
OCR MEI Notes

(Year 2)

Department of Mathematics



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Disclaimer

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Siddhartha Nath is the founder of *PolyNath* and the author of a plethora of top-selling books (*Quantitative Finance*, *MAT*, *NLP*, *TMUA/CTMUA*).

He has an undergraduate First-Class Honours BSc Mathematics with Statistics from Imperial College London (ICL) and a postgraduate Distinction MSc Computational Statistics and Machine Learning from University College London (UCL). He has taken his education and amassed over 2 years of industry experience, having worked as a Data Scientist for PayPal and DataSpartan.

In terms of academic teaching, he is an extremely dedicated and supportive tutor, having taught for over 5 years at public and private schools, for GCSE, A-Levels and University preparation. He possesses a wealth of knowledge in all UK-based Mathematics and Computer Science admission tests, with the following accomplishments:

- Achieved a perfect score of 9.0/9.0, in the Cambridge TMUA, resulting in a top 10% ranking.
- Achieved a commendable score of 63/100, in the Oxford MAT, resulting in a top 25% ranking.
- Achieved a high score of 1, 1, 1 in the Cambridge STEP I, II and III.

Outside of academia, he enjoys music, dance, watching sports and creating content.



Preface

This document includes a summary of notes of Chapter 1 - Proof (OCR MEI Year 2).

Happy Learning!

Chapter 1

Proof

Summary Checklist

1.
 - \Rightarrow means *if...then, implies, therefore* etc...
 - \Leftarrow means *only if..., is implied by, follows from* etc...
 - \Leftrightarrow means *implies and is implied by, is equivalent to* etc...
2. The converse of $A \Rightarrow B$ is $B \Rightarrow A$.
3.
 - If $A \Leftarrow B$, A is a **necessary** condition for B .
 - If $A \Rightarrow B$, A is a **sufficient** condition for B .
4. The methods of proof are:
 - Direct Argument
 - Exhaustion
 - Contradiction
 - Counter-Example

Below are 2 examples, demonstrating a complete *thought-process* breakdown of how to approach a *proof*:

Example

Consider the following example

"Prove that $n^3 - n$ is a multiple of 6."

We do this via **direct argument**.

1. **Firstly, is this true?**

When $n = 1$, $1^3 - 1 = 0$, which is divisible by 6. For $n = 2$, $2^3 - 2 = 6$, which is divisible by 6. For $n = 3$, $3^3 - 3 = 24$, which is divisible by 6. So far, it seems to hold. Note, for shorthand, we will use $|$ to denote *divides* i.e. $2|6$ means 2 divides 6.

2. **Secondly, how do we prove divisibility?**

To show divisibility, the number should be divisible by all the *prime* factors of the number i.e. prime factorisation tree. Hence, to be divisible by 6, we should prove that the number is divisible by 2 and 3.

3. **Thirdly, how do we use algebra to demonstrate these divisibility properties?**

- Since we are thinking about factors, we perhaps should do some *factorisation*...

$$\begin{aligned}n^3 - n &= n(n^2 - 1) \\ &= n(n - 1)(n + 1)\end{aligned}$$

- Logically, $2|n^3 - n$ because we have a product 2 consecutive integers between the 3 factors and thus one of them must be divisible by 2.
- Logically, $3|n^3 - n$ because we have a product 3 consecutive integers between the 3 factors and thus one of them must be divisible by 3.
- Alternatively, we could have looked at cases for n i.e. when $n = 2k$ so even or $n = 2k + 1$ so odd, and we would arrive at the same conclusion.

4. **Finally, how do we conclude the proof?**

Hence, since $2|n^3 - n$ and $3|n^3 - n$, then $2 \cdot 3 = 6|n^3 - n$, a trivial property of divisibility and factors.

Note: $n^3 - n$ does not only have 3 factors due to its factorisation - $n(n - 1)$ is also a factor, $n(n + 1)$ too etc... but this is not focus of the problem.

A follow up question is

"Prove that $n^3 + 11n$ is a multiple of 6."

1. **Firstly, is this true?**

When $n = 1$, $1^3 + 11(1) = 12$, which is divisible by 6. For $n = 2$, $2^3 + 11(2) = 30$, which is divisible by 6. For $n = 3$, $3^3 + 11(3) = 42$, which is divisible by 6. So far, it seems to hold.

2. **Secondly, how do we prove divisibility?**

We already know from before that we should show this is divisible by both 2 and 3.

3. **Thirdly, how do we use algebra to demonstrate these divisibility properties?**

- Since we are thinking about factors, we perhaps should do some *factorisation*...

$$n^3 + 11n = n(n^2 + 11)$$

PROBLEM:

We cannot do the same trick as before because we do not seem to have consecutive factors i.e. we do not have a nice factorisation. We could try cases, but after some thought, it would be difficult to reason about divisibility by 3. This is usually the point where we stop and give up... But no! **There was a reason for why we did the first part of the question!**

4. **Fourthly, how can I use previous knowledge to help?**

- A trivial property of divisibility is that if $k|A$ and $k|B$, then $k|A + B$ i.e. if you add multiples of k together, the resulting number is also divisible by k .
- Hence, since $6|n^3 - n$, one can spot that to get $n^3 + 11n$, we must simply add $12n$.
- And voila! $6|12n$ because 6 is a factor of 12, regardless of n , and hence using the rule before, we can conclude that since $6|n^3 - n$ and $6|12n$, we must have $6|(n^3 - n) + (12n) = n^3 + 11n$.

Note: For advanced material, it is easy to tackle this using *modulo arithmetic*, but this is not covered in the syllabus.

Example

Consider the following example

"Prove that $\sqrt{3}$ is irrational."

We do this via **contradiction**.

1. **Firstly, is this true?**

We know when taking square roots of numbers which are not square numbers themselves, that we end up with a decimal value, which typically is never ending i.e. infinite and thus at first glance, this seems true.

2. **Secondly, how do we prove irrationality?**

We prove irrationality by looking at the negation of the statement i.e. **assuming** rationality. We then follow algebraic steps and gather information, to which if we end up concluding something which is **false**, we then know that we were **wrong** to assume rationality in the first place.

3. **Thirdly, how do we use algebra to demonstrate rationality?**

- Since we assume $\sqrt{3}$ is rational, this means it can be expressed as a fraction $\frac{a}{b}$, where a and b are positive integers with no common factors i.e. $\text{HCF}(a, b) = 1$.
- Given that $\sqrt{3} = \frac{a}{b}$, we can arrange to obtain

$$3 = \left(\frac{a}{b}\right)^2 = \frac{a^2}{b^2}$$
$$\therefore a^2 = 3b^2$$

The right hand side is divisible by 3 and hence so must the left i.e. $3|a^2$. This also means that $3|a$ i.e. $a = 3k$. This step should also be proven rigorously but the main reason for this is because 3 is prime and thus if $3|a^2$, the prime factorisation of a^2 must contain 3^2 , which implies $3|a$.

- Substituting this back in gives us

$$(3k)^2 = 3b^2$$
$$\Rightarrow 9k^2 = 3b^2$$
$$\therefore 3k^2 = b^2$$

The left hand side is divisible by 3 and hence so must the right i.e. $3|b^2$. This also means that $3|b$ i.e. $b = 3p$.

- And voila! We have arrived at a contradiction since the $\text{HCF}(a, b) = \text{HCF}(3k, 3p) = 3$ but, by construction of the statement, we should have $\text{HCF}(a, b) = 1$.

Note: It is not so clear on an alternative method to prove this. If we tried to do this via *direct argument*, we would need more knowledge about square root representations, but this is not covered in the syllabus.