# MA 205: Complex Analysis Endsem TSC

Siddhant Midha

11th September 2022

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- Feel free to stop us and ask questions.
- Given the rough time limit of two hours, we will not be able to cover everything done in the lectures.
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- Finally, if you notice some mistake, do let us know.

### Table of Contents

- Preliminaries
- 2 Functions, Continuity, Differentiability
- 3 Power Series
- 4 Integrals and all that
- 5 Singularities and Residues

### Definition (Open Disks)

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Recall:  $z \in \mathbb{C}$  is a limit point of  $\Omega \subset \mathbb{C}$  if there exists a sequence  $z_n \in \Omega$ ,  $z_n \neq z$ , such that  $z_n \to z$ .

#### Connectedness

#### Definition (Connected)

A subset  $S \subseteq \mathbb{C}$  is said to be connected if given any 2 points  $x, y \in S$ , there exists a continuous path joining them. i.e, a continuous function  $f: [0,1] \to S$  such that f(0) = x and f(1) = y.

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### Definition (Domain)

A open and connected subset of  $\mathbb{C}$  is called a domain.

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# Sequences and Convergence

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A sequence  $z_n$  is said to be converging to some  $z \in \mathbb{C}$  if  $\forall \epsilon > 0$ ,  $\exists N_{\epsilon} \in \mathbb{N}$  s.t.

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#### Theorem

If  $z_n = x_n + \iota y_n$  is a sequence in  $\mathbb{C}$ , then

$$z_n \rightarrow z = x + \iota y \Leftrightarrow x_n \rightarrow x \text{ and } y_n \rightarrow y$$

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# Continuity

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$$\lim_{z\to z_0}f(z)=f(z_0)$$

Equivalently<sup>a</sup>, f is continuous at  $z_0$  if for all sequences  $(z_n)_n$   $(z_n \in \Omega)$  such that  $z_n \to z_0$  we have  $f(z_n) \to f(z_0)$ .

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- f is said to be continuous if it is continuous at all  $z_0 \in \Omega$ .
- f is continuous iff u and v are continuous.

### Definition (Complex Differentiability (CD))

Let  $\Omega \subset \mathbb{C}$  be open. A function  $\Omega \to \mathbb{C}$  is said to be complex-differentiable at  $z_0 \in \mathbb{C}$  if the limit

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exists. If it does, we denote it by  $f'(z_0)$ .

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Let  $\Omega \subset \mathbb{C}$  be open. A function  $\Omega \to \mathbb{C}$  is said to be holomorphic on  $\Omega$  if it is complex differentiable at each  $z_0 \in \Omega$  and the derivative f' is continuous on  $\Omega$ . We denote  $f \in C^1(\Omega)$ .

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- A function holomorphic on  $\mathbb C$  is said to be entire.
- Holomorphic at a point  $\implies$  CD at a point. Reverse?
- **Remark**: A function can be CD at a point and not holomorphic at the same point. Consider  $f(z) = |z|^2$ .

If  $f: \Omega \to A$  and  $g: \Omega \to B$  are holomorphic on  $\Omega$ , then,

•  $c_1f + c_2g$  is holomorphic on  $\Omega$ , and  $(c_1f + c_2g)' = c_1f' + c_2g'$ .

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- If  $h: A \to \mathbb{C}$  is holomorphic on A, then  $h \circ f(z) := h(f(z))$  is holomorphic on  $\Omega$ , and  $(h \circ f)'(z) = h'(f(z))f'(z)$ .

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- For  $z_0 \in \Omega$  s.t.  $g(z_0) \neq 0$ , f/g is holomorphic at  $z_0$ , and,

$$\left(\frac{f}{g}\right)(z_0) = \frac{f'(z_0)g(z_0) - g'(z_0)f(z_0)}{g(z_0)^2}$$

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### Real Differentiability

 $F:\Omega_R\to\mathbb{R}^2$  is differentiable at  $(x,y)\in\Omega$  if there exists a  $2\times 2$  matrix DF(x,y) such that

$$\lim_{h,k\to 0} \frac{\left|\left| \begin{pmatrix} u(x+h,y+k) \\ v(x+h,y+k) \end{pmatrix} - \begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix} - DF(x,y) \begin{pmatrix} h \\ k \end{pmatrix} \right|\right|}{\left|\left| \begin{pmatrix} h \\ k \end{pmatrix} \right|\right|} = 0$$

If so, we have

$$DF(x,y) = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}$$

### Theorem (The CR Equations)

Let  $f(z) = u(x,y) + \iota v(x,y)$  be defined on some open set  $\Omega$ . Suppose that  $f'(z_0)$  exists for some point  $z_0 = x_0 + \iota y_0 \in \Omega$ . Then the first order partial derivatives of u and v exist at that point  $(x_0, y_0)$  and satisfy the CR equations

$$u_x = v_y$$
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- [2020 Quiz] Consider,

$$f(z) := \begin{cases} \frac{\overline{z}^2}{z} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases}$$

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We shall see that together they are sufficient to show CD.

