Equity premium and correlated shocks

Siddarth Venkatesh

October 01, 2024

Abstract

Quisque facilisis auctor sapien. Pellentesque gravida hendrerit lectus. Mauris rutrum sodales sapien. Fusce hendrerit sem vel lorem. Integer pellentesque massa vel augue. Integer elit tortor, feugiat quis, sagittis et, ornare non, lacus. Vestibulum posuere pellentesque eros. Quisque venenatis ipsum dictum nulla. Aliquam quis quam non metus eleifend interdum. Nam eget sapien ac mauris malesuada adipiscing. Etiam eleifend neque sed quam. Nulla facilisi. Proin a ligula. Sed id dui eu nibh egestas tincidunt. Suspendisse arcu.

1 Model

1.1 The basic problem

I start by examining the basic consumption-saving problem. The individual maximizes their discounted lifetime utility from the consumption stream $\{C_t\}_{t=0}^{T}$, where $T = \infty$ in the infinite-horizon model

$$\max_{\{C_t\}_{t=0}^{T}} \mathbb{E}_t \sum_{t=0}^{T} \beta^t u(C_t) \tag{1}$$

subject to the period-wise constraints

$$C_t + A_{t+1} = W_t + Y_t$$

where A_{t+1} is the total level of investment in assets incoming in period $t+1$, W_t the total monetary wealth at the beginning of the period, and Y_t is the current-period income. Income can be modelled as centered around an "expected" permanent income

$$Y_t = \zeta_t P_t \tag{2}$$

where ζ_t is a transitory mean-one shock. Permanent income, P_t, itself grows according to

$$P_t = \Gamma_t P_{t-1} \eta_t \tag{3}$$

where Γ_t is the predictable component of the growth of permanent income, and η_t is a mean-one shock. Further, individuals can also invest, in a perfectly divisible manner, between a risk-free (bond) and a risky (equity) asset. If the consumer chooses to hold κ_{t+1} share of the savings in the risky asset in period t (that is, their portfolio at the start of period $t+1$ contains κ_{t+1} share of the risky asset), wealth in period $t+1$ is determined by

$$W_{t+1} = \overbrace{(\mathsf{R} + \kappa_{t+1}(\Re_{t+1} - \mathsf{R}))}^{\mathcal{R}_{t+1}} A_{t+1} \tag{4}$$

$$\Re_{t+1} = \Re \nu_{t+1} \tag{5}$$

where $\Re_{t+1}$ is the return on the investments made in the risky asset in period t, ν_{t+1} is a mean-one shock, and $\mathcal{R}_{t+1}$ is the effective rate of return on assets stemming from the portfolio optimization decision κ_{t+1}. We can then use equation (4) rewrite the period budget constraint as

$$W_{t+1} = \mathcal{R}_{t+1}(W_t + Y_t - C_t)$$

Allowing $M_t = W_t + Y_t$ to denote the total current level of monetary resources

$$M_{t+1} = \mathcal{R}_{t+1}(M_t - C_t) + Y_{t+1}$$

Writing the problem from equation (1) in Bellman form with the added assumption that the period utility from consumption assumes a CRRA form,

$$V_t(M_t,\, P_t) = \max_{\{C_t,\, \kappa_{t+1}\}} \frac{C_t^{1-\rho}}{1-\rho} + \beta\, \mathbb{E}_t \left[V_{t+1}(\mathcal{R}_{t+1}(M_t - C_t) + Y_{t+1},\, P_{t+1}) \right] \tag{6}$$

Normalize all period t variables by the permanent income P_t (since A_t is determined in period $t-1$, it is normalized by P_{t-1}), and denote these new variables in lowercase (i.e. $c_t = C_t/P_t$). Letting $\mathcal{G}_{t+1} = \Gamma_{t+1}\eta_{t+1}$,

$$m_{t+1} = \frac{\mathcal{R}_{t+1}}{\mathcal{G}_{t+1}}(m_t - c_t) + \zeta_{t+1}$$

