Portfolio choice with correlated income growth and equity return shocks

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Abstract

Portfolio choice models normally predict that the portfolio share of equity declines with wealth, while the poor invest all their savings in equity. I examine an infinite-horizon consumption-saving problem where the agent has to decide how to invest their savings between a risky and risk-free asset. Following Viceira (2001), I model permanent income growth shocks as correlated with returns on equity. I find that with moderate levels of relative risk aversion, this correlation lowers the optimal portfolio share of equity at target wealth levels (Kimball's (1991) temperance motive), while inverting the relationship between wealth and equity portfolio share. I find that, under standard calibrations, neither the model with borrowing constraints nor the one with positive unemployment probability can fully explain the equity premium observed in U.S. data without high levels of risk aversion. The calibrated temperance motive may observe the observed equity portfolio share in conjunction with other popular explanations documented in the literature.

Link to code: https://github.com/sidd3888/equity-premium-correlated-shocks

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The multivariate lognormal discretization code I use to solve this model is available in the HARK toolkit.

1 Introduction

The standard buffer stock model (Deaton 1991, Carroll 1992) predicts that consumers engage in precautionary saving either because of borrowing constraints or the possibility of unemployment. Despite this, under a canonical formulation with independent shocks to income and returns on equity, the model counterintuitively predicts that poor consumers should invest their entire savings in equity, while wealthy consumers should save in the safe asset.[1] The predictions of lifetime portfolio optimization models by Merton (1969), Samuelson (1969) suggest that the major component in the willingness to invest in the risky asset arises from a regular income stream (Heaton & Lucas 1997).

Kimball (1991) showed that consumers may display a temperance motive to moderate exposure to other risks when faced with income uncertainty, even if these risks are statistically independent. Koo (1999) shows that an increase in the volatility of permanent income growth shocks has a significant temperance effect, though transitory income shocks have a negligible temperance effect. This does not change a key prediction of the model, however, that the poor should invest all their savings in equity, and the portfolio share of equity declines with wealth. Furthermore, given the equity premium and the volatility of returns on stock observed in the data, the baseline model suggests that even the wealthy should invest all their savings in stock.

One explanation for the modest portfolio share of equity observed in the data is that future consumption is positively correlated with returns on equity. Constantinides et al. (2002) use an overlapping generations (OLG) model to explore this idea. In their model, there are three generations of individuals: the young, middle-aged, and old/retired. Retired individuals receive a labor income of zero, which means that the future consumption of the middle-aged is highly correlated with the returns on equity. On the other hand, the middle-aged face uncertain wage income, implying that the future consumption of the young depends on more than just the returns on equity, lowering their correlation. Their argument is that a positive correlation implies that the realization of low marginal utility of consumption coincides with high returns, and vice versa. Thus, the low portfolio share of equity is a consequence of the inability of the young to participate in the stock market due to borrowing constraints. This paper departs from their modeling assumptions by retaining income uncertainty and introducing correlations between permanent income growth shocks and equity returns to correlate future consumption and equity returns.

This paper primarily focuses on an infinitely-lived agent[2] who consumes a single good

[1] The one exception is that with zero income events, extremely poor consumers invest a lower fraction of their savings in equity than consumers at the target level of wealth.

[2] The setup is well-suited to life-cycle modeling, but the insights derived in this paper are from the

and maximizes the discounted sum of utility from consumption. The agent faces three risks: the return on equity, permanent income growth, and transitory income. The model can accommodate pairwise correlations between all three shocks, though the focus of this paper will be on the case where permanent income shocks and equity returns are correlated, while transitory income shocks are independent of both. The agent decides their consumption (therefore, saving) and the portfolio share of equity in every period. The flexibility of the model in terms of the distribution of the shocks allows us to establish that a correlation between transitory income shocks and equity returns has a negligible effect on the portfolio share of equity for all agents but those with near-zero savings.

There is ample justification in the literature to model permanent income growth as correlated with equity returns while modeling transitory income shocks as independent of the two. Campbell (1996) shows that there is a high correlation between the present value of human capital and market returns, despite assuming that the contemporaneous correlation between wage income and stock returns. However, the source of this covariance in his model is due to a common time-variant discount factor applied to both calculations. Baxter & Jermann (1997) also find that the correlation between the returns to human capital and physical capital are high, even as labor and capital income growth rates may not exhibit a high correlation. With independent shocks across time, permanent income growth proportionally affects the expected present discounted value of human capital, while the independent transitory shock allows for the low correlation in actual labor income growth and returns on equity.

There have been papers that have studied correlations between labor income growth and equity returns previously in a life-cycle setting. Bodie et al. (1992) modelled the labor supply decision as endogenous, and made after the current return to equity has been determined. They show that agents vary their labor supply ex-post to cushion themselves against greater risks taken in their investment decisions. Subsequent papers study this relationship with an exogenous income process and a positive relationship between income growth and equity returns. Benzoni et al. (2007) explore a model with cointegrated labor income and stock market returns in a continuous-time setting. Bagliano et al. (2014) on the other hand study a discrete-time model with a permanent income shock that has both an aggregate and an idiosyncratic component, and two risky assets. The closest paper to mine is by Viceira (2001), whose model is based on a similar buffer-stock setting with correlations between permanent income growth and risky asset returns. The difference here is that I focus on the problem of the infinitely-lived agent, whose income is subject to transitory and permanent income shocks, and who may face an artificial borrowing constraint instead of possible spells of unemployment that otherwise serves as the motive

predictions of the infinite-horizon model.

for precautionary saving.

The imposition of the artificial borrowing constraint serves multiple purposes. First, in a model trying to understand the long-term equity holdings of consumers, it abstracts away from short-selling behavior. Then, it avoids any discontinuities in the problem at the point where savings are zero. Most lucratively, however, the consumer who faces artificial borrowing constraints tends to save very little at their target wealth and therefore invests none of their savings in equity, lending support to the well-studied argument that borrowing constraints among the young, for instance, can cause reduced or non-participation in the stock market (Constantinides et al. 2002, Haliassos & Hassapis 1998, Kogan et al. 2007, Jang & Park 2015, Harenberg 2018). I also artificially restrict the portfolio share of equity to between 0 and 1, which implies that consumers cannot hold debt-financed equity positions. Davis et al. (2006) argue that borrowing costs can be a major deterrent against investing in equity using loans, and it is beyond the scope of this paper to examine the differences in rates of return and mechanisms available for consumers to borrow versus save.

The findings of this paper can be divided into three major strands. First, a positive correlation between permanent income growth shocks and returns on equity does have a negative effect on the portfolio share of equity of an agent with a moderate level of risk aversion ($\rho = 4$). This effect is primarily observed among the poor who save very little, and gradually disappears for extremely wealthy agents. As wealth tends to infinity, the portfolio share of equity tends to the value in the case with uncorrelated income and return shocks. Second, for high levels of correlation between the shocks, the portfolio share of equity is actually increasing in wealth, unlike the model with uncorrelated shocks. Third, with the artificial borrowing constraint, agents do not invest in equity or invest extremely small proportions of their savings in it around the target level of wealth, while the threat of unemployment better explains the portfolio share of equity around target levels of wealth conditional on participation in the stock market. Though the model generates insightful predictions for an equity premium set at 3 percent with the standard deviation of the logged shock to the return set at 15 percent, calibrating the model to parameters relevant to data on U.S. equity returns as reported in Mehra (2006) necessitates an elevated risk aversion parameter of around 7 to generate modest portfolio shares of equity around target wealth. Section 5.1 discusses other explanations for the equity portfolio share that, in conjunction with correlated shocks to permanent income and returns, can favorably revise the predictions of the model without requiring a large coefficient of relative risk aversion.

The paper is structured as follows. Section 2 sets up the model and discusses the solution method used. Section 3 examines predictions from the model with the artificial no-borrowing constraint. Section 4 relaxes the borrowing constraint and introduces the possibility of unemployment in the income process. Section 5 provides a few comments and concludes.

