MA 109 D2 T1
Week Five Recap

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- Note $m(f) \leq m_i(f) \leq M_i(f) \leq M(f) \;\; \forall i$, and define the sums,

$$L(f, \mathcal{P}) := \sum_{i=1}^{n} m_i(f)(x_i - x_{i-1}) \; ; \; U(f, \mathcal{P}) := \sum_{i=1}^{n} M_i(f)(x_i - x_{i-1}) \text{ (Draw!)}$$

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- Define the *main players*

$$L(f) := \sup\{L(f, \mathcal{P}) \mid \mathcal{P} \text{ is a partition of } [a, b]\}$$
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- *If $\mathcal{P}$ is partition of $[a,b]$, and $\mathcal{P}^*$ is a refinement of $\mathcal{P}$, then*

$$L(f,\mathcal{P}) \leq L(f,\mathcal{P}^*) \quad \text{and} \quad U(f,\mathcal{P}^*) \leq U(f,\mathcal{P}),$$

- *If $\mathcal{P}_1$ and $\mathcal{P}_2$ are partitions of $[a,b]$, then $L(f,\mathcal{P}_1) \leq U(f,\mathcal{P}_2)$.*

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- *$L(f) \leq U(f)$.*

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Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Then, f is said to be Riemann Integrable on $[a, b]$ if $U(f) = L(f)$. Further, we say that the quantity $U(f)$ $(= L(f))$ is the *Riemann Integral* of f on $[a, b]$, and denote it as

$$U(f) = L(f) = \int_a^b f(x)dx$$

The Riemann Condition

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Given a bounded real function f on $[a, b]$, it holds that f is RI as per the previous definition **if and only if** the following holds,

$$\forall \epsilon > 0, \text{ there exists a partition } \mathcal{P} \text{ of } [a, b]$$

$$\text{such that } U(f, \mathcal{P}) - L(f, \mathcal{P}) < \epsilon$$

Two Funny Functions

- The Dirichlet Function, denoted $1_{\mathbb{Q}} : [0, 1] \to \mathbb{R}$, defined as,

$$1_{\mathbb{Q}} := \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

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- The Thomae Function, denoted $\mathrm{T} : [0,1] \to \mathbb{R}$, defined as,

$$\mathrm{T}(x) := \begin{cases} \frac{1}{q} & \text{if } x \in \mathbb{Q}, x \neq 0, x = p/q \text{ with } p, q \text{ coprime} \\ 0 & \text{if } x = 0 \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

Properties

Proposition (Domain Additivity)

Let $f : [a, b] \to \mathbb{R}$ be a bounded function and let $c \in (a, b)$. Then f is integrable on $[a, b]$ if and only if f is integrable on $[a, c]$ and on $[c, b]$. In this case,

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

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Proposition (Order Relations)

Let $f, g : [a, b] \to \mathbb{R}$ be integrable. (i) If $f \leq g$, then $\int_a^b f(x)dx \leq \int_a^b g(x)dx$. (ii) The function $|f|$ is integrable and $\left|\int_a^b f(x)dx\right| \leq \int_a^b |f|(x)dx$.

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Proposition (Algebraic and order relations)

Let $f, g : [a, b] \to \mathbb{R}$ be integrable functions. Then

- $f + g$ is integrable and $\int_a^b (f+g)(x)dx = \int_a^b f(x)dx + \int_a^b g(x)dx$,
- rf is integrable for any $r \in \mathbb{R}$ and $\int_a^b (rf)(x)dx = r\int_a^b f(x)dx$,
- fg is integrable,
- if there is $\delta > 0$ such that $|f(x)| \geq \delta$ and all $x \in [a, b]$, then $1/f$ is integrable,
- if $f(x) \geq 0$ for all $x \in [a, b]$, then for any $k \in \mathbb{N}$, the function $f^{1/k}$ is integrable.

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The Fundamental Theorem of Calculus

The FTC connects together the notions of integrability and differentiability. That is,

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Also, a helpful definition.

Definition (Antiderivative)

Given a function $f : D \to \mathbb{R}$, we say that f has a antiderivative on D if there exists a differentiable function $F : D \to \mathbb{R}$ such that,

$$F'(x) = f(x) \ \forall x \ \in \ D$$

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- If f is continuous at $c \in [a, b]$, then F is differentiable at c. Further, $F'(c) = f(c)$.

That is, if your f is continuous, the proposed F is an antiderivative. Thus a continuous function on an interval always possesses an antiderivative.

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for *any* antiderivative F of f. This also relates to the fact that two antiderivatives differ by a constant.

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Given $f : [a, b] \to \mathbb{R}$, and a tagged partition $(\mathcal{P}, t)$ of $[a, b]$, define

$$R(f, \mathcal{P}, t) := \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1})$$

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Note that, $L(f, \mathcal{P}) \leq R(f, \mathcal{P}, t) \leq U(f, \mathcal{P})$.

