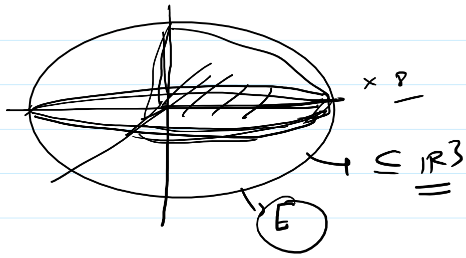
Question Four (a)

Compute the volume of the solid enclosed by the ellipsoid:

$$\left[\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,\right]$$

where a, b, c are given real numbers.



$f: E \to \mathbb{R}$ $\quad \left(f(x,y,z)=1\right) \quad \forall (x,y,z) \in E$

$$\begin{pmatrix} x = au \\ y = bv \\ z = cp \end{pmatrix} \qquad \begin{pmatrix} u \\ v \\ p \end{pmatrix} \xrightarrow{h} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$(J) = \begin{bmatrix} a & & \\ & b & \\ & & c \end{bmatrix}$$

$$\det(J) = abc$$

(E^*)

$$E = \left\{ (x,y,z) \;\middle|\; \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} \leq 1 \right\}$$

$$E^* = \left\{ (u,v,p) \;\middle|\; u^2+v^2+p^2 \leq 1 \right\}$$

$$\left(\iiint_E f\right) = \iiint_{E^*} (f \circ h)\,|J| \qquad f \circ h = 1 \qquad |J| = abc$$

$$= \iiint_{E^*} (abc)$$

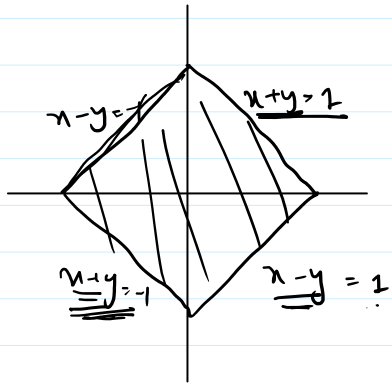
$$= (abc)\left[\iiint_{E^*} 1\right]$$

$$= (abc) \times \frac{4\pi}{3} = \frac{4\pi}{3}abc$$

Question Four (b)

Find the volume of the region under the graph of $f(x, y) = e^{x+y}$ over the region

$$D := \{(x, y) \in \mathbb{R}^2 \mid |x| + |y| \le 1\}.$$



$$f : D \to \mathbb{R}$$

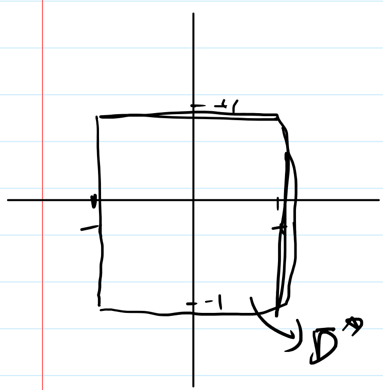
$$f(x,y) := e^{x+y}$$

$$\rightarrow \begin{cases} u = x+y \\ v = x-y \end{cases} \qquad \begin{pmatrix} u \\ v \end{pmatrix} \xrightarrow{h} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\rightarrow \begin{cases} x = \left(\frac{u+v}{2}\right) = h_1(u,v) \\ y = \left(\frac{u-v}{2}\right) = h_2(u,v) \end{cases} \qquad J[h](u,v) = \begin{bmatrix} \frac{\partial h_1}{\partial u} & \frac{\partial h_1}{\partial v} \\ \frac{\partial h_2}{\partial u} & \frac{\partial h_2}{\partial v} \end{bmatrix}$$

$$J = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

$$|\det(J)| = \left|\left(-\frac{1}{2}\right)\right| = \frac{1}{2}$$



$$f(u,v) = e^{u}$$

$$\left[\iint_D (e^{u}) \times \frac{1}{2}\right]$$

$$= \int_{-1}^{+1}\int_{-1}^{+1} \left(\frac{e^{u}}{2}\right) du\,dv$$

$$= \frac{1}{2} \times \left(\int_{-1}^{1} e^{u}\,du\right)\left(\int_{-1}^{1} dv\right)$$

$$= (e - e^{-1})$$

Question Seven (a)

