider the curvature of ellipse given by $r(s) = (a\cos(s), b\sin(s))$

Now consider the mean curvature expression given by Eq. 9.

Now $$\left(x(s), y(s)\right) = \left(a\cos(s), b\sin(s)\right)$$

$$(\dot{x}) = (-a\sin(s))$$

$$(\dot{y}) = (\mathrm{b}\cos(s))$$

$$(\ddot{x}) = (-a\cos(s))$$

$$(\ddot{y}) = (-\mathrm{b}\sin(s))$$

Now, $$\kappa(s) = \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{\left(x\dot{^2} + \dot{y}^2\right)^{3/2}}$$

Substituting the values in equation 9 we get,

$$\kappa(s) = \frac{ab}{(b^2cos^2(s) + a^2sin^2(s))^{3/2}}$$

The mean curvature expression is also given as follows:

$$\kappa(s) = -\frac{\|r'(s) \times r''(s)\|}{\|r'(s)\|^3}$$

2. 128 marker particles without correction.

128 marker particles are used to simulate the evolution of the interface. Time steps are varied and 3 simulations are performed. The evolution of the interface is studied for these time steps. The interface initially at t=0 is as shown in Fig. 1. There are 128 marker particles spaced uniformly from each other over the interface.

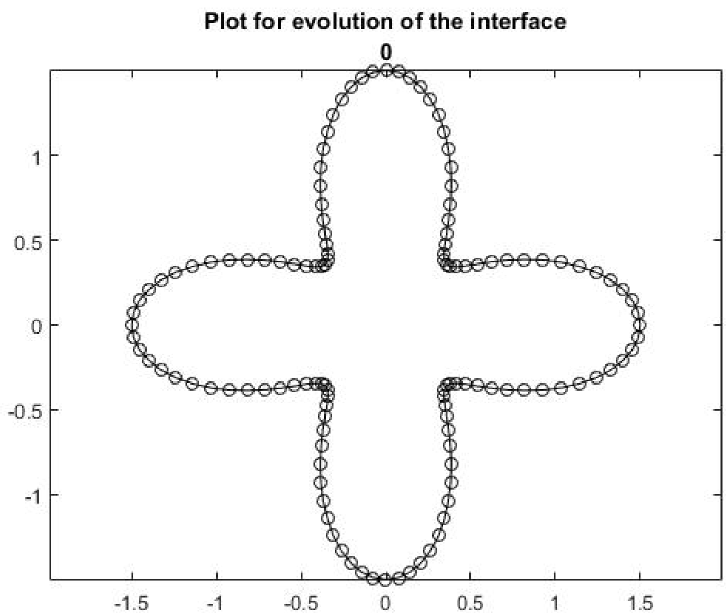
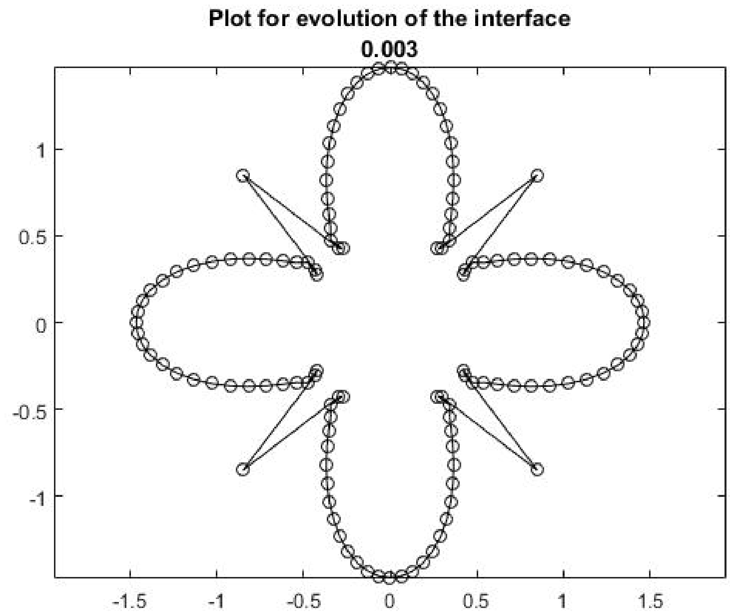


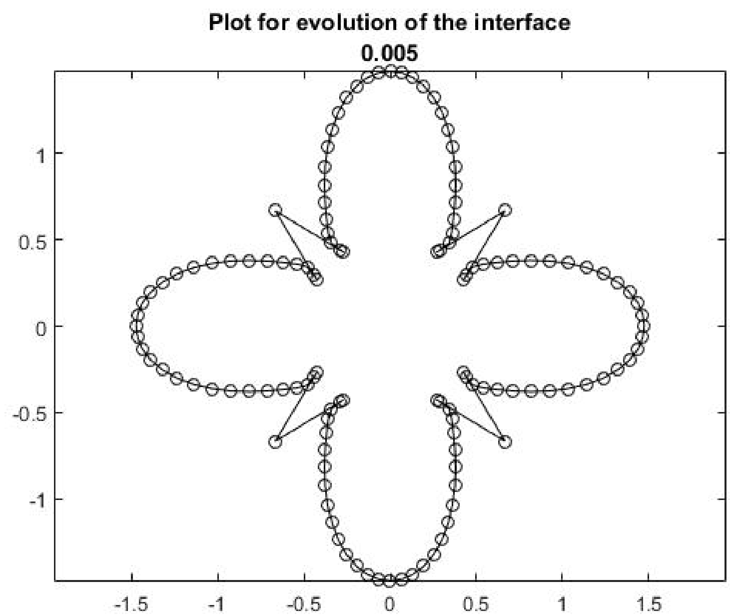
Figure 1 Interface at time t=0, 128 Marker Particles

2.1 Case-1- Time step of 10^{-3} 128 Marker Particles:

In this case, the evolution of the interface is performed with a time step of 10^{-3}. No correction method is used to simulate this system. The system evolution was set from 0 to 0.5 with the time step of 10^{-3}. The evolution of the interface was plotted at different times to study the movement of particles. Successive plots were taken at different time limits to study the evolution of the interface as shown in Fig. 2. It can be observed from Fig. 2(a) that at t=0.003, the particles at each of the 4 side corners move normally outward. At t=0.005 this particle now moves little inward as seen in Fig. 2(b). This particle continues moving inward as time moves forward as seen in Fig. 2(c) and 2(d) which are plotted at t=0.015 and t=0.025. It can be observed in Fig. 2(e) that the particles which moved outwards are now completely aligned and other particles are moving trying to arrange themselves. However, at t=0.5 Fig. 2(f) the particles have come to steady state and continue to be at steady state at t=2, t=10 Fig. 2(g) and 2(h). Clearly the interface does not form a circular shape.

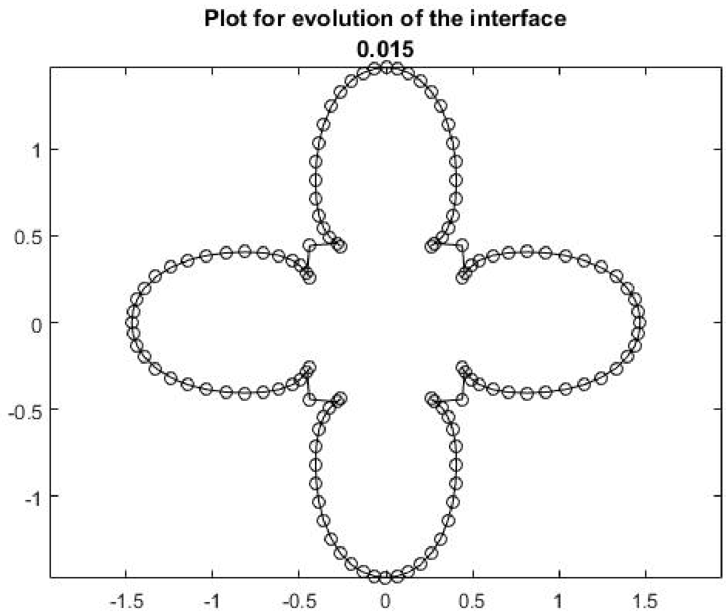


(a)

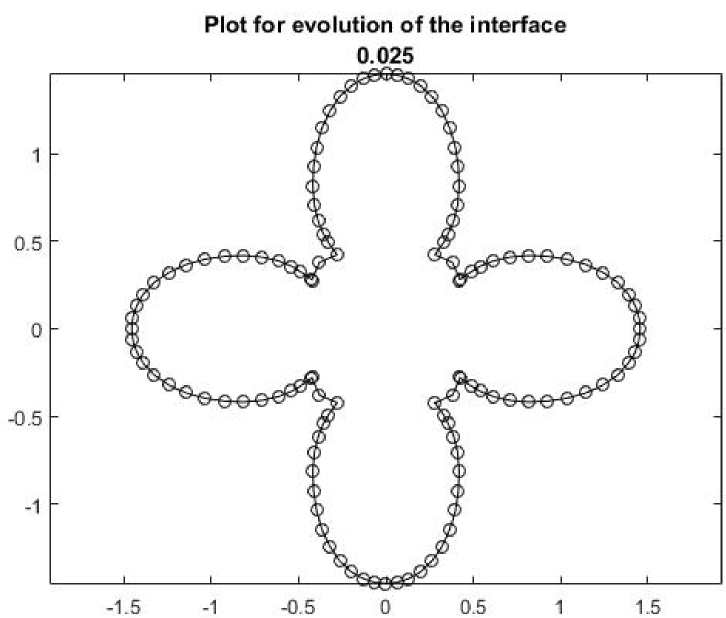


(b)

Figure 2 (a) Interface with 128 marker particles dt=10^{-3}, t=0.003, (b) Interface with 128 marker particles dt=10^{-3}, t=0.005

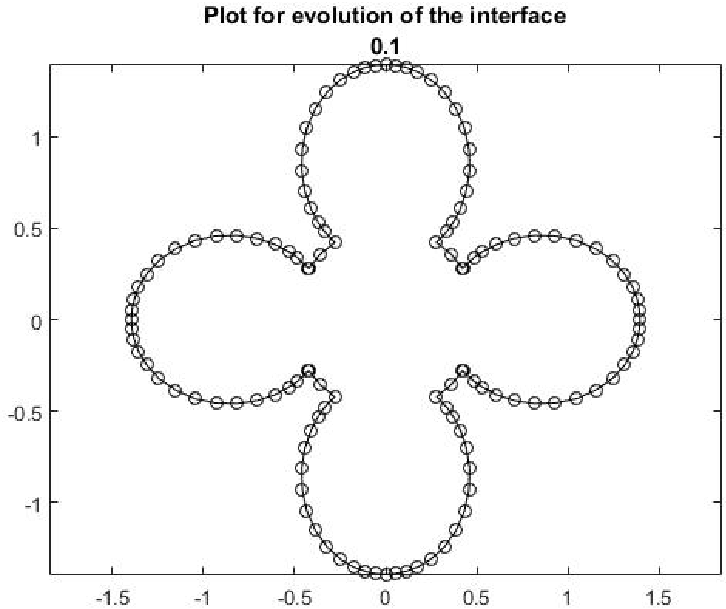


(c)

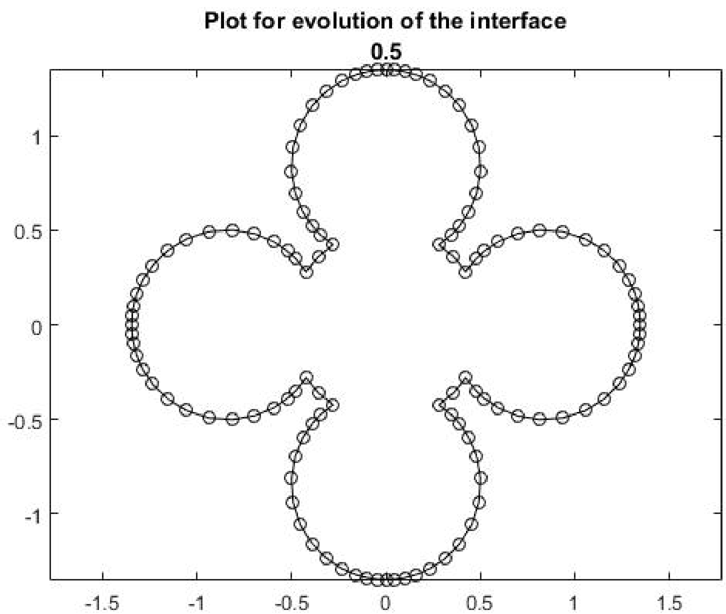


(d)

