MA 205 Autumn 2022
Tutorial Sheet 4

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1. Assume that f is an entire function and that there exists a point $\zeta_0 \in \mathbb{C}$ such that

$$|f(z) - \zeta_0| \geq 1 \text{ for all } z \in \mathbb{C}.$$

Conclude that f is constant.
Hint: Consider $g(z) = \dfrac{1}{f(z) - \zeta_0}$.

Sol.
Two ways,

- Given that $|f(z) - \zeta_0| \geq 1$ for all $z \in \mathbb{C}$ holds, we have that $f(z) \neq \zeta_0$. Thus, we have that the function $g : \mathbb{C} \to \mathbb{C}$ defined as

  $$g(z) := \frac{1}{f(z) - \zeta_0}$$

  is entire. But, we have that

  $$|g(z)| \leq 1$$

  Thus, g is bounded. By Liouville's Theorem, we are done.

- Note that f omits the entire disk $D(\zeta_0, 1)$. By Picard's Little Theorem, we are done.

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2. Assume that f is an entire function and satisfies the estimate

$$|f(z)| \leq C(1 + |z|)^n \text{ for all } z \in \mathbb{C},$$

for some positive integer n. Conclude that f is a polynomial in z of degree $\leq n$.
Hint: Apply Cauchy Inequalities in the disk $D(0, R)$, and let $R \to \infty$ to show that $f^{(n+1)}(z) = 0$ for all $z \in C$.

Sol. Consider the disk $D(0, R)$. Since f is entire, we can expand it around the origin as

$$f(z) = \sum_n a_n z^n$$

where, the the radius of convergence is ∞. Further recall that

$$a_n = \frac{f^{(n)}(0)}{n!} \ (\#)$$

Now, by Cauchy's Inequality, we have

$$\begin{aligned} |f^{(n+1)}(0)| &\leq \frac{(n+1)!M_R}{R^{n+1}} \\ &\leq \frac{(n+1)!C(1+R)^n}{R^{n+1}} \\ &\leq \frac{(n+1)!C(1/R+1)^n}{R^1} \end{aligned}$$

Take $R \to \infty$, and see that $f^{n+1}(0)$ is zero. Similarly, $f^m(0)$ is zero for all $m \geq n+1$. This along with (#) shows that

$$f(z) = a_0 + a_1 z + \dots a_n z^n$$

Thus, f is a polynomial with degree atmost n. ■

3. Let $R \geq 1$ and n, m be positive integers. Show

$$\left| \int_{|z|=R} \frac{z^n}{z^m - 1}\ dz \right| \leq 2\pi \frac{R^{n+1}}{R^m - 1}.$$

Sol. We have,

$$\begin{aligned} \left| \int_{|z|=R} \frac{z^n}{z^m - 1}\ dz \right| &\leq 2\pi R \times \max_{|z|=R} \left| \frac{z^n}{z^m - 1} \right| \\ &\leq 2\pi R \times \frac{R^n}{R^m - 1} \\ &\leq 2\pi \frac{R^{n+1}}{R^m - 1} \end{aligned}$$

as, we have that $||z^m| - 1| \leq |z^m - 1| \implies \frac{z^n}{z_m - 1} \leq \frac{R^n}{R^m - 1}$.

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4. Let n be a positive integer. Show

$$\int_0^{2\pi} \cos^{2n}\theta\ d\theta = \frac{2\pi}{4^n} \frac{(2n)!}{(n!)^2}.$$

Hint: Compute $\displaystyle\int_{|z|=1} \left(z + \frac{1}{z}\right)^{2n} \frac{1}{z}\ dz.$

Sol. Let the desired integral be I. Consider,

$$\begin{aligned} \int_{|z|=1} \left(z + \frac{1}{z}\right)^{2n} \frac{1}{z}\ dz &= \int_0^{2\pi} \left(e^{\iota\theta} + \frac{1}{e^{\iota\theta}}\right)^{2n} \frac{1}{e^{\iota\theta}}\ \iota e^{\iota\theta} d\theta \\ &= \int_0^{2\pi} (2\cos\theta)^{2n}\ \iota d\theta \\ \implies I &= \frac{1}{4^n \iota} \int_{|z|=1} \left(z + \frac{1}{z}\right)^{2n} \frac{1}{z}\ dz \\ &= \frac{1}{4^n \iota} \int_{|z|=1} \left(\frac{(z^2+1)^{2n}}{z^{2n+1}}\right)\ dz \\ &= \frac{1}{4^n \iota} \times \frac{2\pi\iota}{(2n)!} \frac{d^{2n}}{dz^{2n}} (z^2+1)^{2n}\ |_{z=0} \end{aligned}$$

Where the last point followed from the CIF. Now,

$$\begin{aligned} (1+z^2)^{2n} &= \sum_{k=0}^{2n} \binom{2n}{k} z^{2k} \\ \implies \frac{d^{2n}}{dz^{2n}} (z^2+1)^{2n}{}_{z=0} &= \left(\frac{2n!}{n!}\right)^2 \\ \implies \frac{1}{4^n \iota} \times \frac{2\pi\iota}{(2n)!} \frac{d^{2n}}{dz^{2n}} (z^2+1)^{2n}\ |_{z=0} &= \frac{2\pi}{4^n} \frac{(2n)!}{(n!)^2} \end{aligned}$$

5. Locate and classify the singularities for the following functions:

1. $\dfrac{1}{\sin\left(\frac{1}{z}\right)}$,
2. $\dfrac{z^5 \sin\left(\frac{1}{z}\right)}{1+z^4}$,
3. $\dfrac{z^2+z+1}{z^3-11z+13}$.

Sol.

1. Non-isolated singularity at $z = 0$ - consider the sequence $z_n = 1/(n\pi)$. Isolated singularity at $1/(n\pi)$ for all $n \in \mathbb{Z}/\{0\}$ (Why isolated?).
2. Essential singularity at $z = 0$ (z is Complex, $\sin 1/z$ is not bounded). Isolated singularity at $z = e^{\iota\pi/4}, e^{\iota 3\pi/4}, e^{\iota 5\pi/4}, e^{\iota 7\pi/4}$.
3. Isolated singularities at the three real roots of the denominator polynomial.

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