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(1)

Q1) Show that the stationary point (zero gradient) of function

$$f(x_1, x_2) = 2x_1^2 - 4x_1x_2 + 1.5x_2^2 + x_2$$

is saddle (with indefinite Hessian)

$\rightarrow \dfrac{\partial f(x_1,x_2)}{\partial x_1} = 4x_1 - 4x_2 = 0 \qquad -(1)$

$\rightarrow \dfrac{\partial f(x_1, x_2)}{\partial x_2} = -4x_1 + 3x_2 + 1 = 0 \qquad -(2)$

convert eq (1) & (2) into matrix and find x_1 x_2 with gauss-elimination

$$\begin{bmatrix} 4 & -4 \\ -4 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

R_1

$$\left[\begin{array}{cc|c} 4 & -4 & 0 \\ -4 & 3 & -1 \end{array}\right] \xrightarrow{R_1 \to R_1/4} \left[\begin{array}{cc|c} 1 & -1 & 0 \\ -4 & 3 & -1 \end{array}\right]$$

$R_2 \to R_2 + 4R_1$

$$\left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & -1 & -1 \end{array}\right] \xrightarrow{R_2 \to R_2/-1} \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 1 & 1 \end{array}\right]$$

$R_1 \to R_1 + R_2$

$$\left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 1 \end{array}\right]$$

Back to equation from

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\boxed{x_1 = 1 \qquad x_2 = 1}$

stationary point $x^* = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Ans

calculate Hessian

$$H = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_2 x_1} & \frac{\partial^2 f}{\partial x_2^2} \end{bmatrix}$$

$$\Rightarrow \nabla^2 f(x_1, x_2) = \begin{bmatrix} 4 & -4 \\ -4 & 3 \end{bmatrix}$$

calculate the eigen value of Hessian Matrix

$$|H - \lambda I| = 0$$

$$\begin{vmatrix} 4-\lambda & -4 \\ -4 & 3-\lambda \end{vmatrix} = 0 \Rightarrow (4-\lambda)(3-\lambda) - 16 = 0$$

$$\Rightarrow \lambda^2 - 7\lambda - 4 = 0$$

$$\text{Roots} = -\left(\frac{-7+\sqrt{65}}{2}\right), \quad \frac{7+\sqrt{65}}{2}$$

$$\uparrow \qquad\qquad \uparrow$$

$$\underline{\underline{\lambda_1}} \qquad\qquad \underline{\underline{\lambda_2}}$$

Since one of eigen value positive and another one is negative so the stationary point is a saddle point

Direction of Downslope

Taylor's expansion from saddle point $x^* = [1 \quad 1]^T$

$$f(x,y) = f(1,1) + \nabla f\Big|_{(1,1)}^{T} (x - x^*] + \frac{1}{2}(x - x^*)^T H\Big|_{(1,1)} (x - x^*)$$

Let so

$$f(x,y) = f(1,1) + 0 + \frac{1}{2}\begin{bmatrix} x_1 - 1 \\ x_2 - 1 \end{bmatrix}^T \begin{bmatrix} 4 & -4 \\ -4 & 3 \end{bmatrix} \begin{bmatrix} x_1 - 1 \\ x_2 - 1 \end{bmatrix}$$

Let say $x_1 - 1 = \partial x_1$ & $x_2 - 1 = \partial x_2$

$$f(x,y) = f(1,1) + \frac{1}{2}[\partial x_1 \ \ \partial x_2]\begin{bmatrix} 4 & -4 \\ -4 & 3 \end{bmatrix}\begin{bmatrix} \partial x_1 \\ \partial x_2 \end{bmatrix}$$

$$f(x,y) = f(1,1) + \frac{1}{2}\left[(4\partial x_1 - 4\partial x_2) \ \ (-4\partial x_1 + 3\partial x_2)\right]\begin{bmatrix} \partial x_1 \\ \partial x_2 \end{bmatrix}$$

$$f(x,y) = f(1,1) + \frac{1}{2}\left[\partial x_1(4\partial x_1 - 4\partial x_2) + \partial x_2(-4\partial x_1 + 3\partial x_2)\right)$$

$$f(x,y) = f(1,1) + \frac{1}{2}\left((4\partial^2 x_1 - 4\partial x_2 \partial x_1) + (3\partial^2 x_2 - 4\partial x_1 \partial x_2)\right)$$

$$f(x,y) = f(1,1) + \frac{1}{2}(4\partial^2 x_1 + 3\partial^2 x_2 - 8\partial x_1 \partial x_2)$$

$$f(x,y) = f(1,1) + \frac{1}{2}(2\partial x_1 - \partial x_2)(2\partial x_1 - 3\partial x_2)$$

$$f(x,y) - f(1,1) = \frac{1}{2}(2\partial x_1 - \partial x_2)(2\partial x_1 - 3\partial x_2) < 0$$

to get the downslope $(2\partial x_1 - \partial x_2) < 0$ and $(2\partial x_1 - 3\partial x_2) > 0$

OR

$(2\partial x_1 - \partial x_2) > 0$ and $(2\partial x_1 - 3\partial x_2) < 0$

Problem 2

Find the point in the plane $x_1 + 2x_2 + 3x_3 = 1$ in $\mathbb{R}^3$ that is nearest to the ploint $(-1\ 0\ 1)^T$. Is this a convex problem?