Figure 2 (c) Interface with 128 marker particles dt=10^{-3}, t=0.015, (d) Interface with 128 marker particles dt=10^{-3}, t=0.025

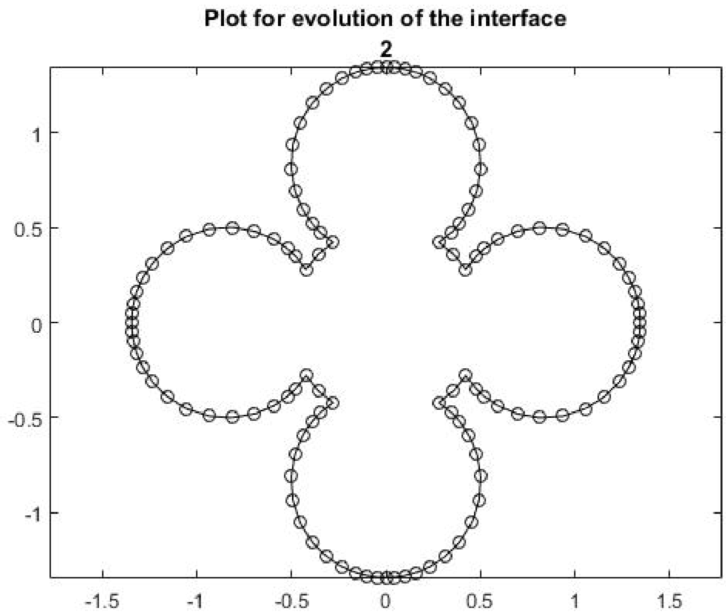


(e)

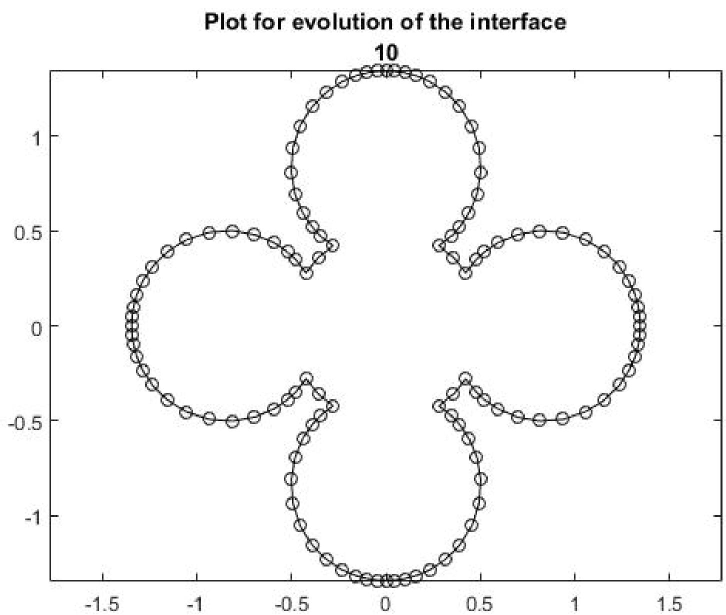


(f)

Figure 2 (e) Interface with 128 marker particles dt=10^{-3}, t=0.1, (f) Interface with 128 marker particles dt=10^{-3}, t=0.5



(g)

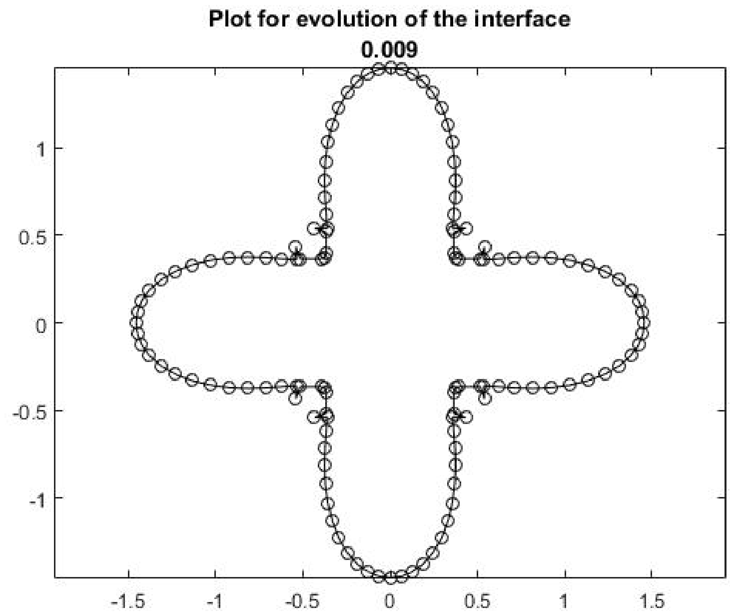


(h)

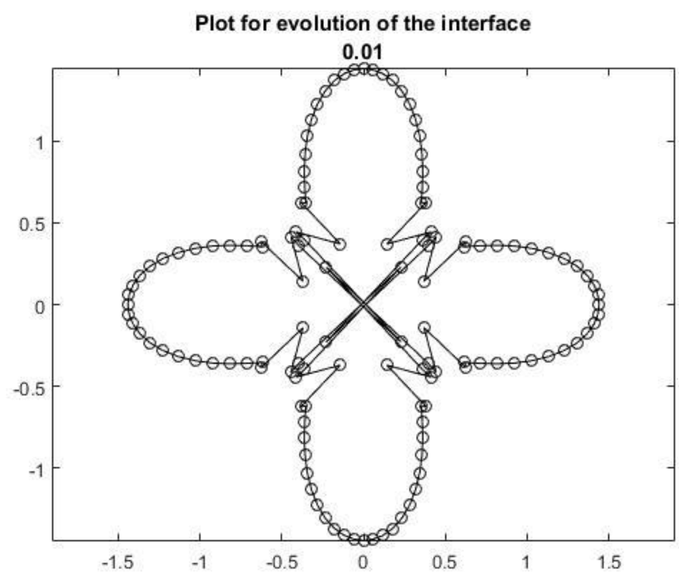
Figure 2 (e) Interface with 128 marker particles dt=10^{-3}, t=2, (f) Interface with 128 marker particles dt=10^{-3}, t=10

2.2 Case-1- Time step of 5 x 10^{-4} 128 Marker Particles:

In second case, we keep the number of marker particles same however, we change the time steps. We reduce the time steps to 5 x 10^{-4}. The plots were generated at same time iterations as in above. At t=0.009, Fig. 3(a) we observe that the particles have started moving and some particles have moved out of the original interface while some have tried to merge with adjacent particles. At t=0.01, the particles at the corner have moved to the center and accordingly other particles have changed their position Fig. 3(b). In Fig. 3(c), at t=0.015 we see that now the particles are trying to move so the particles which moved out are at same position. At t=0.025, Fig. 3(d) there is further abrupt behavior of the particles. At t=0.1, Fig. 3(e) there is further abrupt behavior of the particles and this continues without taking the interface to steady state. Fig. 3(f)-3(h).

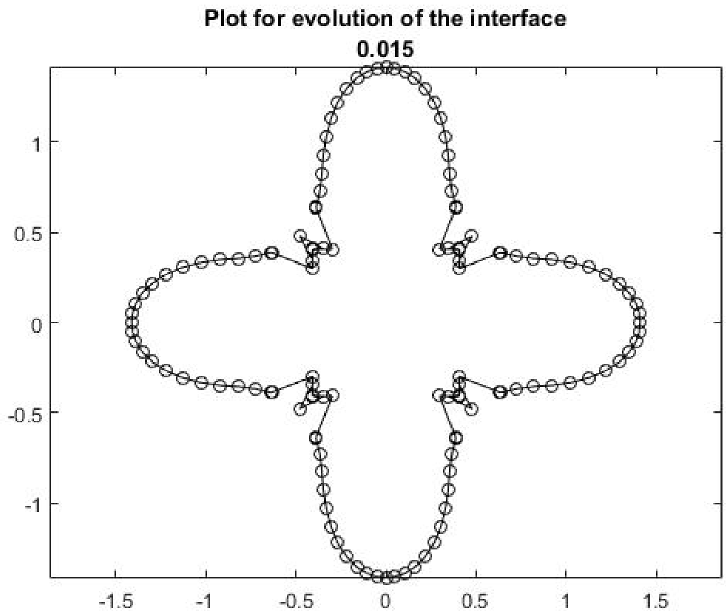


(a)

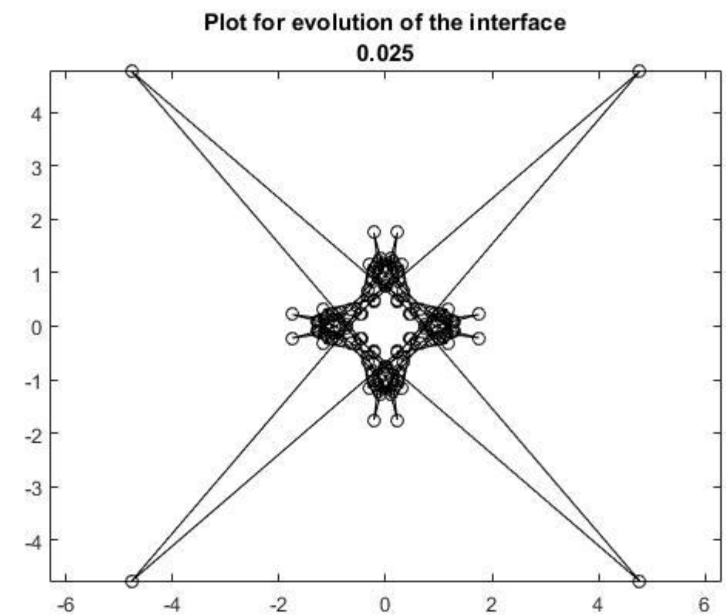


(b)

Figure 3 (a) Interface with 128 marker particles dt=5x10^{-4}, t=0.009, (b) Interface with 128 marker particles dt=5x10^{-4}, t=0.01

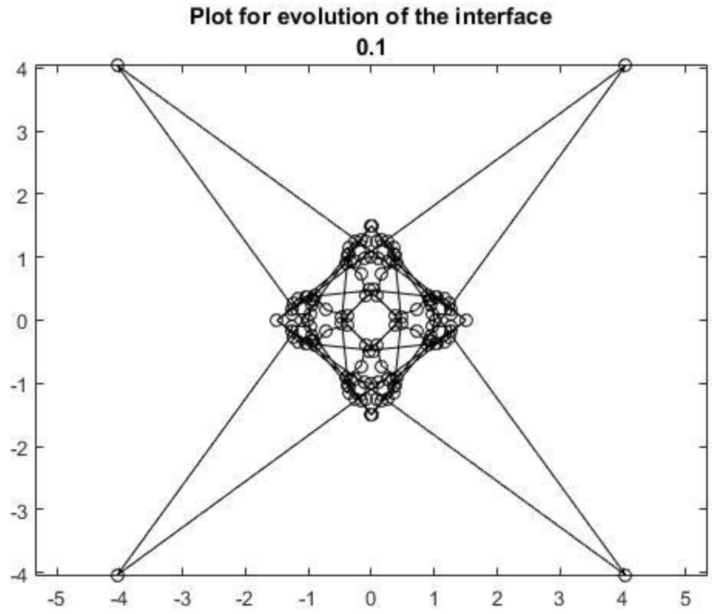


(c)

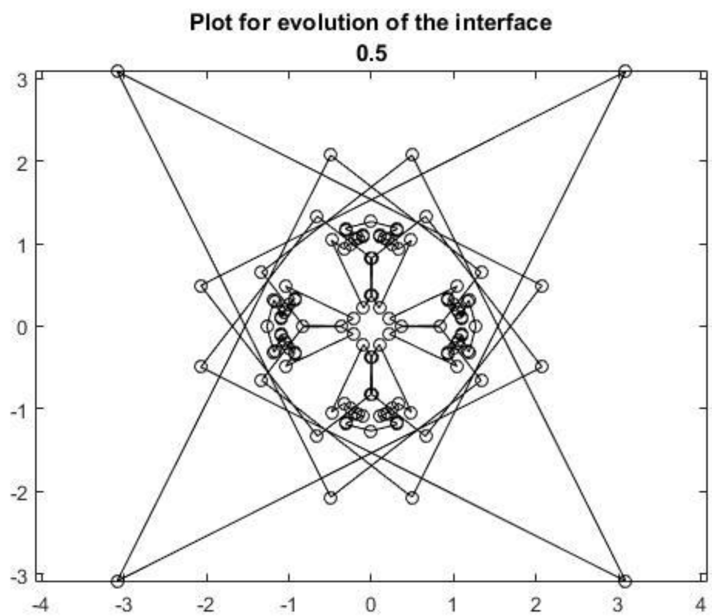


(d)

