Precautionary saving and portfolio choice with joint income and equity return processes

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Abstract

Portfolio choice models normally predict that the portfolio share of equity declines with wealth, while the poor invest all their savings in equity. I examine an infinite-horizon consumption-saving problem where the agent has to decide how to invest their savings between a risky and risk-free asset. Following Viceira (2001), I model permanent income growth shocks as correlated with returns on equity. I find that with moderate levels of relative risk aversion, this correlation lowers the optimal portfolio share of equity at target wealth levels (Kimball's (1991) temperance motive), while inverting the relationship between wealth and equity portfolio share. I find that, under standard calibrations, neither the model with borrowing constraints nor the one with positive unemployment probability can fully explain the equity premium observed in U.S. data without high levels of risk aversion. The calibrated temperance motive may explain the observed equity portfolio share in conjunction with other popular explanations documented in the literature.

Link to code: https://github.com/sidd3888/precautionary-saving-portfolio-choice

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The multivariate lognormal discretization code I use to solve this model is available in the HARK toolkit.

1 Introduction

The standard buffer stock model (Deaton 1991, Carroll 1992) predicts that consumers engage in precautionary saving either because of borrowing constraints or the possibility of unemployment. Despite this, under a canonical formulation with independent shocks to income and returns on equity, the model counterintuitively predicts that poor consumers should invest their entire savings in equity, while wealthy consumers should save in the safe asset.[1] The predictions of lifetime portfolio optimization models by Merton (1969), Samuelson (1969) suggest that the major component in the willingness to invest in the risky asset arises from a regular income stream (Heaton & Lucas 1997).

Kimball (1991) showed that consumers may display a temperance motive to moderate exposure to other risks when faced with income uncertainty, even if these risks are statistically independent. Koo (1999) shows that an increase in the volatility of permanent income growth shocks significantly amplifies the temperance effect, though transitory income shocks have negligible influence. This does not change a key prediction of the model, however, that the poor should invest all their savings in equity, and the portfolio share of equity declines with wealth. Furthermore, given the equity premium and the volatility of returns on stock observed in the data (Mehra & Prescott 1985), the baseline model suggests that even the wealthy should invest all their savings in stocks.

One explanation for the modest portfolio share of equity observed in the data is that future consumption is positively correlated with returns on equity. Constantinides et al. (2002) use an overlapping generations (OLG) model to explore this idea. In their model, there are three generations of individuals: the young, middle-aged, and old/retired. Retired individuals receive a labor income of zero, which means that the future consumption of the middle-aged is highly correlated with the returns on equity. On the other hand, the middle-aged face uncertain wage income, implying that the future consumption of the young depends on more than just the returns on equity, lowering their correlation. Their argument is that a positive correlation implies that the realization of low marginal utility of consumption coincides with high returns, and vice versa. Thus, the low portfolio share of equity is a consequence of the inability of the young to participate in the stock market due to borrowing constraints. This paper departs from their modeling assumptions by retaining income uncertainty and introducing correlations between permanent income growth shocks and equity returns to correlate future consumption and equity returns. A positive correlation between these shocks reduces the potential of equity risk to serve as a hedge against fluctuations in future consumption due to permanent income growth risk.

This paper primarily focuses on an infinitely-lived agent[2] who consumes a single good and maximizes the discounted sum of utility from consumption. The agent faces three risks: shocks to the return on equity, permanent income growth, and transitory income. The model can accommodate pairwise correlations between all three shocks, though the focus of this paper will be on the case where permanent income shocks and equity returns are correlated, while transitory

[1]The one exception is that with zero income events, arbitrarily poor consumers invest the same fraction of their savings in equity as the arbitrarily wealthy.

[2]The setup is well-suited to life-cycle modeling, but the insights derived in this paper are from the predictions of the infinite-horizon model.

income shocks are independent of both. The agent decides their consumption (therefore, saving) and the portfolio share of equity in every period. The flexibility of the model in terms of the distribution of the shocks allows us to establish that a correlation between transitory income shocks and equity returns has a negligible effect on the portfolio share of equity for all agents but those with near-zero savings.

There is ample justification in the literature to model permanent income growth as correlated with equity returns while modeling transitory income shocks as independent of the two. Campbell (1996) shows that there is a high correlation between the present value of human capital and market returns, despite finding that the contemporaneous correlation between wage income and stock returns is low. However, the source of this covariance in his model is due to a common time-varying discount factor applied to both calculations. Baxter & Jermann (1997) also find that the correlation between the returns to human capital and physical capital are high, even as labor and capital income growth rates may not exhibit a high correlation. With independent shocks across time, permanent income growth proportionally affects the expected present discounted value of human capital, while the independent transitory shock allows for the low correlation in actual labor income growth and returns on equity.

There have been papers that have studied correlations between labor income growth and equity returns previously in a life-cycle setting. Bodie et al. (1992) modelled the labor supply decision as endogenous, and made after the current return to equity has been determined. They show that agents vary their labor supply ex-post to cushion themselves against greater risks taken in their investment decisions. Subsequent papers study this relationship with an exogenous income process and a positive relationship between income growth and equity returns. Benzoni et al. (2007) explore a model with cointegrated labor income and stock market returns in a continuous-time setting. Bagliano et al. (2014) on the other hand study a discrete-time model with a permanent income shock that has both an aggregate and an idiosyncratic component, and two risky assets. The closest paper to mine is by Viceira (2001), whose model is based on a similar buffer-stock setting with correlations between permanent income growth and risky asset returns. The difference here is that I focus on the problem of the infinitely-lived agent, whose income is subject to both transitory and permanent income shocks, and who may face an artificial borrowing constraint instead of possible spells of unemployment that otherwise serves as the motive for precautionary saving. My focus, thus, is on portfolio choice across the wealth distribution and around the target wealth, as opposed to over the life-cycle.[3]

The saving behavior of the agent who faces an artificial borrowing constraint is slightly different as compared to that of the agent who faces a positive probability of zero income. While both factors serve as precautionary saving motives, the consumer who faces artificial borrowing constraints tends to save very little at their target wealth and therefore invests none of their savings in equity, which lends support to the well-studied argument that borrowing constraints among the young, for instance, can cause reduced or non-participation in the stock market (Constantinides et al. 2002, Haliassos & Hassapis 2002, Kogan et al. 2007, Jang & Park 2015, Harenberg 2018). I also artificially restrict the portfolio share of equity to between 0 and 1, which implies that consumers cannot supplement their portfolio with debt-financed equity positions. Davis et al. (2006) argue that borrowing costs can be a major deterrent against investing in equity

[3] Fagereng et al. (2020) point out that the composition of the portfolio even within the class of risky assets is heterogeneous across the wealth distribution. This affects the rate of return on the risky component of the portfolio, which, in turn, also affects portfolio choice.

using loans, and it is beyond the scope of this paper to examine the differences in rates of return and mechanisms available for consumers to borrow versus save.

The findings of this paper can be divided into three major strands. First, a positive correlation between permanent income growth shocks and returns on equity does have a negative effect on the portfolio share of equity of an agent with a moderate level of risk aversion ($\rho = 4$). When faced with the no-borrowing constraint, this effect is most prominently observed among the poor, who save very little and invest all of their savings in the risk-free asset. Proposition 1 also shows that this effect is indeed discontinuous in the covariance between permanent income growth and equity return shocks. In fact, poor agents invest either all or none of their savings in equity, and they invest none if the covariance between income growth and equity return exceeds the equity premium times the inverse of the expected risky rate, scaled by the RRA parameter. This effect is absent in the model where a positive zero-income probability serves as the motive for precautionary saving instead of the borrowing constraint. This effect gradually disappears in wealth, as the limiting portfolio share of equity approximately tends to the optimal portfolio share in the models by Merton (1969), Samuelson (1969), which is independent of the correlation between income and equity return shocks. Proposition 3 formalizes this result for both the cases where the agent faces a no-borrowing constraint and a positive probability of zero-income.

Second, in a closely related effect, the optimal portfolio share of equity is increasing in wealth for above-threshold levels of covariance between the permanent income growth and equity return shocks. In particular, Proposition 2 shows that in the model with the no-borrowing constraint, this effect is perfectly tied with the threshold level of covariance that makes the poor invest in the risk-free asset. Numerical approximations and further analysis reveal why this conclusion eludes the model with zero-income events. In particular, as the agent becomes arbitrarily poor, the incidence of the zero-income event on utility differences becomes infinitely pronounced. Since the zero-income event occurs independently of all other shocks, the agent accords highest weight to sequences of zero-income shocks in the limit, causing portfolio choice behavior to revert to that in the Merton-Samuelson model.

Third, with the artificial borrowing constraint, agents do not invest in equity or invest extremely small proportions of their savings in it around the target level of wealth, while the threat of unemployment better explains the portfolio share of equity around target levels of wealth conditional on participation in the stock market. Though the model generates insightful predictions for an equity premium set at 3 percent with the standard deviation of the logged shock to the return set at 15 percent, calibrating the model to parameters implied by data on U.S. equity returns as reported in Mehra (2006) necessitates an elevated risk aversion parameter of around 7 to generate modest portfolio shares of equity around target wealth.

The covariance of permanent income growth and returns on equity is, of course, one among multiple explanations posited to resolve the equity premium puzzle. A section of the literature argues that habit-formation makes consumers more averse to the volatility in consumption across time-periods induced by the risk on equity return (Constantinides 1990, Abel 1990, Detemple & Zapatero 1991, Campbell & Cochrane 1999, Otrok et al. 2002). Another strand of the literature focuses on the low participation rates in the stock market despite the large equity premium and uses market frictions like fixed participation costs to explain increasing participation with age and wealth (Cocco 2005, Gomes & Michaelides 2005, Alan 2006, Khorunzhina 2013, Fagereng et al. 2017). Modest portfolio shares of equity are also explained by heterogeneous and pessimistic beliefs about the returns on stocks. The evidence suggests that beliefs about the volatility of returns on equity are exaggerated, and portfolio decisions are correlated with the reported beliefs about the stock market (Dominitz & Manski 2007, Hurd et al. 2011, Amromin & Sharpe 2014, Ameriks et al. 2020, Velásquez-Giraldo 2024). Yet another explanation revolves around heterogeneity in preferences or departures from expected utility (Guvenen 2009, Haliassos & Bertaut 1995, Haliassos & Hassapis 2001, Routledge & Zin 2010, Schreindorfer 2020).[4] Section 6 discusses how some of these explanations may tie in with the model in this paper to favorably revise the predictions of the model without requiring a large coefficient of relative risk aversion.

Finally, numerically solving a model with three potentially correlated shocks presents its own computational challenges. Numerical integration, particularly given a joint distribution of more than two random variables, can be painfully slow. To that end, one of the contributions of this paper is to provide a simple algorithm to discretize a multivariate lognormal distribution to compute expectations in the Euler equation. This algorithm is detailed in Section 2.5.

The paper is structured as follows. Section 2 sets up the model and provides a preliminary discussion of the solution to the model. Section 3 provides the main results and analysis contained in the paper. Section 5 examines the implications of the calibrated temperance motive for portfolio choice in the model. Section 6 provides a few comments and concludes.

2 Model

2.1 The basic problem

Consider the standard consumption-saving problem. The agent maximizes their discounted lifetime utility from the consumption stream $\{C_t\}_{t=0}^{T}$. The primary focus in this paper is on the infinitely-lived agent, so we will let $T = \infty$ for the most part of the analysis. Their problem can then be written as

$$\max_{\{C_t\}_{t=0}^{T}} \mathbb{E}_0 \sum_{t=0}^{T} \beta^t u(C_t) \tag{1}$$

subject to the period-wise constraints

$$C_t + A_{t+1} = W_t + Y_t$$

where A_{t+1} is the total stock of assets at the end of period t, W_t the total monetary wealth at the beginning of period t, and Y_t is labor income in period t. Labor income is modeled exogenously, where Y_t is centered around an "expected" permanent income

$$Y_t = \theta_t P_t \tag{2}$$

where θ_t is a mean-one transitory shock. Permanent income, P_t, itself evolves according to the process

$$P_t = \Gamma_t P_{t-1} \psi_t \tag{3}$$

[4] See Barucci (2003) for a detailed exploration of the assumptions imposed under the standard rational paradigm and the puzzles that stem from relaxing them.

where Γ_t is the predictable component of the growth of permanent income, and ψ_t is a mean-one shock. When considering the infinite-horizon problem, I will simplify the permanent income prcess with $\Gamma_t = \Gamma$ for all t.

