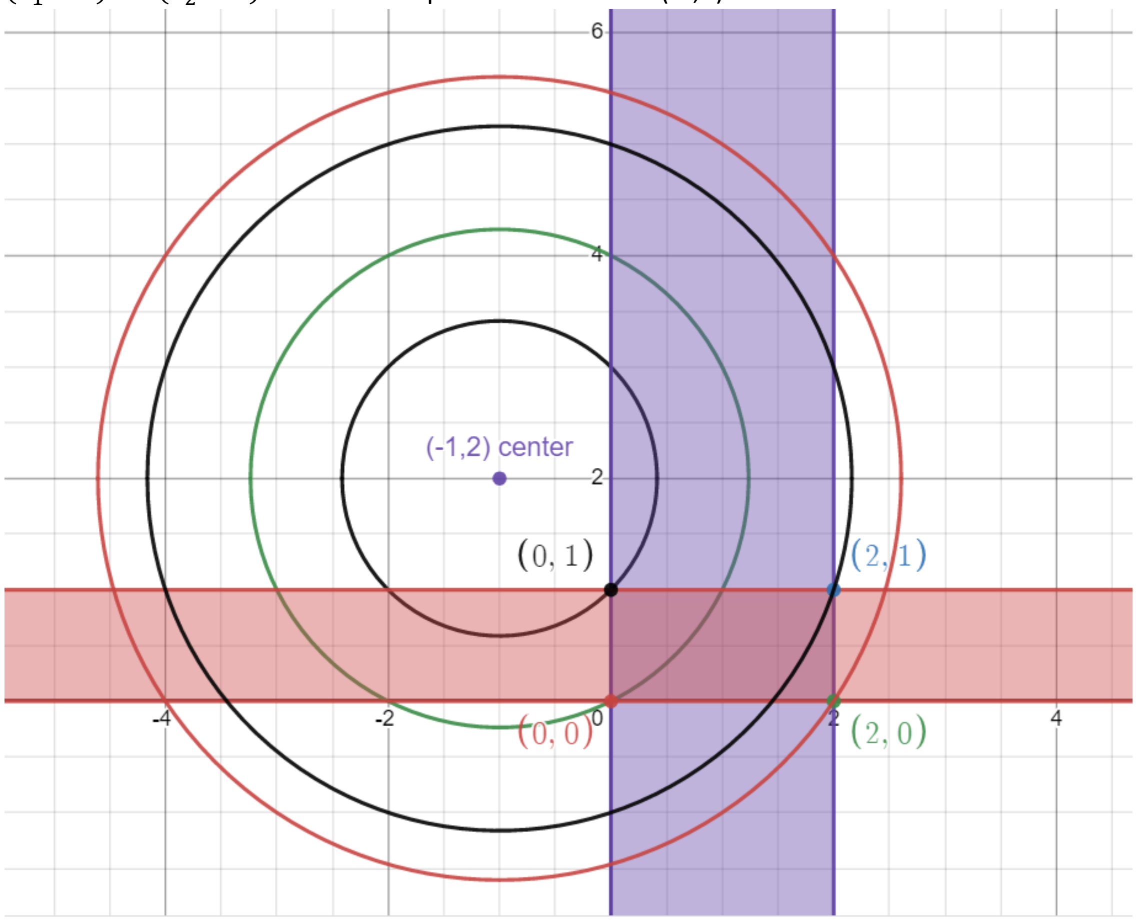
1. **Sketch graphically the problem**

$$\min_{x_1,x_2} f(x) = (x_1+1)^2 + (x_2-2)^2$$
$$subjected\ to: g_1 = x_1 - 2 \le 0, \qquad g_3 = -x_1 \le 0$$
$$g_2 = x_2 - 1 \le 0, \qquad g_4 = -x_2 \le 0$$

$(x_1+1)^2 + (x_2-2)^2$ is a circle equation with center (-1,2)



$$L = (x_1+1)^2 + (x_2-2)^2 + \mu_1(x_2-2) + \mu_2(x_1-1) + \mu_3(-x_1) + \mu_4(-x_2)$$

Conditions for μ

$if\ x_2 - 2 = 0, \quad then\ \mu_1 > 0 \qquad if\ x_2 - 1 < 0, \quad then\ \mu_1 = 0$