Figure 3 (c) Interface with 128 marker particles dt=$5x10^{-4}$, t=0.015, (d) Interface with 128 marker particles dt=$5x10^{-4}$, t=0.025

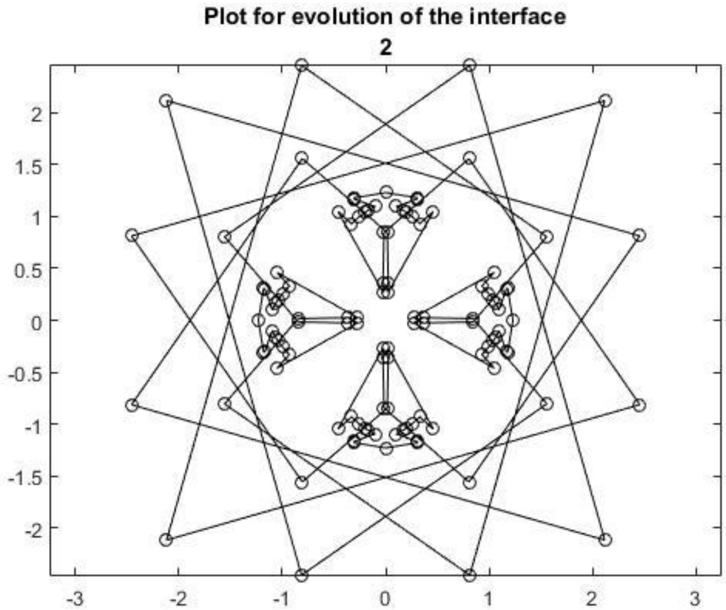


(e)

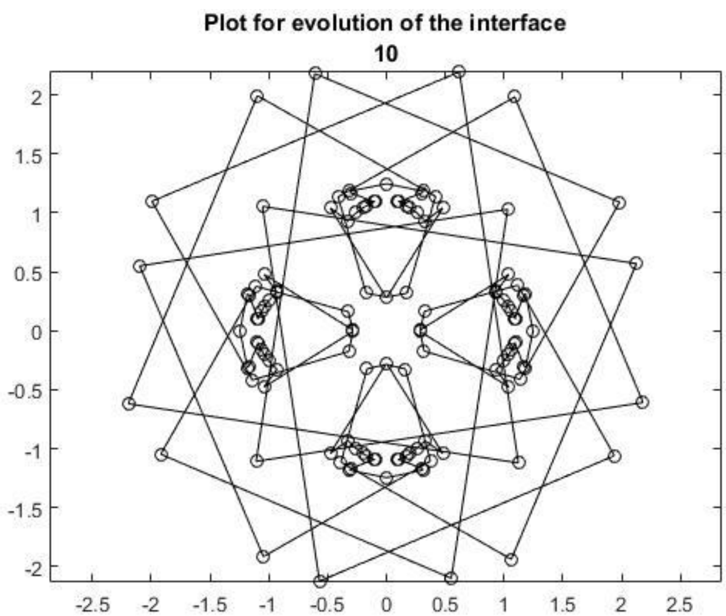


(f)

Figure 3 (e) Interface with 128 marker particles dt=5x10^{-4}, t=0.1, (f) Interface with 128 marker particles dt=5x10^{-4}, t=0.5



(g)

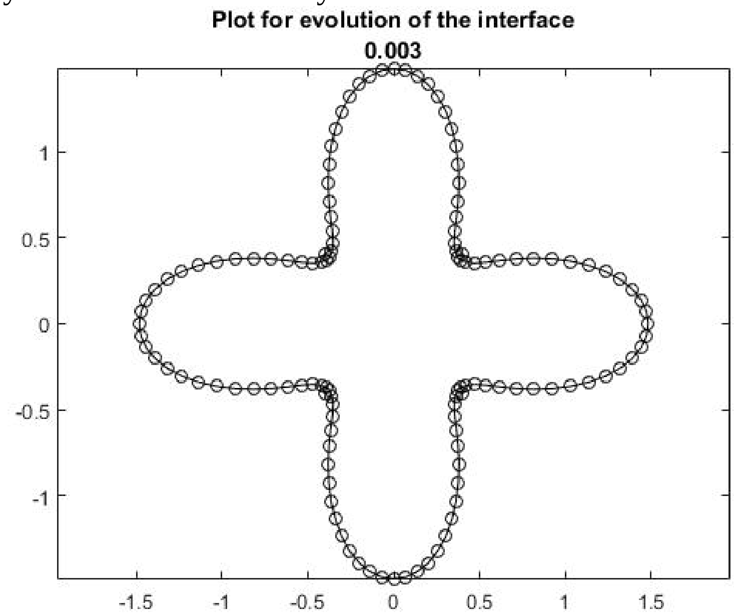


(h)

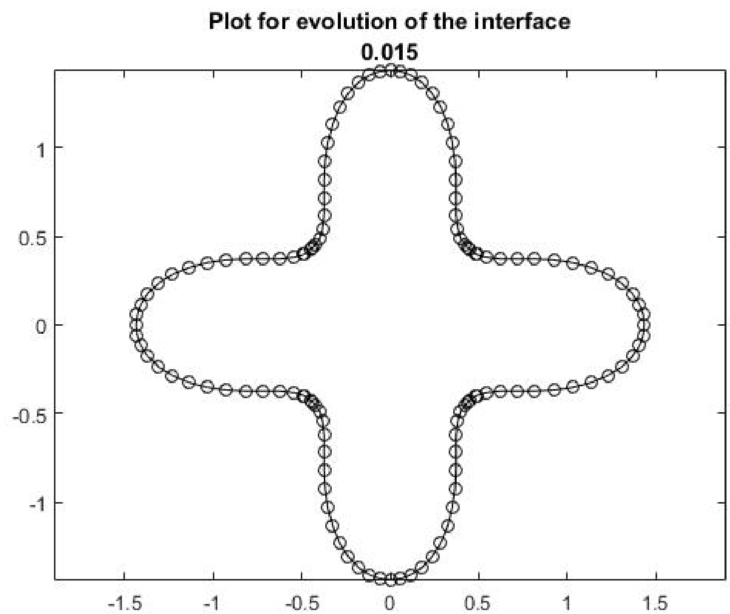
Figure 3 (f) Interface with 128 marker particles dt=5x10^{-4}, t=2, (h) Interface with 128 marker particles dt=5x10^{-4}, t=10

2.3 Case-3- Time step of 2.5 x 10^{-4} and 128 Marker Particles:

In Case 3, we again keep the same number of Marker particles i.e. 128. Here we have further reduced the time steps to 2.5 x 10^{-4}. We again study the evolution of the interface at various time intervals till the interface reaches steady state. In this case, the interface achieves the circular shape as it reaches steady state. As seen in Fig. 4(a), at t=0.003, the particles start to move outwards normally. Further, at t=0.015, Fig. 4(b), the particles move more outward. At t=0.025, the particles have moved more outwards as seen in Fig. 4(c). At t=0.1, Fig. 4(d) we see that the particles have significantly moved normally outwards at uniform velocity. Further, at t=0.2 (Fig. 4(e)), we see that the interface is slowly approaching towards steady state and finally at t=0.5 (Fig. 4(f)), it achieves steady state and forms circular shape. By plotting the same evolution at t=2 Fig. 4(g) and t=5 Fig. 4(h) we see that there is no change and thus can conclude that the interface is fully evolved and is at steady state.

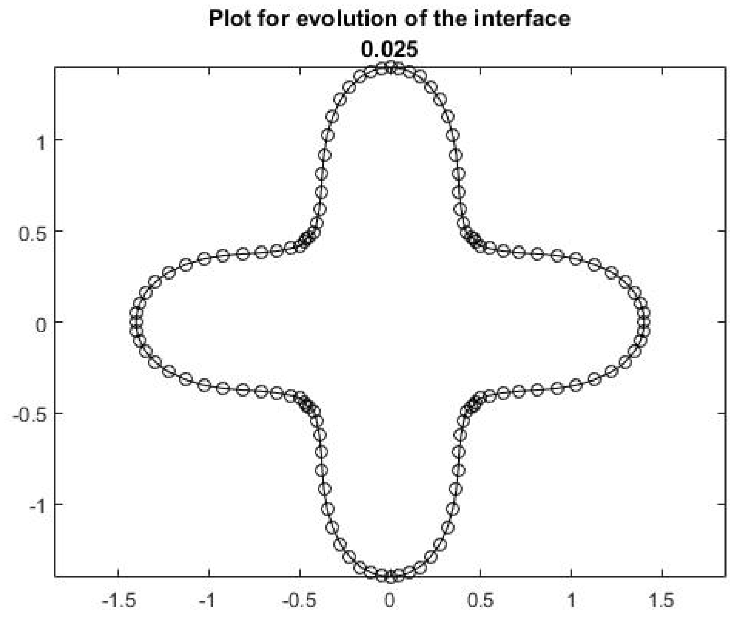


(a)

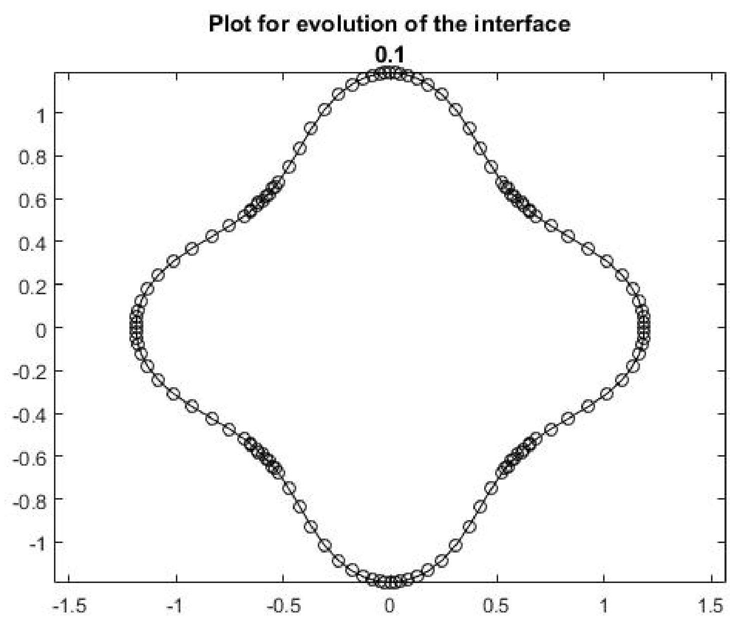


(b)

Figure 4 (a) Interface with 128 marker particles dt=2.5x10^{-4}, t=0.003, (b) Interface with 128 marker particles dt=2.5x10^{-4}, t=0.015

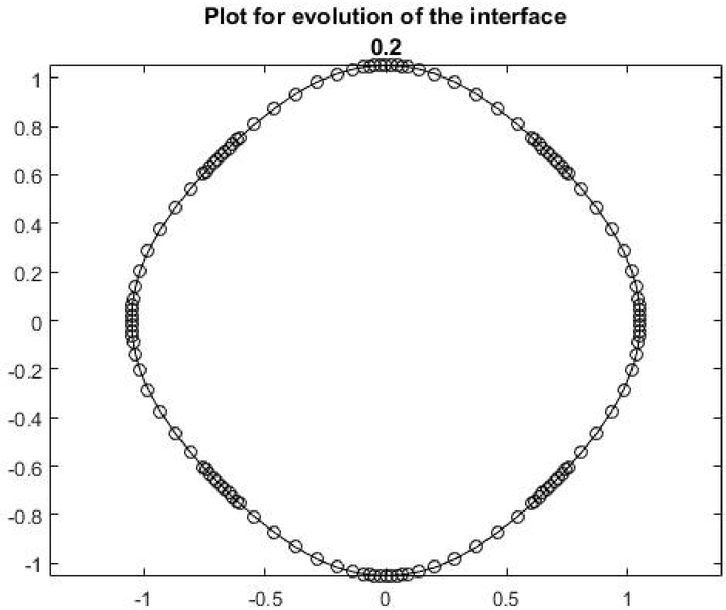


(c)