2 Model

2.1 The basic problem

I start by examining the basic consumption-saving problem. The individual maximizes their discounted lifetime utility from the consumption stream $\{C_t\}_{t=0}^T$, where $T = \infty$ in the infinite-horizon model

$$\max_{\{C_t\}_{t=0}^T} \mathbb{E}_0 \sum_{t=0}^{T} \beta^t u(C_t) \tag{1}$$

subject to the period-wise constraints

$$C_t + A_{t+1} = W_t + Y_t$$

where A_{t+1} is the total level of investment in assets incoming in period $t+1$, W_t the total monetary wealth at the beginning of the period, and Y_t is the current-period income. Income can be modelled as centered around an "expected" permanent income

$$Y_t = \zeta_t P_t \tag{2}$$

where ζ_t is a transitory mean-one shock. Permanent income, P_t, itself grows according to

$$P_t = \Gamma_t P_{t-1} \eta_t \tag{3}$$

where Γ_t is the predictable component of the growth of permanent income, and η_t is a mean-one shock. Further, individuals can also invest, in a perfectly divisible manner, between a risk-free (bond) and a risky (equity) asset. If the consumer chooses to hold κ_{t+1} share of the savings in the risky asset in period t (that is, their portfolio at the start of period $t+1$ contains κ_{t+1} share of the risky asset), wealth in period $t+1$ is determined by

$$W_{t+1} = \overbrace{(\mathsf{R} + \kappa_{t+1}(\mathfrak{R}_{t+1} - \mathsf{R}))}^{\mathcal{R}_{t+1}} A_{t+1} \tag{4}$$

$$\mathfrak{R}_{t+1} = \mathfrak{R}\nu_{t+1} \tag{5}$$

where $\Re_{t+1}$ is the return on the investments made in the risky asset in period t, ν_{t+1} is a mean-one shock, and $\mathcal{R}_{t+1}$ is the effective rate of return on assets stemming from the portfolio optimization decision κ_{t+1}. The choice of κ_{t+1} is restricted to the interval [0, 1], thus precluding debt-financed equity positions. I impose an artificial borrowing constraint on the consumer that limits their borrowing as a proportion of their permanent income,

$$A_{t+1} \geq \underline{a} P_t \qquad (\underline{a} \leq 0)$$

It is reasonable to model borrowing limits as dependent on permanent income, which is something that is observed practically. In the context of this problem, it also aids in preserving its homotheticity. I set $\underline{a} = 0$ to focus purely on how savings are allocated between the two assets. We can then use equation (4) rewrite the period budget constraint as

$$W_{t+1} = \mathcal{R}_{t+1}(W_t + Y_t - C_t)$$

Allowing $M_t = W_t + Y_t$ to denote the total current level of monetary resources

$$M_{t+1} = \mathcal{R}_{t+1}(M_t - C_t) + Y_{t+1}$$

Writing the problem from equation (1) in Bellman form with the added assumption that the period utility from consumption assumes a CRRA form,

$$V_t(M_t,\, P_t) = \max_{\{C_t,\, \kappa_{t+1}\}} \frac{C_t^{1-\rho}}{1-\rho} + \beta\, \mathbb{E}_t \left[V_{t+1}(\mathcal{R}_{t+1}(M_t - C_t) + Y_{t+1},\, P_{t+1}) \right] \tag{6}$$

Normalize all period t variables by the permanent income P_t (since A_t is determined in period $t-1$, it is normalized by P_{t-1}), and denote these new variables in lowercase (i.e. $c_t = C_t/P_t$). Letting $\mathcal{G}_{t+1} = \Gamma_{t+1}\eta_{t+1}$,

$$m_{t+1} = \frac{\mathcal{R}_{t+1}}{\mathcal{G}_{t+1}}(m_t - c_t) + \zeta_{t+1}$$

The Bellman formulation then becomes

$$v_t(m_t) = \max_{c_t,\, \kappa_{t+1}} \frac{c^{1-\rho}}{1-\rho} + \beta\, \mathbb{E} \left[(\mathcal{G}_{t+1})^{1-\rho} v_{t+1} \left(\frac{\mathcal{R}_{t+1}}{\mathcal{G}_{t+1}}(m_t - c_t) + \zeta_{t+1} \right) \right] \tag{7}$$

and the consumption Euler equation is then given by[3]

$$c_t^{-\rho} = \beta\,\mathbb{E}_t\left[\mathcal{R}_{t+1}(\mathcal{G}_{t+1}c_{t+1})^{-\rho}\right] \tag{8}$$

The optimality condition for the portfolio share, κ_{t+1} is slightly trickier, given that it is bounded by $[0,\,1]$. Finally, note that portfolio share is irrelevant when $a_{t+1} = 0$, which means that we can only pin it down for when $a_{t+1} \neq 0$. In that case, a choice of κ_{t+1} is optimal if

$$\begin{cases} \mathbb{E}_t\left[(\Re_{t+1} - \mathsf{R})(\mathcal{G}_{t+1}c_{t+1})^{-\rho}\right] = 0 & \kappa_{t+1} \in (0,\,1) \\ \mathbb{E}_t\left[(\Re_{t+1} - \mathsf{R})(\mathcal{G}_{t+1}c_{t+1})^{-\rho}\right] \geq 0 & \kappa_{t+1} = 1 \\ \mathbb{E}_t\left[(\Re_{t+1} - \mathsf{R})(\mathcal{G}_{t+1}c_{t+1})^{-\rho}\right] \leq 0 & \kappa_{t+1} = 0 \end{cases} \tag{9}$$

Now see that whenever the optimal portfolio decision is to hold a mixture of both the safe and the risky asset, the consumption Euler equation can be reduced to

$$\begin{aligned} c_t^{-\rho} &= \beta\,\mathbb{E}_t\left[(\mathsf{R} + \kappa_{t+1}(\Re_{t+1} - \mathsf{R}))(\mathcal{G}_{t+1}c_{t+1})^{-\rho}\right] \\ &= \beta\left[\mathbb{E}_t\left[\mathsf{R}(\mathcal{G}_{t+1}c_{t+1})^{-\rho}\right] + \kappa_{t+1}\,\mathbb{E}_t\left[(\Re_{t+1} - \mathsf{R})(\mathcal{G}_{t+1}c_{t+1})^{-\rho}\right]\right] \\ &= \beta\mathsf{R}\Gamma_{t+1}^{-\rho}\,\mathbb{E}_t\left[(\eta_{t+1}c_{t+1})^{-\rho}\right] && \text{(from (9))} \end{aligned}$$

I have hitherto remained silent on the exact distribution followed by the shocks to income and wealth. First, I assume that shocks are independent across time periods. Then, I model the shocks to be jointly lognormally distributed. In particular,

$$\log(\eta_t,\,\nu_t,\,\zeta_t) \sim \mathcal{N}(\mu,\,\Sigma)$$

[3] Use the following steps to show that under optimality, $u'(c_t) = v'(m_t)$

$$\begin{aligned} u'(c_t) &= \beta\,\mathbb{E}\left[\mathcal{G}_{t+1}^{-\rho}\mathcal{R}_{t+1}v'(m_{t+1})\right] \\ v'(m_t) &= v_m(m_t,\,c_t(m_t)) + \frac{\partial c_t}{\partial m_t}v_c(m_t,\,c_t(m_t)) \\ &= v_m(m_t,\,c_t(m_t)) + \frac{\partial c_t}{\partial m_t}\left[u'(c_t) - \beta\,\mathbb{E}\left[\mathcal{G}_{t+1}^{-\rho}\mathcal{R}_{t+1}v'(m_{t+1})\right]\right] \\ &= v_m(m_t,\,c_t(m_t)) \\ &= \beta\,\mathbb{E}\left[\mathcal{G}_{t+1}^{-\rho}\mathcal{R}_{t+1}v'(m_{t+1})\right] \end{aligned}$$

Then using the same conditions under optimality for period $t+1$, replace $v'(m_{t+1})$ with $u'(c_{t+1})$.

where

$$\Sigma = \begin{bmatrix} \sigma_\eta^2 & \omega_{\eta,\,\nu} & \omega_{\eta,\,\zeta} \\ \omega_{\nu,\,\eta} & \sigma_\nu^2 & \omega_{\eta,\,\nu} \\ \omega_{\zeta,\,\eta} & \omega_{\nu,\,\eta} & \sigma_\zeta^2 \end{bmatrix}$$

$$\mu = \begin{bmatrix} -\sigma_\eta^2/2 \\ -\sigma_\nu^2/2 \\ -\sigma_\zeta^2/2 \end{bmatrix}$$

The marginal distributions of each of the shocks ensure that $\mathbb{E}_t[\eta] = \mathbb{E}_t[\nu] = \mathbb{E}_t[\zeta] = 1$. Meanwhile, $\omega_{x,\,y}$ captures the covariance of any two variables x and y among the three.