The agent can also invest, in a perfectly divisible manner, between a risk-free and a risky (equity) asset. If the consumer chooses to hold ς_{t+1} share of their savings in the risky asset in period t (that is, their portfolio at the start of period $t+1$ contains ς_{t+1} share of the risky asset), wealth in period $t+1$ is determined by

$$W_{t+1} = \overbrace{\left(\mathsf{R} + \varsigma_{t+1}(\Re_{t+1} - \mathsf{R})\right)}^{\mathscr{R}_{t+1}} A_{t+1} \tag{4}$$

$$\Re_{t+1} = \Re\eta_{t+1} \tag{5}$$

where $\Re > \mathsf{R}$ is the expected return on the risky asset, η_{t+1} is a mean-one shock, and $\mathscr{R}_{t+1}$ is the effective rate of return on assets stemming from the portfolio allocation decision ς_{t+1}. The choice of ς_{t+1} is restricted to the interval [0, 1], thus precluding debt-financed equity positions. We can then use equation (4) rewrite the period budget constraint as

$$W_{t+1} = \mathscr{R}_{t+1}(W_t + Y_t - C_t)$$

Allowing $M_t = W_t + Y_t$ to denote the total current level of monetary resources

$$M_{t+1} = \mathscr{R}_{t+1}(M_t - C_t) + Y_{t+1}$$

2.2 Income and wealth uncertainty

I know describe the exact distribution followed by the shocks to income and wealth. There are, particularly, three uncertainties of interest. ψ and η have been introduced before. The third is ζ, of which θ is a function. The specifications of interest of this function will be detailed in section 2.4. I model the shocks $\{(\psi_t, \eta_t, \zeta_t)\}_{t\geq 0}$ as independently and identically distributed multivariate lognormal random variables. That is,

$$\log(\psi_t, \eta_t, \zeta_t) \sim \mathcal{N}(\mu, \Sigma)$$

where

$$\Sigma = \begin{bmatrix} \sigma_\psi^2 & \omega_{\psi,\eta} & \omega_{\psi,\zeta} \\ \omega_{\eta,\psi} & \sigma_\eta^2 & \omega_{\eta,\zeta} \\ \omega_{\zeta,\psi} & \omega_{\zeta,\eta} & \sigma_\zeta^2 \end{bmatrix}$$

$$\mu = \begin{bmatrix} -\sigma_\psi^2/2 \\ -\sigma_\eta^2/2 \\ -\sigma_\zeta^2/2 \end{bmatrix}$$

The marginal distributions of each of the shocks ensure that $\mathbb{E}_t[\psi] = \mathbb{E}_t[\eta] = \mathbb{E}_t[\zeta] = 1$. Meanwhile, $\omega_{x,y}$ captures the covariance of any two variables x and y among the three.

2.3 Optimal behavior

In this paper, I restrict my attention to an agent with CRRA utility. That is, the agent's period utility from consumption is given by

$$u(C_t) = \frac{C_t^{1-\rho}}{1-\rho}$$

We can then write the problem in Bellman form:

$$V_t(M_t,\, P_t) = \max_{\{C_t,\, \varsigma_{t+1}\}} \frac{C_t^{1-\rho}}{1-\rho} + \beta\, \mathbb{E}_t \left[V_{t+1}(\mathscr{R}_{t+1}(M_t - C_t) + Y_{t+1},\, P_{t+1}) \right] \tag{6}$$

Normalize all period t variables by the permanent income P_t (since A_t is determined in period $t-1$, it is normalized by P_{t-1}), and denote these new variables in lowercase (i.e. $c_t = C_t/P_t$). Letting $\mathscr{G}_{t+1} = \Gamma_{t+1}\psi_{t+1}$,

$$m_{t+1} = \frac{\mathscr{R}_{t+1}}{\mathscr{G}_{t+1}}(m_t - c_t) + \theta_{t+1}$$

The Bellman formulation then becomes

$$v_t(m_t) = \max_{c_t,\, \varsigma_{t+1}} \frac{c^{1-\rho}}{1-\rho} + \beta\, \mathbb{E} \left[(\mathscr{G}_{t+1})^{1-\rho} v_{t+1} \left(\frac{\mathscr{R}_{t+1}}{\mathscr{G}_{t+1}}(m_t - c_t) + \theta_{t+1} \right) \right] \tag{7}$$

and the consumption Euler equation is then given by[5]

$$c_t^{-\rho} = \beta\, \mathbb{E}_t \left[\mathscr{R}_{t+1} (\mathscr{G}_{t+1} c_{t+1})^{-\rho} \right] \tag{8}$$

The optimality condition for the portfolio share, ς_{t+1} is slightly trickier, given that it is bounded by $[0,\, 1]$. Finally, note that portfolio share is irrelevant when $a_{t+1} = 0$, which means that we can only pin it down for when $a_{t+1} \neq 0$. In that case, a choice of ς_{t+1} is optimal if[6]

$$\begin{cases} \mathbb{E}_t \left[(\Re_{t+1} - \mathsf{R})(\mathscr{G}_{t+1} c_{t+1})^{-\rho} \right] = 0 & \varsigma_{t+1} \in (0,\, 1) \\ \mathbb{E}_t \left[(\Re_{t+1} - \mathsf{R})(\mathscr{G}_{t+1} c_{t+1})^{-\rho} \right] \geq 0 & \varsigma_{t+1} = 1 \\ \mathbb{E}_t \left[(\Re_{t+1} - \mathsf{R})(\mathscr{G}_{t+1} c_{t+1})^{-\rho} \right] \leq 0 & \varsigma_{t+1} = 0 \end{cases} \tag{9}$$

[5] Use the envelope theorem to see that, $u'(c_t) = v'(m_t)$, and

$$v'(m_t) = \beta\, \mathbb{E} \left[\mathscr{G}_{t+1}^{-\rho} \mathscr{R}_{t+1} v'(m_{t+1}) \right]$$

Then using the same conditions under optimality for period $t+1$, $v'(m_{t+1}) = u'(c_{t+1})$.

[6] A sufficient condition for this formulation of optimality is that $v(m_t)$ is single-peaked over possible values of ς_{t+1} for every $m_t \in \mathbb{R}_+$. Holding c_t fixed, the problem of interest is

$$\mathbb{E} \left[\psi_{t+1}^{1-\rho} v_{t+1}(m_{t+1}) \right] = \int_{\eta_{t+1} < \frac{\mathsf{R}}{\Re}} \psi_{t+1}^{1-\rho} v_{t+1}(m_{t+1}) d\mu + \int_{\eta_{t+1} > \frac{\mathsf{R}}{\Re}} \psi_{t+1}^{1-\rho} v_{t+1}(m_{t+1}) d\mu$$

where μ is the multivariate lognormal measure. The first integral is strictly increasing in ς_{t+1}, while the second integral is strictly decreasing in it. In the first integral, for every realization of the $t+1$ shocks, increasing ς_{t+1} leads to a decrease in future wealth. Since the value function is concave, the utility of wealth decreases at an accelerating rate. By the same argument, the increase in the value of the second integral occurs at decelerating rate. This means that the sum of the two integrals is single-peaked over ς_{t+1}.

Now see that whenever the optimal portfolio decision is to hold a mixture of both the safe and the risky asset, the consumption Euler equation can be reduced to

$$\begin{aligned}
c_t^{-\rho} &= \beta\,\mathbb{E}_t\left[(\mathsf{R} + \varsigma_{t+1}(\Re_{t+1} - \mathsf{R}))(\mathscr{G}_{t+1}c_{t+1})^{-\rho}\right] \\
&= \beta\left[\mathbb{E}_t\left[\mathsf{R}(\mathscr{G}_{t+1}c_{t+1})^{-\rho}\right] + \varsigma_{t+1}\,\mathbb{E}_t\left[(\Re_{t+1} - \mathsf{R})(\mathscr{G}_{t+1}c_{t+1})^{-\rho}\right]\right] \\
&= \beta\mathsf{R}\Gamma_{t+1}^{-\rho}\,\mathbb{E}_t\left[(\psi_{t+1}c_{t+1})^{-\rho}\right] && \text{(from (9))}
\end{aligned}$$

2.4 Motives for precautionary saving

I extend the models by Deaton (1991) and Carroll (1992), along with their distinct approaches to precautionary saving. The former model is based on the idea that the agent is unable to borrow to smooth consumption, while the latter model imposes a zero-income risk, due to which the agent chooses not to borrow. I examine both scenarios to see how portfolio choice is affected by the joint distribution of income and equity return shocks. Going forward, I shall refer to the former as the no-borrowing constraint (NBC) model, and the latter as the no-income risk (NIR) model.

In the NBC model, the agent is faced with an artificial no-borrowing constraint,

$$A_{t+1} \geq 0$$

which implies that $a_{t+1} \geq 0$. Along with this, the transitory income shock $\theta_{t+1} = \zeta_{t+1}$, which means that θ_{t+1} is simply the mean-one lognormal variable from the joint distribution described previously. The agent with the NBC faces a kink in the consumption function at the point where the borrowing constraint begins to bind. At this point, the agent simply consumes the totality of their wealth. Lemma 1, echoing Deaton (1991), provides a useful characterization of this kink that will appear in later results.

Lemma 1. *For some $\tilde{m} > 0$, the optimal consumption function for the NBC agent satisfies*

$$\begin{cases} c(m) = m & m \leq \tilde{m} \\ c(m) < m & m > \tilde{m} \end{cases}$$

Proof. Consider the problem of the NBC agent without the borrowing constraint for the current period only. Under such conditions, $c(m) > 0$ for all $m \geq 0$, as $\lim_{c\to 0} u'(c) = \infty$ for $\rho > 0$. Since $c(m) - m$ is continuous, and $\lim_{c\to\infty} u'(c) = 0$, $c(m) < m$ for $m > \tilde{m}$. We know $\tilde{m} > 0$ because $c(0) > 0$. However, since $c(m) - m$ is decreasing, $c(m) = m$ for $m = \tilde{m}$. Since the borrowing constraint binds, for $m < \tilde{m}$, $c(m) = m$. □

In the NIR model, the agent faces no such constraint on borrowing, though they may potentially encounter a zero-income event, i.e. their labor income may be zero in some periods. Particularly, their transitory income is determined in each period by

$$\theta_{t+1} = \begin{cases} 0 & \text{with probability } \wp \\ \zeta_{t+1}/(1-\wp) & \text{with probability } 1-\wp \end{cases}$$

where the resolution of the zero-income event is independent of any of the other shocks. In the finite-horizon version of the model, consumers may face zero-income events till period T with positive probability, which means that no-default condition at the end of life is enough to ensure that consumers do not borrow.[7] On the other hand, we can introduce a standard no-Ponzi condition in the infinite horizon version of the model (which is the limit of the finite horizon problem) to the same effect. As such, I numerically solve for the optimal policy functions only over a domain of positive savings.

One of the possible contentions with modelling the zero-income event as independent of asset return shocks could be that recessions are periods of layoffs, and therefore increasing unemployment. It would then be natural for the probability of the zero-income event to be greater conditional on lower returns on equity. While this is a valid line of reasoning, it is currently beyond the scope of this paper to introduce this added layer of complexity. Another interpretation that can be accorded to zero-income events are major unforeseen expenditures such as on healthcare, which can be approximated as situations where the consumer's net disposable income to spend on the consumption good is near zero. In such situations, they must consume using their savings.

2.5 Numerical solution

The first part of solving the model is to address the problem of efficiently computing expectations of the marginal utilities of consumption decisions in future periods. Under the current formulation, the shocks to income and returns are drawn independently in each time period, which means that it is enough to discretize these shocks using their single-period distribution. I use an equiprobable approximation of a truncated version (at 3 standard deviations of the underlying standard normal) of these lognormal variables.[8]

Here are the steps involved in discretizing the distribution:

1. Choose a suitable truncation of the distribution in each dimension by choosing an interval $[p_{min}, p_{max}] \subseteq [0, 1]$
2. Divide the interval given by $\left[\Phi^{-1}(p_{min}), \Phi^{-1}(p_{max})\right]$ into n intervals of $\frac{p_{max}-p_{min}}{n}$ probability each, $I = \left\{\left[\Phi^{-1}\left(\frac{(i-1)p_{max}+(n-i+1)p_{min}}{n}\right), \Phi^{-1}\left(\frac{ip_{max}+(n-i)p_{min}}{n}\right)\right]\right\}_{i=1}^{n}$
3. Decompose the covariance matrix Σ using the Cholesky decomposition and obtain a matrix L such that $LL^T = \Sigma$
4. Then construct the random variables $Y = \mu + LZ$, where $Z \sim \mathcal{N}(0, I)$, to get $Y \sim \mathcal{N}(\mu, \Sigma)$
5. Construct the set I^3 and , and compute the conditional expectation of the vector of shocks $X = \exp(Y)$ in each set of I^3, yielding the set of equiprobable atoms $S = \left\{(\psi, \eta, \zeta)_j\right\}_{j=1}^{n^3}$

[7] Carroll & Shanker (2024) discuss that the NBC model can be interpreted as the limit of the NIR model as $\wp$ tends to 0.

