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Q1) Show that the stationary point (zero gradient) of function

$$f(x_1, x_2) = 2x_1^2 - 4x_1x_2 + 1.5x_2^2 + x_2$$

is saddle (with indefinite Hessian)

$$\rightarrow \frac{\partial f(x_1,x_2)}{\partial x_1} = 4x_1 - 4x_2 = 0 \quad -(1)$$

$$\rightarrow \frac{\partial f(x_1,x_2)}{\partial x_2} = -4x_1 + 3x_2 + 1 = 0 \quad -(2)$$

convert eq (1) & (2) into matrix and find x_1 x_2 with gauss-elimination

$$\begin{bmatrix} 4 & -4 \\ -4 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 4 & -4 & 0 \\ -4 & 3 & -1 \end{array}\right] \xrightarrow{R_1 \to R_1/4} \left[\begin{array}{cc|c} 1 & -1 & 0 \\ -4 & 3 & -1 \end{array}\right]$$

$R_2 \to R_2 + 4R_1$

$$\left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & -1 & -1 \end{array}\right] \xrightarrow{R_2 \to R_2/-1} \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 1 & 1 \end{array}\right]$$

$R_1 \to R_1 + R_2$

$$\left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 1 \end{array}\right]$$

Back to equation from

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\boxed{x_1 = 1 \quad x_2 = 1}$$

∴ Stationary point $x^* = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Ans

calculate Hessian
$$H = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_2 x_1} & \frac{\partial^2 f}{\partial x_2^2} \end{bmatrix}$$

$$\Rightarrow \nabla^2 f(x_1 x_2) = \begin{bmatrix} 4 & -4 \\ -4 & 3 \end{bmatrix}$$

calculate the eigen value of Hessian matrix

$$|H - \lambda I| = 0$$

$$\begin{vmatrix} 4-\lambda & -4 \\ -4 & 3-\lambda \end{vmatrix} = 0 \Rightarrow (4-\lambda)(3-\lambda) - 16 = 0$$

$$\Rightarrow \lambda^2 - 7\lambda - 4 = 0$$

$$\text{Roots} = -\left(\frac{-7+\sqrt{65}}{2}\right), \quad \frac{7+\sqrt{65}}{2}$$

$$\quad\quad\uparrow\quad\quad\quad\quad\quad\uparrow$$
$$\quad\quad\lambda_1\quad\quad\quad\quad\quad\lambda_2$$

Since one of eigen value positive and another one is negative so the stationay point is a saddle point

Direction of Downslope

Taylor's expansion from saddle point $x^* = [1 \quad 1]^T$

$$f(x,y) = f(1,1) + \nabla f\Big|_{(1,1)}^T (x-x^*] + \frac{1}{2}(x-x^*)^T H\Big|_{(1,1)} (x-x^*)$$

~~let~~ ~~So~~

$$f(x,y) = f(1,1) + 0 + \frac{1}{2}\begin{bmatrix} x_1 - 1 \\ x_2 - 1 \end{bmatrix}^T \begin{bmatrix} 4 & -4 \\ -4 & 3 \end{bmatrix}\begin{bmatrix} x_1 - 1 \\ x_2 - 1 \end{bmatrix}$$

let say $x_1 - 1 = \partial x_1$ & $x_2 - 1 = \partial x_2$

$$f(x,y) = f(1,1) + \frac{1}{2}\begin{bmatrix} \partial x_1 & \partial x_2 \end{bmatrix}\begin{bmatrix} 4 & -4 \\ -4 & 3 \end{bmatrix}\begin{bmatrix} \partial x_1 \\ \partial x_2 \end{bmatrix}$$

$$f(x,y) = f(1,1) + \frac{1}{2}\begin{bmatrix} (4\partial x_1 - 4\partial x_2) & (-4\partial x_1 + 3\partial x_2) \end{bmatrix}\begin{bmatrix} \partial x_1 \\ \partial x_2 \end{bmatrix}$$

