MA 109 D2 T1
Week Two Recap

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The Idea

Transition: $f : \mathbb{N} \to \mathbb{R} \longrightarrow f : \mathbb{R} \to \mathbb{R}$.

We defined and spent time understanding convergence of sequences. Now, we will apply those properties to talk about functions over $\mathbb{R}$.

Continuity

We give *two* definitions.

Definitions (Continuity)

Let A be a subset of $\mathbb{R}$. Let $f : A \to \mathbb{R}$ be a function, and let $x_0 \in A$. We say that f is *continuous* at x_0 if,

1. For any sequence x_n in A which converges to x_0, we have that $y_n := f(x_n)$ converges to $f(x_0)$. That is,

$$\boxed{x_n \to x_0 \implies f(x_n) \to f(x_0)}$$

2. For any $\epsilon > 0$, there exists $\delta > 0$ such that

$$\boxed{|x - x_0| < \delta \implies |f(x) - f(x_0)| < \epsilon}$$

Further, we have that, both the definitions are **equivalent**! Can you give a proof?

Limits

We give *two* definitions.

Definitions (Limit)

Let $x_0 \in \mathbb{R}$, and let $A \subset \mathbb{R}$ such that $N_r(x_0) \subset A$ for some $r > 0$. Let $f : A/\{x_0\} \to \mathbb{R}$ be a function. We say that f has a limit at x_0 if there exists $L \in \mathbb{R}$ such that

1. If x_n is any sequence in $A/\{x_0\}$ which converges to x_0, then $y_n := f(x_n)$ converges to L. That is,

$$\boxed{x_n \to x_0 \implies f(x_n) \to L}$$

2. For any $\epsilon > 0$, there exists $\delta > 0$ such that

$$\boxed{0 < |x - x_0| < \delta \implies |f(x) - L| < \epsilon}$$

Finishing up continuity; a relation

We defined limits for the case where the domain of definition contains $N_r(x_0)$ for some $r > 0$. We can make similar definitions for domains containing only $(x_0 - r, x_0)$ or $(x_0, x_0 + r)$ for some $r > 0$. These lead to left-hand and right-hand limit respectively. Also, note that we do not require continuity for the existence of a limit. But, we need the latter for the former. We finish up with the following proposition.

Proposition

Let $x_0 \in \mathbb{R}$, and let $A \subset \mathbb{R}$ such that $N_r(x_0) \subset A$ for some $r > 0$. The function $f : A \to \mathbb{R}$ is continuous at x_0 iff $\lim_{x \to x_0} f(x)$ exists and is equal to $f(x_0)$. That is,

$$\boxed{\text{Continuity at } x_0 \Leftrightarrow \lim_{x \to x_0} f(x) = f(x_0)}$$

Questions

Define the Dirichlet Function, denoted $1_{\mathbb{Q}} : \mathbb{R} \to \mathbb{R}$, as,

$$1_{\mathbb{Q}} := \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

We will show that this is nowhere continuous.

1. Let $x \in \mathbb{Q}$. Define
$$x_n := x + \frac{\sqrt{2}}{n}$$
Note that $x_n \notin \mathbb{Q}\ \forall n$. Thus, $f(x_n) = 0 \forall n$. But, $f(x) = 1$. Thus, $x_n \to x$ but $f(x_n) \not\to f(x)$. Thus, f is discontinuous at all rationals.
2. Let $x \notin \mathbb{Q}$.Define,
$$x_n := \frac{\lfloor 10^n x \rfloor}{10^n}$$
Note that $x_n \in \mathbb{Q}\ \forall n$. By a similar argument, we see that $x_n \to x$ but $f(x_n) \not\to f(x)$. Thus, f is discontinuous at all irrationals.

Hence, we conclude that f is continuous nowhere. $\square$

Questions

How about a less intimidating example? Can you show continuity of $f(x) := 5x + 3$ using

- The $\epsilon - N$ way
- The $\epsilon - \delta$ way

Further, note that,

- The $\epsilon - N$ way is usually good for disproving continuity.
- The $\epsilon - \delta$ way is usually good for showing continuity.

Also attempt the following: Just like we made precise the handwavy definition of a sequence 'going to infinity', give a definition using the $\epsilon - \delta$ way for a function 'going to infinity' at some $x \in \mathbb{R}$.

Basic Topology

Definition (Open Neigbourhoods)

For any $x \in \mathbb{R}$, and for any $\epsilon \in \mathbb{R}_+$ define the open neighbourhood, denoted $N_x(\epsilon)$ as

$$N_x(\epsilon) := \{y \in \mathbb{R} \mid |x - y| < \epsilon\}$$

Definition (Open Sets)

Any $U \subset \mathbb{R}$ is called an *open* subset of $\mathbb{R}$ if for all $u \in U$ there exists a $\epsilon > 0$ such that $N_u(\epsilon) \subset U$.

A subset which is said to be closed if its complement is open. When we talk about differentiability, we do so using open sets.

Differentiability

Definition (Differentiability)

Let $U \subset \mathbb{R}$ be an **open** interval, and let $f : U \to \mathbb{R}$ be a function. Let $c \in U$ and let $\epsilon_0 > 0$ be such that $N_c(\epsilon_0) \subset U$[a]. Define $d(\cdot; f, c) : (-\epsilon_0, \epsilon_0)/\{0\} \to \mathbb{R}$ as,

$$d(h; f, c) := \frac{f(c+h) - f(c)}{h}$$

Then, we say that f is differentiable at c if $\lim_{h \to 0} d(h; f, c)$ exists. If it does, we denote it by $f'(c)$.

[a] we know that this exists due to U being open!

Questions:

- Differentiability $\implies$ Continuity? Yes
- Thus, if a function is not continuous at a point, it cannot be differentiable a that point.
- Continuity $\implies$ Differentiability? No