Smoothed Analysis of Probabilistic Roadmaps

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Abstract

The probabilistic roadmap algorithm is a leading heuristic for robot motion planning. It is extremely efficient in practice, yet its worst case convergence time is unbounded as a function of the input's combinatorial complexity. We prove a smoothed polynomial upper bound on the number of samples required to produce an accurate probabilistic roadmap, and thus on the running time of the algorithm, in an environment of simplices. This sheds light on its widespread empirical success.

1 Introduction

Smoothed analysis It is well-documented that many geometric algorithms that are extremely efficient in practice have exceedingly poor worst-case performance guarantees. Smoothed analysis [15] addresses this issue by observing that geometric inputs often contain a small amount of random noise, such as with point clouds generated by a laser scanner [12]. It can be argued that small degrees of randomness creep into geometric inputs even if they are created by a human modeler [14]. By this reasoning, finely tuned worst-case examples have a low probability of arising and should not disproportionately skew theoretical measures of algorithm performance.

Smoothed analysis [15] measures the maximum over inputs of the expected running time of the algorithm under slight random perturbations of those inputs. For example, let $A \in \mathbb{R}^{n\times d}$ specify a set of n points in $\mathbb{R}^d$, and let $f_X(A)$, where $f_X : \mathbb{R}^{n\times d} \mapsto \mathbb{R}$, be a measure of the performance of algorithm X on A. Then the smoothed performance of X is

$$\max_{A\in\mathbb{R}^{n\times d}} \mathrm{E}_{R\sim\mathcal{N}} \left[f_X(A + \|A\|R) \right],$$

where $\|A\|$ denotes the Frobenius norm of A and $\mathcal{N} = N(0, \sigma^2 I_{n\times d})$ is a Gaussian distribution in $\mathbb{R}^{n\times d}$ with mean 0 and variance σ^2. The parameter σ controls the magnitude of the random perturbation, and as it varies from 0 to ∞ the smoothed performance measure interpolates between worst-case and average-case performance.

Probabilistic roadmaps The probabilistic roadmap (PRM) algorithm revolutionized robot motion planning [8, 10]. It is a simple heuristic that exhibits rapid performance and has become the standard algorithm in the field [4, 5, 13]. Yet its worst-case running time is *unbounded* as a function of the input's combinatorial complexity. The basic algorithm for constructing a probabilistic roadmap is as follows:

> Sample uniformly at random a set of points, called milestones, from the *configuration space* $\mathcal{C}$ of the robot. Keep only those milestones that lie in the *free configuration space* $\mathcal{C}_{\mathsf{free}}$.[1] Let V be the resulting point set. For every $u, v \in V$, if the straight line segment between u and v lies entirely in $\mathcal{C}_{\mathsf{free}}$, add $\{u, v\}$ to the set of edges E, initially empty. The graph $G = (V, E)$ is the probabilistic roadmap.

Given such a roadmap G, a motion between two points p, q in $\mathcal{C}_{\mathrm{free}}$ can be constructed as follows:

> Find a milestone p' (resp., q') in V that is visible from p (resp., from q). If p' and q' lie in different connected components of G, report that there is no feasible motion between p and q. Otherwise plan the motion using a path in G that connects p' and q'.

The above PRM construction and query algorithms can be efficiently implemented in very general settings. The outstanding issue is what the number of samples should be to guarantee (in expectation) that G accurately represents the connectivity of $\mathcal{C}_{\mathrm{free}}$. Clearly, for the algorithm to be accurate there should be a milestone visible from any point in $\mathcal{C}_{\mathrm{free}}$, and there should be a bijective correspondence between the set of connected components of G and the set of connected components of $\mathcal{C}_{\mathrm{free}}$. Unfortunately, the number of random samples required to guarantee this can be made arbitrarily large even for very simple configuration spaces [5].

A number of theoretical analyses provide bounds for the number of samples under assumptions on the

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[1] A robot's *configuration space* is the set of physical positions it may attain (which may or may not coincide with obstacles), parametrised by its degrees of freedom (so a robot with d degrees of freedom has a d-dimensional configuration space). The robot's *free configuration space* is the subset of these positions which do *not* coincide with obstacles, i.e. are possible in real life. These terms are standard in the motion planning literature.

structure of $\mathcal{C}_{\text{free}}$ such as goodness [2, 9], expansiveness [6], and the existence of high-clearance paths [7]. However, none of these assumptions were justified in terms of realistic motion planning problems. In practice, the number of random samples is chosen ad hoc.

