OCR MEI Notes

(Year 2)

Department of Mathematics

Education for Everyone

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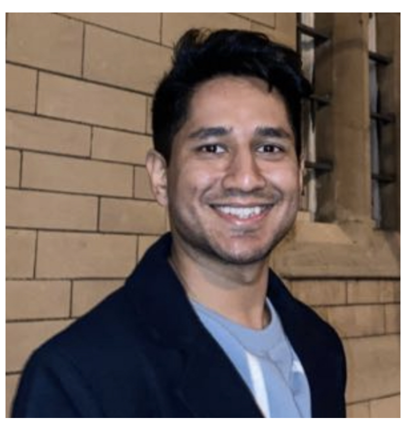
Siddartha Nath is the founder of *PolyNath* and the author of a plethora of top-selling books (*Quantitative Finance, MAT, NLP, TMUA/CTMUA*).

He has an undergraduate First-Class Honours BSc Mathematics with Statistics from Imperial College London (ICL) and a postgraduate Distinction MSc Computational Statistics and Machine Learning from University College London (UCL). He has taken his education and amassed over 2 years of industry experience, having worked as a Data Scientist for PayPal and DataSpartan.

In terms of academic teaching, he is an extremely dedicated and supportive tutor, having taught for over 5 years at public and private schools, for GCSE, A-Levels and University preparation. He possesses a wealth of knowledge in all UK-based Mathematics and Computer Science admission tests, with the following accomplishments:

- Achieved a perfect score of 9.0/9.0, in the Cambridge TMUA, resulting in a top 10% ranking.
- Achieved a commendable score of 63/100, in the Oxford MAT, resulting in a top 25% ranking.
- Achieved a high score of $1, 1, 1$ in the Cambridge STEP I, II and III.

Outside of academia, he enjoys music, dance, watching sports and creating content.



Preface

This document includes a summary of notes of Chapter 1 - Proof (OCR MEI Year 2).

Happy Learning!

Chapter 1

Proof

Summary Checklist

1. - $\Rightarrow$ means *if...then*, *implies*, *therefore* etc...
   - $\Leftarrow$ means *only if...*, *is implied by*, *follows from* etc...
   - $\iff$ means *implies and is implied by*, *is equivalent to* etc...
2. The converse of $A \Rightarrow B$ is $B \Rightarrow A$.
3. - If $A \Leftarrow B$, A is a **necessary** condition for B.
   - If $A \Rightarrow B$, A is a **sufficient** condition for B.
4. The methods of proof are:
   - Direct Argument
   - Exhaustion
   - Contradiction
   - Counter-Example

Below are 2 examples, demonstrating a complete *thought-process* breakdown of how to approach a *proof*:

Example

Consider the following example

"Prove that $n^3 - n$ is a multiple of 6."

We do this via **direct argument**.

1. **Firstly, is this true?**

   When $n = 1$, $1^3 - 1 = 0$, which is divisible by 6. For $n = 2$, $2^3 - 2 = 6$, which is divisible by 6. For $n = 3$, $3^3 - 3 = 24$, which is divisible by 6. So far, it seems to hold. Note, for shorthand, we will use | to denote *divides* i.e. $2|6$ means 2 divides 6.

2. **Secondly, how do we prove divisibility?**

   To show divisibility, the number should be divisible by all the *prime* factors of the number i.e. prime factorisation tree. Hence, to be divisible by 6, we should prove that the number is divisible by 2 and 3.

3. **Thirdly, how do we use algebra to demonstrate these divisibility properties?**

   - Since we are thinking about factors, we perhaps should do some *factorisation*...

   $$\begin{aligned} n^3 - n &= n\left(n^2 - 1\right) \\ &= n\left(n - 1\right)\left(n + 1\right) \end{aligned}$$

   - Logically, $2|n^3 - n$ because we have a product 2 consecutive integers between the 3 factors and thus one of them must be divisible by 2.
   - Logically, $3|n^3 - n$ because we have a product 3 consecutive integers between the 3 factors and thus one of them must be divisible by 3.

4. **Finally, how do we conclude the proof?**

   Hence, since $2|n^3 - n$ and $3|n^3 - n$, then $2 \cdot 3 = 6|n^3 - n$, a trivial property of divisibility and factors.

A follow up question is

"Prove that $n^3 + 11n$ is a multiple of 6."

1. **Firstly, is this true?**

   When $n = 1$, $1^3 + 11\left(1\right) = 12$, which is divisible by 6. For $n = 2$, $2^3 + 11\left(2\right) = 30$, which is divisible by 6. For $n = 3$, $3^3 + 11\left(3\right) = 42$, which is divisible by 6. So far, it seems to hold.

2. **Secondly, how do we prove divisibility?**

   We already know from before that we should show this is divisible by both 2 and 3.

