4 Partial differential equations

In this section we will consider second order partial differential equations (pde's). The three main equations we will study are

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} &= c^2\nabla^2 u, \quad \textbf{Wave equation,} \\ \frac{\partial u}{\partial t} &= \kappa\nabla^2 u, \quad \textbf{Heat equation}, \\ \nabla^2\Phi &= f(\mathbf{r}), \quad \textbf{Laplace/Poisson equation}. \end{aligned}$$

4.1 The wave equation

For this discussion we will restrict ourselves to the one-dimensional version of the equation, namely

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}. \tag{1}$$

4.1.1 Solution on a finite domain in x : the method of separation of variables

Consider the following problem. Solve (1) for $u(x,t)$ subject to the boundary conditions

$$u(0,t) = u(L,t) = 0, \quad (t \geq 0) \tag{2}$$

and the initial conditions

$$u(x,0) = f(x), \quad (0 \leq x \leq L), \tag{3}$$

$$\frac{\partial u}{\partial t}(x,0) = 0, \quad (0 \leq x \leq L). \tag{4}$$

We seek a solution to (1) of the 'separated-variables' form

$$u(x,t) = X(x)T(t).$$

Substituting this expression into (1) we get

$$X(x)T''(t) = c^2 X''(x)T(t).$$

Dividing both sides by c^2XT we can write this as

$$\frac{T''(t)}{c^2T(t)} = \frac{X''(x)}{X(x)}.$$

We observe that the left-hand-side above only depends on t, while the right-hand-side is dependent only on x. The expression must hold for all $t > 0$ and $x \in [0,L]$. The only way this can be satisfied is if the left and right hand sides are equal to a constant. In other words we must have

$$\frac{X''}{X} = K = \frac{T''}{c^2T}.$$

We have therefore reduced our pde to two second order ordinary differential equations which should be easier to solve, particularly as they have constant coefficients. Now let's consider applying the boundary conditions. These imply that

$$X(0) = X(L) = 0,$$

with $X(x)$ satisfying

$$X'' - KX = 0. \tag{5}$$

The form of the general solution depends on whether K is positive, zero or negative. Let's consider those cases in turn.

(i) $K > 0$. In this case the general solution of (5) is

$$X = A\cosh(\sqrt{K}x) + B\sinh(\sqrt{K}x).$$

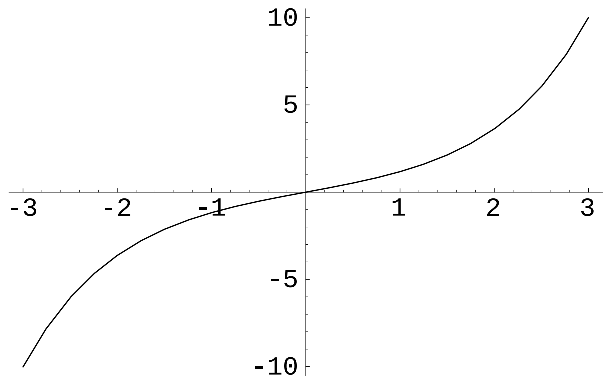

Figure 1: $\sinh x$ versus x

However if we then apply the boundary conditions, we have

$$\begin{aligned} X(0) &= 0 \Rightarrow A = 0, \\ X(L) &= 0 \Rightarrow B = 0 \text{ or } \sinh(\sqrt{K}L) = 0. \end{aligned}$$

Neither of these options are acceptable: the first leads to X identically zero, while a consideration of the graph of $\sinh x$ (figure 1) shows that no such positive value for K exists. We therefore conclude that for these boundary conditions the constant K cannot be positive.

(ii) $K = 0$. In this case the solution of (5) is

$$X = ax + b.$$

Again, if we apply $X(0) = X(L) = 0$ we see that no non-zero solution is possible.

(iii) $K < 0$. It is convenient to set $K = -\lambda^2$. Now the general solution of (5) is

$$X = A\cos\lambda x + B\sin\lambda x.$$

Applying the boundary conditions

$$X(0) = 0 \Rightarrow A = 0,$$

as before, but now the condition $X(L) = 0$ leads to

$$\sin\lambda L = 0,$$

for which there are an infinite number of solutions:

$$\lambda = \frac{n\pi}{L}, \quad n = \pm 1, \pm 2, \ldots. \tag{6}$$

The solution for X is therefore

$$X = B_n \sin(n\pi x/L).$$

Now we turn to the corresponding equation for T which takes the form

$$T'' + c^2\lambda^2 T = 0,$$

and therefore has the general solution

$$T = C\cos(\lambda ct) + D\sin(\lambda ct).$$

The initial condition (4) implies that $T'(0) = 0$ and this leads to $D = 0$. Substituting for λ from (6) we are left with

$$T = C_n \cos(n\pi ct/L).$$

A solution that satisfies the wave equation and conditions (2), (4) is therefore

$$y_n = XT = \beta_n \sin(n\pi x/L)\cos(n\pi ct/L)$$

where we have introduced $\beta_n = B_n C_n$. Since the wave equation is linear it follows that any linear combination of the solutions for different n is also a solution. The most general solution is therefore of the form

$$u(x,t) = \sum_{n=-\infty}^{\infty} \beta_n \sin(n\pi x/L)\cos(n\pi ct/L),$$

which can be expressed more succinctly as

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin(n\pi x/L)\cos(n\pi ct/L), \tag{7}$$

where we have written $b_n = \beta_n - \beta_{-n}$. We have one more initial condition to apply - the condition that $u = f(x)$ when $t = 0$. Imposing this, we see that $f(x)$ is related to the unknown coefficients b_n in the following way:

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right), \quad (0 < x < L).$$

We recognize this as a half-range Fourier sine series for $f(x)$. Our Fourier series studies of section 2 have shown us that

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\, dx, \tag{8}$$

and so can be computed for a given $f(x)$. The required solution to the wave equation is therefore given by the infinite sum (7), with the coefficients b_n calculated from (8).