Contributions This paper initiates the use of smoothed analysis to explain the success of PRM. We model the configuration space using a set of n simplices in $\mathbb{R}^d$ whose vertices are subject to Gaussian perturbation with variance σ^2. We prove a smoothed upper bound on the required number of milestones that is polynomial in n and $\frac{1}{\sigma}$. The result extends to all γ-smooth perturbations, see below.

In order to achieve this bound we define a space decomposition called the locally orthogonal decomposition. Previously known decompositions, like the vertical decomposition [3, 11] and the "castles in the air" decomposition [1] turn out to be unsuitable for our purpose. We prove that for the roadmap to accurately represent the free configuration space it is sufficient that a milestone is sampled from every cell of this decomposition (Corollary 2). We then prove a smoothed lower bound on the volume of every decomposition cell (Corollary 5). This leads to the desired bound on the number of milestones (Theorem 6).

Our result is only the first step towards a convincing theoretical justification of PRM. The analysis is quite challenging already for the simple representation of the configuration space using independently perturbed simplices. We suggest extensions to more general configuration space models as future work.

2 Bounding the Number of Milestones

Notation Two objects are ε-close (ε-distant) if the shortest distance between them is at most (at least) ε. $A \oplus B$ denotes the Minkowski sum of A and B. $B_d(r)$ denotes the d-ball of radius r, and $S_{d-1}(r)$ denotes its boundary (the $(d-1)$-sphere of radius r).

The model Let Σ be a fixed, convex, polyhedral bounding box for $\mathcal{C}_{\text{free}}$ in $\mathbb{R}^d$, where d is assumed to be constant. This is the domain from which the milestones are sampled by the PRM algorithm. Let D be the diameter and D_{in} be the inner diameter of Σ (the inner diameter of a region is the diameter of the largest ball contained completely within the region). Let $\mathcal{S}$ be a set of n $(d-1)$-simplices in Σ. These are the $\mathcal{C}$-space obstacles in our model. Thus $\mathcal{C}_{\text{free}} = \Sigma \setminus \bigcup_{s \in \mathcal{S}} s$.

A probability distribution $\mathcal{D}$ on $\mathbb{R}^d$ with density function $\mu(.)$ is said to be γ-smooth, for some $\gamma \in \mathbb{R}$, if

1. $\mu(x) \leq \gamma$ for all $x \in \mathbb{R}^d$, and
2. given any hyperplane H, a point distributed under $\mathcal{D}$ is almost surely not on H.

A symmetric d-variate Gaussian distribution with variance σ^2 (covariance matrix $\sigma^2 I_d$) is $\Theta\left(\frac{1}{\sigma^d}\right)$-smooth. We assume that each vertex of each simplex in $\mathcal{S}$ is independently perturbed according to a γ-smooth distribution within the domain.

We note that these simplices may also be thought of as boundary elements of full-dimensional polyhedral obstacles. Our upper bound on the the number of samples required to build an accurate roadmap applies verbatim, since we will discard those samples which fall in the interior of these polyhedra. However, our analysis is then not completely realistic because our perturbation model destroys the connectivity of these boundaries — an improved model and its analysis form a possible avenue of future work (see the Conclusion section).

The locally orthogonal decomposition The locally orthogonal decomposition $\boxtimes(\mathcal{S})$ of $\mathcal{S}$ is the arrangement of the following two collections of hyperplanes:

- $\mathrm{Aff}(s)$ for each $s \in \mathcal{S}$.
- The hyperplane orthogonal to s that is spanned by f (i.e. contains f, since f is of lower dimension), for each $s \in \mathcal{S}$ and each facet f of s.

Hyperplanes of the second type are called *walls*. A facet of $\boxtimes(\mathcal{S})$ is *bound* if it is contained in some $s \in \mathcal{S}$, otherwise it is *free*. In the following, the decomposition is assumed to be restricted to Σ. The second property of γ-smooth distributions ensures that under our perturbation model, $\boxtimes(\mathcal{S})$ is almost surely in general position. We readily obtain the following lemma and its obvious corollary.