# **Necessary and Sufficient Condition**

#### Theorem

Let  $f(z) = u(x,y) + \iota v(x,y)$  be defined on some open set  $\Omega$  and let  $F: \Omega_R \to \mathbb{R}^2$  be the corresponding real function. For some  $z_0 = x_0 + \iota y_0 \in \Omega$ , if

- **1** F is differentiable at  $(x_0, y_0)$ .
- 2 The  $DF(x_0, y_0)$  is of the following form

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

(equivalently, the CR equations are satisfied at  $(x_0, y_0)$ )

Then, we have that  $f'(z_0)$  exists and equals  $a + \iota b$ . Further, the converse holds.

#### **Another Sufficient Condition**

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- 2 the CR equations are satisfied at  $(x_0, y_0)$

Then, we have that  $f'(z_0)$  exists.

This turns out to be easier to check.

#### Harmonic Functions

#### Definition (Harmonic Function)

A function  $g:\Omega_R\subset\mathbb{R}^2\to\mathbb{R}$  is said to be harmonic if it has continuous partial derivatives of the first and second order, and satisfies

$$\triangle g(x,y) = g_{xx}(x,y) + g_{yy}(x,y) = 0 \ \forall (x,y) \in \Omega_R$$

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#### Theorem

If a function  $f(z) = u(x, y) + \iota v(x, y)$  is CD in a domain  $\Omega$ , then u and v are harmonic in  $D_R$ .

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- Preliminaries
- 2 Functions, Continuity, Differentiability
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- Fact: Absolute Convergence ⇒ Convergence.

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The word 'expression' signifies that the series/power series may or may not be meaningful (read convergent).



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Usually, we allow for  $R = 0, \infty$  for convenience.

ano comments on the boundary!

### Follow up theorem

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- When the limit exists, the limsup is equal to the limit.

#### Theorem

**1** The power series  $\sum_n a_n(z-z_0)^n$  defines a holomorphic function  $f:D(z_0,R)\to\mathbb{C}$ ,  $f(z):=\sum_n a_n(z-z_0)^n$  where R is the RoC.

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This leads to the statement

Analytic  $\Longrightarrow$  Holomorphic

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### Monotone Convergence Theorem (MCT)

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- **1** Necessary Condition for Convergence If  $\sum_n z_n$  converges, then  $|z_n| \to 0$  as  $n \to \infty$ . (Recall the tutorial question about  $\sum nz^n$ )
- **Necessary & Sufficient Condition for Convergence** Note that  $|z_n| \ge 0$  (thus  $s_n := \sum_{k=0}^n |z_k|$  is monotonic increasing), we have that  $\sum_n |z_n|$  converges iff  $s_n = \sum_{k=0}^n |z_k|$  is bounded above. (follows from the MCT, recall the tutorial question about  $\sum z^n/n^2$ )

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#### Theorem,

If f is holomorphic in a domain (thus, connected), and  $f' \equiv 0$  in that region, then f is a constant.

11th September 2022

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Given a continuous function g on a domain  $\Omega$ , we have that

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### Cauchy Integral Theorem

Let  $\Omega$  be a bounded domain in  $\mathbb{C}$ , with piecewise smooth boundary  $\partial\Omega$  and  $f\in C^1(\bar{\Omega})$  is holomorphic on  $\Omega$ . Then,

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$$f(z) = \frac{1}{2\pi\iota} \int_{\partial\Omega} \frac{f(\eta)}{\eta - z} d\eta$$

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Note that, in these theorems we are dealing with  $\partial\Omega$  being traversed anticlockwise. Also, we do *not* need  $f\in C^1(\bar{\Omega})$ , as holomorphicity of f guarantees holomorphicity and thus continuity of f'.

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#### CIT - Aliter

If  $f:\Omega\to\mathbb{C}$  is holomorphic, and  $\Omega$  is a **simply connected** domain, then for every closed piecewise smooth curve  $\gamma$  within  $\Omega$  we have,

$$\int_{\gamma} f(z)dz = 0$$

Questions?

