2 Fourier series

2.1 Introduction

We will start off with some motivation for our study. Consider a thin metal bar of length 1 which is maintained at zero temperature at the ends $x = 0$ and $x = 1$. At time $t = 0$ the temperature $T(x)$ of the bar is measured - suppose the temperature distribution is $f(x)$. We wish to find the subsequent temperature of the bar, i.e. find $T(x,t)$ for $t > 0$ and $0 < x < 1$.

It can be shown that the temperature satisfies the **heat conduction equation**

$$\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2}.$$

(We will study this **partial differential equation** in the final part of the course). The constant κ is related to the properties of the bar. Let's consider the boundary and initial conditions that need to be applied. Since the heat equation is second order in space and first order in time we would expect to need two space (boundary) conditions and one time (initial) condition. As mentioned above, the temperature is fixed at the ends $x = 0$ and $x = 1$ so this means

$$T(0,t) = T(1,t) = 0 \text{ for all } t.$$

The initial condition imposes the initial shape of the temperature distribution, so we have

$$T(x,0) = f(x), \quad (0 \le x \le 1),$$

with $f(x)$ known. A possible solution to this problem that satisfies both the equation and the two boundary conditions is

$$T_n = b_n \sin(n\pi x) e^{-\kappa n^2 \pi^2 t}$$

where b_n is a constant and n is any integer. (You should check that the form for T_n given here satisfies the heat conduction equation). Each individual T_n is known as a **mode**. Since the heat equation is linear we can take a linear combination of these solutions to form a 'general' solution

$$T(x,t) = \sum_{n=1}^{\infty} b_n \sin(n\pi x) e^{-\kappa n^2 \pi^2 t},$$

where we have assumed that the form of the b_n's is such that the infinite series converges. This solution has to be consistent with the initial temperature distribution we imposed, and so applying the initial condition:

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(n\pi x), \quad (0 \le x \le 1). \tag{1}$$

We conclude that 'any' function $f(x)$ can be written as a superposition of trigonometric oscillations (in this case just sine waves) of appropriate amplitudes (the b_n) and 'frequencies' $n\pi$. The right hand side above is known as the Fourier series of $f(x)$ (in this case we actually have what is known as a 'half-range' series - we will come back to this later).

The main problem with (1) is it is by no means obvious how we obtain the so-called Fourier coefficients b_n for a given shape function $f(x)$, i.e. can we make b_n the subject of the expression (1)? The answer is yes, and in a surprisingly simple way, but first of all we need to introduce some ideas and definitions.

2.2 Orthonormal systems

A sequence of integrable functions $\{\phi_i\}_{i=1}^{\infty}$ on an interval $[a,b]$ is called **orthogonal** if

$$\int_a^b \phi_i \phi_j \, dx = 0 \text{ for } i \ne j.$$

If, in addition

$$\int_a^b \phi_i^2 \, dx = 1 \text{ for all } i,$$

the system is said to be orthonormal.

Example

The functions

$$\frac{1}{\sqrt{2\pi}},\ \ \frac{1}{\sqrt{\pi}}\cos nx,\ \ \frac{1}{\sqrt{\pi}}\sin nx,\ \ (n=1,2,\ldots)$$

form an orthonormal system over the interval $[-\pi, \pi]$.

To see this we consider the individual cases. First, if we consider the orthogonality we see that

$$\begin{aligned}
\int_{-\pi}^{\pi} \frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{\pi}}\cos nx\, dx &= \int_{-\pi}^{\pi} \frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{\pi}}\sin nx\, dx = 0,\\
\int_{-\pi}^{\pi} \frac{1}{\sqrt{\pi}}\cos nx\,\frac{1}{\sqrt{\pi}}\cos mx\, dx &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos[(n-m)x] + \cos[(n+m)x]\, dx = 0,\ \ (n \neq m)\\
\int_{-\pi}^{\pi} \frac{1}{\sqrt{\pi}}\cos nx\,\frac{1}{\sqrt{\pi}}\sin mx\, dx &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \sin[(n+m)x] - \sin[(n-m)x]\, dx = 0,\ \ (n \neq m)\\
\int_{-\pi}^{\pi} \frac{1}{\sqrt{\pi}}\sin nx\,\frac{1}{\sqrt{\pi}}\sin mx\, dx &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos[(n-m)x] - \cos[(n+m)x]\, dx = 0,\ \ (n \neq m).
\end{aligned}$$

Now for the orthonormal part:

$$\begin{aligned}
\int_{-\pi}^{\pi} (\frac{1}{\sqrt{2\pi}})^2 dx &= 1,\\
\int_{-\pi}^{\pi} (\frac{1}{\sqrt{\pi}}\cos nx)^2\, dx &= \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{1}{2} + \frac{1}{2}\cos(2nx)\, dx = \frac{1}{2\pi}\left[x + \frac{1}{2n}\sin(2nx)\right]_{-\pi}^{\pi} = 1,\\
\int_{-\pi}^{\pi} (\frac{1}{\sqrt{\pi}}\sin nx)^2\, dx &= \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{1}{2} - \frac{1}{2}\cos(2nx)\, dx = \frac{1}{2\pi}\left[x - \frac{1}{2n}\sin(2nx)\right]_{-\pi}^{\pi} = 1.
\end{aligned}$$

Thus the system is shown to be orthonormal. We can write two of the key results as

$$\int_{-\pi}^{\pi} \cos mx \cos nx\, dx = \int_{-\pi}^{\pi} \sin mx \sin nx\, dx = \pi\delta_{mn},$$

where δ_{mn} is the Kronecker delta introduced in Section 1 of the course.

Exercise

Show that the functions $(\sqrt{2/\pi})\sin nx, n = 1, 2, \ldots$ form an orthonormal system on the interval $[0, \pi]$.

2.3 Periodic functions

A function f is periodic with period T if

$$f(x+T) = f(x)\ \text{ for all values of } x.$$

Obviously $2T, 3T, \ldots$ will also satisfy this condition, but the period is defined to be the smallest value of T.

Example

We have

$$\begin{aligned}
\sin nx &= \sin(nx + 2\pi) \equiv \sin[n(x + 2\pi/n)],\\
\cos nx &= \cos(nx + 2\pi) \equiv \cos[n(x + 2\pi/n)],
\end{aligned}$$

and so $\sin nx, \cos nx$ both have period $2\pi/n$. Note that the period gets shorter as the frequency n increases.

Now consider the finite sum

$$\sum_{n=1}^{N} a_n \cos nx + b_n \sin nx.$$

This is a sum of functions with individual periods $2\pi, 2\pi/2, 2\pi/3, 2\pi/4$ etc. The overall period is therefore 2π, i.e. determined by the $n = 1$ mode.

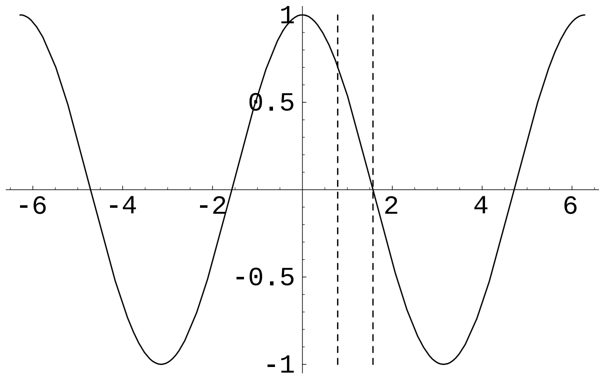

Figure 1: The function $\cos x$ and the lines $x = \pi/4, x = \pi/2$.

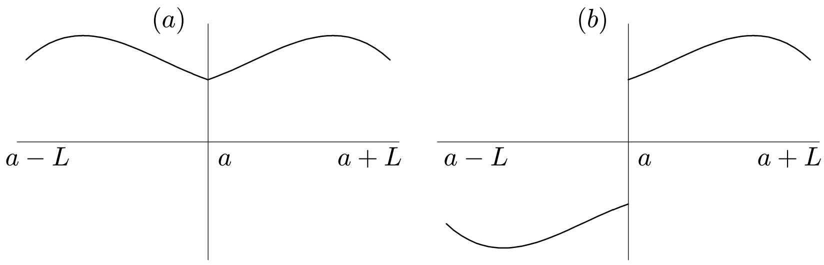

Figure 2: (a) A function which is even about $x = a$; (b) a function which is odd about $x = a$.

Exercise

Find the period of the function

$$\cos 3x \sin 4x.$$

2.4 Odd and even functions

These are defined as follows.

A function $f(x)$ is **even** about $x = a$ if $f(a + x) = f(a - x)$ for all x,
A function $f(x)$ is **odd** about $x = a$ if $f(a + x) = -f(a - x)$ for all x.

Example

Since $\cos(x) = \cos(-x)$ and $\sin(x) = -\sin(-x)$, we have that $\cos x$ is even about $x = 0$ and $\sin x$ is odd about $x = 0$. Note that if we consider $a \neq 0$, these properties change. For example $\cos x$ is neither odd nor even about $x = \pi/4$, and $\cos x$ is *odd* about $x = \pi/2$ (see figure 1).

2.4.1 Multiplication of odd and even functions

It follows from the definition of odd and even functions, that if $f(x)$ is odd (even) about $x = a$ and $g(x)$ is even (odd) about $x = a$ then the product $f(x)g(x)$ is odd. Similarly the product of two odd functions is even and the product of two even functions is even, (n.b. odd $\times$ odd is even, odd $\times$ even is odd, unlike for odd and even numbers!)

2.4.2 Integration of odd/even functions

A useful property of odd and even functions that we will make use of is what happens if you integrate them over a range, with the midpoint of the range being the line of symmetry.