The Bellman formulation then becomes

$$v_t(m_t) = \max_{c_t,\, \kappa_{t+1}} \frac{c^{1-\rho}}{1-\rho} + \beta\, \mathbb{E} \left[(\mathcal{G}_{t+1})^{1-\rho} v_{t+1} \left(\frac{\mathcal{R}_{t+1}}{\mathcal{G}_{t+1}}(m_t - c_t) + \zeta_{t+1} \right) \right] \tag{7}$$

and the consumption Euler equation is then given by[1]

$$c_t^{-\rho} = \beta\, \mathbb{E}_t \left[\mathcal{R}_{t+1} (\mathcal{G}_{t+1} c_{t+1})^{-\rho} \right] \tag{8}$$

while the first order condition for κ_{t+1} is given by

$$(m_t - c_t)\, \mathbb{E}_t \left[(\Re_{t+1} - \mathsf{R})\, (\mathcal{G}_{t+1} c_{t+1})^{-\rho} \right] = 0 \tag{9}$$

[1]Use the following steps to show that under optimality, $u'(c_t) = v'(m_t)$

$$\begin{aligned}
u'(c_t) &= \beta\, \mathbb{E} \left[\mathcal{G}_{t+1}^{-\rho} \mathcal{R}_{t+1} v'(m_{t+1}) \right] \\
v'(m_t) &= v_m(m_t,\, c_t(m_t)) + \frac{\partial c_t}{\partial m_t} v_c(m_t,\, c_t(m_t)) \\
&= v_m(m_t,\, c_t(m_t)) + \frac{\partial c_t}{\partial m_t} \left[u'(c_t) - \beta\, \mathbb{E} \left[\mathcal{G}_{t+1}^{-\rho} \mathcal{R}_{t+1} v'(m_{t+1} \right] \right] \\
&= v_m(m_t,\, c_t(m_t)) \\
&= \beta\, \mathbb{E} \left[\mathcal{G}_{t+1}^{-\rho} \mathcal{R}_{t+1} v'(m_{t+1}) \right]
\end{aligned}$$

Then using the same conditions under optimality for period $t+1$, replace $v'(m_{t+1})$ with $u'(c_{t+1})$.

If we can show that under optimality, $m_t - c_t \neq 0$, then*

$$\mathbb{E}_t\left[(\Re_{t+1} - \mathsf{R})(\mathcal{G}_{t+1}c_{t+1})^{-\rho}\right] = 0$$

Now see that from equation (8) we get

$$\begin{aligned}
c_t^{-\rho} &= \beta\,\mathbb{E}_t\left[(\mathsf{R} + \kappa_{t+1}(\Re_{t+1} - \mathsf{R}))(\mathcal{G}_{t+1}c_{t+1})^{-\rho}\right] \\
&= \beta\left[\mathbb{E}_t\left[\mathsf{R}(\mathcal{G}_{t+1}c_{t+1})^{-\rho}\right] + \kappa_{t+1}\,\mathbb{E}_t\left[(\Re_{t+1} - \mathsf{R})(\mathcal{G}_{t+1}c_{t+1})^{-\rho}\right]\right] \\
&= \beta\mathsf{R}\Gamma_{t+1}^{-\rho}\,\mathbb{E}_t\left[(\eta_{t+1}c_{t+1})^{-\rho}\right] && \text{(from (9))}
\end{aligned}$$

Note that $\mathbb{E}_t\left[(\eta_{t+1}c_{t+1})^{-\rho}\right] > (\mathbb{E}_t\, c_{t+1})^{-\rho}$, so we require $\limsup_{n\to\infty} \beta\mathsf{R}\Gamma_{t+1}^{-\rho} < 1$ as the impatience condition for the existence of a steady-state normalized consumption.

