Problem - 2

consider a moon lander with state $[h, v, m]^T$ to have the following dynamics

$$\begin{cases} \dot{h}(t) = v(t) \\ \dot{v}(t) = -g + a(t)/m(t) \\ \dot{m}(t) = -K a(t) \end{cases}$$

here h is the altitude, v is the velocity, and m is the mass of the moon Lander, $a(t) \in [0, 1]$ is the thrust, and k is a constant fuel burning rate. let the initial state be $[h_0, v_0, m_0]^T$, and the target be $h(t^*) = 0$ and $v(t^*) = 0$ at terminal time t^*. Derive the optimal control Policy for minimal fuel consumption.

~~Assumption~~

some aditional constraints that

* hight can not be negative $h(t) \geq 0$
* mass can not be negative $m(t) \geq 0$

The goal here is to minimize the fule consumption, which mean we need to maximizing the mass of moon Lander.

$$\max \; m(T).$$

* $m(T)$ is the remaning fuel

where τ is the first time $h(\tau) = v(\tau) = 0$
as we can see $a(t) = -\frac{\dot{m}(t)}{k}$, so if we

minimize the total applied thurst befor landing is as equal as the maxamizing the mass of moon lander. which give us the minimal fuel consumption. so,

$$\min_{a(t)} \int_0^{\tau} a(t)\, dt = \frac{m_0 - m(\tau)}{k}$$

In, terms of general notation the state vectors

$$f = \begin{bmatrix} v \\ -g + a/m \\ -ka \end{bmatrix} \qquad L = a$$

Hence the Hamiltonian

$$H = -L + \lambda^T f \Rightarrow -a + \lambda_1 v + \lambda_2\left(-g + \frac{a}{m}\right) + \lambda_3(-ka)$$

$$a^* = \underset{a \in [0,1]}{\arg\max}\ H.$$

$$\boxed{a^* = \underset{a \in [0,1]}{\arg\max}\ \left(-1, \frac{\lambda_2}{m} - \lambda_3 k\right) a + \lambda_1 v - \lambda_2 g}$$

Thus the

$$a(t) = \begin{cases} 0 & b \le 0 \\ 1 & b > 0 \end{cases}$$

where $b = -1 + \frac{\lambda_2}{m} - \lambda_3 K$

The guess policy $a(t) = \begin{cases} 0 & \text{for } t \in [0, t^*] \\ 1 & \text{for } t \in [t^*, \tau] \end{cases}$

In order to prove the guess of optimal policy we need to show that the b is either monotonically increasing or decreasing.

so $\dot{b} = \frac{\dot{\lambda}_2}{m} - \frac{\lambda_2 \dot{m}}{m^2} - \dot{\lambda}_3 K$

and we know acc^n to adjoint equation

$$\dot{\lambda}_1 = \frac{-\partial H}{\partial h} = 0$$

$$\dot{\lambda}_2 = \frac{-\partial H}{\partial v} = -\lambda_1$$

$$\dot{\lambda}_3 = \frac{-\partial H}{\partial m} = \frac{\lambda_2 a}{m^2}$$

Putting all the $\dot{\lambda}$ value in $\dot{b}$

we have

$$\dot{b} = -\frac{\lambda_1}{m} - \frac{\lambda_2}{m^2}(-ka) - \left(\frac{\lambda_2 a}{m^2}\right) k$$

$$* \quad // \quad \boxed{\dot{b} = -\frac{\lambda_1}{m}}$$

I will use this $\dot{b}$ at end

Now as we have switching condition when to engine/thrust start or when to stop acc^n to our guess, so here first I define the dynamics when the engin is off no thrust, $t \in [0, t^*]$

thus, the dynamics become

$$\begin{cases} \dot{h}(t) = v(t) \\ \dot{v}(t) = -g \\ \dot{m}(t) = 0 \end{cases}$$

here $\dot{m}(t) = 0$ $\Rightarrow \int_0^{t^*} \dot{m}(t)\,dt = \int_0^{t^*} 0\,dt$

$$\Rightarrow m = 0 + c_1$$

$$\Rightarrow m = c_1$$

and we know at starting $m = m_0$.