3. **Thirdly, how do we use algebra to demonstrate these divisibility properties?**

   - Since we are thinking about factors, we perhaps should do some *factorisation*...

     $$n^3 + 11n = n\left(n^2 + 11\right)$$

     PROBLEM:

     We cannot do the same trick as before because we do not seem to have consecutive factors i.e. we do not have a nice factorisation. This is usually the point where we stop and give up... But no! **There was a reason for why we did the first part of the question!**

4. **Fourthly, how can I use previous knowledge to help?**

   - A trivial property of divisibility is that if $k|A$ and $k|B$, then $k|A+B$ i.e. if you add multiples of k together, the resulting number is also divisible by k.
   - One should spot that to get $n^3 + 11n$, we must simply add $12n$.

5. **Finally, how do we conclude the proof?**

   As $6|12n$ because 6 is a factor of 12, regardless of n, using the rule before, we can conclude that since $6|n^3 - n$ and $6|12n$, we must have $6|\left(n^3 - n\right) + (12n) = n^3 + 11n$.

What happens if we need to prove divisibility from first principles, where factorisation or prior knowledge base is not available?

To achieve this, we follow a 2-step approach.

- Obtain the prime factors of the divisor.
- Proceed via cases with the prime factors via exhaustion.

Take a look back at $n^3 + 11n$ - focusing on divisibility by 2 and 3, we do a case breakdown by covering all possibilities of what n could be

1. **Divisibility by 2:**

   - $n = 2k$: $n^3 + 11n = 8k^3 + 22k = 2k\left(4k^2 + 11\right) \to$ **divisible by 2**.
   - $n = 2k+1$: $n^3 + 11n = 8k^3 + 12k^2 + 6k + 1 + 22k + 1 = 2\left(4k^3 + 6k^2 + 14k + 1\right) \to$ **divisible by 2**.

   This says that for all n, broken into multiples/non-multiples of 2, $n^3 + 11n$ is always divisible by 2.

2. **Divisibility by 3:**

- $n = 3k$: $n^3 + 11n = 27k^3 + 33k = 3k\left(9k^2 + 11\right) \rightarrow$ **divisible by 3**.
- $n = 3k + 1$: $n^3 + 11n = 27k^3 + 27k^2 + 42k + 12 = 3\left(9k^3 + 9k^2 + 14k + 4\right) \rightarrow$ **divisible by 3**.
- $n = 3k + 2$: $n^3 + 11n = 27k^3 + 54k^2 + 69k + 30 = 3\left(9k^3 + 18k^2 + 33k + 10\right) \rightarrow$ **divisible by 3**.

This says that for all n, broken into multiples/non-multiples of 3, $n^3 + 11n$ is always divisible by 3.

3. Hence, since $2|n^3 + 11n$ and $3|n^3 + 11n$, then $2 \cdot 3 = 6|n^3 + 11n$, a trivial property of divisibility and factors.

Note: We do not need to test say $2k+2$ or $3k+3$ because they are equivalent constructions i.e. $2k + 2 = 2(k + 1) \equiv 2p$ and $3k + 3 = 3(k + 1) \equiv 3q$. This links to an equivalent derivation, known as *modulo arithmetic*, which is not covered in the syllabus.

Example

Consider the following example

"Prove that $\sqrt{3}$ is irrational."

We do this via **contradiction**.

1. **Firstly, is this true?**

   We know when taking square roots of numbers which are not square numbers themselves, that we end up with a decimal value, which typically is never ending i.e. infinite and thus at first glance, this seems true.

2. **Secondly, how do we prove irrationality?**

   We prove irrationality by looking at the negation of the statement i.e. **assuming** rationality. We then follow algebraic steps and gather information, to which if we end up concluding something which is **false**, we then know that we were **wrong** to assume rationality in the first place.

3. **Thirdly, how do we use algebra to demonstrate rationality?**

   - Since we assume $\sqrt{3}$ is rational, this means it can be expressed as a fraction $\frac{a}{b}$, where a and b are positive integers with no common factors i.e. $\text{HCF}(a, b) = 1$.

- Given that $\sqrt{3} = \frac{a}{b}$, we can arrange to obtain

$$3 = \left(\frac{a}{b}\right)^2 = \frac{a^2}{b^2}$$
$$\therefore a^2 = 3b^2$$

  The right hand side is divisible by 3 and hence so must the left i.e. $3|a^2$. This also means that $3|a$ i.e. $a = 3k$. This step should also be proven rigorously but the main reason for this is because 3 is prime and thus if $3|a^2$, the prime factorisation of a^2 must contain 3^2, which implies $3|a$.

- Substituting this back in gives us

$$(3k)^2 = 3b^2$$
$$\Rightarrow 9k^2 = 3b^2$$
$$\therefore 3k^2 = b^2$$

  The left hand side is divisible by 3 and hence so must the right i.e. $3|b^2$. This also means that $3|b$ i.e. $b = 3p$.

- And voila! We have arrived at a contradiction since the $\text{HCF}\,(a,b) = \text{HCF}\,(3k,3p) = 3$ but, by construction of the statement, we should have $\text{HCF}\,(a,b) = 1$.

Note: It is not so clear on an alternative method to prove this. If we tried to do this via *direct argument*, we would need more knowledge about square root representations, but this is not covered in the syllabus.