[8] The code for this algorithm is available as part of a contribution I made to the HARK toolkit.

Computing expectations of functions of these shocks can now be reduced to the following approximation

$$\mathbb{E}\left[g(\psi,\,\eta,\,\zeta)\right] \approx n^{-3}\sum_{j=1}^{n^3} g(\psi_j,\,\eta_j,\,\zeta_j)$$

Given a method to compute expectations over a multivariate lognormal distribution, the next step is to compute the optimal decision rules. To do so, I sequentially apply the endogenous grid method to a two-stage representation of the portfolio choice problem (stage 2) and the consumption-saving problem (stage 1), and iteratively compute the infinite horizon policy rule (Carroll 2006, 2024).[9]

3 Results

3.1 Revisiting the excess return equation

Appendix A.2 shows, similar to Mehra & Prescott's (1985) original derivation, that the excess return equation that determines the optimal portfolio allocation can be approximated as

$$\mathbb{E}_t\left[\Re_{t+1}\right] - \mathsf{R} \approx \rho \mathrm{cov}(\Delta \ln C_{t+1},\, \Re_{t+1}) \tag{10}$$

Appendix A.4 further provides an approximation of the covariance of log consumption growth and equity returns, which allows us to derive an approximation of the optimal equity share rule. The subsequent sections will use this approximation to derive certain properties of the optimal equity share rule.

As a preface to those results, we can make certain observations on the basis of equation (10). First, since consumption growth is closely related to permanent income growth, a large increase in the covariance between permanent income growth shocks and shocks to $\Re_{t+1}$ would then imply that the excess return on equity is less than the covariance between consumption growth and the return on equity, scaled by the relative risk aversion coefficient. One thing to note here is that this approximation depends solely on the covariance, and not the correlation between permanent income growth and the risky rate of return. Proposition 1 shows that the optimal equity share at low wealth levels depends solely on the covariance of income shocks and equity returns, the average rates of return, and the relative risk aversion coefficient. The particularly surprising element here, is that this is independent of the volatility of equity. Closely linked to this is the observation is that optimal equity share is increasing in wealth, in contrast to the decreasing share observed in the standard model. This is shown in Proposition 2.

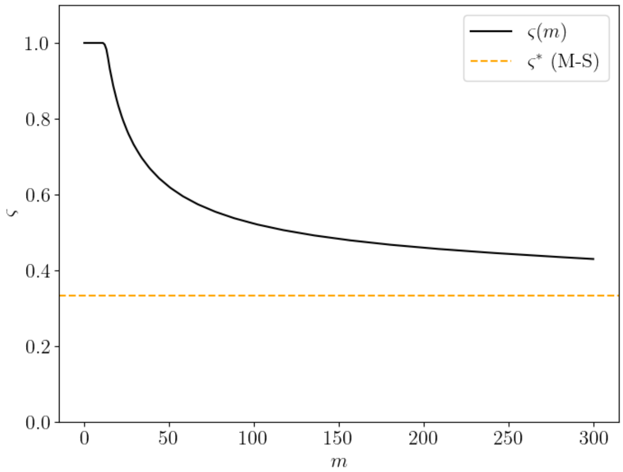
The other aspect is how the optimal equity share behaves as the wealth to permanent income ratio grows arbitrarily large. Proposition 3 shows that the optimal equity share converges to a value independent of the covariance between income shocks and equity returns. In particular, the optimal equity share converges to the Merton-Samuelson limit, which is the optimal equity share in the absence of any income.[10]

[9]See Appendix B.1 for a detailed explanation of the computational algorithm.
[10]See Carroll (2024) for a discussion.

3.2 Baseline results

I start by looking at the baseline NBC model with uncorrelated shocks to income and asset returns. I set the equity premium at 3 percent for this part of the analysis, and the standard deviation of the logged shock to the equity return at 15 percent, i.e. $\sigma_\eta = 0.15$. I let all other parameters be as in Table 1. Figure 1 shows that the optimal portfolio allocation $\varsigma(m)$ is 1 at low values of m and decreases to an asymptotic value, as specified by the Merton-Samuelson model, as m tends to infinity.

Figure 1. Optimal portfolio share with uncorrelated shocks



The analytic expression for the asymptotic portfolio share (the Merton-Samuelson share) shows that $\varsigma^* = \lim_{m\to\infty} \varsigma(m)$ is increasing in the equity premium. Since the consumer is risk-averse, increasing the volatility of the returns to equity will decrease its attractiveness, thus reducing the optimal value of ς^* upon an increase in σ_η. However, this is a largely counterintuitive prediction, keeping in mind that those with low wealth-to-income ratios are predicted to invest their entire income in equity. This is a result of the fact that the wealthy agent consumes largely out of wealth, and their consumption exceeds their income, while the poorer agent consumes largely out of income.

3.3 Optimal equity share at low wealth levels

The first result, stemming from the NBC model, is that optimal portfolio share is either 0 or 1 for low wealth levels.

Proposition 1. *Under the no-borrowing constraint, for some $m^* > 0$, the optimal portfolio rule is given as*

$$\varsigma(m) = \begin{cases} 0 & \text{if } \omega_{\psi,\eta} + \frac{c'(1)}{c(1)}\omega_{\zeta,\eta} < \tilde{\omega} \\ 1 & \text{if } \omega_{\psi,\eta} + \frac{c'(1)}{c(1)}\omega_{\zeta,\eta} > \tilde{\omega} \end{cases}$$

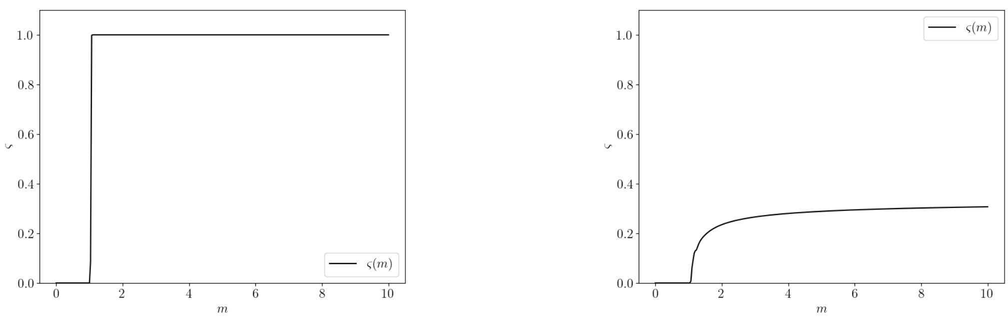
for all $m < m^*$, where $\tilde{\omega} \cong \frac{\Re - \mathsf{R}}{\rho \Re}$.

Proof: See Appendix A.5

The idea behind the result above is that for low levels of wealth, the NBC agent consumes their entire wealth. As a result, changing the portfolio share does not affect the covariances between future consumption growth and equity returns. For agents who save small but negligible amounts of their wealth, the excess return on equity cannot make up for the imbalance caused by large covariances between income shocks (whether permanent or transitory) and equity return shocks. As such, low wealth NBC agents faced with such uncertainty invest their savings in the risk-free asset. A key observation to make from the above result is that the threshold on covariance between income shocks and equity returns for the NBC agent to exhibit such behavior is independent (at least directly) of the volatility of returns on equity. The only role the volatility of equity plays, then, is through the bounds it imposes on $\omega_{\psi,\eta}$ and $\omega_{\zeta,\eta}$.

The behavior described in the result above can be seen in the figure below.

Figure 2. Individuals with low wealth invest all their savings in the risk-free asset



(a) Transitory shock **(b)** Permanent shock

Figure 2 shows that agents with normalized wealth less than 1 (somewhere between 1 and 2 to be precise), invest all their savings in the risk-free asset, irrespective of whether asset returns are correlated with transitory or permanent income shocks. At low levels of savings, notice that θ_{t+1} comprises the major component of m_{t+1}, and high covariance between η_{t+1} and ζ_{t+1} implies that low values of θ_{t+1} go with low values of η_{t+1}. Since the marginal utility of future consumption is high at low values of m_{t+1}, which coincides with low values of $\Re_{t+1}$, greater weight is placed on instances with low asset returns when taking the expectations in equation (9). This lowers the optimal portfolio share of the risky asset, in this case to 0. The other situation is when η is correlated with ψ. Given the equation of m_{t+1}, this actually reduces the variability in $c(m_{t+1})$. However, the positive correlation between η and ψ implies that when $\Re_{t+1} - \mathsf{R}$ is negative, $\mathscr{G}_{t+1}$ is low, implying that $\mathscr{G}_{t+1}^{-\rho}$ is higher. Thus, the instances of negative return are weighted higher in the excess return equation. Supposing that η and ψ are perfectly correlated, if $\varsigma = 1$, then $m_{t+1} - \theta_{t+1}$ becomes a constant, and the higher weight accorded to instances with negative return implies that the expectation becomes negative. On the other hand, if $\varsigma = 0$, negative values of $\Re_{t+1} - \mathsf{R}$ are coupled with low values of $\mathscr{G}_{t+1}$ and therefore higher m_{t+1}, implying that the lower marginal utility of normalized consumption under negative excess returns makes $\varsigma = 0$ closer to optimality.

The second aspect is how the optimal portfolio share looks for slightly higher levels of wealth. While the kink in the consumption function occurs at $m < 1$, agents consume almost all of their monetary resources and save next to nothing upto $m \approx 1.5$ (see Appendix B.2). As such, the high MPC causes variability in future monetary resources to translate into variability in future consumption at an almost one-to-one level. After a certain threshold, however, the MPC sharply falls, and the concavity of the consumption function ensures that it continues to fall. Moreover, due to the diminishing marginal utility of consumption and the very low magnitude of the marginal-marginal-utility of consumption, variability in m_{t+1} translates to very little variability in $c(m_{t+1})^{-\rho}$. The analysis of the finite horizon model in section 3.5 highlights the particular relevance of the MPC channel on this effect.

One additional change in the portfolio share rule observed in the case of covarying permanent income shocks is that after the threshold value of wealth for $\varsigma(m) = 0$ has passed, the optimal portfolio share is not only low, but increasing in wealth. This is formally proved in Proposition 2. The result also shows that for a reasonable collection of values of the equity premium, risk-free rate, and volatility of equity returns, the optimal portfolio share is less than the Merton-Samuelson share for all wealth levels if and only if the covariance between permanent income shocks and equity returns is greater than $\tilde{\omega}$.

Proposition 2. *Under the no-borrowing constraint, with $\omega_{\zeta,\eta} = 0$,*

(i) the optimal portfolio share is increasing in wealth (m) if $\omega_{\psi,\eta} > \sigma_\eta^2$

(ii) if $\omega_{\psi,\eta} < \sigma_\eta^2$, it is increasing (decreasing) in wealth if and only if $\omega_{\psi,\eta} > \tilde{\omega}$ $(\omega_{\psi,\eta} < \tilde{\omega})$, where $\tilde{\omega} \cong \frac{\Re - \mathsf{R}}{\rho \Re}$. Consequently, for plausible values of $\Re$, R, and σ_η, the optimal portfolio share is less than (greater than) the Merton-Samuelson share for all wealth levels if and only if $\omega_{\psi,\eta} > \tilde{\omega}$ $(\omega_{\psi,\eta} < \tilde{\omega})$.

