On open and closed sets in $\mathbb{R}^n$

note: your syllabus is defined strictly by the lectures, slides, and tutorial sheets only

def (Neighborhood)

Let $a \in \mathbb{R}^n$.

Define, $N_\gamma(a) := \{x \mid x \in \mathbb{R}^n, |a-x| < \gamma\}$

This is called "an open ball of radius γ" or "a neighborhood".

def (Interior point)

Let $A \subseteq \mathbb{R}^n$

A point $a \in A$ is called an interior point of A if,

$\exists\ \gamma > 0$, s.t. $N_\gamma(a) \subset A$

def. (limit point)

Let $A \subseteq \mathbb{R}^n$

A point $a \in \mathbb{R}^n$ is called a limit point of A, if,

$\forall\ \gamma > 0$, $\exists\ x \in A$, $x \neq a$, s.t.

$x \in N_\gamma(a)$

Def (closed)

$A \subseteq \mathbb{R}^n$ is closed in $\mathbb{R}^n$ if all limit points of A are in A.

Def (open)

$A \subseteq \mathbb{R}^n$ is open in $\mathbb{R}^n$ if all points of A are interior points.

fact#1 $A \subseteq \mathbb{R}^n$ is closed in $\mathbb{R}^n$ $\Longleftrightarrow$ A^c is open in $\mathbb{R}^n$

(where, A^c is the complement of A)

Q

i) What are the limit points of
a) $(0,1)$
b) $[0,1)$
c) $(1,0]$
d) $[1,0]$

ii) Let $A \subseteq \mathbb{R}^n$. Can A be
a) both open and closed
b) neither open nor closed

Choose suitable $n \in \mathbb{N}$ and give an example for both.

hint: can go back to i) for some part of ii)

iii) Try to prove the **fact#1**.

iv) let $n \in \mathbb{N}$, $a \in \mathbb{R}^n$, $r \in \mathbb{R}$, $r > 0$
Show that $N_r(a)$ is open in $\mathbb{R}^n$.

Now, as it happen, the notions of "open" and "closed" sets can be extended to more general situations.

Much like $\mathbb{R}^n$ equipped with the 'distance' $|\cdot|$, we can have any set X with some notion of distance d, and we call (X, d) a metric space if certain conditions are satisfied (google!)

(there are even more general definitions, related to topology)

Now, note the 'in $\mathbb{R}^n$' highlighted above. We can talk about a set $A \subset \mathbb{R}^n$ being open or closed in some other subset B of $\mathbb{R}^n$, by suitably modifying the definitions above. (note: to talk about A being open/closed in B, we need A to be a subset of B)

Here we have another fact:

fact #2 Let $B \subseteq \mathbb{R}^n$. A subset A of B is open in B $\iff$ (if and only if -- i.e., double implication) $A = B \cap M$ for some $M \subseteq \mathbb{R}^n$, s.t. M is open in $\mathbb{R}^n$.

given fact#2, can you show that for a set to be open in an open set (the second one being open in $\mathbb{R}^n$), we need the former to also be open in $\mathbb{R}^n$?

Given this, how might we reformulate the definition below, such that we do not use U_1 and U_2?

Definition: A subset D of $\mathbb{R}^n$ is called connected if it cannot be written as a disjoint union of two non-empty subsets $D_1 \cup D_2$, with $D_1 = D \cap U_1$ and $D_2 = D \cap U_2$, where U_1 and U_2 are open sets.

(now recall the D_1 and D_2 I defined in the recap pertaining to the unit disk - see that one of them is open in D, the other is not)