Example

Solve the wave equation subject to

$$\begin{aligned} u(0,t) &= u(L,t) = 0 \text{ for } t \geq 0, \\ u(x,0) &= 0 \text{ for } 0 \leq x \leq L, \\ \frac{\partial u}{\partial t}(x,0) &= g(x) \text{ for } 0 \leq x \leq L. \end{aligned}$$

4.1.2 Uniqueness of solution of the wave equation

Consider the following problem for $u(x,t)$ (using subscripts for derivatives):

$$\begin{aligned} u_{tt} &= c^2 u_{xx}, \quad (0 < x < L, \quad t > 0), \\ u(x,0) &= f(x), \quad u_t(x,0) = g(x), \\ u(0,t) &= u(L,t) = 0 \quad (t > 0). \end{aligned}$$

Let u_1, u_2 be two possible solutions and consider the difference $U = u_1 - u_2$. Then U satisfies

$$\begin{aligned} U_{tt} &= c^2 U_{xx}, \quad (0 < x < L, \quad t > 0), \\ U(x,0) &= 0, \quad U_t(x,0) = 0, \\ U(0,t) &= U(L,t) = 0 \quad (t > 0). \end{aligned}$$

Note that these initial and boundary conditions imply that

$$U_x(x,0) = 0, \quad U_t(0,t) = U_t(L,t) = 0.$$

Introduce

$$E(t) = \frac{1}{2}\int_0^L \left(U_t^2 + c^2 U_x^2\right)\, dx.$$

and note that $E(0) = 0$. Our solution will be unique provided we can show that $U \equiv 0$. Now

$$dE/dt = \int_0^L \left(U_t U_{tt} + c^2 U_x U_{xt} \right) \, dx = \left[c^2 U_x U_t \right]_0^L + \int_0^L \left(U_t U_{tt} - c^2 U_t U_{xx} \right) \, dx,$$

after integrating by parts. The integral on the RHS is zero since $U_{tt} = c^2 U_{xx}$, while the integrated term is zero on application of the boundary conditions. We therefore have

$$dE/dt = 0,$$

and so E =constant= $E(0) = 0$. Hence we must have U_t and U_x identically zero and so U is at most a constant. However the initial condition then tells us that $U \equiv 0$ and hence the solution is unique.

4.1.3 Solution on an infinite domain: use of Fourier transforms

Suppose we have to solve the following problem:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \text{for } -\infty < x < \infty, \ t > 0,$$

$$u(x,0) = 4e^{-5|x|} \quad \text{for } -\infty < x < \infty, \tag{9}$$

$$\frac{\partial u}{\partial t}(x,0) = 0. \tag{10}$$

with u bounded as $x \to \pm\infty$ for all t.

We take the Fourier transform in x of the differential equation, using property (vii) from section 3.2. This gives

$$\frac{\partial^2 \widehat{u}}{\partial t^2} = -c^2 \omega^2 \widehat{u},$$

where $\widehat{u}(\omega, t)$ is the Fourier transform of $u(x,t)$. The general solution is

$$\widehat{u}(\omega, t) = A(\omega) \cos(\omega c t) + B(\omega) \sin(\omega c t).$$

The initial condition (10) implies that $\partial \widehat{u}/\partial t = 0$ at $t = 0$, and so we conclude that $B(\omega) = 0$. Then applying condition (9) we see that

$$A(\omega) = \mathcal{F}\{4e^{-5|x|}\} = \frac{40}{25 + \omega^2},$$

using a result we saw on Problem Sheet 6. Hence we have

$$\widehat{u}(\omega, t) = \frac{40}{25 + \omega^2} \cos(\omega c t).$$

We can invert this by using the convolution theorem, since $\widehat{u}$ is the product of two terms we know the individual inverses of. We proceed as follows.

$$\begin{aligned} u(x,t) &= \mathcal{F}^{-1}\left\{ \frac{40}{25+\omega^2} \cos(\omega c t) \right\} \\ &= \mathcal{F}^{-1}\left\{ \frac{40}{25+\omega^2} \right\} * \mathcal{F}^{-1}\left\{ \cos(\omega c t) \right\} \\ &= 4e^{-5|x|} * \mathcal{F}^{-1}\left\{ \cos(\omega c t) \right\}. \end{aligned} \tag{11}$$

Now to find the inverse transform of the second term we can use the result from section 3.5.3 that

$$\mathcal{F}\{\cos(\omega_0 x)\} = \pi \delta(\omega + \omega_0) + \pi \delta(\omega - \omega_0),$$

where δ is the Dirac delta function. Then using the symmetry formula (property (vi) in section 3.2) it follows that

$$\mathcal{F}\{\pi \delta(x + \omega_0) + \pi \delta(x - \omega_0)\} = 2\pi \cos(-\omega_0 \omega),$$

and hence

$$\mathcal{F}^{-1}\{\cos(\omega_0\omega)\} = \frac{1}{2}\delta(x+\omega_0) + \frac{1}{2}\delta(x-\omega_0).$$

Using this result in (11):

$$\begin{aligned} u(x,t) &= 4e^{-5|x|} * \frac{1}{2}(\delta(x+ct)+\delta(x-ct)) \\ &= 2\int_{-\infty}^{\infty} e^{-5|x-s|}\delta(s+ct)\,ds + 2\int_{-\infty}^{\infty} e^{-5|x-s|}\delta(s-ct)\,ds \\ &= 2e^{-5|x+ct|} + 2e^{-5|x-ct|}, \end{aligned}$$

with the last line following from the sifting property of the delta function (section 3.5.3).