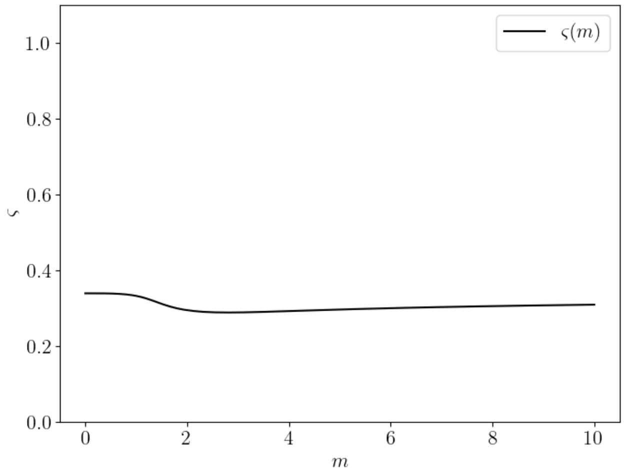
Proof: See Appendix A.6

Proposition 2 shows that if $\sigma_\psi \geq \sigma_\eta$, the optimal portfolio share is increasing in wealth under two possible conditions: either $\omega_{\psi,\eta}$ exceeds σ_η^2 or, if $\sigma_\eta^2 > \tilde{\omega}$ and $\omega_{\psi,\eta} > \tilde{\omega}$. If $\sigma_\psi < \sigma_\eta$, then $\omega_{\psi,\eta} < \sigma_\eta^2$ by definition. As such, portfolio share can be increasing in wealth if and only if $\tilde{\omega} < \sigma_\eta \sigma_\psi$ and $\omega_{\psi,\eta} > \tilde{\omega}$. The first observation to make is that this is the same threshold by which the optimal portfolio share is 0 or 1 for low wealth levels in the NBC model. The second, and perhaps more stark observation to make here, is that the optimal portfolio share is less than the Merton-Samuelson share for all wealth levels if and only if $\omega_{\psi,\eta} > \tilde{\omega}$. Since the Merton-Samuelson share represents the optimal level of investment in equity in the absence of any income, the implication of the result is that when the covariance between shocks to income growth and equity returns is high enough, agents exhibit greater caution than even when they have no labor income.

Given these findings, the natural question to ask is whether this behavior prevails in the NIR model. As seen in Figure 3, the optimal portfolio share for low wealth levels asymptotically converges to the Merton-Samuelson share. This is in sharp contrast to the NBC model, where the optimal portfolio allocation rule is monotonic in wealth.

A close examination of the Figure 3 and 2b shows that the optimal portfolio share of equity in the NIR model is almost identical to that in the NBC model for ratios of wealth to permanent income greater than 2. The divergence in the optimal portfolio share for low wealth levels from the previous model can be attributed to the fact that the zero-income event, where $\theta_{t+1} = 0$, has more extreme consequences for low wealth levels, and features prominently in the expectation calculation in equation (9).

Figure 3. Arbitrarily poor invest the Merton-Samuelson share



Since the excess return equation can be thought of as a weighted average of the excess return on equity, we must observe the nature of events that are weighted the highest. Note that as wealth becomes smaller, the marginal utility of consumption contingent on the realization of the zero-income event becomes arbitrarily large. This is so, because future consumption tends to 0 as wealth tends to 0. As such, the instances with $\theta_{t+1} = 0$ are weighted the highest, and are asymptotically accorded full weight in the excess return calculation. As current wealth tends to 0, next-period wealth also tends to 0 under the realization of the zero-income event. By the same argument, optimal consumption and portfolio allocation behavior in the next period will reflect an arbitrarily large weight accorded to the realization of the zero-income event in period $t+2$. By repeated application of this argument, the optimal portfolio share should be akin to the model with no labor income asymptotically. This is the description of the Merton-Samuelson model, where consumption is purely a cake-eating problem. Thus, the optimal portfolio allocation behavior also tends to the Merton-Samuelson share, as shown in Figure 3.

3.4 Optimal equity share at high wealth levels

The next question of interest is how the portfolio allocation rule behaves as the wealth-to-permanent-income ratio grows arbitrarily large. The following proposition shows that the optimal equity share, in both the NBC and NIR models, converges to the Merton-Samuelson share.

Proposition 3. *In both the NBC and NIR models, the optimal portfolio share does not depend on $\omega_{\psi,\eta}$ or $\omega_{\zeta,\eta}$ as $m \to \infty$. Furthermore,*

$$\lim_{m \to \infty} \varsigma(m) \approx \frac{\Re - \mathsf{R}}{\rho \sigma_\eta^2}$$

Proof: See Appendix A.7

Proposition 3 shows that the optimal portfolio share is not only independent of the covariance of the income process with returns on equity, but also tends to a value that is characteristic of the model in which there is no income. The explanation for this behavior can be found by examining the optimal portfolio share condition, given by:

$$\mathbb{E}_t \left[(\Re_{t+1} - \mathsf{R})(\mathscr{G}_{t+1} c(m_{t+1}))^{-\rho} \right] = 0$$

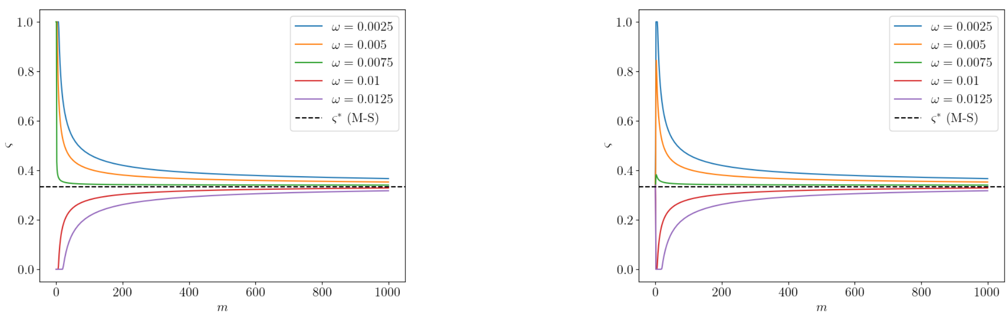
where

$$m_{t+1} = \frac{\mathscr{R}_{t+1}}{\mathscr{G}_{t+1}}(m_t - c_t) + \theta_{t+1}$$

Unquestionably, the covariances between the shocks affect the realized values of m_{t+1}, and therefore the realizations of future consumption given by $c(m_{t+1})$. However, as m_t becomes arbitrarily large, two things occur. First, the agent consumes primarily out of their wealth, as opposed to their income. As such, the marginal utility of consumption becomes less sensitive to the realization of the income shocks. After all, as wealth becomes arbitrarily large, a small deviation in income produces an even smaller deviation in consumption, depending on the limiting MPC, which is an insignificant percentage deviation in consumption. Since the approximate excess return equation highlights that the optimal portfolio share is determined by equating the scaled covariance of consumption growth and equity returns with the equity premium, changing the income process does not affect the optimal portfolio share by much.

Second, as wealth grows large, the ratio of labor income and present discounted value of human capital to assets tends to 0. As such, in the limit, irrespective of the covariances between the income shocks and asset returns, the consumer decides their portfolio share as if income is not a consideration. In other words, they behave like an individual who has no labor income, and derives all income from the return on their investments (Carroll 2024).

Figure 4. Optimal portfolio share is unaffected by the income process



(a) NBC model **(b)** NIR model

Figure 4 shows the different optimal portfolio allocation rules for various values of $\omega_{\psi,\eta}$. While they all converge to the same limiting portfolio share in wealth, their functional forms are indeed very dissimilar. Given the extremely large savings required to observe such convergence, which is unlikely to be observed of any agent in the model, a change in the correlation between the shocks would indeed affect the equity holdings of even the right-tail of the wealth distribution. To do so, we would require a closer look at the savings behavior implied by the model, and the natural distribution generated under optimal behavior. Section 3.6 elaborates further on how optimal portfolio allocation behavior differs in the NBC and NIR models around the target level of wealth.

3.5 Next-to-last period in finite-horizon

Due to the convergence properties of the consumption function, we know that the optimal consumption rule in the finite-horizon model for periods sufficiently away from the last period closely approximate the infinite horizon consumption rule. As such, if the next period's consumption is similar to the infinite-horizon consumption, the optimal portfolio allocation rule should also be similar to the infinite-horizon rule. On the other end of this discussion is the period that is next to last.

The consumer in the last period knows that their optimization problem in the last period boils down to maximizing utility from current-period consumption, which implies that $c_T(m_T) = m_T$. The first useful feature of this is that it provides us with a consumption function for which we have an analytical expression, which allows us to rewrite the optimal portfolio allocation condition as:

$$\mathbb{E}_{T-1}\left[(\Re_T - \mathsf{R})(\mathscr{R}_T a_T + \Gamma\psi_T\theta_T)^{-\rho}\right] = 0$$

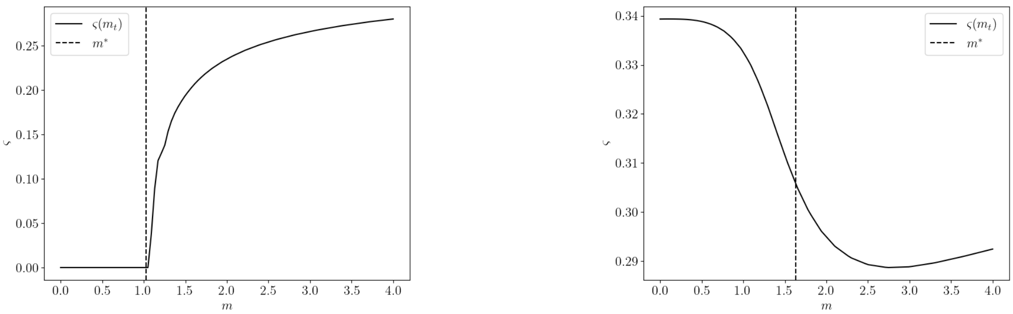
First, note that a permanent income growth and transitory income shock are equivalent in the last period, so to analyze one is to analyze the other. While the coincidence of negative values of $\Re_T - \mathsf{R}$ and small values of $\mathscr{G}_T$ still holds true, the MPC out of total monetary resources is a constant 1. As a result, the only channel through which the portfolio choice problem differs at high m_{T-1} as opposed to low is the marginal utility of future consumption. Figure B.2 shows that the optimal portfolio allocation is nearly identical to that in the infinite horizon problem, showing that the effect of correlations between permanent, as opposed to transitory, income shocks and asset returns for moderate values of m_{T-1} is due to the low MPC out of transitory income. However, it also reiterates the point that the limiting MPC implied by the consumption function in the infinite-horizon problem does not affect the portfolio choice of the extremely wealthy.

3.6 Behavior around target wealth

Till now, I have looked at the predictions of the model based on ad-hoc or asymptotic categorizations of poor and wealthy. However, the ability of the model to explain portfolio allocation decisions of individuals also depends on the saving behavior predicted by the model. Deaton (1991) showed that under the satisfaction of a growth impatience condition, individuals save to achieve a target wealth, m^*. Assuming that a wealth distribution of agents facing idiosyncratic shocks would be centered around this target level, it is informative to examine how agents behave at the target.

Figure 5a shows that at the target level of wealth, the optimal portfolio share accorded to equity is actually 0. This is because, under the no-borrowing constraint, the target level of wealth implies very little saving, or $a_{t+1} \approx 0$, where the optimal portfolio share was found to be 0. If the consumer faces a negative shock to log transitory income, they still have some savings left to allow them to remain at the target level of wealth. When a consumer faces a positive shock to log transitory income, they begin participating in the stock market, and invest a small proportion of their savings in equity. However, from the target wealth, with all savings in the risk-free asset, a consumer's normalized wealth in the next period cannot exceed 1.4 under the current parameterization of the truncated distribution used to model log shocks to income. Then, a

Figure 5. Optimal portfolio share at target wealth



(a) NBC model

(b) NIR model

large proportion of agents in the wealth distribution should invest no more than 20 percent of their savings in equity, which, of course, would be an extreme prediction.

A reason for these extreme findings is the binding no-borrowing constraint, and the consumer's heavy dependence on the transitory income for consumption. In fact, if the consumer experiences a negative shock to log transitory income, their consumption would drastically fall, as the target wealth lies just above the kink in the consumption function, and the MPC rises to 1 once the no-borrowing constraint binds. This means that the agent will begin saving once again only after they experience a positive transitory income shock, which prevents them from investing in equity at low levels of savings. In light of this, it can be observed that though the optimal portfolio share quickly rises to 1 in the case of a positive correlation between asset return and transitory income shocks, the target wealth actually lies below this region, implying that for levels of covariance high enough, the equity share at target wealth drops to 0.

Figure 5b shows the most significant way in which the zero-income event affects the distribution. As opposed to the case with the no-borrowing constraint, consumers hold a much greater proportion of their permanent income in their savings. As such, they would want to hold some of their savings in equity at the target level of wealth, which means that a distribution of agents facing idiosyncratic shocks would also be centered around a reasonable portfolio share. However, one thing to note is that the optimal portfolio share at the target level of wealth is decreasing in wealth. This means that upon being close to the long-run savings target, the consumer would increase their equity holdings if they are faced with a negative shock to transitory income and decrease it if the shock is positive. Given the absence of any serial correlation in transitory income shocks, this prediction is rather counterintuitive.

