Beliefs in echo chambers: Characterization and identification

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Abstract

We introduce an interactive model of belief formation and transmission within echo chambers. Individuals in our model have subjective core beliefs, but these are not the same as the beliefs underlying their behavior. An individual's *behavioral beliefs* incorporate her core beliefs but are also influenced by the behavioral beliefs of others within her echo chamber. Therefore, echo chambers feature interactive behavioral beliefs with any individual's beliefs both being influenced by as well as influencing the beliefs of others in her echo chamber. The *echo chamber representation* of a profile of such behavioral beliefs that we propose captures the steady state of this process of interaction and influence. We characterize the model based on two axioms. The first emphasizes the need for conformity when it comes to assessments about certain events, while the second highlights the possibility of differing behavioral beliefs about uncertain events and the potential for everyone to exercise influence. We also analyze when the model permits exact identification, i.e., when can an analyst draw on the profile of behavioral beliefs to uniquely identify the composition of the echo chambers in society, the core beliefs of different individuals, and the degree to which each of them is immune to influence. Further, we relate our model to leading empirical themes like elite influence, naive updating and stickiness of beliefs within echo chambers.

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1 Introduction

Research across several disciplines has shown that *echo chambers* that form in society impact a range of important social and economic outcomes like polarization, populism, inequality, and asset prices in financial markets (Barberá 2020; Cookson et al. 2023; McCarty et al. 2006). A key reason echo chambers drive these outcomes is the beliefs that form within them and how they get transmitted. This paper looks at beliefs formed within echo chambers. Specifically, we propose a theory of interactive beliefs within echo chambers that capture how individual beliefs are formed based both on subjective assessments of uncertainty as well as the influence cast by popular perceptions within one's echo chamber that encourages conformism.

The key idea underlying our theory is the following. We consider a society that is partitioned into a set of echo chambers. Each individual in society has some *core beliefs* over the states of the world capturing her subjective assessment of uncertainty. However, her behavior is not determined exclusively by her core beliefs. Rather, her *behavioral beliefs* also draw on the overall beliefs prevailing within her echo chamber. Specifically, we model an individual's behavioral beliefs as a weighted average of her core beliefs and the average behavioral beliefs in her echo chamber. The weight put on the former measures the degree to which she is immune from her echo chamber's influence and is an important behavioral parameter. Echo chambers, therefore, feature interactive behavioral beliefs with an individual's beliefs both being influenced by and at the same time influencing the beliefs of others in her echo chamber. The *echo chamber representation* of a profile of such behavioral beliefs that we introduce in this paper captures the steady state of this process of interaction and influence.

The substantive work that we undertake in this paper focuses on two sets of questions. First, we provide a thorough decision-theoretic analysis of the model in terms of looking at its characterization and identification. Second, we highlight the empirical content of the model by showing its implications for leading echo chamber related themes that have been highlighted in the literature like elite influence, naive updating and stickiness of beliefs within echo chambers.

Our first set of decision-theoretic results provide a characterization of the model. Specifically, suppose an analyst has data on a profile of behavioral beliefs in society that she gathers, say, by observing individual's betting behavior. When can she conclude that this profile of beliefs has an echo chamber representation? In other words, are these beliefs consistent with some partitioning of society into echo chambers, and each individual in society having some subjective core beliefs and idiosyncratic immunity to influence such that everyone's behavioral beliefs are formed through the process of interaction and influence outlined above. We show that a class of echo chamber representations capturing a well-defined notion of maximum influence (maximal influence echo chamber representation) is characterized by two simple yet meaningful axioms.

The first axiom is called *certainty conformism*. It says that for any decision maker (DM) to assign probability one to an event according to her behavioral beliefs, i.e., behave as if she is certain about it, she must have the confirmation of at least someone else who does likewise, with this process being mutual. This captures a key behavioral restriction of the echo chamber model. No DM can behave as if she is sure about an event on her own. She can do so only when there are others (her revealed friends) who do so. This therefore generates conformity when it comes to sure events. At the same time our second axiom, called *subjective assessment of uncertainty*, makes the point that inter-personal interactions in a maximal influence echo-chamber representation is not just about conformity. Rather, such interactions also incorporate the possibility of individuals having different behavioral beliefs about uncertain events from that of their friends and thus the potential for everyone to exercise influence. Specifically, for any individual there exists at least some event where her behavioral belief about that event differs from the average behavioral belief about it amongst her friends.

Together, these two axioms connect interesting themes across the literatures on social influence and decisions under risk. Keeping with the former, the axioms reiterate that social influence is not just about conformity, but also about inter-personal differences that open the scope for differential influence. When it comes to the latter, the axioms highlight the point that sure events may be viewed qualitatively differently by DMs than events over which they are uncertain. The need for conformity in the model is restricted only to sure events and doesn't necessarily spill over to events about which people maintain a degree of uncertainty. This qualitative distinction between sure events and uncertain events resonates with similar distinctions that have been made in the context of theories like rank dependent utility and prospect theory, which draw on non-linear probability weighting, in especially pronounced ways around unit (and zero) probability events, and in such observed phenomenon like the certainty effect.

Our second set of decision-theoretic results pertain to the identification of the model. Suppose the behavioral beliefs of individuals in society is consistent with an echo chamber representation. Such a representation is based on several parameters. First, there is the partitioning of society into echo chambers. Second, there are the core beliefs of different individuals in society. Third, is the degree to which each individual is immune to influence. Is it possible to identify all these parameters uniquely from the profile of behavior beliefs? We provide a thorough analysis of this question. We show that in general an echo chamber representation is not uniquely identified. We provide a full classification of the structure of multiplicity of such representations. At the same time, we are able to establish that maximal influence echo chamber representations are indeed uniquely identified, which adds to the appeal of this class of representations.

Thereafter, we relate our model to empirical observations regarding echo chambers that have been reported in the literature. Our model distinguishes between individuals more susceptible to influence and those less susceptible. We show that individuals who are influenced less, in turn, are the ones who have the most influence on beliefs within an echo chamber, and vice versa. Therefore, a key insight that the model develops is the disproportionate effect a few can have on the many within an echo chamber—a form of elite influence highlighted in the context of societal polarization (Wojcieszak et al. 2021). Moreover, whether a group can act on new information crucially depends on who receives this information. For instance, if individuals more susceptible to influence receive new information, then not only does the new information not get incorporated into their behavior to any great extent, but even to the extent that it does, it fails to impact the group's beliefs and behavior significantly. A consequence is that groups may have sticky beliefs that do not respond to information.

The literature on echo chambers is primarily divided between examining the process of segregation into homogeneous groups and the transmission of beliefs and biases through these groups thereafter. Segregation into various groups is attributed to a variety of reasons, including economic, social, and cultural (Levy and Razin 2019). Baccara and Yariv (2016) examine the conditions under which segregation results either in homogeneous groups or polarization. Our model assumes the segregation of society into echo chambers to be exogenous, and we shift our focus to the transmission of beliefs within existing clusters, and provide behavioral identification and characterization of this process.

Acemoglu et al. (2014) and Golub and Jackson (2010) characterize convergence to the truth in a network where individuals learn from their peers' beliefs. Whereas the former incorporate endogenous network formation and communication of private signals into their model, the latter analyze the DeGroot (1974) model. Levy and Razin (2015a, 2015b) examine the effects of correlation neglect between signals on polarization and political outcomes. In particular, they argue that it is possible to achieve better outcomes compared to a rational electorate. Martinez and Tenev (2022) similarly argue that echo chambers could improve the process of learning if the quality of various sources of information is uncertain. A common characteristic of this literature is that information is transmitted within echo chambers by assuming the possibility of directly sharing signals amongst peers, whereas we restrict the channel of influence to observed behavior. On these lines, Eyster and Rabin (2010) model herding behavior based on sequential observation of actions. They find that in settings where agents assume that the observed actions are based solely on private information, they may converge to incorrect actions with confidence.

There is a substantial body of recent literature that studies the process of social influence from a choice theoretic perspective (Fershtman and Segal 2018; Kashaev and Lazzati 2019; Lazzati 2020; Borah and Kops 2018; Chambers et al. 2019; Chambers et al. 2021; Cuhadaroglu 2017). In particular, the structure of our model draws inspiration from Fershtman and Segal (2018). Whereas we model influence through belief transmission, they look at influence in tastes. They consider two sets of preferences for each individual, represented by her core and behavioral utilities, and an influence function such that her behavioral utilities can be represented as a function of her core utility and others' behavioral utilities. Analogous to their model, core beliefs are private and behavioral beliefs are observable in our model.

The rest of the paper is organized as follows. The next section introduces our setup and formally defines an echo chamber representation. Section 3 discusses the behavioral foundation of the model. Section 4 details the analysis of the identification of the model. Finally, in Section 5, we elaborate on some properties and empirical content of the model. Proofs of all results are available in the Appendix.

2 Setup

2.1 Primitives

Let S be a finite set of states, with any subset of S referred to as an event. Our stylized society consists of a set $I = \{1, \ldots, n\}$ of individuals. Each individual $i \in I$ is probabilistically sophisticated and her behavior over uncertain prospects is guided by a probability distribution π_i over the state space. We refer to this probability determining her behavior as her *behavioral beliefs*. We assume that an analyst or outside observer is able to observe these behavioral beliefs, say, from observing her betting behavior. Accordingly, the profile of behavioral beliefs $(\pi_i)_{i\in I}$ is data for the analyst.

2.2 Echo chamber representation

We now lay out our theoretical representation of interactions in the model that determine the behavioral beliefs of individuals. To that end, assume that the set of individuals in I

are partitioned into the sets $\mathcal{E} = \langle E_1, \dots, E_k \rangle$, with each element of the partition denoting an echo chamber (or chamber, for short) in society. To keep the setup meaningful, we assume that none of the echo chambers is a singleton. For any $i \in I$, we let $E(i)$ denote the echo chamber to which individual i belongs.

An individual's behavioral beliefs are formed both from her independent perception about the underlying uncertainty as well as the influence cast by the echo chamber she is a part of. Specifically, we assume that any such decision maker (DM), $i \in I$, is endowed with some *core beliefs* μ_i on S. Behavioral beliefs, of course, depend on core beliefs. But additionally, the working of influence within her echo chamber implies that her behavioral beliefs may be influenced by her perception of the overall beliefs prevailing in her echo chamber. We consider the average behavioral belief prevailing within her echo chamber, $\frac{1}{|E(i)|}\sum_{j \in E(i)} \pi_j$, as an aggregate summary statistic capturing this aspect of influence. One may think of this aggregate as the *echo* generated in the process of interactive belief formation.[1] We assume that the dependence on the two takes a linear weighted average form, with the weights determined by a parameter $\alpha_i \in (0, 1)$ that captures the degree to which this DM is immune to influence, i.e., higher is α_i, the less susceptible is this DM to influence. Specifically, for any event $A \subseteq S$, we assume that her behavioral belief is given by,

$$\overbrace{\pi_i(A)}^{\textbf{Behavioral belief}} = \alpha_i \underbrace{\mu_i(A)}_{\textbf{Core belief}} + (1 - \alpha_i) \underbrace{\frac{1}{|E(i)|} \sum_{j \in E(i)} \pi_j(A)}_{\textbf{Echo}}$$

This, therefore, makes behavioral beliefs interactive within an echo chamber. Our representation of the profile of behavioral probabilities $(\pi_i)_{i \in I}$ provides a formal statement of these interactions. It captures the steady state of this process of interactions by requiring mutually consistent behavioral beliefs within an echo chamber.

