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Investigative design of missile longitudinal dynamics using LQR-LQG controller in presence of measurement noise and inaccurate model

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Abstract. The paper presents the design and development of a Linear Quadratic Gaussian (LQG) controller for a longitudinal dynamics of a high maneuvering missile. The Linear Quadratic Regulator (LQR) in association with Linear Quadratic Estimator (LQE) is realized through extensive numerical simulations to address the plant dynamics associated with measurement noise and inaccurate plant model. The LQR gives the minimum cost function to the proposed optimal solution for a missile control design application. The LQR and LQE design has been tested for its robustness subjecting it to parametric variation of disturbances and plant uncertainties. The study brings out the efficacy of the LQG controller over traditional methods in missile control system design.

Keywords. Missile longitudinal dynamics; Linear Quadratic Regulator (LQR); Linear Quadratic Estimator (LQE); Linear Quadratic Gaussian (LQG).

1. Introduction

Longitudinal autopilot design for an aerospace vehicle is a critical design process in achieving the terminal performance objectives in neutralizing a potential target. This becomes extremely challenging in applications of agile missiles where precision target engagement is demanded from the vehicle. The design of multi-variable missile longitudinal autopilot for practical missile applications, considering all the stringent design constraints becomes complex due to various factors, some of which are the parametric uncertainties, environmental disturbances, and system non-linearities. Quite often, the sensor data required in the feedback loop is not available a priori, due to a corrupted signal and complete loss of data during flight. In addition to the sensor being noisy, there are aggravated problems when the plant is inaccurate. The control system has to take additional burden imposed due to the plant inaccuracies. When this goes out of limit, especially in tactical missile cases, leads to catastrophic design failures and loss of mission. The present study is focused on alleviating this issue leading to a realizable, robust flight controller for agile missile applications.

This study has been carried out in a modern control design framework, using state-space formulation. An optimal controller design of longitudinal dynamics of the missile has been proposed which caters to the in-flight disturbances and noise, considering both scalar disturbances and gaussian noise. Extensive numerical simulation has been carried out due to the unpredictable nature of the plant. This controller design methodology enables a controller to include all possible flight disturbances experienced during an actual flight to be synthesized which is optimal with respect to a specified quadratic performance index in the presence of internal noise and external disturbance in the system. The study leads to the design of a system with an optimal solution requiring minimal control effort, by translating into a problem of minimizing a quadratic cost function, realized using LQR design method, and considering the actual flight disturbances, implemented using LQE method, and thereby adhering to the required performance objectives.

In real-world flight tests of aerospace vehicles, the system is subjected to varied noise and external disturbances, which are propagated into the system with time. Ignoring the effects of such disturbance often leads to system instability.

A significant body of work exists in this area and various researchers have studied this problem and the related work is summarized as follows. John Doyle, in his seminal work[1], with just three words of abstract *'There are none'* in context to robustness aspects of LQ controllers, emphasized the LQG based design must be tested for stability margins for specific control designs.

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In another pioneering research paper [2] on multi-variable feedback control systems, it has been stated that the design of LQG-based controller should include aspects of nominal stability, stability-robustness, consideration of modeling errors, and acceptable performance subject to application-specific performance trade-offs.

[3] gives a chronological history and evolution of modern control theory with a summary of critical works of various eminent researchers in context to LQR-LQG based controllers. Saptarshi Das *et al* [4] presents the design of the LQG controller and shows the vulnerability of the controller even to a small amount of disturbance due to the fact that the dynamics of the filter are slower than the overall missile dynamics. Though the proposed solution provides higher stability margins and improved tracking performance, with the increase of Q, the initial controller effort at $t = 1s$ suddenly becomes large which may cause actuator saturation. Investigation on boost phase of longitudinal autopilot for a ballistic missile has been carried out by Mohsen Ahmed *et al* in [5] using gain-scheduled LQR controller which shows improvements over classical methods. However, the controller is shown to be sensitive in off-nominal flight conditions and parameter uncertainties. YU Jianqiao *et al* [6] introduced integrators to augment the plant dynamics of the faster modes of LQG controller, which mostly limits the application of LQG controllers for real engineering problems. Kisabo *et al* [7] studied pitch control of a rocket with a novel LQG/LTR control algorithm. A technique of adding constant gain to the reference is proposed in order to reduce the steady-state error of the system output. Javad Tayebi *et al* [8] discusses attitude stabilization of an agile microsatellite with CMG using LQR-LQG. It is shown that this method could stabilize satellites from the initial condition, with large angles and with more accuracy in comparison to feedback quaternion and PID controllers. The result also shows the effect of filtering the noisy signal in the LQG controller. Kulcsar B *et al* [9] carried out an application of LQG control theory through a study of linear time-invariant aircraft dynamic system, which was described in time and frequency domain. The uncontrolled dynamics were stabilized with LQR, LQG, and LQG/LQR controllers. Seong *et al* [10] present a stabilization loop design of a two-axis gimbal system which has been compared with the standard PI controller. The LQG controller is shown to improve the GM and PM of the gimbal system in both the pitch and yaw axis. Brown *et al* [11] present the advantages of LQG/LTR controller in the design of controller for a non-minimum phase system, viz. tail-controlled missile. The controller is shown to meet design goals in terms of minimum rise-time (68% in 200 ms) with minimum overshoot ($\leq 25\%$). The eigenstructure of the plant is modified by the relocation of the plant poles and partial reassignment of the components of the Eigenvector. Comparison in both time and frequency have been made with less coupling when the controller incorporates a careful selection of inner-loop feedback. Sheperd *et al* [12] carried out a study that utilizes LQG/LTR methodology to design a command following autopilot with the loop shaping methodology in LQG/LTR. Here, the pitch and yaw channel are addressed as a multi-variable, coupled system. Various possibilities of self-aware aerospace vehicles, their flight conditions, and re-orient flight plans as per autopilot demands have been discussed by Alliare *et al* [13]. Even though the state estimation is presented to augment data using various sensors, the aspects of parameter uncertainties and system stability have not been addressed. A comparison of approaches and performance with H-∞ and optimal LQG are made in the design of optimal and robust longitudinal flight control of a canard-configured high-performance aircraft by Fatima Shoaib *et al* [14]. Monte-Carlo results presented indicated LQG is more robust than H-infinity by mixed sensitivity approach and loop shaping method in the presence of the disturbance due to wind gusts. Wang *et al* [15] discuss the computation of static and dynamic derivatives using the RANS method and validation of data with wind tunnel results. A reasonable agreement between CFD and experimental data has been found in the region of interest. Even though the 6-DoF simulation for missile guidance and control presented by Xu *et al* [16] is able to precisely control and track the target, the fundamental limitation of system performance and guidance error introduced due to noise influence and disturbance rejection have not been addressed in the paper. In the paper on Heading control of an unmanned surface vehicle by T Asfihani *et al* [17], it is shown that the LQG controller outperforms the conventional and adaptive PID in terms of settling time response. However, the paper does not address the robustness and stability aspects of the proposed controller. Munadi *et al* [18] has proven in their paper that a PI-L1 adaptive controller for missile longitudinal dynamic addresses various aspects like performance, robustness, and adaptability in uncertainty conditions. However, this method fails to address stability and robustness aspects of high-speed, fast-response real-time aerospace applications. [19] discusses simulation results of implementing a pitch, roll, and sideslip angle controller using LQR and LQG for an aircraft application. It is demonstrated in the paper that LQG gives good performance even in the presence of process noise and measurement errors. As stated in all the above papers, the design of various controllers using classical and modern techniques has been addressed. In some studies, the autopilot has been designed in presence of noise and disturbances. However, it has been observed that the design does not cater to robustness and meets the criteria required for a high-performance aerospace vehicle.