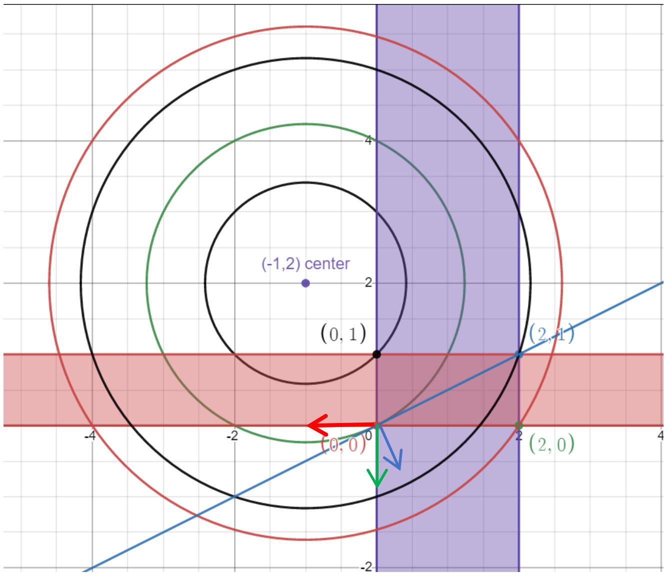
$if\ x_1 - 1 = 0, \quad then\ \mu_2 > 0 \qquad if\ x_1 - 2 < 0, \quad then\ \mu_2 = 0$

$if\ -x_1 = 0, \quad then\ \mu_3 > 0 \qquad if\ -x_1 < 0, \quad then\ \mu_3 = 0$

$if\ -x_2 = 0, \quad then\ \mu_4 > 0 \qquad if\ -x_2 < 0, \quad then\ \mu_4 = 0$

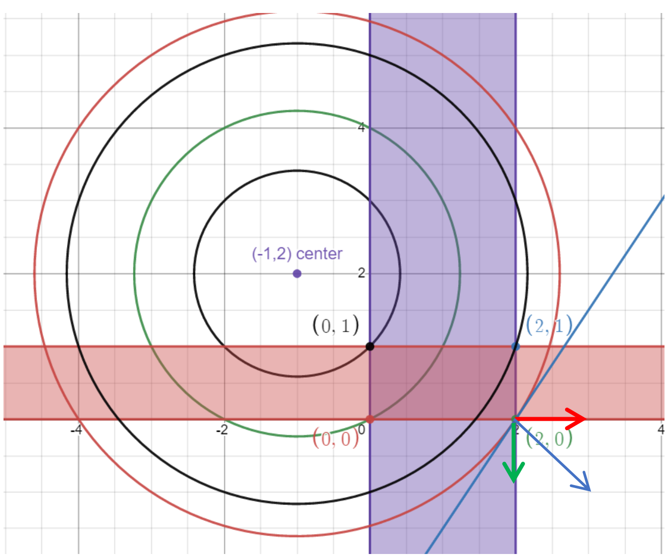
For point (0,0) the active constraints are g_3 and g_4

So $\nabla g_3 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$ and $\nabla g_4 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$



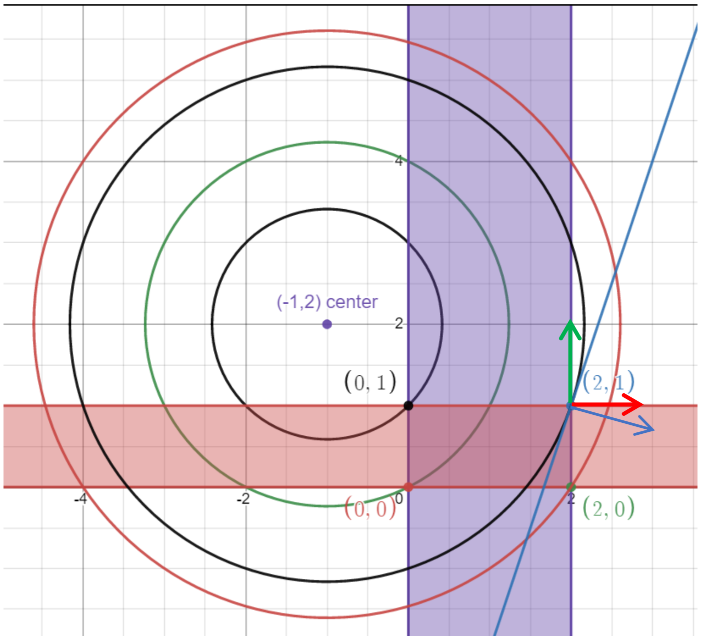
For point (2,0) the active constraints are g_1 and g_4

So $\nabla g_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\nabla g_4 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$