$$f(x,y) = f(1,1) + \frac{1}{2}[\partial x_1(4\partial x_1 - 4\partial x_2) + \partial x_2(-4\partial x_1 + 3\partial x_2))$$

$$f(x,y) = f(1,1) + \frac{1}{2}((4\partial^2 x_1 - 4\partial x_2 \partial x_1) + (3\partial^2 x_2 - 4\partial x_1 \partial x_2))$$

$$f(x,y) = f(1,1) + \frac{1}{2}(4\partial^2 x_1 + 3\partial^2 x_2 - 8\partial x_1 \partial x_2)$$

$$f(x,y) = f(1,1) + \frac{1}{2}(2\partial x_1 - \partial x_2)(2\partial x_1 - 3\partial x_2)$$

$$f(x,y) - f(1,1) = \frac{1}{2}(2\partial x_1 - \partial x_2)(2\partial x_1 - 3\partial x_2) < 0$$

$\boxed{\text{to get the downslope } (2\partial x_1 - \partial x_2) < 0 \text{ and } (2\partial x_1 - 3\partial x_2) > 0 \text{ OR } (2\partial x_1 - \partial x_2) > 0 \text{ and } (2\partial x_1 - 3\partial x_2) < 0}$

Problem 2

Find the point in the plane $x_1 + 2x_2 + 3x_3 = 1$ in $\mathbb{R}^3$ that is nearest to the point $(-1 \; 0 \; 1)^T$. Is this a convex problem?

$\Rightarrow$ Distance b/w two point $(x_1+1)^2 + (x_2+0)^2 + (x_3-1)^2$

need to $\min_{x_1 x_2 x_3} \; (x_1+1)^2 + x_2^2 + (x_3-1)^2$

that subjected to

$$x_1 + 2x_2 + 3x_3 = 1$$

To make Problem unconstrained let substitituting $x_1 = 1-(2x_2+3x_3)$ in distance equation

$$f = (1-2x_2-3x_3+1)^2 + x_2^2 + (x_3-1)^2$$

$$\frac{\partial f}{\partial x_2} = -4(2-2x_2-3x_3) + 2x_2 \Rightarrow \boxed{10x_2 + 12x_3 - 8 = 0}$$

$$\frac{\partial f}{\partial x_3} = -6(2-2x_2-3x_3) + 2(x_3-1) \Rightarrow \boxed{-14 + 12x_2 + 20x_3 = 0}$$

$\boxed{x_2 = \frac{-1}{7}}$ and $\boxed{x_3 = 11/14}$

put x_2 and x_3 in x_1 equation

$\boxed{x_1 = -15/14}$

Calculate the Hessian of unconstrained function

$$H = \begin{bmatrix} \frac{\partial^2 f}{\partial x_2^2} & \frac{\partial^2 f}{\partial x_2 \partial x_3} \\ \frac{\partial^2 f}{\partial x_3 \partial x_2} & \frac{\partial^2}{\partial x_3^2} \end{bmatrix} \Rightarrow \begin{bmatrix} 10 & 12 \\ 12 & 20 \end{bmatrix}$$

eigen value of H matrix

$$\begin{vmatrix} 10-\lambda & 12 \\ 12 & 20-\lambda \end{vmatrix} = (20-\lambda)(10-\lambda) - 144$$

$$= 200 - 20\lambda - 10\lambda + \lambda^2 - 144$$

$$= \lambda^2 - 30\lambda + 56$$

$$= (\lambda - 28)(\lambda - 2) = 0$$

$$\boxed{\begin{aligned} \lambda_1 &= 28 \\ \lambda_2 &= 2 \end{aligned}}$$

H is +ve definite Thus the Problem is convex

$\boxed{\text{for Part b see code}}$

Problem (3)

Prove that a hyperplane is convex set.