2.2 Solving the model

2.2.1 Discretizing the joint distribution

The first part of solving the model is to address the problem of efficiently computing expectations of the marginal utilities of consumption decisions in future periods. Under the current formulation, the shocks to income and returns are drawn independently in each time period, which means that it is enough to discretize these shocks using their single-period distribution. I use an equiprobable approximation of a truncated version (at 3 standard deviations) of these lognormal variables.[4]

Here are the steps involved in discretizing the distribution:

1. Choose a suitable truncation of the distribution in each dimension by choosing an interval $[p_{min},\, p_{max}] \subseteq [0,\, 1]$
2. Divide the interval given by $[\Phi^{-1}(p_{min}),\, \Phi^{-1}(p_{max})]$ into n intervals of $\frac{p_{max}-p_{min}}{n}$ probability each, $I = \left\{ \left[\Phi^{-1}\left(\frac{(i-1)p_{max}+(n-i+1)p_{min}}{n} \right),\, \Phi^{-1}\left(\frac{ip_{max}+(n-i)p_{min}}{n} \right) \right] \right\}_{i=1}^{n}$
3. Decompose the covariance matrix Σ using the Cholesky decomposition and obtain a matrix L such that $LL^T = \Sigma$
4. Then construct the random variables $Y = \mu + LZ$, where $Z \sim \mathcal{N}(0,\, I)$, to get $Y \sim \mathcal{N}(\mu,\, \Sigma)$
5. Construct the set I^3 and , and compute the conditional expectation of the vector of shocks $X = \exp(Y)$ in each set of I^3, yielding the set of equiprobable atoms

[4]The code for this algorithm is available as part of a contribution I made to the HARK toolkit.

$$S = \left\{ (\eta, \nu, \zeta)_j \right\}_{j=1}^{n^3}$$

Computing expectations of functions of these shocks can now be reduced to the following operation

$$\mathbb{E}\left[g(\eta, \nu, \zeta)\right] = n^{-3} \sum_{j=1}^{n^3} g(\eta_j, \nu_j, \zeta_j)$$

2.2.2 Computing optimal decisions

I solve the model using a sequential application of the endogenous grid method (Carroll 2006), dividing a period into two subperiods, the first stage involving a consumption decision (c), and the second involving the portfolio optimization problem (κ).

Construct a grid of assets $\mathcal{A} = [\underline{a} = a_1 < a_2 < ... < a_k = \overline{a}]$. To solve the problem pertaining to any period t, observe from equation (9) that whenever $a_i \neq 0$, the optimal share of risky assets is given by the choice of $\hat{\kappa}_{t+1}(a_i) \in [0, 1]$ such that

$$n^{-3} \Gamma_{t+1}^{-\rho} \sum_{j=1}^{n^3} (\Re \nu_i - \mathsf{R})(\eta_j c_{t+1}(m_{ij}))^{-\rho} = 0 \tag{10}$$

where

$$m_{ij} = \frac{\mathsf{R} + \hat{\kappa}_{t+1}(a_i)(\Re \nu_j - \mathsf{R})}{\Gamma_{t+1} \eta_j} a_i + \zeta_j$$

The problem then becomes a root-finding operation pertaining to a function of $\hat{\kappa}$, which, given a policy function c_{t+1}, yields an optimal level of $\hat{\kappa}$ for each a_i. Denote this pair as $(a, \hat{\kappa})_i$, and the resulting effective return $\mathsf{R} + \hat{\kappa}_i(\Re \nu_j - \mathsf{R})$ for each value of the shocks as $\mathcal{R}_{ij}$.

For each *end-of-period* outcome $(a, \hat{\kappa})_i$, given c_{t+1}, we can use the consumption Euler equation to get

$$\left[\hat{c}_t(a_i, \hat{\kappa}_i)\right]^{-\rho} = \beta \Gamma_{t+1}^{-\rho} n^{-3} \sum_{j=1}^{n^3} \mathcal{R}_{ij} (\eta_j c_{t+1}(m_{ij}))^{-\rho}$$

where $\hat{c}$ denotes that this yields a consumed function of the assets and portfolio share. This function is then given by

$$\hat{c}_t(a_i, \hat{\kappa}_i) = \left[\beta \Gamma_{t+1}^{-\rho} n^{-3} \sum_{j=1}^{n^3} \mathcal{R}_{ij} (\eta_j c_{t+1}(m_{ij}))^{-\rho} \right]^{-\frac{1}{\rho}}$$

Now we have a vector of $\hat{c}_i$ corresponding to each $(a, \hat{\kappa})_i$. Since $m_t = c_t + a_{t+1}$, we can construct the grid $\mathcal{M}$ with each $m_i \in \mathcal{M}$ given by $m_i = \hat{c}_i + a_i$, where $\hat{c}_i = \hat{c}_t(a_i, \hat{\kappa}_i)$. We can now rewrite $c_t(m_i) = \hat{c}_t(a_i, \hat{\kappa}_i)$ and $\kappa_{t+1}(m_i) = \hat{\kappa}_{t+1}(a_i)$, and interpolate to get the policy functions $(c_t(m), \kappa_{t+1}(m)) = g_t(m)$ for period t.[5] In the finite-horizon case, the model can be solved using $c_T(m) = m$ as the initial policy function and iterating backwards till period 0. For the infinite-horizon case, I use a guess $c_0(m)$ to obtain a sequence of guesses $\{c_k(m), \kappa_k(m)\}_{k=0}^{K}$ that converge to the true policy functions $c(m)$ and $\kappa(m)$. Since my focus is not on life-cycle applications, I solve each model with a constant permanent growth factor Γ.

3 Portfolio share with borrowing constraints

3.1 Predictions from the infinite-horizon model

I first analyze the findings from the infinite-horizon model on three fronts, the optimal portfolio share of the risky-asset for high-wealth individuals, the shape of the optimal portfolio share as a function of normalized monetary resources, and the effect of covariances between the income and asset shocks on the consumption function.

3.1.1 Optimal portfolio share at high-wealth levels

I start by looking at the baseline model with uncorrelated shocks to income and asset returns. I set the equity premium at 3 percent for this part of the analysis, and the standard deviation of the logged shock to the equity return at 15 percent, i.e. $\sigma_\nu = 0.15$. I let all other parameters be as in Table 1. Figure 1 shows that the optimal portfolio allocation $\kappa(m)$ is 1 at low values of m and decreases to an asymptotic value, as specified by the Merton-Samuelson model, as m tends to infinity.

It is obvious that this asymptotic value of κ is increasing in the equity premium. Since the consumer is risk-averse, increasing the volatility of the returns to equity will decrease its attractiveness, thus reducing the optimal value of κ upon an increase in σ_ν. The only question, then, is the effect of a positive covariance between income shocks and asset returns on optimal asymptotic portfolio share.

Figure 2 shows how optimal portfolio share responds to a positive correlation between

[5] While I use linear interpolation by default, cubic-spline interpolation yields similar results for the consumption function. However, due to drastic directional changes in the optimal portfolio share, spline interpolation sometimes suggests a portfolio share outside the interval [0, 1].

Figure 1. Optimal portfolio share with uncorrelated shocks

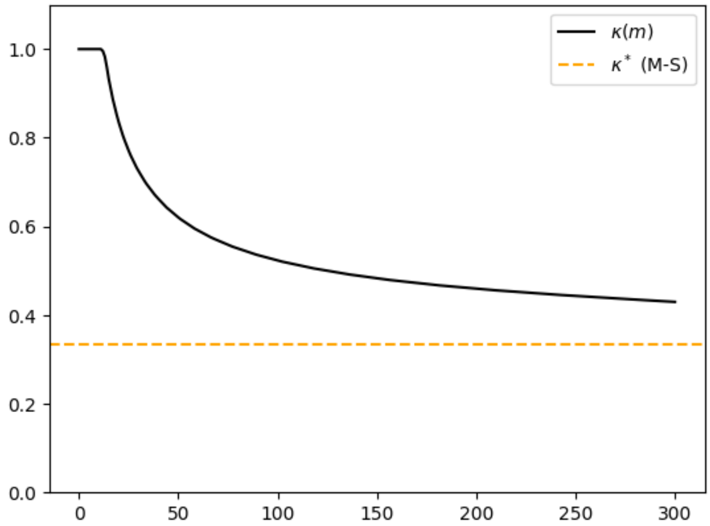
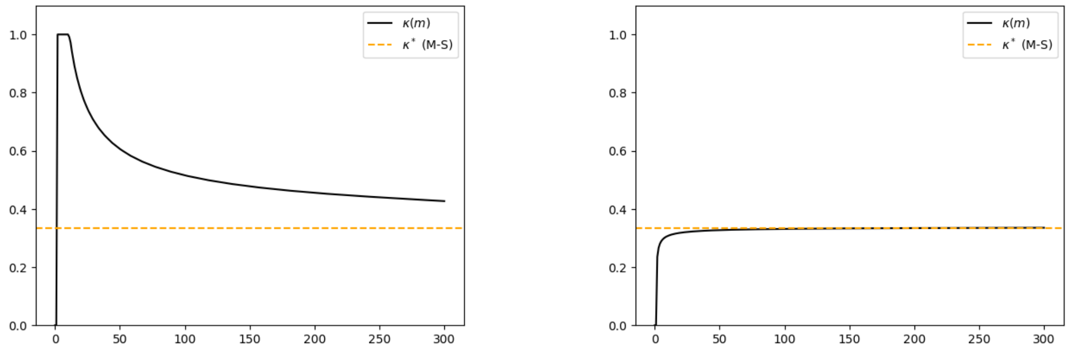


Figure 2. Optimal portfolio share with a positive correlation between income shocks and asset returns



(a) Transitory shock ($\omega_{\nu,\zeta} = 0.014$) **(b)** Permanent shock ($\omega_{\eta,\nu} = 0.008$)

either of the income shocks and asset returns. Figure 2a depicts the case when the transitory shock is correlated with asset returns. Since the optimal portfolio allocation is unaffected for all but the lowest values of m (see section 3.1.2), the asymptotic level of κ remains the same as in the model with uncorrelated shocks. Figure 2b shows the portfolio allocation rule under a moderate correlation between permanent income and asset return shocks. Even with a completely different optimal portfolio allocation rule, for high levels of wealth, the optimal portfolio share tends to similar values as in the case with no correlations.

