The EPR Experiment and Bell's Inequality

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The Objective of the EPR experiment

- The EPR experiment was designed to show that the *non-classical* properties of quantum mechanics were not self-consistent and argue that quantum mechanics in a sense was not a *complete theory*.
- EPR believed that any complete physical theory must represent any *element of reality*.
- It was claimed that for a certain physical property to be an *element of reality*, it was sufficient that it be possible to predict with certainty the value the property will have immediately before the measurement.
- The goal was to show that quantum mechanics in this sense lacked some *element of reality*
- This experiment was more of a thought experiment and did not use the laws of quantum mechanics.

The Setup

- Charlie has prepared two particles and hands one of them to Alice and the other one to Bob.
- Alice has two measurement apparatuses, by which she can measure the physical properties P_Q and P_R. Similarly Bob can measure properties P_S and P_T. Suppose that the particle has the objective value Q for the property P_Q which is revealed by the measurement and likewise for other properties.
- Alice and Bob choose which measurement they want to perform randomly at the time they receive the particle and independently of each other and perform the experiment at the same time. For simplicity, let the values of $P, Q, R,$ and S be $+1$ or -1.

Classical Viewpoint: CHSH Inequality

- We are going to look at the quantity $QS + RS + RT - QT$. Since P, Q, R, S take values either $+1$ or -1, this quantity can only take values $+2$ or -2.
- Let $p(q, r, s, t)$ be the probability that before the measurement, the system is in the state $Q = q, R = r, S = s,$ and $T = t$. We have

$$E(QS + RS + RT - QT) = \sum_{q,r,s,t} p(q, r, s, t).(qs + rs + rt - qt) \leq 2$$

- Distributing the expectation, we get a form of the *Bell's inequality*

$$E(QS) + E(RS) + E(RT) - E(QT) \leq 2$$

- While deriving this, two assumptions were made (Where? Discuss.):
  (a) Realism: The property P_Q has the value Q independent of observation.
  (b) Locality: Alice's experiment does not affect Bob's measurement because they are done at the same time.

Where is the local realism assumption made?

- We can only measure one of P_Q, P_R and one of P_S, P_T. However, when saying that $QS + RS + RT - QT$ can only take values ± 2, we end up having to assign values ± 1 to the quantities not measured in the measurement. We have assigned ± 1 values to all four properties simultaneously.
- We assumed there was a joint probability distribution which governs the possible outcomes of all measurements Alice and Bob might perform.This is the hidden variable hypothesis. It assumes that if the values of the hidden variables (p,q,r,s) are exactly known, the outcome of any measurement can be predicted with certainty.
- We assume above that the randomness caused is coming only from not knowing the hidden variables and which value they take from their domain. If that is known, we can predict with certainty the result of the measurement.

Quantum Mechanical Viewpoint

- Let us assume that Charlie has prepared two qubits in the state $|\psi\rangle = \frac{|01\rangle - |10\rangle}{2}$ and send the first qubit to Alice and the second qubit to Bob.
- Suppose that they perform measurements using the following observables: $Q = Z_1$, $R = X_1$, $S = \frac{-Z_2 - X_2}{\sqrt{2}}$, $T = \frac{Z_2 - X_2}{\sqrt{2}}$.
- From simple calculation, we see that the average values for these observables satisfy $\langle QS \rangle = \langle RS \rangle = \langle RT \rangle = -\langle QT \rangle = \frac{1}{\sqrt{2}}$.
- This leads to

$$\langle QS \rangle + \langle RS \rangle + \langle RT \rangle - \langle QT \rangle = 2\sqrt{2}$$

which is in apparent violation of Bell's inequality.

How does Quantum Mechanics escapes local realism?

- If the properties on which measurements are to be performed are chosen, then nothing is said about the other properties.
- The assumption that a measurement of the first particle (i.e. qubit) by Alice does not affect the measurement Bob makes is not taken into consideration. In fact, the very opposite is true here as the two qubits are taken to be in a "maximally entangled" state.
- This property of entanglement completely violates locality.
- A physical quantity is never given a value, but a state which might lead to a multiple values upon measurement, violating realism.

Results and Conclusion of the EPR experiment

- We perform the experiment multiple times using photons and average to get the expectation values of the concerned expression.
- The results were in favor of the quantum prediction.
- This leads us to question the reality and locality assumptions we made while deriving the Bell's inequality and implies that the world is not locally realistic.
- We now discover a tool called entanglement which is profoundly useful in QIQC.