Classical design procedures are best suited for linear single-input, single-output systems with zero initial conditions. One can achieve the acceptable controller design by the trial-and-error method. Many complex multi-input, multi-output applications, such as the design of state-of-the-art missile design which demands high maneuverability, fast turning rate time constants with minimal control demand, are not amenable to classical control techniques. These problems can be alleviated by incorporating optimal control techniques [20–22].

In the present paper, a novel LQG technique is proposed to address the design of a longitudinal controller for an agile missile. The design is optimized in the presence of disturbance and noise. The performance of the controller and robustness studies are carried out in the time and frequency domain. The results are very encouraging and hence presented in this paper.

The layout of the paper is as follows, Section 2 formulates the problem along with the challenges faced in missile control design where noise and disturbances are propagated into the system dynamics. Section 3 discusses the methodology implemented for the design of the LQG controller to address the noise and disturbance states. Section 4 presents the results, discussion, and analysis from numerical simulation. Section 5 concludes the paper with recommendations and a scope for future work.

2. Methodology

The paper explores the design and development of a novel LQG control algorithm through numerical simulation and analysis, addressing a longitudinal dynamics of a high maneuvering missile flying in cruise conditions, as shown in figure 1. The missile has been perturbed from its intended flight path from initial state $x(t)$ to a perturbed position $x_0 + x(t)$ in finite time t. The objective of the present study is to design a controller to bring the flight state to the trim condition in the shortest time possible. The system efficacy is verified w.r.t. LQR performance as a benchmark. The objective of the study is to investigate the applicability of LQG-based algorithms addressing the noisy, disturbance signals in the design of missiles demanding stringent performance criteria.

2.1 *Problem definition*

With the experiences gathered from several flight tests, it has been learned from a statistical set of data that flight vehicles with low MoI and mass properties (eg. very short-range air-defence missile, surface-to-surface, air-to-surface missile applications) inherently suffer from external flight disturbances. Added to that, the control grade inertial measurement unit package has inherent sensor noise and drifts which further strain the missile autopilot and hence demanding more control effort. The missile plant being studied here has an $I_{xx} = 0.1\ kg - m^2$ about the x-axis, which is vulnerable to disturbances even with small magnitudes of cross-wind and gust.

Putting things in perspective, a strategic missile, which weighs of the order of 10 *tonnes*, has an I_{xx} of the order of $1000\ kg - m^2$. This class of flight vehicles is highly robust in terms of the external disturbances and the autopilot demands are benign and the angle of attack (α) requirements are met within low-to-medium ranges. The linear theory holds good in such applications. However, while investigating the flight dynamics of a tactical missile plant, which demands high maneuverability and high α ranges to meet the desired lateral acceleration in y- and z- planes, requiring very high turning rate time constants. Here, the linear theory ceases to exist in such extreme flight conditions.

In the present study, with the missile plant shown in table 1, the $I_{xx} \approx 0.1\, kg - m^2$ and the PID gain tuning is insufficient when severe disturbances are encountered during a flight when launched in a stormy environment. Thus, the tactical missile design demands all-weather operation which can be launched under any circumstances and perform satisfactorily at severe environmental conditions. The present study addresses this critical requirement in the design of a practical modern controller, with the realizable Q and R matrices demanding optimal control effort at critical trajectory requirements.

2.2 *Missile mathematical model and plant dynamics*

The missile 6-DoF are expressed in terms of translational and rotational equations of motion. The translational kinetic equations of the missile are shown in Eqn. (1).

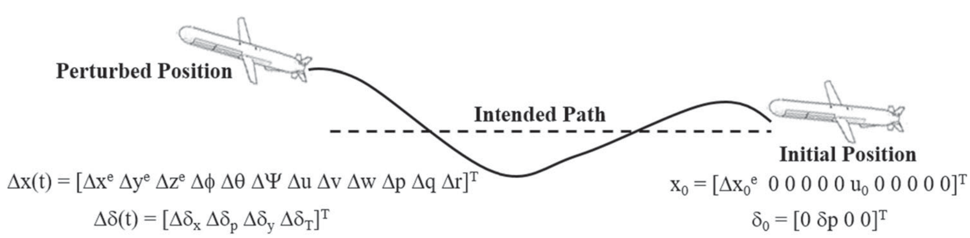

Figure 1. Vehicle perturbed position due to flight disturbance.

Table 1. Plant parameters.