$\Rightarrow$ Distance b/w two point $(x_1+1)^2 + (x_2+0)^2 + (x_3-1)^2$

need to $\min\limits_{x_1 x_2 x_3} \quad (x_1+1)^2 + x_2^2 + (x_3-1)^2$

that subjected to

$$x_1 + 2x_2 + 3x_3 = 1$$

To make Problem unconstrained let substituting $x_1 = 1-(2x_2+3x_3)$ in distance equation

$$f = (1-2x_2-3x_3+1)^2 + x_2^2 + (x_3-1)^2$$

$$\frac{\partial f}{\partial x_2} = -4(2-2x_2-3x_3) + 2x_2 \Rightarrow \boxed{10x_2 + 12x_3 - 8 = 0}$$

$$\frac{\partial f}{\partial x_3} = -6(2-2x_2-3x_3) + 2(x_3-1) \Rightarrow \boxed{-14 + 12x_2 + 20x_3 = 0}$$

$$\boxed{x_2 = \frac{-1}{7}} \quad \text{and} \quad \boxed{x_3 = 11/14}$$

put x_2 and x_3 in x_1 equation

$$\boxed{x_1 = -15/14}$$

Calculate the Hessian of unconstrained function

$$H = \begin{bmatrix} \frac{\partial^2 f}{\partial x_2^2} & \frac{\partial^2 f}{\partial x_2 \partial x_3} \\ \frac{\partial^2 f}{\partial x_3 \partial x_2} & \frac{\partial^2}{\partial x_3^2} \end{bmatrix} \Rightarrow \begin{bmatrix} 10 & 12 \\ 12 & 20 \end{bmatrix}$$

eigen value of H matrix

$$\begin{vmatrix} 10-\lambda & 12 \\ 12 & 20-\lambda \end{vmatrix} = (20-\lambda)(10-\lambda) - 144$$

$$= 200 - 20\lambda - 10\lambda + \lambda^2 - 144$$

$$= \lambda^2 - 30\lambda + 56$$

$$= (\lambda - 28)(\lambda - 2) = 0$$

$$\boxed{\begin{matrix} \lambda_1 = 28 \\ \lambda_2 = 2 \end{matrix}}$$

H is +ve definite Thus the Problem is convex

for Part b see code

Problem (3)

Prove that a hyperplane is convex set.

Let define the Hyperplane H

$$H = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n \mid \overbrace{a_1 x_1 + a_2 x_2 \dots a_n x_n}^{a^T x} = c \right\}$$

where $a_1, \dots a_n \neq 0$ and $c \in \mathbb{R}$

Now let consider $x_1\ x_2 \in H$

When $x_1 \in H \Rightarrow a^T x_1 = c$

when $x_2 \in H \Rightarrow a^T x_2 = c$

Now eq of line segement joint x_1 and x_2

$v = \lambda x_1 + (1-\lambda)x_2 \quad : \lambda \in [0, 1]$

thus

$$a^T v = a^T (\lambda x_1 + (1-\lambda) x_2)$$

$$= a^T \lambda x_1 + a^T (1-\lambda) x_2)$$

$$= \lambda a^T x_1 + (1-\lambda) (a^T x_2)$$

$$= \lambda c + (1-\lambda) c$$

$$\boxed{a^T v = c}$$

$v \in H$ i.e $\boxed{\lambda x_1 + (1-\lambda) x_2 \in H}$: $0 \le \lambda \le 1$

Hence H is a convex set

Problem 4

consider the following illumination problem:

$$\min_{p} \max_{k} \{ h(a_k^T p, I_t) \}$$

subjected to $0 \leq p_i \leq p_{max}$

where $p = [p_1 \dots p_n]^T$ ... $h(I, I_t)$ is defined as follows

$$h(I, I_t) = \begin{cases} I_t / I & \text{if } I < I_t \\ I / I_t & \text{if } I_t < I \end{cases}$$

a) show that the problem is convex

let take $h(a^T p, I_t)$

gradient $\quad g = \frac{\partial h}{\partial p} = \frac{\partial h}{\partial I} \frac{\partial a^T p}{\partial p} = h' a$

$$H = \frac{\partial^2 h}{\partial p^2} = \frac{\partial h'}{\partial I} \frac{\partial a^T p}{\partial p} = h'' a a^T \qquad \boxed{h'' \geq 0}$$

to prove the problem is convex we need to show the H is P.S.D

let λ be an eigenvalue of aa^T and q be the eigenvector.