Tsirelson's Inequality

Theorem

Suppose $Q = \vec{q}.\vec{\sigma}$*,*$R = \vec{r}.\vec{\sigma}$*,*$S = \vec{s}.\vec{\sigma}$*, and* $T = \vec{t}.\vec{\sigma}$ *where* $\vec{q}$*,*$\vec{r}$*,*$\vec{s}$*, and* $\vec{t}$ *are real unit vectors in three dimensions. Then*

$$\langle Q \otimes S \rangle + \langle R \otimes S \rangle + \langle R \otimes T \rangle - \langle Q \otimes T \rangle \leq 2\sqrt{2}$$

Proof of theorem. We start off by proving some lemmas.

Lemma

$(\vec{v}.\vec{\sigma}).(\vec{v}.\vec{\sigma}) = I$ *for any three dimensional unit vector* v*.*

Proof of Lemma 1. Let $\vec{v} = v_1\hat{i} + v_2\hat{j} + v_3\hat{k}$. Then

$$\vec{v}.\vec{\sigma} = \begin{bmatrix} v_3 & v_1 - iv_2 \\ v_1 + iv_2 & -v_3 \end{bmatrix}$$

.

Squaring, we have

$$(\vec{v}.\vec{\sigma})^2 = \begin{bmatrix} v_1^2 + v_2^2 + v_3^2 & 0 \\ 0 & v_1^2 + v_2^2 + v_3^2 \end{bmatrix}$$

Using the fact that v is a unit vector, we obtain the result.

Lemma

$$(Q \otimes S + R \otimes S + R \otimes T - Q \otimes T)^2 = 4I + [Q, R] \otimes [S, T]$$

Proof of Lemma 2. We rewrite this as

$$((Q + R) \otimes S + (R - Q) \otimes T)^2 = 4I + (QR - RQ) \otimes (ST - TS)$$

After squaring, this leads to

$$(Q + R)^2 \otimes S^2 + (Q - R)^2 \otimes T^2 + (R^2 - Q^2) \otimes ST + (R^2 - Q^2) \otimes TS = 4I$$

Lemma 1 implies that $Q^2 = I$, $R^2 = I$, $S^2 = I$, and $T^2 = I$.

Using these we get

$$(2I + QR + RQ) \otimes I + (2I - RQ - QR) \otimes I + 0 + 0 = 4I$$

or $4I = 4I$.

Lemma

$\langle [Q, R] \rangle = 2ib$ where $b \in \mathbb{R}$ and $|b| \leq 1$.
$\langle [S, T] \rangle = 2id$ where $d \in \mathbb{R}$ and $|d| \leq 1$.

Proof of Lemma 3.

$$\langle [Q, R] \rangle = \langle \psi | QR | \psi \rangle - \langle \psi | RQ | \psi \rangle$$

As Q, R are hermitian we have

$$\langle [Q, R] \rangle = \langle \psi | QR | \psi \rangle - \langle \psi | R^* Q^* | \psi \rangle = \langle \psi | QR | \psi \rangle - (\langle \psi | QR | \psi \rangle)^*$$

Now as QR is unitary, the inner product is preserved so that

$$\langle \psi | \psi \rangle = 1 \implies |\langle \psi | QR | \psi \rangle|^2 = 1$$

Let $\langle\psi| QR |\psi\rangle = a + ib$ where $a, b \in \mathbb{R}$. Now we know that $a^2 + b^2 = 1$ so we have $|b| \leq 1$. Similar analysis can be done for $\langle\psi| ST |\psi\rangle = c + id$.

Back to Main Proof.

Let $M = Q \otimes S + R \otimes S + R \otimes T - Q \otimes T$

$$\left\langle M^2 \right\rangle = \langle 4I + [Q, R] \otimes [S, T] \rangle = 4 - 4bd$$

By Cauchy-Schwarz inequality and lemma 3 we have

$$\langle M \rangle \leq \sqrt{\langle M^2 \rangle} = \sqrt{4 - 4bd} \leq 2\sqrt{2}$$

QED.

Implication on CHSH inequality. This result implies that the violation of the CHSH inequality we discussed is the maximum possible violation of the Bell's inequality in quantum mechanics.