Definition 1. *The profile of behavioral probability measures $(\pi_i)_{i \in I}$ on S has an echo chamber representation if there exists a partition $\mathcal{E} = \langle E_1, E_2, ..., E_k \rangle$ of I, and for each $i \in I$:*

- *a core probability measure μ_i on S, and*
- *an immunity from influence parameter $\alpha_i \in (0, 1)$*

[1] To understand better the idea behind the echo, suppose there is a change in the core beliefs of a DM i, say, due to fresh information she receives. This would clearly impact her behavioral beliefs and, in doing so, impact the behavioral beliefs of all others in her chamber, given the interactive nature of these beliefs. But, at the same time, the change induced in the behavioral beliefs of others would in turn bounce back like an echo and further influence i's behavioral beliefs, and so forth.

such that $(\pi_i)_{i\in I}$ *can be defined as a solution to the system of equations,*

$$\pi_i(A) = \alpha_i\mu_i(A) + (1-\alpha_i)\frac{1}{|E(i)|}\sum_{j\in E(i)}\pi_j(A), \ \ i \in I,$$

Like in any equilibrium or steady state notion, we close the interactions by assuming that individuals hold correct expectations about the behavioral beliefs of others in their echo chamber. This allows them to correctly forecast the average behavioral belief about any event in their echo chamber.

We also focus on a subclass of echo chamber representations that identify the maximum extent of influence consistent with a given profile of behavioral beliefs. Observe that there are two channels that mediate the scope of influence in an echo chamber representation. The first is through the magnitude of the immunity of influence parameters $(\alpha_i)_{i\in I}$, the lower these are the greater is the scope of influence on individuals in an echo chamber. Accordingly, if there are two echo chamber representations with the same echo chamber partitioning and immunity to influence parameters $(\alpha_i)_{i\in I}$ and $(\tilde{\alpha}_i)_{i\in I}$, respectively, then we can say that the first representation doesn't fully capture the scope of influence consistent with the data if $\tilde{a}_i \leq \alpha_i$, for all i, holding strictly for some i. The second channel through which influence work is the size of echo chambers, the larger these are the more the number of individuals any given influence is influenced by and influences. Therefore, if under two echo chamber representations, the echo chamber partitioning in the second is a coarsening of the first, then we say that the first representation doesn't fully capture the scope of influence on this dimension. This motivates the following definition.

Definition 2.

Remark 1 (Existence)**.** It is straightforward to establish that the steady state notion captured by the echo chamber representation doesn't suffer from concerns about non-existence. That is, given a collection $(\mu_i)_{i\in I}$ of core beliefs, it is immediate to establish that there exists a collection of behavioral beliefs $(\pi_i)_{i\in I}$ that simultaneously satisfy the system of equations:

$$\pi_i(A) = \alpha_i\mu_i(A) + (1-\alpha_i)\frac{1}{|E(i)|}\sum_{j\in E(i)}\pi_j(A), \ \ i \in I$$

To see this, note that the equation determining the behavioral beliefs of individual i, depends only on the beliefs of the individuals belonging to $E(i)$. Thus, it is sufficient to prove existence for a single echo chamber. Given a chamber E, subtract both sides by the average behavioral belief, and sum over all $i \in E$ to yield:

$$\frac{1}{|E|}\sum_{j\in E}\pi_j(A) = \sum_{i\in E}\frac{\alpha_i\mu_i(A)}{\sum_{j\in E}\alpha_j}$$

Substituting this expression in the earlier system of equations, we get:

$$\pi_i(A) = \alpha_i\mu_i(A) + (1-\alpha_i)\sum_{j\in E}\frac{\alpha_j\mu_j(A)}{\sum_{l\in E}\alpha_l}$$

This process can be used for each echo chamber, and the resulting collection $(\pi_i)_{i\in I}$ satisfies the system of equations. Since this is also true of all $(\pi_i)_{i\in I}$ that solve the equations, given a collection of $(\mu_i)_{i\in I}$, $(\alpha_i)_{i\in I}$ and partition $\mathcal{E} = \langle E_1, ..., E_k\rangle$, the resultant $(\pi_i)_{i\in I}$ must additionally be unique.

2.3 Maximal influence echo chamber representation

An echo chamber representation of a profile of behavioral beliefs need not be unique. We will discuss in detail later the structure of multiplicity of such representations. Given this multiplicity, one of the things we would like to do is focus on a subclass of representations in which the potential for influence is "maximized" in the sense that there does not exist another EC representation of the same profile of behavioral beliefs which is rationalized by a greater degree of influence. To understand this, say a profile of behavioral probabilities admits two EC representations and the echo chamber partitioning under the first is a coarsening of that under the second. Then, the first representation admits a greater potential for influence because everyone who any individual influences under the second representation, she influences under the first, but there are individuals who she doesn't influence under the second that she does under the first. In that case, the second representation doesn't incorporate the maximum possible influence that can potentially be rationalized by this profile of behavioral beliefs and the refinement we are proposing discounts it on this ground. It is also possible that two representations may have the same echo chamber partitioning. But, say, under the first, the immunity to influence parameter of all individuals is smaller than that under the second. In this case too the first representation admits a greater potential for influence than the second because everyone under it is putting more weight on the average behavioral belief in their cluster and less on their core beliefs compared to the second. Our refinement picks this up and once gain discounts the second representation on the ground that it underestimates the potential for influence consistent with this profile of behavioral beliefs. We first illustrate these observations using a couple of examples before presenting the formal definition.

Example 1 Consider a society with four individuals, $I = \{1, 2, 3, 4\}$, and a state space with three states, $S = \{s_1, s_2, s_3\}$. Let the behavioral beliefs of each of the individuals be as follows:

	π_1	π_2	π_3	π_4
s_1	0.3	0.2	0.3	0.5
s_2	0.2	0.6	0.4	0.1
s_3	0.5	0.2	0.3	0.4

The partition $\mathcal{E} = \langle \{1, 2\}, \{3, 4\} \rangle$, immunity from influence parameters $\alpha_1 = 0.6$, $\alpha_2 = 0.5$, $\alpha_3 = 0.3$, $\alpha_4 = 0.7$ and core beliefs:

	μ_1	μ_2	μ_3	μ_4
s_1	$1/3$	$7/45$	$1/15$	$19/35$
s_2	$1/15$	$8/10$	$3/4$	$1/28$
s_3	$3/5$	$1/20$	$11/60$	$59/140$

form an echo chamber representation of the stated profile of behavioral beliefs. However, the same behavioral beliefs can be represented under the trivial partition $\mathcal{E} = \langle I \rangle$, with the immunity from influence parameters $\alpha_1 = 0.4$, $\alpha_2 = 0.5$, $\alpha_3 = 0.2$, $\alpha_4 = 0.8$ and core beliefs given by:

	μ_1	μ_2	μ_3	μ_4
s_1	$21/80$	$3/40$	$1/5$	$87/160$
s_2	$1/80$	$7/8$	$7/10$	$7/160$
s_3	$29/40$	$1/20$	$1/10$	$33/80$

The trivial partition, is, of course, a coarsening of $\langle \{1, 2\}, \{3, 4\} \rangle$, which means that the first representation doesn't fully capture the potential for influence that can be rationalized by this profile of behavioral beliefs.

Example 2 Consider a society with three individuals, $I = \{1, 2, 3\}$, and a state space with two states, $S = \{s_1, s_2\}$. Let the behavioral beliefs of each of the individuals be as follows:

	π_1	π_2	π_3
s_1	0.4	0.5	0.7
s_2	0.6	0.5	0.3

Consider the trivial partition $\mathcal{E} = \langle I \rangle$ and immunity from influence parameters $\alpha_1 = 0.4$, $\alpha_2 = 0.3$, $\alpha_3 = 0.6$ and core beliefs:

	μ_1	μ_2	μ_3
s_1	$1/5$	$19/45$	$73/90$
s_2	$4/5$	$26/45$	$17/90$

These parameters together constitute an echo chamber representation of the profile of behavioral beliefs. However, keeping the partition unchanged, changing the immunity from influence parameters to $\tilde{\alpha}_1 = 0.3$, $\tilde{\alpha}_2 = 0.2$, $\tilde{\alpha}_3 = 0.5$, and core beliefs to:

	$\tilde{\mu}_1$	$\tilde{\mu}_2$	$\tilde{\mu}_3$
s_1	$4/45$	$11/30$	$13/15$
s_2	$41/45$	$19/30$	$2/15$

gives another echo chamber representation, where for each i, $\tilde{\alpha}_i < \alpha_i$, which means that the scope for influence is greater in the second representation than the first. As such, the first representation underestimates the potential for influence that is consistent with this profile of behavioral beliefs.

To summarize the content of the examples, the refinement we are introducing discounts any echo chamber representation $(\mathcal{E}, (\mu_i)_{i\in I}, (\alpha_i)_{i\in I})$ for which there exists another representation $(\tilde{\mathcal{E}}, (\tilde{\mu}_i)_{i\in I}, (\tilde{\alpha}_i)_{i\in I})$ where $\tilde{\mathcal{E}}$ is a coarsening of $\mathcal{E}$. The refinement also discounts any representation $(\mathcal{E}, (\mu_i)_{i\in I}, (\alpha_i)_{i\in I})$ if there exists another representation $(\mathcal{E}, (\tilde{\mu}_i)_{i\in I}, (\tilde{\alpha}_i)_{i\in I})$ with the same echo chamber partition but $\tilde{\alpha}_i \leq \alpha_i$, for all i, holding strictly for some i. We refer to the representations that survive this refinement as maximal influence echo chamber representations. Formally:

Definition 3. *An EC representation $(\mathcal{E}, (\mu_i)_{i\in I}, (\alpha_i)_{i\in I})$ of $(\pi_i)_{i\in I}$ is maximal influence (MIEC) if there does not exist any other EC representations $(\tilde{\mathcal{E}}, (\tilde{\mu}_i)_{i\in I}, (\tilde{\alpha}_i)_{i\in I})$ of $(\pi_i)_{i\in I}$ such that for some $i \in I$, either:*

1. *$E(i) \subsetneq \tilde{E}(i)$, or*
2. *$\tilde{E}(i) = E(i)$ and $\tilde{\alpha}_i < \alpha_i$*

In the next two sections, we focus on the characterization and identification of MIEC representations and show that there are interesting insights to be gained when we contrast them with the general class of EC representations.

3 Characterization

3.1 Maximal influence echo chamber representation

We now show that maximal influence echo chamber representations can be characterized by two axioms. The first of these axioms spells out the key constraint that influence imposes on a DM's behavioral beliefs. It says that a DM is constrained by influence to think of an event as sure only if there exists at least some other individual who does likewise, with this influence being mutual. In other words, even if an individual fundamentally thinks of an event as sure by her core beliefs, it is not guaranteed to translate into her behavior unless it receives conformity from others, with this process of conformity being mutual.

Axiom A1 (Certainty conformism)**.** *For all $i \in I$, there exists $j \in I$, $j \neq i$, such that, for any event $A \subseteq S$, $\pi_i(A) = 1$ if and only if $\pi_j(A) = 1$.*

The first axiom incorporates the idea of conformism, which the literature identifies as a key marker of social influence. When specialized to our current context, such conformism is seen on behavioral probabilistic judgments about sure events.