4.1.4 D'Alembert's solution for the wave equation

The particular solution we obtained by separation of variables can be rewritten as

$$u(x,t) = \frac{1}{2}\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}(x+ct)\right) + \frac{1}{2}\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}(x-ct)\right),$$

Similarly, the solution we obtained by Fourier transforms also depends on the combination of variables $x-ct$ and $x+ct$. The functional dependence on these quantities indicates that in both cases the solution is the sum of a left-travelling $(x+ct)$ and right-travelling $(x-ct)$ wave, with both waves propagating at speed c. This observation provides us with some motivation for the following study which results in the derivation of the general solution of the wave equation.

We introduce new variables

$$\xi = x+ct, \quad \eta = x-ct.$$

The partial derivatives transform as follows:

$$\begin{aligned} \frac{\partial}{\partial x} &= \frac{\partial \xi}{\partial x}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial x}\frac{\partial}{\partial \eta} = \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}, \\ \frac{\partial}{\partial t} &= \frac{\partial \xi}{\partial t}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial t}\frac{\partial}{\partial \eta} = c\frac{\partial}{\partial \xi} - c\frac{\partial}{\partial \eta}. \end{aligned}$$

We can then calculate

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial x}(\frac{\partial u}{\partial x}) = \left(\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}\right)\left(\frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta}\right) \\ &= \frac{\partial^2 u}{\partial \xi^2} + 2\frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{\partial^2 u}{\partial \eta^2}, \end{aligned}$$

and

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} &= \frac{\partial}{\partial t}(\frac{\partial u}{\partial t}) = \left(c\frac{\partial}{\partial \xi} - c\frac{\partial}{\partial \eta}\right)\left(c\frac{\partial u}{\partial \xi} - c\frac{\partial u}{\partial \eta}\right) \\ &= c^2\left(\frac{\partial^2 u}{\partial \xi^2} - 2\frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{\partial^2 u}{\partial \eta^2}\right). \end{aligned}$$

Under this transformation the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$$

becomes

$$-4c^2\frac{\partial^2 u}{\partial \xi \partial \eta} = 0,$$

The equation is said to be in its **canonical form**. This equation can be integrated once with respect to ξ to give

$$\frac{\partial u}{\partial \eta} = f'(\eta)$$

where f' is an arbitrary function of η. Integrating again, this time with respect to η, we obtain

$$u = f(\eta) + g(\xi),$$

with g an arbitrary function of ξ. The general solution of the wave equation therefore has the form

$$u = f(x - ct) + g(x + ct),$$

and so can always be written as the sum of right and left travelling waves.

N. B. Here (and in the uniqueness proof earlier) we are assuming that $\partial^2 u/\partial\xi\partial\eta = \partial^2 u/\partial\eta\partial\xi$. A sufficient condition for this is that all partial derivatives of u up to order two are continuous.

4.2 The Heat Equation

In one dimension this is the partial differential equation

$$\frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2}. \tag{12}$$

Again we will look at some straightforward methods of solution for simple geometries.

4.2.1 Solution on a finite domain: separation of variables

Suppose we have boundary conditions of the form

$$u(0, t) = u(L, t) = 0 \text{ for } t > 0, \tag{13}$$

and an initial condition

$$u(x, 0) = f(x) \text{ for } 0 \le x \le L. \tag{14}$$

In a similar way to the wave equation we can seek a separated-variables solution of the form

$$u(x, t) = X(x)T(t).$$

Substitution into (12) leads to

$$XT' = \kappa X''T,$$

so that

$$\frac{X''}{X} = \frac{T'}{\kappa T}.$$

As in the case of the wave equation we now see that the left-hand-side is a function of x only, and the right-hand-side depends only on t. Therefore we must have, for some constant λ^2 :

$$\frac{X''}{X} = \frac{T'}{\kappa T} = -\lambda^2.$$

The sign of the separation constant λ^2 depends on the type of boundary conditions we impose. In our particular case since we are imposing (13) we require $X(0) = X(L) = 0$ which can only be accomplished if $\lambda^2 > 0$ (otherwise the solutions for X will be of exponential form). We therefore have

$$X(x) = A\cos\lambda x + B\sin\lambda x$$

with $A = 0$ and $\lambda = n\pi/L$. The corresponding solution for T arises from

$$\frac{T'}{T} = -\lambda^2\kappa = -\frac{n^2\pi^2\kappa}{L^2},$$

and hence

$$T(t) = Ce^{-n^2\pi^2\kappa t/L^2}.$$

Putting the two components of our solution together and summing over all modes we obtain

$$u(x, t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) \exp\left(-\frac{n^2\pi^2\kappa t}{L^2}\right). \tag{15}$$

Finally, applying the initial condition (14) we have

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right), \quad (0 < x < L),$$

which we recognize as a half-range Fourier sine series for $f(x)$ with coefficients

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\,dx. \tag{16}$$

Therefore the required solution of the heat equation subject to (13) and (14) is given by (15) and (16). The method can be adapted to accommodate boundary conditions on $\partial u/\partial x$, rather than u. Uniqueness of solution can also be established in a similar way as for the wave equation. [See problem sheet 7].