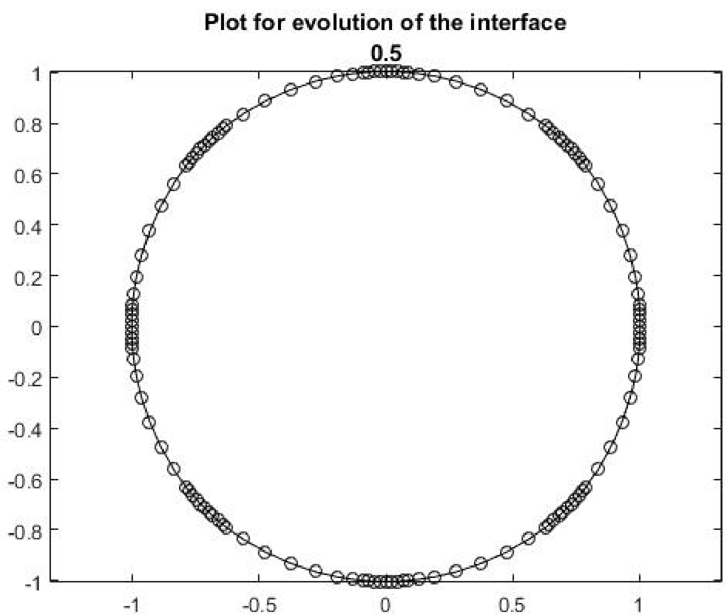


(d)

Figure 4 (c) Interface with 128 marker particles dt=2.5×10^{-4}, t=0.025, (d) Interface with 128 marker particles dt=2.5×10^{-4}, t=0.1

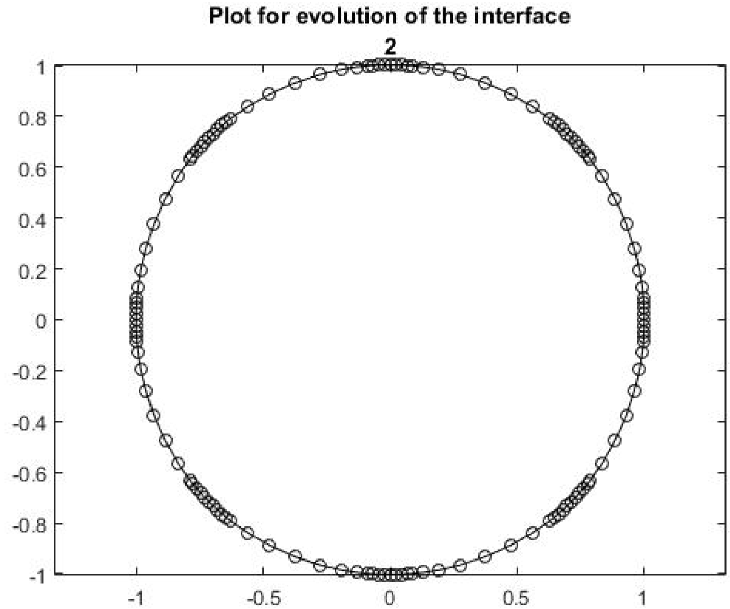


(e)

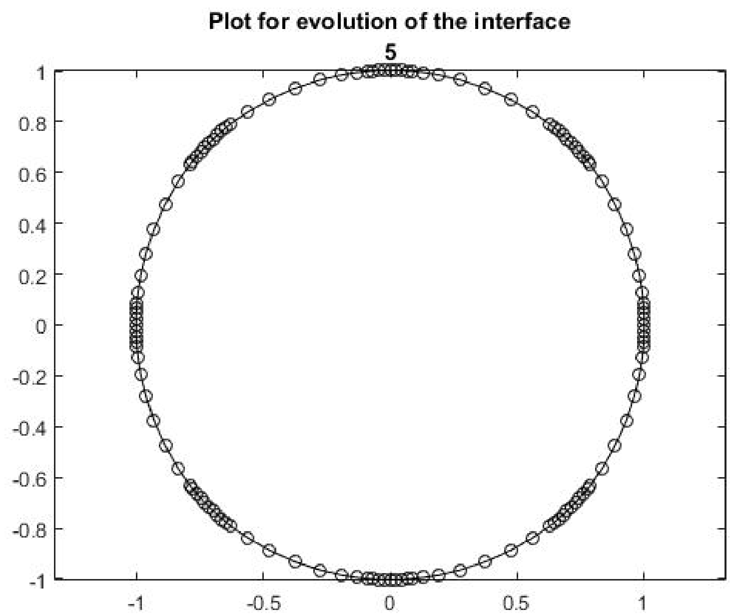


(f)

Figure 4 (e) Interface with 128 marker particles dt=2.5×10^{-4}, t=0.2, (f) Interface with 128 marker particles dt=2.5×10^{-4}, t=0.5



(g)



(h)

Figure 4 (g) Interface with 128 marker particles dt=2.5x10^{-4}, t=2, (h) Interface with 128 marker particles dt=2.5x10^{-4}, t=5

Thus it can be concluded that as we decrease the time-step, the movement of the particles become uniform and the interface achieves steady state as lower time steps. However, the particles are not uniformly spaced from each other even at steady state.

3.1 Case-1- Time step of 10^{-3} and 129 Marker Particles:

In this cases we have increased the number of marker particles. Now the number of marker particles on the interface are 129. The interface at time t=0 is as show in Fig. 5 below.

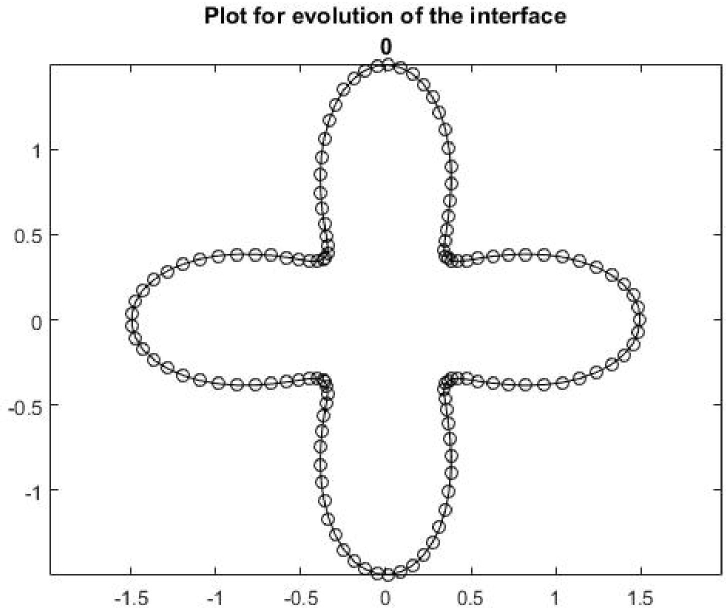
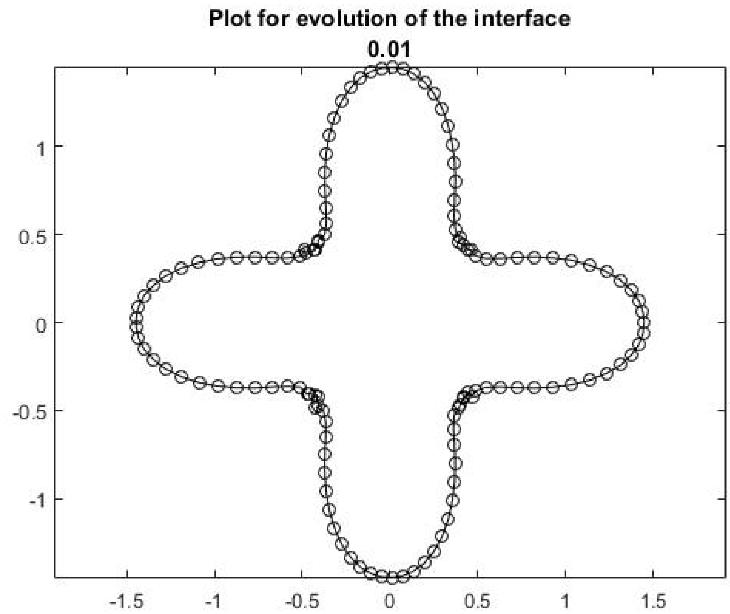
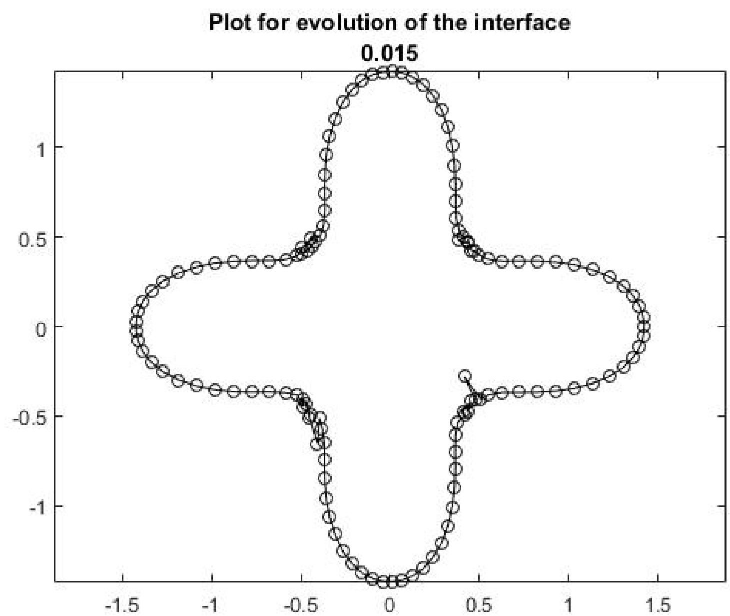


Figure 5 Interface at t=0 , 129 Marker Particles

In this case also, the same methodology was followed as in Case 2.1 – Case 2.3. No correction was performed. The plots were obtained at various times to study the evolution of the system and was compared. Comparison was also done with varying the time steps. Now the time step is 10^{-3}. In this case, the particles do not move uniformly. Due to the addition of 1 extra marker particle, the system becomes unstable and the interface tries to accommodate the additional particle. Furthermore, the system does not give a perfect circle after t=0.5. Hence, the evolution was studied at t=5, but the system becomes worse. The system does not achieve steady state Fig. 6(a-h).

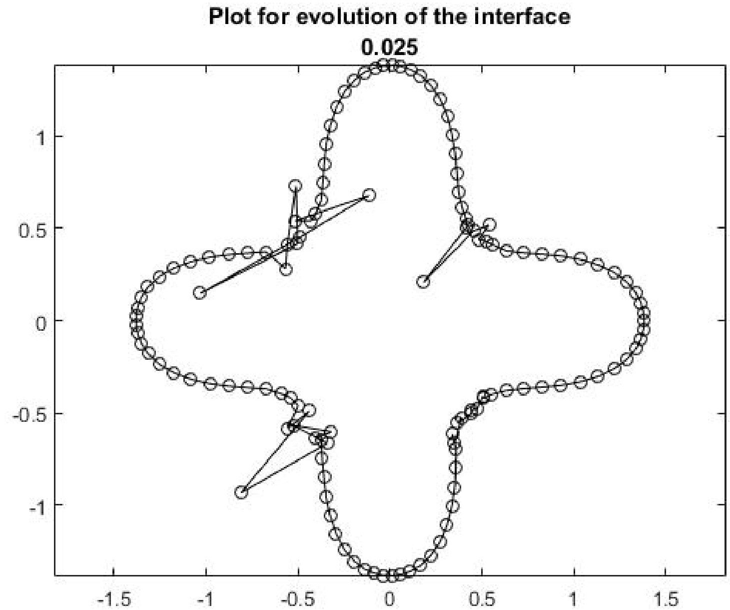


(a)

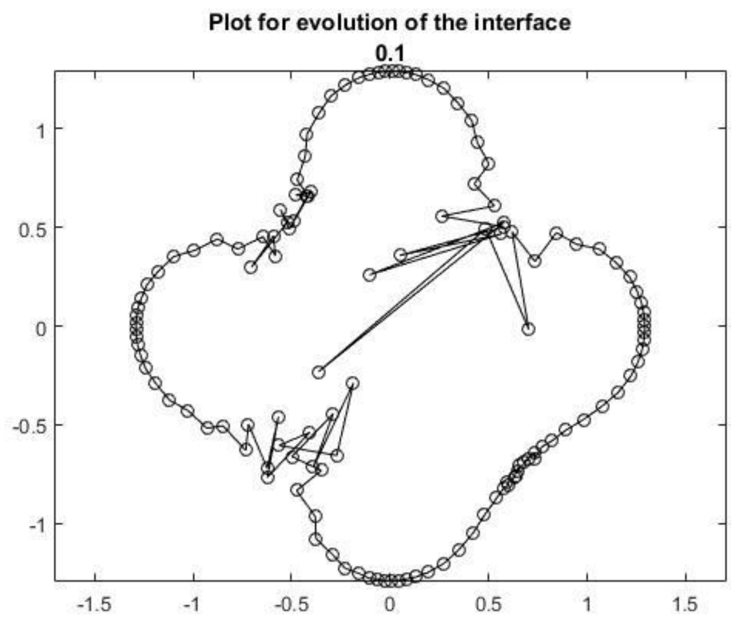


(b)