Our second axiom, on the other hand, reinforces the message that inter-personal interactions of influence need not be just about conformity. There is scope for any individual influencing the beliefs of others she interacts with and not just mimicking them. To present this axiom, we first introduce a definition. We define the revealed neighborhood of any $i \in I$ by

$$N(i) = \{j \in I \setminus \{i\} : \pi_i(A) = 0 \iff \pi_j(A) = 0, \text{for any event } A\}$$

The idea behind this revealed elicitation is quite straightforward. An individual is presumably connected to those individuals who she seeks conformity from. Hence, these individuals form her neighborhood in the social network of interacting beliefs. Observe that, the certainty conformism axiom guarantees that $N(i)$ is non-empty for any i. Now for any $J \subseteq I$, and any event A, denote the average behavioral belief about A amongst individuals in J by

$$\overline{\pi}_J(A) = \frac{1}{|J|} \sum_{j \in J} \pi_j(A)$$

Axiom A2 (Subjective assessment of uncertainty)**.** *For all $i \in I$, there exists an event A such that $\overline{\pi}_{N(i)}(A) \neq \overline{\pi}_{N(i) \cup \{i\}}(A)$*

The axiom highlights the point that any $i \in I$ is not simply a passive recipient of the probabilistic judgments of her neighbors. Rather, her presence may also change the average beliefs prevailing in her neighborhood in as much as there exists at least some event in which the average belief in her neighborhood with and without her is different. Equivalently, on this event, i's behavioral belief is different from the average behavioral belief in her neighborhood, i.e., $\pi_i(A) \neq \overline{\pi}_{N(i)}(A)$. Observe that, given what the first axiom says, such influence that i casts is necessarily for events that she and everyone in her neighborhood share a degree of uncertainty over. The axiom can also be seen in the light of "maximality" of influence. It implies that all individuals cast some influence on the formation of beliefs. There is no individual who is simply the recipient of influence.

We can now present our behavioral characterization result.

Theorem 1. *The behavioral probability profile $(\pi_i)_{i\in I}$ has a maximal influence echo chamber representation if and only if it satisfies certainty conformism and subjective assessment of uncertainty.*

Proof: Please refer to Appendix Section A.4

3.2 Disagreements on fundamental certainty

The way we have set-up the model, the primitives do not impose any restriction on the core beliefs of individuals, neither within an echo chamber nor across chambers. In reality, one would imagine that these beliefs may be inter-related to some extent. Presumably, a necessary condition for individuals coming together and forming an echo chamber is some minimal agreement in their core beliefs, especially when it comes to perceptions about certainty, with this agreement not shared by those outside the chamber. This is in the spirit of belief-based homophily. At the same time, for these echo chambers to retain their salience, there has to be pathways of influence within them and it is this influence that serves as a glue holding them together. But for influence to operate, there also needs to be disagreements within the echo chamber that opens up the scope for such influence. In other words, we would imagine that the functioning of echo chambers incorporate both inter-chamber and intra-chamber disagreements over core beliefs. It is, therefore, interesting to note that one of the things that the characterization of the maximal influence echo chamber model brings to the forefront is that such notions of disagreements are embedded in this class of representations. We now make precise this important property that these representations possess.

To state these observations formally, we introduce some terminology. We say that i is fundamentally certain about an event A if $\mu_i(A) = 1$. We say that a chamber E is fundamentally certain about an event A if $\mu_i(A) = 1$, for all $i \in E$. We now introduce two conditions pertaining to disagreements on fundamental certainty (DFC), which we show hold true for maximal influence echo chamber representations. The first condition provides a statement about differing views on certainty across echo chambers.

Condition 1 (DFC-Inter)**.** *For all chambers E and E', there exists an event that one of the chambers is fundamentally certain about, but the other chamber is not.*

As mentioned above, one may think of this as a constitutive condition of echo chambers—the fact that, at a fundamental level, echo chambers are formed by people who share a certain view of "reality" that is not necessarily shared by those outside the echo chamber. The condition above expresses this idea in a fairly weak form. At the same time, as noted earlier, just because echo chambers form presumably doesn't mean that there is complete agreement on matters of certainty within an echo chamber. Our next condition captures this viewpoint, leaving open the possibility that there is room for disagreement and influence within an echo chamber.

Condition 2 (DFC-Intra)**.** *For all $i \in I$, there exists an event she is fundamentally certain about, but someone in her chamber is not*

Theorem 2. *Suppose $(\mathcal{E} = \langle E_1, \ldots, E_k \rangle, (\mu_i, \alpha_i)_{i \in I})$ is an echo chamber representation of $(\pi_i)_{i \in I}$. $(\mathcal{E} = \langle E_1, \ldots, E_k \rangle, (\mu_i, \alpha_i)_{i \in I})$ is a maximal influence echo chamber representation iff it satisfies DFC-Inter and DFC-Intra.*

Proof: Please refer to Appendix Section A.6

The result establishes that DFC-Inter and DFC-Intra are properties of maximal influence echo chamber representations and not of echo chamber representations in general. This adds to the appeal of this sub-class of representations as they implicitly capture intuitive notions underling echo chamber formation and salience, even though we do not explicitly model this in our set-up.

3.3 Echo chamber representation

A final detail regarding characterization that the reader may be interested in is about the characterization of echo chamber representations. It turns out such representations are characterized simply by certainty conformism.

Theorem 3. *The behavioral probability profile $(\pi_i)_{i \in I}$ has an echo chamber representation if and only if it satisfies certainty conformism.*

Proof: Please refer to Appendix Section A.3

4 Identification

We now address the question about the behavioral identification of the model parameters. That is, suppose $(\pi_i)_{i\in I}$ has a maximal influence echo chamber representation. Is this representation unique, or are multiple such representations possible? The following result shows that a desirable feature of such a representation is that it is uniquely identified.

Theorem 4. *If $(\mathcal{E} = \langle E_1, \ldots, E_k\rangle, (\mu_i, \alpha_i)_{i\in I})$ and $(\tilde{\mathcal{E}} = \langle \tilde{E}_1, \ldots, \tilde{E}_l\rangle, (\tilde{\mu}_i, \tilde{\alpha}_i)_{i\in I})$ are maximal influence echo chamber representations of $(\pi_i)_{i\in I}$, then $\mathcal{E} = \tilde{\mathcal{E}}$, and for each $i \in I$, $\mu_i = \tilde{\mu}_i$, and $\alpha_i = \tilde{\alpha}_i$.*

Proof: Please refer to Appendix Section A.5

This property regarding the identification of maximal influence echo chamber representations contrasts sharply with echo chamber representations where we don't impose this refinement. Such representations may not be precisely identified.

We first examine the plausible partitions of society that might feature in echo chamber representations of a profile of behavioral probabilities $(\pi_i)_{i\in I}$. If there exists a representation under the partition $\mathcal{E}$, then we first show that if $j \in E(i)$ for $j \neq i$, then $j \in N(i)$. That is, $E(i)$ must be a subset of $N(i) \cup \{i\}$ for all $i \in I$. As an implication, $N(i) \cup \{i\}$ is also the largest possible echo chamber to which i may belong. Furthermore, we also show that the model does not inherently restrict a multiplicity of partitions (barring that echo chambers be non-singleton) of the society for which there exist collections of core beliefs and immunity from influence parameters that together form distinct echo chamber representations. This idea is formalized in proposition 1 and the corollary that follows.

Proposition 1. *Let $\mathcal{E} = \langle E_1, \ldots, E_k\rangle$ be the partition of society such that $E(i) = N(i) \cup \{i\}$ for all $i \in I$. If $\tilde{\mathcal{E}} = \left\langle \tilde{E}_1, \ldots, \tilde{E}_\ell \right\rangle$ is a partition of society for which there exists an echo chamber representation of the beliefs $(\pi_i)_{i\in I}$, then $\tilde{E}(i) \subseteq E(i)$. Further, if $\tilde{E}(i) \subseteq E(i)$ and $|\tilde{E}(i)| \geq 2$ for all $i \in I$, then there exists an echo chamber representation of $(\pi_i)_{i\in I}$ under the partition $\tilde{\mathcal{E}}$.*

Proof: Please refer to Appendix Section A.1

We can then identify a condition on behavioral beliefs for when partitions must indeed be unique in the absence of other restrictions on the model.

Corollary 1. *For all echo chamber representations $\left(\mathcal{E}, (\alpha_i)_{i\in I}, (\mu_i)_{i\in I}\right)$ and $\left(\tilde{\mathcal{E}}, (\tilde{\alpha}_i)_{i\in I}, (\tilde{\mu}_i)_{i\in I}\right)$ of $(\pi_i)_{i\in I}$, the partition is unique, or $\mathcal{E} = \tilde{\mathcal{E}}$, iff $|N(i)| < 3$ for all $i \in I$.*

While we have shown that partitions are generally non-unique, there exists a class of behavioral beliefs such that the choice of partition for any echo chamber representation is uniquely specified. Does this extend to the other parameters of the model, namely the core beliefs and immunity from influence parameters? To examine this, we first identify the range of viable choices for the immunity from influence parameter for each $i \in I$ and establish a lower bound on these choices, with the upper bound always being 1.

Proposition 2. *Suppose the collection of behavioral beliefs $(\pi_i)_{i\in I}$ has an echo chamber representation under the partition $\mathcal{E} = \langle E_1, ..., E_\ell \rangle$, with $E(i)$ denoting i's echo chamber under $\mathcal{E}$. Then $(\pi_i)_{i\in I}$ has an echo chamber representation with influence parameters $(\alpha_i)_{i\in I}$ iff $\alpha_i \in \left[1 - \min \frac{\pi_i(A)}{\bar{\pi}_{E(i)}(A)},\, 1\right) \cap (0,\, 1)$ for all $i \in I$.*

Proof: Please refer to Appendix Section A.2

The first key insight we can gain from this result is that once we identify a suitable partition $\mathcal{E}$, restrictions on the parameters of any $i \in I$ depend only on $E(i)$. However, while the bounds on these parameters are dependent on $E(i)$, the exact choice is independent of all other individuals. Particularly, what this means is that if, under some partition $\mathcal{E}$, $(\alpha_i)_{i\in I}$ and $(\tilde{\alpha}_i)_{i\in I}$ are two collections of immunity from influence parameters that constitute echo chamber representations of $(\pi_i)_{i\in I}$, then $(\alpha_i,\, \tilde{\alpha}_{-i})$ and $(\tilde{\alpha}_i,\, \alpha_{-i})$ are also valid choices, given any $i \in I$. Finally, as evident from the result, immunity from influence parameters are always non-unique.

Corollary 2. *If $(\pi_i)_{i\in I}$ has an echo chamber representation, then it has uncountably infinite representations.*

5 Properties of the model

In this section, we highlight some important properties and empirical content of the model. These pertain to the scope of elite influence, naive updating of beliefs with the possibility of sticky beliefs, and the polarization of beliefs. We find that agents who are least susceptible to influence tend to exert the most influence on the average behavioral belief in their echo chambers. We also outline a mechanism of updating beliefs where an individual's core beliefs are updated in the manner of a Bayesian, though their behavioral beliefs are updated using a naive weighted average. As a result, DMs are unable to fully incorporate their own private information or that revealed in others' behavior, causing beliefs in echo chambers to be sticky. We synthesize these findings to pose a problem of interest in information design. Finally, we show that polarization across echo chambers can be attributed to a failure to communicate information across chambers, that could otherwise mitigate misinformation due to noisy signals.

5.1 Elite Influence

A theme that has featured prominently in recent times is that of elite influence. For instance, it has been pointed out in the context of partisan politics that each side of the partisan divide has elites who have a disproportionate influence on their respective sides. In other words, when there is influence at play, it is typically marked by a great degree of heterogeneity in terms of the ability to influence. Such an effect shows up in our model. The key feature in this regard that our model demonstrates is that within each echo chamber, the individuals who are the least influenced (high α-s) happen to be the ones who end up having the greatest influence in terms of shaping beliefs within their echo chamber.