If we suppose $f(x)$ is even about $x = a$ then

$$\begin{aligned}\int_{a-L}^{a+L} f(x)\,dx &= \int_{a-L}^{a} f(x)\,dx + \int_{a}^{a+L} f(x)\,dx\\ &= \int_{-L}^{0} f(u+a)\,du + \int_{a}^{a+L} f(x)\,dx\\ &= \int_{-L}^{0} f(a-u)\,du + \int_{a}^{a+L} f(x)\,dx\\ &= -\int_{a+L}^{a} f(x)\,dx + \int_{a}^{a+L} f(x)\,dx\\ &= 2\int_{a}^{a+L} f(x)\,dx,\end{aligned}$$

which is obvious if you think of integration as the area under the curve (see figure 2a). Similarly if $g(x)$ is odd about $x = a$ then

$$\int_{a-L}^{a+L} g(x)\,dx = 0,$$

because the areas under the curve between $a - L$ & a, and a & $a + L$ are equal and opposite (figure 2b). We will use these properties when studying half-range Fourier series later.

2.5 Full-range Fourier series

We begin by considering Fourier series over the range $[-\pi, \pi]$. We will generalize this later. Let $f(x)$ be a periodic function with period 2π. The Fourier series for $f(x)$ is the representation of $f(x)$ as a series in $\sin nx$ and $\cos nx$ of the form

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\{a_n \cos nx + b_n \sin nx\}, \tag{2}$$

where a_n $(n > 0)$ and b_n $(n \geq 1)$ are constants to be found.

2.5.1 Finding the Fourier coefficients

We will assume that the series may be integrated term by term, i.e. that integration and summation are interchangeable. First, if we integrate both sides of (2) from $x = -\pi$ to $x = +\pi$ we have

$$\int_{-\pi}^{\pi} f(x)\,dx = \pi a_0$$

$$\Rightarrow \underline{a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx.} \tag{3}$$

Now, if we go back to (2) but this time multiply by $\cos mx$ and integrate from $-\pi$ to $+\pi$. The first term on the right-hand-side is

$$\frac{1}{2}a_0\int_{-\pi}^{\pi} \cos mx\,dx = 0.$$

The second term is

$$\sum_{n=1}^{\infty} a_n \int_{-\pi}^{\pi} \cos mx \cos nx\,dx = \sum_{n=1}^{\infty} a_n \pi \delta_{mn} = \pi a_m,$$

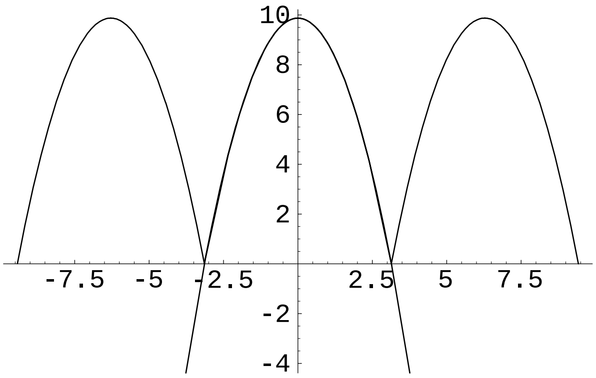

Figure 3: The function $\pi^2 - x^2$ and its Fourier series representation.

using the result for the integral calculated earlier in section 2.2. Similarly the third term is

$$\sum_{n=1}^{\infty} b_n \int_{-\pi}^{\pi} \cos mx \sin nx \, dx = 0.$$

We are thus left with the result that

$$\underline{a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \, \cos nx \, dx, \quad (n = 1, 2, \ldots).} \tag{4}$$

Note that, in view of (3), we can also use (4) for $n = 0$ (provided we write $\frac{1}{2}a_0$ in (2)). We can find the b_n coefficients in a similar way. We go back to (2) and multiply by $\sin mx$ and integrate. This time the first two terms on the right-hand-side vanish, leaving

$$\int_{-\pi}^{\pi} f(x) \, \sin mx \, dx = \sum_{n=1}^{\infty} b_n \int_{-\pi}^{\pi} \sin mx \sin nx \, dx = \sum_{n=1}^{\infty} b_n \pi \delta_{mn} = \pi b_m,$$

and hence

$$\underline{b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin \, nx \, dx, \quad (n = 1, 2, \ldots).} \tag{5}$$

Note that if $f(x)$ is even about $x = 0$ then the integrand in (5) is odd and hence $b_n = 0$. Similarly if $f(x)$ is odd about $x = 0$ then $a_n = 0$ (including a_0). These results can be quoted when appropriate.

It can be seen that we only require $f(x)$ to be integrable over the range $-\pi$ to π. This means that f could be discontinuous and still have a Fourier series representation. In fact all we require is that it is piecewise continuous, i.e. it has a finite number of finite discontinuities. It is also important to realize that provided you only want to represent a function by a Fourier series over a specific range, the function itself need not be periodic as the next example shows.

Example

Obtain the Fourier series for the function $\pi^2 - x^2$ over the range $-\pi \le x \le \pi$.

We write

$$\pi^2 - x^2 = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} \{a_n \cos nx + b_n \sin nx\},$$

and note that the right-hand side is a periodic function of period 2π. Using the formulae for the Fourier coefficients derived above:

$$\begin{aligned} a_0 &= \frac{1}{\pi} \int_{-\pi}^{\pi} (\pi^2 - x^2) \, dx = \frac{4}{3}\pi^2, \\ a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} (\pi^2 - x^2) \cos nx \, dx = \cdots = -\frac{2}{\pi n^2} [x \cos nx]_{-\pi}^{\pi} = (-1)^{n+1} \frac{4}{n^2}, \end{aligned}$$

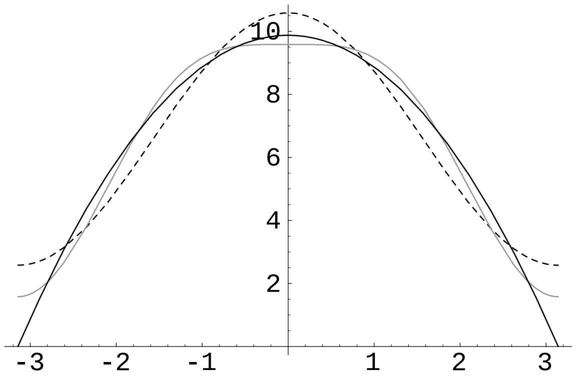

Figure 4: The solid line is $\pi^2 - x^2$, the dashed line is the Fourier series with 2 terms, and the grey line is the Fourier series with 3 terms.

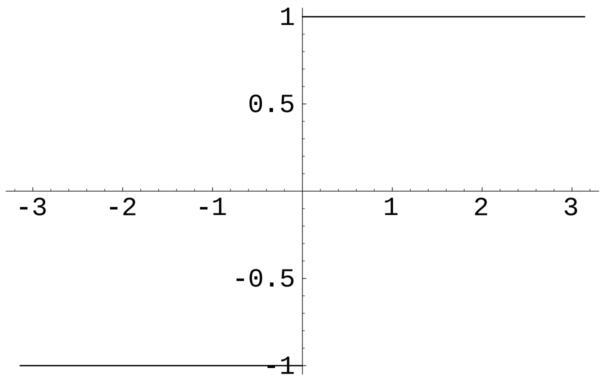

Figure 5: The rectangular-wave function.

where we have integrated by parts twice, and used that $\sin n\pi = 0$ and $\cos n\pi = (-1)^n$. Since $\pi^2 - x^2$ is even about $x = 0$ we can say without further calculation that $b_n = 0$ for all n. We therefore have the result that

$$\pi^2 - x^2 = \frac{2}{3}\pi^2 + 4\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\cos nx}{n^2}, \quad (-\pi \le x \le \pi).$$

We can see that the infinite series converges absolutely by comparison with $\sum_{n=1}^{\infty}(1/n^2)$. In figures 3,4 we show the original parabola, the Fourier series with an infinite number of terms, and approximations resulting from retaining a finite number of terms - these are known as 'partial sums'. Although the original function is a smooth parabola, the Fourier series, when extended outside $[-\pi, \pi]$ is not smooth, as it is of course a periodic extension of its form within $[-\pi, \pi]$.

Exercise (non-smooth function)

Obtain the Fourier series for the function $f(x) = |x|$ over the range $[-\pi, \pi]$.

Example (discontinuous function)

Derive the Fourier series for the rectangular wave

$$f(x) = \begin{cases} -1, & (-\pi < x < 0), \\ +1, & (0 < x < \pi). \end{cases}$$

The function is sketched in figure 5. First we observe that $f(x)$ is odd about $x = 0$ so that we have $a_n = 0$ for all n, and we can write

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx,$$

where

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \\ &= \frac{2}{\pi}\int_{0}^{\pi} (1)\,\sin nx\,dx, \end{aligned}$$

since the integrand is even about $x = 0$. It follows that

$$\begin{aligned} b_n &= \frac{2}{n\pi}((-1)^{n+1}+1) \\ &= \begin{cases} 0 & (n \text{ even}), \\ 4/n\pi & (n \text{ odd}). \end{cases} \end{aligned}$$

Thus we can express the rectangular wave as

$$f(x) = \frac{4}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots\right), \quad (-\pi < x < \pi). \tag{6}$$

2.5.2 Calculating infinite sums using Fourier series

We can obtain interesting results by substituting in various values of x. For example if we consider the rectangular wave example and choose $x = \pi/2$, we know that $f(x) = 1$ from the original definition of x, and hence, putting $x = \pi/2$ into the series:

$$1 = \frac{4}{\pi}\left(\sin\frac{\pi}{2} + \frac{1}{3}\sin\frac{3\pi}{2} + \frac{1}{5}\sin\frac{5\pi}{2} + \cdots\right).$$

Rearranging, we conclude:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots.$$

(We can check that the series on the right converges by using the alternating series test).

In the parabola example we can substitute $x = 0$ and $x = \pi$ and obtain the results

$$\begin{aligned} \frac{\pi^2}{12} &= 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots, \\ \frac{\pi^2}{6} &= 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots. \end{aligned}$$

2.5.3 The behaviour of Fourier series at a discontinuity

In the previous example, the rectangular wave is discontinuous at $x = 0$ with $\lim_{x\to 0+} f(x) = 1$ and $\lim_{x\to 0-} f(x) = -1$. We see, from (6), that the value of the Fourier series is zero, i.e. the average of the right hand and left hand limits. This result is true in general as we shall see shortly. The Fourier series (6) also has the value zero at $x = \pm\pi$. At these values of x there is a discontinuity in the periodically-extended function. Another issue concerns how the partial sums converge to the mean value. In figure 6 we plot the partial sums for (6) with 1, 2 and 10 terms retained in the series. There is also an animation of this I will show. It can be observed that the partial sums do not converge smoothly to the mean value at a point of discontinuity. This is known as **Gibbs phenomenon,** which we will discuss in more detail shortly.