The explanation for this behavior can be found by examining the optimal portfolio share condition, given by:

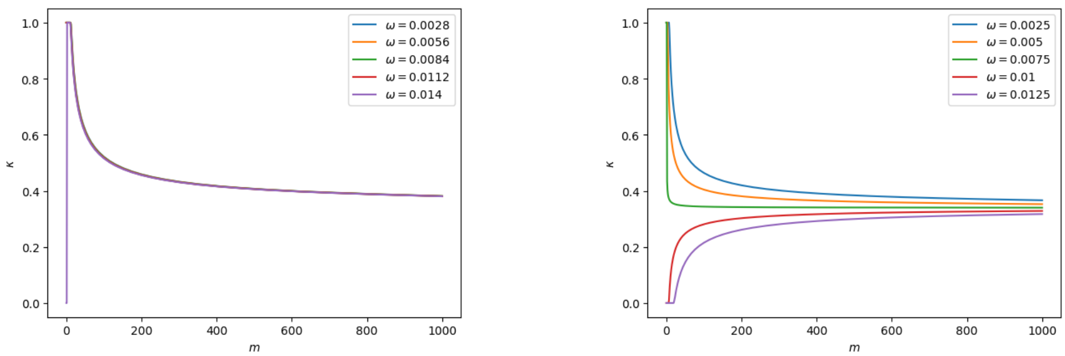
$$\mathbb{E}_t\left[(\Re_{t+1} - \mathsf{R})(\mathcal{G}_{t+1} c(m_{t+1}))^{-\rho}\right] = 0$$

where

$$m_{t+1} = \frac{\mathcal{R}_{t+1}}{\mathcal{G}_{t+1}}(m_t - c_t) + \zeta_{t+1}$$

Unquestionably, the covariances between the shocks affect the realized values of m_{t+1}, and therefore the realizations of future consumption given by $c(m_{t+1})$. However, as m_t becomes arbitrarily large, the ratio of labor income and present discounted value of human capital to assets tends to 0. As such, in the limit, irrespective of the covariances between the income shocks and asset returns, the consumer decides their portfolio share as if income is not a consideration. In other words, they behave like an individiual who has no labor income, and derives all income from the return on their investments (Carroll 2024).

Figure 3. Optimal portfolio share tends to the same value as $m \to \infty$



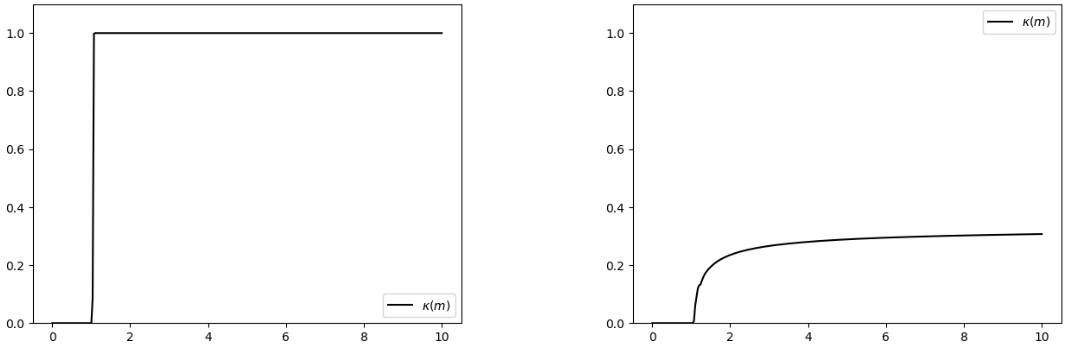
(a) Transitory shock ($\omega_{\nu,\zeta} = 0.014$) **(b)** Permanent shock ($\omega_{\eta,\nu} = 0.008$)

Figure 3 provides a rough idea of how the optimal portfolio rule behaves for different levels of correlation between income shocks and asset returns. Figure 3a shows that correlations between transitory income shocks and asset returns hardly affect the portfolio allocation decision. However, while the portfolio allocation rule is indeed affected significantly by permanent income and asset return shock correlations, as seen in Figure 3b, the share limit as $m \to \infty$ appears to be unaffected, as the portfolio allocation rules seem to converge. However, given the extremely large savings required to observe such convergence, which is unlikely to be observed of any agent in the model, a change in the correlation between the shocks would indeed affect the distribution of equity holdings. Section 3.4 elaborates further on how the equity portfolio share at the target level of wealth is indeed extreme in this model.

3.1.2 Optimal portfolio share at low wealth

While optimal portfolio shares at large wealth levels are not affected much by the correlation between shocks, we can observe a stark change in the share of the risky asset among individuals whose wealths are less than twice of their permanent incomes.

Figure 4. Individuals with low wealth invest all their savings in the risk-free asset



(a) Transitory shock **(b)** Permanent shock

Figure 4 shows that agents with normalized wealth less than 2 (somewhere between 1 and 2 to be precise), invest all their savings in the risk-free asset, irrespective of whether asset returns are correlated with transitory or permanent income shocks. At low levels of savings, notice that ζ_{t+1} comprises the major component of m_{t+1}, and high covariance between ν_{t+1} and ζ_{t+1} implies that low values of ζ_{t+1} go with low values of ν_{t+1}. Since the marginal utility of future consumption is high at low values of m_{t+1}, which coincides with low values of $\Re_{t+1}$, greater weight is placed on instances with low asset returns when taking the expectations in equation (9). This lowers the optimal portfolio share of the risky asset, in this case to 0. The other situation is when ν is correlated with η. Given the equation of m_{t+1}, this actually reduces the variability in $c(m_{t+1})$. However, the positive correlation between ν and η implies that when $\Re_{t+1} - \mathsf{R}$ is negative, $\mathcal{G}_{t+1}$ is low, implying that $\mathcal{G}_{t+1}^{-\rho}$ is higher. Thus, the instances of negative return are weighted higher in the excess return equation. Supposing that ν and η are perfectly correlated, if $\kappa 1$, then $m_{t+1} - \zeta_{t+1}$ becomes a constant, and the higher weight accorded to instances with negative return implies that the expectation becomes negative. On the other hand, if $\kappa = 0$, negative values of $\Re_{t+1} - \mathsf{R}$ are coupled with low values of $\mathcal{G}_{t+1}$ and therefore higher m_{t+1}, implying that the lower marginal utility of normalized consumption under negative excess returns makes $\kappa = 0$ closer to optimality.

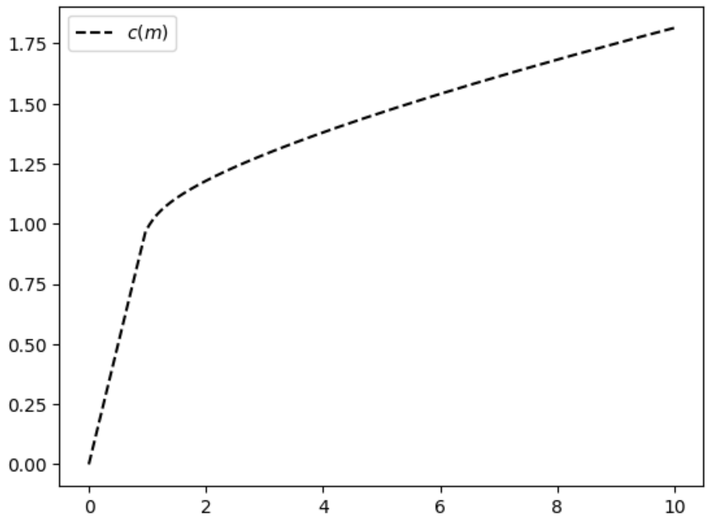
The second aspect is how the optimal portfolio share looks for slightly higher levels of wealth. Upto $m \approx 1.5$, agents consume almost all of their monetary resources and save next to nothing (see section 3.1.3). As such, the high MPC out of consumption causes variability in future monetary resources to translate into variability in future consumption at an almost one-to-one level. After a certain threshold, however, the MPC sharply falls, and the concavity of the consumption function ensures that it continues to fall. Moreover, due to the diminishing marginal utility of consumption and the very low magnitude of the marginal-marginal-utility of consumption, variability in m_{t+1} translates to very little variability in $c(m_{t+1})^{-\rho}$. The analysis of the finite horizon model in section 3.2 highlights the particular relevance of the MPC channel on this effect.