The nature of linear influence in the model gives the average behavioral belief in the echo chamber a unique structure. Recall the following expression for this average belief we derived in Remark 1:

$$\overline{\pi}_i(A) = \sum_{j \in E(i)} \frac{\alpha_j}{\sum_{k \in E(i)} \alpha_k} \mu_j(A)$$

That is, the average belief in an echo chamber can be represented as the weighted average of core beliefs, with the weights capturing relative influence. In particular, the weight attached to i's core belief is given by $\frac{\alpha_i}{\sum_{k \in E(i)} \alpha_k}$. It is relative because it depends on the ratio of the DM's own α_i to the sum of all α_j in her chamber. It also measures the degree of influence as the more a DM is immune to influence, the greater the weight placed on her core belief in the determination of the average behavioral belief. However, the more others are immune from influence, the more they influence the average belief, thus reducing the relative influence exhibited by the DM. A way of capturing the influence exhibited by a DM is the difference between her core beliefs and the average behavioral beliefs, which is expressed as follows.

$$|\overline{\pi}_i(A) - \mu_i(A)| = \left| \frac{1}{\sum_{k \in E(i)} \alpha_k} \left(\sum_{j \in E(i) \setminus \{i\}} \alpha_j (\mu_j(A) - \mu_i(A)) \right) \right|$$

This is a measure of her influence because it captures how close average behavioral beliefs in her echo chamber are pulled towards her core beliefs. Note that α_i appears only in the denominator, which means the difference is decreasing in α_i, and for any collection $(\mu_i)_{i \in I}$, the average behavioral belief is influenced more by i's core belief if she is more immune to influence. Note also, that this difference depends on the relative influence exhibited by others. As α_j increases for $j \neq i$, the average belief moves closer to j's core belief, and $|\overline{\pi}_i(A) - \mu_j(A)|$ decreases. Since individual behavioral beliefs are a weighted average of core beliefs and average behavioral beliefs, each individual's behavioral belief moves closer to j's core belief. Thus, $|\pi_i(A) - \mu_j(A)|$ decreases too. In this manner, individuals who are more immune to influence and exhibit greater influence on the echo chamber act as a group of elites, whose core beliefs play a large role in determining average behavioral beliefs. This feature of elite influence also manifests itself in the way private information is incorporated into behavior and transmitted to others in the echo chamber.

5.2 Belief Updating

5.2.1 Mechanism

We now examine the ways in which beliefs are updated in our model upon the arrival of new information. The key assumption we maintain in the model is that this information is conveyed to any individual through private signals that cannot be shared with anyone else in the echo chamber. This means that the core probabilities of only the information recipient get updated, and the response within the echo chamber is purely driven by the behavioral changes she exhibits. Let C be a finite set of signals. Each agent observes the signal c conditional on the realized state being s with the probability $\phi_i^s(c)$. Let $\mathcal{S}$ denote the set of non-empty subsets of the state space S. For each i, define a function $\sigma_i : C \to \mathcal{S}$, such that $\sigma_i(c) = \{s \in S : \phi_i^s(c) > 0\}$. For all $c \in C$, we assume that $\mu_i(\sigma_i(c)) > 0$, that is, $\phi_i^s(c) > 0$ for some $s \in S$ such that $\mu_i(s) > 0$.

Define a source as the collection of conditional distributions $\Phi_i = \{\phi_i^s | s \in S\}$. Each individual knows the conditional distributions of her source and observes the realized signal produced by it. Then, given c^i as the signal received by i, her posterior core belief is given by:

$$\mu_i(A|c^i) = \frac{\sum_{s \in A} \mu_i(s)\phi_i^s(c^i)}{\sum_{s' \in S} \mu_i(s')\phi_i^{s'}(c^i)}$$

As remarked earlier, while signals cannot be transmitted to others in the echo chamber, the way behavioral beliefs are updated throughout the network is through changes in observed behavior. For each $i \in I$, denote her updated core belief $\mu_i(.|c^i)$ by μ_i' and their behavioral belief by π_i'. Then, the change observed in each individual's behavioral belief is given by:

$$\pi_i'(A) - \pi_i(A) = \alpha_i(\mu_i'(A) - \mu_i(A)) + (1 - \alpha_i) \sum_{j \in E(i)} \frac{\alpha_j(\mu_j'(A) - \mu_j(A))}{\sum_{k \in E(i)} \alpha_k}$$

5.2.2 Incorporation of private information

First, consider the case that individual i is the only one who receives private information in the echo chamber, which will happen as long as $\phi_j^s = \phi_j$ for all $s \in S$, $j \neq i$, $j \in E(i)$. In this case, the ratio of the change in her behavioral beliefs to the change in core beliefs can be written as:

$$\frac{\pi_i'(A) - \pi_i(A)}{\mu_i'(A) - \mu_i(A)} = \alpha_i \left(1 + \frac{1-\alpha_i}{\sum_{k \in E(i)} \alpha_k}\right)$$

Note that this ratio is increasing in α_i and is equal to 0 for $\alpha_i = 0$ and 1 for $\alpha_i = 1$. Note also, that it is bounded below by α_i. Taking the second derivative will reveal that it is concave in α_i. To interpret this, we can decompose the effect of α_i on this ratio into the direct effect $D(\alpha_i) = \alpha_i$, and the echo effect defined as:

$$I(\alpha_i) = \frac{\alpha_i(1-\alpha_i)}{\sum_{j \in E(i)} \alpha_j}$$

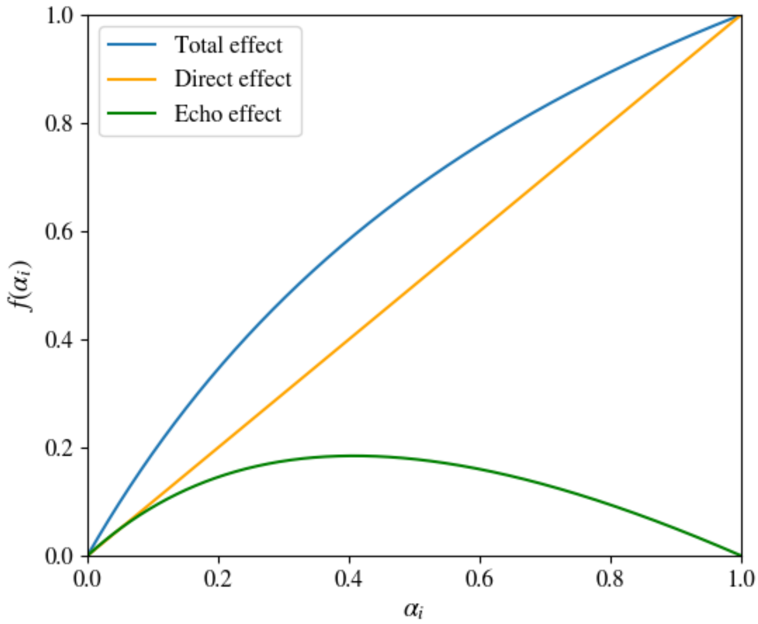
The echo effect captures the impact of the change in her beliefs on others' behavioral beliefs, which she then incorporates into her own behavioral beliefs. While $D'(\alpha_i) = 1$ for all $\alpha_i \in (0,\, 1)$, $I''(\alpha_i) < 0$ and $I'(\alpha_i) < 0$ for some $\alpha_i \in (0,\, 1)$. While $D(\alpha_i)$ captures the effect of μ_i on π_i through the DM's immunity from influence, $I(\alpha_i)$ captures the effect of the DM's core beliefs that influence her through the behavioral beliefs of others in her echo chamber. Since $I(0) = I(1) = 0$, there exists some $a \in (0,\, 1)$ such that $I'(a) = 0$. Denote by $\overline{a} = \sum_{j \in E(i) \setminus \{i\}} \alpha_j$. Then we can uniquely define a:[2]

$$a = \sqrt{\overline{a}(\overline{a}+1)} - \overline{a}$$

We know that $I'(\alpha_i) > 0$ for $\alpha_i < a$ and $I'(\alpha_i) < 0$ for $\alpha_i > a$. What this means is that for $\alpha_i < a$, the rise in the DM's influence on her echo chamber's behavioral beliefs outweighs the fall in the influence of the echo chamber's behavioral beliefs on her own. For high α_i, the direct effect is high, but the echo effect starts decreasing as $1 - \alpha_i$ goes to 0. We can also note that for $\alpha_i \in (0,\, 1)$, $D(\alpha_i) + I(\alpha_i)$ is never 1. Figure 1 depicts the direct and echo effects for a chamber with 4 individuals and $\overline{a} = 0.9$.

[2]To do so, we must find the roots of the $I'(\alpha_i)$, the solution to which is given by $\sum_{j \in E(i) \setminus \{i\}} \alpha_j = -\frac{\alpha_i^2}{2\alpha_i - 1}$. This function is invertible in the domain $(0,\, 0.5)$.

Figure 1. Direct and Echo effect for $|E(i)| = 4$ and $\overline{a} = 0.9$



5.2.3 Belief updation and transmission

We now focus attention on the transmission of updated beliefs. Since DMs in our model require those in their chamber to agree on sure and null events, they also require agreement on new information they receive for it to translate into behavior fully. Without such agreement, beliefs may be sticky and fail to incorporate the full extent of the available information.

Proposition 3 (Sticky Beliefs)**.** *For any* $i \in I$, $\pi_i(\sigma_i(c^i)\,|\,c^i) = 1$ *if and only if* $\mu_j(\sigma_i(c^i)\,|\,c^j) = 1$, *for all* $j \in E(i)$.

Proof: Please refer to Appendix Section A.7

For Bayesian DMs whose beliefs are solely based on private information, the way they update their beliefs upon receiving new information would imply that if $\mathbb{P}(c\,|\,s) = 0$ and $\mathbb{P}(c) > 0$, then $\mathbb{P}(s\,|\,c) = 0$, while for s such that $\mathbb{P}(s) = 0$, $\mathbb{P}(s\,|\,c) = 0$. However, in our model, $\pi_i(A) = 0$ if and only if $\overline{\pi}_i(A) = 0$, which holds if and only if $\mu_j(A) = 0$ for all $j \in E(i)$. Since DMs seek the validation of all members of their echo chamber, they are unable to assign beliefs according to their private information, which would demand that they rule out all states $s \notin \sigma_i(c^i)$. Instead, they assign 0 probability to only those states that are ruled out by everyone else or are null states according to all their priors. In other words:

$$\pi_i\left(S \setminus \cup_{j\in E(i)}\sigma_j(c^j)\right) = 0$$

A special case of this is when the signal received by the DM suggests that she should rule out a state, but not only does she assign it a positive probability, her posterior belief on the state can actually be greater than that under her prior. Since DMs in our model seek their echo chamber's validation when assigning 0 probability to any event, they are unable to rule states out solely on private information. However, if high-influence individuals believe that such a state is more likely given the signals they receive, the DM may end up considering the state to be more likely even though her private information implies otherwise. Denote by π_i' and μ_i' the posterior behavioral and core beliefs of individual i. Then, $\pi_i'(A) > \pi_i(A)$ for some A such that $A \cap \sigma_i(c^i) = \emptyset$ if the following inequality is satisfied.

$$\frac{\alpha_i(1 + \sum_{k\in E(i)\setminus\{i\}} \alpha_k)}{1-\alpha_i}\mu_i(A) < \sum_{k\in E(i)\setminus\{i\}} \alpha_k(\mu_k'(A) - \mu_k(A))$$

This inequality suggests that this effect is most likely to be prevalent for individuals with low α_i and for states where their prior core belief was low. While low immunity from influence makes it more difficult for DMs to incorporate their private information, the seeking of validation from all members in the cluster, i.e. $\pi_i(A) = 0 \iff \pi_j(A) = 0$ for $j \in E(i)$, comes in the way of the DM assigning $\pi_i(A)$ equal to 0 if $\mu_j(A|c^j) - \mu_j(A)$ is strongly positive for some $j \in E(i)$ with a high α_j. In that case, the DM ends up acting against her private information.

We can now extend the statement of Proposition 3 to how the behavioral beliefs of an entire echo chamber are updated when new information is received. Let E_k be some echo chamber and denote by S_0^i the set of states such that $\mu_i(s_0) = 0$ for all $s_0 \in S_0^i$.

