Non-Hermitian (NH) Topology – What Changes?

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EE 787 Course Project
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- Why? Topological Origins? Let's see!

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$$w = \sum_{n=1}^{N} \int_{-\pi}^{\pi} \frac{dk}{2\pi} \partial_k \arg E_n(k) \tag{3}$$

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$$w = \begin{cases} 1 & |J_R| < |J_L| \, ; \\ -1 & |J_R| > |J_L| \, . \end{cases} \tag{6}$$

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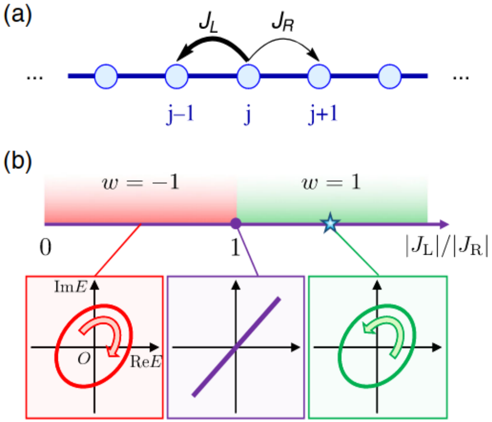

Figure 1: Hatano Nelson Model: Single Band Topology!

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- $\sum_n |\psi_L^n\rangle \langle \psi_R^n| = 1$

(Assuming no degenerecies for simplicity)

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- Eigenvectors *coalesce* at these points.

Exceptional Points (EPs) [MA19]

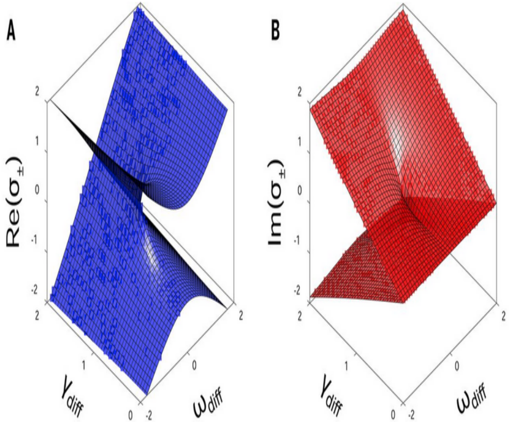

Figure 2: Exceptional Points Visualized

EPs $\rightarrow$ Monopoles and Mobius Strips! [ZW20]

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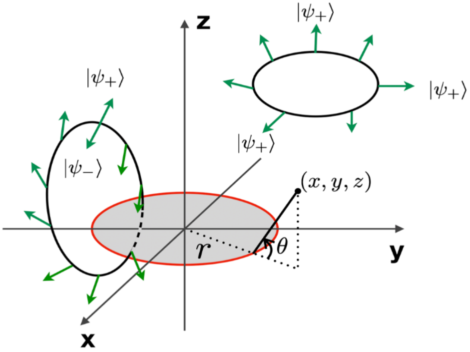

Figure 3: Hermitian Mobius Strip

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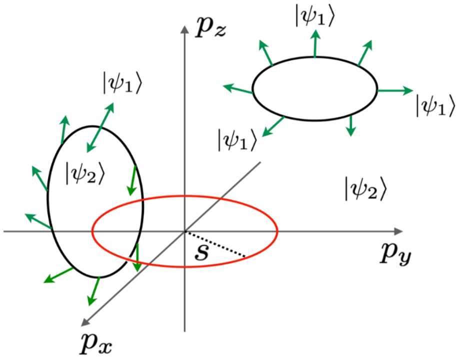

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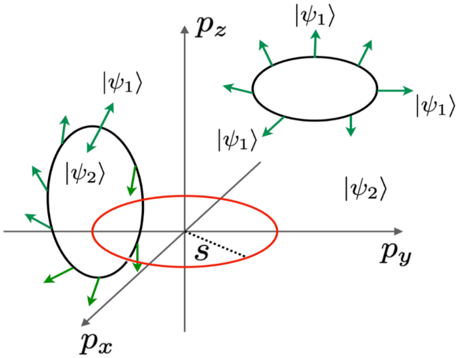

Figure 4: NH Mobius Strip

Branch Cuts! $f(z) := \sqrt{z}$ is <u>not</u> holomorphic on $\mathbb{C}$.

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Where, $\omega_{mn}(t) = (E_m(t) - E_n(t))/\hbar$.