4 Distribution of wealth and portfolio share

The results in the previous section highlight the nuances of the how consumers in the model optimally allocate their savings between the risk-free asset and equity. However, another informative facet of the model is the stationary distribution over wealth and portfolio share that the behavior of agents gives rise to. The distribution of wealth and portfolio share is important in that it provides a richer picture of how the precautionary saving motives of agents interact with the portfolio choice problem.

Given the form of the policy function, note that under either of the specifications, computing a wealth distribution of agents is tantamount to obtaining a distribution over the portfolio share of equity. To do so, we can simulate the trajectories of N agents (assuming a suitably large N) over a substantial number of periods to approximate a stationary distribution of wealth and portfolio shares. The first thing we need to do prior to this exercise is to formalize the manner in which shocks are generated.

Firstly, returns on equity are common to all agents. The shocks to permanent income growth and transitory income, however, can be modelled as idiosyncratic. Then, a consumer's wealth transition is given by

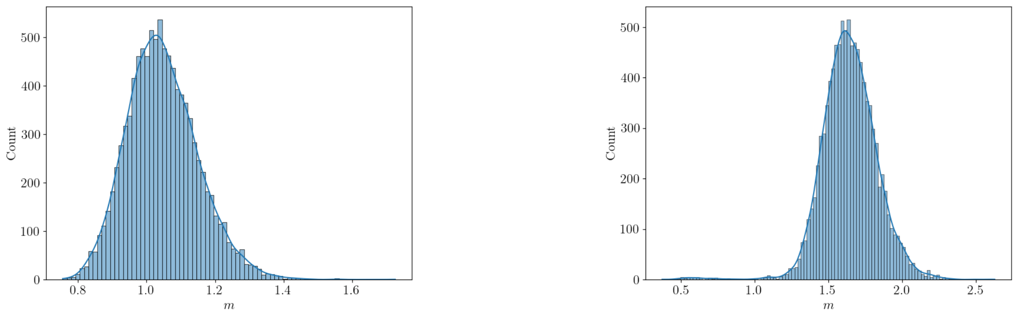
$$m_{i,t+1} = \frac{\mathsf{R} + \varsigma(m_{i,t})(\Re\eta_{t+1} - \mathsf{R})}{\Gamma\psi_{i,t+1}}(m_{i,t} - c(m_{i,t})) + \theta_{i,t+1}$$

Note that the shocks to income experienced by all the individuals are independent of each other only when the shocks to income are uncorrelated to the common shock, which is the return on equity. However, the shocks to permanent and transitory income can be modelled as conditionally independent upon the realization of the shock to the return on equity. Then, the transition of wealth in the economy can be simulated by generating an asset return shock for each period, and generating N values for the permanent income growth shock from the conditional distribution of ψ and ζ, given the realization of η. In the NBC model, the generated values of ζ are exactly the values of θ. In the NIR model, however, we can independently generate N draws of a Bernoulli random variable with probability $1 - \wp$ to determine which agents experience a zero-income event, thereby determining the value of θ. Given the realization of the shocks, the transition will be specified by $m_{i,t+1}$ as defined above.

For the general statement of the problem, my simulation algorithm proceeds as follows. I first generate a single draw from the marginal distribution of the return on equity. Then, given the realized value of $\ln(\eta)$, I calculate the conditional distribution of the bivariate normal variables $(\ln(\psi), \ln(\zeta))$ as in Anderson (2003, p.34), and instantiate the derived bivariate lognormal. I then use the draws from this distribution and the Bernoulli draws to determine $\psi_{i,t+1}$ and $\theta_{i,t+1}$, for each i. Finally, I calculate the transition of wealth for each agent, and repeat the process for a large number of periods ($T = 120$) to approximate the stationary distribution of wealth and *incoming* portfolio shares. That is, the distribution over portfolio shares of equity stemming from previous period wealth, which reflects each agent's beginning-of-period asset holdings. I look at the baseline parameterization used in Section 3.

Figure 6 shows the distribution of wealth-to-permanent income among individuals in the NBC and NIR models. As is already well understood, the stationary distribution of wealth is centered around the target level of wealth in both models, though these savings targets are different, due to the differences in how agents respond to borrowing constraints and zero-income events. Given this stationary distribution, we can define a stationary distribution over portfolio shares of equity, which is a function of the wealth-to-permanent income ratio for each individual. The stationary distribution of portfolio shares in shown in Figure 7. Both distributions are radically different. Due to the nature of the optimal portfolio share rule in the NBC model, there is a high rate of clustering around 0. That is, many agents do not participate in the stock market, so long as they do not face a large transitory income shock in the current period. In the NIR model, however, the distribution is more spread out, reflecting the smoothness of the portfolio

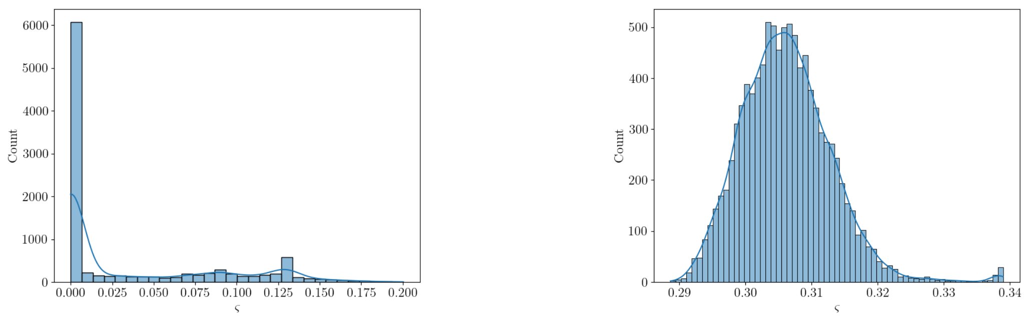
Figure 6. Stationary wealth distribution in the NBC and NIR models



(a) NBC model **(b)** NIR model

share rule around the target level of wealth. The joint distribution of wealth and equity portfolio

Figure 7. Stationary distribution of portfolio shares in the NBC and NIR models



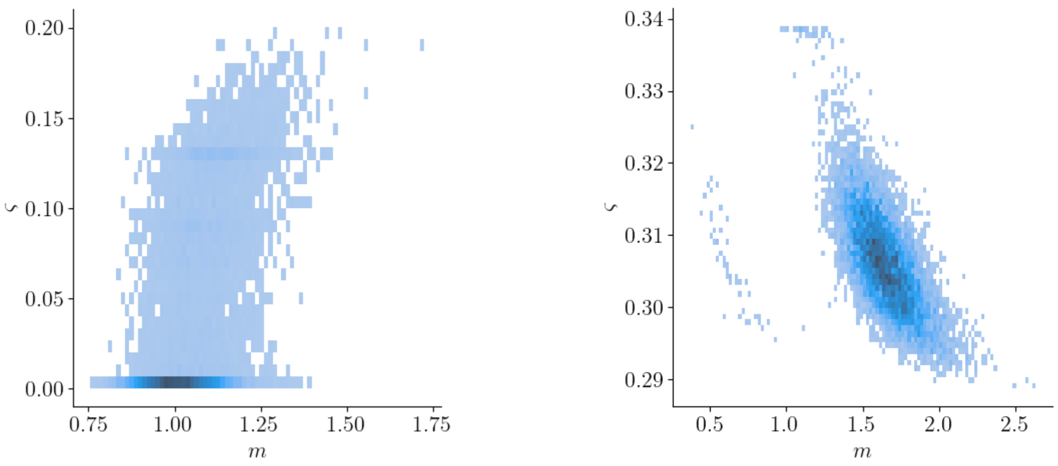
(a) NBC model **(b)** NIR model

share going into the next period is rather trivial, as it is concentrated along the optimal portfolio share rule, weighted according to the stationary distribution of wealth. On the other hand, there is merit in examining the joint distribution of current portfolio holdings (which are determined by the previous period's wealth), and current wealth. This distribution is shown in Figure 8. As predictable, this distribution exhibits the relationship between wealth and portfolio share according to the optimal portfolio share rule around target wealth, but also reflects the dispersion arising from idiosyncratic shocks in the next period. Furthermore, the distribution in the NIR model also reflects the small proportion of agents who are faced with zero-income events, who, despite their lower wealth, invest similar amounts in equity as those with higher wealth.

5 Calibration

While the previous sections distill the primary insights from the model with an artificial parameterization of asset returns, both in terms of the equity premium and the variability of the returns from equity, I now look at how the model responds to being calibrated to parameters documented in the literature about U.S. data. Since the primary determinant of optimal portfolio allocation in the model that is of interest to us is the covariance between permanent income shocks and shocks to the return on the risky asset ($\omega_{\psi,\eta}$), I vary this parameter while holding

Figure 8. Stationary joint distribution of wealth and portfolio shares in the NBC and NIR models



(a) NBC model **(b)** NIR model

the others constant at documented values. To begin, Mehra & Prescott (1985) estimate that the historical real rate of return on equity in the U.S. is 7.67 percent, while the return on a relatively risk-free securities over the same period was 1.31 percent.[11] Furthermore, the Sharpe ratio for these assets was calculated to be 0.37. Since η is a mean-one lognormal, we know that:

$$\sigma_\eta^2 = \log\left(\left(\frac{\Re - \mathsf{R}}{0.37\Re}\right)^2 + 1\right)$$

I follow Carroll (1992) and set the standard deviations of logged permanent and transitory income shocks to 10 percent. Following the same paper, I set permanent income growth at 3 percent and the probability of the zero income event as 0.5 percent. I also set $\beta = 0.93$ and $\omega_{\varsigma,\psi} = \omega_{\varsigma,\eta} = 0$. I then solve the model using a baseline of $\rho = 4$ for different values of $\omega_{\psi,\eta}$. The full choice of parameters is then given in Table 1.

Table 1. Parameters used to solve the model

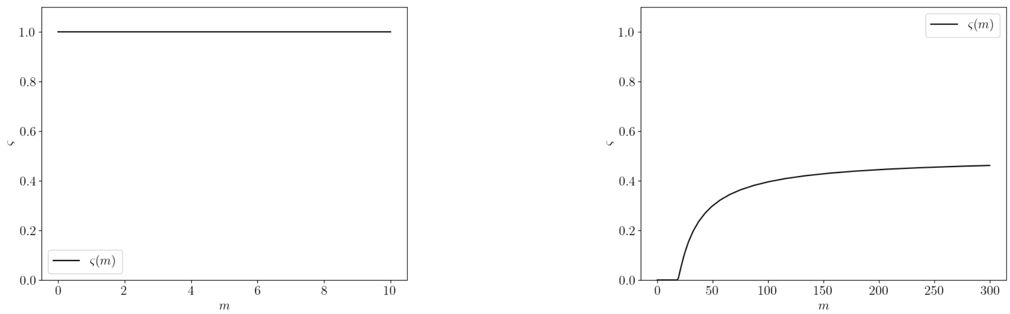
Parameter	Value	Source
ρ	4	
β	0.93	
Γ	1.03	Carroll (1992)
$\Re$	1.0767	Mehra (2006)
R	1.0131	Mehra (2006)
σ_η^2	0.011	Mehra (2006)
σ_ψ^2	0.01	Carroll (1992)
σ_ς^2	0.01	Carroll (1992)
$\wp$	0.005	Carroll (1992)

The first thing we can see under this new parameterization is that even with highly collinear shocks ($\text{corr}(\log\eta, \log\psi) \approx 1$), the optimal portfolio share is 1. This is because despite the high

[11] The original data was later updated till 2005, which forms the source of these estimates. See Mehra (2006) for details.

covariance, the excess return of more than 6 percent and the relatively low volatility, with a standard deviation of under 10.5 percent for the logged shock to returns, makes it difficult to justify holding the risk-free asset. In fact, Figure 9 shows that the equity share of portfolio falls to realistic levels under the no-borrowing constraint only when ρ is as large as 12. This number is close to the benchmark by Schreindorfer (2020), whose model incorporates disappointment averse preferences and has agents exhibit levels of relative risk aversion of close to 10.

