ECE 1505 Convex Optimization
Prof. Wei Yu

Homework Set #3

1. Problem 5.12 of Boyd and Vandenberghe
2. Problem 5.27 of Boyd and Vandenberghe
3. Problem 5.35 of Boyd and Vandenberghe
4. Problem 5.42 of Boyd and Vandenberghe
5. **Strong Duality for LP**
   Consider an LP:

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \geq b \\ & x \geq 0 \end{array}$$

   Find the dual of the LP and argue that:

   (a) if the primal is unbounded, then the dual is infeasible;

   (b) if the primal is infeasible, then the dual is either infeasible or unbounded.

   Note that strong duality holds for an LP if either the primal or the dual is feasible. In other words, for LPs the only exception to strong duality occurs when $p^* = +\infty$ and $d^* = -\infty$.

6. **Game Theory**
   Consider a minimax problem with bilinear objective and separable polyhedral constraints on the variables as follows:

$$\begin{array}{ll} \min\limits_{x} \max\limits_{y} & x^T P y \\ \text{subject to} & Ax \leq b \\ & Cy \leq d \end{array}$$

   Assume that the feasible set is non-empty and bounded. Show that the optimal value of the above is identical to the optimal value of the following max-min problem

$$\begin{array}{ll} \max\limits_{y} \min\limits_{x} & x^T P y \\ \text{subject to} & Ax \leq b \\ & Cy \leq d \end{array}$$

   by proving the following:

(a) Show that the min-max problem has the same optimal value as the following minimization problem (over both x and λ):

$$\begin{array}{ll} \text{minimize} & d^T\lambda \\ \text{subject to} & C^T\lambda = P^T x \\ & Ax \leq b \\ & \lambda \geq 0 \end{array}$$

(b) Show that the max-min problem has the same optimal value as the following maximization problem (over both y and ν):

$$\begin{array}{ll} \text{maximize} & -b^T\nu \\ \text{subject to} & A^T\nu + Py = 0 \\ & Cy \leq d \\ & \nu \geq 0 \end{array}$$

(c) Show that the above minimization problem and the maximization problem have the same optimal value. Thus, min-max equals max-min.

7. **Optimal Control of a Unit Mass Revisited**

In the last assignment, you solved an optimal control problem with a quadratic (i.e., $||\cdot||_2$) objective. Here, we consider two different objectives, $||\cdot||_1$ and $||\cdot||_\infty$

$$\| \mathbf{p} \|_1 = \sum_{i=1}^{10} |p_i|, \qquad \| \mathbf{p} \|_\infty = \max_{i=1,\ldots,10} |p_i|$$

$||\cdot||_1$ can be a good measure of fuel consumption; $||\cdot||_\infty$ is the *peak* of the force vector $\mathbf{p}$.

(a) Consider a minimum $||\mathbf{p}||_1$ problem with the same setup as in Q6(a) of the last assignment. Find the optimal solution using `MATLAB`. Plot the optimal force, position, and velocity. Please write down both the primal and dual solutions. Note that

- `linprog` returns both primal and dual optimal solutions.
- the solution is called *bang-bang* control.

(b) In this part of the problem, we will verify the solution you've found using a different method. This will also give a different interpretation for Lagrange multiplier.

i. Verify that for any vector v and w, we always have $|w^T v| \leq ||v||_\infty ||w||_1$.

ii. Let z be any solution of $Az = y$, explain why for any λ, we must have

$$||z||_1 \geq \frac{|\lambda^T y|}{||A^T\lambda||_\infty}$$

Therefore, if you wish to minimize $||z||_1$ subject to $Az = y$ and you are able to identify a pair of (z, λ) for which the above inequality is satisfied with equality, then this z must be optimal.

iii. Set λ to be the Lagrange multiplier associated with the equality constraint in part (a). Use the above inequality to directly verify that the *bang-bang* solution is optimal.

(c) Repeat part (a) for the $||\cdot||_\infty$ minimization problem. Does the solution make sense? Submit all `MATLAB` codes.

You can actually verify the $||\cdot||_\infty$ solution using the method in part (b) as well, but you don't need to do this explicitly for this homework. You will notice that $||\cdot||_1$ and $||\cdot||_\infty$ are duals of each other. In general, l_p and l_q spaces are duals of each other if $\frac{1}{p}+\frac{1}{q}=1$. (Thus, l_2 space is the dual of itself.)