2.6 Convergence of Fourier series

We would like to study what happens to the Fourier series for $f(x)$ as we include more and more terms. How do we approach the function $f(x)$ and what happens if $f(x)$ is discontinuous as in the previous example? To do this we need to use the following theorem, which we quote without proof.

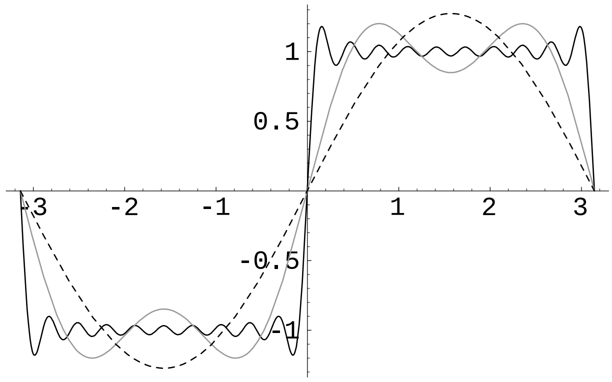

Figure 6: The partial sums of the Fourier series for the rectangular wave. Dotted line: 1 term in series; grey line: 2 terms in series; solid line: 10 terms in series.

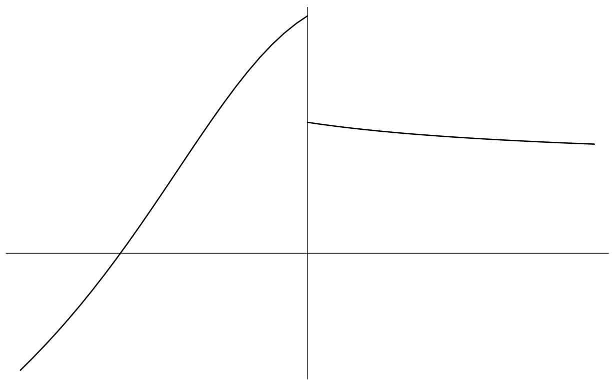

Figure 7: A discontinuous function with finite left and right hand derivatives at the point of discontinuity.

2.6.1 Riemann's Lemma

If $g(x)$ is an integrable function in the interval $[a,b]$, then

$$\lim_{\alpha\to\infty}\int_a^b g(x)\sin\alpha x\,dx = 0.$$

2.6.2 Behaviour of the partial sums

We now consider the partial sums of a given Fourier series, $S_N(x)$, defined over the range $-\pi < x < \pi$ as

$$S_N(x) = \frac{1}{2}a_0 + \sum_{n=1}^{N}\{a_n\cos nx + b_n\sin nx\},$$

$$\text{with } a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(u)\cos nu\,du, \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(u)\sin nu\,du.$$

We will suppose that the function $f(x)$ may have finite discontinuities at certain values of x. At such discontinuities we define the left-hand and right-hand limits as

$$f(x-) = \lim_{\varepsilon\to 0-} f(x+\varepsilon), \quad f(x+) = \lim_{\varepsilon\to 0+} f(x+\varepsilon).$$

(Of course, if there is no discontinuity then these limits are equal). We will also assume that the left-hand and right-hand derivatives exist at these points (see figure 7). Turning now to the expression for S_N, substituting for the coefficients, interchanging the order of summation and the integration, and using $\cos nx \cos nu + \sin nx \sin nu \equiv \cos n(u-x)$ we get

$$S_N(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(u)\{\frac{1}{2} + \sum_{n=1}^{N} \cos n(u-x)\}\, du.$$

It can be shown that

$$1 + 2\sum_{n=1}^{N} \cos n\theta = \sin[(N+1/2)\theta]/\sin(\theta/2).$$

[Problem Sheet 5]. We can therefore rewrite our expression for the partial sum as

$$S_N(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(u)\frac{\sin[(N+\frac{1}{2})(u-x)]}{\sin[\frac{1}{2}(u-x)]}\, du = \frac{1}{2\pi}\int_{-\pi-x}^{\pi-x} f(x+s)\frac{\sin[(N+\frac{1}{2})s]}{\sin\frac{1}{2}s}\, ds,$$

upon making the substitution $s = u - x$. We want to investigate the behaviour of $S_N(x)$ as $N \to \infty$. We can't use Riemann's lemma directly because $g(s) = f(x+s)/\sin\frac{1}{2}s$ is not integrable over a range containing $s = 0$. We will therefore split the range of integration to deal with the problem near $s = 0$ separately, and we also let $\alpha = N + 1/2$. We write

$$S_N(x) = \frac{1}{2\pi}\left(\int_{-\pi-x}^{-\delta} + \int_{-\delta}^{0} + \int_{0}^{\delta} + \int_{\delta}^{\pi-x}\right)\left\{f(x+s)\frac{\sin\alpha s}{\sin\frac{1}{2}s}\right\} ds \equiv I_1 + I_2 + I_3 + I_4$$

with $\delta > 0$. The integrals I_1 and I_4 tend to zero by Riemann's lemma. Now consider the second integral:

$$I_2 = \frac{1}{2\pi}\int_{-\delta}^{0} f(x+s)\frac{\sin\alpha s}{\sin\frac{1}{2}s} ds \equiv \frac{1}{\pi}\int_{-\delta}^{0} f(x+s)\frac{\sin\alpha s}{s}\, ds + \frac{1}{2\pi}\int_{-\delta}^{0} f(x+s)\left\{\frac{1}{\sin\frac{1}{2}s} - \frac{2}{s}\right\}\sin\alpha s\, ds. \quad (7)$$

The second integral on the right hand side in (7) tends to zero as $\alpha \to \infty$ by Riemann's lemma, as the term in curly brackets is well-behaved at $s = 0$. Turning to the first integral on the right-hand-side of (7):

$$\frac{1}{\pi}\int_{-\delta}^{0} f(x+s)\frac{\sin\alpha s}{s}\, ds \equiv \frac{1}{\pi}\int_{-\delta}^{0}\left\{\frac{f(x+s) - f(x-)}{s}\right\}\sin\alpha s\, ds + \frac{f(x-)}{\pi}\int_{-\delta}^{0}\frac{\sin\alpha s}{s}\, ds \quad (8)$$

Again, the term in curly brackets is well-behaved at $s = 0$ because as $s \to 0-$ this term is simply the left-hand derivative which we are assuming is well-defined. Therefore the first term on the right-hand-side of (8) tends to zero by Riemann's lemma. We are therefore left with

$$\lim_{\alpha\to\infty} I_2 = \frac{f(x-)}{\pi}\lim_{\alpha\to\infty}\int_{-\delta}^{0}\frac{\sin\alpha s}{s}\, ds = \frac{f(x-)}{\pi}\int_{0}^{\infty}\frac{\sin q}{q}\, dq = \frac{1}{2}f(x-),$$

since the standard integral $\int_0^\infty (\sin q)/q\, dq = \pi/2$. [Problem sheet 6]. In a similar way we can show that

$$\lim_{\alpha\to\infty} I_3 = \frac{1}{2}f(x+).$$

Putting all this together:

$$\lim_{N\to\infty} S_N(x) = \frac{1}{2}[f(x+) + f(x-)].$$

We have therefore shown that at a point of discontinuity the Fourier series converges to the average value of f, which is what we saw explicitly in the rectangular wave example.

2.6.3 Gibbs phenomenon

From our computations of the Fourier series for the rectangular wave (figure 6) we saw that the series behaves strangely near $x = 0$, overshooting the required jump by the same amount for all N. Let's examine the specific series in more detail. Recall that the Fourier series in question is

$$f(x) = \frac{4}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots\right), \quad (-\pi < x < \pi).$$

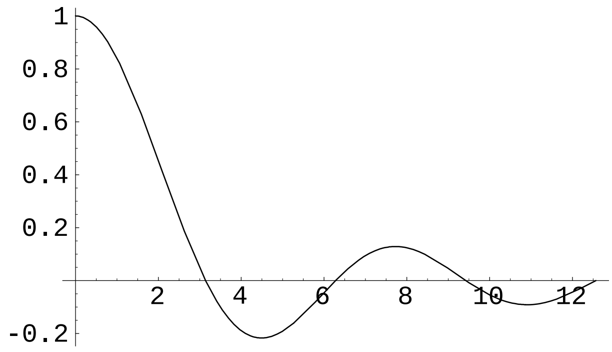

Figure 8: A plot of $\sin(x)/x$ for $x > 0$.

The sum of the first N terms is

$$S_N(x) = \frac{4}{\pi}\sum_{n=1}^{N}\frac{\sin[(2n-1)x]}{2n-1}$$

and this can be rewritten in the form

$$S_N(x) = \frac{2}{\pi}\int_0^x \frac{\sin 2N\theta}{\sin\theta}\,d\theta$$

(see problem Sheet 5). Then

$$S_N(x) = \frac{2}{\pi}\int_0^x \left\{\frac{1}{\sin\theta} - \frac{1}{\theta}\right\}\sin 2N\theta\,d\theta + \frac{2}{\pi}\int_0^x \frac{\sin 2N\theta}{\theta}\,d\theta.$$

The first term tends to zero as $N \to \infty$ by Riemann's lemma. Making the substitution $\xi = 2N\theta$ in the second integral:

$$S_N(x) \sim \frac{2}{\pi}\int_0^{2Nx}\frac{\sin\xi}{\xi}\,d\xi \text{ as } N \to \infty.$$

If we fix $x > 0$ and let $N \to \infty$ we see that $S_N(x) \to 1$ (recall $\int_0^\infty (\sin x)/x\,dx = \pi/2$). This is what we expect since the Fourier series is representing the function $f(x) = 1$ when $x > 0$. Similarly if we fix $x < 0$ and let $N \to \infty$ we get $S_N(x) \to -1$. Now let's find the value of x at which $S_N(x)$ is maximum. If we think about the area under the curve $(\sin\xi)/\xi$ (see figure 8) we see that the maximum value of the integral occurs when $2Nx = \pi$. At this point we can compute

$$\int_0^\pi (\sin x)/x\,dx \simeq 1.852 \text{ and so } S_N(\pi/2N) \simeq 1.179,$$

a value which is *independent* of N. This means there is always an overshoot (by a factor 1.179) but that overshoot occurs closer and closer to the discontinuity at $x = 0$ as we increase the number of terms in the Fourier series.

