MA 205 Autumn 2022
Tutorial Sheet 2

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1. Using the geometric series, or otherwise, obtain the following

1. $$\sum_{n=0}^{\infty}(-1)^n z^{2n} = \frac{1}{1+z^2} \quad \text{for } |z| < 1.$$

2. For any $z_0 \in \mathbb{C}$ and $z_0 \neq 1$
$$\sum_{n=0}^{\infty}\left(\frac{1}{1-z_0}\right)^{n+1}(z-z_0)^n = \frac{1}{1-z} \quad \text{for } |z-z_0| < |1-z_0|.$$

Sol.
For both parts, denote the power series by P.

1. Define $\alpha := -z^2$. Then we have,
$$P = \sum_{n=0}^{\infty}\alpha^n$$
We know, this geometric series converges to $1/(1-\alpha)$ for $\alpha : |\alpha| < 1$. Thus, P converges to $1/(1+z^2)$ for $z : |z| < 1$.

2. Similar to the previous question, define $\alpha := (z-z_0)/(1-z_0)$. Then we have
$$P = \frac{1}{1-z_0}\sum_{n=0}^{\infty}\alpha^n$$
Thus for $|z-z_0| < |1-z_0|$, P converges to $1/(1-z)$.

$\blacksquare$

2. Find the radius of convergence of the power series $\sum_{n=0}^{\infty}(1+i)^n z^n$.

Sol.
Identifying this series with $\sum_n a_n(z-z_0)^n$, we have that $a_n = (1+i)^n$ and $z_0 = 0$. Now, $|a_n|^{1/n} = \sqrt{2}$. Thus we have $\limsup |a_n|^{1/n} = \sqrt{2}$. By the Convergence Theorem, we have
$$R = \frac{1}{\sqrt{2}}$$

$\blacksquare$

3. Show the following

1. The power series $\sum_{n=0}^{\infty} nz^n$ converges if and only if $|z| < 1$.

2. The power series $\sum_{n=0}^{\infty} \frac{z^n}{n^2}$ converges if and only if $|z| \leq 1$.

3. The power series $\sum_{n=0}^{\infty} \frac{z^n}{n}$ converges for any $|z| < 1$ but not at $z = 1$.

Sol.

Note that for showing an *iff* condition about convergence, the convergence theorem as done in class is **not** enough. Why? It does not talk about the nature of the series on the boundary of the disk of convergence.

1. Here, we have $z_0 = 0$ and $a_n = n$. Thus we have, $|a_n|^{1/n} = n^{1/n}$. Since this sequences converges to the limit 1, we have that,

$$R = \frac{1}{\limsup n^{1/n}} = \frac{1}{\lim n^{1/n}} = 1$$

Thus, now we know that the series converges for $z : |z| < 1$, and does not converge for $z : |z| > 1$.

Further, we need to show that it does not converge for any $z : |z| = 1$. We shall do this by contradiction. Assume there exists some $z : |z| = 1$ such that this series converges at z. Now, this means that the sequence

$$s_k := \sum_{n=0}^{k} nz^n$$

converges. Since convergence $\implies$ cauchy, we have, $\forall \epsilon > 0\ \exists N \in \mathbb{N}$ s.t.

$$k > l - 1 > N \implies |\sum_{n=l}^{k} nz^n| < \epsilon$$

Now take $k = l$. Thus, $\forall \epsilon > 0 \exists N \in \mathbb{N}$ s.t.

$$k > N \implies |nz^n| < \epsilon$$

This means that $|nz^n|$ converges to zero. But, this is not true (Why?). We reach a contradiction, and thus are done.

2. The analysis for the RoC remains similar, and is omitted. We have $R = 1$. We need to show that the series converges at *all* points on the boundary of the disk of convergence. We shall show that this series converges absolutely for all $z : |z| = 1$, thereby showing a sufficient condition for convergence. Given any $z : |z| = 1$, we have that $|z^n/n^2| = 1/n^2$. Thus, we will show the convergence of $\sum_n 1/n^2$.

Now, let

$$s_n := 1 + \sum_{k=2}^{n} \frac{1}{k^2}$$

and note that for $k > 1$,

$$\frac{1}{k^2} < \frac{1}{k(k-1)} = \frac{1}{k-1} - \frac{1}{k}$$

Thus,

$$s_n < 1 + (1 - \frac{1}{n}) = 2 - \frac{1}{n} < 2$$

Thus s_n is bounded. Further, it is easy to see that s_n is monotonically increasing. By the monotone convergence theorem, s_n converges. And by definition of convergence of a series, the series $1/n^2$ converges. Thus we have absolute convergence, which implies convergence.

3. The analysis for the RoC is similar and is omitted. We have, $R = 1$. Thus we need to show that the series does not converge at $z = 1$, i.e., $\sum_n 1/n$ is divergent. Similar to the previous question, define,

$$s_n := \sum_{k=1}^{n} \frac{1}{k}$$

And note that s_n is monotonically increasing. We can disprove convergence by showing that it is *not* bounded. For $n = 2^l$, we have

$$s_{2^l} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots > 1 + \frac{1}{2} + 2 \cdot \frac{1}{4} + 4 \cdot \frac{1}{8} + \cdots > 1 + \frac{l}{2}$$

Thus, we are done.

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4. Show that in polar coordinates, the Cauchy-Riemann equations take the form:

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{1}{r}\frac{\partial u}{\partial \theta} = -\frac{\partial v}{\partial r},$$

where $f = u + iv$.

Sol.