In that light, see that the argument for why optimal κ is low when permanent income shocks and returns are correlated does not crucially depend on the value of a_{t+1} being extremely low, and the optimal κ is always lower than the asymptotic value. However, when looking at the case with correlation between ζ and ν, note that the only channel through which there is any effect is the marginal utility of consumption, or $c(m_{t+1})^{-\rho}$. By the discussion above, this has less of an effect at higher values of m_{t+1}, and the optimal portfolio allocation rule resembles the one in the model with uncorrelated shocks.

3.1.3 Optimal consumption policy

While the properties of the consumption function in the buffer stock model are well understood, I reiterate some important features to augment that arguments provided above. Figure 5 shows the optimal consumption function.

Figure 5. Optimal consumption function in the buffer-stock model



Under the artificial no-borrowing constraint, the $c(m) \leq m$, which implies that c is both defined only for $m \geq 0$ and has a kink at the point where the borrowing constraint begins to bind. For this region, the MPC is 1, implying that for low values of m, as remarked earlier, the total savings rate is extremely low. However, the MPC sharply falls beyond this point. The code shows that the optimal consumption policy in the models with correlated shocks remains mostly unchanged. The low MPC for high values of m thus leads to the ineffectiveness of transitory shocks in affecting optimal portfolio choice.

3.2 Next-to-last period in finite-horizon

Due to the convergence properties of the consumption function, we know that the optimal consumption rule in the finite-horizon model for periods sufficiently away from the last period closely approximate the infinite horizon consumption rule. As such, if the next period's consumption is similar to the infinite-horizon consumption, the optimal portfolio allocation rule should also be similar to the infinite-horizon rule. On the other end of this discussion is the period that is next to last.

The consumer in the last period knows that their optimization problem in the last period boils down to maximizing utility from current-period consumption, which implies that $c_T(m_T) = m_T$. The first useful feature of this is that it provides us with a consumption function for which we have an analytical expression, which allows us to rewrite the optimal portfolio allocation condition as:

$$\mathbb{E}_{T-1}\left[(\Re_T - \mathsf{R})(\mathcal{R}_T a_T + \mathcal{G}_T \zeta_T)^{-\rho}\right]$$

First, note that a permanent income growth and transitory income shock are identical in the last period, so to analyze one is to analyze the other. While the coincidence of negative values of $\Re_T - \mathsf{R}$ and small values of $\mathcal{G}_T$ still holds true, the MPC out of total monetary resources is a constant 1. As a result, the only channel through which the portfolio choice problem differs at high m_{T-1} as opposed to low is the marginal utility of consumption. Figure A.1 shows that the optimal portfolio allocation is nearly identical to that in the infinite horizon problem, showing that the effect of correlations between permanent, as opposed to transitory, income shocks and asset returns for moderate values of m_{T-1} is due to the low MPC out of transitory income. However, it also reiterates the point that the low MPC implied by the consumption function in the infinite-horizon problem plays a negligible role in affecting the portfolio choice of the extremely wealthy.

3.3 Revisiting the excess return equation

Appendix A.2 shows that the excess return equation that determines optimal portfolio allocation can be approximated by

$$\mathbb{E}_t\left[\Re_{t+1}\right] - \mathsf{R} \approx \rho \mathrm{cov}(\Delta \ln C_{t+1}, \Re_{t+1}) \tag{11}$$

around steady state values of normalized consumption. Since real consumption change is approximately proportional to permanent income growth around the normalized steady state, a large increase in the covariance between permanent income growth shocks and shocks to $\Re_{t+1}$ would then imply that the excess return on equity is less than the covariance between consumption growth and the return on equity, scaled by the relative risk aversion coefficient. One thing to note here is that this approximation depends solely on the covariance, and not the correlation between permanent income growth and the risky rate of return. In fact, for fixed values of expected rates of return and volatility of the risky asset, one can observe that the optimal portfolio allocation rule does not depend much on the volatility of permanent income growth, just its covariance with the risky rate. For instance, under the baseline parameters for equity returns and various levels of permanent income growth volatility, one can observe that poor consumers make the switch from equity to the safe asset at approximately the same level of covariance between logged permanent income growth and equity return shocks.

A second aspect that can be analyzed about the optimal portfolio allocation rule is that for high enough covariance between permanent income growth and asset return shocks, the optimal portfolio share of equity is increasing in wealth, rather than decreasing as before. To understand this, we can look at the version of equation (11) that also takes into account the expected consumption growth,

$$\mathbb{E}_t\left[\Re_{t+1}\right] - \mathsf{R} \approx \left(1 - \rho\, \mathbb{E}_t\left[\Delta \ln C_{t+1}\right]\right)^{-1} \rho \mathrm{cov}(\Delta \ln C_{t+1}, \Re_{t+1})$$

At wealth levels lower than at which the normalized consumption steady state is attained, $\mathbb{E}_t \Delta \ln C_{t+1}$ is large,[6] making equity a less desirable asset. On the other hand, for large values of m_t, as will be elborated on in the next section, agents engage in dissaving behavior (in normalized terms), implying that wealthy agents consume more out of wealth. In such a case, the covariance between consumption growth and equity returns become positive irrespective of the covariance between permanent income growth and equity returns. With $\Delta \ln C_{t+1}$ decreasing in wealth due to the previously mentioned dissaving behavior, the identical portfolio share limit across the parameterizations of the model can be explained by a convergence in the consumption growth rate as $m \to \infty$ and a decreasing component of permanent income growth in the covariance between consumption growth and equity returns.

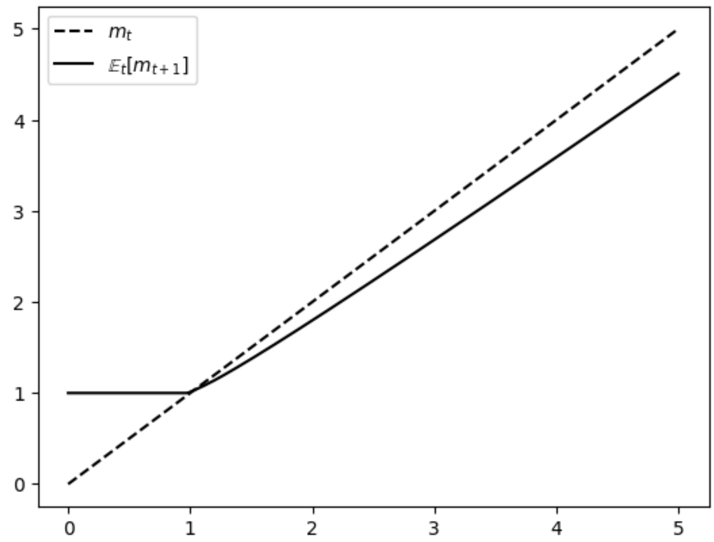
3.4 Behavior around target wealth

Till now, I have looked at the predictions of the model based on ad-hoc categorizations of poor and wealthy. However, the ability of the model to explain portfolio allocation decisions of individuals also depends on the saving behavior predicted by the model.