Corollary 3. $\pi_i(\sigma_i(c^i)\,|\,c^i) = 1$ *for all* $i \in E_k$ *if and only if for any* i, $j \in E_k$

$$\sigma_i(c^i) \setminus S_0^i \subseteq \sigma_j(c^j)$$

Proof: Please refer to Appendix Section A.8

Corollary 3 asserts that agents can assign non-zero probability to only those states that are not ruled out by the signals received by everyone else in the echo chamber. The result follows directly from Proposition 3 in that it guarantees that $\mu_i(\sigma_j(c^j)\,|\,c^i) = 1$ for all $j \in E(i)$ for every choice of $i \in E_k$. This is also equivalent then to all non-null states in the posterior being in the intersection of all $\sigma_j(c^j)$ for $j \in E(i)$. That means that every agent in the echo chamber receives the same information about non-null states. Contrarily, in the presence of heterogenous information, DMs are unable to fully incorporate their private information. Instead, they require that everyone's observed beliefs agree with their private information, which is only possible if everyone receives the same private information. The only potential disagreements in signals are then restricted to the domain of null states, which do not affect posterior beliefs.

Remark 2 (A problem in information design)**.** Consider a scenario where a designer wishes to convey information to agents about the realized state. Say, for instance, she knows that the realized state is in $A \subset S$. That is, $A^c = S \setminus A$ will occur with 0 probability. However, she can convey this information only to $m < |E(i)|$ agents. Suppose her objective is to minimize $\bar{\pi}'(A^c)$. Through the unique identification of parameters and the specification of the average behavioral belief as in Remark 1, this is akin to solving the following problem:

$$\min_{\left|\left\{j \in E(i): \mu_j'(A^c) \neq \mu_j(A^c)\right\}\right| \leq m} \sum_{j \in E(i)} \alpha_j \mu_j'(A^c)$$

It is clear that she must pick the agents with the highest $\alpha_j[\mu_j(A^c) - \min_{c^j \in C_j} \mu_j(A^c|c^j)]$ to provide the information. However, this problem can be extended to consider conveying information to limited individuals across chambers and incentives for the designer in terms of changing betting behaviour with respect to these beliefs rather than the beliefs themselves.

5.3 Confirmation Bias and Voting

Consider a simple voting model, where there are two policies, L and R. Individuals $I = \{1, ..., 2n\}$ are divided into two groups of equal size, E_L and E_R. Policy L can be thought of as the in-group policy for E_L, and likewise with policy R for E_R. Each individual votes for one of the policies, and one of the two policies is implemented by simple majority. Ties are broken by a coin toss.

There are two states of the world, $S = \{\ell, r\}$. If the realized state of the world is ℓ and policy L is implemented, then individuals receive $\overline{u}$. Likewise if policy R is implemented and the realized state is r. If L and R are implemented with the realized state being r and ℓ respectively, individuals receive the utility $\underline{u}$. Assume $\overline{u} > \underline{u}$.

Suppose each individual $i \in L$ (resp. $i \in R$) has the core belief $\mu_i = \mu_L$ (resp. $\mu_i = \mu_R$) over S. Consequently, $\pi_i = \bar{\pi}_{E(i)} = \mu_{E(i)}$, where $E(i) \in \{L, R\}$. For simplicity, assume $\mu_L(\ell) = \mu_R(r) = q > \frac{1}{2}$. Individuals receive signals, $C = \{c_\ell, c_r\}$, from a common source Φ of accuracy $\rho \geq \frac{1}{2}$. That is, $\phi^\ell(c_\ell) = \phi^r(c_r) = \rho$. Upon receiving a signal, the updated core beliefs of individuals are given by:

$$\mu_L(\ell \mid c_\ell) = \frac{\rho q}{\rho q + (1-\rho)(1-q)} = \mu_R(r \mid c_r)$$
$$\mu_L(\ell \mid c_r) = \frac{(1-\rho) q}{(1-\rho) q + \rho(1-q)} = \mu_R(r \mid c_\ell)$$

Then the updated behavioral beliefs of individual i, π_i', upon observing the signal c^i, are given by

$$\pi_i'(s) = \alpha_i \mu_i(s \mid c^i) + (1-\alpha_i) \sum_{j \in E(i)} \frac{\alpha_j}{\sum_{j' \in E(i)} \alpha_{j'}} \mu_j(s \mid c^j)$$

Suppose individuals vote for a policy after receiving a signal, and simply choose the policy that they believe maximizes their subjective expected utility with respect to π_i'. Then, they vote for L if $\pi_i'(\ell) > \frac{1}{2}$, and likewise for R.[3] If individuals deem both states equally likely, suppose they vote for their in-group policy. For simplicity, suppose $\alpha_i = \alpha$ for all $i \in I$.

The first observation we can make is that individuals update their core beliefs to reflect that the out-group policy may be more beneficial if and only if the accuracy of the signal (ρ) outweighs the strength of the prior in-group belief (q).

Lemma 1. *$\mu_L(r \mid c_r) > \frac{1}{2}$ (resp. $\mu_R(\ell \mid c_\ell)$) iff $\rho > q$*

The natural next observation to make is that while the accuracy of the signal is enough to revise the core belief on the out-group state to greater than $\frac{1}{2}$, the same is true for behavioral beliefs only if enough people in the echo chamber have received the out-group signal.

Lemma 2. *Let X_ℓ^L (resp. X_r^R) denote the number of individuals in E_L who received the signal c_ℓ. If $\alpha \leq \frac{1}{2}$ and $\rho \neq 1$, there exists $\hat{x}(\alpha, \rho, q) \in [0, 1)$ such that conditional on receiving the signal c_r, $\pi_i'(\ell) < \frac{1}{2}$ iff $\frac{X_\ell^L}{n} < \hat{x}(\alpha, \rho, q)$ for $i \in E_L$.*

The proof of Lemma 2 shows that the expression for $\hat{x}$ when $\rho > q$ is:

$$\hat{x}(\alpha, \rho, q) = \frac{1}{1-\alpha} \frac{\frac{1}{2} - \frac{(1-\rho)q}{(1-\rho)q + \rho(1-q)}}{\frac{\rho q}{\rho q + (1-\rho)(1-q)} - \frac{(1-\rho)q}{(1-\rho)q + \rho(1-q)}}$$

$\hat{x}$ is clearly increasing in α and decreasing in q, suggesting that confirmation bias is stronger when individuals are more susceptible to influence or when in-group prior beliefs are more extreme. The numerator and denominator of $\hat{x}$ are both increasing in ρ.

[3]See that if this were modelled as a game, then voting for the policy they find is more likely to give them higher utility is a weakly dominant strategy.

However, the partial derivative of $\hat{x}$ with respect to ρ shows that $\hat{x}$ is indeed increasing in ρ.[4] The effect of ρ and q on this expression works through two channels. First, the updated core belief depends on the original belief and the accuracy of the signal. The lower the accuracy of the signal and the stronger the original belief, the more reluctant an individual is to assign a low degree of belief to the in-group state.

The next lemma tells us that if individuals are relatively immune from influence, they are able to incorporate private information fully without the need for perfect signals.

Lemma 3. *If $\alpha > \frac{1}{2}$, there exists $q < \rho^*(\alpha,\, q) < 1$ such that $\hat{x}(\alpha,\, \rho^*(\alpha,\, q),\, q) = 1$.*

Using this insight, the first question to ask is whether a sub-optimal policy gets voted in. Naturally, a policy is sub-optimal if it gives individuals utility $\underline{u}$, which, in turn, is state-contingent. The following proposition shows that unless signals are perfectly accurate (and therefore, precise), sub-optimal policies are implemented with a probability greater than they would in the absence of echo chambers. Define $\rho^*(\alpha,\, q)$ as $\min \operatorname{argmax}_\rho \hat{x}(\alpha,\, \rho,\, q)$.

Proposition 4. *Suppose $q < \rho < \rho^*(\alpha,\, q)$. Then, sub-optimal policies are implemented with a positive probability $\lambda > \underline{\lambda}$, where $\underline{\lambda}$ is the probability of implementing sub-optimal policies based on core beliefs. Furthermore, $\lambda \to \underline{\lambda}$ as $\hat{x} \to 0$ or $\hat{x} \to 1$.*

The second question to look at is what happens to voting behavior as society becomes arbitrarily large. The next proposition shows that for various ranges of ρ, voting outcomes converges in probability to either implementing the optimal policy with probability 1, or a coin toss, which is the same as when no signals are conveyed.

Proposition 5. *Let λ denote the probability of implementing a sub-optimal policy. For each α, q, there exists $\tilde{\rho}(\alpha,\, q) \in (q,\, 1)$ such that as $n \to \infty$, if:*

1. *$\rho \leq \tilde{\rho}(\alpha,\, q)$, then $\lambda \to \frac{1}{2}$*
2. *$\rho > \tilde{\rho}(\alpha,\, q)$, then $\lambda \to 0$*

5.4 Behavioral Probabilities as a Steady State

Remark 1 establishes that for every collection of parameters, $(\mu_i)_{i\in I}$, $(\alpha_i)_{i\in I}$, and $\mathcal{E} = \langle E_1,\, ...,\, E_k\rangle$, there exists a unique profile of behavioral beliefs $(\pi_i)_{i\in I}$ generated by the model. We now detail a way in which agents may arrive at these beliefs.

[4]Observe that $\frac{\partial \hat{x}(\alpha,\rho,q)}{\partial \rho} = \frac{(2q-1)(2\rho^2-2\rho+1)}{2(1-q)q(2\rho-1)^2}$. Since $q > \frac{1}{2}$ and $\rho > q$ by assumption, $2\rho - 1 > 2q - 1 > 0$. Since $\rho \leq 1$, $q < 1$, implying that $0 < 1 - q < q$. Lastly, $2\rho^2 - 2\rho + 1$ is positive for all $\rho \in \mathbb{R}$.

Let π_i^t denote the behavioral beliefs of agent i in period t, $i \in I$ and $t \in \mathbb{Z}_+$. Suppose that agents hold some initial beliefs π_i^0 and update their behavioral beliefs by

$$\pi_i^{t+1}(A) \equiv \alpha_i \mu_i(A) + (1-\alpha_i)\frac{1}{|E(i)|}\sum_{j \in E(i)} \pi_j^t(A)$$

This model suggests that individuals update their beliefs partly (with $1-\alpha_i$ weight) using an unweighted DeGroot (1974) style rule of thumb that averages the behavioral beliefs in their cluster. However, their beliefs also remain anchored (with α_i weight) to their core beliefs.

A plausible guess for initial beliefs may be that individuals start out with their core beliefs before they begin to update their beliefs by observing their peers. Nevertheless, the result that follows shows that even without any such assumptions, agents' beliefs converge to the behavioral beliefs generated by the echo chamber model.

Proposition 6. *Suppose the profile of behavioral beliefs $(\pi_i)_{i \in I}$ admits an echo chamber representation with parameters $\mathcal{E} = \langle E_1, ..., E_k \rangle$, $(\alpha_i)_{i \in I}$, and $(\mu_i)_{i \in I}$. Let individuals hold some initial beliefs $(\pi_i^0)_{i \in I}$ and define for each $i \in I$ their beliefs in period $t+1$ as*

$$\pi_i^{t+1}(A) \equiv \alpha_i \mu_i(A) + (1-\alpha_i)\frac{1}{|E(i)|}\sum_{j \in E(i)} \pi_j^t(A)$$

Then for each $i \in I$, $\lim_{t \to \infty} \pi_i^t = \pi_i$.

Proof: Please refer to Appendix Section A.9

The proof demonstrates that this process of updating can be accommodated by the model of social influence by Friedkin and Johnsen (1990), and shows that the parameters in our model can be fashioned in the style of theirs to demonstrate that beliefs determined through such a process converge to the unique stationary distribution of the process, particularly the behavioral beliefs under our model.