Let define the Hyperplane H

$$H = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n \;\middle|\; \overbrace{a_1 x_1 + a_2 x_2 \dots a_n x_n}^{a^T x} = c \right\}$$

where $a_1, \dots a_n \neq 0$ and $c \in \mathbb{R}$

Now let consider $x_1, x_2 \in H$

When $x_1 \in H \Rightarrow a^T x_1 = c$

when $x_2 \in H \Rightarrow a^T x_2 = c$

Now eq of line segement joint x_1 and x_2

$$v = \lambda x_1 + (1-\lambda) x_2 \quad : \lambda \in [0, 1]$$

thus

$$a^T v = a^T (\lambda x_1 + (1-\lambda) x_2)$$

$$= a^T \lambda x_1 + a^T (1-\lambda) x_2)$$

$$= \lambda a^T x_1 + (1-\lambda) (a^T x_2)$$

$$= \lambda c + (1-\lambda) c$$

$$\boxed{a^T v = c}$$

$v \in H$ i.e $\boxed{\lambda x_1 + (1-\lambda) x_2 \in H}$: $0 \leq \lambda \leq 1$

Hence H is a convex set

Problem 4

consider the following illumination problem:

$$\min_{p} \max_{k} \{ h(a_k^T p, I_t) \}$$

Subjected to $0 \le p_i \le p_{max}$

where $p = [p_1, \dots p_n]^T$ ... $h(I, I_t)$ is defined as follows

$$h(I, I_t) = \begin{cases} I_t / I & \text{If } I < I_t \\ I / I_t & \text{If } I_t < I \end{cases}$$

a) show that the problem is convex

let take $h(a^T p, I_t)$

gradient $g = \dfrac{\partial h}{\partial p} = \dfrac{\partial h}{\partial I} \dfrac{\partial a^T p}{\partial p} = h' a$

$H = \dfrac{\partial^2 h}{\partial p^2} = \dfrac{\partial h'}{\partial I} \dfrac{\partial a^T p}{\partial p} = h'' a a^T$ $\boxed{h'' \geq 0}$

To prove the problem is convex we need to show the H is P.S.D

let λ be an eigenvalue of aa^T and q be the eigenvector.

Then $(aa^T) q = \lambda q \Rightarrow q^T a a^T q = \lambda q^T q$

$\lambda = \dfrac{q^T a a^T q}{q^T q}$ Let say $w = a^T q$ $w^T = q^T a$

~~...~~

$\lambda = \dfrac{w^T w}{q^T q}$ Consider the ith element $\boxed{\begin{matrix} w_i^T w_i = w_i^2 \\ q_i^T q_i = q_i^2 \end{matrix}}$

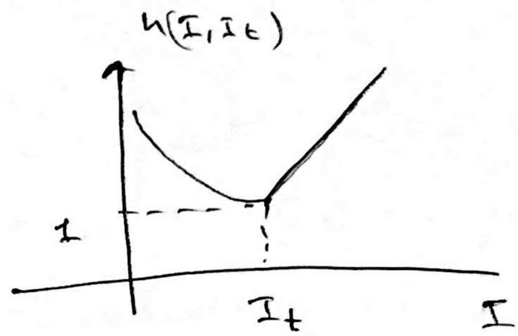
We can observed that $\lambda \geq 0$ ~~since~~ since all the value is squared so the H is P.S.D, mean that $h(a^T p, I_t)$ is convex, but not strically convex & $\max_k \{ h(a_k^T p, I_t) \}$ is convex function

If we look the conditions

$$h(I, I_t) = \frac{I}{I_t} \quad \text{when} \quad I_t < I$$

By substituting $I = a_k^T P = a_{k1} P_1 + \dots a_{kn} P_n$

$$h(I, I_t) = \frac{a_{k1} P_1 + \dots a_{kn} P_n}{I_t}$$



and $h(I, I_t) = \dfrac{I_t}{I}$ when $I < I_t$

$$\therefore h(I, I_t) = \frac{I_t}{a_{k1} P_1 + \dots a_{kn} P_n}$$

$$\boxed{h''(I, I_t) = \begin{cases} 2I_t / I^3 & I \leq I_t \\ 1/I_t & I_t \leq I \end{cases}}$$

so the function $h(I, I_t)$ to be convex when $I \geq 0$, mean $a_k^T p \geq 0$,

this is valid for all k^{th} term in $h(a_k^T, I_t)$ as by property we know the max of convex set is convex then $\boxed{\max_k \{ h(a_k^T P, I_t) \} \text{ is a convex function.}}$ when $a_k^T p \geq 0$