Figure 6 (a) Interface with 129 marker particles dt=10^{-3}, t=0.01, (b) Interface with 129 marker particles dt=10^{-3}, t=0.015

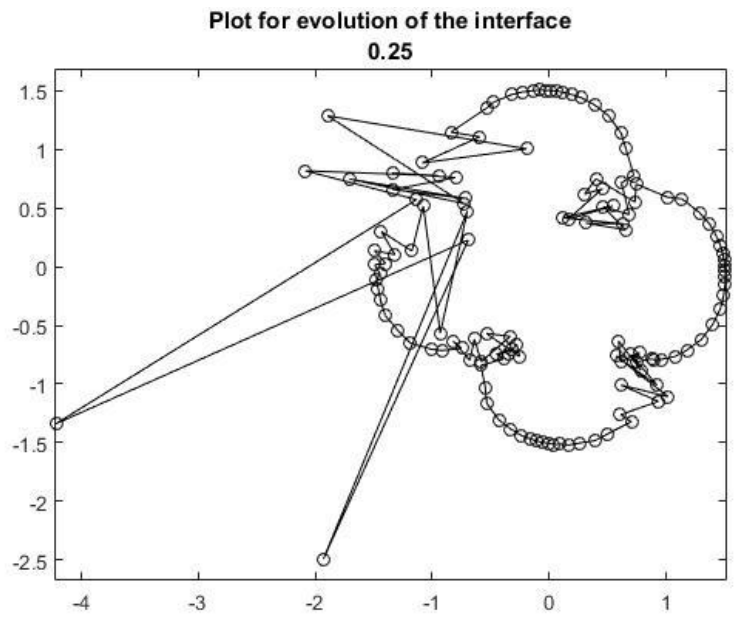


(c)



(d)

Figure 6 (c) Interface with 129 marker particles dt=10^{-3}, t=0.025, (d) Interface with 129 marker particles dt=10^{-3}, t=0.1



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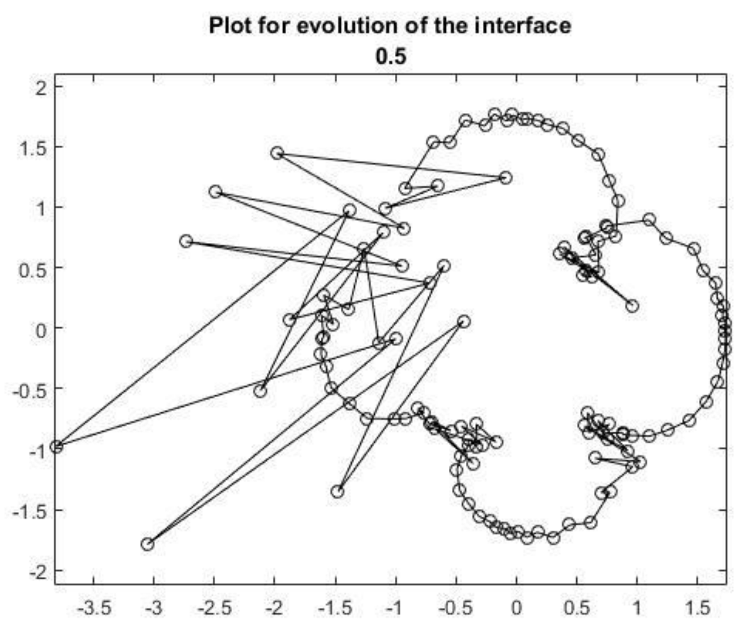
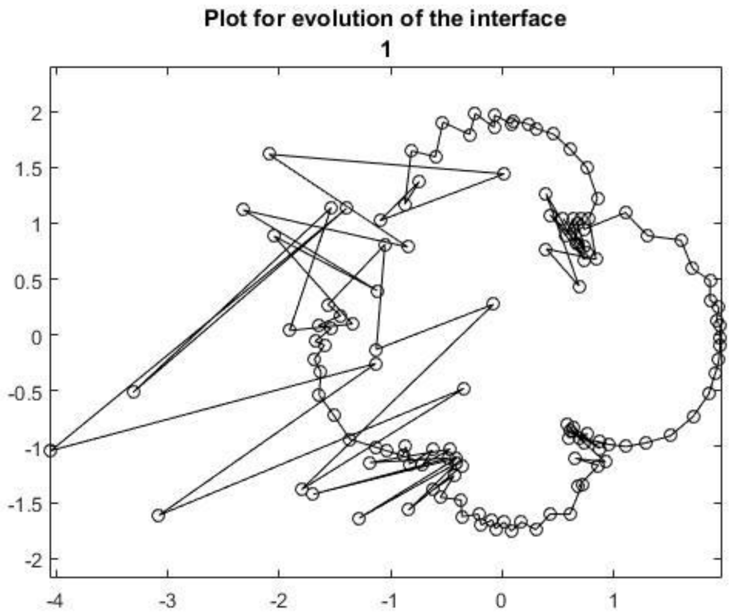
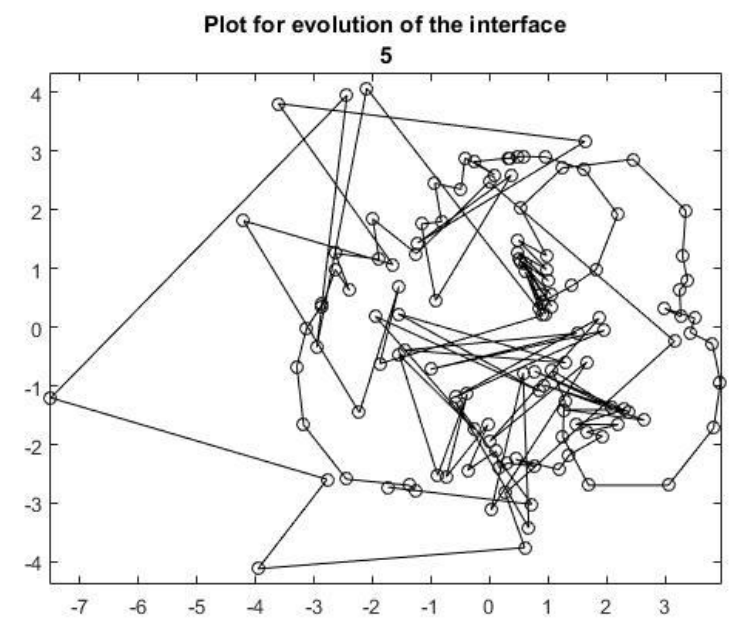


(f)

Figure 6 (e) Interface with 129 marker particles dt=10^{-3}, t=0.25, (f) Interface with 129 marker particles dt=10^{-3}, t=0.5



(g)

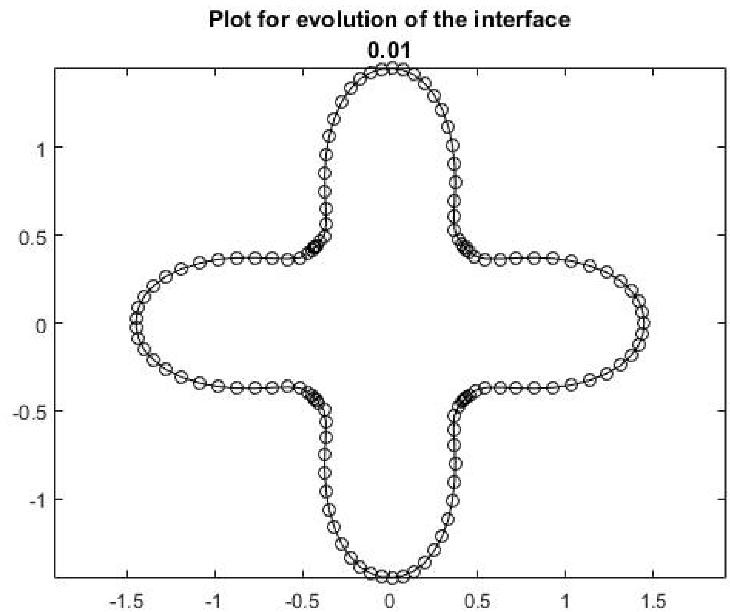


(h)

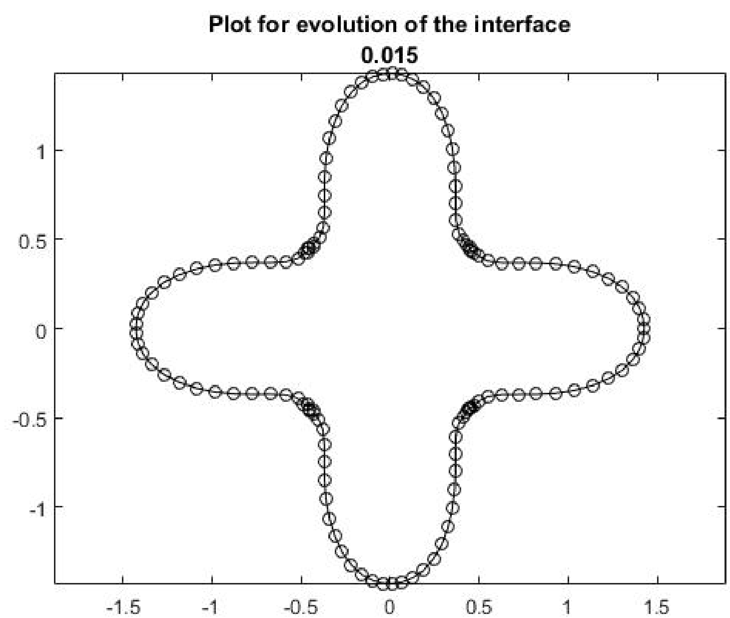
Figure 6 (g) Interface with 129 marker particles dt=10^{-3}, t=1, (h) Interface with 129 marker particles dt=10^{-3}, t=5

3.2 Case-2- Time step of 5×10^{-4} and 129 Marker Particles:

Now we reduce the time step to 5×10^{-4} by keeping the number of the marker particles. Again we plot the evolution at the times defined in above case. In this case, we can observe that the system is again finding it difficult to accommodate the additional 129th particle while evolving. Again, we observe same evolution as in Case above. However, initially the particles were stable till t=0.03. But after that the particles were not able to accommodate the extra particle. The system was run for t=5 but as we increase the time the system breaks even further. The evolution does not go to steady state. Fig. 7(a-h).