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- Berry phase is real only if ([ZW19])

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- $$\boxed{\mathbf{A}_m = \iota \langle \psi_L^m(\mathbf{R}) \mid \nabla_{\mathbf{R}} \psi_R^m(\mathbf{R}) \rangle} \tag{17}$$
- $$\boxed{\mathbf{B}_m = \iota \langle \nabla \psi_L^m \mid \times \mid \nabla \psi_R^m \rangle} \tag{18}$$
- Berry phase is real only if ([ZW19])
$$\langle \psi_R^m \mid (\nabla_{\mathbf{R}} X) \mid \psi_R^m \rangle = 0 \tag{19}$$

Breakdown of the BBC

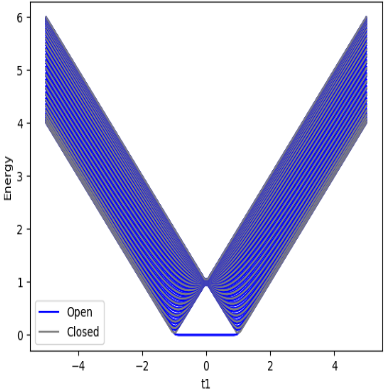

Figure 5: SSH Chain: Standard Energy Plot

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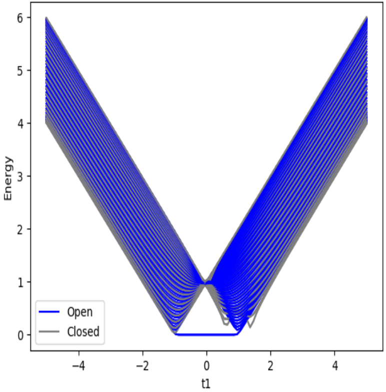


Figure 6: SSH Chain: Non-reciprocity with $\Delta = 0.2$

NHSE!

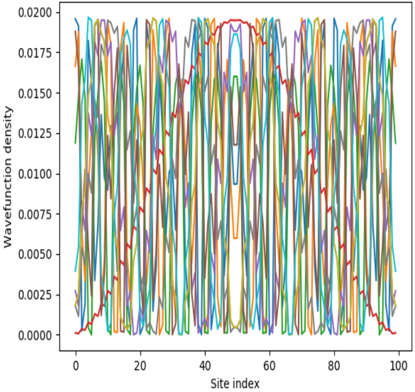
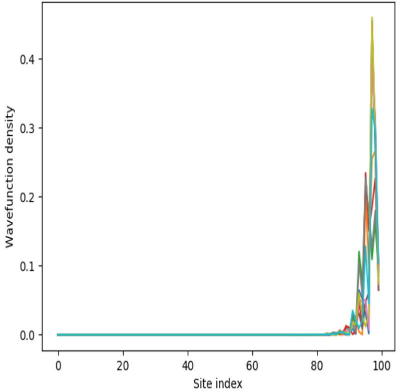


Figure 7: SSH Chain: Standard Density Plot

NHSE!



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Now what?

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Theorem 2 (GBZ Theorem)

The GBZ is that closed contour in the complex plane which encircles the pole of order m at the origin and exactly m zeros of the polynomial $P(z) - Ez^m$ for any $E \in \mathbb{C}$.

GBZ: Consequences [ZYF20] [Gon+18] [YM19]

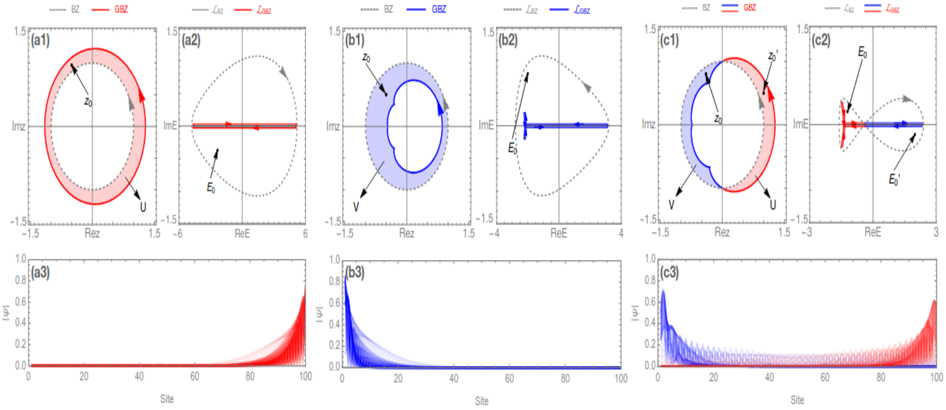

Figure 9: [ZYF20] GBZ, BZ and spectra for different hamiltonians $H(z) = z^{-2}/5 + 3z^{-1} + 2z$, $H(z) = z^{-2}/5 + z^{-1} + 2z$ and $H(z) = 2z^{-2}/5 + z^{-1} + z$

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NHSE almost always!