Parameter	Values
Mass	70 kg
I_{xx}	0.1 $kg-m^2$
I_{yy}	10.3 $kg-m^2$
I_{zz}	10.1 $kg-m^2$

$$\begin{aligned} X - mg\sin\theta &= m(\dot{u} + qw - rv) \\ Y - mg\sin\phi\cos\theta &= m(\dot{v} + ru - pw) \\ Z - mg\cos\phi\cos\theta &= m(\dot{w} + pv - qu) \end{aligned} \tag{1}$$

Similarly, rotational EoM of the missile are written as shown in Eqn. (2).

$$\begin{aligned} L &= I_{xx}\dot{p} \\ M &= I_{yy}\dot{q} + (I_{xx} - I_{zz})pr \\ N &= I_{zz}\dot{R} + (I_{yy} - I_{xx})pq \end{aligned} \tag{2}$$

In the present study, longitudinal dynamics of a missile has been normalized and shown in Eqn. (3):

$$\begin{aligned} m\dot{u} &= X_u u + X_w w + X_q q - mg\theta\cos\theta_e + X_\eta \eta \\ m\dot{w} &= Z_u u + Z_w w + Z_q q - mg\theta\sin\theta_e + Z_\eta \eta \\ I_y\dot{q} &= M_u u + M_w w + M_q q + M_\eta \eta \\ \dot{\theta} &= q \end{aligned} \tag{3}$$

The MIMO systems in modern control systems are represented by state-space equations of Eqn. (4)

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned} \tag{4}$$

Here, the longitudinal dynamics shown in Eqn. (3) are linearized about the trim condition and represented in state-space form as given in Eqn. (5).

$$\begin{bmatrix} \dot{u} \\ \dot{w} \\ \dot{q} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} x_u & x_w & x_q & x_\theta \\ z_u & z_w & z_q & z_\theta \\ m_u & m_w & m_q & m_\theta \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} u \\ w \\ q \\ \theta \end{bmatrix} + \begin{bmatrix} x_\eta \\ z_\eta \\ m_\eta \\ 0 \end{bmatrix} \delta_P \tag{5}$$

Numerically, Eqn. (5) can be expressed in state-space form as shown in Eqn. (6):

$$\begin{aligned} A &= \begin{bmatrix} -0.0089 & -0.1474 & 0 & -9.75 \\ -0.0216 & -0.3601 & 5.9470 & -0.151 \\ 0 & -0.00015 & -0.0224 & 0.0006 \\ 0 & 0 & 1 & 0 \end{bmatrix} \\ B &= \begin{bmatrix} 9.748 \\ 3.77 \\ -0.034 \\ 0.01 \end{bmatrix}; \; C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}; \; D = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \end{aligned} \tag{6}$$

The matrices A, B, C and D of Eqn. (6) correspond to those of Eqn. (4).

2.3 *Problem formulation*

The states of the missile system due to flight disturbances (flight scenario 1–4) are shown in figure 2 and the step response is given in figure 3. It can be observed that all system states are unstable and diverge with time.

In modern control theory, it is known that if the system dynamics, i.e. matrices A, B of the plant are controllable, it is possible to manipulate the eigenvalues of the closed-loop system $[A - BK_r]$ through the choice of full-state feedback controller formulation $u = -K_r x$. This assumes, all the full state measurements are available, i.e., $C = I$ and $D = 0$, such that $y = x$. The problem arises as the system dimension increases and it gets difficult to achieve full-state controllability and observability. However, if the system is observable, it is possible to build a full-state estimation from the sensor measurement data.

In the process of placing appropriate eigenvalues for the closed-loop poles, both for the strongly and weakly unstable states, the system may be driven towards instability in the presence of even a small time lag affecting the overall robustness of the system. Thus, the objective is to arrive at the best gain matrix K stabilizing all the states of the system without expending much control effort in doing so. A fine balance between the stability of the closed-loop system and the control effort is the objective.

To achieve optimal philosophy, it requires the availability of all the full-state measurements from the system, which is a rarity. There are also significant computational and processing time constraints adding to the time delays bringing down the system robustness. In addition, it is (i) often difficult to access the sensing points, (ii) sensor accuracy, (iii) computational complexity, viz. the data from rate gyros and accelerometers have their inherent errors and introduce inaccuracies into the system, apart from, the

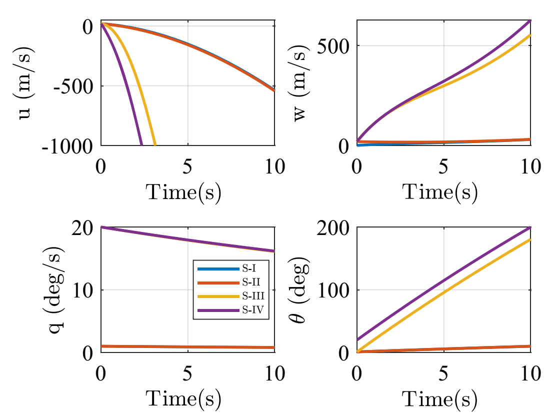

Figure 2. System states due to initial disturbances (Scenario I to IV).

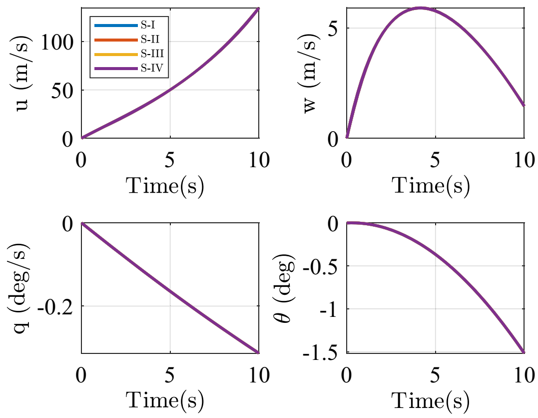

Figure 3. Step response of unstable system dynamics.

process noise and measurement noise. At the same time, the established missile dynamics may have various uncertainties from unmodeled higher dynamics, parameter variation, and linearization of non-linear elements. The uncertainties considered in the present study are the model perturbation/uncertainties such as aerodynamic stability coefficient variation ($\pm 3\%$), i.e. pitching moment (C_m) and normal force (C_N) stability derivatives as $f(\alpha)$, actuator perturbations ($\pm 10\%$) in an additive sense. The missile is subjected to environmental disturbances in form of a wind gust, which induces side-slip and affects the actual measurements.

The LQG approach presented in the paper uses the separation principle for optimal control which leads to an estimator-regulator pair. The LQG controller is used with a Gaussian Noise. The LQG controller has a Linear Quadratic Regulator and Linear Quadratic Estimator (LQE) in cascade.

The best solution is to estimate the error, where it is not possible to sense accurate data due to the reasons mentioned above. The objective of the present study is to design a longitudinal autopilot to obtain the fastest time response while maintaining acceptable stability margins and robustness of the system.