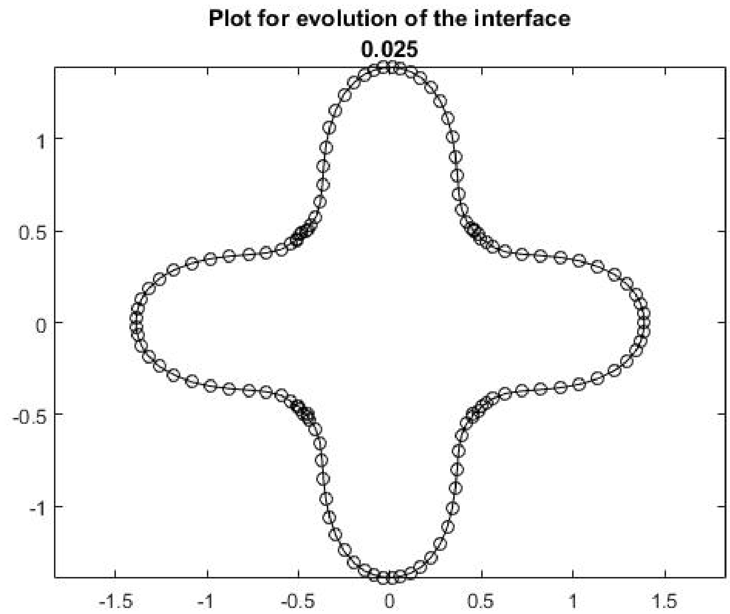


(a)

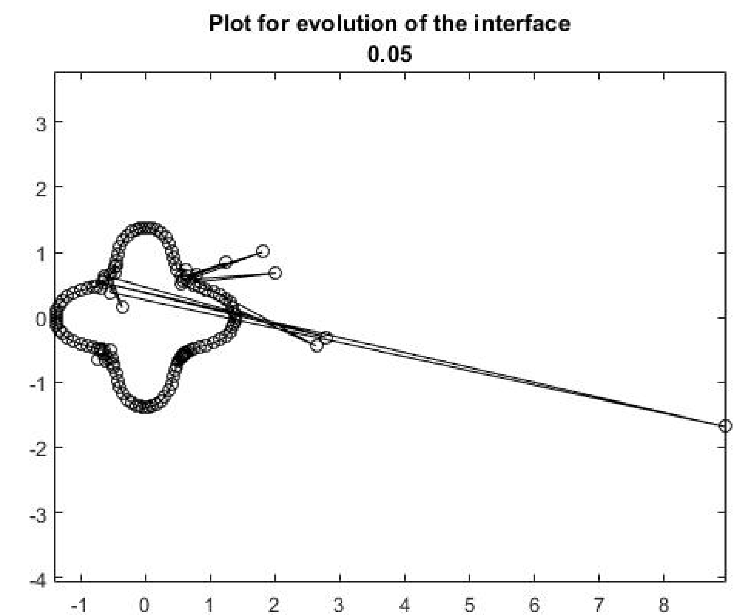


(b)

Figure 7 (a) Interface with 129 marker particles dt=5x10^{-4}, t=0.01, (b) Interface with 129 marker particles dt=5x10^{-4}, t=0.015

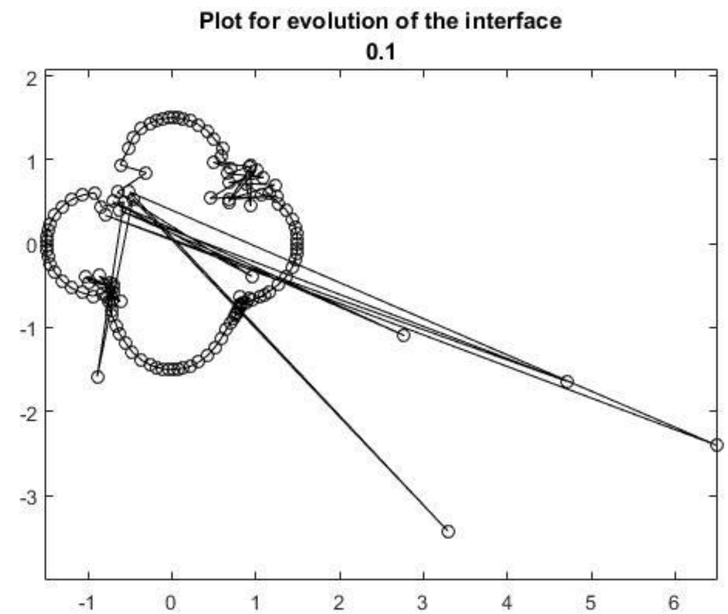


(c)

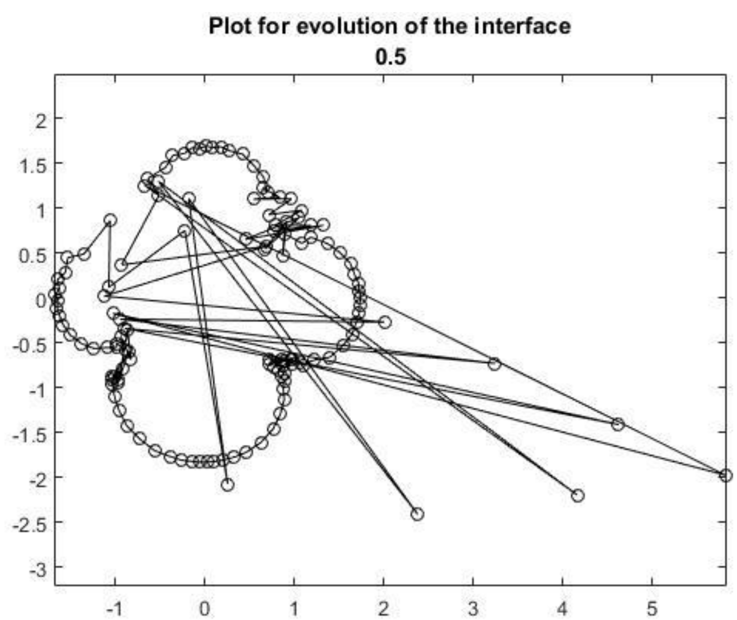


(d)

Figure 7 (c) Interface with 129 marker particles dt=5x10^{-4}, t=0.025, (d) Interface with 129 marker particles dt=5x10^{-4}, t=0.05

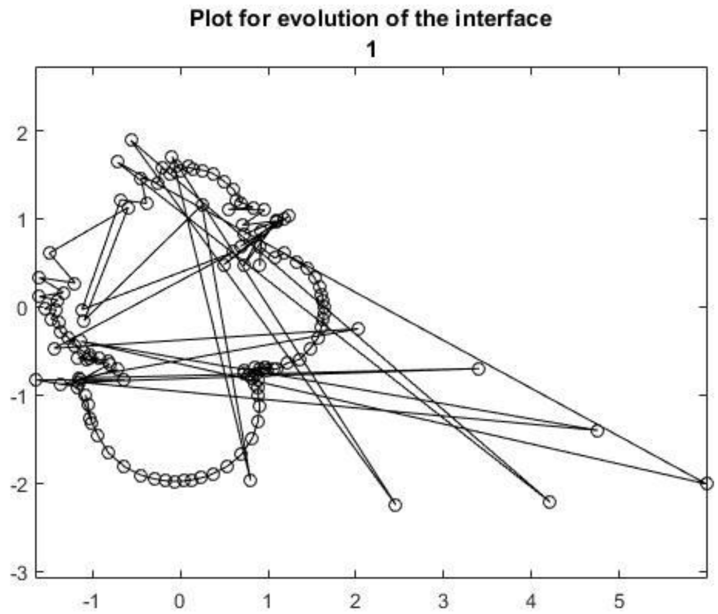


(e)

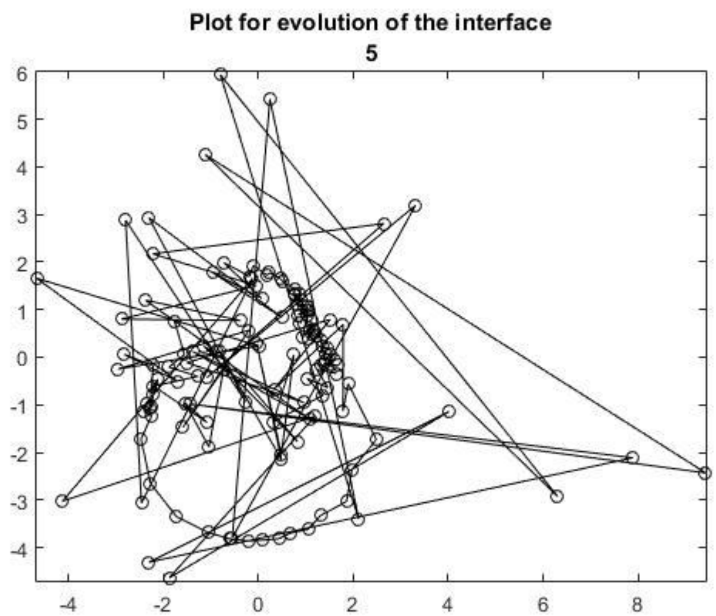


(f)

Figure 7 (e) Interface with 129 marker particles dt=5x10^{-4}, t=0.1, (f) Interface with 129 marker particles dt=5x10^{-4}, t=0.5



(g)



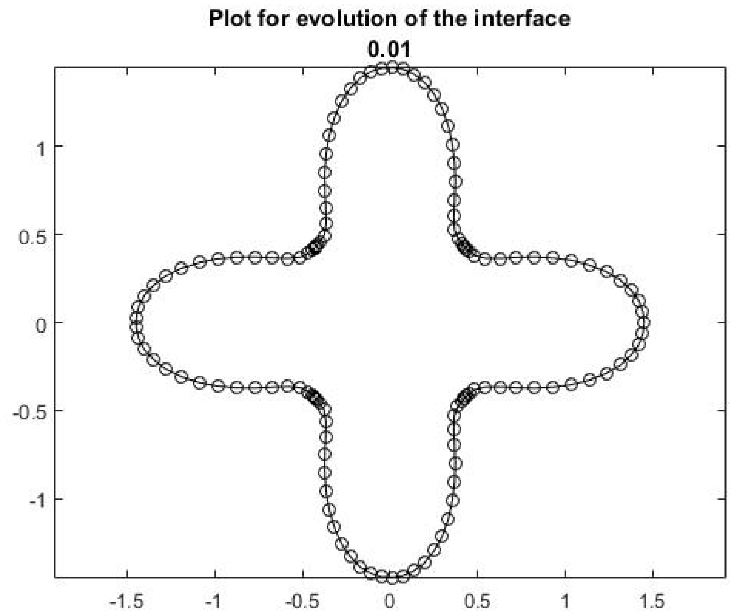
(h)

Figure 7 (g) Interface with 129 marker particles dt=5x10^{-4}, t=1, (h) Interface with 129 marker particles dt=5x10^{-4}, t=5

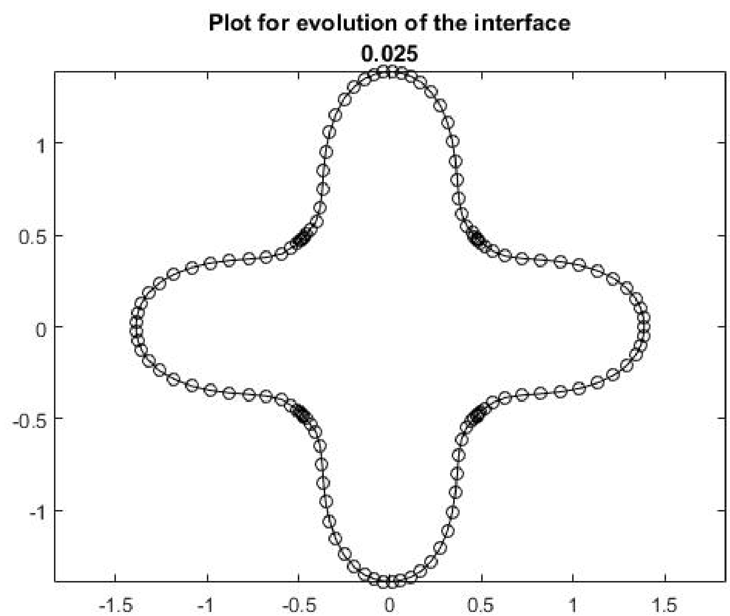
3.3 Case-2- Time step of 2.5 x 10^{-4} and 129 Marker Particles:

Now by keeping the number of marker particles same we decrease the time step to 2.5 x 10^{-4}. We again study the evolution of the system for same time as in above cases. We observe significant improvement in this case. The system has actually accommodated the additional 129^{th} particle. Unlike in case 3.1 and

3.2, the 129th particle never moves out of the system. The evolution takes place uniformly and no oscillations are observed. Also the system starts to go to steady state after t=0.25. We can see that the system is fully evolved at t=0.3 and remains same at t=5 Fig-8(a-h). However, the particles are not uniformly spaced and the evolution takes lot of time to go to steady state. Thus by adding correction factor, we may increase the accuracy as well as reduce the time. Compared to cases 3.1 and 3.2 we can conclude that as we reduce the time step, the system oscillations are completely ruled out and system goes to steady state

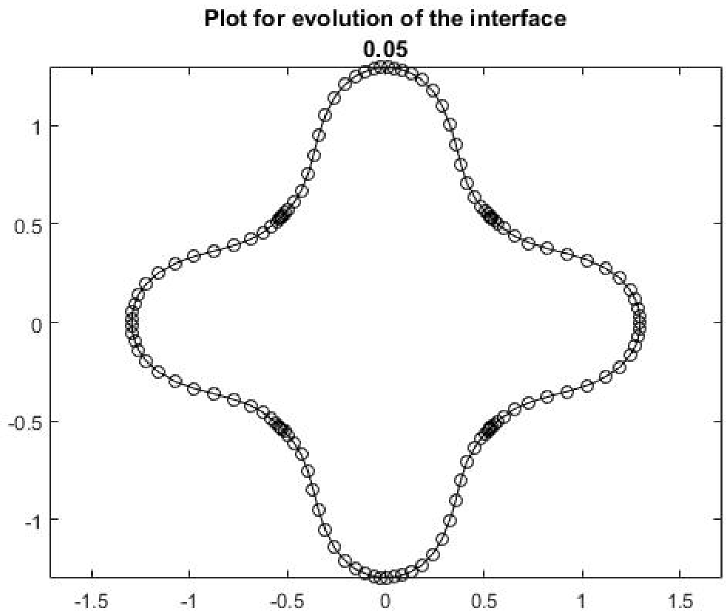


(a)