Figure 9. U.S. figures requires extremely high RRA to explain the equity premium

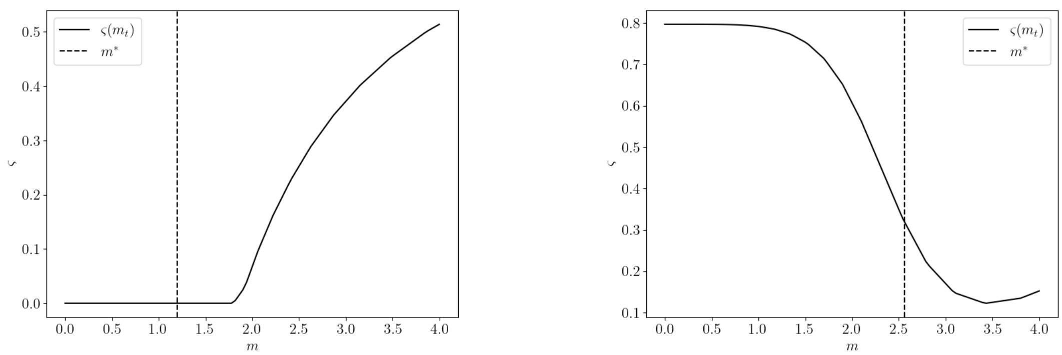


(a) $\rho = 4$ **(b)** $\rho = 12$

This result is well-contextualized in light of Proposition 1, which characterizes the threshold on covariance between income and asset returns that would make the poor invest in the risk-free asset. Given the values of σ_η and σ_ψ in Table 1, $\omega_{\eta,\psi}$ is bounded above by a little over 0.01. Even assuming nearly perfect correlation between equity returns and permanent income growth, $\tilde{\omega}$, as defined in Proposition 1, needs an RRA (ρ) of approximately 6 to make the poor invest in the risk-free asset. However, despite this high RRA, the portfolio share quickly converges toward the Merton-Samuelson bound, which is relatively close to 1.

The limiting value of the portfolio share of equity is not very different under the model with zero-income events. In fact, the limiting portfolio share of equity is identical between the two models. What changes is how we can explain the equity share around the target wealth.

Figure 10. Portfolio share around target wealth under the no-borrowing constraint and zero-income events



(a) $\rho = 7$, NBC **(b)** $\rho = 7.5$, Zero Income Events

Figure 10 shows that for $\rho = 7.5$, optimal portfolio share around the target wealth actually falls to around 30%. Meanwhile, in the model with the no-borrowing constraint, the equity share at target is 0. This is so, even as the limiting values of share holdings under this parameterization are really high. As such, while the model with zero income events makes for a better approximation around the target wealth, it performs identically to the model with the borrowing constraint for high m_t and worse by predicting that the poor will invest close to the share limit in equity, particularly as $m_t \to 0$. In any case, a value for ρ greater than 7 does not produce suitable implications for consumption-savings behavior.

These findings show that while these models do not explain the equity premium perfectly, the introduction of the correlation between permanent income shocks and asset returns has produced a significant improvement in how optimal portfolio decisions fit the data at relatively reasonable levels of risk aversion. Furthermore, while the model with the no-borrowing constraint accurately prohibits the extremely poor from investing in equity, the introduction of the zero income events ensures that consumers engage in precautionary saving and invest some of their fairly substantial savings in equity as a result.

6 Discussion

Section 5 shows that while a positive covariance between permanent income growth shocks and equity return shocks ($\omega_{\psi,\eta}$) does explain the preference for saving in the safe asset to some extent, the expected returns on equity ($\Re$) in the data are too high at relatively low volatility (σ_η). As such, even with a very high correlation between permanent income growth shocks and equity returns, we require unrealistically high levels of relative risk aversion to explain the portfolio allocation observed in the data. A particular problem area is that the equity share limit for wealthy individuals is extremely high, at close to 80 percent, even with $\rho = 7$. While portfolio shares dip to reasonable numbers with zero income events around the target wealth, the model cannot explain the portfolio choice decisions of consumers outside a small neighborhood of the target wealth in the distribution and non-participation in the stock market.

One approach used to address this problem is to incorporate non-expected utility preferences. Haliassos & Hassapis (2001) examine how various models of decision-making under risk improve predictions on equity holdings. While they conclude that changing preferences alone is not sufficient to account for the equity premium, they show that Kreps & Porteus (1978) preferences and, to a greater extent, Rank-Dependent Utility (Quiggin 1982) provide more realistic predictions on portfolio composition. Similarly, Schreindorfer (2020) shows that disappointment averse preferences (Gul 1991, Routledge & Zin 2010) can help explain the equity premium at a much lower level of relative risk aversion than with expected utility preferences under their model. However, the risk aversion coefficient necessary with expected utility preferences in their model is 34, meaning that despite the marked improvement, the new risk aversion coefficient is as high as 10.

Another explanation for the lower portfolio share of equity in the data than predicted in the model is pessimism and heterogeneity in beliefs about stock returns. Haliassos & Bertaut (1995) argue that in addition to correlations between labor income and asset returns and departures from expected utility, factors such as informational frictions provide a good explanation for the equity premium. While they note that a lack of knowledge about the stock market constrained participation, Dominitz & Manski (2007) find that agents also hold exaggerated beliefs about the possibility of negative nominal stock returns. Velásquez-Giraldo (2024) incorporates estimated beliefs from survey data into a life-cycle model and finds that stock market participation and conditional equity portfolio share can be explained by heterogeneous beliefs with a high average belief about the volatility of returns to equity. For consumers who believe that stock returns are extremely volatile, a high, or even moderate correlation between permanent income shocks and asset returns should ensure that they do not participate in the stock market, whereas the conditional distribution over equity portfolio share would then be determined by those who believe the stock market is not as volatile, though possibly more than actually observed in the data.

As far as non-participation in the stock market is concerned, minimum investment limits and fixed costs for participation have also been studied as probable obstacles. In the current model, with the parameterization as in section 5 stock market non-participation is observed at target wealth solely due to the no-borrowing constraint, and cannot be seen with the zero income event. However, a fixed participation cost would preclude consumers with very little wealth from investing their savings in equity, thereby generating a non-participation effect among low-wealth consumers. One limitation to this approach is that it cannot explain non-participation across wealth levels. Andersen & Nielsen (2011), Briggs et al. (2021) show that consumers who experience windfall gains do not see significantly higher participation rates, and that some of them even liquidate inheritances received in the form of stock.[12]

A Mathematical Appendix

A.1 Approximation of expectations

I recurrently use the following approximation of the expectation of a function of a random variable X around its expectation, μ_X.

$$\begin{aligned} f(X) &\approx f(\mu_X) + f'(\mu_X)(X-\mu_X) + \frac{1}{2}f''(\mu_X)(X-\mu_X)^2 \\ \mathbb{E}\left[f(X)\right] &\approx f(\mu_X) + \frac{1}{2}f''(\mu_X)\,\mathrm{var}(X) \end{aligned}$$

Accordingly, we can compute the expectation of a function of a random vector $X = (X_1, \ldots, X_k)$ around its expectation vector $\mu_X = (\mu_{X,1}, \ldots, \mu_{X,k})$

$$\begin{aligned} f(X) &\approx f(\mu_X) + (\nabla f)(\mu_X)\cdot(X-\mu_X) + \frac{1}{2}\,\mathrm{tr}\big(H(f)(\mu_X)(X-\mu_X)(X-\mu_X)'\big) \\ \mathbb{E}\left[f(X)\right] &\approx f(\mu_X) + \frac{1}{2}\,\mathrm{tr}(H(f)(\mu_X)\Sigma_X) \\ &= f(\mu_X) + \frac{1}{2}\sum_{i=1}^{k}\sum_{j=1}^{k}\left.\frac{\partial^2 f(X)}{\partial X_i \partial X_j}\right|_{\mu_X}\mathrm{cov}(X_i,\, X_j) \end{aligned}$$

[12] Note that individuals who experience windfall gains do not experience an increase in permanent labor income, which means that their normalized wealth must also increase. Thus, variance in wealth due to the variance in permanent income alone does not capture such individuals.

where Σ_X is the covariance matrix of X, $(\nabla f)(\mu_X)$ is the gradient of f, and $H(f)(\mu_X)$ is the Hessian matrix of f, both evaluated at μ_X.

A.2 Approximating the excess return equation

We want to approximate the excess return equation

$$\mathbb{E}_t\left[(\Re_{t+1} - \mathsf{R})(\mathscr{G}_{t+1}c_{t+1})^{-\rho}\right] = 0$$

Note that $\mathscr{G}_{t+1}c_{t+1} = C_{t+1}$. We can then divide both sides of the equation by C_{t+1} and get

$$\mathbb{E}_t\left[(\Re_{t+1} - \mathsf{R})\left(\frac{C_{t+1}}{C_t}\right)^{-\rho}\right] = 0$$

The first approximation we can apply is that

$$\frac{C_{t+1}}{C_t} \approx 1 + \Delta \ln C_{t+1}$$

for small enough values of $\Delta \ln C_{t+1}$. Then observe that $(1+x)^\alpha \approx 1 + \alpha x$ for small x. Put together,

$$\mathbb{E}_t\left[(\Re_{t+1} - \mathsf{R})\left(1 - \rho\Delta \ln C_{t+1}\right)\right] \approx 0$$

We can then expand this to

$$\mathbb{E}_t\left[(\Re_{t+1} - \mathsf{R})\left(1 - \rho\Delta \ln C_{t+1}\right)\right] = \mathbb{E}_t\left[(\Re_{t+1} - \mathsf{R})\right]\mathbb{E}_t\left[1 - \rho\Delta \ln C_{t+1}\right] - \operatorname{cov}(\rho\Delta \ln C_{t+1}, \Re_{t+1})$$
$$\mathbb{E}_t\left[\Re_{t+1}\right] - \mathsf{R} \approx \frac{\rho \operatorname{cov}(\Delta \ln C_{t+1}, \Re_{t+1})}{1 - \rho\,\mathbb{E}_t\left[\Delta \ln C_{t+1}\right]}$$

Around steady state values of normalized consumption, $\mathbb{E}_t \Delta \ln C_{t+1} \approx \mathbb{E}_t \Delta \ln P_{t+1} = \gamma - \frac{\sigma_\psi^2}{2} \approx 0$, where $\gamma = \ln \Gamma$. This holds true more generally as well. Then

$$\mathbb{E}_t\left[\Re_{t+1}\right] - \mathsf{R} \approx \rho \operatorname{cov}(\Delta \ln C_{t+1}, \Re_{t+1})$$

A.3 Approximations of covariances

We want to approximate the covariances of functions of two random variables X and Y around their expectations μ_X and μ_Y. To do so:

$$\begin{aligned}
\operatorname{cov}(f(X),\, g(Y)) &= \mathbb{E}\left[f(X)g(Y)\right] - \mathbb{E}\left[f(X)\right]\mathbb{E}\left[g(Y)\right] \\
\mathbb{E}\left[f(X)g(Y)\right] &\approx f(\mu_X)g(\mu_Y) + \frac{1}{2}\left[f''(\mu_X)g(\mu_Y)\operatorname{var}(X) + f(\mu_X)g''(\mu_Y)\operatorname{var}(Y)\right] \\
&\quad + f'(\mu_X)g'(\mu_Y)\operatorname{cov}(X,\, Y) \\
\mathbb{E}\left[f(X)\right]\mathbb{E}\left[g(Y)\right] &\approx f(\mu_X)g(\mu_Y) + \frac{1}{2}\left[f''(\mu_X)g(\mu_Y)\operatorname{var}(X) + f(\mu_X)g''(\mu_Y)\operatorname{var}(Y)\right] \\
&\quad + \frac{1}{4}f''(\mu_X)g''(\mu_Y)\operatorname{var}(X)\operatorname{var}(Y) \\
\operatorname{cov}(f(X),\, g(Y)) &\approx f'(\mu_X)g'(\mu_Y)\operatorname{cov}(X,\, Y) - \frac{1}{4}f''(\mu_X)g''(\mu_Y)\operatorname{var}(X)\operatorname{var}(Y)
\end{aligned}$$

Using a similar application of the expectations formulae, it can be observed that

$$\begin{aligned}
\mathbb{E}\left[f(X_1,\, X_2,\, X_3)g(X_3)\right] \approx f(\mu_X)g(\mu_{X,3}) &+ \frac{1}{2}g(\mu_{X,3})\sum_{i=1}^{3}\sum_{j=1}^{3}\operatorname{cov}(X_i,\, X_j)\left.\frac{\partial^2 f(X)}{\partial X_i \partial X_j}\right|_{\mu_X} \\
&+ g'(\mu_{X,3})\sum_{i=1}^{3}\left.\frac{\partial f(X)}{\partial X_i}\right|_{\mu_X}\operatorname{cov}(X_i,\, X_3) \\
&+ \frac{1}{2}g''(\mu_{X,3})f(\mu_X)\operatorname{var}(X_3)
\end{aligned}$$