I have hitherto remained silent on the exact distribution followed by the shocks to income and wealth. First, I assume that shocks are independent across time periods. Then, I model the shocks to be jointly lognormally distributed. In particular,

$$\log(\zeta_t,\, \eta_t,\, \nu_t) \sim \mathcal{N}(\mu,\, \Sigma)$$

where

$$\Sigma = \begin{bmatrix} \sigma_\zeta^2 & \omega_{\zeta,\,\eta} & \omega_{\zeta,\,\nu} \\ \omega_{\zeta,\,\eta} & \sigma_\eta^2 & \omega_{\eta,\,\nu} \\ \omega_{\zeta,\,\nu} & \omega_{\eta,\,\nu} & \sigma_\nu^2 \end{bmatrix}$$

$$\mu = \begin{bmatrix} -\sigma_\zeta^2/2 \\ -\sigma_\eta^2/2 \\ -\sigma_\nu^2/2 \end{bmatrix}$$

The marginal distributions of each of the shocks ensure that $\mathbb{E}_t[\eta] = \mathbb{E}_t[\nu] = \mathbb{E}_t[\zeta] = 1$. Meanwhile, $\omega_{x,\,y}$ captures the covariance of any two variables x and y among the three.

1.2 Solving the model

1.2.1 Discretizing the joint distribution

The first part of solving the model is to address the problem of efficiently computing expectations of the marginal utilities of consumption decisions in future periods. Under the current formulation, the shocks to income and returns are drawn independently in each time period, which means that it is enough to discretize these shocks using their single-period distribution. I use an equiprobable approximation of a truncated version (at 3 standard deviations) of these lognormal variables.

Here are the steps involved in discretizing the distribution:

1. Choose a suitable truncation of the distribution in each dimension by choosing an interval $[p_{min},\, p_{max}]$
2. Divide the interval given by $[\Phi^{-1}(p_{min}),\, \Phi^{-1}(p_{max})]$ into n intervals of $\frac{p_{max}-p_{min}}{n}$ probability each, $I = \left\{ \left[\Phi^{-1}\left(\frac{(i-1)p_{max}+(n-i+1)p_{min}}{n} \right),\, \Phi^{-1}\left(\frac{ip_{max}+(n-i)p_{min}}{n} \right) \right] \right\}_{i=1}^{n}$
3. Decompose the covariance matrix Σ using either the Cholesky decomposition, the eigendecomposition, or the positive-definite square root, to obtain a matrix A such that $AA^T = \Sigma$
4. Then construct the random variables $Y = \mu + AZ$, where $Z \sim \mathcal{N}(0,\, I)$, to get $Y \sim \mathcal{N}(\mu,\, \Sigma)$
5. Construct the set I^3 and , and compute the conditional expectation of the vector of shocks $X = \exp(Y)$ in each set of I^3, yielding the set of equiprobable atoms $S = \left\{ (\eta,\, \nu,\, \zeta)_j \right\}_{j=1}^{n^3}$

Computing expectations of functions of these shocks can now be reduced to the following operation

$$\mathbb{E}\left[g(\eta,\,\nu,\,\zeta)\right] = n^{-3}\sum_{j=1}^{n^3} g(\eta_j,\,\nu_j,\,\zeta_j)$$

1.2.2 Computing optimal decisions

Finite-Horizon

I solve the model using a sequential application of the endogenous grid method, dividing a period into two subperiods, the first stage involving a consumption decision (c), and the second involving the portfolio optimization problem (κ).

To solve the problem pertaining to any period t, construct a grid of assets $\mathcal{A} = [\underline{a} = a_1 < a_2 < ... < a_k = \overline{a}]$. First, observe from equation (9) that whenever $a_i \neq 0$, the optimal share of risky assets is given by the choice of $\hat{\kappa}_{t+1}(a_i) \in [0,\,1]$ such that

$$n^{-3}\Gamma_{t+1}^{-\rho}\sum_{j=1}^{n^3}(\Re\nu_i - \mathsf{R})(\eta_j c_{t+1}(m_{ij}))^{-\rho} = 0 \tag{10}$$

where

$$m_{ij} = \frac{\mathsf{R} + \hat{\kappa}_{t+1}(a_i)(\Re\nu_j - \mathsf{R})}{\Gamma_{t+1}\eta_j}a_i + \zeta_j$$

The problem then becomes a root-finding operation pertaining to a function of $\hat{\kappa}$, which, given a policy function c_{t+1}, yields an optimal level of $\hat{\kappa}$ for each a_i. Denote this pair as $(a,\,\hat{\kappa})_i$, and the resulting effective return $\mathsf{R} + \hat{\kappa}_i(\Re\nu_j - \mathsf{R})$ for each value of the shocks as $\mathcal{R}_{ij}$.