and since there is no thrust untill t^*
so at t^* $m = m_0$, so $c_1 = m_0$

$$\boxed{m = m_0}$$

→ $\dot{v} = -g$

$$\int \dot{v} = \int -g$$

$$v = -gt + c_2$$

we know at time $t = 0$ $v = v_0$.

$$\boxed{c_2 = v_0} \quad \boxed{v = -gt + v_0}$$

→ $\dot{h} = v$

$$\int \dot{h} = \int v = \int -gt + v_0$$

$$h = -g\frac{t^2}{2} + v_0 t + c_3$$

at $t = 0$, ~~[illegible]~~ $h = h_0$

$$h_0 = c_3$$

$$\boxed{h = -\frac{1}{2}gt^2 + tv_0 + h_0}$$

$$\text{so} \begin{cases} h_{free}(t) = -\frac{1}{2} g t^2 + t v_0 + h_0 \\ v_{free}(t) = -gt + v_0 \\ m_{free}(t) = m_0 \end{cases}$$

now we are considering the another condition when $a=1$, mean engine is on

$\therefore$ the following dynamics at $t \in [t^*, \tau]$

$$\begin{cases} \dot{h}(t) = v(t) \\ \dot{v}(t) = -g + \boxed{1/m(t)} \\ \dot{m}(t) = -K \end{cases}$$

$\rightarrow \int \dot{m}(t) = \int -k$

$m = -Kt + c_1$

we know at $t = t^*$, $m = m_0$.

$m_0 = -Kt^* + c_1$

$c_1 = m_0 + Kt^*$

$m = m_0 + K(-t + t^*)$

$$// \boxed{m = m_0 + K(t^* - t)}$$

$$\int \dot{v} = \int -g + 1/m$$

$$\int \dot{v} = \int -g + 1/(m_0 + k(t^* - t))$$

$$v = -gt + \frac{\log(m_0 + k(t^* - t))}{k} + c_1$$

we know at $t = \tau$ $v(\tau) = 0$.

$$0 = -g(\tau) + \frac{\log(m_0 + k(t^* - \tau))}{k} + c_1$$

$$c_1 = g\tau - \frac{\log(m_0 + k(t^* - \tau))}{k}$$

So $v = -gt + \frac{\log(m_0 + k(t^* - t))}{k} + g(\tau) - \frac{\log(m_0 + k(t^* - \tau))}{k}$

$$\boxed{v = g(\tau - t) + \frac{1}{k} \log\left[\frac{m_0 + k(t^* - \tau)}{m_0 + k(t^* - t)}\right]}$$

$\rightarrow$ $\dot{h} = v$

$$\int \dot{h} = \int v$$

$$h = \int g(\tau - t) + \frac{1}{K} \log \left[\frac{m_0 + K(t^* - \tau)}{m_0 + K(t^* - t)}\right]$$

$\cancel{\Rightarrow -g\frac{t^2}{2}}$

$$\Rightarrow g\left(\tau t - \frac{t^2}{2}\right) + \frac{1}{K} B$$

I am solving B here.

$$B = \int \log \left[\frac{m_0 + K(t^* - \tau)}{m_0 + K(t^* - t)}\right]$$

$$= \int \log(m_0 + K(t^* - \tau)) - \log(m_0 + K(t^* - t))$$

$$= \log(m_0 + K(t^* - \tau))t - \left(-t - \log(K(t^* - t) + m_0)v \frac{(K(t^* - t) + m_0)v}{K}\right) + c_1$$

Due to limited space in my note book i calulate it on side. and after calculation

$$h = g\left(\tau t - \frac{t^2}{2}\right) + \left(-\frac{\tau - t}{K} - \frac{1}{K^2} \log\left(\frac{A}{e}\right)\right).$$

$$(m_0 - K(t - t^*) - \frac{1}{2} g \tau^2$$

where $A = m_0 + k(t^* - t)$

$e = m_0 + k(t^* - t)$

So.