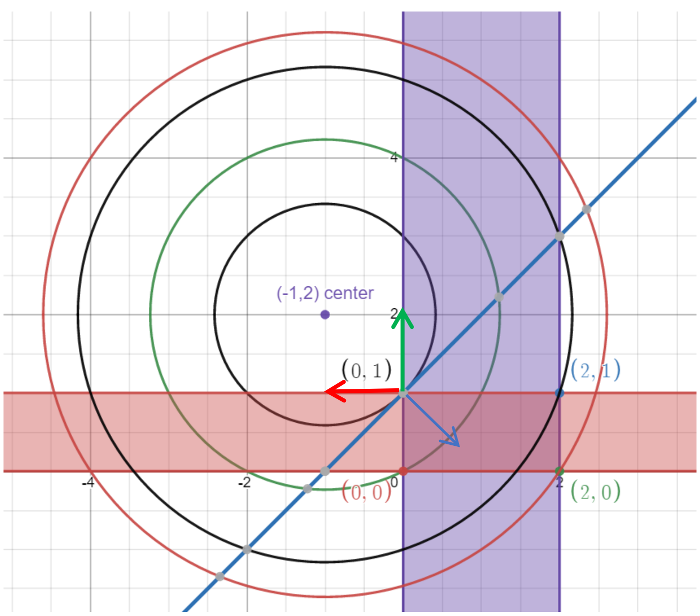
For point (2,1) the active constraints are g_1 and g_2

So $\nabla g_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ *and* $\nabla g_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$



For point (0,1) the active constraints are g_3 and g_2

So $\nabla g_3 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$ *and* $\nabla g_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$



As we can see at point (0,1) all directions are ascent and there is no feasible descent direction, Hence the $x_* = (0,1)^T$ *is the minimizer*.

We can also check (0,1) point by Applying the KKT Necessary and Sufficient conditions.

- Necessary conditions:
  - The g_3 and g_2 are the active constrains mean μ_3 *and* $\mu_2 > 0$ and μ_1 *and* μ_4 *equal to* 0

$$\nabla f - \mu^T \nabla g = 0$$
$$\begin{bmatrix} 2(x_1+1) \\ 2(x_2-2) \end{bmatrix} + \begin{bmatrix} -\mu_3 \\ \mu_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 2(0+1) \\ 2(1-2) \end{bmatrix} + \begin{bmatrix} -\mu_3 \\ \mu_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 2-\mu_3 \\ -2+\mu_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

  from here we have $\mu_3 = 2$ *and* $\mu_2 = 2$ that satisfied the KKT necessary conditions.

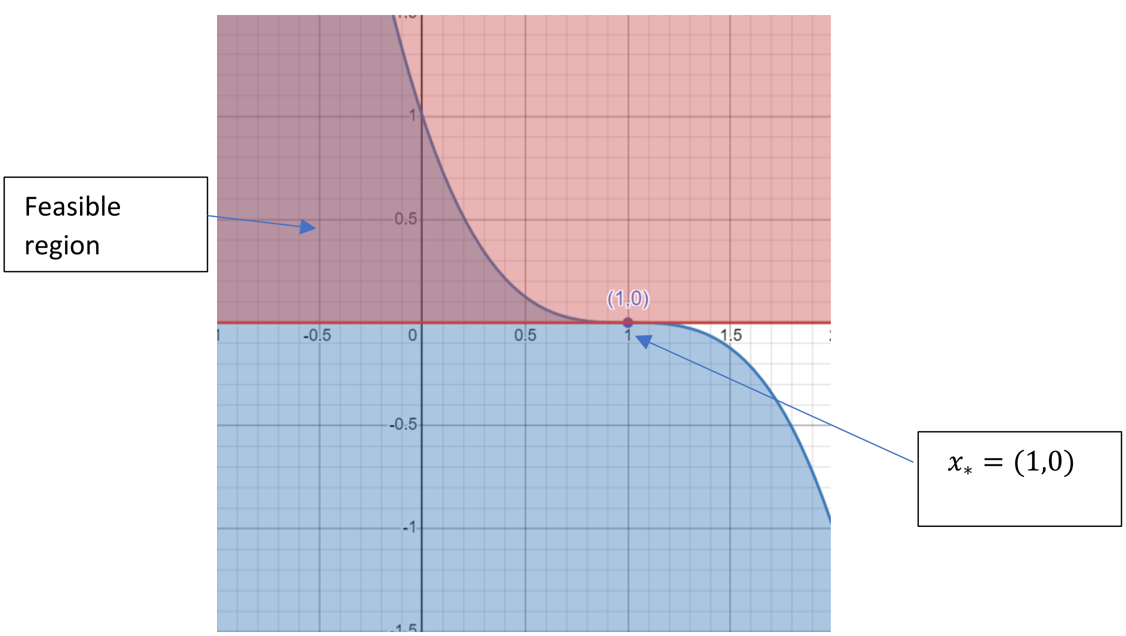
- Sufficient conditions:
  - The Hessian of Lagrangian $= \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$, $\lambda_1 = 2$ *and* $\lambda_2 = 2$