Remark 1.1. *The choice of the grid $\mathcal{A}$, at least for the next-to-last period, will be determined while keeping in mind the natural borrowing constraint given by the no-default condition. Suppose an individual has decided to invest all their savings in the risky asset, and is hit by the worst possible set of shocks in the next period. Then the individual should not be able to borrow to the extent that they may default* *on their debt. The natural borrowing constraint is then given by*

$$\underline{a} = \frac{-\zeta_{\min}\Gamma_T\eta_{\min}}{\Re\nu_{\min}}$$

An alternative choice would be to simplify the grid by using an artificial no-borrowing constraint in the grid. Note, however, that the calculation of the optimal consumption function is not constrained by this unless the borrowing constraint is imposed on the model. I then use this lower bound to generate an exponentially spaced grid of assets.

For each *end-of-period* outcome $(a,\,\hat{\kappa})_i$, given c_{t+1}, we can use the consumption Euler equation to get

$$[\hat{c}_t(a_i,\,\hat{\kappa}_i)]^{-\rho} = \beta\Gamma_{t+1}^{-\rho}n^{-3}\sum_{j=1}^{n^3}\mathcal{R}_{ij}(\eta_j c_{t+1}(m_{ij}))^{-\rho}$$

where $\hat{c}$ denotes that this yields a "consumed" function. However, as $\hat{\kappa}_i$ is the value that solves equation (10)

$$[\hat{c}_t(a_i,\,\hat{\kappa}_i)]^{-\rho} = \beta\Gamma_{t+1}^{-\rho}\mathsf{R}n^{-3}\sum_{j=1}^{n^3}(\eta_j c_{t+1}(m_{ij}))^{-\rho}$$

The consumed function is then given by

$$\hat{c}_t(a_i,\,\hat{\kappa}_i) = \left[\beta\Gamma_{t+1}^{-\rho}\mathsf{R}n^{-3}\sum_{j=1}^{n^3}(\eta_j c_{t+1}(m_{ij}))^{-\rho}\right]^{-\frac{1}{\rho}}$$

Now we have a vector of c_i corresponding to each $(a,\,\hat{\kappa})_i$. Since $m_t = c_t + a_{t+1}$, we can construct the grid $\mathcal{M}$ with each $m_i \in \mathcal{M}$ given by $m_i = c_i + a_i$, where $c_t(a_i,\,\hat{\kappa}_i)$. We can now rewrite $c_t(m_i) = \hat{c}_t(a_i,\,\hat{\kappa}_i)$ and $\kappa_{t+1}(m_i) = \hat{\kappa}_{t+1}(a_i)$, and interpolate to get the policy functions $(c_t(m),\,\kappa_{t+1}(m)) = g_t(m)$ for period t. Finally, we can use $c_T(m) = m$ as a starting point and recursively obtain the estimated policy functions.

Infinite-Horizon

Since it is impossible to store an infinite sequence of values for the predictable growth factors Γ_t, I simplify the infinite-horizon problem by assuming that $\Gamma_t = \Gamma$ for all periods t, i.e. $\{\Gamma_t\}_{t=0}^{\infty}$ is a constant sequence. Then the Bellman equation can be simplified to

$$v(m) = \max_{c,\kappa'} \frac{c^{1-\rho}}{1-\rho} + \beta\Gamma^{1-\rho}\,\mathbb{E}\left[(\eta')^{1-\rho} v\left(\frac{\mathsf{R} + \kappa'(\Re' - \mathsf{R})}{\Gamma\eta'}(m-c) + \zeta'\right)\right]$$

We can use the same process as in the finite-horizon case, where I use the first-order conditions to find the policy functions. The only difference, now, is that instead of using c_{t+1} to find κ_{t+1} and c_t, I now use an initial guess about the time-invariant c, say c_0, to compute c_1 and κ_1. I use this newly computed c_1 to obtain c_2 and κ_2. This process continues, till the policy functions converge to an approximation of $c(m)$ and $\kappa(m)$.