Lemma 1 *Let c_1 and c_2 be two cells of $\boxtimes(\mathcal{S})$ that are incident at a free facet. Then for any $p_1 \in c_1$ and $p_2 \in c_2$, the line segment between p_1 and p_2 is disjoint from all $s \in \mathcal{S}$.*

Corollary 2 *If a milestone is placed in each cell of $\boxtimes(\mathcal{S})$ then any two points that can be connected by a path in $\mathcal{C}_{\text{free}}$ can also be connected by a piecewise linear path whose only internal vertices are milestones.*

Volume bound Corollary 2 implies that it suffices to place a milestone in every cell of $\boxtimes(\mathcal{S})$. To show that this can be accomplished with a polynomial number of samples we prove a high-probability lower bound on the volume of each cell of $\boxtimes(\mathcal{S})$. This is achieved with the help of the following simple lemma.

Lemma 3 *Let $\mathcal{A}(\mathcal{H})$ be the arrangement induced by a set of hyperplanes $\mathcal{H}$. If every vertex v of $\mathcal{A}(\mathcal{H})$*

is ε-distant from every hyperplane $H \in \mathcal{H}$ for which $v \notin H$, then the volume (k-dimensional measure) of any k-face of the arrangement is at least $\varepsilon^k/k!$, for $1 \le k \le d$.

Lemma 3 implies that volume bounds can be proved through vertex-hyperplane separation bounds. Accordingly, Section 3 is devoted to proving the following theorem:

Theorem 4 *Consider a vertex v and a hyperplane H of $\boxtimes(\mathcal{S})$ such that $v \notin H$, and let $\Delta := \min\{1, D_{\text{in}}\}$. Given $\varepsilon \in [0, \Delta)$, v is ε-close to H with probability at most*

$$O\left(\varepsilon^{1-\alpha} \max\{\gamma, \gamma^{d^2}\}\right)$$

for any $\alpha > 0$.

Note that all terms involving only the constants d and D are subsumed into the $O(.)$ notation. The number of hyperplanes in $\boxtimes(\mathcal{S})$ is $O(n)$ and the number of vertices of $\boxtimes(\mathcal{S})$ is $O(n^d)$. A union bound and an application of Lemma 3 thus yield the following corollary to Theorem 4.

Corollary 5 *Each cell of $\boxtimes(\mathcal{S})$ has volume at least ε with probability at least $1-\omega$ if*

$$\varepsilon \;\le\; \min\left\{K\, \omega^{\frac{d}{1-\alpha}}\; n^{-\frac{d(d+1)}{1-\alpha}} \left(\max\{\gamma, \gamma^{d^2}\}\right)^{-\frac{d}{1-\alpha}}, \frac{\Delta^d}{d!}\right\}$$

for any $\alpha > 0$ and an appropriate constant K.

If each cell of $\boxtimes(\mathcal{S})$ has volume at least ε, standard probability theory implies that the expected number of samples sufficient for placing a milestone in every cell is $O\left(\frac{1}{\varepsilon}\log\frac{1}{\varepsilon}\right)$. Applying Corollary 5, we conclude that with high probability, a set of $\mathrm{Poly}(n, \gamma)$ samples from Σ is expected to place a milestone in every cell of $\boxtimes(\mathcal{S})$. This yields our main theorem, which we state in the special case of Gaussian perturbations.

Theorem 6 *Let a free configuration space be defined by n $(d-1)$-simplices in $\mathbb{R}^d$ within a fixed domain. If independent Gaussian perturbations of variance σ^2 are applied to the simplex vertices then the expected number of uniformly chosen random samples required to construct an accurate probabilistic roadmap is polynomial in n and $\frac{1}{\sigma}$.*

3 Distance Bounds

This section forms the technical bulk of the analysis and is devoted to proving Theorem 4. The one-dimensional case admits a simple proof, so we assume $d \ge 2$ in the balance of this paper. The hyperplane H can be of three types, which we analyse separately:

1. The affine span of $s \in \mathcal{S}$.
2. A wall spanned by a facet of $s \in \mathcal{S}$.
3. A hyperplane defining the boundary of Σ.