3. Controller design

The main idea in the LQR control design philosophy is to minimize the quadratic Jacobian cost function matrix.

In the present study, instead of measuring full state x, the system states have been estimated which are accompanied by noise. The full state estimation is theoretically possible, as long as [A, C] is observable. The estimation depends on the degree of observability as quantified by observability Gramian. Kalman filter is the bedrock for designing a full-state estimator, making an optimal balance between the competing effects of measurement noise, disturbances, and model uncertainty. The efficacy of the LQG controller presented in this paper, establishes a full-state estimator from the Kalman filter, working in conjunction with an optimal LQR controller.

The Kalman filter considers the disturbance and noise are zero-mean Gaussian processes with known co-variances, Eqn. (7):

$$\begin{aligned} E(\xi_d(t)\xi_d(\tau)^*) &= V_d\delta(t-\tau) \\ E(\xi_n(t)\xi_n(\tau)^*) &= V_n\delta(t-\tau) \end{aligned} \tag{7}$$

where E is the expected value, δ is the Dirac delta function, matrices V_d and V_n are positive semi-definite with entries containing the co-variances of the disturbance and noise terms, ξ_d is the external disturbance and ξ_n is the process noise.

Here, in the design solution presented, the LQR control gain, K_{LQR} and the Kalman filter gain, (K_f) are designed independently. The resulting sensor feedback will remain optimal and still retain the closed-loop eigenvalues when combined. The efficacy of the LQG controller depends essentially on two prerequisites – (i) accurate model of the system, (ii) magnitude of noise and disturbances.

3.1 *Design of LQR controller for missile longitudinal dynamics*

The open-loop step response of the plant is verified and found to be unstable for the given initial disturbance as shown in figure 3. It can be noted that all the state variables are not converging to a steady-state, even after 50*s*. Further investigation shows the eigenvalues are in RHP of the s-plane and controllability computation indicates that the rank of the system is 4 which suggests that all four state variables are controllable. Extensive studies are carried out by the author in [25] on stand-alone LQR controller implementation of the missile dynamics. The computed optimal values of a gain matrix, K_r and eigenvalues of the $[A - BK_r]$ are shown in figure 6 bringing out the minimum energy demand from the system. These values of Q and R are arrived at after carrying out perturbation studies shown in figures (4 and 5). However, in presence of noise, the system performance deteriorates and is driven to instability as shown in figure 7. The system also exhibits non-minimum phase (NMP) behavior, which is typical of tail-controlled missiles, where, pitch deflection in a negative direction would give rise to the positive angle of attack. This poses additional challenges for robust control.

3.2 *Observability and kalman filter*

The missile in-flight is vulnerable to measurement noise and sensors' inherent characteristics that are inevitable. In

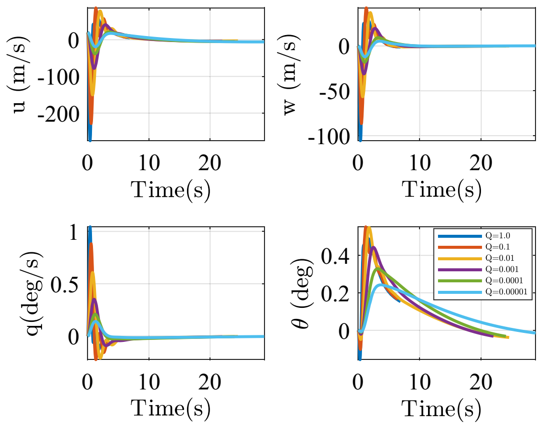

Figure 4. LQR state response with Q parameterization.

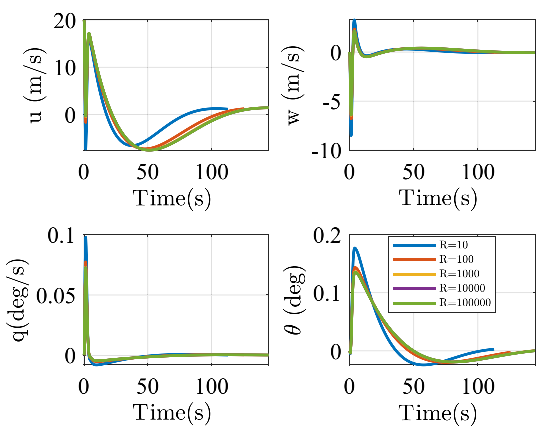

Figure 5. LQR state response with R parameterization.

this paper, a solution is proposed through a design of a novel estimator, with fundamental principles derived from Kalman filter [23, 24]. The goal is now to put the estimator in a cascade with the LQR and stabilize the unstable dynamics. If any of the internal states considered here is not measurable in the actual scenario, it is possible to estimate the system state, provided observability of the system state is ensured. Thus, the linear system is represented by Eqn. (4), if controllable and observable, an optimal full-state feedback controller using LQR can be developed. At the same time, an optimal linear quadratic estimator (LQE) can be designed, which, when combined using matrices A & C would result in an optimal system. The block diagram of LQE is given in figure 8. This may not appear to be explicit in the formulation but the LQR gains are used to choose the eigenvalues of A and the LQE gains are used to place the eigenvalues of the estimator dynamics and when combined, the same dynamics are retained. The LQR control minimizes the cost function, and the estimator handles the Gaussian disturbances. While designing, care needs to be taken as it makes the system vulnerable to robustness issues and can lead to instability, if the system is over-sensitive at some flight conditions.

The observability of a system has been computed using Eqn. (8).

$$N = [C^T, A^T C^T, (A^T)^2 C^T, \ldots, (A^T)^{n-1} C^T] \tag{8}$$

where, the plant matrix N is of the rank n.

