11.30 am to 11.30 am. students will have to ...

Croatia ①

[.5]

1. Not true $\boxed{\cancel{3}}$

Let $x_n = \sqrt{n}$. Then $x_{n+1} - x_n = \sqrt{n+1} - \sqrt{n} = \dfrac{1}{\sqrt{n+1}+\sqrt{n}} \longrightarrow 0$ as $n \to \infty$.

But $\{x_n\}$ does not converge. [3.5]

2. Let $f : [0,1] \longrightarrow (0,1)$ be continuous and onto.

Observe: $\sup_{x\in[0,1]} f(x) = 1$ & $\inf_{x\in[0,1]} f(x) = 0$ (Since $f$ is onto) [1]

But since $f$ is continuous on $[0,1]$, it must attains its sup / inf. That is, $\exists\, x_1 \in [0,1]$ s.t. $f(x_1) = \sup_{x\in[0,1]} f(x) = 1$

&, $\exists\, x_2 \in [0,1]$ s.t. $f(x_2) = \inf_{x\in[0,1]} f(x) = 0$ [2]

— which is not possible ($\because f : [0,1] \longrightarrow (0,1)$) [1]

4.

Croatia (2)

3. By Taylor's Thm.

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f^{(3)}(0) + \frac{x^4}{4!} f^{(4)}(c)$$

for some $0 < c < x$

[1]

$f(x) = \log(1+x)$ $\quad f'(x) = \frac{1}{1+x}$, $\quad f^{(2)}(x) = -\frac{1}{(1+x)^2}$

$\therefore$ ~~$f(x) =$~~

$f^{(3)}(x) = \frac{2}{(1+x)^3}$, $\quad f^{(4)}(x) = -\frac{3!}{(1+x)^4}$.

$$\therefore f(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \cdot \frac{1}{(1+c)^4}$$

[1]

$$\left|\log(1+x) - \left(x - \frac{x^2}{2} + \frac{x^3}{3}\right)\right| = \left|\frac{x^4}{4} \cdot \frac{1}{(1+c)^4}\right|$$

$$\leq \frac{x^4}{4} \quad \left(\because \frac{1}{1+c} < 1\right)$$

[1].

4.

$f:[0,1]\longrightarrow\mathbb{R}$ by $f(x)=\begin{cases}1 & \text{if } x\in\mathbb{Q}\cap[0,1]\\ 0 & \text{if } x\notin\mathbb{Q}\cap[0,1]\end{cases}$

Let $P=\{\underset{x_0}{\overset{}{0}}, x_1, x_2, \ldots, x_n=1\}$ be a partition of $[0,1]$.

Then, $U(P;f)=\sum_{i=1}^{n} 1\,(x_i-x_{i-1})=1$

& $L(P;f)=\sum_{i=0}^{n} 0\cdot(x_i-x_{i-1})=0$ [1]

Hence,

$U(f)=1$

& $L(f)=0\neq U(f)$

$\therefore$ f is not Riemann integrable [1].

5.

Define, $F(x)=\int_0^x f(t)\,dt-x^2$ [2]

F is cont. on $[0,1]$ & diff. on $(0,1)$ (by FTC-1) [1]

$F(1)=\int_0^1 f(t)\,dt-1=0=F(0)$.

$\therefore$ By Rolle's thm. $\exists c\in(0,1)$ s.t. $F'(c)=0$

i.e., $f(c)-2c=0$

i.e., $f(c)=2c$. [2]

11:30 am to 11:50 am students will have to scan their answers and ...

Croatia (4)

6. $f(x,y) = \begin{cases} \dfrac{3x^2y - y^3}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0). \end{cases}$

(a) $|f(x,y) - f(0,0)| = \left| \dfrac{y(3x^2-y^2)}{x^2+y^2} \right| \leq |y| \dfrac{3x^2+y^2}{x^2+y^2}$ (triangle inequality)

$\leq 3|y|$

Let $\epsilon > 0$, [1]

$\therefore |f(x,y) - f(0,0)| < \epsilon$ if $3|y| < \epsilon$

i.e. if $|y| < \epsilon/3$

Let $\delta = \epsilon/3$. Then, if $\sqrt{x^2+y^2} < \delta$ (i.e. if $|y| < \delta$)

$\Rightarrow |f(x,y) - f(0,0)| < \epsilon$.

[1].

$\therefore f$ is continuous at $(0,0)$.

(b). Let $x \neq 0$

$f_y(x,0) = \lim_{k \to 0} \dfrac{f(x,k) - f(x,0)}{k} = \lim_{k \to 0} \dfrac{3x^2k - k^3}{k(x^2+k^2)}$

$= \lim_{k \to 0} \dfrac{3x^2 - k^2}{x^2+k^2} = 3$

[1].

(c). $f_y(0,0) = \lim_{k \to 0} \dfrac{f(0,k) - f(0,0)}{k} = -1$ [1]

Croatia 

Therefore, consider $\left\{\left(\frac{1}{n}, 0\right)\right\}_n \longrightarrow (0,0)$.

but $\left\{f_y\left(\frac{1}{n}, 0\right)\right\}_n = \{3, 3, \cdots\} \longrightarrow 3 \neq -1 = f_y(0,0)$.

