MA 205 Autumn 2022
Tutorial Sheet 1

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1. Show for all complex numbers z and w that

$$||z| - |w|| \leq |z - w|$$

Sol.
First, let us see the following lemmas.

Lemma 1.1. *For any* $x \in \mathbb{C}$ *we have that* $\mathrm{Re}(x) \leq |x|$.

Proof. Done in class. $\square$

Recall, $|x|^2 = x\bar{x}$ for all $x \in \mathbb{C}$. Thus, we have,

$$\begin{aligned}
|z - w|^2 &= (z - w)\overline{(z - w)} \\
&= (z - w)(\bar{z} - \bar{w}) \\
&= z\bar{z} - z\bar{w} - w\bar{z} + w\bar{w} \\
&= |z|^2 + |w|^2 - 2\mathrm{Re}(z\bar{w})
\end{aligned}$$

Where we use that $x + \bar{x} = 2\mathrm{Re}(x)$ for all $x \in \mathbb{C}$. Also we have that

$$\begin{aligned}
||z| - |w||^2 &= |z|^2 + |w|^2 - 2|z||w| \\
&= |z|^2 + |w|^2 - 2|z||\bar{w}| \\
&= |z|^2 + |w|^2 - 2|z\bar{w}|
\end{aligned}$$

Where we used that $|x||y| = |xy|$ and $|x| = |\bar{x}|$ for all $x, y \in \mathbb{C}$.
Now since $z\bar{w} \in \mathbb{C}$, by the lemma above, we have

$$\begin{aligned}
\mathrm{Re}(z\bar{w}) &\leq |z\bar{w}| \\
2\mathrm{Re}(z\bar{w}) &\leq 2|z\bar{w}| \\
-2\mathrm{Re}(z\bar{w}) &\geq -2|z\bar{w}| \\
|z|^2 + |w|^2 - 2\mathrm{Re}(z\bar{w}) &\geq |z|^2 + |w|^2 - 2|z\bar{w}| \\
|z - w|^2 &\geq ||z| - |w||^2
\end{aligned}$$

Since $|z - w|, ||z| - |w|| \geq 0$, we get

$$|z - w| \geq ||z| - |w||$$

Alternatively, if we assume the triangle inequality (proved in class) we see that

$$|z| = |z - w + w| \leq |z - w| + |w| \implies |z - w| \geq |z| - |w|$$

And similarly, $|z - w| = |w - z| \geq |w| - |z|$. Thus, $|z - w| \geq ||z| - |w||$. $\blacksquare$

2. Show for all complex numbers z and w that

$$|z+w|^2 + |z-w|^2 = 2(|z|^2 + |w|^2)$$

Sol.
As in the first question, we have

$$|z-w|^2 = z\bar{z} - z\bar{w} - w\bar{z} + w\bar{w}$$

And similarly,

$$|z+w|^2 = z\bar{z} + z\bar{w} + w\bar{z} + w\bar{w}$$

Combining the two, we get

$$|z+w|^2 + |z-w|^2 = 2(|z|^2 + |w|^2)$$

$\blacksquare$

3. Show for all complex numbers $z \neq 0$ that $\text{Re}(z) > 0$ if and only if $\text{Re}(1/z) > 0$.
Sol.

- Brute Force. Let $z = a + \iota b$. We have,

$$\begin{aligned}
\frac{1}{z} &= \frac{1}{a + \iota b} \\
&= \frac{a - \iota b}{(a + \iota b)(a - \iota b)} \\
&= \frac{a - \iota b}{a^2 + b^2} \\
&= \frac{a}{a^2 + b^2} + \iota(\frac{-b}{a^2 + b^2})
\end{aligned}$$

  Since $a^2 + b^2 > 0$ $(z \neq 0)$, we have

$$a > 0 \implies \frac{a}{a^2 + b^2} > 0$$

  Similarly, one can let $1/z = c + \iota d$ and repeat the exact procedure. $\blacksquare$

- Polar Form. If we let $z = r \exp \iota\theta$ for some $r, \theta \in \mathbb{R}$ and $r > 0$ (since $z \neq 0$). Thus we have

$$\frac{1}{z} = \frac{1}{r} \exp -\iota\theta$$

  Now using that fact that $\exp \iota\alpha = \cos\alpha + \iota \sin\alpha \forall \alpha \in \mathbb{R}$, we see that

$$\begin{aligned}
\text{Re}(z) &= r\cos\theta \\
\text{Re}(1/z) &= (1/r)\cos\theta
\end{aligned}$$

  Since $\text{Re}(z) > 0$ we have $\cos\theta > 0$ and we are done with the first implication. Again, we can repeat the process for $1/z$ and establish the reverse implication.

