MA 205: $\mathbb{C}$omplex Analysis
Endsem TSC

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11th September 2022

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- Finally, if you notice some mistake, do let us know.

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Open and Closed Sets

Definition (Open Disks)

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Recall: $z \in \mathbb{C}$ is a limit point of $\Omega \subset \mathbb{C}$ if there exists a sequence $z_n \in \Omega$, $z_n \neq z$, such that $z_n \to z$.

Connectedness

Definition (Connected)

A subset $S \subseteq \mathbb{C}$ is said to be connected if given any 2 points $x, y \in S$, there exists a continuous path joining them. i.e, a continuous function $f : [0, 1] \to S$ such that $f(0) = x$ and $f(1) = y$.

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Definition (Domain)

A open and connected subset of $\mathbb{C}$ is called a domain.

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Sequences and Convergence

Definition (Sequences)

A sequence in $\mathbb{C}$ is a function $f : \mathbb{N} \cup \{0\} \to \mathbb{C}$. We denote $z_n = f(n)$.

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Theorem

If $z_n = x_n + \iota y_n$ is a sequence in $\mathbb{C}$, then

$$z_n \to z = x + \iota y \Leftrightarrow x_n \to x \text{ and } y_n \to y$$

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A function $f : \Omega \subset \mathbb{C} \to \mathbb{C}$ is said to be continuous at a point $z_0 \in \Omega$ if

$$\lim_{z \to z_0} f(z) = f(z_0)$$

Equivalently[a], f is continuous at z_0 if for all sequences $(z_n)_n$ $(z_n \in \Omega)$ such that $z_n \to z_0$ we have $f(z_n) \to f(z_0)$.

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[a] The $\epsilon - \delta$ continuity definition $\Leftrightarrow$ the sequential definition

- f is said to be continuous if it is continuous at all $z_0 \in \Omega$.
- f is continuous iff u and v are continuous.

Differentiability

Definition (Complex Differentiability (CD))

Let $\Omega \subset \mathbb{C}$ be open. A function $\Omega \to \mathbb{C}$ is said to be complex-differentiable at $z_0 \in \mathbb{C}$ if the limit

$$\lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}$$

exists. If it does, we denote it by $f'(z_0)$.

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- Differentiability $\implies$ Continuity.

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- **Remark**: A function can be CD at a point and not holomorphic at the same point. Consider $f(z) = |z|^2$.

Properties

If $f : \Omega \to A$ and $g : \Omega \to B$ are holomorphic on Ω, then,

- $c_1 f + c_2 g$ is holomorphic on Ω, and $(c_1 f + c_2 g)' = c_1 f' + c_2 g'$.

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- If $h : A \to \mathbb{C}$ is holomorphic on A, then $h \circ f(z) := h(f(z))$ is holomorphic on Ω, and $(h \circ f)'(z) = h'(f(z))f'(z)$.

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- If $h : A \to \mathbb{C}$ is holomorphic on A, then $h \circ f(z) := h(f(z))$ is holomorphic on Ω, and $(h \circ f)'(z) = h'(f(z))f'(z)$.
- For $z_0 \in \Omega$ s.t. $g(z_0) \neq 0$, f/g is holomorphic at z_0, and,

$$\left(\frac{f}{g}\right)(z_0) = \frac{f'(z_0)g(z_0) - g'(z_0)f(z_0)}{g(z_0)^2}$$

Real Differentiability

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Real Differentiability

$F : \Omega_R \to \mathbb{R}^2$ is differentiable at $(x,y) \in \Omega$ if there exists a 2×2 matrix $DF(x,y)$ such that

$$\lim_{h,k \to 0} \frac{|| \begin{pmatrix} u(x+h, y+k) \\ v(x+h, y+k) \end{pmatrix} - \begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix} - DF(x,y) \begin{pmatrix} h \\ k \end{pmatrix} ||}{|| \begin{pmatrix} h \\ k \end{pmatrix} ||} = 0$$

If so, we have

$$DF(x,y) = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}$$

Necessary Conditions for (complex) differentiability

Theorem (The CR Equations)

Let $f(z) = u(x, y) + \iota v(x, y)$ be defined on some open set Ω. Suppose that $f'(z_0)$ exists for some point $z_0 = x_0 + \iota y_0 \in \Omega$. Then the first order partial derivatives of u and v exist at that point (x_0, y_0) and satisfy the CR equations

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Let f be defined on some open set Ω be differentiable at some $z = (x + \iota y) \in \Omega$. Then, the real counterpart F will be differentiable at $(x, y) \in \Omega_R$.

