Distance Measures for Quantum Information

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Classical Measures of Distance

We have two different measures of similarity or distance between two probability distributions/states $\{p_x\}$ and $\{q_x\}$ over the same index set x. They are -

1. Trace Distance $= D(p_x, q_x) = \frac{1}{2}\sum_x |p_x - q_x|$
   Properties:
   - It is a metric on probability distributions.
   - $D(p_x, q_x) = \max_S |\sum_{x\in S} p_x - \sum_{x\in S} q_x|$ over all subsets S of the index set $\{x\}$, $\rightarrow$ relation to distinguishability.
2. Fidelity $= F(p_x, q_x) = \sum_x \sqrt{p_x q_x}$
   Properties:
   - It is not a metric but $\cos^{-1}(F)$ is a metric
   - It can be geometrically interpreted as the inner product between the unit vectors $\sqrt{p_x}$ and $\sqrt{q_x}$

Note that these are *static* measures of distance between probability distributions.

Now let us consider a dynamic measure of distance which encapsulates how well information is preserved by a physical process i.e. noise.

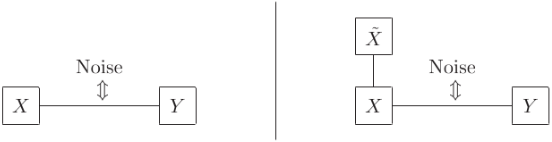

Figure 9.2. Given a Markov process $X \to Y$ we may first make a copy of X, $\tilde{X}$, before subjecting X to the noise which turns it into Y.

Suppose you have the state X and you subject it to a Markov process to get Y. A natural measure would be $P(X \neq Y)$. Now make a perfectly correlated copy $\hat{X}$ of X. It turns out trace has an intimate relation with this dynamic measure of distance which is

$$D(X, \hat{X}) = P(X \neq Y)$$

Quantum Trace Distance

Define the quantum trace distance between two density operators as

$$D(\rho, \sigma) = \frac{1}{2} Tr(|\rho - \sigma|)$$

where $|A| = \sqrt{A^T A}$ (positive square root)
We have,

- The tuple $(Dens(\mathcal{H}), D)$ is a metric space.
- For commuting states,

$$D(\rho, \sigma) = D(\lambda_i, \mu_i)$$

Quantum Trace Distance

Bloch sphere. Let $\rho \equiv \vec{r}, \sigma \equiv \vec{s}$. Then,

$$D(\rho, \sigma) = \frac{|\vec{r} - \vec{s}|}{2}$$

1. Converts to euclidean distance.
2. Hints towards rotation invariance.
3. Helpful visualization.

Relation to distinguishability via measurement.

$$D(\rho, \sigma) = max_{P:P \leq I} tr(P(\rho - \sigma))$$

Proof. Key step, $\rho - \sigma \rightarrow Q - S$.

More properties

Quantum Trace distance as an upper bound

Let $\{E_m\}$ be any POVM, and let $p_m = Tr(\rho E_m)$ and $q_m = Tr(\sigma E_m)$. Then the following holds,

$$D(\rho, \sigma) = max_{E_m} D(p_m, q_m)$$

Proof - Show the inequality, and then show existence.

Contractiveness

Suppose $\mathcal{E}$ is a TP map. Then the following holds

$$D(\mathcal{E}(\rho), \mathcal{E}(\sigma)) \leq D(\rho, \sigma) \forall \rho, \sigma$$

Proof - Along similar lines, use the previous properties. Corollary:

$$D(\rho_A, \sigma_A) \leq D(\rho_{AB}, \sigma_{AB})$$

Quantum Fidelity

- Define the quantum fidelity as

$$F(\rho, \sigma) = Tr\sqrt{\rho^{1/2}\sigma\rho^{1/2}}$$

- Observe,
  1. If ρ and σ commute, then

$$F(\rho, \sigma) = F(\lambda_i, \mu_i)$$

  2. $F(|\psi\rangle, \rho) = \sqrt{\langle\psi| \rho |\psi\rangle}$
  3. $F(U\rho U^*, U\sigma U^*) = F(\rho, \sigma)$

More on Fidelity

Theorem (Uhlmann's Theorem)

Suppose $\rho, \sigma \in Dens(\mathcal{H}_Q)$. Let R be a copy of Q. Then,

$$F(\rho, \sigma) = \max_{|\psi\rangle, |\varphi\rangle} |\langle \psi \mid \varphi \rangle|$$

where $|\psi\rangle$ and $|\varphi\rangle$ are purifications of ρ and σ in $\mathcal{H}_Q \otimes \mathcal{H}_R$.