Find

$$\lim_{r\to\infty} \iint_{D(r)} e^{-(x^2+y^2)} d(x,y), \quad \to r^2$$

where $D(r)$ equals:

(a) $\{(x,y) : x^2+y^2 \le R^2\}. = D$

$$x = r\cos(\theta)$$
$$y = r\sin(\theta)$$

$$\begin{pmatrix} r \\ \theta \end{pmatrix} \xrightarrow{h} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\left(|J| = r \right)$$

$$D^* = \{ (r,\theta) \mid 0 < r < R \ , \ \theta \in [0, 2\pi] \}$$

$$\iint_{D^*} r e^{-r^2} d(r,\theta)$$

$$= \int_0^{2\pi} \int_0^R r e^{-r^2} dr\, d\theta$$

$$= (2\pi) \times \int_0^R r e^{-r^2} dr$$

$$= (-\pi) \times \int_0^R -2r e^{-r^2} dr$$

$$= \pi \left(1 - e^{-R^2} \right)$$

$D(R)$

$$\left[I(R) = \pi\left(1 - e^{-R^2}\right) \right]$$

$$\left[\lim_{R\to\infty} I(R) = \boxed{\pi} \right]$$

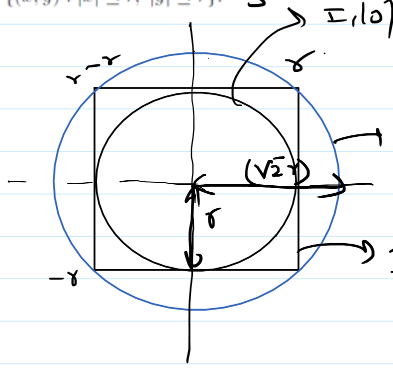
Question Seven (c)

Find

$$\lim_{r\to\infty} \iint_{D(r)} e^{-(x^2+y^2)} d(x, y),$$

where $D(r)$ equals:

(c) $\{(x, y) : |x| \le r,\ |y| \le r\}. = S$



Sandwich Thm

$I_2(r) = \iint_S f$

$I_1(r) = \iint_{D(r)} f$

$I_3(r) = \iint_{D(\sqrt{2}r)} f$

$\underline{(I_1(r) \le I_2(r) \le I_3(r))}$ $\underline{\underline{\text{why?}}}$

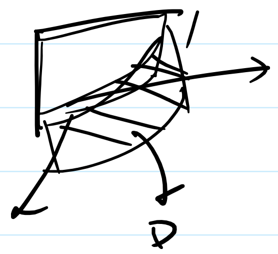
$\underline{\underline{\forall r}}$

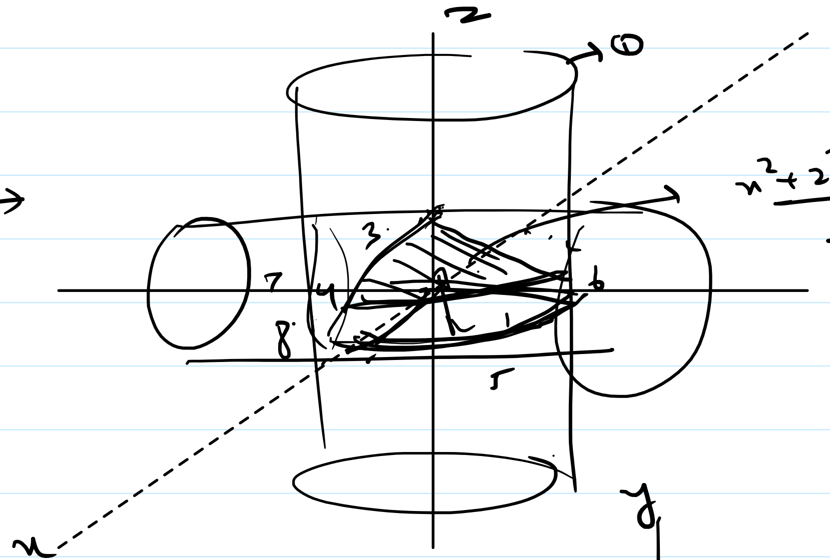
$\lim_{r\to\infty} I_1(r) = K = \lim_{r\to\infty} I_3(r)$