The computed observability matrix [AC] for the plant is given in eqn.(9):

$$[AC] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -0.0089 & -0.1474 & 0 & -9.75 \\ -0.0216 & -0.3601 & 5.947 & -0.151 \\ 0 & -0.0001 & -0.0224 & 0.0006 \\ 0 & 0 & 1 & 0 \\ 0.0033 & 0.0544 & -10.6266 & 0.109 \\ 0.008 & 0.132 & -2.4257 & 0.2685 \\ 0 & 0.0001 & 0.0002 & 0 \\ 0 & -0.0001 & -0.0224 & 0.0006 \\ -0.0012 & -0.0185 & 0.6705 & -0.0464 \\ -0.0029 & -0.0483 & 1.1077 & -0.0991 \\ 0 & 0 & 0.0003 & 0 \\ 0 & 0.0001 & 0.0002 & 0 \end{bmatrix} \tag{9}$$

The state space equation with disturbance and noise can be expressed as shown in Eqn. (10) and (11) respectively.

$$\begin{aligned} \dot{x} &= Ax + Bu + \xi_d \\ y &= Cx + Du + \xi_n \\ \hat{y} &= C\hat{x} + Du \end{aligned} \tag{10}$$

The matrices ξ_d and ξ_n are positive semi-definite with entries containing the co-variances of the disturbance and noise terms. It is possible to obtain an estimate $\hat{x}$ of the full state x from measurements of the input u and output y, via the estimator dynamical system represented by Eqn. (11).

$$\begin{aligned} \frac{d}{dt}\hat{x} &= A\hat{x} + Bu + K_f(y - \hat{y}) \\ \hat{y} &= C\hat{x} + Du \end{aligned} \tag{11}$$

The matrices A, B, C, and D are obtained from the system model, and the filter gain is determined using Eqn. (12).

$$K_f = YC^* \xi_n \tag{12}$$

where, Y is the solution to Algebraic Riccati Equation (ARE), Eqn. (13).

$$YA^* + AY - YC^*\xi_n^{-1}CY + \xi_d = 0 \tag{13}$$

This solution is commonly referred to as a full-state estimator concerning the cost function shown in Eqn. (14):

$$J = \lim_{t\to\infty} E((x(t) - \hat{x}(t))^*(x(t) - \hat{x}(t))) \tag{14}$$

This cost function implicitly includes the effects of disturbance and noise, which are required to determine the optimal balance between aggressive estimation and noise attenuation. Thus, the filter is referred to as linear quadratic estimation (LQE) and has a dual formulation to the LQR optimization. The estimator gains K_f obtained is given in Eqn. (15).

$$K_f = \begin{bmatrix} 4.3396 & -0.4832 & -0.1784 & -0.9733 \\ -0.4832 & 2.7606 & 0.8095 & 0.2665 \\ -0.1784 & 0.8095 & 0.5219 & 0.1303 \\ -0.9733 & 0.2665 & 0.1303 & 0.4747 \end{bmatrix} \tag{15}$$

Linear Quadratic Estimator (LQE) is shown in figure 8. Substituting the output estimate $\hat{y}$ yields Eqns. (16) and (17):

$$\frac{d}{dt}\hat{x} = (A - K_fC)\hat{x} + K_fy + (B - K_fD)u \tag{16}$$

$$\frac{d}{dt}\hat{x} = (A - K_fC)\hat{x} + [K_f(B - K_fD)]\begin{bmatrix} y \\ u \end{bmatrix} \tag{17}$$

The eigenvalues of the filter are provided by $[A - K_fC]$. It can be shown that x converges to the full-state estimate $\hat{x}$ asymptotically, provided the eigenvalues of the estimator are stable and the model faithfully captures the true system dynamics.

3.3 *LQG controller design*

The full-state estimator is used in conjunction with the feedback control law LQR control algorithm. Here, the LQR gain K_r and the estimator gains K_f have been independently designed and the overall system provides a combination of feedback and estimator system. Combining the LQR full-state feedback in cascade with the estimator results in the Linear Quadratic Gaussian (LQG) controller.

The LQG controller system, whose schematic, given in figure 9, can be defined as shown in Eqn. (18):

$$\frac{d}{dt}\hat{x} = (A - K_fC - BK_r)\hat{x} + K_fy \tag{18}$$

The control input signal is defined as per Eqn. (19):

$$u = -K_r\hat{x} \tag{19}$$

The LQG controller is optimal concerning the following ensemble-averaged version of the cost function shown in Eqn. (20):

$$J = \int_{t_0}^{t_f} [X^*QX + u^*Ru]dt \tag{20}$$

Since the controller $u = -K_r\hat{x}$ is the state estimate and the cost function is considered as an average of combinations of disturbance ξ_d and noise ξ_n.

The estimation error ε can be defined as in Eqn. (21):

$$\varepsilon = x - \hat{x} \tag{21}$$

The state-space representation of the system can be formulated as per Eqn. (22):

$$\begin{aligned} \dot{x} &= Ax + Bu \\ &= Ax - BK_r\hat{x} + \xi_d \end{aligned} \tag{22}$$

The state estimate can be defined as per Eqn. (23):

$$\begin{aligned} \hat{x} &= x - (x - \hat{x}) \\ &= Ax - BK_rx + BK_r(x - \hat{x}) + \xi_d \end{aligned} \tag{23}$$

Thus, the estimation error can be solved as per Eqn. (24):

$$\dot{\varepsilon} = (A - K_fC)\varepsilon + \xi_d - K_f\xi_n \tag{24}$$

The closed-loop system of LQG may be written as per Eqn. (25):

$$\frac{d}{dt}\begin{bmatrix} x \\ \varepsilon \end{bmatrix} = \begin{bmatrix} A - BK_r & BK_r \\ 0 & A - K_fC \end{bmatrix}\begin{bmatrix} x \\ \varepsilon \end{bmatrix} + \begin{bmatrix} I & 0 \\ I & K_f \end{bmatrix}\begin{bmatrix} \xi_d \\ \xi_n \end{bmatrix} \tag{25}$$

From the above system, it is evident that the closed-loop eigenvalues of the LQG are provided by the eigenvalues of $[A - BK_r]$ and $[A - K_fC]$ which is a combination of the eigenvalues obtained by LQR and LQE independently. In the estimator design, the disturbance and noise are exogenous inputs to the system. The system dynamics are derived from ε and x and the eigenvalues of the coupled system are derived, i.e. $[A - BK_r]$ and $[A - K_fC]$ and the eigenvalues of state are stabilized by LQR control.

In control theory, this is called the 'separation principle', which essentially states the LQR controller and the estimator can be designed in isolation and combined so that the overall system retains the property of each.

However, in practical systems with multiple loops, even a small time-lag, and model uncertainty may result in degradation of robustness and drive the system towards instability. Thus, designing the controller, ensuring and handling the robustness issues becomes a part of the LQG design methodology. Keeping the classical design techniques in perspective, such as PID, where the faster inner-loop and slower outer-loop control is designed assuming separation of timescales, whereas LQG design is able to handle MIMO systems with overlapping timescales and multi-objective cost-functions with minimal complexity in the implementation and minimally burdening the computational resources.