[6]Relative to the average level of permanent income growth

Deaton (1991) showed that under the satisfaction of a growth impatience condition, individuals save to achieve a target wealth, m^*. Assuming that a wealth distribution of agents facing idiosyncratic shocks would be centered around this target level, it would be informative to examine how agents behave at the target. Figure 6 shows how the

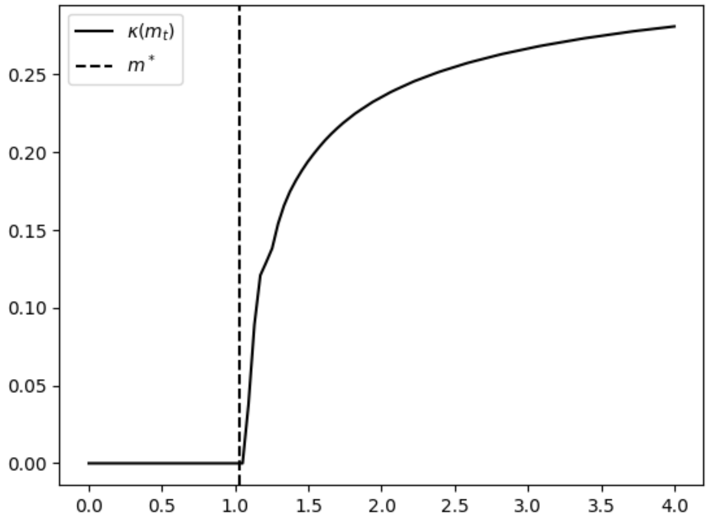
Figure 6. Expected next period normalized wealth conditional on current period wealth



dynamics of normalized wealth play out from period to period. For low values of m_t, where savings are zero, next-period wealth is entirely determined by the transitory income shock. Since the expected value of the shock is 1, future wealth is expected to be 1 for all low values of current wealth. Once the no-borrowing constraint stops binding, future wealth is increasing in current wealth. However, it is clear that expected future wealth is lower than current wealth for all values of current wealth slightly greater than 1. In fact, it can be numerically shown that the target level of saving is 2.8 percent of permanent income.

Figure 7 shows that at the target level of wealth, the optimal portfolio share accorded to equity is actually 0. This is because, under the no-borrowing constraint, the target level of wealth implies very little saving, or $a_{t+1} \approx 0$, where the optimal portfolio share was found to be 0. If the consumer faces a negative shock to log transitory income, they still have some savings left to allow them to remain at the target level of wealth. When a consumer faces a positive shock to log transitory income, they begin participating in the stock market, and invest a small proportion of their savings in equity. However, from the target wealth, with all savings in the risk-free asset, a consumer's normalized wealth in the next period cannot exceed 1.4 under the current parameterization of the truncated distribution used to model log shocks to income. Then, a large proportion of agents in the wealth distribution should invest no more than 20 percent of their savings in equity, which, of course, would be an extreme prediction. It would, instead, be more intuitive if

Figure 7. Optimal portfolio allocation at the target level of wealth



there was a smoother distribution over equity portfolio share.

A reason for these extreme findings is the binding no-borrowing constraint, and the consumer's heavy dependence on the transitory income for consumption. In fact, if the consumer experiences a negative shock to log transitory income, their consumption would drastically fall, as the target wealth lies just above the kink in the consumption function, and the MPC rises to 1 once the no-borrowing constraint binds. This means that the agent will begin saving once again only after they experience a positive transitory income shock, which prevents them from investing in equity at low levels of savings. In light of this, it can be observed that though the optimal portfolio share quickly rises to 1 in the case of a correlation between asset return and transitory income shocks, the target wealth actually lies below this region, implying that for levels of correlation high enough, the equity share at target wealth drops to 0.

In the next section, I modify the income process and relax the no-borrowing constraint to address this problem.

4 Zero income events

Until now, the version of the model examined saw agents save purely due to the no-borrowing constraint, as opposed to a need to save due to any form of uncertainty. As long as agents did not borrow, they were guaranteed positive wealth in the next period due to the next period's income. In this section, in the style of Carroll (1997), I relax the no-borrowing constraint, and instead introduce a zero-income event that induces the precautionary saving motive. Let the income process now be modelled as

$$P_t = \Gamma_t \eta_t P_{t-1}$$
$$Y_t = P_t \psi_t$$

where ψ_t is modelled as

$$\psi_t = \begin{cases} 0 & \text{with probability } \xi \\ \zeta_t & \text{with probability } (1-\xi) \end{cases}$$

and the realization of the zero-income event is independent of the realization of the any of the other three shocks. The normalized intertemporal budget constraint can then be rewritten as

$$m_t = \frac{\mathcal{R}_t}{\mathcal{G}_t}(m_{t-1} - c_{t-1}) + \psi_t$$

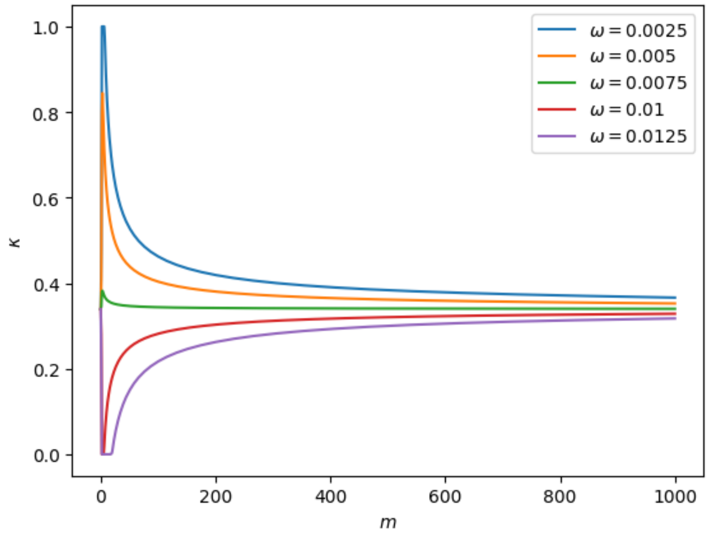
Where the introduction of ψ_t makes the only difference. The joint distribution of (η, ν, ζ) remains unchanged. In the finite-horizon version of the model, consumers may face zero-income events till period T, which means that no-default condition at the end of life is enough to ensure that consumers do not borrow. On the other hand, we can introduce a standard no-Ponzi condition in the infinite horizon version of the model to the same effect. As such, I numerically solve for the optimal policy functions only over a domain of positive savings.

One of the possible contentions with modelling the zero-income event as independent of asset return shocks could be that recessions are periods of layoffs, and therefore increasing unemployment. It would then be natural for the probability of the zero-income event to be greater conditional on lower returns on equity. While this is a valid line of reasoning, it is currently beyond the scope of this paper to introduce this added layer of complexity. Another interpretation that can be accorded to zero-income events are major unforeseen expenditures such as on healthcare, which can be approximated as situations where the consumer's net disposable income to spend on the consumption good is near zero. In such situations, they must consume using their savings.

4.1 Results

Following the same parameterization as in the previous section, with the added parameter ξ set at 0.005 (see Table 1), we can compute the optimal portfolio allocation rule for various settings of $\omega_{\eta,\nu}$. These can be seen in Figure 8.

Figure 8. Optimal portfolio allocation with zero-income events and income growth and asset return covariance

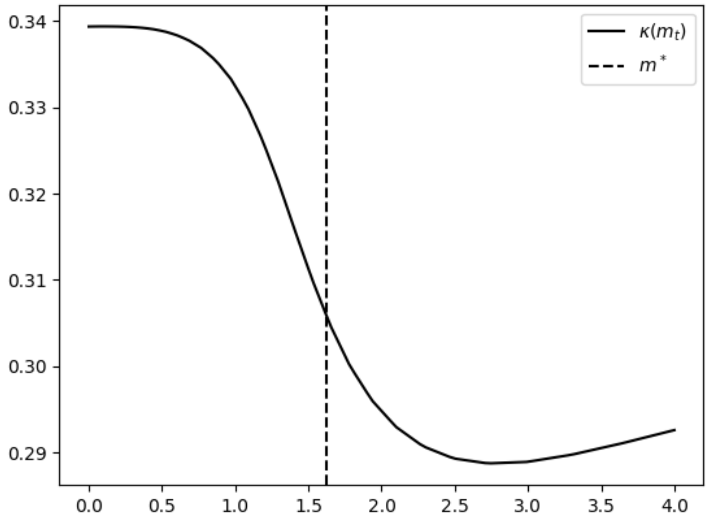


Like in the case with the no-borrowing constraint, as the correlation between the income growth and asset return shocks increases, low and moderate wealth individuals begin investing a greater proportion of their wealth in the safe asset. A departure, however, is that irrespective of the correlation, individuals well below the target wealth level choose to invest a shade above 30 percent of their wealth in equity. This, of course, does not reconcile with real-world observations, where the extremely poor often do not enter the equity market. On the other hand, like in the model with the no-borrowing constraint, the limiting value of the optimal portfolio allocation as $m \to \infty$ is largely unaffected by the correlation parameter.

A closer looks reveals that the equity portfolio share tends to the Merton-Samuelson limit as $m \to 0$ as well. This is because as m goes to 0, the marginal utility of consumption conditional on the realization of the zero income event grows arbitrarily large. As such, the weight placed on a realization such that the ratio of labor income to wealth is 0 grows arbitrarily large when solving the optimization problem implied by the excess return equation. This causes the portfolio share of equity to tend to the optimal share in the model without income.