$\lim_{r\to\infty} I_2(r) = K = \pi$

Question Eight

8. Find the volume common to the cylinders $x^2 + y^2 = a^2$ and $x^2 + z^2 = a^2$ using double integral over a region in the plane. (Hint: Consider the part in the first octant.)

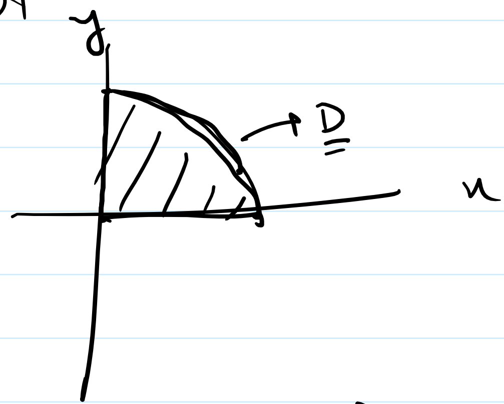




$x^2 + z^2 = a^2$

$z = \sqrt{a^2 - x^2}$

f



$$D := \{(x,y) \mid x^2+y^2 \leq a^2,\ x,y \geq 0\}$$

$$f : D \to \mathbb{R}$$

$$f(x,y) := \sqrt{a^2 - x^2}$$

$$\left(\iint_D f\right) \times 8 = \text{desired volume}$$

T 1

$$\int_0^a \int_0^{\sqrt{a^2-x^2}} f(x,y)\, dy\, dx$$

$$\int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{a^2-x^2}\, dy\, dx$$

a

T 2

$$\int_0^a \int_0^{\sqrt{a^2-y^2}} f(x,y)\, dx\, dy$$

$$\int_0 \int_0 \sqrt{a^2-x^2}\, dy\, dx$$

$$\int_0^a \sqrt{a^2-x^2}\,(y)_0^{\sqrt{a^2-x^2}}\,dx = \int_0^a (a^2-x^2)\,dx$$

$$= a^2\int_0^a dx - \int_0^a x^2\,dx$$

$$= a^3 - \frac{a^3}{3} = \left(\frac{2a^3}{3}\right)$$

Done?

$$\text{final volume} = 8 \times \frac{2a^3}{3} = \left(\frac{16}{3}\right)a^3$$

Question Nine

$$\iint_D (x^2+y^2)\,d(x,y)$$

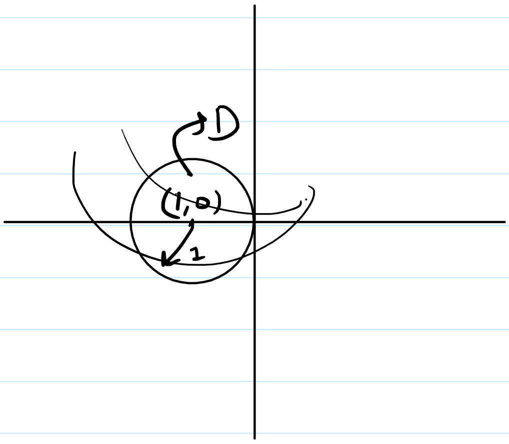
9. Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$ above the region $x^2 + y^2 = 2x$ in $x - y$ plane.