We have

$$x = r\cos\theta$$
$$y = r\sin\theta$$

Thus,

$$\frac{\partial}{\partial r} = \frac{\partial x}{\partial r}\frac{\partial}{\partial x} + \frac{\partial y}{\partial r}\frac{\partial}{\partial r} = \cos\theta\frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial y}$$

Similarly, we obtain

$$\frac{\partial}{\partial \theta} = -r\sin\theta\frac{\partial}{\partial x} + r\cos\theta\frac{\partial}{\partial y}$$

Now, we will apply this on u and use the CR equations in x and y:

$$\begin{aligned}\frac{\partial u}{\partial r} &= \cos\theta\frac{\partial v}{\partial y} - \sin\theta\frac{\partial v}{\partial x} \\ &= \frac{1}{r}\frac{\partial v}{\partial \theta}\end{aligned}$$

Similarly, we will obtain:

$$\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

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5. Show that the logarithm function defined by

$$\log z = \log r + i\theta \quad \text{where } z = re^{i\theta} \quad \text{with } -\pi < \theta < \pi$$

is holomorphic in the region $r > 0$ and $-\pi < \theta < \pi$.

Sol.

We can use the form of the CR equations we obtained in the last question. Let $f(z) = \mathtt{log}(z)$. Thus, we get $u = \mathtt{log}(r)$, $v = \theta$. Using the polar form of CR equations, we see that they are satisfied everywhere on the domain. Now, recall that satisfaction of the CR equation alongwith continuity of the partial derivatives of u and v implies complex differentiability. Further, it is easy to see that u_x and v_x are continuous on the domain. Thus, the function is complex differentiable and its derivative $(= u_x + \iota v_x)$ is continuous on the domaain. Hence, holomorphic. ■

6. Consider the differential operators

$$\frac{\partial}{\partial z} := \frac{1}{2}\left(\frac{\partial}{\partial x} + \frac{1}{i}\frac{\partial}{\partial y}\right) \quad \text{and} \quad \frac{\partial}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x} - \frac{1}{i}\frac{\partial}{\partial y}\right).$$

If f is holomorphic at z_0, then show that

$$\frac{\partial f}{\partial \bar{z}}(z_0) = 0 \quad \text{and} \quad f'(z_0) = \frac{\partial f}{\partial z}(z_0).$$

Sol.
We have

$$x = \frac{z+\bar{z}}{2}$$
$$y = \frac{z-\bar{z}}{2i}$$

Let $f = u + iv$. We will use the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$
$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right) - \frac{i}{2}\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)$$
$$= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)$$
$$= \frac{df}{dz}$$

Similarly, we obtain

$$\frac{\partial f}{\partial \bar{z}} = 0$$

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7. Suppose f is holomorphic in $\mathbb{C}$. Show that

1. $\mathrm{Re}(f)$ is constant implies f is constant,
2. $|f|$ is constant implies f is constant.

Sol.
Consider the following lemma:

Lemma 7.1. *If $f' \equiv 0$ on a connected domain Ω, then f is a constant on Ω*

Proof. Done in class. □

1. We have $\mathrm{Re}(f) = c$ where c is some constant. Assume $f = u + iv$, here we have $u = c$. Since our function is holomorphic, it satisfies the CR equations. Thus,

   $$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = 0$$

   Now, recall, $f'(z) = u_x + \iota v_x$. Conclude that $f' \equiv 0$ and thus by the lemma f is a constant.

2. We have $|f| = c$ for some $c \geq 0$. If $c = 0$, we are done as $f \equiv 0$. If not, let $f = u + iv$. Thus, $u^2 + v^2 = c^2$. This gives us

   $$u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x} = 0$$
   $$u\frac{\partial u}{\partial y} + v\frac{\partial v}{\partial y} = 0$$

   Using the Cauchy-Riemann equations, we obtain

   $$u\frac{\partial u}{\partial x} - v\frac{\partial u}{\partial y} = 0$$
   $$u\frac{\partial u}{\partial y} + v\frac{\partial u}{\partial x} = 0$$

Solving these as linear equations in u_x and u_y, we obtain

$$u_x = u_y = 0$$

Now proceeding similarly as the first part, we are done.

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8. Suppose f is holomorphic in a (connected) region Ω. Prove that any two antiderivatives (or primitives) of f differ by a constant.

Sol.
Assume F_1 and F_2 are two primitives for f. This gives us

$$F_1' = F_2' = f$$

Now, if we consider $F_1 - F_2$, we obtain

$$(F_1 - F_2)' = 0$$

Using the lemma, we can say that $F_1 - F_2 = c$ where c is some constant. Thus, we have shown that the two primitives of f differ by a constant.

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9. For $n \in \mathbb{Z}$(integers) evaluate the integrals

$$I_n = \int_\gamma z^n \, dz,$$

where $\gamma = e^{it}$, $0 \leq t \leq 2\pi$ is a parametrization of the unit circle.

Sol.
The parametrization of the curve is given as $\gamma(t) = e^{it}, t \in [0, 2\pi]$.

$$\begin{aligned} I_n &= \int_\gamma z^n dz \\ &= i\int_0^{2\pi} e^{i(n+1)t} dt \\ &= \frac{1}{n+1}(1-1) \\ &= 0 \end{aligned}$$

Thus, we have obtained that $I_n = 0 \, \forall n \neq -1$.
For n=-1, we obtain $I_n = 2\pi i$ (Done in class). Another way to do this is to note that for $n \neq -1$, our function admits a primitive in the region bounded by γ. ■