Similarly,

$$\begin{aligned}
\mathbb{E}\left[f(X)\right]\mathbb{E}\left[g(X_3)\right] \approx f(\mu_X)g(\mu_{X,3}) &+ \frac{1}{2}g(\mu_{X,3})\sum_{i=1}^{3}\sum_{j=1}^{3}\operatorname{cov}(X_i,\, X_j)\left.\frac{\partial^2 f(X)}{\partial X_i \partial X_j}\right|_{\mu_X} \\
&+ \frac{1}{4}g''(\mu_{X,3})\operatorname{var}(X_3)\sum_{i=1}^{3}\sum_{j=1}^{3}\left.\frac{\partial^2 f(X)}{\partial X_i \partial X_j}\right|_{\mu_X} \\
&+ \frac{1}{2}f(\mu_X)\operatorname{var}(X_3)g''(\mu_{X,3})
\end{aligned}$$

Then

$$\begin{aligned}
\operatorname{cov}(f(X),\, g(X_3)) \approx g'(\mu_{X,3})&\sum_{i=1}^{3}\left.\frac{\partial f(X)}{\partial X_i}\right|_{\mu_X}\operatorname{cov}(X_i,\, X_3) \\
&- \frac{1}{4}g''(\mu_{X,3})\operatorname{var}(X_3)\sum_{i=1}^{3}\sum_{j=1}^{3}\left.\frac{\partial^2 f(X)}{\partial X_i \partial X_j}\right|_{\mu_X}
\end{aligned}$$

A.4 Approximation of covariance of consumption and equity returns

We want to approximate the expression

$$\operatorname{cov}(\Delta \ln C_{t+1},\, \Re_{t+1})$$

which features in the approximate excess return equation. We can start by noting that C_t is a known quantity, so $\operatorname{cov}(\Delta \ln C_{t+1},\, \Re_{t+1}) = \operatorname{cov}(\ln C_{t+1},\, \Re_{t+1})$. Thereafter

$$\begin{aligned}
C_{t+1} &= \Gamma \psi_{t+1} P_t c_{t+1} \\
\ln C_{t+1} &= \ln \Gamma + \ln P_t + \ln \psi_{t+1} + \ln c_{t+1} \\
\operatorname{cov}(\ln C_{t+1},\, \Re_{t+1}) &= \Re \left[\, \operatorname{cov}(\ln \psi_{t+1},\, \eta_{t+1}) + \operatorname{cov}(\ln c_{t+1},\, \eta_{t+1})\right]
\end{aligned}$$

Note that $\mathbb{E}\left[\ln \psi_{t+1}\right] = -\frac{\sigma_\psi^2}{2}$ and $\mathbb{E}\left[\ln \eta_{t+1}\right] = -\frac{\sigma_\eta^2}{2}$. We can then approximate

$$\begin{aligned}
\operatorname{cov}(\ln \psi_{t+1},\, \eta_{t+1}) &= \operatorname{cov}(\ln \psi_{t+1},\, \exp(\ln \eta_{t+1})) \\
&\approx \exp\left(-\frac{\sigma_\eta^2}{2}\right) \operatorname{cov}(\ln \psi_{t+1},\, \ln \eta_{t+1}) \\
&\approx \left(1 - \frac{\sigma_\eta^2}{2}\right) \omega_{\psi,\,\eta} \\
&\approx \omega_{\psi,\,\eta}
\end{aligned}$$

Now observe that with m_t known and $c_t = c(m_t)$ subsequently determined, c_{t+1} can be written as $\hat{c}(\psi_{t+1},\, \theta_{t+1},\, \eta_{t+1})$, where

$$\hat{c}(\psi_{t+1},\, \theta_{t+1},\, \eta_{t+1}) \equiv c\left(\frac{\mathsf{R} + \varsigma(\Re \eta_{t+1} - \mathsf{R})}{\Gamma \psi_{t+1}}\,(m_t - c_t) + \theta_{t+1}\right)$$

Define $\bar{m}$ as

$$\bar{m} = \frac{\mathsf{R} + \varsigma(\Re - \mathsf{R})}{\Gamma}\,(m_t - c_t) + 1$$

Then, setting $f \equiv \ln \circ \hat{c}$ and g as the identity function, we can approximate

$$\begin{aligned}
\operatorname{cov}(\ln \circ \hat{c}(\psi_{t+1},\, \theta_{t+1},\, \eta_{t+1}),\, \eta_{t+1}) \approx\; & \operatorname{var}(\eta_{t+1}) \frac{c'(\bar{m})}{c(\bar{m})} \frac{\varsigma \Re}{\Gamma} (m_t - c_t) \\
&- \operatorname{cov}(\psi_{t+1},\, \eta_{t+1}) \frac{c'(\bar{m})}{c(\bar{m})} \frac{\mathsf{R} + \varsigma(\Re - \mathsf{R})}{\Gamma} (m_t - c_t) \\
&+ \operatorname{cov}(\theta_{t+1},\, \eta_{t+1}) \frac{c'(\bar{m})}{c(\bar{m})}
\end{aligned}$$

Now we can see that

$$\begin{aligned}
\operatorname{cov}\big(e^X,\, e^Y\big) &\approx \exp\left(-\frac{\sigma_X^2 + \sigma_Y^2}{2}\right) \operatorname{cov}(X,\, Y) - \frac{1}{4}\sigma_X^2 \sigma_Y^2 \exp\left(-\frac{\sigma_X^2 + \sigma_Y^2}{2}\right) \\
&\approx \operatorname{cov}(X,\, Y) \qquad \text{(for small } \sigma_X \text{ and } \sigma_Y\text{)}
\end{aligned}$$

For the NBC model, $\theta_{t+1} = \zeta_{t+1}$, so $\operatorname{cov}(\theta_{t+1}, \eta_{t+1}) \approx \omega_{\zeta,\eta}$. However, even in the NIR model, note that $\mathbb{E}[\theta_{t+1}]$ and $\mathbb{E}[\eta_{t+1}]$ are unchanged, while

$$\mathbb{E}[\theta_{t+1}\eta_{t+1}] = \wp \cdot 0 + (1-\wp)\frac{\mathbb{E}[\zeta_{t+1}\eta_{t+1}]}{1-\wp} = \mathbb{E}[\zeta_{t+1}\eta_{t+1}]$$

Thus,

$$\operatorname{cov}(\Delta \ln C_{t+1}, \Re_{t+1}) \approx \Re \left(\omega_{\psi,\eta} + \frac{c'(\bar{m})}{c(\bar{m})}\left((m_t - c_t)\left(\frac{\varsigma \Re \sigma_\eta^2 - (\mathsf{R} + \varsigma(\Re - \mathsf{R}))\omega_{\psi,\eta}}{\Gamma}\right) + \omega_{\zeta,\eta}\right)\right)$$

A.5 Proof of Proposition 1

Note from Appendix A.4, that

$$\rho \operatorname{cov}(\ln \Delta C_{t+1}, \Re_{t+1}) \approx \rho\Re \left(\omega_{\psi,\eta} + \frac{c'(\bar{m})}{c(\bar{m})}\left((m_t - c_t)\left(\frac{\varsigma \Re \sigma_\eta^2 - (\mathsf{R} + \varsigma(\Re - \mathsf{R}))\omega_{\psi,\eta}}{\Gamma}\right) + \omega_{\zeta,\eta}\right)\right)$$

By Lemma 1, there exists $\tilde{m} > 0$ such that $c(m) = m$ for $m \leq \tilde{m}$. For such m,

$$\rho \operatorname{cov}(\ln \Delta C_{t+1}, \Re_{t+1}) \approx \rho\Re \left(\omega_{\psi,\eta} + \frac{c'(1)}{c(1)}\omega_{\zeta,\eta}\right)$$

as $a_{t+1} = 0$ and $\bar{m} = \mathbb{E}[\theta_{t+1}] = 1$. Since $\rho \operatorname{cov}(\Delta \ln C_{t+1}, \Re_{t+1})$ is continuous in m,[13] if $\rho \operatorname{cov}(\ln \Delta C_{t+1}, \Re_{t+1}) > \Re - \mathsf{R}$ for $m < \tilde{m}$, it is also true for $m < \tilde{m} + \epsilon$, for $\epsilon > 0$ and any $\varsigma \in [0, 1]$. Thus, under the conditions for optimality, $\varsigma(m) = 0$ for all such m. Likewise, $\varsigma(m) = 1$ for $m < \tilde{m} + \epsilon$ if $\rho \operatorname{cov}(\ln \Delta C_{t+1}, \Re_{t+1}) < \Re - \mathsf{R}$ for $m' < \tilde{m}$. Defining m^* as the supremum of such m, we have the binary policy rule, as in the result. The only thing left to prove is that $\tilde{\omega} = \frac{\Re - \mathsf{R}}{\rho\Re}$. That is obvious upon dividing both sides of the excess return equation by $\rho\Re$.

A.6 Proof of Proposition 2

Note from Appendix A.4, the excess return equation is approximated as

$$\Re - \mathsf{R} \approx \rho\Re \left(\omega_{\psi,\eta} + \frac{c'(\bar{m})}{c(\bar{m})}\left((m_t - c_t)\left(\frac{\varsigma \Re \sigma_\eta^2 - (\mathsf{R} + \varsigma(\Re - \mathsf{R}))\omega_{\psi,\eta}}{\Gamma}\right) + \omega_{\zeta,\eta}\right)\right)$$

Note,

$$\bar{m}'(m) = \frac{\mathsf{R} + \varsigma(\Re - \mathsf{R})}{\Gamma} > 0$$

and

$$\frac{d}{dm}\frac{c'(\bar{m})(m - c(m))}{c(\bar{m})} = \frac{c(\bar{m})(c''(\bar{m})(m - c(m))\bar{m}'(m) + c'(\bar{m})(1 - c'(m))) - c'(\bar{m})^2(m - c(m))\bar{m}'(m)}{c(\bar{m})^2}$$

[13] Carroll & Shanker (2024) show that c is twice continuously differentiable in m.

By the concavity of c, the above derivative is negative. Supposing that $\omega_{\zeta,\eta} = 0$,[14] we only care about the positivity or negativity of the term

$$\varsigma\Re\sigma_\eta^2 - (\mathsf{R} + \varsigma(\Re - \mathsf{R}))\omega_{\psi,\eta}$$

This term is positive if and only if

$$\begin{aligned}\omega_{\psi,\eta} &< \frac{\varsigma\Re\sigma_\eta^2}{\mathsf{R} + \varsigma(\Re - \mathsf{R})} \\ &\leq \sigma_\eta^2\end{aligned}$$

Therefore, if $\omega_{\psi,\eta} \geq \sigma_\eta^2$, this term is non-positive, and $\text{cov}(\ln \Delta C_{t+1}, \Re_{t+1})$ is increasing in wealth. As such, the optimal portfolio share is decreasing in wealth. On the other hand, if $\omega_{\psi,\eta} < \sigma_\eta^2$, the term is positive when $\varsigma = 1$, and negative when $\varsigma = 0$. Thus, there exists a threshold ς such that the optimal portfolio share is increasing in wealth below this portfolio share. This threshold is given by

$$\varsigma < \frac{\mathsf{R}\omega_{\psi,\eta}}{\Re\sigma_\eta^2 - (\Re - \mathsf{R})\omega_{\psi,\eta}}$$

Now, suppose $\omega_{\psi,\eta} = \tilde{\omega} = \frac{\Re - \mathsf{R}}{\rho\Re}$. Then

$$\frac{\mathsf{R}\frac{\Re - \mathsf{R}}{\rho\Re}}{\Re\sigma_\eta^2 - \frac{(\Re - \mathsf{R})^2}{\rho\Re}} = \frac{\mathsf{R}(\Re - \mathsf{R})}{\rho\Re^2\sigma_\eta^2 - (\Re - \mathsf{R})^2}$$

Since the threshold value is increasing in $\omega_{\psi,\eta}$, the optimal portfolio share is increasing in wealth for $\omega_{\psi,\eta} > \tilde{\omega}$ as long it is below this share share. Likewise, the optimal portfolio share is decreasing in wealth for $\omega_{\psi,\eta} < \tilde{\omega}$ as long as it is above this share. As such, we must only show that in each case, the optimal portfolio share lies on either side of this threshold.