This theorem is quite nice for showing properties of the fidelity. Also, we have

1. $F(\rho, \sigma) = \min_{\{E_m\}} F(p_m, q_m)$
2. $F(\mathcal{E}(\rho), \mathcal{E}(\sigma)) \geq F(\rho, \sigma)$
3. $A(\rho, \sigma) := \arccos F(\rho, \sigma)$ is a metric on $Dens(\mathcal{H})$

Relationship between Distance Measures

Quantum trace distance and fidelity are qualitatively equivalent measures of distance. In case of pure states, they are exactly equal.
Proof. Let the two pure states be $|a\rangle$ and $|b\rangle$. Let $|a\rangle = |0\rangle$ and $|b\rangle = cos(\theta)\,|0\rangle + sin(\theta)\,|1\rangle$. For these, $F(a,b) = |cos(\theta)|$ and $D(a,b) = |sin(\theta)| = \sqrt{1 - F(a,b)^2}$.

Theorem

$1 - F(\rho,\sigma) \leq D(\rho,\sigma) \leq \sqrt{1 - F(\rho,\sigma)^2}$ *for any two states* ρ,σ*.*

Proof. To prove the right side of the inequality, consider purifications $|\psi\rangle, |\phi\rangle$ chosen such that $F(\rho,\sigma) = \langle\psi|\phi\rangle = F(\psi,\phi)$. Since $|\psi\rangle, |\phi\rangle$ are pure states and tace distance is non- increasing under partial trace

$$D(\rho,\sigma) \leq D(\psi,\phi) \leq \sqrt{1 - F(\rho,\sigma)^2}$$

The left side of the inequality can be proved using POVMs and simple mathematical manipulation.

How well does $\mathcal{E}$ preserve QI?

Consider the system to be in $|\psi\rangle$ initially. Suppose it undergoes evolution under $\mathcal{E}$. The information preserved can be estimated by

$$F(|\psi\rangle, \mathcal{E}(|\psi\rangle\langle\psi|)$$

If we talk about the channel alone, we can minimize over all states [1]

$$F_{min} = min_{|\psi\rangle} F(|\psi\rangle, \mathcal{E}(|\psi\rangle\langle\psi|)$$

Illustrations

- Depolarizing Channel
- Phase Damping Channel
- Gate Fidelities

[1] pure states are enough, because $F_{min} \leq F(\rho, \mathcal{E}(\rho)) \forall \rho$ can be shown

Defining a quantum information source - Attempt (1)

Ensemble Oriented

A QIS is an entity which produces states ρ_i with probability p_i for all $i \in I$.

With this and the previous slide, we can talk about how well a source is preserved under a channel with the following quantity

$$\bar{F} := \sum_j p_j F(\rho_j, \mathcal{E}(\rho_j))^2$$

This is called the *ensemble average fidelity*. Provided $\bar{F} \approx 1$ one can be confident that the source is being preserved by the channel.

Defining a quantum information source - Attempt (2)

This notion is inspired from the idea of converting dynamic distance to correlation between a system and its copy. The correlation here translates to entanglement – A channel which preserves information well is one which preserves entanglement well.

Entanglement Oriented

A QIS (Q,R) is a system Q in some state which is entangled to some environment R. WLOG it is in the state $\rho = Tr_R(|RQ\rangle\langle RQ|)$.

We define *entanglement fidelity*,

$$\begin{aligned} F_e(\rho, \mathcal{E}) &:= F(|RQ\rangle, |RQ'\rangle) \\ &= \langle RQ| [(I_R \otimes \mathcal{E})(|RQ\rangle\langle RQ|)] |RQ\rangle \end{aligned}$$

where $\rho = Tr_R(|RQ\rangle\langle RQ|)$. It can be shown that only the choice of ρ and $\mathcal{E}$ affect the EF, not the choice of purification.

Computation & Properties

1. There exists a nice formula to compute the F_e. Let E_i be kraus elements of $\mathcal{E} \otimes I_R$

$$\begin{aligned} F_e(\mathcal{E}, \rho) &= \langle RQ| \, \rho_{RQ'} \, |RQ\rangle \\ &= \sum_i | \langle RQ| \, E_i \, |RQ\rangle |^2 \\ &= \sum_i |tr(\rho E_i)|^2 \end{aligned}$$

2. $F_e(\rho, \mathcal{E}) \leq (F(\rho, \mathcal{E}(\rho))^2$. Intuitively, attempt (2) is stronger than attempt (1). It is tougher to preserve the entanglement and the state than just the state.
3. F_e is convex. Now, we have,

$$F_e(\sum_j p_j \rho_j, \mathcal{E}) \leq \sum_j p_j F_e(\rho_j, \mathcal{E}) \leq \sum_j p_j F(\rho_j, \mathcal{E}(\rho_j))^2$$

Thus, $F_e \leq \bar{F}$!

Concluding Remarks

Thus, any quantum channel $\mathcal{E}$ which does a good job of preserving the entanglement between a source described by a density operator and a reference system will automatically do a good job of preserving an ensemble source described by probabilities p_j and states ρ_j such that $\rho = \sum_j p_j \rho_j$. In this sense the notion of a quantum source based on entanglement fidelity is a more stringent notion than the ensemble definition.