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Further for a partion $\mathcal{P} := \{x_0 = a < x_1 < x_2 < \cdots < x_{n-1} < x_n = b\}$, define $||\mathcal{P}|| := \max\{x_i - x_{i-1}\}$ to be the *norm* of that partition.

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$$|R(f, \mathcal{P}, t) - R| < \epsilon,$$

whenever $\|\mathcal{P}\| < \delta$

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whenever $\|\mathcal{P}\| < \delta$ and for all choice of tags t. In this case R is called the Riemann integral of the function f on the interval $[a, b]$.

A Definition

Further for a partion $\mathcal{P} := \{x_0 = a < x_1 < x_2 < \cdots < x_{n-1} < x_n = b\}$, define $||\mathcal{P}|| := \max\{x_i - x_{i-1}\}$ to be the *norm* of that partition.

Definition (RS Def I)

A function $f : [a, b] \to \mathbb{R}$ is said to be Riemann integrable if there exists some $R \in \mathbb{R}$ such that for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$|R(f, \mathcal{P}, t) - R| < \epsilon,$$

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A weaker definition

Definition (RS Def II)

A function $f : [a, b] \to \mathbb{R}$ is said to be Riemann integrable if there exists some $R \in \mathbb{R}$ such that for every $\epsilon > 0$ there exists a $\delta > 0$ **and a** partition $\mathcal{P}$ such that for every tagged refinement $(\mathcal{P}', t')$ of $\mathcal{P}$ with $\|\mathcal{P}'\| \leq \delta$ we have,

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- $||\mathcal{P}^*|| \leq ||\mathcal{P}||$
- $L(f, \mathcal{P}) \leq L(f, \mathcal{P}^*) \leq R(f, \mathcal{P}^*, t^*) \leq U(f, \mathcal{P}^*) \leq U(f, \mathcal{P})$

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Theorem (Tying things up)

The three definitions, viz. The RI, RS Def I, and the RS Def II are **equivalent**.

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The RI definition, along with the RC help us in proving things about integrals. The RS Def (II) helps us in computing integrals (rigorously).

Application: Area

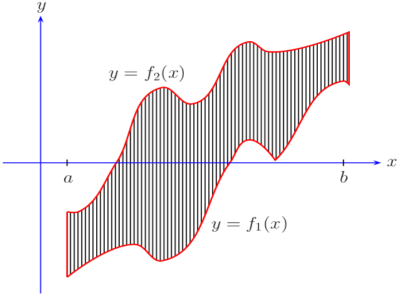

Figure: Area: Type 1

Application: Area

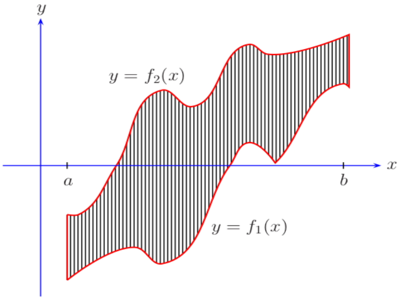

Figure: Area: Type 1

We compute the area as,

$$A = \int_a^b (f_2(x) - f_1(x))dx$$

Application: Area

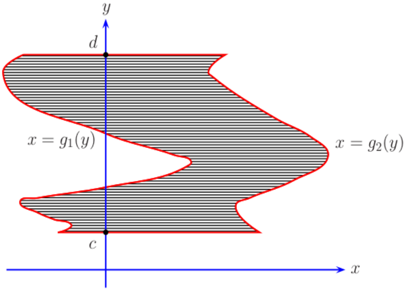

Figure: Area: Type 2

Application: Area

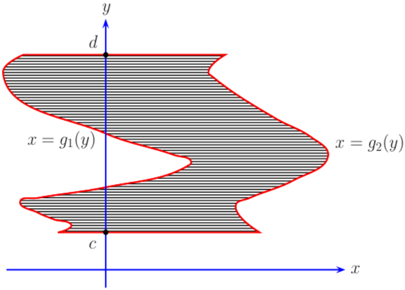

Figure: Area: Type 2

We compute the area as,

$$A = \int_c^d (g_2(y) - g_1(y))dy$$

Application: Area

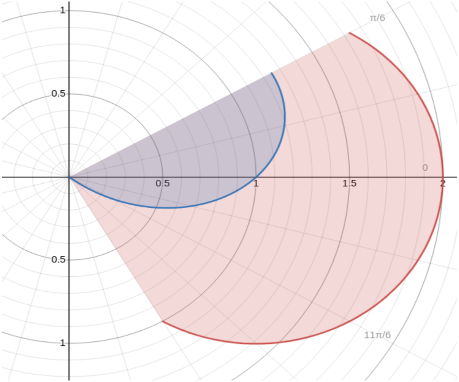

Figure: Area: Type 3, $\rho_1(\theta) = 2\cos\theta$, $\rho_2(\theta) = \cos^2\theta + \sin\theta$