  Here Hessian of Lagrangian is positive definite everywhere. Therefore, $x_* = (0,1)^T$ is the global minimum.

2. Graph the problem

$$\min f - x_1$$
$$subject\ to: g_1 = x_2 - (1-x_1)^3 \le 0\ and\ x_2 \ge 0$$

First let's convert $x_2 \ge 0$ *to* $-x_2 \le 0$



As we can see at point $x_* = (1,0)^T$ is the solution

Checking the KKT conditions at $x_* = (1,0)^T$

$$L = -x_1 + \mu_1(x_2 - (1 - x_1)^3) + \mu_2(-x_2)$$

- Necessary conditions:
  - The g_1 and g_2 are the active constrains mean μ_1 and $\mu_2 > 0$

$$\nabla f - \mu^T \nabla g = 0$$

$$\begin{bmatrix}-1\\0\end{bmatrix} + \mu_1 \begin{bmatrix}3(1-x_1)^2\\1\end{bmatrix} + \mu_2 \begin{bmatrix}0\\-1\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

at $x_* = (1,0)$,

$$\begin{bmatrix}-1\\0\end{bmatrix} + \mu_1 \begin{bmatrix}3(1-1)^2\\1\end{bmatrix} + \mu_2 \begin{bmatrix}0\\-1\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

$$\begin{bmatrix}-1\\0\end{bmatrix} + \mu_1 \begin{bmatrix}0\\1\end{bmatrix} + \mu_2 \begin{bmatrix}0\\-1\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

$$-1 = 0: contradict$$
$$\mu_1 - \mu_2 = 0$$

So, the point $x_* = (1,0)$ is not a KKT point because this is not a regular point.

3. Find a local solution to the problem

$$\max f = x_1x_2 + x_2x_3 + x_1x_3$$

$$subjected\ to\ h = x_1 + x_2 + x_3 - 3 = 0$$

You can use either the reduced gradient or the Lagrangian method.

- Solving it using Lagrangian Method:

$$L = -f + \lambda h$$
$$L = -(x_1x_2 + x_2x_3 + x_1x_3) + \lambda(x_1 + x_2 + x_3 - 3)$$
$$\nabla_x L = \begin{bmatrix}-x_2 - x_3 + \lambda\\-x_1 - x_2 + \lambda\\-x_1 - x_3 + \lambda\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$
$$\nabla_\lambda L = x_1 + x_2 + x_3 = 0$$

We have 4 unknown and 4 equations so by solving the system of linear equation

$$x_1 = 1, \quad x_2 = 1, \quad x_3 = 1, \quad \lambda = 2$$

- Check Sufficient conditions:
  - The Hessian of Lagrangian $L_{xx} = \begin{bmatrix} 0 & -1 & -1 \\ -1 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix}$
  - The eigen value of Hessian of Lagrangian is $\lambda_1 = -2\,, \lambda_2 = 1, \lambda_3 = 1.$
    - As we can see all the eigen value of Hessian are not positive

But if we check the second order condition which is, $dx^T L_{xx} dx$

$$\partial x L_{xx} \partial x : Second\ order\ pertubations$$

$$\partial x^T L_{xx} \partial x \ = [\partial x_1\ \partial x_2\ \partial x_3] \begin{bmatrix} 0 & -1 & -1 \\ -1 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix} \begin{bmatrix} \partial x_1 \\ \partial x_2 \\ \partial x_3 \end{bmatrix}$$

$$= -2\partial x_1 \partial x_2 - 2\partial x_1 \partial x_3 - 2\partial x_2 \partial x_3$$