Hence, $f_y$ is not continuous at $(0,0)$. [1]

10(a). given, $x(t) = 2\cos t - \cos 2t$

& $y(t) = 2\sin t - \sin 2t$.

$$\text{Area}(S) = 2\pi \int_0^{\pi} y(t) \sqrt{x'(t)^2 + y'(t)^2}\, dt$$

$$= 2\pi \int_0^{\pi} 2\sin t(1-\cos t) \sqrt{(-2\sin t + 2\sin 2t)^2 + (2\cos t - 2\cos 2t)^2}\, dt$$

$$= 2\pi \int_0^{\pi} 2\sin t(1-\cos t) \sqrt{4 + 4 - 8\sin t \sin 2t - 8\cos t \cos 2t}\, dt$$

$$= 4\pi \int_0^{\pi} \sin t(1-\cos t) \cdot \sqrt{8(1-\cos t)}\, dt$$

$$= 4\pi \cdot 4 \int_0^{\pi} \sin t \cdot 2\sin^2 t/2 \cdot \sin t/2\, dt$$

$$= 64\pi \int_0^{\pi} \cos \tfrac{t}{2} \sin^4 t/2\, dt$$

$$= 128\pi \int_0^1 y^4\, dy = \boxed{\frac{128\pi}{5}}.$$

(b). By shell method, the volume is

$$2\pi \int_0^1 x \cdot e^{x^2}\, dx \qquad x^2 = y.$$

$$= 2\pi \int_0^1 \frac{e^y}{2}\, dy = \boxed{\pi(e-1)}$$

(c).

$$f(x,y) = x^2 e^y + \cos xy. \qquad v = \left(\frac{3}{5}, \frac{4}{5}\right).$$

$$\nabla_v f(1,2) = \nabla f(1,2)\cdot v \qquad \nabla f(x,y) = \left(2x e^y - y\sin xy,\ x^2 e^y - x\sin xy\right).$$

$$= \left(2e^2 - 2\sin 2,\ e^2 - \sin 2\right)\cdot\left(\frac{3}{5}, \frac{4}{5}\right)$$

$$= \frac{6}{5}\left(e^2 - \sin 2\right) + \frac{4}{5}\left(e^2 - \sin 2\right)$$

$$= \boxed{2\left(e^2 - \sin 2\right)}.$$

(d). $f(x,y) = 2x^2 + xy + y^2$. Find unit vector $v = (v_1, v_2)$ s.t.

$$\nabla f = (4x+y,\ x+2y)$$

$\nabla_v f(1,1) = \nabla f(1,1)\cdot v$ is lowest

$$\therefore v = -\frac{\nabla f(1,1)}{\|\nabla f(1,1)\|} = -\frac{(5,3)}{6} = \boxed{\left(-\frac{5}{6}, -\frac{3}{6}\right)}$$

France

8

3. $f(x) = \ln(1+x)$

$f'(x) = \frac{1}{1+x}$ ; $f^{(2)}(x) = -\frac{1}{(1+x)^2}$ ; $f^{(3)}(x) = \frac{2}{(1+x)^3}$ ;

$f^{(4)}(x) = -\frac{3!}{(1+x)^4}$ ; $f^{(5)}(x) = \frac{4!}{(1+x)^5}$ .

$\therefore f(x) = f(0) + x f^{(1)}(0) + \frac{x^2}{2!} f^{(2)}(0) + \frac{x^3}{3!} f^{(3)}(0) + \frac{x^4}{4!} f^{(4)}(0) + \frac{x^5}{5!} f^{(5)}(c)$

for $0 < c < x$.

[1]

$\therefore \left| \ln(1+x) - \left( x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \right) \right| = \frac{x^5}{5} \cdot \frac{1}{(1+c)^5}$ [1]

$\leq \frac{x^5}{5}$ $(\because c > 0)$.

[1]

4. $f : [0,1] \longrightarrow \mathbb{R}$ by $f(x) = \begin{cases} 2 & \text{if } x \in \mathbb{Q} \cap [0,1] \\ 1 & \text{if } x \notin \mathbb{Q} \cap [0,1]. \end{cases}$

Let $P = \{0 = x_0, x_1, \ldots, x_n = 1\}$ be a partition of $[0,1]$.

Then, $U(P;f) = \sum_{i=1}^{n} 2(x_i - x_{i-1}) = 2$

$L(P;f) = \sum_{i=1}^{n} 1(x_i - x_{i-1}) = 1$. —— [1]

$\therefore U(f) = 2 \neq 1 = L(f)$ —— [1].

Franco (9)

5. Define, $F(x) = \int_0^x f(t)\,dt - x^3$ [2]

Then, $F$ is continuous on $[0,1]$ & diff. on $(0,1)$ (by FTC-1).

Also, $F(1) = 0 = F(0)$. $F'(x) = f(x) - 3x^2$ [1]

$\therefore$ By Rolle's thm. $\exists\, c \in (0,1)$ s.t.