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i.e., $CD \implies RD$.

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  CR equations are satisfied at zero, but it is not CD at zero.
- We shall see that together they are sufficient to show CD.

Necessary and Sufficient Condition

Theorem

Let $f(z) = u(x, y) + \iota v(x, y)$ be defined on some open set Ω and let $F : \Omega_R \to \mathbb{R}^2$ be the corresponding real function. For some $z_0 = x_0 + \iota y_0 \in \Omega$, if

1. F is differentiable at (x_0, y_0).
2. The $DF(x_0, y_0)$ is of the following form
$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$
(equivalently, the CR equations are satisfied at (x_0, y_0))

Then, we have that $f'(z_0)$ exists and equals $a + \iota b$. Further, the converse holds.

Another Sufficient Condition

Theorem

Let $f(z) = u(x, y) + \iota v(x, y)$ be defined on some open set Ω. For some $z_0 = x_0 + \iota y_0 \in \Omega$, if

1. the partial derivatives of u and v exist in some neighbourhood of (x_0, y_0) and are continuous at (x_0, y_0), and
2. the CR equations are satisfied at (x_0, y_0)

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2. the CR equations are satisfied at (x_0, y_0)

Then, we have that $f'(z_0)$ exists.

This turns out to be easier to check.

Definition (Harmonic Function)

A function $g : \Omega_R \subset \mathbb{R}^2 \to \mathbb{R}$ is said to be harmonic if it has continuous partial derivatives of the first and second order, and satisfies

$$\triangle g(x, y) = g_{xx}(x, y) + g_{yy}(x, y) = 0 \ \forall (x, y) \in \Omega_R$$

Definition (Harmonic Function)

A function $g : \Omega_R \subset \mathbb{R}^2 \to \mathbb{R}$ is said to be harmonic if it has continuous partial derivatives of the first and second order, and satisfies

$$\triangle g(x, y) = g_{xx}(x, y) + g_{yy}(x, y) = 0 \ \forall (x, y) \in \Omega_R$$

Theorem

If a function $f(z) = u(x, y) + \iota v(x, y)$ is CD in a domain Ω, then u and v are harmonic in D_R.

Summarizing ...

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The word 'expression' signifies that the series/power series may or may not be meaningful (read convergent).

The Convergence Theorem

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Usually, we allow for $R = 0, \infty$ for convenience.

[a] no comments on the boundary!

Follow up theorem

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The *radius of convergence* of a power series as defined before is given as

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Herein, we allow for $R = 0, \infty$ by letting $1/0 = \infty, 1/\infty = 0$.

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- Thus, the lim-sup always exists. The limit might not.
- When the limit exists, the limsup is equal to the limit.

Power series are holomorphic

Theorem

1. The power series $\sum_n a_n(z - z_0)^n$ defines a holomorphic function $f : D(z_0, R) \to \mathbb{C}$, $f(z) := \sum_n a_n(z - z_0)^n$ where R is the RoC.

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This leads to the statement

$$\textbf{Analytic} \implies \textbf{Holomorphic}$$

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Let $\sum_n z_n$ be a complex series. Note the following,

1. **Necessary Condition for Convergence** If $\sum_n z_n$ converges, then $|z_n| \to 0$ as $n \to \infty$. (Recall the tutorial question about $\sum nz^n$)
2. **Necessary & Sufficient Condition for Convergence** Note that $|z_n| \geq 0$ (thus $s_n := \sum_{k=0}^{n} |z_k|$ is monotonic increasing), we have that $\sum_n |z_n|$ converges iff $s_n = \sum_{k=0}^{n} |z_k|$ is bounded above. (follows from the MCT, recall the tutorial question about $\sum z^n/n^2$)

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- If f is holomorphic on an open set containing Image(γ), then,

$$\int_\gamma f'(z)dz = f(\gamma(b)) - f(\gamma(a))$$

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Theorem

If f is holomorphic in a domain (thus, connected), and $f' \equiv 0$ in that region, then f is a constant.

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Cauchy Integral Theorem

Let Ω be a bounded domain in $\mathbb{C}$, with piecewise smooth boundary $\partial\Omega$ and $f \in C^1(\bar{\Omega})$ is holomorphic on Ω. Then,

$$\int_{\partial\Omega} f(z)dz = 0$$

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$$f(z) = \frac{1}{2\pi\iota} \int_{\partial\Omega} \frac{f(\eta)}{\eta - z} d\eta$$

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Note that, in these theorems we are dealing with $\partial\Omega$ being traversed anticlockwise.