Then $(aa^T) q = \lambda q \Rightarrow q^T a a^T q = \lambda q^T q$

$\lambda = \frac{q^T a a^T q}{q^T q}$ Let say $w = a^T q \qquad w^T = q^T a$

$\lambda = \frac{w^T w}{q^T q}$ consider the ith element $\boxed{\begin{matrix} w_i^T w_i = w_i^2 \\ q_i^T q_i = q_i^2 \end{matrix}}$

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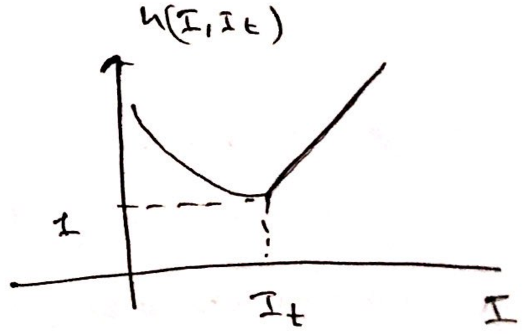
We can observed that $\Delta \geq 0$ ~~since~~ since all the value is squared so the H is P.S.D, mean that $h(a^T p, I_t)$ is convex, but not strically convex & $\max_k \{h(a^T_k p, I_t)\}$ is convex function

If we look the conditions

$$h(I, I_t) = \frac{I}{I_t} \quad \text{when} \quad I < I_t$$

By substituting $I = a^T_k p = a_{k_1} p_1 + \dots a_{kn} p_n$

$$h(I, I_t) = \frac{a_{k_1} p_1 + \dots a_{kn} p_n}{I_t}$$



and $$h(I, I_t) = \frac{I_t}{I} \quad \text{when} \quad I_t < I$$

$$\therefore h(I, I_t) = \frac{I_t}{a_{k_1} p_1 + \dots a_{kn} p_n}$$

so the function $h(I, I_t)$ to be convex when $I \geq 0$, mean $a^T_k p > 0$,

This is valid for all k^{th} term in $h(a^T_k, I_t)$ as by property we know the max of convex set is convex

then $\boxed{\max_k \{h(a^T_k p, I_t)\} \text{ is a convex function.}}$ when $a^T_k p > 0$

The constrain $0 \leq P_i \leq P_{max}$ is also the convex constrein Because ~~P~~ P_i in the two half plain $P_i \geq 0$, and $P_i \leq P_{max}$

So we can Now say that the $\boxed{\text{Problem is convex Problem}}$

(B) Any of 10 lamps from the n lamps less than p^* power

$\rightarrow$ $P_1 + P_2 \dots P_{10} \leq p^*$

$\vdots$

$P_{n-10} + P_{n-9} \dots + P_n \leq p^*$

We have C^n_{10} combination of lamps that power ~~are~~ less than p^*

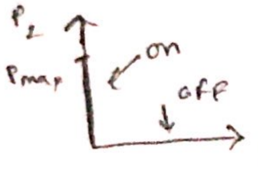
In vector $[1, \dots 0]\begin{bmatrix} P_1 \\ \vdots \\ P_n \end{bmatrix} \leq p^*$

So by looking we can say the formulate castrine is linear combination, which still point the Problem is ~~[illegible]~~ convex and the nature of fesible Sol^n does not change the convexity of ~~the~~ problem

So $\boxed{\text{It has a unique sol}^n}$

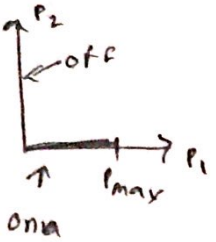
(C) we we wouldn't Required more than 10 lamps to be switched on, we don't say how many sol^n we have Because we have n number of way to start lamps

as example consider two lamps P_1 and P_2



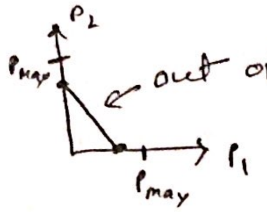
P_2 on and P_1 off

convex set



P_1 on and P_2 off

convex set



when we choose ~~[illegible]~~ Both and the point any location B/w P_1 & P_2 the set Become non-convex

By using the constrain not more than 10 lampe to be swiched on the orignal problem become non-convex optimization problem and we can-not tell how many local sol^n we have, $\boxed{\text{so it can not have a unique } sol^n}$

Problem (5)

$$c^*(y) = \max_x \{xy - c(x)\}$$

consider the ith element of problem

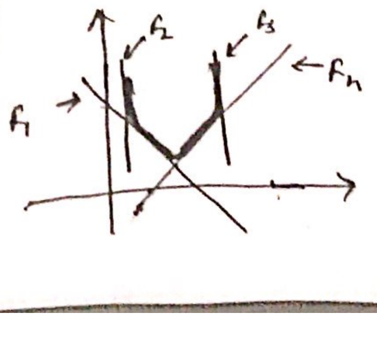
$$f_i = (x_i y_i - c(x_i))$$

calculating the gradient

$$g = \frac{\partial f}{\partial y_i} = x_i$$

$$H = \frac{\partial^2 f}{\partial y_i^2} = 0$$ { The Hessian 0 mean the function is Linear function and convex function }

Now we have Bunch of funtion sets which are lineare and By the property the max of convex set is a convex so we $\boxed{\text{can say that } \max_x \{xy - c(x)\} \text{ is convex with respect to } y}$



Graphacilly

$\boxed{c^*(y) \text{ is convex}}$