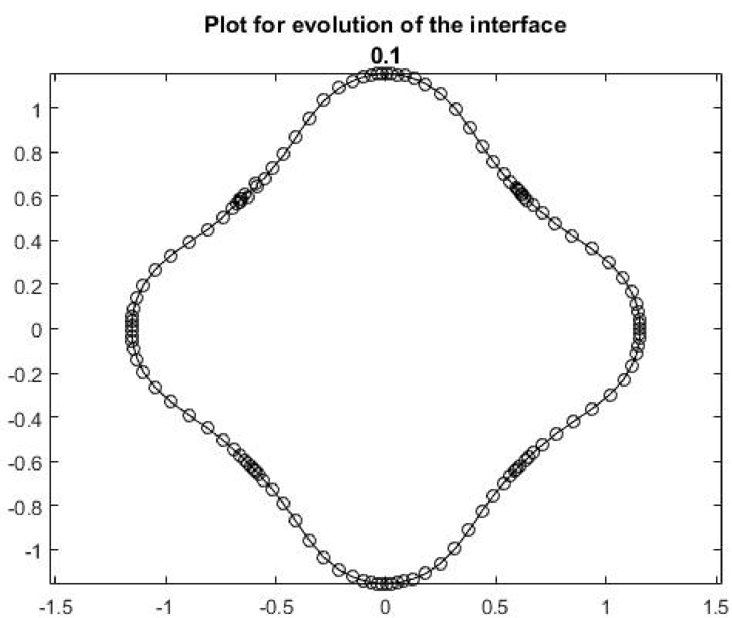


(b)

Figure 8 (a) Interface with 129 marker particles dt=2.5x10^{-4}, t=0.01, (h) Interface with 129 marker particles dt=2.5x10^{-4}, t=0.025

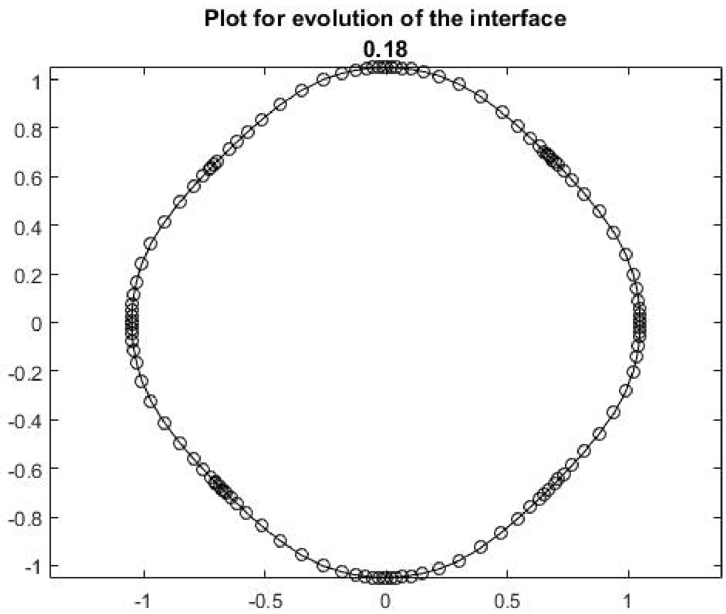


(c)

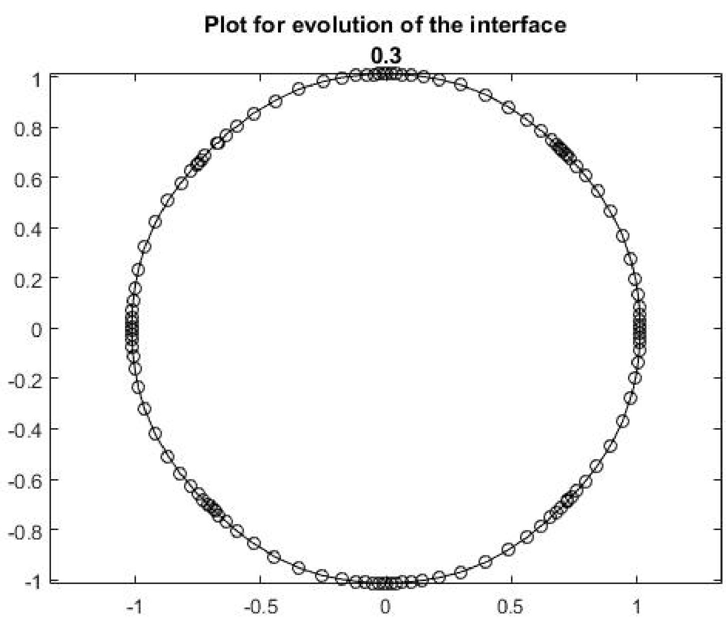


(d)

Figure 8 (c) Interface with 129 marker particles dt=$2.5x10^{-4}$, t=0.05, (d) Interface with 129 marker particles dt=$2.5x10^{-4}$, t=0.1

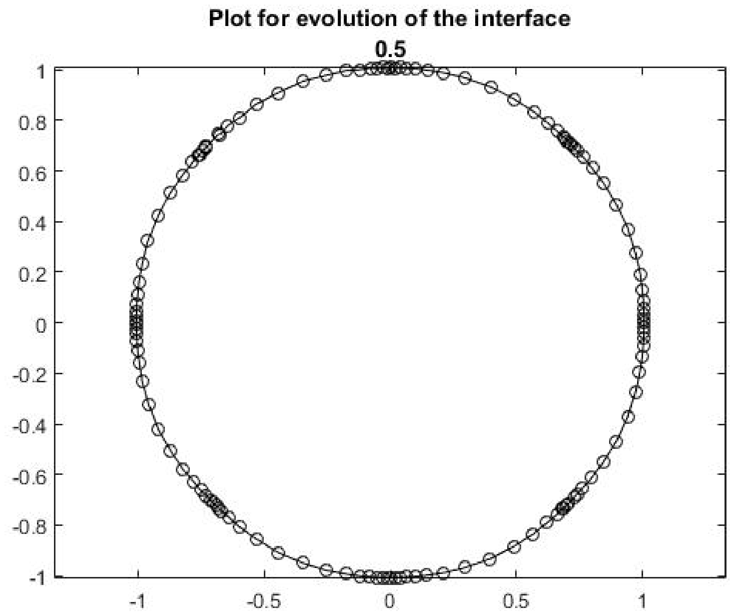


(e)

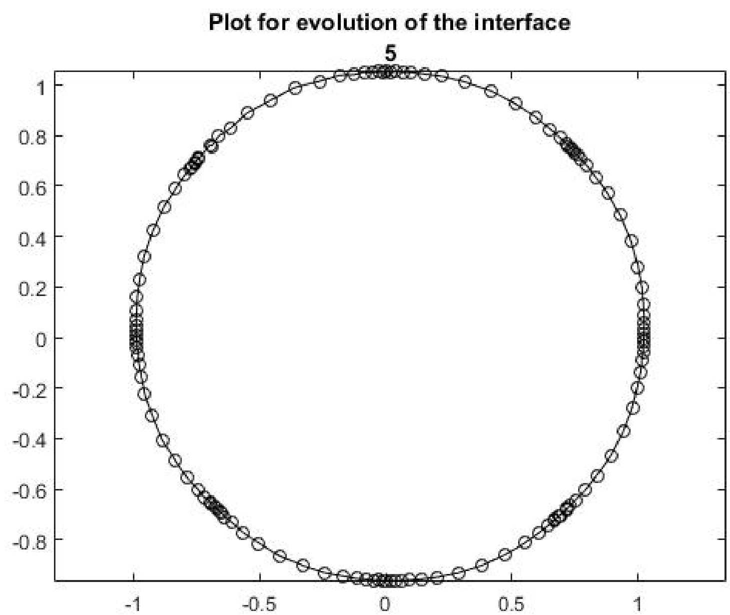


(f)

Figure 8 (e) Interface with 129 marker particles dt=2.5x10^{-4}, t=0.18, (f) Interface with 129 marker particles dt=2.5x10^{-4}, t=0.3



(g)



(h)

Figure 8 (g) Interface with 129 marker particles dt=2.5×10^{-4}, t=0.5, (h) Interface with 129 marker particles dt=2.5×10^{-4}, t=5

In Q2, there was no problem of accommodating the odd number of particles and the interface always remained closed. However, in Q3 the 129th particle was not accommodated thus the interface broke and did not remain closed. For t=2.5×10^{-4} both the cases, the interface went to steady state. However, the particle did not arrange themselves at equal distance from each other.

4. Time step of 10^{-3} and 129 Marker Particles with Re-parametrization:

As the stability breaks in case 3.1, we try to correct the same by performing re-parametrization for case 3.1. We keep the number of particles same as 129. The time steps are set at $t=10^{-3}$. In re-parametrization, we repartition the interface periodically so the marker particles are equally spaced. We have re-parametrized the interface by arc length.

Given $\left(x(s,t), y(s,t)\right)$, we define

$$\sigma(s) = \int_0^s (x_s^2(\tau,t) + y_s^2(\tau,t))^{1/2} d\tau$$

We define new parameterization $(\tilde{x}(\sigma,t), \tilde{y}(\sigma,t))$ where $(\tilde{x}, \tilde{y})$ are defined on $0 \le \sigma \le \sigma(L)$ and

$$\tilde{x}(\sigma(s),t) = x(s,t), \qquad \tilde{y}(\sigma(s),t) = y(s,t)$$

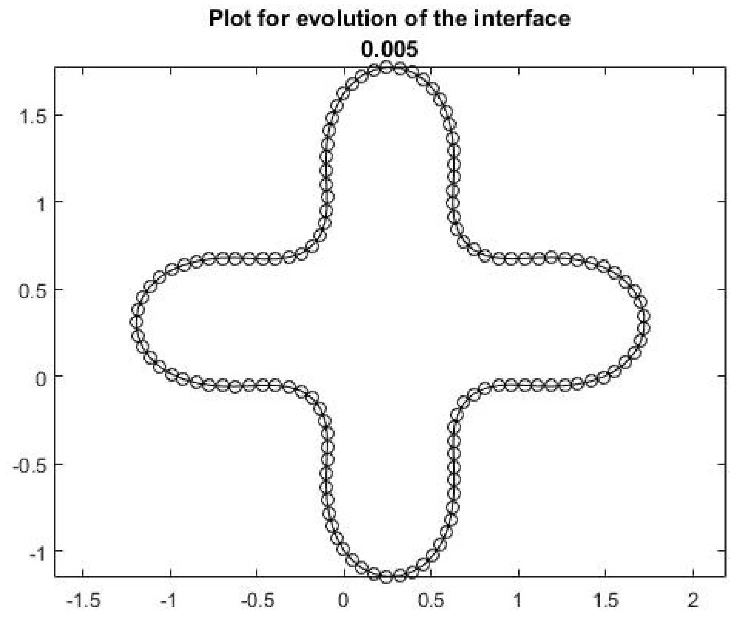
$$\frac{d}{ds}\left(\tilde{x}\big(\sigma(s),t\big)\big), \tilde{y}(\sigma(s),t)\right) = \frac{d}{ds}(x(s,t), y(s,t))$$

$$\left(\tilde{x}_\sigma\big(\sigma(s),t\big)\big), \tilde{y}_\sigma(\sigma(s),t)\right)\frac{d\sigma}{ds} = (x_s(s,t), y_s(s,t))$$

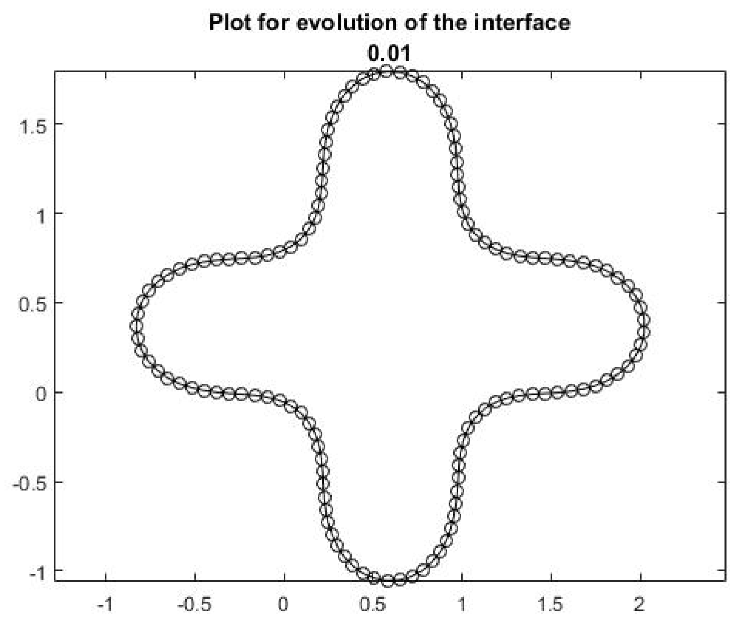
$$\left(\tilde{x}_\sigma\big(\sigma(s),t\big)\big), \tilde{y}_\sigma(\sigma(s),t)\right) = \frac{1}{(x_s^2(s,t) + y_s^2(s,t))^{1/2}}(x_s(s,t), y_s(s,t))$$

$$\left\|\left(\tilde{x}\big(\sigma(s),t\big)\big), \tilde{y}(\sigma(s),t)\right)\right\|=1$$

It was observed that the system goes to steady state faster as compared to cases 3.1-3.3. It is also observed that the nodes are equally spaced at steady state. The system goes to perfect steady state at (Fig. 9(f)). In this case, there are no oscillations involved when the system evolves. The system achieves steady state faster compared to all the above cases. The distance between the particles is maintained same as we have reparametrized.