Application: Area

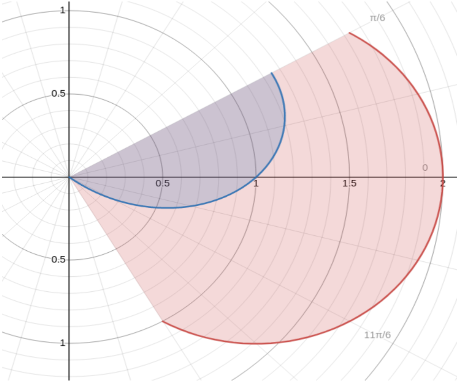

Figure: Area: Type 3, $\rho_1(\theta) = 2\cos\theta$, $\rho_2(\theta) = \cos^2\theta + \sin\theta$

We compute the area as,

$$A = \frac{1}{2}\int_{\alpha}^{\beta} (\rho_2(\theta)^2 - \rho_1(\theta)^2) d\theta$$

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Definition (Curve)

A *curve* is a tuple $C = (x(t), y(t))$ wherein $x, y : [a, b] \to \mathbb{R}$ are continuous functions.

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Steps:

- Sum of lengths, $\sum_{i=1}^{n} \sqrt{\left(x\left(t_i\right) - x\left(t_{i-1}\right)\right)^2 + \left(y\left(t_i\right) - y\left(t_{i-1}\right)\right)^2}$

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- Apply MVT, $\sum_{i=1}^{n} \sqrt{\left(x'\left(s_i\right)\right)^2 + \left(y'\left(u_i\right)\right)^2}\left(t_i - t_{i-1}\right);$

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- Conclude,

$$\boxed{\text{Arc Length}(C) := \int_{\alpha}^{\beta} \sqrt{x'(t)^2 + y'(t)^2} dt}$$

Application: Surface Area

- Identify the area of a frustum, $A_F = \pi\lambda_2(d_1 + d_2)$

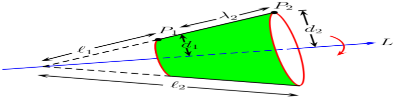

Figure: A frustum

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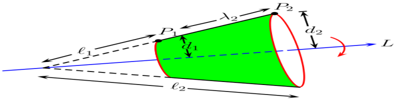

Figure: A frustum

- Use this to form the Riemann Sum, $\pi \sum(\rho(t_{i-1}) + \rho_{t_i})\lambda_i$

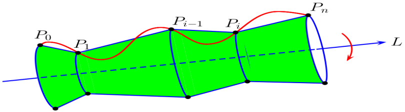

Figure: A general area

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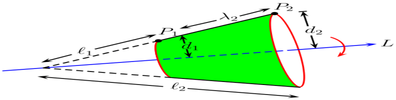

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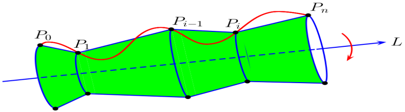

Figure: A general area

Thus,

$$\boxed{\text{Area}(S) := 2\pi\int_{\alpha}^{\beta}\rho(t)\sqrt{x'(t)^2 + y'(t)^2}\,dt,}$$

Application: Volume

Same procedure – partition, form a Riemann Sum, conclude hoping that the area function A is integrable.

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We will deal with solids of revolution. Two methods, funny names,

1. Washer,
2. Shell.

Washer Method

Slices take **perpendicular to rotation axis**.

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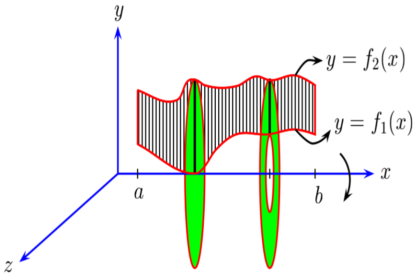

Figure: Washers and Disks

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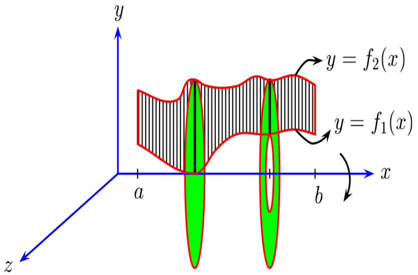

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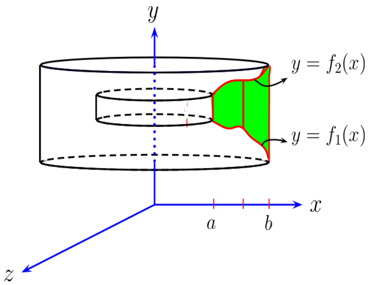

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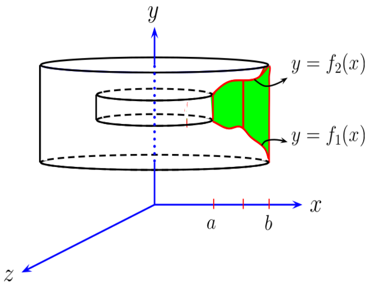

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