$$\begin{cases} h_p = g\left(\tau t - \frac{t^2}{2}\right) - \frac{\tau - t}{k} - \frac{1}{k^2}\operatorname{Log}\left(\frac{A}{e}\right)\left(m_0 + k(t^* - t)\right) - \frac{1}{2}g\tau^2 \\ v_p = g(\tau - t) + \frac{1}{k}\operatorname{Log}\left[\frac{m_0 + k(t^* - \tau)}{m_0 + k(t^* - t)}\right] \\ m_p = m_0 + k(t^* - t) \end{cases}$$

now let put $t = t^*$ in equation when $a = 1$ (thrust is on) an $a = 0$ (thrust is off)

when $a=0$

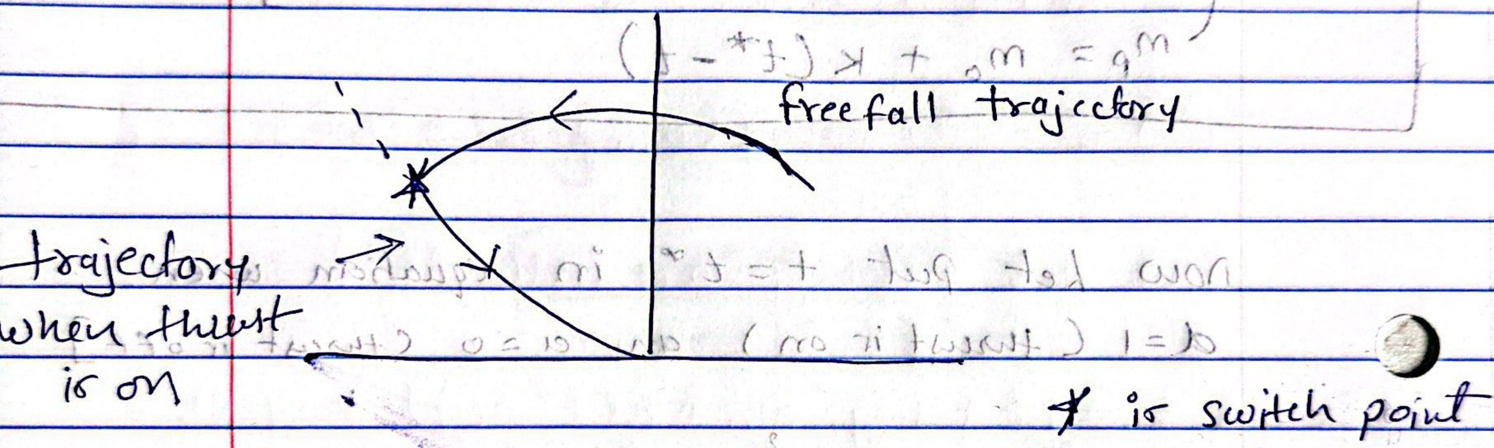
$$\begin{cases} h_{free}(t^*) = -\frac{1}{2}g t^{*2} + t^* v_0 + h_0 \\ v_{free}(t^*) = -g t^* + v_0 \\ m_{free}(t^*) = m_0 \end{cases}$$

when $a=1$

$$\begin{cases} h_p(t^*) = -g(t^* - \tau^2) + \frac{t^* - \tau}{k} - \frac{m_0}{k^2}\operatorname{Log}\left(\frac{m_0 + k(t^* - \tau)}{m_0}\right) \\ v_p(t^*) = g(\tau - t^*) + \frac{1}{k}\operatorname{Log}\left[\frac{m_0 + k(t^* - \tau)}{m_0}\right] \\ m_p(t^*) = m_0 \end{cases}$$

By using the above equations equating the $V_{free}(t^*) = v_p(t^*)$ and $h_{free}(t^*) = h_p(t^*)$ we can determine the t^* and τ value.

Now we can vizulize the system as



Now we back to the $\dot{b}$ equation,

$$\dot{b} = \frac{-\lambda_1}{m}$$

if we consider the λ is a -ve so the $\dot{b}$ is monotonically increasing function. because $\dot{b}$ is become +ve

The reason λ is not choose +ve because we can not use thrus before the swichig point if this is a case so the velocity at the end can not be zero.

so $a(t) = \begin{cases} 1 & \text{if } b > 0 \text{ on } (t^*, \tau] \\ 0 & \text{if } b < 0 \text{ on } [0\ t^*) \end{cases}$

our earlier guess of $a(t)$ does indeed the pontryagian maximum principle.

also the optimal control just change once from 0 to 1. and the $b < 0$ on $[0\ t^*)$, $b > 0$ $(t^*, \tau]$