2.7 Parseval's theorem

If $f(x)$ is represented by a Fourier series as

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\{a_n\cos nx + b_n\sin nx\}, \quad (-\pi \le x \le \pi),$$

then

$$\frac{1}{\pi}\int_{-\pi}^{\pi}[f(x)]^2 dx = \frac{1}{2}a_0^2 + \sum_{n=1}^{\infty}\{a_n^2 + b_n^2\}.$$

Proof

We write $[f(x)]^2$ as

$$[f(x)]^2 = \left[\frac{1}{2}a_0 + \sum_{n=1}^{\infty}\{a_n \cos nx + b_n \sin nx\}\right]\left[\frac{1}{2}a_0 + \sum_{m=1}^{\infty}\{a_m \cos mx + b_m \sin mx\}\right],$$

noting the m index in the second factor. Expanding this out we get

$$[f(x)]^2 = \frac{1}{4}a_0^2 + \frac{1}{2}a_0\sum_{n=1}^{\infty}\{a_n \cos nx + b_n \sin nx\} + \frac{1}{2}a_0\sum_{m=1}^{\infty}\{a_m \cos mx + b_m \sin mx\}$$

$$+\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(a_n a_m \cos nx \cos mx + a_n b_m \cos nx \sin mx + b_n a_m \sin nx \cos mx + b_n b_m \sin nx \sin mx).$$

Now we integrate both sides of this expression with respect to x between $-\pi$ and π. Most of the terms integrate to zero, leaving

$$\begin{aligned}\int_{-\pi}^{\pi}[f(x)]^2 dx &= \frac{\pi}{2}a_0^2 + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(a_n a_m \pi\delta_{nm} + b_n b_m \pi\delta_{mn}) \\ &= \frac{\pi}{2}a_0^2 + \sum_{n=1}^{\infty}\pi(a_n^2 + b_n^2).\end{aligned}$$

Hence result.

Example

Compute the Fourier series for $\cos(x/2)$ over $[-\pi, \pi]$. Use Parseval's theorem to deduce the value of

$$\sum_{n=1}^{\infty}\frac{1}{(4n^2-1)^2}.$$

First we observe that $\cos(x/2)$ is even about $x = 0$ and hence $b_n = 0$ for all n. We can therefore write

$$\cos\left(\frac{x}{2}\right) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}a_n \cos nx, \quad (-\pi \le x \le \pi).$$

The Fourier coefficients are.

$$\begin{aligned}a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi}\cos(x/2)\,dx = \frac{2}{\pi}\int_0^{\pi}\cos(x/2)\,dx \\ &= \frac{2}{\pi}\left[2\sin(x/2)\right]_0^{\pi} = \frac{4}{\pi},\end{aligned}$$

and

$$\begin{aligned}a_n &= \frac{2}{\pi}\int_0^{\pi}\cos(x/2)\cos nx\,dx \\ &= \frac{1}{\pi}\int_0^{\pi}\cos[(n+1/2)x] + \cos[(n-1/2)x]\,dx \\ &= \frac{1}{\pi}\left(\frac{\sin(\pi/2+n\pi)}{(n+1/2)} + \frac{\sin(-\pi/2+n\pi)}{(n-1/2)}\right) \\ &= \frac{(-1)^n}{\pi}\left(\frac{2}{2n+1} - \frac{2}{2n-1}\right) \\ &= \frac{4(-1)^{n+1}}{\pi(4n^2-1)}.\end{aligned}$$

Therefore using Parseval's theorem

$$\begin{aligned}\frac{1}{\pi}\int_{-\pi}^{\pi}[\cos(x/2)]^2\,dx &= \frac{1}{2}a_0^2+\sum_{n=1}^{\infty}a_n^2\\ &= \frac{8}{\pi^2}+\frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(4n^2-1)^2}.\end{aligned}$$

But the left-hand-side is

$$\frac{1}{\pi}\int_{-\pi}^{\pi}\left(\frac{1}{2}+\frac{1}{2}\cos x\right)\,dx=1.$$

Therefore we conclude that

$$\frac{8}{\pi^2}+\frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(4n^2-1)^2}=1,$$

and hence

$$\sum_{n=1}^{\infty}\frac{1}{(4n^2-1)^2}=\frac{\pi^2-8}{16}.$$

2.8 Fourier series over a general interval

The use of Fourier series would obviously be limited if it were confined to the range $[-\pi,\pi]$. However the theory is easily generalized to an arbitrary interval by observing that $\sin(n\pi x/L),\cos(n\pi x/L)$ have period $2L/n$ and the set of functions

$$\frac{1}{\sqrt{2L}},\ \frac{1}{\sqrt{L}}\cos\left(\frac{n\pi x}{L}\right),\ \frac{1}{\sqrt{L}}\sin\left(\frac{n\pi x}{L}\right),\quad (n=1,2,\ldots)$$

are orthonormal over the interval $[a,a+2L]$ where a is any real number, i.e. we have

$$\int_a^{a+2L}\cos\left(\frac{n\pi x}{L}\right)\,dx=\int_a^{a+2L}\sin\left(\frac{n\pi x}{L}\right)\,dx=0,$$

$$\int_a^{a+2L}\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)\,dx=0,$$

$$\int_a^{a+2L}\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)\,dx=\int_a^{a+2L}\cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)\,dx=L\delta_{mn}.$$

Using these results we can then represent a function $f(x)$ over the interval $[a,a+2L]$ in the form

$$f(x)=\frac{1}{2}a_0+\sum_{n=1}^{\infty}\{a_n\cos\left(\frac{n\pi x}{L}\right)+b_n\sin\left(\frac{n\pi x}{L}\right)\}.$$

By proceeding in a similar fashion to the $[-\pi,\pi]$ case we can establish that the corresponding Fourier coefficients are given by

$$\begin{aligned}a_n &= \frac{1}{L}\int_a^{a+2L}f(x)\cos\left(\frac{n\pi x}{L}\right)\,dx,\quad n=0,1,2,\ldots,\\ b_n &= \frac{1}{L}\int_a^{a+2L}f(x)\sin\left(\frac{n\pi x}{L}\right)\,dx,\quad n=1,2,\ldots.\end{aligned}$$

Obviously, these formulae reduce to the $[-\pi,\pi]$ case when $a=-\pi$ and $L=\pi$. It follows that if $f(x)$ is integrable over a finite interval, a Fourier series can be found for $f(x)$ in this interval.

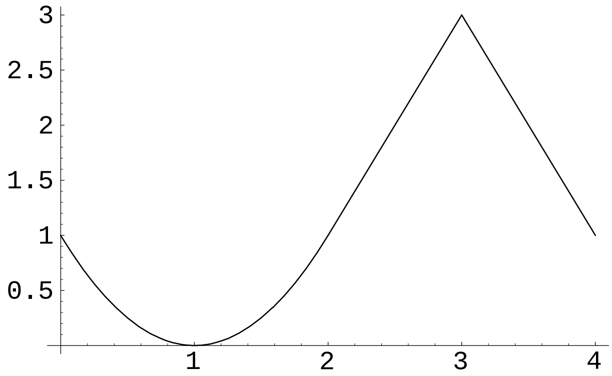

Figure 9: A function defined as $(x-1)^2$ for $0 \le x \le 2$, $2x-3$ for $2 \le x \le 3$ and $9-2x$ for $3 \le x \le 4$.

2.8.1 Parseval's theorem for Fourier series over a general interval

It's straightforward to generalize Parseval's result for the $[-\pi, \pi]$ interval to the interval $[a, a+2L]$:

$$\frac{1}{L}\int_a^{a+2L}[f(x)]^2 dx = \frac{1}{2}a_0^2 + \sum_{n=1}^{\infty}\{a_n^2 + b_n^2\}.$$

Example

Find the Fourier expansion of the periodic function whose definition over one period is

$$f(x) = \begin{cases} (x-1)^2, & (0 \le x \le 2), \\ 2x-3, & (2 \le x \le 3), \\ 9-2x, & (3 \le x \le 4). \end{cases}$$

The function is sketched in figure 9. First we see that the period $2L = 4$ in this case, and so $L = 2$. We therefore use our formulae with $a = 0$ and $L = 2$, so that

$$\begin{aligned} a_0 &= \frac{1}{2}\int_0^4 f(x)\,dx \\ &= \frac{1}{2}\int_0^2 (x-1)^2 dx + \frac{1}{2}\int_2^3 (2x-3)\,dx + \frac{1}{2}\int_3^4 (9-2x)\,dx \\ &= \cdots = 7/3. \end{aligned}$$

Similarly

$$a_n = \frac{1}{2}\int_0^4 f(x)\cos(n\pi x/2)\,dx, \quad b_n = \frac{1}{2}\int_0^4 f(x)\sin(n\pi x/2)\,dx.$$

After some algebra we obtain

$$a_n = \frac{8}{n^2\pi^2}\cos\left(\frac{3n\pi}{2}\right), \quad b_n = \frac{8}{n^3\pi^3}\left(n\pi\sin\left(\frac{3n\pi}{2}\right) - 1 + \cos(n\pi)\right).$$

2.9 Half-range Fourier series

The series we have looked at so far are known as **full-range Fourier series** in view of the fact that the function is represented by the Fourier series over one full period of the series. Suppose we have a function

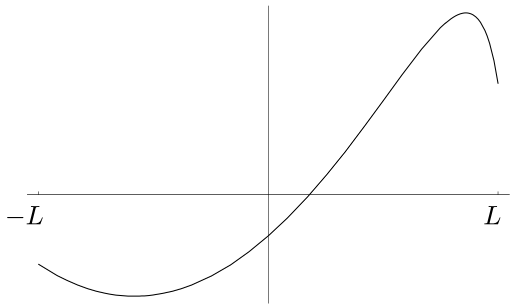

Figure 10: A function defined over $[-L, L]$.