$$x^2 + y^2 = 2x$$

$$x^2 - 2x + y^2 = 0$$

$$x^2 - 2x + 1 + y^2 = 1$$

$$(x-1)^2 + y^2 = 1$$



$$(x-1) = r\cos\theta$$

$$y = r\sin\theta$$

$$\rightarrow \begin{cases} x = r\cos\theta + 1 \\ y = r\sin\theta \end{cases}$$

$$|J| = r$$

$$D := \{(x,y) \mid (x-1)^2 + y^2 \le 1\}$$

$$\binom{r}{\theta} \to \binom{x}{y}$$

$$D^* := \{(r,\theta) \mid 0 < r < 1,\ \theta \in [0, 2\pi]\}$$

$$(f \circ h)(r,\theta) = (1 + r\cos\theta)^2 + (r\sin\theta)^2$$

$$= r^2 + 1 + 2r\cos\theta$$

$$\rightarrow \iint_{D^*} (r^2 + 1 + 2r\cos\theta) \times r$$

$$\int_0^{2\pi}\int_0^1 (r^3 + r + 2r^2\cos\theta)\,dr\,d\theta$$

$$\left(\int_0^{2\pi} d\theta\right)\left(\int_0^1 (r^3 + r)\,dr\right) + \underbrace{\int_0^{2\pi}\int_0^1 2r^2\cos\theta\,dr\,d\theta}_{=0}$$

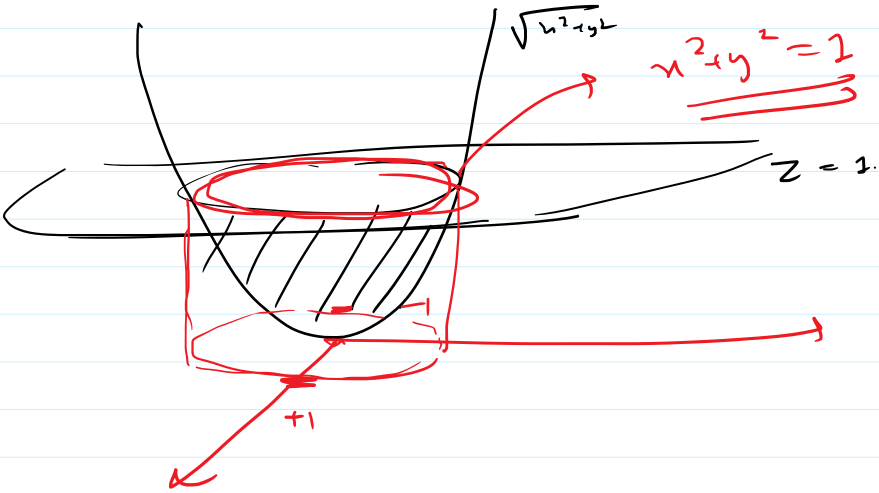
$$2\pi\left[\frac{1}{4} + \frac{1}{2}\right]$$

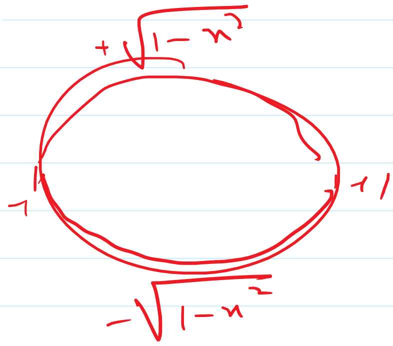
$$= 2\pi \times \left[\frac{1+2}{4}\right] = \boxed{\frac{3\pi}{2}}$$

Question Ten

Express the solid $D = \{(x, y, z) | \sqrt{x^2 + y^2} \le z \le 1\}$ as

$D = \{(x, y, z) \mid a \le x \le b, \quad \phi_1(x) \le y \le \phi_2(x), \quad \xi_1(x, y) \le z \le \xi_2(x, y)\}.$





$D = \{(x,y,z) \mid -1 \le x \le +1, \ -\sqrt{1-x^2} \le y \le \sqrt{1-x^2}, \ \sqrt{x^2+y^2} \le z \le 1\}$