4.2.2 Solutions on an infinite or semi-infinite domain

Again, as with the wave equation we can use Fourier transforms to help us obtain a solution. Consider for example the problem

$$\begin{aligned}
\frac{\partial u}{\partial t} &= \kappa\frac{\partial^2 u}{\partial x^2} \text{ for } 0 < x < \infty,\ t > 0, \\
u(0,t) &= 0 \text{ for } t \geq 0, && (17)\\
u(x,0) &= f(x) \text{ for } 0 < x < \infty && (18)
\end{aligned}$$

and u bounded as $x \to \infty$ for all t.

Since this problem is posed over a semi-infinite domain we could take either a Fourier cosine or sine transform. Recall from property (ix), section 3.2 that if we take a cosine transform of a second derivative we require a knowledge of $\partial u/\partial x$ when $x = 0$, while if we take a sine transform we need to know u at $x = 0$. In this particular case, in view of (17), we have the latter situation and so we will take a Fourier sine transform of the equation, to give

$$\frac{\partial \widehat{u}_s}{\partial t} = -\omega^2\kappa\widehat{u}_s + \omega\kappa u(0,t),$$

where $\widehat{u}_s(\omega,t)$ is the Fourier sine transform of $u(x,t)$ with respect to x. Substituting for $u(0,t)$ from (17) and integrating, we obtain

$$\widehat{u}_s = B(\omega)e^{-\omega^2\kappa t}.$$

Taking the Fourier sine transform of (18) we obtain $\widehat{u}_s = \widehat{f}_s(\omega)$ on $t = 0$, allowing us to determine $B(\omega) = \widehat{f}_s(\omega)$, and hence

$$\widehat{u}_s = \widehat{f}_s(\omega)e^{-\omega^2\kappa t}.$$

Applying the inversion formula for the Fourier sine transform we can then write the solution in the form

$$u(x,t) = \frac{2}{\pi}\int_0^{\infty} \widehat{f}_s(\omega)e^{-\omega^2\kappa t}\sin(\omega x)\,d\omega,$$

where

$$\widehat{f}_s(\omega) = \int_0^{\infty} f(\xi)\sin(\omega\xi)\,d\xi.$$

4.2.3 Higher dimensions

The methods of separation of variables and Fourier transforms can also be used on the heat equation in two and three dimensions. However the variables in the boundary conditions need to be separated for this technique to work, and so the methods are of limited use for problems with complicated geometries.

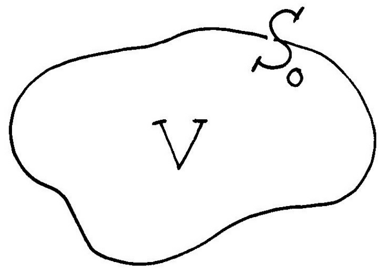

Figure 2: A volume V bounded on its exterior by the surface S_0.

4.3 Laplace's equation and Poisson's equation

We will write Laplace's equation in the form

$$\nabla^2 \phi = 0.$$

We will also study the related equation

$$\nabla^2 \phi = f(\mathbf{r}), \tag{19}$$

where f is a prescribed function of position $\mathbf{r}$ $(= x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$. Equation (19) is known as **Poisson's equation**.

4.3.1 Types of boundary conditions

If we look for a solution in a volume V, then the boundary conditions will be given on the surfaces S which bound V. These boundary conditions are generally of two types:

(i) **Dirichlet boundary conditions**, in which ϕ is given on the boundary;

(ii) **Neumann boundary conditions**, in which the normal derivative $\partial\phi/\partial n$, is prescribed on the boundary.

4.3.2 Solution using separation of variables and transform techniques

The same techniques used for the wave and heat equations will also work on Laplace's equation. The main limitations of these methods are the simple shapes of domain to which they can be applied, and the necessity for the variables in the boundary conditions to separate. There are examples of the use of these techniques on problem sheets 7 and 8.

In what follows we shall consider the three-dimensional form of the equations and apply some of the techniques and theorems we have learned to formulate solutions. We will use a number of results from vector calculus and will also generalize the Dirac delta function we encountered in section 3.

4.3.3 Uniqueness of solution for interior problems

Let ϕ satisfy Poisson's equation

$$\nabla^2 \phi = f(\mathbf{r})$$

in a volume V. The volume is bounded on its exterior by a surface S_0 (figure 2). On the surface we have a Dirichlet boundary condition, i.e. $\phi = p(\mathbf{r})$ on S_0. Then the solution for ϕ is unique.