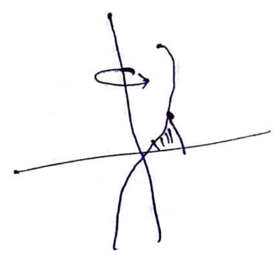
$$F'(c) = 0$$

i.e., $\cancel{\int_0} f(c) - 3c^2 = 0$

i.e., $f(c) = 3c^2$ [2]

7.

(a) The surface area



$$= 2\pi \int_0^{1/2} f(x)\sqrt{1+(f'(x))^2}\,dx$$

$$= 2\pi \int_0^{1/2} x^3\sqrt{1+9x^4}\,dx$$

$1 + 9x^4 = y$.

$\therefore 36x^3\,dx = dy$.

$$= \frac{2\pi}{36}\int_1^{25/16} y^{1/2}\,dy$$

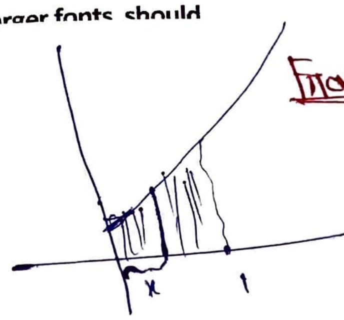
$$= \frac{\pi}{18}\cdot\frac{1}{\frac{3}{2}}\cdot\left(\left(\frac{25}{16}\right)^{3/2} - 1\right)$$

$1 + \frac{9}{16}$

$$= \frac{\pi}{27}\left(\frac{125}{64} - 1\right) = \frac{61\pi}{1728}.$$

(b). Volume (by shell method)

$$= 2\pi \int_0^1 x \cdot e^{2x^2}\, dx$$




$x^2 = y$.

$$= 2\pi \int_0^1 e^{2y}\, \frac{dy}{2}$$

$$= \pi \left[\frac{e^2 - 1}{2}\right] = \boxed{\frac{\pi}{2}\left(e^2 - 1\right)}.$$

(c).

$$\nabla_v f(2,3) = \nabla f(2,3) \cdot v$$

$$= \left(4e^3 - 3\sin 6,\ 4e^3 - 2\sin 6\right) \cdot \left(\frac{3}{5}, \frac{4}{5}\right)$$

$$= \frac{12}{5}e^3 - \frac{9}{5}\sin 6 + \frac{16}{5}e^3 - \frac{8}{5}\sin 6$$

$$= \frac{28}{5}e^3 - \frac{17}{5}\sin 6$$

$$\boxed{= \frac{1}{5}\left(28e^3 - 17\sin 6\right)}.$$

(d).

$$v = -\frac{\nabla f(2,2)}{\|\nabla f(2,2)\|} = -\frac{(10,6)}{\sqrt{136}} = -\frac{(5,3)}{\sqrt{34}}$$

$$= \boxed{\left(\frac{-5}{\sqrt{34}}, -\frac{3}{\sqrt{34}}\right)}$$

Argentina

5. Define:

$$F(x) = \int_0^x f(t)\,dt - x^4$$ [2]

F is cont. and diff. on [0,1] (by FTC-1)

$F(1) = 0 = F(0)$ & $F'(x) = f(x) - 4x^3$. [1]

∴ By Rolle's thm.

$F'(c) = 0$ for some $c \in (0,1)$.

i.e., $f(c) = 4c^3$. [2]

7.

(a) The surface area

$$= 2\pi \int_0^1 f(x)\sqrt{1+(f'(x))^2}\,dx$$

$$= 2\pi \int_0^1 x^3 \sqrt{1+9x^4}\,dx$$

$1+9x^4 = y$

$\therefore 36x^3\,dx = dy$

$$= \frac{2\pi}{36}\int_1^{10} y^{1/2}\,dy$$

$$= \frac{2\pi}{36}\left[(10)^{3/2} - 1\right] \times \frac{2}{3}$$

$$= \frac{\pi}{27}\left[(10)^{3/2} - 1\right].$$

(b) Volume (by shell method):

$$2\pi \int_0^1 x \cdot e^{3x^2}\, dx$$

$x^2 = y$

$2x\,dx = dy$

$$= \pi \int_0^1 e^{3y}\, dy$$

$$\boxed{= \frac{\pi}{3}\left[e^3 - 1\right]}.$$

(c).

$$\nabla_v f(1,1) = \nabla f(1,1) \cdot v$$

$$= (2e - \sin 1,\ e - \sin 1) \cdot \left(\frac{3}{5}, \frac{4}{5}\right)$$

$$= \frac{6}{5}e + \frac{4}{5}e - \frac{7}{5}\sin 1$$

$$\boxed{= 2e - \frac{7}{5}\sin 1}$$

(d).

$$v = -\frac{\nabla f(2,1)}{\|\nabla f(2,1)\|} = -\frac{(9,4)}{\sqrt{97}}$$

$$\boxed{= \left(-\frac{9}{\sqrt{97}}, -\frac{4}{\sqrt{97}}\right)}.$$

81
16
97