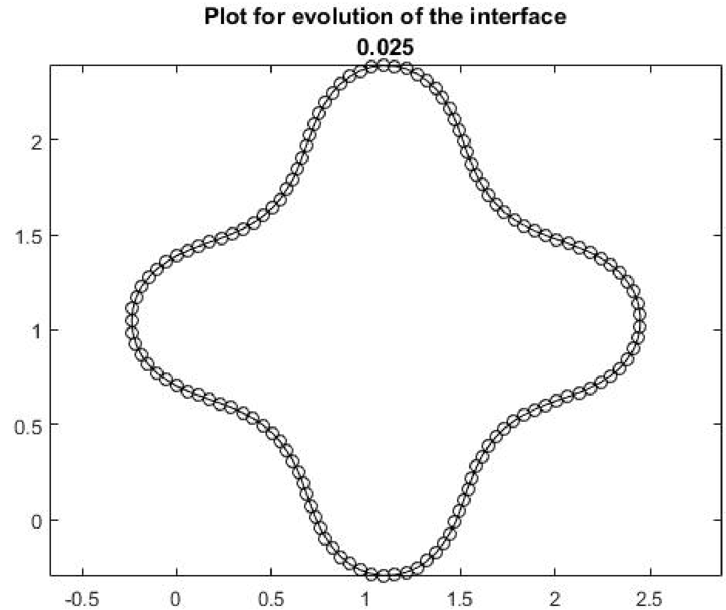


(a)

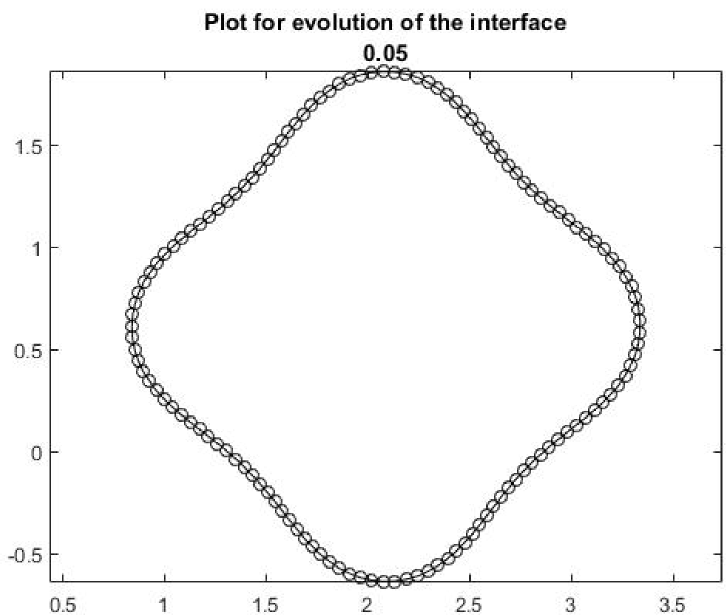


(b)

Figure 9 (a) Reparametrized Interface with 129 marker particles dt=10^{-3}, t=0.005, (b) Interface with 129 marker particles dt=10^{-3}, t=0.01

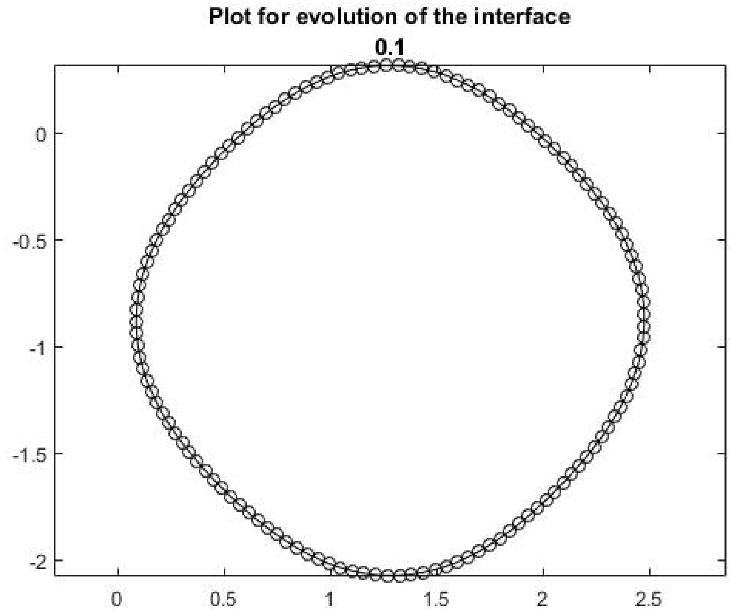


(c)

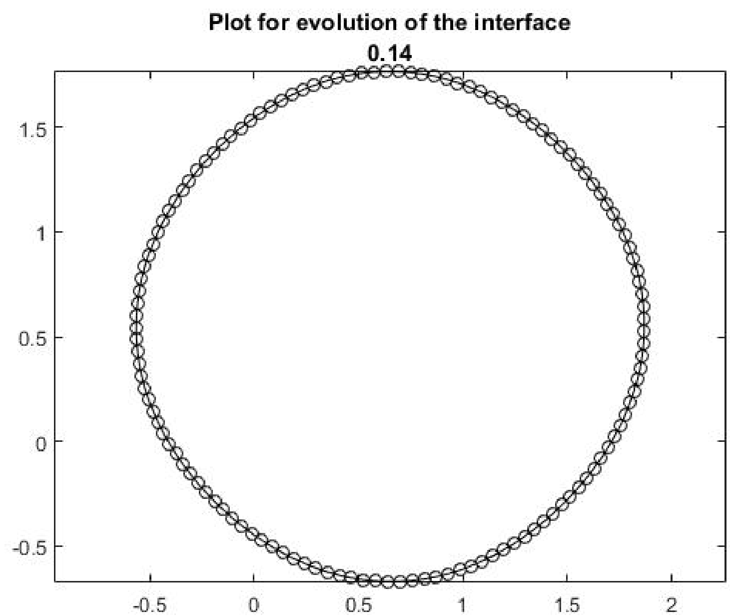


(d)

Figure 9 (c) Reparametrized Interface with 129 marker particles dt=10^{-3}, t=0.025, (d) Interface with 129 marker particles dt=10^{-3}, t=0.05

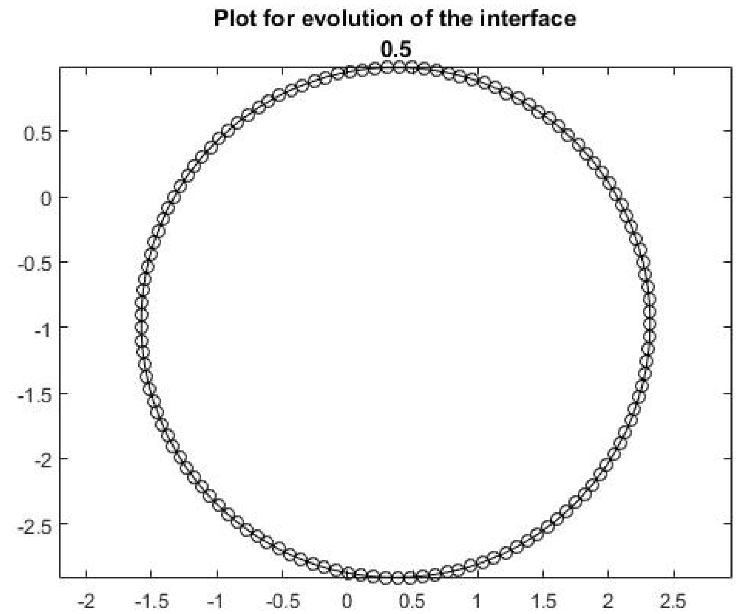


(e)

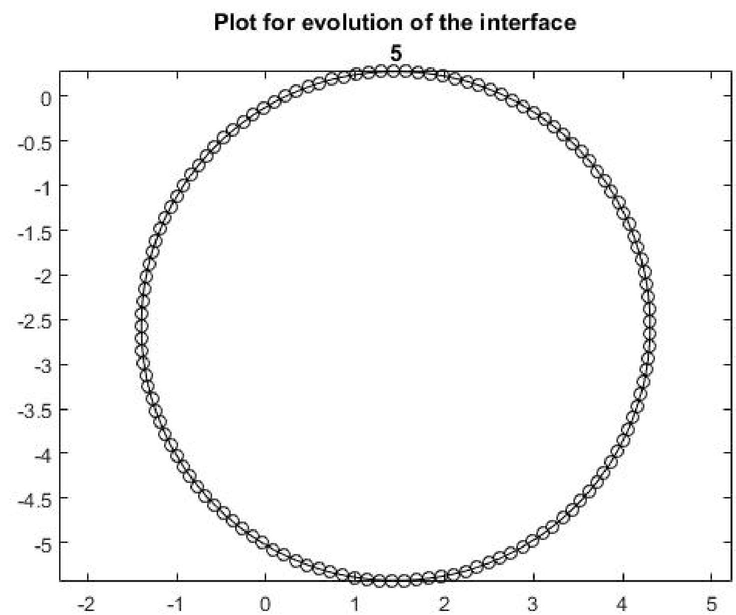


(f)

Figure 9 (e) Reparametrized Interface with 129 marker particles dt=10^{-3}, t=0.1, (f) Interface with 129 marker particles dt=10^{-3}, t=0.14



(g)



(h)

Figure 9 (e) Reparametrized Interface with 129 marker particles dt=10^{-3}, t=0.5, (f) Interface with 129 marker particles dt=10^{-3}, t=5

Conclusion:

The marker particle method was studied for the evolution of the system given by
$r(s) = (1 + 0.5\cos(s))(\cos(s), \sin(s))$

Initially, 128 marker particles were taken and the evolution was studied in three different time steps till the system achieves steady state. The time steps considered were $dt=10^{-3}$, $5x10^{-4}$, $2.5x10^{-4}$. It was observed that the system achieved steady state only at $dt=2.5x10^{-4}$. For other two time steps the system did not achieve steady state. It was observed that at steady state, the interface achieves circular shape. In the next case, we increased the marker particles to 129. We again studied the evolution for same time steps. Similar to Case 1, the system achieved steady state only at $2.5x10^{-4}$. For other two time steps the system was not steady. The interface broke in this case due to additional marker particle. There were lot of oscillations observed. Stability was disturbed. To achieve the stability, we reparametrized the equation of the interface by the arc length. The new achieved interface was simulated at $dt=10^{-3}$. In this case we observed that the system achieved steady state for 129 particles. The distance between the particles was maintained same. Unlike in Case 1 and 2, the particles did not bunch together. The steady state was achieved faster compared to other cases.

References:

1. Class Notes by Dr. Salac
2. Printed Notes by Dr. Salac.
3. http://www.mathworks.com/help/matlab/