$\blacksquare$

4. For $z : |z| < 1$ and $w : |w| < 1$ then show

$$|\frac{w-z}{1-\bar{w}z}| < 1$$

Sol.
We have,

$$\begin{aligned}|z| < 1 \implies |z|^2 < 1 \text{ , } |w| < 1 \implies |w|^2 < 1 \\ &\implies (1-|z|^2)(1-|w|^2) > 0 \\ &\implies 1 + |z|^2|w|^2 - |z|^2 - |w|^2 > 0\end{aligned}$$

Note that (skipping some familiar steps),

$$\begin{aligned}|1-\bar{w}z|^2 &= (1-\bar{w}z)(1-w\bar{z}) \\ &= 1 + \bar{w}zw\bar{z} - \bar{w}z - w\bar{z} \\ &= 1 + |w|^2|z|^2 - 2\text{Re}(z\bar{w})\end{aligned}$$

And,

$$|w-z|^2 = |w|^2 + |z|^2 - 2\text{Re}(z\bar{w})$$

Thus,

$$\begin{aligned}|1-\bar{w}z|^2 - |w-z|^2 &= (1 + |w|^2|z|^2 - 2\text{Re}(z\bar{w})) - (|w|^2 + |z|^2 - 2\text{Re}(z\bar{w})) \\ &= 1 + |w|^2|z|^2 - |w|^2 - |z|^2 > 0\end{aligned}$$

Where the last inequality was established above. Thus,

$$\begin{aligned}|1-\bar{w}z|^2 - |w-z|^2 > 0 \\ |1-\bar{w}z|^2 > |w-z|^2 \\ 1 > |\frac{w-z}{1-\bar{w}z}|^2 \implies |\frac{w-z}{1-\bar{w}z}| < 1\end{aligned}$$

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5. Determine if f is holomorphic and then calculate $f'(z)$

1. $f(z) = \text{Re}(z)$
2. $f(z) = |z|$
3. $f(z) = |z|^2$
4. $f(z) = \frac{z+1}{1+|z|^2}$

Sol.
Since all of the functions here are defined on $\mathbb{C}$, one needs to check holomorphicity on $\mathbb{C}$. Recall, a function f is holomorphic on its domain if it is complex differentiable on each point in the domain and f' is continuous on the domain. Particularly, note that

$$\text{holomorphicity} \implies \text{differentiability}$$

Hence, if a function is not differentiable at even a single point in its domain, then it is not holomorphic on the domain.

For checking differentiability, we shall use two methods:

1. Checking the limit
$$\lim_{h\to 0} \frac{f(z_0+h)-f(z_0)}{h}$$
along simple directions of approach (note that $h \in \mathbb{C}$).

2. Checking whether the *Cauchy Riemann* (CR) equations are satisfied. We know that if a function is complex differentiable at z, it will satisfy the CR equations at z. Thus, this serves as a necessary condition. (What would be a sufficient condition?)

1. Let $z_0 \in \mathbb{C}$
   (a) First way: Consider two possible paths of approach z: $\mathrm{Re}(z) = \mathrm{Re}(z_0)$ and $z : \mathrm{Im}(z) = \mathrm{Im}(Z_0)$. For the first path, we have the limit to be 0 and the limit along the second path is 1. Thus, it is complex differentiable nowhere.
   (b) Second way: We have, $f(z_0 = x_0 + \iota y_0) = \mathrm{Re}(z_0) = x_0 + \iota 0$. Thus, $u(x, y) = x$ and $v(x, y) = 0$. One observes,
   $$u_x = 1 \ , \ v_y = 0$$
   Thus, the CR equations are satisfied nowhere. Thus, it is complex differentiable nowhere.

   Hence, not holomorphic.

2. Let $z_0 \in \mathbb{C}$. We shall start with the second way here (why?). We have, $f(z_0 = x_0 + \iota y_0) = |z_0| = \sqrt{x_0^2 + y_0^2} + \iota 0$. Thus,
$$u(x, y) = \sqrt{x^2 + y^2} \ , \ v(x, y) = 0$$
$$u_x = \frac{x}{\sqrt{x^2 + y^2}} \ , \ v_y = 0$$
Thus the CR equations are not satisfied at any $z \neq 0$. For $z = 0$, one can check that the partial derivative u_x does not exist. One can also use the limit method to see that the limit
$$\lim_{h\to 0} \frac{|h|}{h}$$
does not exist. Hence, complex differentiable nowhere and thus not holomorphic.

3. Similar as the last part for $z \neq 0$. But note that at $z = 0$ the CR equations *are* satisfied. This warrants for a use of the limit method. We have,
$$\lim_{h\to 0} \frac{|h|^2}{h} = \lim_{h\to 0} \bar{h} = 0$$
Thus, f is complex differentiable nowhere except at $z = 0$. Nonetheless, it is not holomorphic. Is it holomorphic *at the point* $z = 0$?

4. Upon simple algebra one sees u_x is not identically equal to v_y. Hence, not holomorphic. As a concrete example, consider $z = 1$. Then, consider the limits along the curves: $|z| = 1$ and $\mathrm{Im}(z) = 0$. The limit along the first curve can be found trivially to be $\frac{1}{2}$. Along the second curve, we can differentiate it as a real function to obtain the limit to be $\frac{-1}{2}$. It can also be checked that CR equations are not satisfied at $z = 1$. Hence, not holomorphic.

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