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Note that, in these theorems we are dealing with $\partial\Omega$ being traversed anticlockwise. Also, we do *not* need $f \in C^1(\bar{\Omega})$, as holomorphicity of f guarantees holomorphicity and thus continuity of f'.

Another way

Another way to state the CIT, which can avoid possible mistakes.

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CIT - Aliter

If $f : \Omega \to \mathbb{C}$ is holomorphic, and Ω is a **simply connected** domain, then for every closed piecewise smooth curve γ within Ω we have,

$$\int_\gamma f(z)dz = 0$$

Questions

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$$\int_{|z|=1} \frac{e^z \sin(z) - z}{z^2 \cos(z)} dz$$

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Answer: $2\pi\iota(e-2)$.

Connecting the dots

Theorem (Morera's Theorem)

Given a continuous function g on a domain Ω, we have that if $\int_\gamma g(z)dz = 0$ for all $\gamma = \partial R$, whenever $R \subset \Omega$ is a rectangle, then, g is holomorphic on Ω.

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Given a function g which is complex differentiable at each point in a domain Ω, we have that $\int_\gamma f(z)dz = 0$ whenever $\gamma = \partial R$ and $R \subset \Omega$ is a rectangle.

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Theorem (Goursat's Theorem)

If a function is complex differentiable at each point in an open set, then it is continuously differentiable in that set. Thus, it is holomorphic on that set.

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- **Holomorphic $\implies$ Analytic**. If f is holomorphic at a point $z_0 \in \mathbb{C}$, then we have that $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$ for all $z : |z - z_0| < r$ for some small r.

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1. [2018 Midsem] If f is a holomorphic function on an open set containing the closed unit disk, and,

$$\int_{|z|=1} f(z)\bar{z}^j dz = 0$$

holds true for all $j = 0, 1, 2 \dots$, then show that $f \equiv 0$.

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Classifying Isolated Singularities - I

[#1] Removable Singularities

An isolated singularity $z \in \mathbb{C}$ of f is said to be removable if there exists a holomorphic function $\tilde{f} : D(z, r) \to \mathbb{C}$ for some $r > 0$ **such that** $\tilde{f}(z) = f(z) \forall z \in D^*(z, r)$.

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Proof? Start with $g(z) := (z - p)^2 f(z)$. Explicitly construct the desired $\tilde{f}$.

Classifying Isolated Singularities - II

[#2] Poles

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A pole of order one is called a *simple pole*.

Classifying Isolated Singularities - III

[#3] Essential Singularities

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Suppose f has an essential singularity at p. Then, for any $r > 0$, f takes on **all possible** complex values, **infinitely often**, with at most a single exception in $D^*(p, r)$.

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Also recall the **Little Picard Theorem**: An entire function can omit at most one complex number.

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$$\frac{(z+2)\cos 1/z}{z-3}$$

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Definition (Meromorphic Function)

A function holomorphic on a domain except possibly for a set of poles, is said to be meromorphic on that domain.

Theorem (Laurent Series)

Suppose f is analytic in the annulus $\mathcal{A}(p, r_1, r_2) := \{z : r_1 < |z - p| < r_2\}$.

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holds for all $z \in \mathcal{A}(p, r_1, r_2)$. Where,

$$a_n := \frac{1}{2\pi\iota} \int_\gamma \frac{f(\eta)}{(\eta - p)^{n+1}} \ \forall n \in \mathbb{Z}$$

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As an immediate consequence, see that

$$\boxed{\int_C f(z)dz = 2\pi\iota \times Res(f, p)}$$

Theorem

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Residues of Meromorphic functions

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$$Res(f, p) = \lim_{z \to p} \frac{1}{(k-1)!} \frac{d^{k-1}}{dz^{k-1}} \left((z-p)^k f(z) \right)$$

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5. $\int_{|z|=1} \frac{1}{\sin 1/z} dz$ Cannot apply the CRT!

The Cauchy Residue Theorem

This is a formal restatement of the earlier conclusion.

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Let γ be a simple closed contour. Let f be analytic inside and on γ except for a finite number of isolated singularities p_k, then,

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$$f(z) = (z - z_0)^m g(z)$$

for some holomorphic g which does not vanish in a neighbourhood of z_0.

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$$f(z) = (z - z_0)^m g(z)$$

for some holomorphic g which does not vanish in a neighbourhood of z_0.

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where, Z is the number of zeros of f inside γ counted with multiplicites and P is the number of poles of f inside γ counted with order.

And that's a wrap!

Thank you, and all the best!