In missile applications, with the actuator, autopilot, and guidance loop working as 3-loop structures, LQG is a good candidate for missile applications. However, care must be taken to address the vulnerability to instability and robustness issues in presence of model imperfections, uncertainties, and time delays [1].

Table 2. Test cases for various flight disturbances.

Flight Scenario	Flight disturbance $[u\ w\ q\ \theta]$
Case-1	[20 1 1 1]
Case-2	[20 20 1 1]
Case-3	[20 20 20 1]
Case-4	[20 20 20 20]

4. Discussion and analysis of results

The flight vehicle is considered to be flying at steady-level flight as shown in figure 1 with initial states as shown $x_0 = [\Delta x_0^e\ 0\ 0\ 0\ 0\ 0\ u_0\ 0\ 0\ 0\ 0\ 0]^T$ with deflection commands as per $\delta_0 = [\delta_{pitch}\ 0\ 0\ 0]^T$. The four longitudinal states, namely, u, w, q and θ have been considered for the analysis along with pitch deflection command δ_{pitch}. The vehicle is deviated from its intended trajectory due to flight disturbances. The disturbances can be manifested into the system dynamics due to various reasons, such as wind gusts, head, tail, and crosswinds affecting the flight path angle and in turn affecting the angle of attack and side-slip angle. Disturbance includes sensor noise and biases from the inertial measurement unit, which gives erroneous feedback on the position and attitude of the missile. For precision attack missile applications, with nearly zero miss distance requirement, this could be catastrophic as the missile deviates from its intended flight trajectory.

Various flight simulation scenarios have been carried out with the flight disturbances. Some of the cases are shown in table 2. The initial flight disturbances are shown in figure 2 and the corresponding step response of the system is shown in figure 3. It has been observed that for all the practical flight disturbances, the system tends to diverge as time progresses for all the states considered.

The missile is subjected to an initial flight disturbance of $[\Delta u_0\ \Delta w_0\ \Delta q_0\ \Delta\theta_0] = [20\ 1\ 1\ 1]$ to analyze and understand its flight behaviour. The missile is found to be unstable as shown in figure 2. The system states do not dampen even after 50 seconds of simulation time and the open-loop system is found to be unstable. LQR design method has been implemented through various parametric studies of Q and R cost function as shown in figure 4. The objective of the parameterization of the Q and R is to expend minimum cost while attaining the stable states of the system. The attempt was made to bring the system dynamics to a trim state in the fastest time possible by a parametric study of Q and R weight matrices through several numerical simulations and it was achieved within 400 *msec* as shown in figure 6.

However, in real-world practical scenarios, the missile experiences a gamut of flight disturbances, both internally and externally while flying towards its designated target. The problem is faced when these disturbances affect the missile trajectory thus deviating its intended mission. figure 7 depicts one such scenario where a typical flight disturbance has corrupted the system states as shown from the state outputs.

The estimator design, shown in figure 8, is implemented to investigate the effect of system noise and measurement disturbances. The controller design, analyzed in the time domain with the LQR-LQG framework, where the controllers are placed in the loop as shown in figure 9. The controller design has been evolved through tuning of the LQR and LQG Q & R matrices along with the *LQG* augmented matrix. Table 3 presents the transient response characteristics of the tunable LQG controller for u/δ_{pitch}. The design goal ensured that the controller is able to trim the state within settling time and steady-state of $\leq$ 600 *msec* and 300 *msec*, respectively. The controller with LQG Con-5 gain matrix has been chosen as the final iteration which meets the desired time-domain specifications. LQG Con-5 design provides the fastest settling time and steady-state with optimal control effort constraints.

As shown in figure 10, the missile is affected in flight by two flight impulses, in the u state, namely, at $t = 2$ to 2.5 secs and $t = 30$ to 30.5 secs. Concurrently, the state is affected by internal system noise. This leads to catastrophic flight failure as the state becomes oscillatory, beyond the permissible limits. Figure 11 presents the result of the estimator design, based on underlying Kalman filter theory.

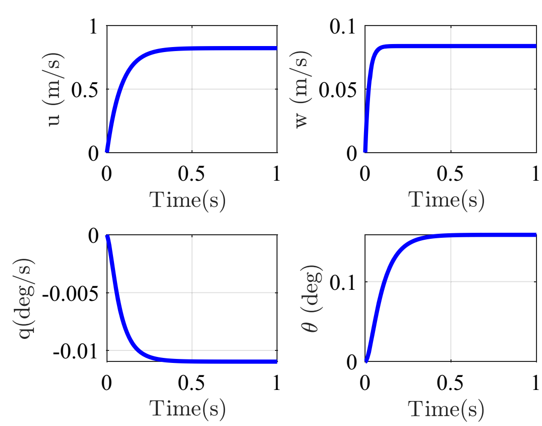

Figure 6. LQR close-loop response with unit step command, $R_{diag} = [0.55\ 0.45\ 0.35\ 0.25]$.

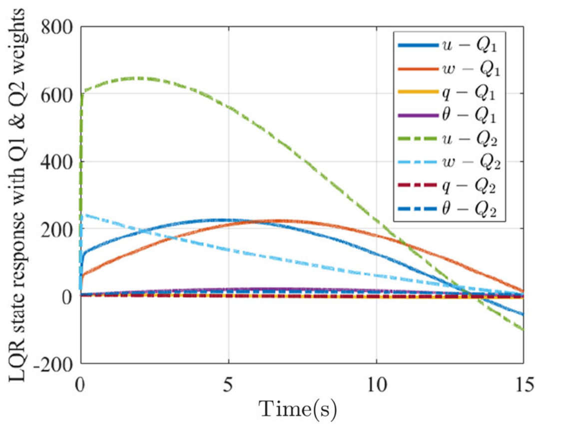

Figure 7. LQR close-loop dynamic response to flight disturbance of $x = [20\ 20\ 1\ 1]$ with Q_1 and Q_2 weights.