Proof. We will suppose there are two solutions and seek a contradiction. Let the two solutions be ϕ_1 and ϕ_2. They must both satisfy Poisson's equation and the same boundary conditions. Forming the difference

$$\Phi \equiv \phi_1 - \phi_2,$$

we must therefore have that

$$\nabla^2 \Phi = 0 \text{ in } V \text{ with } \Phi = 0 \text{ on the boundary } S_0.$$

Now recall from section 1.8.8, Green's first identity with $\psi = \phi = \Phi$:

$$\int_{S_0} \Phi \frac{\partial \Phi}{\partial n}\, dS = \int_V \Phi \nabla^2 \Phi + |\nabla \Phi|^2 \, dV.$$

In view of the boundary conditions, the left-hand side is zero, and since Φ satisfies Laplace's equation throughout V, the first term on the right-hand-side is also zero. This leaves

$$\int_V |\nabla \Phi|^2 \, dV = 0.$$

The volume integral of a positive quantity can only be zero if the integrand is in fact identically zero, and so this implies

$$\nabla \Phi = 0 \text{ throughout } V.$$

This means that Φ is at most a constant throughout V, but because Φ is zero on the boundaries, it follows that Φ is identically zero throughout V. Hence $\phi_1 = \phi_2$ and the solution is unique.

Much the same reasoning applies if ϕ is a complex-valued function of position, if we have Neumann rather than Dirichlet boundary conditions, and also if the volume V has holes in it [problem sheet 8].

Example

Solve $\nabla^2 \Phi = 2$ inside the unit sphere $r \leq 1$ with $\Phi = 1$ on $r = 1$.

We note that the right-hand-side, the geometry and the boundary conditions have radial symmetry. We therefore seek a solution $\Phi = \Phi(r)$ independent of θ, ϕ in spherical polar coordinates. Referring back to section 1.9.10, the form for the Laplacian in spherical polars is

$$\begin{aligned} \nabla^2 \Phi &= \frac{1}{r^2 \sin\theta} \left\{ \frac{\partial}{\partial r}\left(r^2 \sin\theta \frac{\partial \Phi}{\partial r}\right) + \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \Phi}{\partial \theta}\right) + \frac{\partial}{\partial \phi}\left(\frac{1}{\sin\theta}\frac{\partial \Phi}{\partial \phi}\right) \right\} \\ &= \frac{\partial^2 \Phi}{\partial r^2} + \frac{2}{r}\frac{\partial \Phi}{\partial r} + \frac{\cot\theta}{r^2}\frac{\partial \Phi}{\partial \theta} + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \theta^2} + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 \Phi}{\partial \phi^2}, \end{aligned}$$

so that our equation reduces under the conditions of radial symmetry to

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{d\Phi}{dr}\right) = 2.$$

Integrating we obtain

$$r^2 \frac{d\Phi}{dr} = \frac{2}{3} r^3 + C$$

and hence

$$\Phi = \frac{1}{3} r^2 - \frac{C}{r} + D.$$

For Φ to be finite at $r = 0$ we require $C = 0$. Applying $\Phi = 1$ on $r = 1$ gives $D = 2/3$ and so the required solution is

$$\Phi = \frac{1}{3} r^2 + \frac{2}{3}.$$

Due to the uniqueness theorem, we know this is the only possible solution.

4.3.4 Uniqueness of solution for exterior problems

Now suppose we have a situation where there is an inner boundary S_1, but the outer boundary S_0 is taken to infinity so that V is now an unbounded volume (figure 3). Suppose

$$\nabla^2 \phi = f(\mathbf{r})$$

throughout V. Suppose in addition that $\phi = O(1/r), \partial\phi/\partial r = O(1/r^2)$ as $r \to \infty$. Then the solution for ϕ in V is unique.[1]

[1] If $f = O(r^\alpha)$ as $r \to \infty$, this means that $\lim_{r\to\infty} r^{-\alpha} f(r) = K \neq 0$.

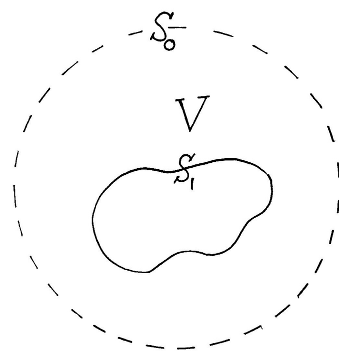

Figure 3: An unbounded volume V with an inner boundary S_1. The dashed line representing the outer boundary is to be considered at infinity.

Proof

Let's start by considering the surface S_0 to be a large sphere of radius R. Suppose there are two solutions ϕ_1, ϕ_2 and form the difference Φ. Proceeding as in the previous proof, using Green's first identity, we have

$$\int_V |\nabla\Phi|^2 \, dV = \sum_{i=0}^{1} \int_{S_i} \Phi \frac{\partial \Phi}{\partial n} \, dS = \int_{S_0} \Phi \frac{\partial \Phi}{\partial n} \, dS,$$

since the integral over S_1 is zero due to the boundary condition that Φ vanishes on S_1. Since S_0 is a sphere of radius R we can write $dS = R^2 \sin\theta d\theta d\phi$ in spherical polar coordinates, and $\partial\Phi/\partial n = \partial\Phi/\partial r$. We therefore have

$$\int_V |\nabla\Phi|^2 \, dV = R^2 \int_0^{2\pi} \int_0^{\pi} \Phi \frac{\partial \Phi}{\partial r} \sin\theta \, d\theta \, d\phi.$$

Because of the assumed behaviour of Φ as $r \to \infty$, the right-hand-side above is of order $1/R$ and hence tends to zero as $R \to \infty$. We therefore see that for the exterior problem

$$\int_V |\nabla\Phi|^2 \, dV = 0,$$

and hence, arguing as in the previous proof, $\Phi = 0$ and hence the solution is unique. The proof extends to Neumann boundary conditions as before, and to the situation where the volume V has any finite number of inner boundaries [problem sheet 8] including no inner boundary.