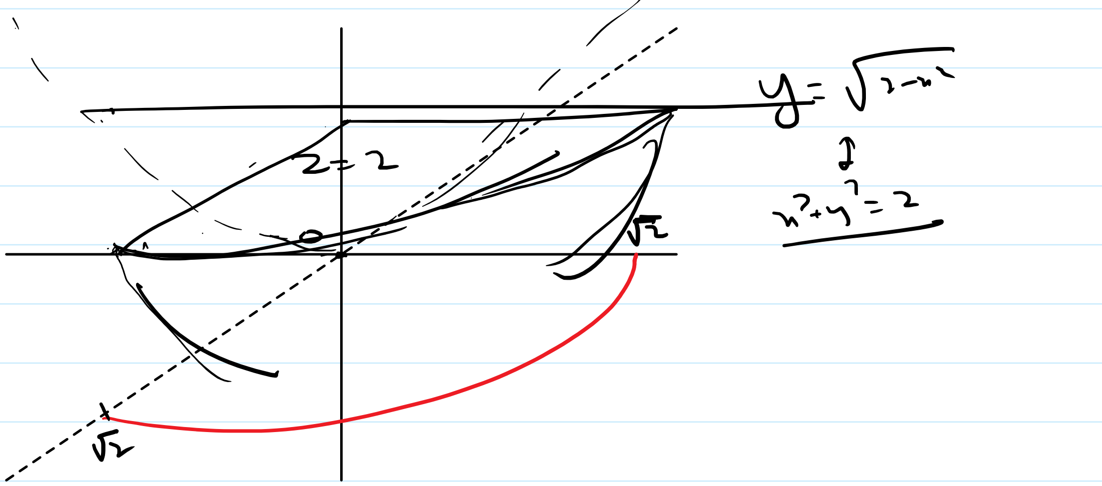
Question Eleven

Evaluate

$$I = \int_0^{\sqrt{2}} \left(\int_0^{\sqrt{2-x^2}} \left(\int_{x^2+y^2}^{2} x \boxed{dz} \right) \boxed{dy} \right) \boxed{dx.}$$

Sketch the region of integration and evaluate the integral by expressing the order of integration as $\boxed{Ixdydz}$

$D := \{(x,y,z) \mid 0 \le x \le \sqrt{2},\ 0 \le y \le \sqrt{2-x^2},\ x^2+y^2 \le z \le 2\}$



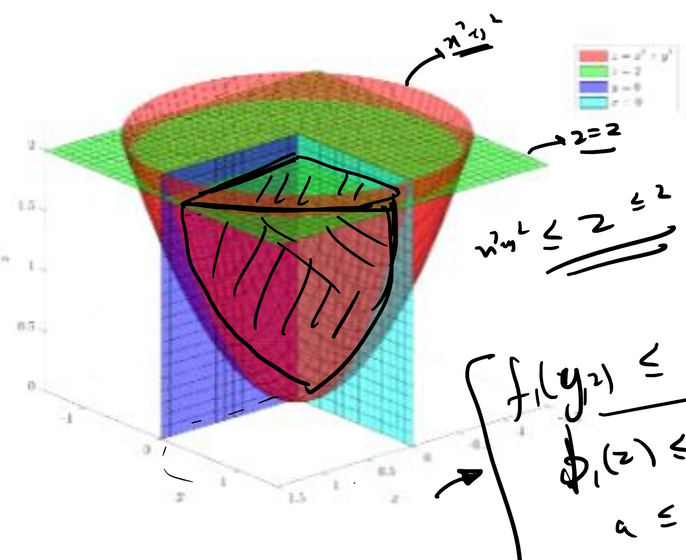


Figure 1: Region of integration for volume in Question 11

$$\begin{cases} f_1(y,z) \le x \le f_2(y,z) \\ \phi_1(z) \le y \le \phi_2(z) \\ a \le z \le b \end{cases}$$