A Appendix

The proofs for all the results in the paper are contained here. We first prove propositions 1 and 2, as they are later used in the proofs of theorems 1, 2, and 4. We then prove theorem 3, followed by theorem 1, 4, and then theorem 2.

A.1 Proof of Proposition 1

Suppose $\left(\tilde{\mathcal{E}},\, (\tilde{\alpha}_i)_{i\in I}\,,\, (\tilde{\mu}_i)_{i\in I}\right)$ is an echo chamber representation of $(\pi_i)_{i\in I}$. Note

$$\pi_i(A) = \tilde{\alpha}_i\tilde{\mu}_i(A) + (1-\tilde{\alpha}_i)\frac{1}{|\tilde{E}(i)|}\sum_{j\in\tilde{E}(i)}\pi_j(A)$$

By the fact that $\pi_j(A) \in [0,\, 1]$ and $\mu_i(A) \in [0,\, 1]$ for all A and j, through $\tilde{\alpha}_i \in (0,\, 1)$, we get $\pi_i(A) = 0$ iff $\pi_j(A) = 0$ for all $j \in \tilde{E}(i)$. This applies to all $i \in I$. Then if $j \in \tilde{E}(i)$, then it must be that $\pi_i(A) = 0 \iff \pi_j(A) = 0$, which means $j \in E(i)$. Thus $\tilde{E}(i) \subseteq E(i)$.

For the converse, suppose $\tilde{E}(i) \subseteq E(i)$ and $|\tilde{E}(i)| \geq 2$ for all $i \in I$. Let $\tilde{\bar{\pi}}_i(A) = \bar{\pi}_{\tilde{E}(i)}(A)$ for notational simplicity. By $\tilde{E}(i) \subseteq E(i)$, we have that $\pi_i(A) = 0$ iff $\pi_j(A) = 0$ for all $j \in \tilde{E}(i)$ iff $\tilde{\bar{\pi}}_i(A) = 0$. Then $\frac{\pi_i(A)}{\tilde{\bar{\pi}}_i(A)} > 0$ for all A such that this ratio is defined, which means that $1 - \min\frac{\pi_i(A)}{\tilde{\bar{\pi}}_i(A)} < 1$. Now choose some $\alpha_i \in \left(1 - \min\frac{\pi_i(A)}{\tilde{\bar{\pi}}_i(A)},\, 1\right)$. Since $\pi_i(S) = \tilde{\bar{\pi}}_i(S)$, we have $\min\frac{\pi_i(A)}{\tilde{\bar{\pi}}_i(A)} \leq 1$, which means $\alpha_i \in (0,\, 1)$.

Defining μ_i as:

$$\mu_i(A) = \frac{\pi_i(A) - (1-\alpha_i)\tilde{\bar{\pi}}_i(A)}{\alpha_i}$$

For A such that $\pi_i(A) = 1$ or $\pi_i(A) = 0$, we have $\mu_i(A) = 1$ and $\mu_i(A) = 0$ respectively. Further, our choice of α_i is such that $\min\frac{\pi_i(A)}{\tilde{\bar{\pi}}_i(A)} > 1 - \alpha_i$, which means $\pi_i(A) - (1 - \alpha_i)\tilde{\bar{\pi}}_i(A) > 0$ for all other A, implying that $\mu_i(A) \geq 0$ for all $A \subseteq S$. For A, B disjoint

$$\begin{aligned}
\mu_i(A\cup B) &= \frac{\pi_i(A\cup B) + (1-\alpha_i)\bar{\pi}_i(A\cup B)}{\alpha_i} \\
&= \frac{\pi_i(A) + \pi_i(B) - (1-\alpha_i)(\bar{\pi}_i(A) + \bar{\pi}_i(B))}{\alpha_i} \\
&= \frac{\pi_i(A) - (1-\alpha_i)\bar{\pi}_i(A)}{\alpha_i} + \frac{\pi_i(B) - (1-\alpha_i)\bar{\pi}_i(B)}{\alpha_i} \\
&= \mu_i(A) + \mu_i(B)
\end{aligned}$$

Thus, μ_i is a probability. Now that we have established that α_i and μ_i are valid choices for the parameters, rearranging the definition for $\mu_i(A)$ gives us $\pi_i(A) = \alpha_i\mu_i(A) + (1 - \alpha_i)\tilde{\bar{\pi}}_i(A)$, which implies that $\left(\tilde{\mathcal{E}},\, (\alpha_i)_{i\in I}\,,\, (\mu_i)_{i\in I}\right)$ is an echo chamber representation of $(\pi_i)_{i\in I}$.

A.2 Proof of Proposition 2

From the representation we have

$$\pi_i(A) = \alpha_i \mu_i(A) + (1-\alpha_i)\overline{\pi}_{E(i)}(A)$$

Rearrange this to obtain

$$\frac{\pi_i(A)}{\overline{\pi}_{E(i)}(A)} = \alpha_i \frac{\mu_i(A)}{\overline{\pi}_{E(i)}(A)} + (1-\alpha_i)$$

Since μ_i and $\overline{\pi}_{E(i)}$ are probabilities, $\frac{\mu_i(A)}{\overline{\pi}_{E(i)}(A)} \geq 0$, which implies $\alpha_i \geq 1 - \min \frac{\pi_i(A)}{\overline{\pi}_{E(i)}(A)}$. Since $\alpha_i \in (0, 1)$ by definition, we have the first part of the result.

For the second, choose α_i in the given set for each $i \in I$. Define μ_i as:

$$\mu_i(A) = \frac{\pi_i(A) - (1-\alpha_i)\overline{\pi}_{E(i)}(A)}{\alpha_i}$$

Note that by the choice of α_i, we have that $\frac{\pi_i(A)}{\overline{\pi}_{E(i)}(A)} \geq 1 - \alpha_i$, which implies $\mu_i(A) \geq 0$ for A such that $\pi_i(A) \neq 0$. For $\pi_i(A) = 0$, we have that $\overline{\pi}_{E(i)}(A) = 0$ by proposition 1, which implies that $\mu_i(A) = 0$. By the argument in proposition 1, μ_i is additive, and the chosen parameters provide an echo chamber representation of $(\pi_i)_{i \in I}$.

A.3 Proof of Theorem 3

We first show that certainty conformism is sufficient for an echo chamber representation.

Define chambers as $E(i) = N(i) \cup \{i\}$. Since $j \in N(i)$ iff $i \in N(j)$, $j \in E(i)$ iff $E(i) = E(j)$. Then we have a valid partition of I. Certainty conformism implies that for all $i \in I$, $N(i) \neq \emptyset$. Thus, $|E(i)| \geq 2$ for all $i \in I$. By proposition 1, there exists an echo chamber representation under this partition.

To show necessity, note that in the absence of certainty conformism, there exists $j \in I$ such that $N(j) = \emptyset$. Then $|N(j) \cup \{j\}| = 1$. However, by proposition 1, if there exists an echo chamber representation with the partition $\mathcal{E}$, then $E(j) \subseteq N(j) \cup \{j\}$, which implies $|E(j)| \leq 1$. However, the definition of echo chambers requires that they are non-singleton, which is a contradiction.

A.4 Proof of Theorem 1

We first show that the axioms are sufficient for a maximal influence echo chamber representation.

Step 1. Defining chambers.

Define $R(i) = N(i) \cup \{i\}$ for each $i \in I$. By A1, $N(i) \neq \emptyset$, which means $|R(i)| \geq 2$ for all $i \in I$. Since $j \in R(i)$ iff $i \in R(j)$, setting $E(i) = R(i)$ gives us a valid partition of I, $\mathcal{E}$. By theorem 3, there exists an echo chamber representation under this partition.

Step 2. Defining core beliefs and immunity from influence parameters.

Given any event A, we can write

$$\begin{aligned}
\overline{\pi}_{R(i)}(A) &= \frac{\pi_i(A)}{|R(i)|} + \frac{|N(i)|}{|R(i)|}\overline{\pi}_{N(i)}(A) \\
\pi_i(A) &= |R(i)|\overline{\pi}_{R(i)}(A) - |N(i)|\overline{\pi}_{N(i)}(A) \\
\pi_i(A) &= (|N(i)| + 1)\overline{\pi}_{R(i)}(A) - |N(i)|\overline{\pi}_{N(i)}(A) \\
\pi_i(A) &= \overline{\pi}_{R(i)}(A) + |N(i)|(\overline{\pi}_{R(i)}(A) - \overline{\pi}_{N(i)}(A))
\end{aligned}$$

By A2, we know that there exists some event A such that $\overline{\pi}_{R(i)}(A) \neq \overline{\pi}_{N(i)}(A)$, and since $N(i) \neq \emptyset$, $\pi_i(A) \neq \overline{\pi}_{R(i)}(A)$. Then either $\pi_i(A) < \overline{\pi}_{R(i)}(A)$ or $\pi_i(A^c) < \overline{\pi}_{R(i)}(A)$. Consequently, $\min \frac{\pi_i(A)}{\overline{\pi}_{R(i)}(A)} < 1$. Choose $\alpha_i = 1 - \min \frac{\pi_i(A)}{\overline{\pi}_{R(i)}(A)} > 1 - 1 = 0$. Furthermore, $\pi_i(A) = 0$ iff $\pi_j(A) = 0$ for all $j \in R(i)$ iff $\overline{\pi}_{R(i)}(A) = 0$, which implies $\frac{\pi_i(A)}{\overline{\pi}_{R(i)}(A)} > 0$ for all A such that this ratio is defined. Then $\alpha_i < 1$, and $\alpha_i \in \left[1 - \min \frac{\pi_i(A)}{\overline{\pi}_{R(i)}(A)},\, 1\right) \cap (0,\, 1)$. By proposition 2, there exists an echo chamber representation with the partition $\mathcal{E}$ and immunity from influence parameters $(\alpha_i)_{i\in I}$. Rearrange the definition of behavioral beliefs in the model to get μ_i as

$$\mu_i(A) = \frac{\pi_i(A) - (1 - \alpha_i)\overline{\pi}_{R(i)}(A)}{\alpha_i}$$

Then $\left(\mathcal{E},\, (\alpha_i)_{i\in I},\, (\mu_i)_{i\in I}\right)$ is an echo chamber representation of $(\pi_i)_{i\in I}$.

Step 3. Establishing that this representation is maximal influence.

By proposition 1, we know that if there exists an echo chamber representation of $(\pi_i)_{i\in I}$ under any partition $\tilde{\mathcal{E}}$, then $\tilde{E}(i) \subseteq R(i) = E(i)$. Thus there does not exist any representation with a coarsening of $\mathcal{E}$. If $\tilde{E}(i) = R(i)$ for all $i \in I$, note by proposition 2 that the corresponding immunity from influence parameter $\tilde{\alpha}_i \geq 1 - \min \frac{\pi_i(A)}{\overline{\pi}_{R(i)}(A)} = \alpha_i$. Then, to show that $\left(\mathcal{E},\, (\alpha_i)_{i\in I},\, (\mu_i)_{i\in I}\right)$, we must ensure that if there exists another echo chamber representation $\left(\tilde{\mathcal{E}},\, (\tilde{\alpha}_i)_{i\in I}\,,\, (\tilde{\mu}_i)_{i\in I}\right)$, such that $\tilde{\mathcal{E}} = \mathcal{E}$ and $\tilde{\alpha}_i = \alpha_i$ for all $i \in I$, then $\tilde{\mu}_i = \mu_i$ for all $i \in I$.