We want the ∂x to be feasible so the feasible perturbation is such that $\frac{\partial h}{\partial x} \partial x = 0$

$$\left[\frac{\partial h}{\partial x_1}, \frac{\partial h}{\partial x_2}, \frac{\partial h}{\partial x_3}\right] \begin{bmatrix} \partial x_1 \\ \partial x_2 \\ \partial x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$[1, 1, 1] \begin{bmatrix} \partial x_1 \\ \partial x_2 \\ \partial x_3 \end{bmatrix} = 0 \rightarrow \partial x_1 + \partial x_2 + \partial x_3 = 0$$

$$\partial x_1 = -\partial x_2 - \partial x_3$$

So,

$$= -2(-\partial x_2 - \partial x_3)\partial x_2 - 2(-\partial x_2 - \partial x_3)\partial x_3 - 2\partial x_2 \partial x_3$$

$$= 2(\partial x_2^2 + \ \partial x_2 \partial x_3 + \partial x_3^2) \ \rightarrow \ 2\left(\partial x_2 + \frac{1}{2}\partial x_3\right)^2 + \frac{3}{4}\partial x_3^2 \geq 0$$

Further, for $\partial x^T L_{xx} \partial x$ to be 0, $\partial x_2\ and\ \partial x_3$ must be 0, if so the ∂x_1 is also 0. Which mean $\partial x = 0$ which is not a perturbation. Therefore, $\partial x^T L_{xx} \partial x > 0$ for any non-zero feasible perturbations

So $x_{1*} = x_{2*} = x_{3*} = 1$ is global maximum to the original problem and if we plug this value in the main f so, $f_* = 3$

4. **Find the solution to**

$$\min f = x_1^2 + x_2^2 + x_3^2$$

$$subjected\ to\ h_1 = \frac{x_1^2}{4} + \frac{x_2^2}{5} + \frac{x_3^2}{25} - 1 = 0$$
$$and\ h_2 = x_1 + x_2 - x_3 = 0$$

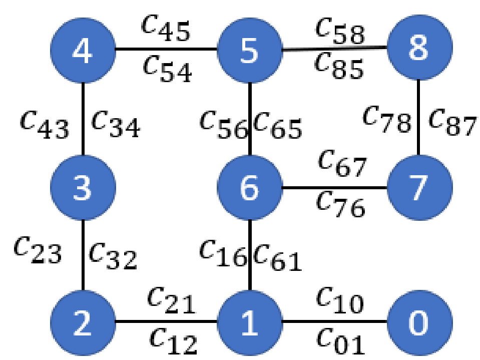
by implementing the generalized reduced gradient algorithm.

Python uploaded into Github :
https://github.com/siddahant/DESOPT/blob/fourth/generalized_reduced_gradient.ipynb

5. Consider the following garbage truck routing problem. Let there be N sites to be visited and consider them as nodes of a graph. The cost of moving from node i to j is c_{ij} if there is an edge between the nodes, or ∞ if there is none. Site 0 is the truck station where the truck starts and returns. Formulate the problem to minimize the total cost while the truck visits all sites and returns to the station

Problem Formulation:

$$\min_{x_{ij}} \sum_{ij}^{N} x_{ij} c_{ij}$$



Where x_{ij} is the movement form node $i \to j$

Forward movement: $x_{ij} = \begin{cases} 1 & if\ i\ connected\ with\ j \\ 0 & if\ \ i\ not\ connected\ with\ j \end{cases}$

Backward movement: $x_{ji} = \begin{cases} 1 & if\ i\ connected\ with\ j \\ 0 & if\ \ i\ not\ connected\ with\ j \end{cases}$

c_{ij} is the cost of moving from node i to j

Forward Move Cost Follows: $x_{ij} = \begin{cases} c_{ji} & if\ i\ connected\ with\ j \\ \infty & if\ \ i\ not\ connected\ with\ j \end{cases}$

Backward Move Cost Follows: $x_{ji} = \begin{cases} c_{ji} & if\ i\ connected\ with\ j \\ \infty & if\ \ i\ not\ connected\ with\ j \end{cases}$

The constraint of objective function as follow:

$\sum x_{ij} \geq N$: the truck needs to visit all the node where N is the number of Nodes

Traffic control: $\sum x_{ij} = \sum x_{ji}\ (times\ in = times\ out)$

There must be a connection between starting to at least one neighbor node

$For\ starting \sum x_{0j} \geq 1\ \forall\ j$
$For\ ending \sum x_{j0} \geq 1\ \forall\ j$

So, the final Problem is:

$$\min_{x_{ij}} \sum_{ij}^{N} x_{ij} c_{ij}$$
$$\sum x_{ij} \geq N$$
$$\sum x_{ij} = \sum x_{ji}$$

$$\sum x_{0j} \geq 1 \;\forall\, j$$

$$\sum x_{j0} \geq 1 \;\forall\, j$$