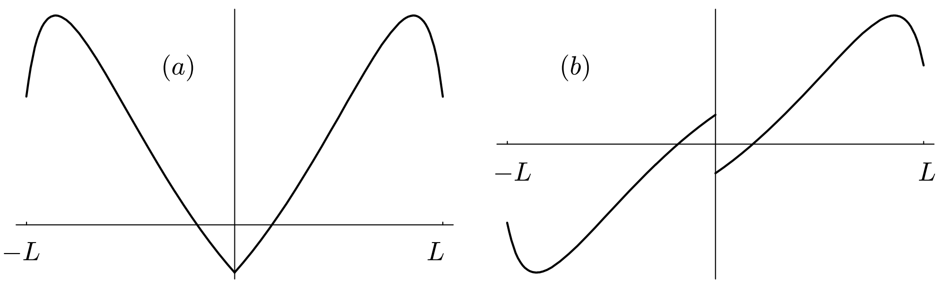

Figure 11: (a) The function $f_1(x)$; (b) the function $f_2(x)$.

$f(x)$ defined over the range $[-L, L]$ (see figure 10). Now consider the following two real functions defined over the same interval:

$$f_1(x) = \begin{cases} f(x), & (0 \le x \le L), \\ f(-x) & (-L \le x \le 0) \end{cases},$$
$$f_2(x) = \begin{cases} f(x), & (0 < x < L), \\ -f(-x) & (-L < x < 0) \end{cases}.$$

These functions are sketched in figure 11. Clearly $f_1(x)$ is even about $x = 0$ and hence the Fourier coefficients $b_n = 0$ in its Fourier series expansion over $[-L, L]$, i.e.

$$f_1(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right), \quad (-L \le x \le L)$$

with

$$\begin{aligned} a_n &= \frac{1}{L}\int_{-L}^{L} f_1(x)\cos\left(\frac{n\pi x}{L}\right) dx \\ &= \frac{2}{L}\int_{0}^{L} f_1(x)\cos\left(\frac{n\pi x}{L}\right) dx, \quad (n = 0, 1, 2, \ldots). \end{aligned}$$

with the last line following from the evenness of the integrand about $x = 0$. In contrast, the function $f_2(x)$ is odd about $x = 0$ and hence the Fourier expansion is simply

$$f_2(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right), \quad (-L < x < L),$$

(note that the interval is open, due to the oddness of $f_2(x)$ about $x = 0$). The coefficients b_n are

$$\begin{aligned} b_n &= \frac{1}{L}\int_{-L}^{L} f_2(x)\sin\left(\frac{n\pi x}{L}\right) dx \\ &= \frac{2}{L}\int_{0}^{L} f_2(x)\sin\left(\frac{n\pi x}{L}\right) dx, \quad (n = 1, 2, \ldots). \end{aligned}$$

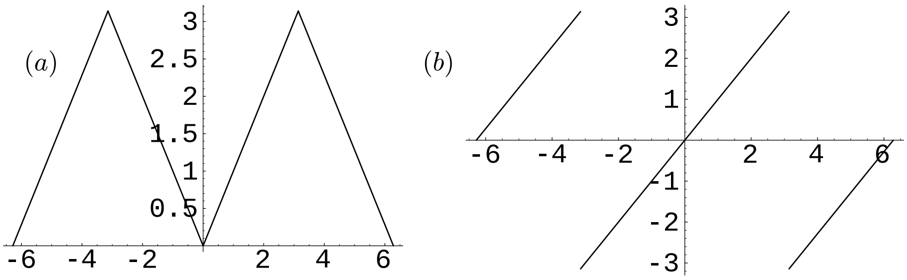

Figure 12: (a) the even extension of the function $f(x) = x$, $(0 < x < \pi)$; (b) the odd extension of the same function.

Now note that by definition, both f_1 and f_2 are equal to f over the range $(0, L)$. There are therefore two ways of representing f over this interval:

$$\begin{aligned}\text{(i) } f(x) &= \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right), \quad (0 \leq x \leq L) \\ a_n &= \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)\,dx, \quad (n = 0, 1, 2, \ldots),\end{aligned}$$

which is known as a **half-range Fourier cosine series**, and

$$\begin{aligned}\text{(ii) } f(x) &= \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right), \quad (0 < x < L) \\ b_n &= \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\,dx, \quad (n = 1, 2, \ldots),\end{aligned}$$

which is a **half-range Fourier sine series**.

2.9.1 Parseval's theorem for half-range series

Again, analogous results to Parseval's formula can be found for half-range series. These are:

$$\frac{2}{L}\int_0^L [f(x)]^2 dx = \begin{cases} \frac{1}{2}a_0^2 + \sum_{n=1}^{\infty} a_n^2, & \text{(Fourier cosine series)} \\ \sum_{n=1}^{\infty} b_n^2, & \text{(Fourier sine series).} \end{cases}$$

Example

Find the half-range Fourier cosine and Fourier sine series for $f(x) = x$ over the range $0 < x < \pi$.

Let's start with the cosine series. Since the range is 0 to π we put $L = \pi$ in our formulae to get

$$x = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx.$$

The Fourier coefficients can be calculated as

$$\begin{aligned} a_0 &= \frac{2}{\pi}\int_0^{\pi} x\,dx = \pi, \\ a_n &= \frac{2}{\pi}\int_0^{\pi} x\cos nx\,dx = \frac{2}{\pi}\left\{\left[x\frac{\sin nx}{n}\right]_0^{\pi} - \int_0^{\pi}\frac{\sin nx}{n}\,dx\right\} \\ &= \frac{2}{\pi n^2}(\cos n\pi - 1). \end{aligned}$$

Therefore we can write

$$x = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n \text{ odd}} \frac{\cos nx}{n^2},$$

or putting $n = 2m - 1$:

$$x = \frac{\pi}{2} - \frac{4}{\pi}\sum_{m=1}^{\infty} \frac{\cos[(2m-1)x]}{(2m-1)^2}, \quad (0 \leq x \leq \pi),$$

note that the end points are included, because this is the even extension of x outside the range $[0, \pi]$. See figure 12a. Now let's derive the Fourier sine series expansion. We write

$$x = \sum_{n=1}^{\infty} b_n \sin nx,$$

where

$$\begin{aligned} b_n &= \frac{2}{\pi}\int_0^{\pi} x \sin nx \, dx \\ &= \text{(by parts)} \\ &= \frac{2}{n}(-1)^{n+1}. \end{aligned}$$

Thus we have the result

$$x = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin nx. \tag{9}$$

Since this is the odd extension of x outside the range 0 to π it is generally only valid in the open interval $(0, \pi)$. However because $f(0) = 0$ in this case the representation is also clearly valid at $x = 0$, and so the region of validity of (9) is $0 \leq x < \pi$. The series is shown in figure 12b over the range $[-2\pi, 2\pi]$.

Exercise

Find the half-range Fourier sine and cosine representations of the function $f(x) = e^x$ for $0 < x < \pi$.

2.10 Integration and differentiation of Fourier series

In our previous example on half-range series we saw that

$$x = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin nx, \quad (0 \leq x < \pi). \tag{10}$$

We can integrate both sides of (10) with respect to x and get the 'correct answer' provided our range of integration lies inside the region of validity of the original series (0 to π in this case). For example, provided $0 < X < \pi$ we can write

$$\begin{aligned} \int_0^X x \, dx &= \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \int_0^X \sin nx \, dx \\ \Rightarrow \quad \frac{1}{2}X^2 &= \sum_{n=1}^{\infty} \frac{2(-1)^n}{n^2}(\cos nX - 1). \end{aligned}$$

The conditions under which we can differentiate are more restrictive. (This is not surprising since differentiating a Fourier series brings out an extra factor n, making the series much less likely to converge). To be able to differentiate we need the function, when extended periodically, to be continuous for all x. This is not true for the series in (10). If we decide to differentiate both sides anyway we get

$$1 = \sum_{n=1}^{\infty} 2(-1)^{n+1} \cos nx \quad \text{Wrong!}$$

This cannot be correct as the general term in the series does not tend to zero and hence the series must diverge. If however, we start with the half-range cosine series representation for x :

$$x = \frac{\pi}{2} - \frac{4}{\pi}\sum_{m=1}^{\infty}\frac{\cos[(2m-1)x]}{(2m-1)^2}, \quad (0 \le x \le \pi),$$

then this is continuous when extended periodically, and so we can differentiate to get

$$1 = \frac{4}{\pi}\sum_{m=1}^{\infty}\frac{\sin[(2m-1)x]}{2m-1},$$

which *is* a valid result, provided $0 < x < \pi$.

