Classical Mechanics

January 13, 2023

1 Introduction

We deal with a system consisting of a single particle under the influence of some external forces. The *generalized coordinates* of this particle are denoted q, and subsequent velocities and acceleration as $\dot{q}$ and $\ddot{q}$ respectively. We take for granted that the dynamics of the said particle are fully specified once we are aware of the q and $\dot{q}$.

1.1 The Lagrangian and the PLA

The previous statement is quantified by what is called the *principle of least action* according to which each system is charecterized by a Langrangian,

$$L(q, \dot{q}, t)$$

That is, if at times 0 and t the generalized coordinates are known to be q_1 and q_2, then the *dynamics* between these times involve a function $q(t)$ (let us work with $q \in$ for now) such that the following quantity

$$S = \int_0^t L(q, \dot{q}, t) dt \tag{1}$$

called the *action* is minimized. The condition for the same is $dS = 0$, which for a perturbation $q(t) \to q(t) + dq(t)$ in the first order (why?) amounts to,

$$\int_0^t \partial_q L dq + \partial_{\dot{q}} L d\dot{q} = 0$$

Integrating the second term by parts and noting that we require $dq(0) = dq(t) = 0$ we have,

$$\int_0^t \frac{d}{dt}(\partial_{\dot{q}} L) - \partial_q L = 0$$

Recognizing that this integral is zero for any perturbation dq and generalizing for $q = (q_1, q_2 \dots q_s)$ we have,

$$\boxed{\frac{d}{dt}(\partial_{\dot{q}_s} L) - \partial_{q_s} L = 0} \tag{2}$$

These s second order differential equations give us the *equations of motion* for the dynamic mechanical problem at hand.
It is perhaps important to note that the lagrangian of a system is far from uniquely defined. It can be seen that, if L is a 'good' lagrangian, then any L' defined as

$$L'(q, \dot{q}, t) = L(q, \dot{q}, t) + \frac{d}{dt} f(q, t) \tag{3}$$

is an equally good choice, because the action will be changed by an amount which is only dependent upon q_1 and q_2, and not the dynamics. This fact shall be useful later.
Another important fact about the lagrangian is that, if for two particles A and B the lagragians for the closed systems are L_A and L_B, then in the limit of large distance between them, the total lagrangian is

$$L = L_A + L_B \tag{4}$$

1.2 Galilean Relativity

The frame of reference chosen to analyze any dynamical mechanics problem is a crucial step. It is known that there exists frames such that with their reference space is homogenous and isotropic and time is homogenous. These are called *inertial frames of reference*. Galilean Relativity dictates that the laws of mechanics are the same in any inertial frame of reference. Further, the following transformation is underwent when changing from a frame S to a frame S' which moves with a uniform velocity $\mathbf{v}$ with respect to S

$$\begin{aligned} \mathbf{r} &\to \mathbf{r} + \mathbf{v}t \\ t &\to t \end{aligned} \tag{5}$$

1.3 Some Lagrangians

1.3.1 The Single Particle Lagrangian

For a free particle in an inertial frame of reference, we see that owing to the homogeneity of space and the lagrangian cannot depend on the coordinates or the direction of the velocity. Thus, it may depend only on the magnitude of the velocty. We write this as,

$$L \equiv L(v^2) \tag{6}$$

The Lagrange's Equation 2 then show that we need the velocity to be constant. Thus, for a free particle under an inertial frame of reference we conclude that the velocity must be constant both in direction and magnitude. But are we done? What is the $L(\cdot)$ function really?
Consider two frames S and S', such that S' moves at a velocity ϵ w.r.t. S. Then, by 3 we need that,

$$L(v^2) - L(v^2 + \epsilon^2 + 2\mathbf{v} \cdot \epsilon) \sim \frac{d}{dt} f(q,t)$$

Up to first order, this demands that,

$$\frac{\partial}{\partial v^2} L(v^2) 2\mathbf{v} \cdot \epsilon \sim \frac{d}{dt} f(q,t)$$

Noting that the velocity has a time derivative already, we conclude a lagrangian of the form

$$L(v^2) = \frac{1}{2} m v^2 \tag{7}$$

The seemingly arbitrary constant m is said to be the mass of the particle. We note two things,

1. The mass cannot be negative! (or the PLA would not work! why?)
2. The invariance under finite velocity reference frames also holds. (why?)