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① [2020 Quiz]

$$\int_{|z|=1} \frac{e^z \sin(z) - z}{z^2 \cos(z)} dz$$

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Answer: 0

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Answer:  $2\pi\iota(e-2)$ .



### Theorem (Morera's Theorem)

Given a continuous function g on a domain  $\Omega$ , we have that if  $\int_{\gamma} g(z)dz = 0$  for all  $\gamma = \partial R$ , whenever  $R \subset \Omega$  is a rectangle, then, g is holomorphic on  $\Omega$ .

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Given a function g which is complex differentiable at each point in a domain  $\Omega$ , we have that  $\int_{\gamma} f(z)dz=0$  whenever  $\gamma=\partial R$  and  $R\subset\Omega$  is a rectangle.

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#### Theorem (Goursat's Theorem)

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$$f^{(n)}(z_0) = \frac{n!}{2\pi\iota} \int_{D(z_0,r)} \frac{f(\eta)}{((\eta - z_0)^{n+1}} d\eta$$

for some small r.

• **Holomorphic**  $\Longrightarrow$  **Analytic**. If f is holomorphic at a point  $z_0 \in \mathbb{C}$ , then we have that  $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$  for all  $z : |z - z_0| < r$  for some small r.

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#### 



Questions?

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**●** [2018 Midsem] If *f* is a holomorphic function on an open set containing the closed unit disk, and,

$$\int_{|z|=1} f(z)\bar{z}^j dz = 0$$

holds true for all  $j = 0, 1, 2 \dots$ , then show that  $f \equiv 0$ .

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$$|f(z)| \le 1 + \sqrt{|z|} \ \forall z \in \mathbb{C}$$

show that f is constant.

#### Theorem

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Suppose  $f:\Omega\to\mathbb{C}$  is holomorphic, where  $\Omega$  is a domain. Further suppose  $f(z_0)=0$ . Then, we have  $f(z_0)=0$  on  $f(z_0)=0$  or  $f(z_0)=0$ 

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- $f \equiv 0 \text{ on } \Omega, \text{ or, }$
- ②  $\exists m \in \mathbb{N}$  and a holomorphic function  $g: \Omega \to \mathbb{C}$  such that  $g(z_0) \neq 0$  and  $f(z) = (z z_0)^m g(z)$  in  $D(r, z_0)$  for some small r.

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## Zeros of Analytic Functions

#### Theorem

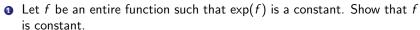
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  - **Identity Principle**: If f, g holomorphic agree on a 'suitable' set of points, then  $f \equiv g$  on  $\Omega$ .

Questions?

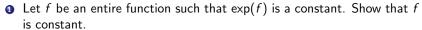
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Then,

- Can g be a constant?
- ② Assume that g is non-constant, then show that  $g(\mathbb{C}) = \mathbb{C}$ .



#### Table of Contents

- Preliminaries
- 2 Functions, Continuity, Differentiability
- 3 Power Series
- 4 Integrals and all that
- 5 Singularities and Residues

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#### [#1] Removable Singularities

An isolated singularity  $z\in\mathbb{C}$  of f is said to be removable if there exists a holomorphic function  $\tilde{f}:D(z,r)\to\mathbb{C}$  for some r>0 such that  $\tilde{f}(z)=f(z)\forall z\in D^*(z,r).$ 

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Also, **fact**: A function has a removable singularity at a point p iff  $\lim_{z\to p} f(z)$  exists.

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Proof? Start with  $g(z) := (z - p)^2 f(z)$ . Explicitly construct the desired  $\tilde{f}$ .

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A pole of order one is called a *simple pole*.

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Also recall the **Little Picard Theorem**: An entire function can omit at most one complex number.

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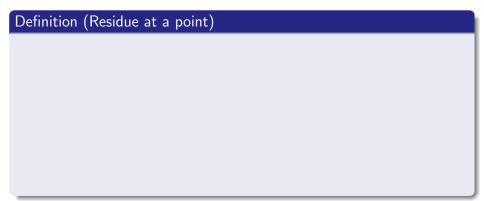
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As an immediate consequence, see that

$$\int_C f(z)dz = 2\pi\iota \times Res(f,p)$$

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Questions?

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$$\frac{\sin z}{z^2(z-\frac{\pi}{2})}$$

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**1** [2020 Quiz] Find the residue at z = 0 of

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#### And that's a wrap!

Thank you, and all the best!