2.11 Exponential form of Fourier series

This alternative representation can sometimes simplify calculations. It is used frequently in engineering applications and writing the formulae in this way provides a clear link to Fourier transforms which we will explore in the next section of the course. Recalling that

$$\cos nx = \frac{1}{2}(e^{inx} + e^{-inx}), \quad \sin nx = -\frac{i}{2}(e^{inx} - e^{-inx}),$$

we can write

$$a_n \cos nx + b_n \sin nx = \frac{1}{2}(a_n - ib_n)e^{inx} + \frac{1}{2}(a_n + ib_n)e^{-inx}.$$

Therefore we can rewrite our Fourier series representation over $[-\pi, \pi]$ in the form

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} c_n e^{inx} + k_n e^{-inx} \tag{11}$$

where

$$\begin{aligned} c_n &= \frac{1}{2}(a_n - ib_n) = \frac{1}{2}\left(\frac{1}{\pi}\right)\int_{-\pi}^{\pi}(f(x)\cos nx - if(x)\sin nx)\,dx \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx, \quad (n = 1, 2, \ldots). \end{aligned}$$

Also note that $c_0 = a_0/2$. Similarly:

$$k_n = \frac{1}{2}(a_n + ib_n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{inx}\,dx \quad (n = 1, 2, \ldots).$$

If we write $c_{-n} = k_n$ we can express this more succinctly as

$$\begin{aligned} f(x) &= \sum_{n=-\infty}^{\infty} c_n e^{inx}, \quad |x| < \pi, \\ c_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx, \quad n = 0, \pm 1, \pm 2, \ldots. \end{aligned}$$

For a function of period $2L$ this is easily generalized to

$$\begin{aligned} f(x) &= \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/L}, \quad |x| < L, \\ c_n &= \frac{1}{2L}\int_{-L}^{L} f(x)e^{-in\pi x/L}dx, \quad n = 0, \pm 1, \pm 2, \ldots. \end{aligned}$$

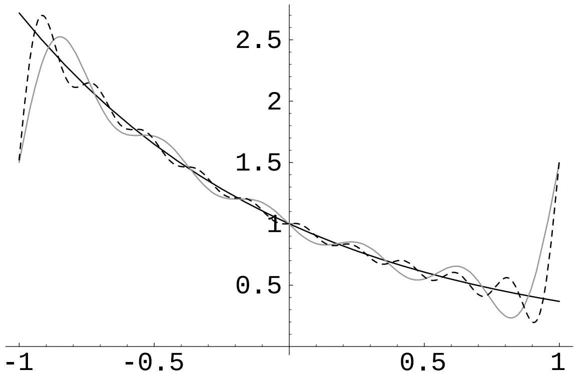

Figure 13: The solid line is the function e^{-x}, the grey line represents the partial sum of the Fourier series with 5 terms, the dotted line has 10 terms.

Example

Find the complex form of the Fourier series of the periodic function whose definition over one period is $f(x) = e^{-x}$, $-1 < x < 1$.

Since $L = 1$ we have

$$\begin{aligned} c_n &= \frac{1}{2}\int_{-1}^{1} e^{-x}e^{-in\pi x}dx \\ &= -\frac{1}{2}\left[\frac{e^{-(1+in\pi)x}}{1+in\pi}\right]_{-1}^{1} \\ &= -\frac{1}{2}\frac{\cos n\pi}{1+in\pi}(e^{-1}-e^{1}) \\ &= \frac{\cos n\pi}{1+in\pi}\sinh 1. \end{aligned}$$

Therefore the complex Fourier series representation is

$$e^{-x} = \sum_{n=-\infty}^{\infty} \frac{(-1)^n(1-in\pi)\sinh 1}{1+n^2\pi^2}e^{in\pi x}, \quad -1 < x < 1.$$

The function e^{-x} and the partial sums of the series with 5 terms and 10 terms are shown together in figure 13. Note the Gibbs phenomenon near $x = \pm 1$ where the series converges to $\frac{1}{2}(e + e^{-1}) = \cosh 1$.

3 Fourier transforms

A key reason for studying this material (and Fourier series) is that we can use these ideas to help us solve certain partial differential equations which we will study in the next section. However there are many other uses for Fourier transforms and they are used particularly by scientists and engineers in the context of signal processing.

We have seen that Fourier series allows us to represent a given function in terms of sine and cosine waves of different amplitudes and frequencies, but this representation is only valid over a finite range of the independent variable. We now wish to study what happens if we take a Fourier series defined over $[-L, L]$ and let $L \to \infty$.

First we write out the Fourier series over the finite range. It is convenient to use the exponential form derived at the end of the previous section:

$$\begin{aligned} f(x) &= \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/L}, \quad |x| < L, \\ c_n &= \frac{1}{2L}\int_{-L}^{L} f(x) e^{-in\pi x/L} dx, \quad n = 0, \pm 1, \pm 2, \ldots . \end{aligned}$$

We define the *angular frequency*

$$\omega_n = n\pi/L$$

and also the *frequency difference*

$$\delta\omega = w_{n+1} - \omega_n = \pi/L.$$

In terms of this new notation the Fourier series becomes

$$f(x) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\left\{\int_{-L}^{L} f(s)e^{-i\omega_n s} ds\right\} e^{i\omega_n x}\delta\omega.$$

Now, as $L \to \infty : \delta\omega \to 0$, and

$$\sum_{n=-\infty}^{\infty} g(\omega_n)\,\delta\omega \to \int_{-\infty}^{\infty} g(\omega)\,d\omega$$

(think about splitting the integral up into strips of width $\delta\omega$). So in the limit $L \to \infty$ we obtain the result that

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty} f(s)e^{-i\omega s} ds\right\} e^{i\omega x} d\omega.$$

We have therefore shown that for a function $f(x)$ defined over $-\infty < x < \infty$:

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\widehat{f}(\omega)e^{i\omega x}\,d\omega, \tag{12}$$

where

$$\widehat{f}(\omega) = \int_{-\infty}^{\infty} f(x)e^{-i\omega x} dx.$$

The function $\widehat{f}(\omega)$ is known as the **Fourier transform** of $f(x)$ and is analogous to the Fourier coefficients in a Fourier series but is now a function of the continuous variable ω, rather than just being computed at discrete values of n as the Fourier coefficients are. The formula (12) is known as the formula for the **inverse Fourier transform** as it enables $f(x)$ to be calculated from a knowledge of the transform function $\widehat{f}(\omega)$.

Some authors use slightly different definitions of Fourier transform (essentially different normalizations), with corresponding changes to the inversion formula. We sometimes also use the notation $\mathcal{F}\{f(x)\}$ instead (or as well as) $\widehat{f}(\omega)$. In order to evaluate the integrals above a necessary condition is that f and its transform decay at $\pm\infty$ - this restriction can be overcome if we allow the use of the Dirac delta function - see later.

Proof of Fourier's integral formula

This is the result

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty}f(s)e^{-i\omega s}ds\right\}e^{i\omega x}d\omega$$

that we arrived at in a non-rigorous way in the previous section. To prove this we need to assume that $f(x)$ is such that

$$\int_{-\infty}^{\infty}|f(x)|\ dx$$

converges. We will also assume that $f(x)$ and $f'(x)$ are continuous for all x, although this condition can be relaxed as we will discuss at the end.

We start by writing the RHS above in the form

$$\begin{aligned}
&\lim_{L\to\infty}\frac{1}{2\pi}\int_{-L}^{L}\left\{\int_{-\infty}^{\infty}f(s)e^{-i\omega s}ds\right\}e^{i\omega x}d\omega\\
\equiv\ &\lim_{L\to\infty}\frac{1}{2\pi}\int_{-L}^{L}\left\{\int_{-\infty}^{\infty}f(s)e^{-i\omega(s-x)}ds\right\}d\omega\\
\equiv\ &\lim_{L\to\infty}\frac{1}{2\pi}\int_{-L}^{L}\left[\left\{\int_{-\infty}^{\infty}f(s)\cos\left[\omega(s-x)\right]ds\right\}-i\left\{\int_{-\infty}^{\infty}f(s)\sin\left[\omega(s-x)\right]ds\right\}\right]d\omega.
\end{aligned}$$

The first integral in curly brackets is even about $\omega=0$, while the second is odd. The above expression therefore simplifies to

$$\lim_{L\to\infty}\frac{1}{\pi}\int_{0}^{L}\left\{\int_{-\infty}^{\infty}f(s)\cos\left[\omega(s-x)\right]ds\right\}d\omega.$$

Because of the absolute convergence of the inner integral we can interchange the order of integration and write this as

$$\begin{aligned}
&\lim_{L\to\infty}\frac{1}{\pi}\int_{-\infty}^{\infty}f(s)\left\{\int_{0}^{L}\cos\left[\omega(s-x)\right]\ d\omega\right\}ds\\
=\ &\lim_{L\to\infty}\frac{1}{\pi}\int_{-\infty}^{\infty}f(s)\frac{\sin[L(s-x)]}{s-x}ds\\
=\ &\lim_{L\to\infty}\frac{1}{\pi}\int_{-\infty}^{\infty}f(x+u)\frac{\sin(Lu)}{u}\ du,
\end{aligned}$$

using the substitution $u=s-x$. We now split the integral into two parts in the form

$$\lim_{L\to\infty}\frac{1}{\pi}\left\{\int_{-\infty}^{\infty}\frac{f(x+u)-f(x)}{u}\sin(Lu)\,du+f(x)\int_{-\infty}^{\infty}\frac{\sin(Lu)}{u}\ du\right\}.$$

The first integral tends to zero as $L\to\infty$ using Riemann's lemma (2.6.1). We then use the substitution $p=Lu$ in the second integral to leave

$$\lim_{L\to\infty}\frac{f(x)}{\pi}\int_{-\infty}^{\infty}\frac{\sin p}{p}dp=f(x),$$

using the fact that $\int_{-\infty}^{\infty}(\sin p)/p\,dp=\pi$ (Problem Sheet 6). **We have therefore proved Fourier's integral formula**. As remarked earlier, we have assumed here that $f(x)$ is continuous at all x. If there is a discontinuity at x_0 (with finite left and right hand derivatives there), the LHS of the formula is replaced by $[f(x_0+)+f(x_0-)]/2$ (analogous to the Fourier series convergence we investigated earlier).

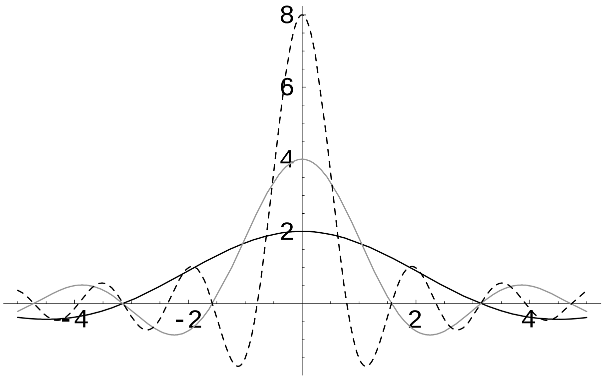

Figure 14: Graph of $(2/\omega)\sin(\omega d)$ for $d = 1\,(\text{black}), 2\,(\text{grey})$ and $4\,(\text{dotted})$.

