What the heck *is* a Hilbert Space?

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June 2, 2022

If you have had any exposure to quantum physics, or quantum computation and information, you might have heard the word 'hilbert space' thrown around. This (very brief) document is devoted to rigorously defining the same. Only definitions are provided.

Definition 1 (Vector Spaces)**.** A *vector space* $(V, +, \times)$ over a field F is a set V equipped with two maps $+$ and $\times$ defined as follows

1. $+ : V \times V \to V$, called *addition*
2. $\times : F \times V \to V$ called *scalar multiplication*

These maps satisfy the following axioms

1. Addition axioms.
   - Commutativity: For all $u, v \in V$, we have $u + v = v + u$.
   - Associativity: For all $u, v, w \in V$, we have $u + (v + w) = (u + v) + w$.
   - Existence of the zero vector: There exists $0 \in V$ such that $0 + v = v + 0$ for all $v \in V$.
   - Existence of additive inverse: For all $u \in V$, there exists $-u \in V$ such that $u + (-u) = 0$.
2. Scalar Product Axioms.
   - Associativity: $(ab)v = a(bv)$ for all $a, b \in F$ and $v \in V$.
   - Action of 1: For the multiplicative identity $1 \in F$, we have $1v = v$ for all $v \in V$.
   - Distributivity 1: $a(u + v) = au + av$ for all $a \in F$ and $u, v \in V$.
   - Distributivity 2: $(a + b)v = av + bv$ for all $a, b \in F$ and $v \in V$.

Note that we do not bother ourselves with what fields are. We will deal with the field $\mathbb{R}$ of real numbers. Now, let us define the notion of *norm* of a vector space.

Definition 2 (Norm)**.** Let V be a vector space over $\mathbb{R}$. A map $||\cdot||_V : V \to \mathbb{R}$ is said to be a norm over V if

1. $||av||_V = |a| \times ||v||_V$ for all $a \in \mathbb{R}$ and $v \in V$.
2. $||v||_V \geq 0$ for all $v \in V$.
3. $||v||_V = 0$ iff $v = 0$.
4. $||v + w||_V \leq ||v||_V + ||w||_V$ for all $v, w \in V$.

A normed vector space $(V, ||\cdot||_V)$ enables us to define a *distance function* as

$$d(x, y) := ||x - y||_V \text{ for all } x, y \in V$$

With this in mind, lets discuss sequences and notions of convergence.

Definition 3 (Sequence)**.** A sequence in a vector space V is a map $u : \mathbb{N} \to V$.

As usual, we denote the elements of a sequence as u_n. With some abuse of notation, we just denote sequences as $\{u_n\}$.

Definition 4 (Convergence of a Sequence)**.** A sequence $\{u_n\}$ in a normed vector space $(V, ||\cdot||_V)$ is said to converge to some $v \in V$ if for all $\epsilon > 0$ there exists $N = N(\epsilon) \in \mathbb{N}$ such that

$$n \geq N \implies ||u_n - u||_V < \epsilon$$

Definition 5 (Cauchy Sequences)**.** A sequence $\{u_n\}$ in a normed vector space $(V, ||\cdot||_V)$ is said to be a cauchy sequence if for every $\epsilon > 0$ if there exists $N = N(\epsilon) \in \mathbb{N}$ such that

$$n, m \geq N \implies ||u_n - u_m||_V < \epsilon$$

There is another map on vector spaces we need, called the *inner products*. Or, the dot product as one might know.

Definition 6 (Inner Products)**.** Let V be a vector space over $\mathbb{R}$. We say that a map $\langle \cdot, \cdot \rangle_V : V \times V \to \mathbb{R}$ is an *inner product* on V if

1. $\langle v, w \rangle_V = \langle w, v \rangle_V$ for all $v, w \in V$.
2. $\langle v, v \rangle_V \geq 0$ for all $v \in V$.
3. $\langle v, v \rangle_V = 0$ iff $v = 0$.
4. $\langle v, aw_1 + bw_2 \rangle_V = a\langle v, w_1 \rangle_V + b\langle v, w_2 \rangle_V$ for all $v, w_1, w_2 \in V$ and $a, b \in \mathbb{R}$.

A vector space with an inner product defined is called an inner product space. For an inner product space, we have a natural choice of norm as $||\cdot||_V := \sqrt{\langle \cdot, \cdot \rangle_V}$ (left to the reader to verify).

Stick with me for one more definition here.

Definition 7 (Complete Space)**.** A normed vector space $(V, ||\cdot||_V)$ is said to be complete (with respect to the norm $||\cdot||_V$) if every cauchy sequence in that space has converges to a vector in V (with respect to the norm $||\cdot||_V$).

Good you've made it this far. Here it is.

Definition 8 (Hilbert Spaces)**.** A (real) *Hilbert Space* is a normed inner product space which is complete with respect to the norm defined by the inner product.

Complex hilbert spaces are defined analogously, with some tweaks here and there.

Well there we go. Time to outsmart physics professors.