3.1 Affine Spans of Simplices

Theorem 7 *Consider a fixed point p in $\mathbb{R}^d$. Given $0 \le k < d$, let $k+1$ points $U = \{u_1, u_2, \ldots, u_{k+1}\}$ be distributed independently and γ-smoothly in Σ. The probability that the affine span of U is ε-close to p is at most*

$$K\varepsilon^{d-k}\gamma^{k+1}$$

for $\varepsilon \ge 0$ and a constant K depending on k, d and D.

Proof. (Sketch) For $k = 0$ the result is trivial. Assume $1 \le k \le \frac{d}{2}$. We will integrate over all k-flats formed by $(k+1)$-tuples of points. For a given u_1, the k-subspace $F - u_1$ of $\mathbb{R}^d$ can be represented as the span of k orthogonal unit vectors $v_1, v_2, \ldots, v_k$. We define a particular onto mapping ϕ between k-tuples of points drawn from S_{d-k} and orthonormal bases for k-flats in $\mathbb{R}^d$ that ensures that the localization of a k-tuple to a differential element ensures a corresponding spatial localization of the image k-flat. With this localization, we can assume that the k-flat F has the same orientation as a fixed subspace F^0, and it can be entirely parametrized by a single point on it, say u_1. The point p is ε-close to the the k-flat iff u_1 lies in $(p + F^0) \oplus B_{d-k}(\varepsilon)$ — the probability of this happening is at most γ times the volume of the latter region. The result follows by integration over k-tuples of points from S_{d-k}. We omit the mathematical details of the mapping ϕ and of the integration. The case $k > \frac{d}{2}$ is handled by considering the orthogonal complement of $F - u_1$. $\square$

The following corollary is immediate.

Corollary 8 *For nonnegative integers k, k' that satisfy $k + k' < d$, consider an arbitrarily distributed k'-flat F in $\mathbb{R}^d$, as well as a set $U = \{u_1, \ldots, u_{k+1}\}$ of γ-smoothly distributed points in Σ, independent of F and of each other. The shortest distance between F and the affine span of U is at most ε with probability at most*

$$K\varepsilon^{d-k-k'}\gamma^{k+1}$$

for $\varepsilon \ge 0$ and a constant K depending on k, d and D.

From Theorem 7 we see that a hyperplane-vertex pair of $\boxtimes(\mathcal{S})$, in which the hyperplane is the affine span of a simplex s, and the vertex v is defined entirely by hyperplanes not associated with s, is ε-close with probability at most polynomial in ε and γ. Specifically, the bound is $K\varepsilon\gamma^d$ for a constant K depending on d and D. The "local orthogonality" of $\boxtimes(\mathcal{S})$ allows us to extend the use of Corollary 8 to the case when the vertex is formed by the intersection of one or more walls supporting s with hyperplanes not associated with s.

3.2 Walls Supporting Simplices

When the hyperplane is a wall spanned by a simplex facet, the analysis is trickier. We divide our work into three cases based on the interdependence of the wall and the vertex. These cases may be summarised as:

1. The wall and the vertex are entirely independent.
2. The wall and the vertex depend on the same simplex but the vertex does not lie in the affine span of that simplex.
3. The wall and the vertex depend on the same simplex and the vertex lies in the affine span of that simplex.

The mathematical details are too involved for this abstract and the reader is encouraged to refer to the full version. We will merely comment that the techniques used are similar to those used to prove Theorem 7, and involve, in the last two cases, a study of the angle between two faces of a simplex, which is where we obtain the mysterious constant α (from the inequality $\log x \le K_\alpha x^\alpha$, for any $x > 1$, $\alpha > 0$ and a constant K_α depending only on α).

3.3 The Boundary of the Domain

Again, the analysis of this final case that deals with hyperplanes constituting the boundary of Σ is omitted due to space limitations. In brief, we use results from Sections 3.1 and 3.2 to show that every vertex of $\boxtimes(\mathcal{S})$ (other than those of the bounding box) follows a smooth distribution and thus prove that the probability of the vertex being ε-close to a boundary hyperplane is at most $K\varepsilon \max\{\gamma, \gamma^{d^2}\}$.

This concludes the proof of Theorem 4.

Acknowledgements

We are grateful to Dan Halperin for crucial early discussions and to Jean-Claude Latombe for his support throughout this project.

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