By the definition of the model, for all $A \subseteq S$:

$$\begin{aligned}\tilde{\mu}_i(A) &= \frac{\pi_i(A) - (1-\tilde{\alpha}_i)\overline{\pi}_{R(i)}(A)}{\tilde{\alpha}_i} \\ &= \frac{\pi_i(A) - (1-\alpha_i)\overline{\pi}_{R(i)}(A)}{\alpha_i} \\ &= \mu_i(A)\end{aligned}$$

Thus $\mu_i = \tilde{\mu}_i$ for each $i \in I$. Then $\left(\tilde{\mathcal{E}},\, (\tilde{\alpha}_i)_{i\in I}\,,\, (\tilde{\mu}_i)_{i\in I}\right)$ is identical to $\left(\mathcal{E},\, (\alpha_i)_{i\in I}\,,\, (\mu_i)_{i\in I}\right)$, which means it cannot be non-maximal. Consequently, it is a maximal influence echo chamber representation of $(\pi_i)_{i\in I}$.

This completes the proof for sufficiency of the axioms.

We now look at the necessity of the two axioms.

Axiom A1: By theorem 3, certainty conformism is necessary for the existence of an echo chamber representation, of which the maximal influence echo chamber representations are a subclass.

Axiom A2: Suppose there exists $i \in I$ such that $\overline{\pi}_{R(i)}(A) = \overline{\pi}_{N(i)}(A)$ for all $A \subseteq S$. Then note

$$\begin{aligned}\pi_i(A) &= \overline{\pi}_{R(i)}(A) + |N(i)|\left(\overline{\pi}_{R(i)}(A) - \overline{\pi}_{N(i)}(A)\right) \\ &= \overline{\pi}_{R(i)}(A)\end{aligned}$$

Then $\pi_i = \overline{\pi}_{R(i)}$. If A1 is violated, there does not exist any echo chamber representation of $(\pi_j)_{j\in I}$, so assume it is satisfied. Let $\left(\mathcal{E},\, (\alpha_i)_{j\in I}\,,\, (\mu_i)_{j\in I}\right)$ be a maximal influence echo chamber representation of $(\pi_j)_{j\in I}$. By proposition 1, we know that $E(i) = R(i)$. By definition, $\alpha_i \in (0,\,1)$. Then there exists $\tilde{\alpha}_i \in (0,\,1)$ such that $\tilde{\alpha}_i < \alpha_i$. Set $\mu_i = \pi_i$. Note that by $\tilde{\mu}_i = \pi_i = \overline{\pi}_{R(i)}$, the parameters $\tilde{\alpha}_i$ and $\tilde{\mu}_i$ imply $\pi_i(A) = \tilde{\alpha}_i\tilde{\mu}_i(A) + (1-\tilde{\alpha}_i)\overline{\pi}_{R(i)}(A)$ for all $A \subseteq S$.

Leaving the parameters associated with all others unchanged, note that $\pi_j(A) = \alpha_j\mu_j(A) + (1-\alpha_j)\overline{\pi}_{R(j)}(A)$ by supposition. Then $(\mathcal{E},\, (\tilde{\alpha}_i,\, \alpha_{-i})\,,\, (\tilde{\mu}_i,\, \mu_{-i}))$ is an echo chamber representation of $(\pi_j)_{j\in I}$. However, as $\tilde{\alpha}_i < \alpha_i$, $\left(\mathcal{E},\, (\alpha_i)_{j\in I}\,,\, (\mu_i)_{j\in I}\right)$ cannot be maximal influence, leading to a contradiction.

A.5 Proof of Theorem 4

Step 1. Show uniqueness of partition.

By theorem 1, if there exists a maximal influence echo chamber representation of $(\pi_i)_{i\in I}$, then $N(i) \neq \emptyset$, which implies that $|R(i)| = |N(i) \cup \{i\}| \geq 2$. Since $\mathcal{E}$, where $E(i) = R(i)$, is a partition of I, by proposition 1, there exists an echo chamber representation of $(\pi_i)_{i\in I}$ with the partition $\mathcal{E}$, and further, there does not exist $\tilde{\mathcal{E}}$, a coarsening of $\mathcal{E}$, such that there exists an echo chamber representation of $(\pi_i)_{i\in I}$ under $\tilde{\mathcal{E}}$.

Step 2. Show uniqueness of immunity from influence parameters.

Again, by the arguments in theorem 1, A2 implies that $\min \frac{\pi_i(A)}{\overline{\pi}_{R(i)}(A)} < 1$, which means that $1 - \min \frac{\pi_i(A)}{\overline{\pi}_{R(i)}(A)} > 0$ for all $i \in I$. Then, by proposition 2, there exists an echo chamber representation such that $\alpha_i = 1 - \min \frac{\pi_i(A)}{\overline{\pi}_{R(i)}(A)}$. Again, by proposition 2, if there exists some other representation with partition $\mathcal{E}$ and immunity from influence parameters $(\tilde{\alpha}_i)_{i\in I}$, $\tilde{\alpha}_i \geq 1 - \min \frac{\pi_i(A)}{\overline{\pi}_{R(i)}(A)} = \alpha_i$. Thus, for any maximal influence echo chamber representation, $\alpha_i = 1 - \min \frac{\pi_i(A)}{\overline{\pi}_{R(i)}(A)}$, which is unique.

Step 3. Show uniqueness of core beliefs.

In the proof of theorem 1, we have shown that if there exist two representations $\left(\mathcal{E},\, (\alpha_i)_{i\in I},\, (\mu_i)_{i\in I}\right)$ and $\left(\tilde{\mathcal{E}},\, (\tilde{\alpha}_i)_{i\in I},\, (\tilde{\mu}_i)_{i\in I}\right)$ such that $\tilde{\mathcal{E}} = \mathcal{E}$ and $\tilde{\alpha}_i = \alpha_i$ for each $i \in I$, then $\tilde{\mu}_i = \mu_i$ as well. Since we have established that for any maximal influence echo chamber representation, the partition must be given by $\mathcal{E}$ with $E(i) = R(i)$ and $\alpha_i = 1 - \min \frac{\pi_i(A)}{\overline{\pi}_{R(i)}(A)}$ for each $i \in I$, it must be that μ_i is also uniquely determined. Thus, we have shown that the maximal influence echo chamber representation for any $(\pi_i)_{i\in I}$ is unique.

A.6 Proof of Theorem 2

First, we prove that an MIEC representation satisfies DFC-Inter and DFC-Intra. Let $\left(\mathcal{E},\, (\alpha_i)_{i\in I},\, (\mu_i)_{i\in I}\right)$ be an MIEC representation of $(\pi_i)_{i\in I}$. As established in the proof of theorem 4, under this representation $E(i) = N(i) \cup \{i\}$ and $\alpha_i = 1 - \min \frac{\pi_i(A)}{\overline{\pi}_{E(i)}(A)}$ for each $i \in I$. From remark 1

$$\overline{\pi}_{E(i)}(A) = \sum_{j\in E(i)} \frac{\alpha_j \mu_j(A)}{\sum_{k\in E(i)} \alpha_k}$$

Then for any $A \subseteq S$, $\pi_j(A) = 1$ for all $j \in E(i)$ iff $\overline{\pi}_{E(i)}(A) = 1$ iff $\mu_j(A) = 1$ for all $j \in E(i)$. Then take any two chambers E and E'. If $i \in E$ and $j \in E'$, note that $j \notin N(i)$, which means there exists A such that $\pi_i(A) = 1$ but $\pi_j(A) \neq 1$, or vice versa. Suppose w.l.o.g. that A^0 is such that $\pi_i(A^0) = 1$ but $\pi_j(A^0) < 1$. Then $\mu_{i'}(A^0) = 1$ for all $i' \in E$, which means E is fundamentally certain about A^0. However, $\pi_j(A^0) < 1$, which implies E' is not fundamentally certain about A^0. Since this is true for all E, E', DFC-Inter is satisfied.

Now by $\alpha_i = 1 - \min \frac{\pi_i(A)}{\overline{\pi}_{E(i)}(A)}$, suppose $A^0 \in \operatorname{argmin} \frac{\pi_i(A)}{\overline{\pi}_{E(i)}(A)}$. Then note that $\frac{\pi_i(A^0)}{\overline{\pi}_{E(i)}(A^0)} = 1 - \alpha_i = \alpha_i \frac{\mu_i(A^0)}{\overline{\pi}_{E(i)}(A^0)} + 1 - \alpha_i$. This implies $\mu_i(A^0) = 0$. However, $\overline{\pi}_{E(i)}(A^0) > 0$ as $\frac{\pi_i(A^0)}{\overline{\pi}_{E(i)}(A^0)}$ is defined, which implies that there exists $j \in E(i)$ such that $\mu_j(A^0) > 0$. Then i is fundamentally certain about $(A^0)^c$ but j is not. Since we can find such an A^0 for each $i \in I$, DFC-Intra is also be satisfied.

Now we show that if $\left(\mathcal{E},\, (\alpha_i)_{i\in I},\, (\mu_i)_{i\in I}\right)$ is a representation that satisfies DFC-Inter and DFC-Intra, it must be an MIEC representation.

By DFC-Inter, for any E, E', there exists an event A such that one chamber is fundamentally certain about it but the other is not. Suppose E is fundamentally certain about A and E' is not. By the earlier arguments, $\pi_i(A) = 1$ for all $i \in E$ and $\pi_j(A) < 1$ for all $j \in E'$. By proposition 1, $E(i) \subseteq N(i) \cup \{i\}$ for all i. On the other hand, if $i \in E$ and $j \in E'$, since $\pi_i(A) = 1$ and $\pi_j(A) < 1$, $j \notin N(i)$. Since this is true for all E, E', $N(i) \cup \{i\} \subseteq E(i)$. Thus, $E(i) = N(i) \cup \{i\}$, which means that there does not exist a coarsening of $\mathcal{E}$ under which partition there exists an EC representation of $(\pi_i)_{i\in I}$.

Then by DFC-Intra, there exists B for each $i \in I$ such that $\mu_i(B) = 1$ but $\mu_j(B) < 1$ for some $j \in E(i)$. Then $\mu_i(B^c) = 0$ and $\mu_j(B^c) > 0$, which means $\overline{\pi}_{E(i)}(B^c) > 0$. Then note that $\frac{\pi_i(B^c)}{\overline{\pi}_{E(i)}(B^c)} = 1 - \alpha_i$, which implies $\alpha_i = 1 - \frac{\pi_i(B^c)}{\overline{\pi}_{E(i)}(B^c)}$. Since $\frac{\pi_i(A)}{\overline{\pi}_{E(i)}(A)} \geq 1 - \alpha_i$ as $\mu_i(A) \geq 0$, $\frac{\pi_i(B^c)}{\overline{\pi}_{E(i)}(B^c)} = \min \frac{\pi_i(A)}{\overline{\pi}_{E(i)}(A)}$. By proposition 2, for any other representation $\left(\mathcal{E},\, (\tilde{\alpha}_i)_{i\in I},\, (\tilde{\mu}_i)_{i\in I}\right)$, $\tilde{\alpha}_i \geq \alpha_i$, which means that $\left(\mathcal{E},\, (\alpha_i)_{i\in I},\, (\mu_i)_{i\in I}\right)$ is an MIEC representation.

A.7 Proof of Proposition 3

Proposition 3 can be proved as follows.

$$\begin{aligned}\pi_i(\sigma_i(c^i)\,|\,c^i) = 1 &\iff \overline{\pi}_i(\sigma_i(c^i)\,|\,(c_1,\, ...,\, c_k)) = 1 \\ &\iff \mu_j(\sigma_i(c^i)\,|\,c^j) = 1\; \forall\, j \in E(i)\end{aligned}$$

A.8 Proof of Corollary 3

Take some state s such that $\mu_i(s) > 0$ and $s \in \sigma_i(c^i)$. Then $\mu_i(s \,|\, c^i) > 0$. This means, however, that $\pi_j(s \,|\, c^j) > 0$ for all $j \in E(i)$. Then, if $\pi_j(\sigma_j(c^j) \,|\, c^j) = 1$ for all $j \in E(i)$, it must be that $s \in \sigma_j(c^j)$. This implies that $\sigma_i(c^i) \setminus S_0^i \subseteq \sigma_j(c^j)$ for all $j \in E(i)$. By a symmetric argument, we can extend this to any i, $j \in E_k$ some echo chamber.