1.3.2 Adding particles

For a *closed* system of countable particles, we write the lagrangian as,

$$L = \sum_i \frac{1}{2} m_i v_i^2 - U(\mathbf{r}_1, \mathbf{r}_2, \dots) \tag{8}$$

where we introduce a *potential energy* function U, which depends on the positions of all the particles. The nature of this Galilean Relativity and this expression tell us that any change in any of the particle coordinate must instantenously change the lagrangian. (For if not, what would go wrong? Consider two frames and see).
We apply the equations 2 to this lagrangian to see an old friend again,

$$m_i \frac{d\mathbf{v_i}}{dt} = -\frac{\partial U}{\partial \mathbf{r_i}} =: \mathbf{F_i} \quad \forall i$$

the *Newton's Equations*!
A point to note is that while we have not been careful with distinguishing amongst generalized coordinates and cartesian ones, we can always perform

$$x_i = f_i(q_1, q_2 \dots q_s) \qquad \dot{x}_i = \sum_k \frac{\partial f_i}{\partial q_k} \dot{q}_k$$

2 Conservation Laws

Given a mechanical system, by now we know that the dynamics are obtained by solving the 2 equations for the $2s$ quantitites, $q_1, \ldots, q_s, \dot{q}_1, \ldots, \dot{q}_s$. Since there are $2s$ second order differential equations, there are $2s$ arbitrary constants. These are called *integrals of motion.* Some of these integrals are of fundamental consequences due to the homogeniety of time and space, and the isotropy of space. These quantities are said to be *conserved.* The important point about these conserved quantitites is that they are additive for non interacting particles. We will see this illustrated through the three conserved quantitites.

2.1 Energy and Homogeneity of Time

We know, by now, that the lagrangian for a closed system does not explicitly depend on time. Thus,

$$\frac{dL}{dt} = \sum_i \frac{\partial L}{\partial q_i}\dot{q}_i + \sum_i \frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i$$

$$\frac{dL}{dt} = \sum_i \frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_i})\dot{q}_i + \sum_i \frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i$$

$$\frac{dL}{dt} = \sum_i \frac{d}{dt}(\frac{dL}{d\dot{q}_i}\dot{q}_i)$$

$$\frac{dL}{dt} = \frac{d}{dt}[\sum_i(\frac{dL}{d\dot{q}_i}\dot{q}_i)]$$

$$\frac{d}{dt}[\sum_i(\frac{dL}{d\dot{q}_i}\dot{q}_i) - L] = 0$$

We define,

$$E \equiv \sum_i(\frac{dL}{d\dot{q}_i}\dot{q}_i) - L \tag{9}$$

as the *energy* of the system. Note that the only property that implied conservation of energy is the lack of explicit dependence of the lagrangian on time. Further, recalling that $L(q,\dot{q}) = T(q,\dot{q}) - U(q)$ and applying the Euler's theorem on homogenous functions (note that T is a quadratic homogenous in $\dot{q}$),

$$E = \sum_i(\frac{dL}{d\dot{q}_i}\dot{q}_i) - L = \sum_i(\frac{dT}{d\dot{q}_i}\dot{q}_i) - L = 2T - (T-U) = T + U$$

Thus we have concluded,

$$\boxed{E = \sum_i \frac{1}{2}m_i v_i^2 + U(\mathbf{r}_1, \mathbf{r}_2 \ldots)}$$

2.2 Momentum and Homogeneity of Space

We impose the physical demand that a closed dynamical system should essentially remain the same if all the components are translated. Note that this does not work if the system is under the influence of some external fields. One can extend this to such systems by including the source of the said fields in the system.
Now, considering an infinitesimal translation by ϵ the lagrangian changes by,

$$dL = \sum_i \frac{\partial L}{\partial \mathbf{r}_a}\cdot d\mathbf{r}_a = \epsilon \cdot \sum_i \frac{\partial L}{\partial \mathbf{r}_i}$$

since this $dL = 0$ for any choice of ϵ, we have,

$$\sum_i \frac{\partial L}{\partial \mathbf{r}_i} = 0$$

$$\sum_i \frac{d}{dt}(\frac{\partial L}{\partial \mathbf{v}_i}) = 0$$

$$\frac{d}{dt}[\sum_i(\frac{\partial L}{\partial \mathbf{v}_i})] = 0$$

Now, we define,

$$\boxed{\mathbf{p} = \sum_i (\frac{\partial L}{\partial \mathbf{v}_i})}$$

We see that,

$$\mathbf{p} = \sum_i m_i \mathbf{v}_i \tag{10}$$

as expected. Moreover, from $\sum_i \frac{\partial L}{\partial \mathbf{r}_i} = 0$ and the fact that $\partial_{\mathbf{r}_i} L = \partial_{\mathbf{r}_i} U = -F_i$, we have,

$$\sum_i \mathbf{F}_i = 0 \tag{11}$$

which for two particles is nothing but the Newton's Third Law! Moreover, in generalized coordinates we have,

1. Generalized momenta, $p_i = \partial L / \partial \dot{q}_i$
2. Generalized forces, $F_i = \partial L / \partial q_i$
3. Thus Lagrange's equations take the form $dp_i/dt = F_i$