Rearranging the excess return equation, the optimal portfolio rule can be written as

$$\varsigma(m) \approx \frac{\Gamma}{\Re\sigma_\eta^2 - (\Re - \mathsf{R})\omega_{\psi,\eta}} \left(\frac{c(\bar{m})}{c'(\bar{m})\,(m - c(m))} \left(\frac{\Re - \mathsf{R}}{\rho\Re} - \omega_{\psi,\eta} \right) - \frac{\mathsf{R}\omega_{\psi,\eta}}{\Gamma} \right)$$

First, note that differentiating the right hand side w.r.t $\omega_{\psi,\eta}$ reveals that it is decreasing in $\omega_{\psi,\eta}$. Thus, the optimal portfolio share is decreasing in $\omega_{\psi,\eta}$. Now, note that the right hand side is decreasing in $\omega_{\psi,\eta}$,[15] so ς is decreasing in $\omega_{\psi,\eta}$. Note, then, for $\omega_{\psi,\eta} = \tilde{\omega}$,

$$\varsigma(m) \approx \frac{\mathsf{R}(\Re - \mathsf{R})}{\rho\Re^2\sigma_\eta^2 - (\Re - \mathsf{R})^2}$$

Therefore, $\varsigma(m)$ is greater than the threshold share if $\omega_{\psi,\eta} < \tilde{\omega}$, and lower otherwise. As such, $\varsigma(m)$ is increasing in wealth for $\omega_{\psi,\eta} > \tilde{\omega}$, and decreasing otherwise. Note that for plausible values of $\Re$, R, and σ_η, $(\Re^2 - 1)\rho\sigma_\eta^2 - (\Re - \mathsf{R})^2 \approx 0$ and $\mathsf{R} \approx 1$. Thus, the threshold value of ς is

[14] This covariance becomes inconsequential with wealth, as $\frac{c'(\bar{m})}{c(\bar{m})}$ sharply falls

[15] Also see that $\text{cov}(\ln C_{t+1}, \Re_{t+1})$ is increasing in $\omega_{\psi,\eta}$, so ς is decreasing in $\omega_{\psi,\eta}$.

approximately

$$
\begin{aligned}
\varsigma(m) &= \frac{\mathsf{R}(\Re - \mathsf{R})}{\rho\Re^2\sigma_\eta^2 - (\Re - \mathsf{R})^2} \\
&= \frac{\mathsf{R}(\Re - \mathsf{R})}{\rho\sigma_\eta^2 + (\Re^2 - 1)\rho\sigma_\eta^2 - (\Re - \mathsf{R})^2} \\
&\approx \frac{\Re - \mathsf{R}}{\rho\sigma_\eta^2}
\end{aligned}
$$

Note that this is the Merton-Samuelson share.

A.7 Proof of Proposition 3

Note from Appendix A.4, that

$$
\rho\,\text{cov}(\ln \Delta C_{t+1}, \Re_{t+1}) \approx \rho\Re\left(\omega_{\psi,\eta} + \frac{c'(\bar{m})}{c(\bar{m})}\left((m_t - c_t)\left(\frac{\varsigma\Re\sigma_\eta^2 - (\mathsf{R} + \varsigma(\Re - \mathsf{R}))\omega_{\psi,\eta}}{\Gamma}\right) + \omega_{\zeta,\eta}\right)\right)
$$

Since the right hand side is differentiable in m, we can determine the approximate limit of the optimal portfolio share from the limit of the right hand side as m approaches infinity. Carroll & Shanker (2024) show that $c'(m) \to \underline{\kappa} \geq 0$ as $m \to \infty$, where $\underline{\kappa} > 0$ subject to the satisfaction of a growth impatience condition. Thus,

$$
\begin{aligned}
\lim_{m\to\infty} \frac{c'(\bar{m})(m - c(m))}{c(\bar{m})} &= \lim_{m\to\infty} \frac{c'(\bar{m})\frac{m - c(m)}{m}}{\frac{c(\bar{m})}{m}} \\
\lim_{m\to\infty} \frac{m - c(m)}{m} &= \lim_{m\to\infty} 1 - c'(m) \\
&= 1 - \underline{\kappa} \\
\lim_{m\to\infty} \frac{c(\bar{m})}{m} &= \lim_{m\to\infty} c'(\bar{m})\bar{m}'(m) \\
\implies \lim_{m\to\infty} \frac{c'(\bar{m})(m - c(m))}{c(\bar{m})} &= \lim_{m\to\infty} \frac{c'(\bar{m})(1 - c'(\bar{m}))}{c'(\bar{m})\bar{m}'(m)} \\
&= \lim_{m\to\infty} \frac{(1 - c'(\bar{m}))}{\bar{m}'(m)} \\
&= \frac{\Gamma}{\mathsf{R} + \varsigma(\Re - \mathsf{R})}
\end{aligned}
$$

Note that $\lim_{m\to\infty} \frac{c'(\bar{m})}{c(\bar{m})} = 0$, so the limit of the covariance term is

$$
\lim_{m\to\infty} \rho\,\text{cov}(\ln \Delta C_{t+1}, \Re_{t+1}) \approx \rho\Re\frac{\varsigma\Re\sigma_\eta^2}{\mathsf{R} + \varsigma(\Re - \mathsf{R})}
$$

Substituting this back into the excess return equation,

$$\begin{aligned}
\Re - \mathsf{R} &\approx \frac{\rho \Re^2 \sigma_\eta^2 \varsigma}{\mathsf{R} + \varsigma(\Re - \mathsf{R})} \\
\mathsf{R}(\Re - \mathsf{R}) + \varsigma(\Re - \mathsf{R})^2 &\approx \rho \Re^2 \sigma_\eta^2 \varsigma \\
\lim_{m \to \infty} \varsigma &\approx \frac{\mathsf{R}(\Re - \mathsf{R})}{\rho \Re^2 \sigma_\eta^2 - (\Re - \mathsf{R})^2}
\end{aligned}$$

Thus, it is evident that the optimal portfolio share is independent of the covariance between income and returns on equity as wealth grows arbitrarily large. By the approximation employed in the proof of Proposition 2, this limit is approximately the Merton-Samuelson share. That is

$$\lim_{m \to \infty} \varsigma \approx \frac{\Re - \mathsf{R}}{\rho \sigma_\eta^2}$$

B Computational Appendix

B.1 Solving the model

I solve the model using a sequential application of the endogenous grid method (Carroll 2006), dividing a period into two subperiods, the first stage involving a consumption decision (c), and the second involving the portfolio optimization problem (ς). This description largely pertains to the finite-horizon version, though the infinite-horizon solution merely replaces periods with a sequence of guesses.

Construct a grid of assets $\mathcal{A} = [\underline{a} = a_1 < a_2 < ... < a_k = \overline{a}]$. To solve the problem pertaining to any period t, observe from equation (9) that whenever $a_i \neq 0$, the optimal share of risky assets is given by the choice of $\hat{\varsigma}_{t+1}(a_i) \in [0, 1]$ such that

$$n^{-3} \Gamma_{t+1}^{-\rho} \sum_{j=1}^{n^3} (\Re \eta_i - \mathsf{R})(\psi_j c_{t+1}(m_{ij}))^{-\rho} = 0 \tag{11}$$

where

$$m_{ij} = \frac{\mathsf{R} + \hat{\varsigma}_{t+1}(a_i)(\Re \eta_j - \mathsf{R})}{\Gamma_{t+1} \psi_j} a_i + \theta_j$$

The problem then becomes a root-finding operation pertaining to a function of $\hat{\varsigma}$, which, given a policy function c_{t+1}, yields an optimal level of $\hat{\varsigma}$ for each a_i. Denote this pair as $(a, \hat{\varsigma})_i$, and the resulting effective return $\mathsf{R} + \hat{\varsigma}_i(\Re \eta_j - \mathsf{R})$ for each value of the shocks as $\mathscr{R}_{ij}$.

For each *end-of-period* outcome $(a, \hat{\varsigma})_i$, given c_{t+1}, we can use the consumption Euler equation to get

$$[\hat{c}_t(a_i, \hat{\varsigma}_i)]^{-\rho} = \beta \Gamma_{t+1}^{-\rho} n^{-3} \sum_{j=1}^{n^3} \mathscr{R}_{ij} (\psi_j c_{t+1}(m_{ij}))^{-\rho}$$

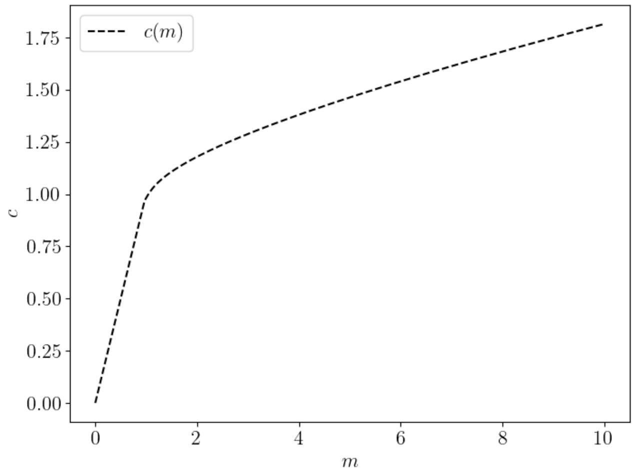
where $\hat{c}$ denotes that this yields a consumed function of the assets and portfolio share. This function is then given by

$$\hat{c}_t(a_i,\,\hat{\varsigma}_i) = \left[\beta\Gamma_{t+1}^{-\rho} n^{-3}\sum_{j=1}^{n^3}\mathscr{R}_{ij}(\psi_j c_{t+1}(m_{ij}))^{-\rho}\right]^{-\frac{1}{\rho}}$$

Now we have a vector of $\hat{c}_i$ corresponding to each $(a,\,\hat{\varsigma})_i$. Since $m_t = c_t + a_{t+1}$, we can construct the grid $\mathcal{M}$ with each $m_i \in \mathcal{M}$ given by $m_i = \hat{c}_i + a_i$, where $\hat{c}_i = \hat{c}_t(a_i,\,\hat{\varsigma}_i)$. We can now rewrite $c_t(m_i) = \hat{c}_t(a_i,\,\hat{\varsigma}_i)$ and $\varsigma_{t+1}(m_i) = \hat{\varsigma}_{t+1}(a_i)$, and interpolate to get the policy functions $(c_t(m),\,\varsigma_{t+1}(m)) = g_t(m)$ for period t.[16] In the finite-horizon case, the model can be solved using $c_T(m) = m$ as the initial policy function and iterating backwards till period 0. For the infinite-horizon case, I use a guess $c_0(m)$ to obtain a sequence of guesses $\{c_k(m),\,\varsigma_k(m)\}_{k=0}^{K}$ that converge to the true policy functions $c(m)$ and $\varsigma(m)$. Since my focus is not on life-cycle applications, I solve each model with a constant permanent growth factor Γ.

B.2 Baseline consumption

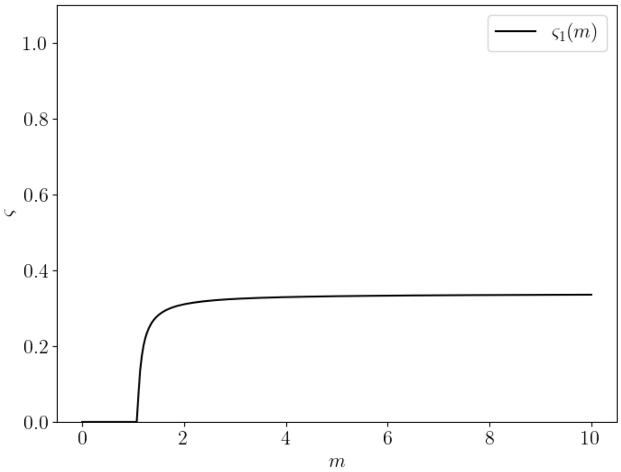
Figure B.1. Optimal consumption function in the buffer-stock model



[16]While I use linear interpolation by default, cubic-spline interpolation yields similar results for the consumption function. However, due to drastic directional changes in the optimal portfolio share, spline interpolation sometimes suggests a portfolio share outside the interval [0, 1].

B.3 Last Period Optimal Portfolio Allocation

Figure B.2. Optimal portfolio allocation in the second-to-last period



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