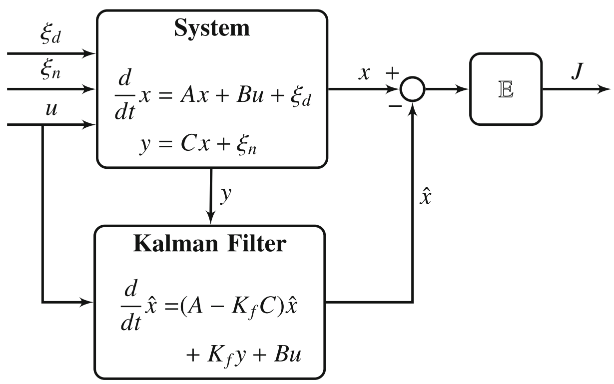

Figure 8. Linear quadratic estimator block diagram.

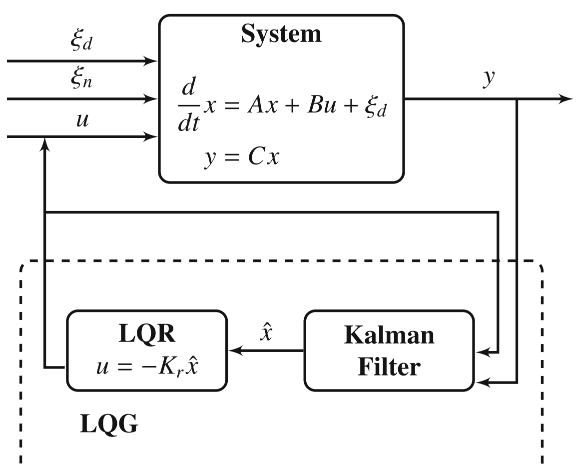

Figure 9. Linear quadratic Gaussian schematic.

The results obtained are seen to appreciably follow the original true state u. The estimator design philosophy is applied to other longitudinal system states and the results are presented in figure 12. The estimated states falls under the allowable state dispersion characteristics, without adding further delay in the autopilot response characteristics. figure 13 represents a typical flight phase during the launch phase, when the system is affected by both the internal Gaussian system noise, viz. IMU sensor and external disturbance. As observed, the estimator is able to effectively attenuate the noise in real-time. The optimal estimator design is seen to effectively follow the actual system states. This has been observed for the other longitudinal states as well. Thus, it can be stated that, when the flight experiences disturbance leading to perturbed system states,leading to deviation in the flight trajectory, due to the reasons mentioned above, the estimator design can be effectively used to filter out the system noise. Figure 14 shows the final system state output implementation with the LQR and LQE controller in the feedback loop, thus addressing the problem of external disturbance and noise in the system. All the system states are observed to settle down within the time specification of $1sec$, i.e. the LQR controller in conjunction with the LQE controller is able to drive the system towards stability from an unstable state addressing both the external and internal noise.

When extreme maneuvers are demanded, the control demand is fulfilled in y and z axes, while maintaining roll stability in the x-axis. Several problems were faced earlier during flight tests, when there were additional disturbances demanding the already strained control fin deflection, leading to saturation of control fin(s) eventually resulting in catastrophic flight failures. The reason was attributed to the limitation in the PID gain scheduling algorithm, which was not able to place the closed-loop poles in the desired location as per extreme flight demands.

In these cases, the control resources available were insufficient when there is a demand concurrently in pitch, yaw, and roll axes along with additional flight disturbances. Thus, the study carried out in this paper justifies the necessity of optimal solution requirements in such applications. The present solution proposed in this paper alleviates this problem with realistic Q and R covariance matrices formulation and the estimator in cascade to handle all possible flight disturbances.

Data insufficiency has been a huge constraint while designing this class of controllers. The use of free gyros earlier restricted us in obtaining the pitch rate q directly from the IMU. The estimator design used here to estimate the state q and q/δ_{pitch} transfer function is analyzed for performance. The Q and R matrices are carefully chosen from a set of successful Q and R covariance matrices and a far-boundary result has been analyzed in this study.

The feasibility of the proposed control strategies is validated through several numerical simulations with parametric variation studies. It has been shown that this method could stabilize the missile in presence of large initial disturbances, where traditional methods tend to underachieve. Results also bring out the filter performance in a noisy environment. LQG control design method is realistic as real-world signals are invariably susceptible to noise and in

Table 3. Transient response characteristics of various tunable LQG controller(s) for u/δ_{pitch}.

Design Specification	LQG Con-1	LQG Con-2	LQG Con-3	LQG Con-4	LQG Con-5
peak amplitude	0.633	0.608	0.586	0.565	0.527
overshoot %	77.6	77.6	77.6	77.6	77.6
at time(s)	0.167	0.161	0.155	0.149	0.139
settling time(s)	0.693	0.666	0.641	0.618	0.577
rise time(s)	0.037	0.036	0.035	0.033	0.031
steady-state(s)	0.356	0.342	0.330	0.318	0.297

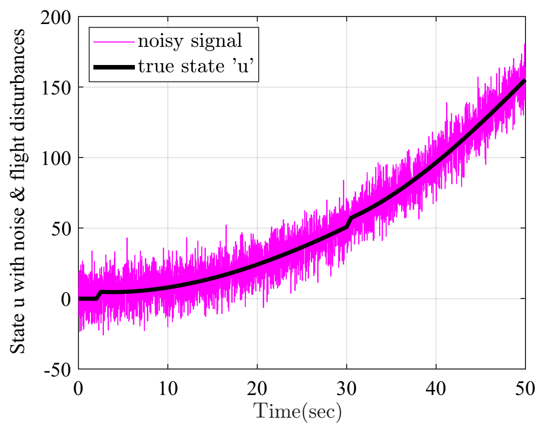

Figure 10. State u along with corrupted signal with two positive impulse disturbance in the longitudinal velocity.

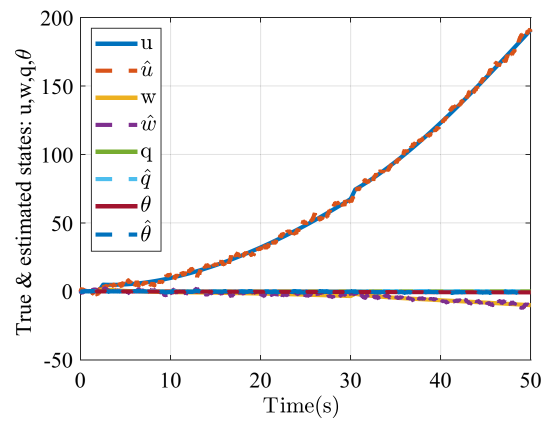

Figure 12. All longitudinal states u, w, q & θ and estimated states.