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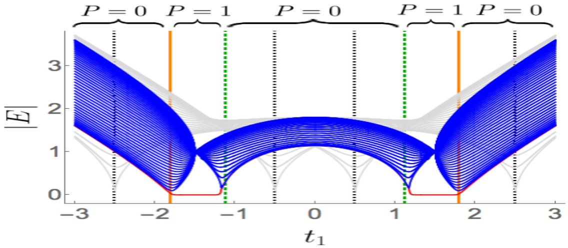

Figure 10: Absolute spectra of NH SSH model. The gray lines indicate the periodic Bloch bands while the qualitatively different spectra in blue (bulk) and red (edge) correspond to the open system. The orange (dark green dashed) vertical lines indicate where $r_L^* r_R = 1$ ($r_L^* r_R = -1$) and the gray dotted-dashed lines correspond to the EPs of the periodic Bloch Hamiltonian. [Kun+18]

Winding Number in NH SSH [Lie18]

$$
\begin{aligned}
H^{\text{hop}} &= v_1 \sum_{n=1}^{N} |n, B\rangle\langle n, A| + v_2 \sum_{n=1}^{N} |n, A\rangle\langle n, B| \\
&+ w_1 \sum_{n=1}^{N-1} |n+1, A\rangle\langle n, B| + w_2 \sum_{n=1}^{N-1} |n, B\rangle\langle n+1, A|
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- Parametrize

$$\left|\psi_{R,K}^{\pm}\right\rangle = \frac{1}{\sqrt{2\sin\theta_k\cos\theta_k}} \begin{pmatrix} e^{-i\phi_k}\cos\theta_k \\ \pm\sin\theta_k \end{pmatrix} \left|\psi_{L,K}^{\pm}\right\rangle = \frac{1}{\sqrt{2\sin\theta_k\cos\theta_k}} \begin{pmatrix} e^{-i\phi_k}\sin\theta_k \\ \pm\cos\theta_k \end{pmatrix} \tag{37}$$

Winding Number in NH SSH: $u = 0$ [Lie18]

- $u = 0 \implies$ CS!
-

$$\sigma_z H(k) \sigma_z = -H(k) \tag{33}$$

$$\left|\psi_R^{\pm}\right\rangle = \sigma_z \left|\psi_R^{\mp}\right\rangle \tag{34}$$

- CS implies a conjugated-pseudo symmetry

$$H^{\dagger} = \sigma_x H^* \sigma_x \tag{35}$$

$$\left|\psi_L^{\pm}\right\rangle = \sigma \left|\psi_R^{\pm}\right\rangle \tag{36}$$

- Parametrize

$$\left|\psi_{R,K}^{\pm}\right\rangle = \frac{1}{\sqrt{2\sin\theta_k \cos\theta_k}} \begin{pmatrix} e^{-i\phi_k}\cos\theta_k \\ \pm\sin\theta_k \end{pmatrix} \left|\psi_{L,K}^{\pm}\right\rangle = \frac{1}{\sqrt{2\sin\theta_k \cos\theta_k}} \begin{pmatrix} e^{-i\phi_k}\sin\theta_k \\ \pm\cos\theta_k \end{pmatrix} \tag{37}$$

Winding Number in NH SSH: $u \neq 0$ [Lie18]

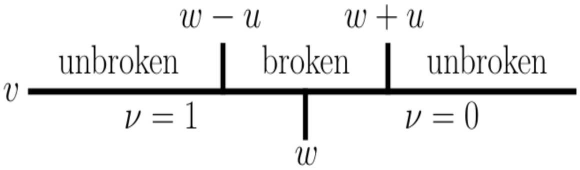

Figure 11: One-dimensional phase diagram in v for the symmetric case $v_1 = v_2 = v \in$, $w_1 = w_2 = w \in$. The system moves in and out of the broken phase as a function of v. ν is the topological index which predicts how many pairs of gapless-real-energy, edge modes exist in the system. The bulk spectrum is gapless in the entire -broken phase. [Lie18]

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