Figure 9 shows the most significant way in which the zero-income event affects the distribution. As opposed to the case with the no-borrowing constraint, consumers would want to hold some of their savings in equity at the target level of wealth, which means that a distribution of agents facing idiosyncratic shocks would also be centered around a reasonable portfolio share. However, an issue one would be hard-pressed to reconcile with the data is that the equity portfolio share is actually decreasing in wealth around the target level, which makes for a counterintuitive prediction.

Figure 9. Optimal portfolio allocation at target wealth with zero-income events



4.2 Calibrating to U.S. Data

While the previous sections distil the primary insights from the model with an artificial parameterization of asset returns, both in terms of the equity premium and the variability of the returns from equity, I now look at how the model responds to being calibrated to parameters documented in the literature about U.S. data. Since the primary determinant of optimal portfolio allocation in the model that is of interest to us is the correlation between permanent income shocks and shocks to the return on the risky asset, I vary this parameter while holding the others constant at documented values. To begin, Mehra & Prescott (1985) estimate that the historical real rate of return on equity in the U.S. is 7.67 percent, while the return on a relatively risk-free securities over the same period was 1.31 percent.[7] Furthermore, the Sharpe ratio for these assets was calculated to be 0.37. Since ν is a mean-one lognormal, we know that:

$$\sigma_\nu^2 = \log\left(\left(\frac{\Re - \mathsf{R}}{0.37\Re}\right)^2 + 1\right)$$

I follow Carroll (1992) and set the standard deviations of logged permanent and transitory income shocks to 10 percent. Following the same paper, I set permanent income growth at 3 percent and the probability of the zero income event as 0.5 percent. I also set $\beta = 0.93$ and $\omega_{\zeta,\eta} = \omega_{\zeta,\nu} = 0$. I then solve the model using a baseline of $\rho = 4$ for different values of $\omega_{\eta,\nu}$. The full choice of parameters is then given in Table 1.

The first thing we can see under this new parameterization is that even with highly

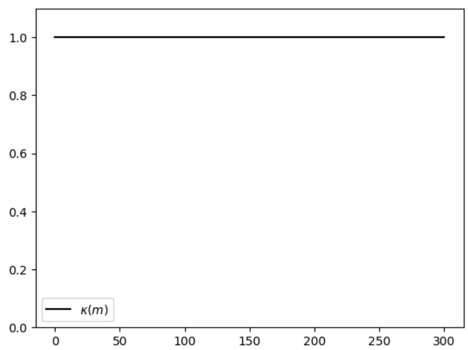
[7] The original data was later updated till 2005, which forms the source of these estimates. See Mehra (2006) for details.

Table 1. Parameters used to solve the model

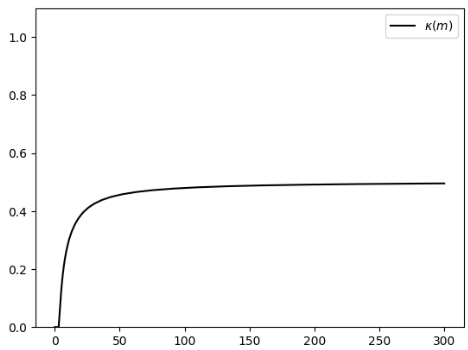
Parameter	Value	Source
ρ	4	
β	0.93	
Γ	1.03	Carroll (1992)
$\Re$	1.0767	Mehra (2006)
R	1.0131	Mehra (2006)
σ_ν^2	0.011	Mehra (2006)
σ_η^2	0.01	Carroll (1992)
σ_ζ^2	0.01	Carroll (1992)
ξ	0.005	Carroll (1992)

collinear shocks (corr($\log \nu$, $\log \eta$) ≈ 1), the optimal portfolio share is 1. This is because despite the high covariance, the excess return of more than 6 percent and the relatively low volatility, with a standard deviation of under 10.5 percent for the logged shock to returns, makes it difficult to justify holding the risk-free asset. In fact, Figure 10 shows that the equity share of portfolio falls to realistic levels under the no-borrowing constraint only when ρ is as large as 12. This number is close to the benchmark by Schreindorfer (2020), whose model incorporates disappointment averse preferences and has agents exhibit levels of relative risk aversion of close to 10.

Figure 10. U.S. figures requires extremely high RRA to explain the equity premium



(a) $\rho = 4$

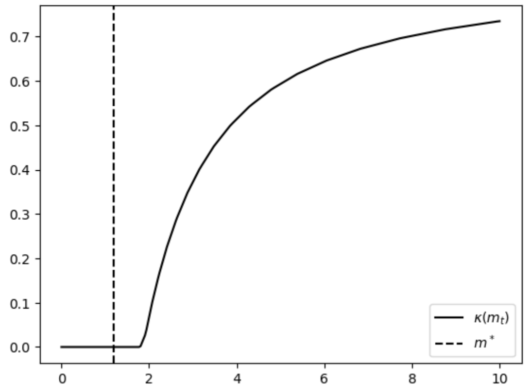


(b) $\rho = 12$

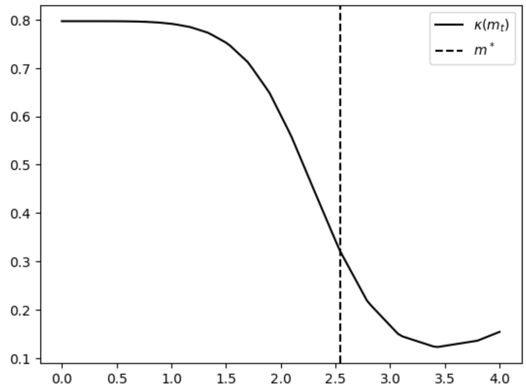
The limiting value of the portfolio share of equity is not very different under the model with zero-income events. In fact, the limiting portfolio share of equity is identical between the two models. What changes is how we can explain the equity share around the target wealth.

Figure 11 shows that for $\rho = 7.5$, optimal portfolio share around the target wealth actually falls to around 30%. Meanwhile, in the model with the no-borrowing constraint, the equity share at target is 0. This is so, even as the limiting values of share holdings under this parameterization are really high. As such, while the model with zero income

Figure 11. Portfolio share around target wealth under the no-borrowing constraint and zero-income events



(a) $\rho = 7$, NBC



(b) $\rho = 7.5$, Zero Income Events

events makes for a better approximation around the target wealth, it performs identically to the model with the borrowing constraint for high m_t and worse by predicting that the poor will invest close to the share limit in equity, particularly as $m_t \to 0$. In any case, a value for ρ greater than 7 does not produce suitable implications for consumption-savings behavior.

These findings show that while these models do not explain the equity premium perfectly, the introduction of the correlation between permanent income shocks and asset returns has produced a significant improvement in how optimal portfolio decisions fit the data at relatively reasonable levels of risk aversion. Furthermore, while the model with the no-borrowing constraint accurately prohibits the extremely poor from investing in equity, the introduction of the zero income events ensures that consumers engage in precautionary saving and invest some of their fairly substantial savings in equity as a result.

5 Discussion

5.1 Reconciling model predictions with data

Section 4.2 shows that while correlations between permanent income growth shocks and equity return shocks do explain the preference for saving in the safe asset to some extent, the returns on equity in the data are too high at relatively low volatility. As such, even with a very high correlation between permanent income growth shocks and equity returns, we require unrealistically high levels of relative risk aversion to explain the portfolio allocation observed in the data. A particular problem area is that the equity share limit for wealthy individuals is extremely high, at close to 80 percent, even with $\rho = 7$. While portfolio shares dip to reasonable numbers with zero income events around the target wealth, the model cannot explain the portfolio choice decisions of consumers outside a small neighbourhood of the target wealth in the distribution and non-participation in the stock market.

One approach used to address this problem is to incorporate non-expected utility preferences. Haliassos & Hassapis (2001) examine how various models of decision-making under risk improve predictions on equity holdings. While they conclude that changing preferences alone is not sufficient to account for the equity premium, they show that Kreps & Porteus (1978) preferences and, to a greater extent, Rank-Dependent Utility (Quiggin 1982) provide more realistic predictions on portfolio composition. Similarly, Schreindorfer (2020) shows that disappointment averse preferences (Gul 1991, Routledge & Zin 2010) can help explain the equity premium at a much lower level of relative risk aversion than with expected utility preferences under their model. However, the risk aversion coefficient necessary with expected utility preferences in their model is 34, meaning that despite the marked improvement, the new risk aversion coefficient is as high as 10.