2.3 Centre of Mass

In this subsection, we take an (apparent) digression from conserved quantities to define and highlight the *center of mass* of a system. Consider a system wherein particles move with velocities $\mathbf{v}_i$ w.r.t. the frame S. Consider frame S' moving with velocity $\mathbf{V}$ w.r.t. S. Now, the velocity of the system as seen by S' is,

$$\mathbf{v}'_i = \mathbf{v}_i - \mathbf{V}$$

Thus the momenta are related by,

$$\mathbf{P}' = \mathbf{P} - \mathbf{V} \sum_i m_i$$

Now, let K' be such that the total momentum in this frame is zero. Thus,

$$\mathbf{P} = \mathbf{V} \sum_i m_i$$

$$\mathbf{P}/(\sum_i m_i) = \mathbf{V}$$

$$\mathbf{V} = \frac{\sum_i \mathbf{v}_i m_i}{\sum_i m_i}$$

This is the velocity of the frame w.r.t. which the system appears to have no momentum. We identify it as the velocity of the center of mass. Further, if we let

$$\mathbf{R} = \frac{\sum_i \mathbf{r}_i m_i}{\sum_i m_i}$$

then we note that $\dot{\mathbf{R}} = \mathbf{V}$. Thus the $\mathbf{R}$ defined is nothing but the center of mass. Note that the law of conservation of momentum implies that the center of mass moves uniformly in a straight line. It is useful to work in the center of mass frame while analyzing the mechanical properties of the system. In this frame, if the the energy of the system is E_i (called the internal energy), then we remark that the energy of a system from a frame in which the velocity of the COM is V, is,

$$E = \frac{1}{2}\mu V^2 + E_i$$

2.4 Angular Momentum and Isotropy of Space

The isotropy of space implies that the properties of the mechanical system are invariant under a rotation in space. We conduct the infinitesimal rotation by $d\phi$, and note that we have,

$$d\mathbf{r} = d\phi \times \mathbf{r}$$

$$d\mathbf{v} = d\phi \times \mathbf{v}$$

Now,

$$dL = \sum_i (\frac{dL}{d\mathbf{r}_i} \cdot d\mathbf{r}_i + \frac{dL}{d\mathbf{v}_i} \cdot d\mathbf{v}_i) = 0$$

$$dL = \sum_i (\frac{dL}{d\mathbf{r}_i} \cdot (d\phi \times \mathbf{r}_i) + \frac{dL}{d\mathbf{v}_i} \cdot (d\phi \times \mathbf{v}_i)) = 0$$

Noting that, $\mathbf{p}_i = dL/d\mathbf{v}_i$ and $dL/d\mathbf{r}_i = d/dt(dL/d\mathbf{v}_i) = \dot{\mathbf{p}}_i$, we have, (after a permutation in the box product)

$$d\phi \cdot (\sum_i (\mathbf{r}_i \times \dot{\mathbf{p}}_i + \mathbf{v}_i \times \mathbf{p}_i))$$

$$d\phi \cdot \frac{d}{dt}[\sum_i \mathbf{r}_i \times \mathbf{p}_i] = 0$$

We define,

$$\mathbf{M} \equiv \sum_i \mathbf{r}_i \times \mathbf{p}_i \tag{12}$$

and call it the *angular momentum* of the system. We make two remarks,

1. The angular momentum of a system depends on the choice of origin. This can be seen by two frames separated by $\mathbf{a}$, leading to $\mathbf{r}_i = \mathbf{r}'_i + \mathbf{a}$ and thus,

$$\mathbf{M} = \sum \mathbf{r}_i \times \mathbf{p}_i = \sum \mathbf{r}'_i \times \mathbf{p}_i + \mathbf{a} \times \sum \mathbf{p}_i = \mathbf{M}' + \mathbf{a} \times \mathbf{P}$$

2. For two frames with constant relative velocity $\mathbf{V}$ (whose origins coincide at an instant)

$$\mathbf{M} = \sum \mathbf{r}_i \times \mathbf{p}_i = \sum m_i \mathbf{r}_i \times \mathbf{v}_i = \sum m_i \mathbf{r}_i \times (\mathbf{v}'_i + \mathbf{V}) = \mathbf{M}' + \mu \mathbf{R} \times \mathbf{V}$$

   Now if the moving frame is the COM frame, then $\mu\mathbf{V} = \mathbf{P}$ w.r.t. the stationery frame, and we have,

$$\mathbf{M} = \mathbf{M}_{COM} + \mathbf{R} \times \mathbf{P} \tag{13}$$

   That is, the angular momentum of a mechanical system is the vector sum of the intrinsic angular momentum (in the COM frame, $\mathbf{M}_{COM}$) and the angular momentum due to the motion as a whole ($\mathbf{R} \times \mathbf{P}$).

Thus, in total there are seven additive integral constants (and none other, in fact) which include energy, the three components of momentum, and the three components of angular momentum.