MA 109 D2 T1
Week Five Recap

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    5. But note that, we will only deal with *interior points* here.

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1. f has a partial derivative w.r.t. x at (x_0, y_0) if

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$$\lim_{h \to 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h}$$
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Definition (Directional Derivatives)

Let $f : D \subset \mathbb{R}^2 \to \mathbb{R}$, and let $(x_0, y_0) \in D$ be an interior point of D. For some $u = (u_1, u_2)^T \in \mathbb{R}^2$, we say that f has a directional derivative along u at (x_0, y_0) if

$$\lim_{t \to 0} \frac{f(x_0 + tu_1, y_0 + tu_2) - f(x_0, y_0)}{t}$$

exists, and denote it by $D_u f(x_0, y_0)$.

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Consider, $f : \mathbb{R}^2 \to \mathbb{R}$ such that $f(x, y) := 0 \forall (x, y) : x \neq 0$ and $y \neq 0$, $f(x, 0) := 1000 \forall x$ and $f(0, y) := 1000 \forall y$.

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1D Definition:

$$\lim_{h \to 0} \frac{|f(x_0 + h) - f(x_0) - f'(x_0)h|}{|h|} = 0$$

What about 2D?

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- Continuity of f at that point.

Alternate Condition

Proposition

Let $f : D \to \mathbb{R}$ and let (x_0, y_0) be an interior point of D. If the partial derivatives of f exist in some neighbourhood of (x_0, y_0) and are continuous at (x_0, y_0) then f is differentiable at (x_0, y_0).

Example

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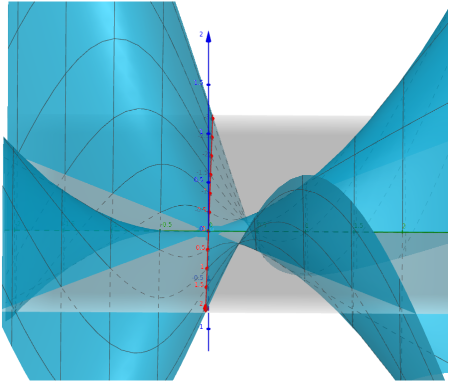

Figure: The graph of f