Example

Solve $\nabla^2\Phi = f(r)$ for $0 \le r < \infty$, where

$$f(r) = \begin{cases} f_0, & r \le a, \\ 0 & r > a. \end{cases}$$

We will assume that Φ is bounded throughout the region and that Φ and $d\Phi/dr \to 0$ as $r \to \infty$ to guarantee a unique solution. As before we can seek a solution with radial symmetry and so

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{d\Phi}{dr}\right) = f(r). \tag{20}$$

Solving for $r \le a$ first, we obtain (after some elementary calculus):

$$\Phi = \frac{1}{6} f_0 r^2 - \frac{A}{r} + B.$$

We need $A = 0$ so that the solution is finite at $r = 0$. Next solving (20) for $r > a$ we obtain

$$\Phi = \frac{C}{r} + D.$$

We require $\Phi \to 0$ as $r \to \infty$ and so we deduce that D must be zero. To find the remaining constants B, C we impose continuity of Φ and $d\Phi/dr$ at $r = a$. This gives

$$\frac{1}{6} f_0 a^2 + B = \frac{C}{a}, \quad \frac{1}{3} f_0 a = -\frac{C}{a^2}.$$

Substituting for B and C we find that the resulting solution for Φ is

$$\Phi = \begin{cases} \frac{1}{6} f_0 r^2 - \frac{1}{2} f_0 a^2, & r \leq a, \\ -\frac{1}{3} f_0 a^3 / r, & r > a. \end{cases}$$

4.3.5 Point sources and the Dirac delta function

Let's consider the result of our previous example in the limit as $a \to 0$. This means that the right-hand side is becoming concentrated at the origin $r = 0$. In addition we will let $f_0 \to \infty$ in such a way that $(4/3)\pi a^3 f_0$ remains equal to a constant - call this K. In this limit the solution we obtained above for Φ for $r > a$ becomes

$$\Phi = -\frac{K}{4\pi r}.$$

We have therefore obtained a solution to

$$\nabla^2 \Phi = f(r),$$

where the right-hand-side has the properties

$$f(r) = \begin{cases} 0, & r \neq 0, \\ \text{`}\infty\text{'}, & r = 0, \end{cases}$$

Another property of the function $f(r)$ can be seen from the following calculation, in which we use the divergence theorem over a sphere radius R centred at the origin:

$$\begin{aligned} \int_V f(r)\, dV &= \int_V \nabla^2 \Phi \, dV = \int_V \nabla \cdot \nabla \Phi \, dV \\ &= \int_{r=R} \frac{\partial \Phi}{\partial r} \, dS \\ &= \int_0^{2\pi} \int_0^{\pi} \frac{K}{4\pi R^2} R^2 \sin\theta \, d\theta \, d\phi \\ &= K. \end{aligned}$$

4.3.6 The delta function with vector argument

The function f/K is the extension of the Dirac delta function studied in section 3 to three dimensions and is denoted by $\delta(r)$. Moreover we can extend this definition so that the argument is a vector, i.e. the function $\delta(\mathbf{r})$. We define

$$\delta(\mathbf{r}) = 0 \text{ for } \mathbf{r} \neq \mathbf{0},$$

and

$$\int_V \delta(\mathbf{r})\, dV = \begin{cases} 1, & \text{if } V \text{ contains the origin} \\ 0, & \text{otherwise.} \end{cases}$$

We claim that the solution of

$$\nabla^2 \Phi = K \delta(\mathbf{r}) \tag{21}$$

is

$$\Phi = -\frac{K}{4\pi |\mathbf{r}|}. \tag{22}$$

This can be verified in the following way. Firstly we have already seen from an example in section 1.4.6 that $\nabla^2(1/r) = 0$ for $r \neq 0$. To check the solution for $r = 0$ we integrate both sides of (21) over a volume V containing $r = 0$ to get

$$\int_V \nabla^2\Phi \, dV = K \int_V \delta(\mathbf{r}) \, dV = K.$$

Using the divergence theorem, the left-hand-side can be rewritten as

$$\int_S \nabla\Phi \cdot \widehat{\mathbf{n}} \, dS.$$

where the origin is interior to the closed surface S which bounds V. But, again from section 1.4.6,

$$\nabla(1/r) = -\mathbf{r}/r^3,$$

where $r = |\mathbf{r}|$. Therefore the surface integral above can be rewritten as

$$\frac{K}{4\pi} \int_S \frac{\mathbf{r} \cdot \widehat{\mathbf{n}}}{r^3} \, dS.$$

The integral here should be familiar as it is equal to 4π using Gauss' flux theorem (1.8.10). The left-hand-side is therefore also equal to K, and we have therefore verified that (22) is indeed a solution to (21).

If we move the source to $\mathbf{r} = \mathbf{r}_0$ then we can easily modify our analysis to show that

A solution of $\nabla^2\Phi = K\delta(\mathbf{r} - \mathbf{r}_0)$ is $\Phi = -K/(4\pi\,|\mathbf{r} - \mathbf{r}_0|)$.

It can also be shown that the sifting property of the delta function carries over to three dimensions, i.e.

$$\int_V g(\mathbf{r})\delta(\mathbf{r} - \mathbf{r}_1) \, dV = g(\mathbf{r}_1),$$

for any continuous function g.