Example

Find the Fourier transform of the rectangular wave

$$f(x) = \begin{cases} 1, & |x| < d, \\ 0, & |x| > d. \end{cases}$$

We have that

$$\begin{aligned} \widehat{f}(\omega) &= \int_{-d}^{d} 1.e^{-i\omega x} dx = \left[\frac{e^{-i\omega x}}{-i\omega}\right]_{-d}^{d} = -\frac{1}{i\omega}(e^{-i\omega d} - e^{i\omega d}) \\ &= \frac{2}{\omega}\sin\omega d. \end{aligned}$$

The graph is shown in figure 14 for $d = 1, 2, 4$. Note that as d gets larger, $\widehat{f}$ becomes more concentrated in the vicinity of $\omega = 0$.

Exercise

Find the Fourier transform of the function

$$f(x) = \begin{cases} e^{-ax}, & x \geqslant 0, \\ 0, & x < 0, \end{cases}$$

where a is a positive constant.

3.1 Fourier cosine and sine transforms

In the same way that we exploited symmetry to define half-range Fourier series, we can similarly define transforms over the range $[0, \infty)$. First if we suppose that $f(x)$ is even about $x = 0$ we can write

$$\begin{aligned} \widehat{f}(\omega) &= \int_{-\infty}^{\infty} f(x)e^{-i\omega x} dx = \int_{-\infty}^{\infty} f(x)(\cos\omega x - i\sin\omega x)\, dx \\ &= 2\int_{0}^{\infty} f(x)\cos\omega x\, dx. \end{aligned}$$

We define

$$\underline{\widehat{f_c}(\omega) = \int_{0}^{\infty} f(x)\cos\omega x\, dx}$$

to be the **Fourier cosine transform** of $f(x)$, so that for an even function $f(x)$:

$$\widehat{f}(\omega) = 2\widehat{f_c}(\omega).$$

Note that $\widehat{f}_c(\omega)$ is even about $\omega = 0$. Using the inversion formula for the regular transform:

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{f}(\omega)e^{i\omega x}d\omega = \frac{1}{\pi}\int_{-\infty}^{\infty} \widehat{f}_c(\omega)e^{i\omega x}\,d\omega.$$

Exploiting the evenness of $\widehat{f}_c(\omega)$, this reduces to

$$\underline{f(x) = \frac{2}{\pi}\int_{0}^{\infty} \widehat{f}_c(\omega)\cos\omega x\,d\omega,}$$

which is the **inversion formula for the Fourier cosine transform**.

In a similar way, by considering $f(x)$ to be odd about $x = 0$, we can define a **Fourier sine transform** and derive the corresponding inversion formula. We obtain the pair of expressions:

$$\underline{\widehat{f}_s(\omega) = \int_{0}^{\infty} f(x)\sin\omega x\,dx,}$$

$$\underline{f(x) = \frac{2}{\pi}\int_{0}^{\infty} \widehat{f}_s(\omega)\sin\omega x\,d\omega}$$

for the transform and its inverse.

3.2 Properties of Fourier transforms

(**i**) The Fourier transform is linear, and so

$$\mathcal{F}\{af(x) + bg(x)\} = a\widehat{f}(\omega) + b\widehat{g}(\omega),$$

where a and b are constants. It follows from this that

$$\mathcal{F}^{-1}\{a\widehat{f}(\omega) + b\widehat{g}(\omega)\} = af(x) + bg(x),$$

where $\mathcal{F}^{-1}$ denotes the inverse transform.

(**ii**) If $a > 0$:

$$\mathcal{F}\{f(ax)\} = \frac{1}{a}\widehat{f}(\frac{\omega}{a}).$$

Proof Starting on the left-hand-side, and making the substitution $s = ax$:

$$\mathcal{F}\{f(ax)\} = \int_{-\infty}^{\infty} f(ax)e^{-i\omega x}dx = \frac{1}{a}\int_{-\infty}^{\infty} f(s)e^{-i(\omega/a)s}ds = \frac{1}{a}\widehat{f}(\frac{\omega}{a}).$$

(**iii**) In a similar way we can establish that

$$\mathcal{F}\{f(-x)\} = \widehat{f}(-\omega).$$

(**iv**) The transform of a shifted function can be calculated as follows (using $s = x - x_0$):

$$\begin{aligned}\mathcal{F}\{f(x-x_0)\} &= \int_{-\infty}^{\infty} f(x-x_0)e^{-i\omega x}dx = \int_{-\infty}^{\infty} f(s)e^{-i\omega(s+x_0)}ds \\ &= e^{-i\omega x_0}\widehat{f}(\omega).\end{aligned}$$

(**v**) This is a similar result, but this time involving a shift in transform space:

$$\mathcal{F}\{e^{i\omega_0 x}f(x)\} = \int_{-\infty}^{\infty} f(x)e^{-i(\omega-\omega_0)x}dx = \widehat{f}(\omega - \omega_0).$$

(**vi**) Symmetry formula. The following result is surprisingly useful. Suppose the Fourier transform of $f(x)$ is $\widehat{f}(\omega)$; change the variable ω to x; then

$$\mathcal{F}\{\widehat{f}(x)\} = 2\pi f(-\omega).$$

Proof. Starting with the inversion formula we have

$$\begin{aligned} f(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{f}(\omega)e^{i\omega x}\,d\omega \\ &\equiv \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{f}(s)e^{isx}\,ds. \end{aligned}$$

If we now let $x = -\omega$:

$$\begin{aligned} f(-\omega) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{f}(s)e^{-i\omega s}\,ds \\ &\equiv \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{f}(x)e^{-i\omega x}\,dx \\ &= \frac{1}{2\pi}\mathcal{F}\{\widehat{f}(x)\}, \end{aligned}$$

as required.

The following results are particularly useful when applying Fourier transforms to partial differential equations.

(**vii**)

$$\mathcal{F}\{d^n f/dx^n\} = (i\omega)^n \widehat{f}(\omega).$$

Proof. This can be established by integration by parts. We assume that all derivatives of f tend to zero as $x \to \pm\infty$.

$$\begin{aligned} \mathcal{F}\{d^n f/dx^n\} &= \int_{-\infty}^{\infty} (d^n f/dx^n)e^{-i\omega x}\,dx \\ &= \left[\left(d^{n-1} f/dx^{n-1}\right) e^{-i\omega x}\right]_{-\infty}^{\infty} + i\omega\int_{-\infty}^{\infty} (d^{n-1} f/dx^{n-1})e^{-i\omega x}\,dx \\ &= i\omega\mathcal{F}\{d^{n-1} f/dx^{n-1}\} \\ &= \cdots \\ &= (i\omega)^n\mathcal{F}\{f\}. \end{aligned}$$

(**viii**) $\mathcal{F}\{xf(x)\} = i\widehat{f}'(\omega)$.

Proof. Considering the left-hand-side:

$$\begin{aligned} \int_{-\infty}^{\infty} f(x)\,xe^{-i\omega x}\,dx &= \int_{-\infty}^{\infty} f(x)\,\frac{d}{d\omega}(ie^{-i\omega x})\,dx \\ &= i\frac{d}{d\omega}\int_{-\infty}^{\infty} f(x)e^{-i\omega x}\,dx \\ &= i\frac{d}{d\omega}\widehat{f}(\omega). \end{aligned}$$

(ix)

$$\begin{aligned} \text{(a) } \mathcal{F}_c\{f'(x)\} &= -f(0) + \omega\widehat{f}_s(\omega), \\ \text{(b) } \mathcal{F}_s\{f'(x)\} &= -\omega\widehat{f}_c(\omega), \\ \text{(c) } \mathcal{F}_c\{f''(x)\} &= -f'(0) - \omega^2\widehat{f}_c(\omega), \\ \text{(d) } \mathcal{F}_s\{f''(x)\} &= \omega f(0) - \omega^2\widehat{f}_s(\omega). \end{aligned}$$

Proof. We prove (a) and (c) and leave the others as exercises. For (a) we have, integrating by parts:

$$\begin{aligned} \mathcal{F}_c\{f'(x)\} &= \int_0^{\infty} f'(x)\cos\omega x\,dx = [f(x)\cos\omega x]_0^{\infty} + \omega\int_0^{\infty} f(x)\sin\omega x\,dx \\ &= -f(0) + \omega\widehat{f}_s(\omega), \end{aligned}$$

as required, while for (c):

$$\begin{aligned}\mathcal{F}_c\{f''(x)\} &= \int_0^\infty f''(x)\cos\omega x\,dx = [f'(x)\cos\omega x]_0^\infty + \omega\int_0^\infty f'(x)\sin\omega x\,dx \\ &= -f'(0) + \omega\mathcal{F}_s\{f'(x)\} \\ &= -f'(0) - \omega^2\widehat{f}_c(\omega),\end{aligned}$$

with the last line following from the use of (b).

(**x**) If $f(x)$ is a complex-valued function and $[f(x)]^*$ is its complex conjugate, then

$$\mathcal{F}\{[f(x)]^*\} = [\widehat{f}(-\omega)]^*.$$

Proof. We have that

$$\widehat{f}(-\omega) = \int_{-\infty}^\infty f(x)e^{i\omega x}dx$$

and so it follows that

$$\begin{aligned}[\widehat{f}(-\omega)]^* &= \int_{-\infty}^\infty [f(x)]^* e^{-i\omega x}dx \\ &= \mathcal{F}\{[f(x)]^*\},\end{aligned}$$

as required.