Solution? :

$$\begin{cases} f_1(y,z) = 0 \\ f_2(y,z) = \sqrt{2-y^2} \\ \phi_1(z) = 0 \end{cases}$$

$$\left\{\begin{array}{l}\phi_1(z) = 0 \\ \phi_2(z) = \sqrt{z} \\ a = 0 \\ b = 2\end{array}\right.$$

Question Twelve (a)

Using suitable change of variables, evaluate the following:

(a)

$$I = \iiint_D (z^2x^2 + z^2y^2)\,dx\,dy\,dz$$

where D is the cylindrical region $x^2 + y^2 \le 1$ bounded by $-1 \le z \le 1$.

(b)

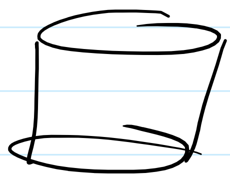
$$I = \iiint_D \exp(x^2 + y^2 + z^2)^{3/2}\,dx\,dy\,dz$$

over the region enclosed by the unit sphere in $\mathbb{R}^3$.

$$D := \{(x,y,z) \mid x^2+y^2 \le 1, \quad -1 \le z \le 1\}$$

cylindrical sub

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$$



$$J = \begin{pmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$|J| = r$$

$$D^* := \{(r,\theta,z) \mid 0 < r \le 1, \quad \theta \in [0,2\pi], -1 \le z \le 1\}$$

$$\begin{pmatrix} r \\ \theta \\ z \end{pmatrix} \xrightarrow{h} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$f(x,y,z) = z^2(x^2+y^2)$$

$$f \circ h(r,\theta,z) = z^2(r^2)$$

$$\rightarrow \left[\int_0^{2\pi}\int_0^1\int_{-1}^1 \left((z^2r^2) \times r\right) dz\,dr\,d\theta\right] \qquad z \to r \to \theta$$

$$= (2\pi) \times \left(\frac{1}{4}\right) \times \frac{2}{3}$$

$$= \left(\frac{\pi}{\ }\right)$$

$$= \left(\frac{\pi}{3}\right)$$

Question Twelve (b)

Using suitable change of variables, evaluate the following:

(a)

$$I = \iiint_D (z^2x^2 + z^2y^2)\,dxdydz$$

where D is the cylindrical region $x^2 + y^2 \le 1$ bounded by $-1 \le z \le 1$.

(b)

$$I = \iiint_D \exp(x^2 + y^2 + z^2)^{3/2}\,dxdydz$$

over the region enclosed by the unit sphere in $\mathbb{R}^3$.

$$\left[\begin{array}{l} D = \{(x,y,z) \mid x^2+y^2+z^2 \le 1\} \\ D^* = \{(r,\theta,\phi) \mid 0 \le r \le 1,\ \theta \in [0,2\pi],\ \phi \in [0,\pi]\} \end{array}\right.$$

$$\left\{\begin{array}{l} x = r\cos\theta\sin\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\phi \end{array}\right.$$

$$f(x,y,z) = e^{(x^2+y^2+z^2)^{3/2}}$$

$$f\circ h(r,\theta,\phi) = \left(e^{r^3}\right)$$

$$|J| = \left(r^2|\sin\phi|\right)$$

$$\int_0^{\pi}\int_0^{2\pi}\int_0^{1} e^{r^3} \times r^2 \sin\phi \, dr\,d\theta\,d\phi$$

$$\left(\int_0^{\pi} \sin\phi\, d\phi\right)\left(\int_0^{2\pi} d\theta\right)\left(\int_0^1 r^2 e^{r^3} dr\right)$$

$$2 \times 2\pi \times \left[\frac{1}{3}\int_0^1 \underbrace{3r^2 e^{r^3}}_{\frac{d}{dr}(e^{r^3})} dr\right]$$

$$\left(e^{r^3}\right)\Big|_0^1 = (e-1)$$

$$2 \times 2\pi \times \frac{(e-1)}{3}$$

$$= \frac{4\pi}{3}(e-1)$$