4.3.7 Green's function

The Green's function $G(\mathbf{r}; \mathbf{r}_0)$ is defined as the solution of

$$\nabla^2 G = \delta(\mathbf{r} - \mathbf{r}_0), \tag{23}$$

subject to some appropriate boundary conditions. For the three-dimensional problems we have studied we have seen that the so-called 'free-space' Green's function, i.e. the function that satisfies (23) and tends to zero as $r \to \infty$ is given by

$$\begin{aligned} G(\mathbf{r}; \mathbf{r}_0) &= -\frac{1}{4\pi\,|\mathbf{r} - \mathbf{r}_0|} \\ &= -\frac{1}{4\pi[(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2]^{1/2}}, \end{aligned}$$

in Cartesian coordinates. As we shall see, knowledge of the Green's function will enable us to write down solutions to Poisson's and Laplace's equation in closed form.

4.3.8 Solutions to Poisson's equation using Green's functions

Suppose that Φ satisfies

$$\nabla^2\Phi = f(\mathbf{r})$$

throughout some volume V. We will denote the boundary of V by the surface ∂V. (If the volume is unbounded we will assume that $\Phi = O(1/r)$ as $r \to \infty$, so that a unique solution is guaranteed). We suppose that the boundary condition is of Dirichlet-type, i.e.

$$\Phi = p(\mathbf{r}) \text{ on } \partial V.$$

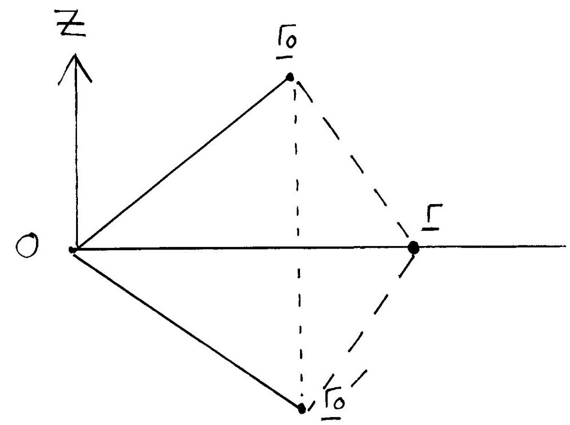

Figure 4: The points $\mathbf{r_0}$ and $\mathbf{r_0'}$ and their position with respect to the plane $z = 0$.

Consider the following associated problem for $G(\mathbf{r}; \mathbf{r}_0)$:

$$\nabla^2 G = \delta(\mathbf{r} - \mathbf{r}_0),$$

in V with

$$G = 0 \text{ on } \partial V.$$

Now take Green's second identity (section 1.8.8) and apply it to the functions Φ and G :

$$\int_V (\Phi \nabla^2 G - G \nabla^2 \Phi)\, dV = \int_{\partial V} (\Phi \frac{\partial G}{\partial n} - G \frac{\partial \Phi}{\partial n})\, dS.$$

The right hand side can be rewritten as

$$\int_{\partial V} p(\mathbf{r}) \frac{\partial G}{\partial n}(\mathbf{r}; \mathbf{r}_0)\, dS,$$

while after substituting for $\nabla^2 G$ and $\nabla^2 \Phi$, the left-hand-side is

$$\int_V \Phi(\mathbf{r})\delta(\mathbf{r} - \mathbf{r}_0) - G(\mathbf{r}; \mathbf{r}_0) f(\mathbf{r})\, dV,$$

which, upon use of the sifting property, becomes

$$\Phi(\mathbf{r}_0) - \int_V G(\mathbf{r}; \mathbf{r}_0) f(\mathbf{r})\, dV.$$

We can therefore write the solution for Φ in the form

$$\Phi(\mathbf{r}_0) = \int_V G(\mathbf{r}; \mathbf{r}_0) f(\mathbf{r})\, dV + \int_{\partial V} p(\mathbf{r}) \frac{\partial G}{\partial n}(\mathbf{r}; \mathbf{r}_0)\, dS. \tag{24}$$

Thus, in principle, if we can find the Green's function we can solve Poisson's equation for Φ.

A similar approach can be taken if Poisson's equation is subject to Neumann conditions on the boundary [Problem Sheet 8].

4.3.9 The method of images

Often it is possible to find the Green's function explicitly by the method of images as shown in the following Laplace equation example.

Suppose we wish to solve

$$\nabla^2 \Phi = 0$$

in the region $z > 0$ with the boundary condition

$$\Phi(x, y, 0) = p(x, y).$$

(We will assume $\Phi = O(1/r)$ as $z \to \infty$ to ensure uniqueness).

