MA 205 Tutorial Batch 3
Recap-5

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Laurent Series

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- The **principal part** of the Laurent Series of f at p is

$$\sum_{n \leq -1} a_n (z - p)^n$$

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As an immediate consequence, see that

$$\boxed{\int_C f(z)dz = 2\pi\iota \times Res(f, p)}$$

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$$Res(f, p) = \lim_{z \to p} \frac{1}{(k-1)!} \frac{d^{k-1}}{dz^{k-1}} \left((z-p)^k f(z) \right)$$

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The Cauchy Residue Theorem

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$$\int_\gamma f(z)dz = 2\pi\iota \sum_k Res(f, p_k)$$

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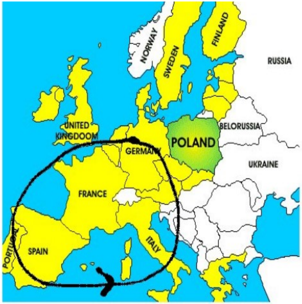

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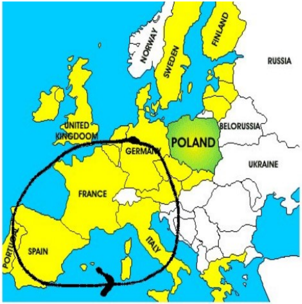

Zero! All the poles are in Eastern Europe.

`credit: MA 205 2021 Lectures`

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... for the recap's of MA 205!