Let us now assume that $\sigma_i \setminus S_0^i \subseteq \sigma_j(c^j)$ for all i, $j \in E_k$. Note that $\mu_i(\sigma_i(c^i) \setminus S_0^i \,|\, c^i) = 1$ for all i, as $\mu_i(s_0 \,|\, c^i) = 0$ if $\mu_i(s_0) = 0$. By the antecedent, $\mu_i(\sigma_j(c^j) \,|\, c^i) = 1$ for all i, j as this is a superset of $\sigma_i(c^i) \setminus S_0^i$. Then it must mean that $\overline{\pi}_i(\sigma_i(c^i)) = 1$ for all $i \in E_k$. This completes the proof.

A.9 Proof of Proposition 6

Since there are a finite number of states, a probability distribution over the state space, S, can be fully described by an $|S|$-dimensional row-vector. Let π_i and μ_i be the vectors corresponding to the behavioral and core beliefs of individual $i \in I$. Keeping with Remark 1, it is enough to examine one echo chamber at a time. Then the core and behavioral beliefs of a chamber, $E = \{1, ..., |E|\}$, can be written in $|E| \times |S|$ matrices, μ_E and π_E.

Let A_E be an $|E| \times |E|$ diagonal matrix with $(A_E)_{ii} = \alpha_i$ and let $I_{|E|}$ be the $|E| \times |E|$ identity matrix. Let $\mathbf{1}_{|E|}$ be the $|E| \times |E|$ matrix of ones. Then

$$\pi_E^{t+1} = A_E \mu_E + |E|^{-1}(I_{|E|} - A_E)\mathbf{1}_{|E|}\pi_E^t$$

Define B as

$$B_E \equiv |E|^{-1}(I_{|E|} - A_E)\mathbf{1}_{|E|}$$

and write

$$\begin{aligned}\pi_E^{t+1} &= A_E \mu_E + B_E \pi_E^t \\ &= (I_{|E|} + B_E + B_E^2 + ... + B_E^t) A_E \mu_E + B_E^{t+1}\pi_E^0\end{aligned}$$

It can easily be verified that $||B_E|| \leq \max_{i \in E}(1 - \alpha_i)$, which implies that

$$\lim_{t\to\infty} \pi_E^t = \left(\sum_{k=0}^{\infty} B_E^k\right) A_E \mu_E + \lim_{t\to\infty} B_E^{t+1}\pi_E^0$$

Since $||B_E|| < 1$, $\lim_{t\to\infty} B_E^{t+1} = 0$, and

$$\begin{aligned}\lim_{t\to\infty} \pi_E^t &= (I_{|E|} - B_E)^{-1} A_E \mu_E \\ &= (I_{|E|} - |E|^{-1}(I_{|E|} - A_E)\mathbf{1}_{|E|})^{-1} A_E \mu_E\end{aligned}$$

irrespective of the initial condition on behavioral beliefs. Now we must simply show that $\lim_{t\to\infty} \pi_E^t$ is indeed π_E as determined by the model. Let $\pi_E^* = \lim_{t\to\infty} \pi_E^t$. Note that

$$\begin{aligned}A_E\mu_E + B_E\pi_E^* &= A_E\mu_E + B_E(I_{|E|} - B_E)^{-1} A_E\mu_E \\ &= (I_{|E|} + B_E(I_E - B_E)^{-1})A_E\mu_E \\ &= \left(I_E + B_E \sum_{k=0}^{\infty} B_E^k\right) A_E\mu_E \\ &= \left(\sum_{k=0}^{\infty} B_E^k\right) A_E\mu_E \\ &= (I_{|E|} - B_E)^{-1} A_E\mu_E \\ &= \pi_E^*\end{aligned}$$

Then, for each $i \in E$ and $A \subseteq S$,

$$\pi_i^*(A) = \alpha_i\mu_i(A) + (1-\alpha_i)\frac{1}{|E|}\sum_{j\in E}\pi_j^*(A)$$

However, Remark 1 established that π_i is the unique solution to the above equation, implying that $\pi_i^* = \pi_i$.

A.10 Proof of Lemma 1

$$\begin{aligned}\mu_L(r|c_r) &> \frac{1}{2} \\ \iff \frac{(\rho(1-q)}{\rho(1-q)+(1-\rho)q} &> \frac{1}{2} \\ \iff \rho(1-q) &> q(1-\rho) \\ \iff \rho &> q\end{aligned}$$

A.11 Proof of Lemma 2

Note that if $i \in L$, then upon receiving the signal c_r, the updated behavioral belief of i is given by

$$\begin{aligned}\pi_i'(\ell) &= \alpha\mu_L(\ell|c_r) + (1-\alpha)\left[\frac{X_\ell^L}{|E_L|}\mu_L(\ell|c_\ell) + \frac{|E_L| - X_\ell^L}{|E_L|}\mu_L(\ell|c_r)\right] \\ &= \frac{(1-\rho)q}{(1-\rho)q+\rho(1-q)} + (1-\alpha)\frac{X_\ell^L}{|E_L|}\left[\frac{\rho q}{\rho q+(1-\rho)(1-q)} - \frac{(1-\rho)q}{(1-\rho)q+\rho(1-q)}\right]\end{aligned}$$

where X_ℓ^L is the random variable that specifies the number of individuals in E_L who have received the signal c_ℓ. Then $\pi_i'(\ell) < \frac{1}{2}$ iff

$$\begin{aligned}\frac{1}{2} &> \frac{(1-\rho)q}{(1-\rho)q+\rho(1-q)} + (1-\alpha)\frac{X_\ell^L}{|E_L|}\left[\frac{\rho q}{\rho q+(1-\rho)(1-q)} - \frac{(1-\rho)q}{(1-\rho)q+\rho(1-q)}\right] \\ \iff \frac{X_\ell^L}{|E_L|} &< \frac{1}{1-\alpha}\underbrace{\frac{\frac{1}{2} - \frac{(1-\rho)q}{(1-\rho)q+\rho(1-q)}}{\frac{\rho q}{\rho q+(1-\rho)(1-q)} - \frac{(1-\rho)q}{(1-\rho)q+\rho(1-q)}}}_{\equiv x^*}\end{aligned}$$

$\alpha \leq \frac{1}{2}$ implies $(1-\alpha)^{-1} \leq 2$. Note that x^* is increasing in ρ. If $\rho > q$, by Lemma 1, x^* is positive and bounded by half, which implies $(1-\alpha)^{-1}x^* \leq 1$. Since $\rho \neq 1$, $x^* < \frac{1}{2}$, so $\hat{x}(\alpha, \rho, q) = (1-\alpha)^{-1}x^* < 1$.

If $\rho \leq q$, note that $\mu_L(\ell, c) \geq \frac{1}{2}$ irrespective of whether $c = c_\ell$ or $c = c_r$. Then for all realizations of signals, $\pi_i'(\ell) \geq \frac{1}{2}$. Let $\hat{x}(\alpha, \rho, q) = 0$. Note that $\frac{X_\ell}{n} \geq 0$ for all realizations of signals. Then $\pi_i'(\ell) < \frac{1}{2} \iff \frac{X_\ell}{n} \leq \hat{x}(\alpha, \rho, q)$ vacuously.

A.12 Proof of Lemma ??

Note from the proof of Lemma 2 that $x^* \in [0, \frac{1}{2}]$ and x^* is continuous in ρ. Since $x^* = 0$ when $\rho = q$ and $x^* = \frac{1}{2}$ when $\rho = 1$, and $0 < 1-\alpha < \frac{1}{2}$, by intermediate value theorem, there exists $\rho^* \in (q, 1)$ such that $x^* = 1-\alpha$ when $\rho = \rho^*$.

References

Acemoglu, Daron, Kostas Bimpikis, and Asuman Ozdaglar. 2014. "Dynamics of information exchange in endogenous social networks." *Theoretical Economics* 9 (1): 41–97.

Baccara, Mariagiovanna, and Leeat Yariv. 2016. "Choosing peers: Homophily and polarization in groups." *Journal of Economic Theory* 165 (September): 152–178.

Barberá, Pablo. 2020. "Social Media, Echo Chambers, and Political Polarization." In *Social Media and Democracy: The State of the Field and Prospects for Reform,* edited by Joshua A Tucker and Nathaniel Persily, 34–55. Cambridge University Press.

Borah, Abhinash, and Christopher Kops. 2018. "Choice via social influence." *Unpublished manuscript.*

Chambers, Christopher P, Tugce Cuhadaroglu, and Yusufcan Masatlioglu. 2019. "Behavioral influence." *Unpublished manuscript.*

Chambers, Christopher P, Yusufcan Masatlioglu, and Christopher Turansick. 2021. "Influence and Correlated Choice." *arXiv preprint arXiv:2103.05084.*

Cookson, J Anthony, Joseph E Engelberg, and William Mullins. 2023. "Echo Chambers." *The Review of Financial Studies* 36 (2): 450–500.

Cuhadaroglu, Tugce. 2017. "Choosing on influence." *Theoretical Economics* 12 (2): 477–492.

DeGroot, Morris H. 1974. "Reaching a Consensus." *Journal of the American Statistical Association* 69 (345): 118–121.

Eyster, Erik, and Matthew Rabin. 2010. "Naïve Herding in Rich-Information Settings." *American Economic Journal: Microeconomics* 2 (4): 221–243.

Fershtman, Chaim, and Uzi Segal. 2018. "Preferences and Social Influence." *American Economic Journal: Microeconomics* 10 (3): 124–142.

Friedkin, Noah E., and Eugene C. Johnsen. 1990. "Social influence and opinions." *The Journal of Mathematical Sociology* 15 (3-4): 193–206. ISSN: 0022-250X. https://doi.org/10.1080/0022250X.1990.9990069.

Golub, Benjamin, and Matthew O Jackson. 2010. "Naïve Learning in Social Networks and the Wisdom of Crowds." *American Economic Journal: Microeconomics* 2 (1): 112–149.

Kashaev, Nail, and Natalia Lazzati. 2019. "Peer effects in random consideration sets." *arXiv preprint arXiv:1904.06742.*

Lazzati, Natalia. 2020. "Codiffusion of Technologies in Social Networks." *American Economic Journal: Microeconomics* 12 (4): 193–228.

Levy, Gilat, and Ronny Razin. 2015a. "Correlation Neglect, Voting Behavior, and Information Aggregation." *American Economic Review* 105 (4): 1634–1645.

Levy, Gilat, and Ronny Razin. 2015b. “Does Polarisation of Opinions Lead to Polarisation of Platforms? The Case of Correlation Neglect.” *Quarterly Journal of Political Science* 10 (3): 321–355.

———. 2019. “Echo Chambers and Their Effects on Economic and Political Outcomes.” *Annual Review of Economics* 11 (1): 303–328.

Martinez, Gabriel, and Nicholas H. Tenev. 2022. *Optimal Echo Chambers,* October.

McCarty, N, K T Poole, and H Rosenthal. 2006. *Polarized America: The Dance of Ideology and Unequal Riches.* MIT Press.

Wojcieszak, Magdalena, Andreu Casas, Xudong Yu, Jonathan Nagler, and Joshua A. Tucker. 2021. “Echo chambers revisited: The (overwhelming) sharing of in-group politicians, pundits and media on Twitter.” *Unpublished manuscript.*