We tackle this problem by introducing an associated *Dirichlet Green's function* that satisfies

$$\nabla^2 G = \delta(\mathbf{r} - \mathbf{r}_0) \text{ for } z > 0, \tag{25}$$

$$G = 0 \text{ on } z = 0. \tag{26}$$

If the boundary condition at $z = 0$ is absent we know that the Green's function is

$$G(\mathbf{r}; \mathbf{r}_0) = -\frac{1}{4\pi |\mathbf{r} - \mathbf{r}_0|}.$$

This solution is referred to as a source singularity of strength 1 situated at $\mathbf{r} = \mathbf{r}_0 = (x_0, y_0, z_0)$. Now let's add another singularity of opposite strength at a location the same distance below the $x - y$ plane, i.e. at $r_0' = (x_0, y_0, -z_0)$ (a mirror image). The modified Green's function is therefore

$$G(\mathbf{r}; \mathbf{r}_0) = -\frac{1}{4\pi |\mathbf{r} - \mathbf{r}_0|} + \frac{1}{4\pi |\mathbf{r} - \mathbf{r}_0'|}. \tag{27}$$

Now, when $z = 0$ we have

$$|\mathbf{r} - \mathbf{r}_0| = |\mathbf{r} - \mathbf{r}_0'|$$

(figure 4), and so we see that $G = 0$ on $z = 0$ as required. Thus the Green's function (27) satisfies equation (25) and the boundary condition (26). Now that we have the Green's function we can apply the result of the previous section (equation (24) with $f = 0$) to obtain the solution for Φ as

$$\Phi(\mathbf{r}_0) = \int_{\partial V} p(x, y) \frac{\partial G}{\partial n}(\mathbf{r}, \mathbf{r}_0)\, dS.$$

In this example ∂V is the plane $z = 0$ and $\partial/\partial n = -\partial/\partial z$ (since n is the outward normal to the volume V). Using our expression for G :

$$G = -(4\pi)^{-1} \left\{ \left((x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2\right)^{-1/2} - \left((x - x_0)^2 + (y - y_0)^2 + (z + z_0)^2\right)^{-1/2} \right\},$$

we find that

$$\frac{\partial G}{\partial z} = -(4\pi)^{-1} \left\{ -(z - z_0) \left((x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2\right)^{-3/2} + (z + z_0) \left((x - x_0)^2 + (y - y_0)^2 + (z + z_0)^2\right)^{-3/2} \right\}.$$

Evaluating this quantity on $z = 0$ we have

$$\left.\frac{\partial G}{\partial z}\right|_{z=0} = -(4\pi)^{-1} 2z_0 \left((x - x_0)^2 + (y - y_0)^2 + z_0^2\right)^{-3/2},$$

and so the solution for Φ can be expressed in the form

$$\Phi(x_0, y_0, z_0) = \frac{z_0}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p(x, y) \left((x - x_0)^2 + (y - y_0)^2 + z_0^2\right)^{-3/2} dx\, dy.$$

Example

Suppose we wish to solve Laplace's equation for $z > 0$ with $\Phi = p(x, y)$ on $z = 0$ and $p(x, y)$ given explicitly as

$$p(x, y) = \begin{cases} 1, & x^2 + y^2 \le 1, \\ 0, & x^2 + y^2 > 1. \end{cases}$$

Using the result derived above with this specific form for p :

$$\begin{aligned} \Phi(x_0, y_0, z_0) &= \frac{z_0}{2\pi} \int_{x^2+y^2 \le 1} \left((x - x_0)^2 + (y - y_0)^2 + z_0^2\right)^{-3/2} dx\, dy. \\ &= \frac{z_0}{2\pi} \int_0^{2\pi} \int_0^1 (\rho^2 + x_0^2 + y_0^2 + z_0^2 - 2x_0\rho\cos\theta - 2y_0\rho\sin\theta)^{-3/2} \rho\, d\rho\, d\theta, \end{aligned}$$

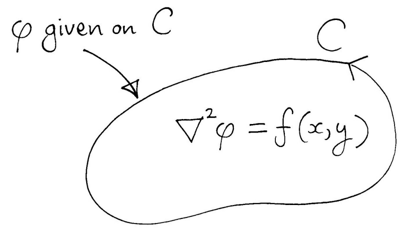

Figure 5: A two-dimensional Poisson problem.

where we have switched to plane polar coordinates (ρ, θ). In particular, the solution along the $z-$axis is

$$\begin{aligned}\Phi(0,0,z_0) &= \frac{z_0}{2\pi}\int_0^{2\pi}\int_0^1 (\rho^2+z_0^2)^{-3/2}\rho\,d\rho\,d\theta \\ &= -z_0[(\rho^2+z_0^2)^{-1/2}]_0^1 \\ &= 1-\frac{z_0}{(1+z_0^2)^{1/2}}.\end{aligned}$$

We can apply the method of images in a similar fashion to solve the same problem but with a Neumann condition on $z=0$ [Problem Sheet 8].

4.3.10 Poisson's equation in two dimensions

This same approach can also be used for two-dimensional problems, although the Green's function has a different (logarithmic) form in this case [see problem sheet 8]. For a Dirichlet problem in which we wish to solve

$$\nabla^2\phi = f(\mathbf{r})$$

in a region R of the $x-y$ plane, with

$$\phi = p(x)$$

on the boundary C of R (figure 5), we consider the Green's function problem

$$\nabla^2 G = \delta(\mathbf{r}-\mathbf{r}_0)$$

in R, with

$$G = 0 \text{ on } C.$$

Applying Green's second identity in 2D (section 1.8.9) we find that

$$\phi(\mathbf{r}_0) = \int_R G(\mathbf{r};\mathbf{r}_0)f(\mathbf{r})\,dx\,dy + \int_C p(x)\frac{\partial G}{\partial n}(\mathbf{r};\mathbf{r}_0)\,ds$$

where now $\mathbf{r}_0 = (x_0, y_0)$. A similar expression can be derived for Neumann boundary conditions. By using the method of images as we did in three dimensions, Green's functions can be found explicitly for certain problems. There is an example on the final problem sheet.