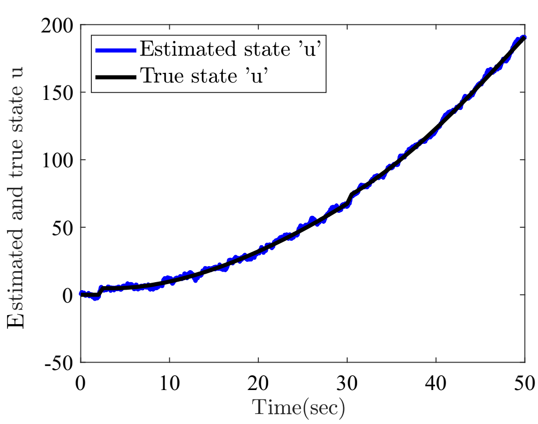

Figure 11. State u and the estimated state $\hat{u}$ after removal of the system noise.

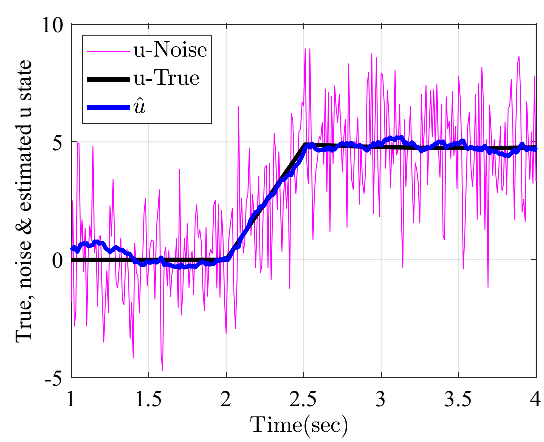

Figure 13. Representation of u-Noise, u-True and $\hat{u}$ during the initial flight phase showing efficacy of the estimator.

most cases, the state information is not available. A real system with limited measurements and limited actuation has been designed with the control design methods proposed in this paper to achieve desirable performance with acceptable robustness.

Various acceptable close-loop eigenvalue matrices for possible flight disturbances are stored in the look-up table which are selected as per the dynamic condition of the flight. A far-boundary case is presented here and this approach ensures both the optimality and the system robustness are preserved.

The problem of plant inaccuracy becomes more prominent during actual flight tests and further post-flight analysis of the flight data. In few instances, it was observed the axial force coefficient (C_{AT}) of the vehicle has suggested 8-10 % reduced drag when compared with wind-tunnel

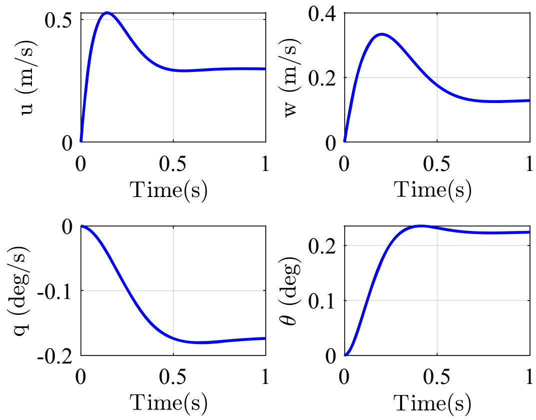

Figure 14. Optimal LQR-LQG controller response to unit step command.

Table 4. Performance comparison of LQG configuration-5 with contemporary literature.

Controller	Con-5 t_s (s)	Ref [4]	Ref [7]	Ref [14]
LQR	0.360	> 2.0	1.82	> 0.5
LQR - LQG	0.577	> 40.0	8.47	> 5.0

data. Thus, this aspect of the design has been addressed through efficient optimal design technique without overburdening the actuator requirements.

As mentioned in table 3, the constraints of turning rate time constant T_α on this missile application calls for quick transient response characteristics. Performance metrics of transient response,viz., the settling time (t_s) has been presented in comparison with other case studies of existing literature is shown in table 4. It can be observed that the LQR-LQG controller in this framework exhibits potential to greatly improve the system performance.

5. Conclusion

In this paper, a novel method of designing a LQR-LQG based controller has been carried out which provides optimal performance in the presence of flight disturbance and noise. The implementation of the controller has been extensively tested for a typical high maneuvering missile application. It has been shown that despite having a relatively inaccurate plant model, the design methodology introduced takes care of all possible flight disturbances. The tuning matrices Q and R are refined to take into account the flight disturbances and inherent noise. The results indicated the design based on the LQR-LQG is superior in all aspects compared to the earlier work based on classical PID controller. The use of state-space formulation has helped in an intuitive understanding of the missile dynamics which resulted in an optimal solution catering to all possible flight disturbances.

The authors believe, through this study of extensive numerical simulations, considering all possible flight perturbations, led to a design of a modern control method that is practically implementable and replaces the popular PID-based controller. The controller performance shows that the algorithm is much robust in handling all flight disturbances compared to all traditional gain scheduling algorithms.

The work carried out in this study showed that modern control methods using LQR-LQG can be successfully applied in the design of agile missile controllers by judicious selection of system cost matrices and estimator design to tackle the system noise and external disturbances. The success of this design methodology has brought out the applicability of modern control methods in the design of real-world, practical state-of-the-art missile control design. Further research areas should focus on robust procedures, improving the stability margins utilizing other sensor outputs to estimate other states to improve the estimator design, and controller design of lateral dynamics.

List of symbols

X, Y, Z	Forces in x, y and z directions
m	Mass of the missile
ϕ	Roll angle
α	Angle of attack
I_{xx}, I_{yy}, I_{zz}	Moment of inertia about x, y and z axes
u, v, w	Components of missile velocity three axes
p, q, r	Roll rate, Pitch rate and Yaw rate
θ	Pitch angle
δ_{pitch}	Pitch control deflection
L, M, N	Rolling moment, Pitch moment, Yawing moment
$x(t)$	State vector
$u(t)$	Control input vector
A, B, C, D	Plant matrix, input matrix, output matrix and feed-forward matrix
C_{AT}	Axial force coefficient
C_N	Normal force coefficient
IMU	Inertial Measurement Unit

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