The constrain $0 \leq P_i < P_{max}$ is also the convex constrain Because ~~#~~ P_i in the two half plain $P_i \geq 0$, and $P_i < P_{max}$

So we can Now say that the $\boxed{\text{Problem is convex problem}}$

(B) Any of 10 Lamps from the n Lamps, Less than p^* power

$\rightarrow$ $P_1 + P_2 \dots P_{10} \leq p^*$

:

$P_{n-10} + P_{n-9} \dots + P_n \leq p^*$

Constrain $\sum_{k=1}^{n} P_i \leq p^*$

where $n = 10$

we have C^n_{10} combination of Lamps that power are Less than p^*

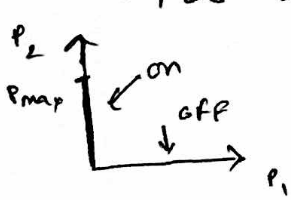
In vector $[1, \dots 0] \begin{bmatrix} P_1 \\ \vdots \\ P_n \end{bmatrix} \leq p^*$

So by looking we can say the formulate castrine is Linear combination, which still point the Problem is convex and the nature of fesible soln does not change the convexity of problem

So $\boxed{\text{It has a unique soln}}$

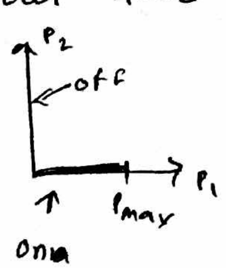
(C) we we would'nt Requried more than 10 lamps to be switched on, we do'nt say how many soln we have Because we have n number of way to start Lamps

as example consider two lamps P_1 and P_2



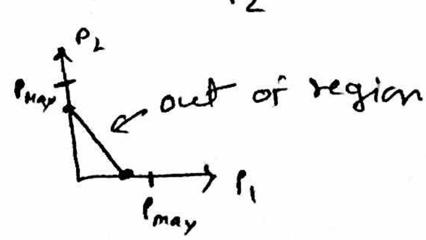
P_2 on and P_1 off

Convex set



P_1 on and P_2 off

Convex set



when we choose Both and the Point any location B/w P_1 & P_2 the set Become non-convex

By using the constrain not more than 10 lampe to (10)

be swiched on the orignal problem become non-convex optimization problem and we can-not tell how many local solⁿ we have, $\boxed{\text{so it can not have a unique sol}^n}$

Constraint $\sum_{k=1}^{n} \text{true}(p_i > 0) \leq 10$

Problem (5)

$$c^*(y) = \max_x \{xy - c(x)\}$$

consider the ith element of problem

$$f_i = (x_i y_i - c(x_i))$$

calculating the gradient

$$g = \frac{\partial f}{\partial y_i} = x_i$$

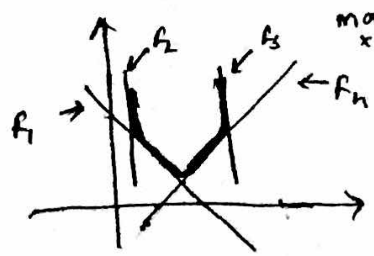
$$H = \frac{\partial^2 f}{\partial y_i^2} = 0$$

{ The Hessian 0 mean the function is Linear function and convex function }

Now we have Bunch of funtion sets which are lineare and By the property the max of convex set is a convex so we $\boxed{\text{can say that } \max_x \{xy - c(x)\} \text{ is convex with respect to } y}$

$$\max_x [x_1 y - c(x_1) \ldots x_n y - c(x_n)]$$

Graphacilly



$\boxed{c^*(y) \text{ is convex}}$