Another explanation for the lower portfolio share of equity in the data than predicted in the model is pessimism and heterogeneity in beliefs about stock returns. Haliassos & Bertaut (1995) argue that in addition to correlations between labor income and asset returns and departures from expected utility, factors such as informational frictions provide a good explanation for the equity premium. While they note that a lack of knowledge about the stock market constrained participation, Dominitz & Manski (2007) find that agents also hold exaggerated beliefs about the possibility of negative nominal stock returns. Velásquez-Giraldo (2024) incorporates estimated beliefs from survey data into a life-cycle model and finds that stock market participation and conditional equity portfolio share can be explained by heterogeneous beliefs with a high average belief about the volatility of returns to equity. For consumers who believe that stock returns are extremely volatile, a high, or even moderate correlation between permanent income shocks and asset returns should ensure that they do not participate in the stock market, whereas the conditional distribution over equity portfolio share would then be determined by those who believe the stock market is not as volatile, though possibly more than actually observed in the data.

As far as non-participation in the stock market is concerned, minimum investment limits and fixed costs for participation have also been studied as probable obstacles. In the current model, with the parameterization as in section 4.2 stock market non-participation is observed at target wealth solely due to the no-borrowing constraint, and cannot be seen with the zero income event. However, a fixed participation cost would preclude consumers with very little wealth from investing their savings in equity, thereby generating a non-participation effect among low-wealth consumers. One limitation to this approach is that it cannot explain non-participation across wealth levels. Andersen & Nielsen (2011), Briggs et al. (2021) show that consumers who experience windfall gains do not see significantly higher participation rates, and that some of them even liquidate inheritances received in the form of stock.

5.2 Distribution of portfolio share

Given the form of the policy function, note that under either of the specifications, computing a wealth distribution of agents is tantamount to obtaining a distribution over the portfolio share of equity. To do so, we can simulate the trajectories of N agents (assuming a suitably large N) over a substantial number of periods to approximate a stationary distribution of wealth and portfolio shares. The first thing we need to do prior to this exercise is to formalize the manner in which shocks are generated.

Firstly, returns on equity are common to all agents. The shocks to permanent income growth and transitory income, however, can be modelled as idiosyncratic. Then, a consumer's wealth transition is given by

$$m_{i,t+1} = \frac{\mathsf{R} + \kappa(m_{i,t})(\Re_{t+1} - \mathsf{R})}{\mathcal{G}_{i,t+1}}(m_{i,t} - c(m_{i,t})) + \zeta_{i,t+1}$$

While transitory income shocks are independent of all else, note that idiosyncratic shocks to permanent income growth are correlated with the aggregate shock to the return on equity. As such, while permanent income growth shocks are not independent across individuals, they can be modelled as conditionally independent upon the realization of the shock to the return on equity. Then, the transition of wealth in the economy can be simulated by generating an asset return shock for each period, and generating N values for the permanent income growth shock from the conditional distribution of η, given the realization of ν. Thereafter, we can independently generate N values for ζ from the specified log-normal distribution. Given the realization of the shocks, the transition will be specified by $m_{i,t+1}$ as above.

5.3 Extension to a life-cycle model

A standard approach found in life-cycle modelling is to include time-varying sequences of permanent income growth factors $\{\Gamma_t\}_{t=\tau}^{T}$ and retirement ages after which income shocks are shut off. Additionally, there may be time-variation in shocks such as unforeseen health expenditures as the consumer ages and survival probabilities. The natural next step is to model transitory shocks to income and correlations between asset returns and permanent income growth shocks as time-variant. The consumption savings problem discussed in this paper can naturally be modified to

$$\max_{\{C_t\}_{t=\tau}^{T}} \mathbb{E}_\tau \sum_{t=\tau}^{T} \beta^{t-\tau} \aleph_\tau^t u(C_t)$$

where $\aleph_\tau^t$ denotes the probability of surviving till period t conditional on being alive in period τ. The budget constraints remain unchanged, while the income process is now distributed according to

$$\ln(\eta_t, \nu_t, \zeta_t) \sim \mathcal{N}\left(-\mathbf{diag}(\Sigma_t)/2, \Sigma_t\right)$$

For simplicity, we can assume that the volatility of logged permanent income growth shocks and asset return shocks are time-invariant, which implies that Σ_t is of the form

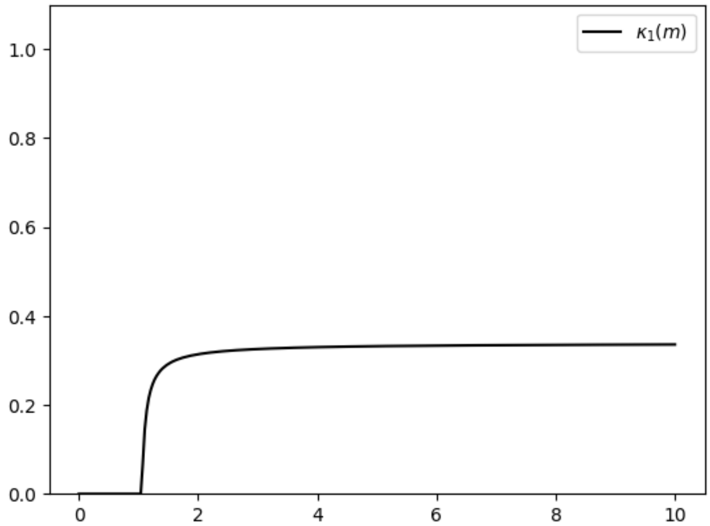
$$\Sigma_t = \begin{bmatrix} \sigma_\eta^2 & \omega_t & 0 \\ \omega_t & \sigma_\nu^2 & 0 \\ 0 & 0 & \sigma_{\zeta,t}^2 \end{bmatrix}$$

The only layer of added complexity in solving this model is a need to store a time-varying sequence of $\{\omega_t, \sigma_{\zeta,t}\}$, which would mean that for every period, we discretize a new distribution over shocks when computing expectations of the Euler equations to solve for the optimal policy functions. The idea of a time-varying distribution of income shocks and asset returns can be found in a life-cycle setting starting with Constantinides et al. (2002). While they directly influence the correlation between consumption and asset returns by introducing and removing wage income uncertainty in different periods of life, the decomposition of income shocks into permanent and transitory components in this model allows to maintain a fairly low contemporaneous correlation between wage income and asset returns, while also inducing a correlation between consumption and asset returns by affecting the present discounted value of human capital.

A Appendix

A.1 Last Period Optimal Portfolio Allocation

Figure A.1. Optimal portfolio allocation in the second-to-last period



A.2 Approximating the excess return equation

We want to approximate the excess return equation

$$\mathbb{E}_t\left[(\Re_{t+1} - \mathsf{R})(\mathcal{G}_{t+1}c_{t+1})^{-\rho}\right] = 0$$

Note that $\mathcal{G}_{t+1}c_{t+1} = C_{t+1}$. We can then divide both sides of the equation by C_{t+1} and get

$$\mathbb{E}_t\left[(\Re_{t+1} - \mathsf{R})\left(\frac{C_{t+1}}{C_t}\right)^{-\rho}\right] = 0$$

The first approximation we can apply is that

$$\frac{C_{t+1}}{C_t} \approx 1 + \Delta \ln C_{t+1}$$

for small enough values of $\Delta \ln C_{t+1}$. Then observe that $(1+x)^\alpha \approx 1 + \alpha x$ for small x. Put together,

$$\mathbb{E}_t\left[(\Re_{t+1} - \mathsf{R})\left(1 - \rho\Delta \ln C_{t+1}\right)\right] \approx 0$$

We can then expand this to

$$\mathbb{E}_t\left[(\Re_{t+1}-\mathsf{R})\left(1-\rho\Delta\ln C_{t+1}\right)\right]=\mathbb{E}_t\left[(\Re_{t+1}-\mathsf{R})\right]\mathbb{E}_t\left[1-\rho\Delta\ln C_{t+1}\right]-\mathrm{cov}(\rho\Delta\ln C_{t+1},\,\Re_{t+1})$$
$$\mathbb{E}_t\left[\Re_{t+1}\right]-\mathsf{R}\approx\frac{\rho\mathrm{cov}(\Delta\ln C_{t+1},\,\Re_{t+1})}{1-\rho\,\mathbb{E}_t\left[\Delta\ln C_{t+1}\right]}$$

Around steady state values of normalized consumption, $\mathbb{E}_t\,\Delta\ln C_{t+1}\approx\mathbb{E}_t\,\Delta\ln P_{t+1}=\gamma-\frac{\sigma_\eta^2}{2}\approx 0$, where $\gamma=\ln\Gamma$. Then

$$\mathbb{E}_t\left[\Re_{t+1}\right]-\mathsf{R}\approx\rho\mathrm{cov}(\Delta\ln C_{t+1},\,\Re_{t+1})$$

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