3.3 Convolution theorem for Fourier transforms

We define the convolution of two functions $f(x), g(x)$, defined over $(-\infty,\infty)$, as

$$f(x)*g(x) = \int_{-\infty}^\infty f(x-u)g(u)\,du.$$

An important result is the so-called convolution theorem:

$$\mathcal{F}\{f*g\} = \widehat{f}(\omega)\widehat{g}(\omega).$$

Proof. Starting on the left-hand-side, we have

$$\int_{x=-\infty}^\infty\left\{\int_{u=-\infty}^\infty f(x-u)g(u)\,du\right\}e^{-i\omega x}dx.$$

Changing the order of integration:

$$\int_{u=-\infty}^\infty g(u)\left\{\int_{x=-\infty}^\infty f(x-u)e^{-i\omega x}dx\right\}du.$$

Now make the substitution $s = x-u$ at fixed u. The double integral becomes

$$\begin{aligned}&\int_{u=-\infty}^\infty g(u)\left\{\int_{s=-\infty}^\infty f(s)e^{-i\omega(s+u)}ds\right\}du \\ =\;& \left(\int_{-\infty}^\infty g(u)e^{-i\omega u}\,du\right)\left(\int_{-\infty}^\infty f(s)e^{-i\omega s}\,ds\right) \\ =\;& \widehat{g}(\omega)\widehat{f}(\omega),\end{aligned}$$

as required.

Example

Find the inverse Fourier transform of the function

$$\frac{1}{(4+\omega^2)(9+\omega^2)}.$$

Setting

$$\widehat{f}(\omega) = 1/(4+\omega^2),\ \widehat{g}(\omega) = 1/(9+\omega^2),$$

we have (from problem sheet 6) that

$$f(x) = (1/4)e^{-2|x|},\ g(x) = (1/6)e^{-3|x|}.$$

Thus, by the convolution theorem:

$$\begin{aligned}
\mathcal{F}^{-1}\left\{\frac{1}{(4+\omega^2)(9+\omega^2)}\right\} &= f(x) * g(x) \\
&= \frac{1}{24}\int_{-\infty}^{\infty} e^{-2|x-u|}e^{-3|u|}du \\
&= \cdots \\
&= \frac{1}{20}e^{-2|x|} - \frac{1}{30}e^{-3|x|}.
\end{aligned}$$

Note that there are other ways to compute the inverse, e.g. we could decompose the original function into partial fractions and invert term-by-term.

3.4 The energy theorem

This is the analogous result to Parseval's theorem for Fourier series. It states that if $f(x)$ is a real-valued function, then

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\widehat{f}(\omega)\right|^2 d\omega = \int_{-\infty}^{\infty}[f(x)]^2\,dx.$$

Proof. Properties (iii) and (x) of the Fourier transform give

$$\mathcal{F}\{[f(-x)]^*\} = [\widehat{f}(\omega)]^*.$$

Since we are assuming f to be real, this simplifies to

$$\mathcal{F}\{f(-x)\} = [\widehat{f}(\omega)]^*.$$

If we now use the convolution theorem with $\widehat{g}(\omega) = [\widehat{f}(\omega)]^*$, we have

$$\mathcal{F}\{f(x) * f(-x)\} = \widehat{f}(\omega)[\widehat{f}(\omega)]^* = \left|\widehat{f}(\omega)\right|^2.$$

Using the inverse transform

$$f(x) * f(-x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\widehat{f}(\omega)\right|^2 e^{i\omega x}d\omega.$$

The left-hand side is

$$\int_{-\infty}^{\infty} f(u+x)f(u)\,du.$$

In particular, setting $x=0$, we obtain

$$\int_{-\infty}^{\infty}[f(u)]^2du = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\widehat{f}(\omega)\right|^2 d\omega,$$

which is the required result.

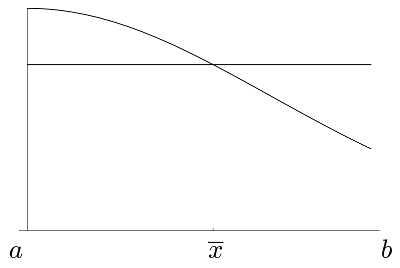

Figure 15: An illustration of the mean-value theorem for integrals. The area under the horizontal line between $x = a$ and $x = b$ is equal to the area under the curve. The value $\overline{x}$ is the x coordinate of the point of intersection of the straight line and the curve.

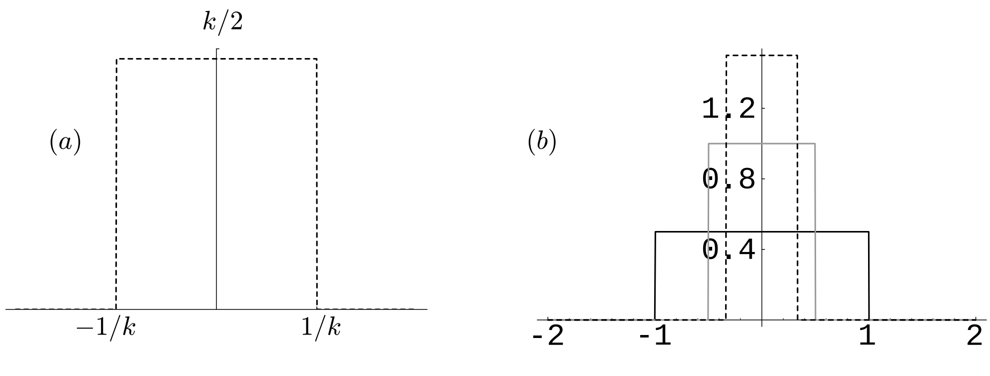

Figure 16: (a) The function $f_k(x)$ used to define the delta function. (b) $f_k(x)$ for different values of k.

3.5 The Dirac delta-function (impulse function)

Before we define this we need to be aware of the following theorem.

3.5.1 Mean-value theorem for integrals

If $g(x)$ is continuous on $[a, b]$ then

$$\int_a^b g(x)\, dx = (b - a)g(\overline{x})$$

for at least one $\overline{x}$ with $a \le \overline{x} \le b$. The proof follows from the regular mean-value theorem for G say, by defining $g = G'$. Geometrically this means that the area under the curve is equivalent to that of a rectangle with length equal to the interval of integration (see figure 15).

3.5.2 Definition of the delta function

Consider the following step-function:

$$f_k(x) = \begin{cases} k/2, & |x| < 1/k, \\ 0, & |x| > 1/k \end{cases} .$$

This function is sketched in figure 16a. Clearly we can see that an important property of this function is that

$$\int_{-\infty}^{\infty} f_k(x)\, dx = 1.$$

As k increases, $f_k(x)$ gets taller and thinner (see figure 16b). We define the delta function to be

$$\delta(x) = \lim_{k \to \infty} f_k(x),$$

although, of course, this limit doesn't exist in the usual mathematical sense. Effectively $\delta(x)$ is infinite at $x = 0$ and zero at all other values of x. The key property however, is that

$$\int_{-\infty}^{\infty} \delta(x)\, dx = 1.$$

The delta function is most useful in how it interacts with other functions.

3.5.3 Sifting property of the delta function

Consider

$$\int_{-\infty}^{\infty} g(x)\delta(x)\, dx,$$

where g is a continuous function over $(-\infty, \infty)$. Using our definition of the delta-function we can rewrite this as

$$\begin{aligned} \lim_{k\to\infty} \int_{-\infty}^{\infty} g(x) f_k(x)\, dx &= \lim_{k\to\infty} \int_{-1/k}^{1/k} \frac{k}{2} g(x)\, dx \\ &= \lim_{k\to\infty} \frac{k}{2} g(\overline{x}) \left(\frac{1}{k} - \left(-\frac{1}{k}\right)\right), \end{aligned}$$

for some $\overline{x}$ in $[-1/k, 1/k]$, using the mean-value theorem for integrals. Clearly, as $k \to \infty$, we must have $\overline{x} \to 0$. The the expression above simplifies to

$$g(0)\frac{k}{2}\frac{2}{k} = g(0).$$

We have therefore established that for any continuous function g :

$$\int_{-\infty}^{\infty} g(x)\delta(x)\, dx = g(0).$$

This result can easily be generalized to

$$\int_{-\infty}^{\infty} g(x)\delta(x-a)\, dx = g(a).$$

Example

Find the Fourier transform of $\delta(x)$.

We have

$$\begin{aligned} \mathcal{F}\{\delta(x)\} &= \int_{-\infty}^{\infty} \delta(x) e^{-i\omega x} dx \\ &= e^{-i\omega 0} \\ &= 1, \end{aligned}$$

using the sifting property. From this we can deduce that the inverse Fourier transform of 1 is $\delta(x)$. From this last result, and using the inversion formula, we see that an alternative representation of the delta function is

$$\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{\pm i\omega x}\, d\omega$$

with the $\pm$ arising from the observation that $\delta(x)$ is an even function of x about $x = 0$. If we interchange the variables we can also write

$$\delta(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{\pm i\omega x}\, dx.$$

If we are prepared to work in terms of delta-functions, we can now take the Fourier transforms of functions that do not decay as $x \to \pm\infty$.

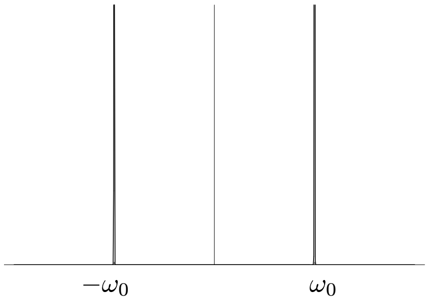

Figure 17: The Fourier transform of the function $\cos\omega_0 x$.

Example

Find the Fourier transform of $\cos\omega_0 x$.

$$\begin{aligned}\mathcal{F}\{\cos\omega_0 x\} &= \int_{-\infty}^{\infty}\frac{1}{2}\left(e^{i\omega_0 x}+e^{-i\omega_0 x}\right)\, e^{-i\omega x}dx\\ &= \frac{1}{2}\int_{-\infty}^{\infty} e^{-i(\omega-\omega_0)x}dx+\frac{1}{2}\int_{-\infty}^{\infty} e^{-i(\omega+\omega_0)x}dx\\ &= \pi\delta(\omega-\omega_0)+\pi\delta(\omega+\omega_0),\end{aligned}$$

which is a two-spiked 'function' (see figure 17).

Exercise

Find the Fourier transform of $\sin\omega_0 x$.