MA 205: $\mathbb{C}$omplex Analysis
Endsem TSC

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Welcome!

Welcome to the (~~first?~~ second!) TSC for $\mathbb{C}$omplex Analysis 2022! Before we start, here are some things to note,

- These slides, along with tutorial solutions and some other material can be found at this page – tinyurl.com/ma-205-22.
- Feel free to stop us and ask questions.
- Given the rough time limit of two hours, we will not be able to cover everything done in the lectures.
- It follows that this session is purely a supplementary one, not a compensation for the lectures.
- Finally, if you notice some mistake, do let us know.

Table of Contents

Open and Closed Sets

Definition (Open Disks)

For any $z \in \mathbb{C}$, and for any $r > 0$ define the open disk, denoted $B(z, r)$ as

$$B(z, r) := \{z_1 | d(z, z_1) < r\}$$

Definition (Open Sets)

A subset $S \subseteq \mathbb{C}$ is said to be open if for all $z \in S$ there exists an $r > 0$ such that $B(z, r) \subset S$.

Definition (Closed Sets)

A subset $S \subseteq \mathbb{C}$ is said to be closed if its complement is open. Equivalently, a set is closed if it contains all of its limit points.

Recall: $z \in \mathbb{C}$ is a limit point of $\Omega \subset \mathbb{C}$ if there exists a sequence $z_n \in \Omega$, $z_n \neq z$, such that $z_n \to z$.

Connectedness

Definition (Connected)

A subset $S \subseteq \mathbb{C}$ is said to be connected if given any 2 points $x, y \in S$, there exists a continuous path joining them. i.e, a continuous function $f : [0, 1] \to S$ such that $f(0) = x$ and $f(1) = y$.

Definition (Domain)

A open and connected subset of $\mathbb{C}$ is called a domain.

Questions

Questions?

1. $\mathbb{C}$ minus the non-zero real numbers, is
   - Open? No. Closed? No!
   - Connected? Yes.
2. [2021 Tutorial] Let A be any countable subset of $\mathbb{C}$. Is $\mathbb{C}/A$ connected?
3. $(0,1) \subset \mathbb{R}$
   - Open? Yes.
   - Closed? No.
4. $(0,1) \subset \mathbb{C}$
   - Open? No.
   - Closed? No.

Sequences and Convergence

Definition (Sequences)

A sequence in $\mathbb{C}$ is a function $f : \mathbb{N} \cup \{0\} \to \mathbb{C}$. We denote $z_n = f(n)$.

Definition (Convergence)

A sequence z_n is said to be converging to some $z \in \mathbb{C}$ if $\forall \epsilon > 0$, $\exists N_\epsilon \in \mathbb{N}$ s.t.

$$n > N_\epsilon \implies |z - z_n| < \epsilon$$

Theorem

If $z_n = x_n + \iota y_n$ is a sequence in $\mathbb{C}$, then

$$z_n \to z = x + \iota y \Leftrightarrow x_n \to x \text{ and } y_n \to y$$

Table of Contents

Continuity

Definition (Continuity)

A function $f : \Omega \subset \mathbb{C} \to \mathbb{C}$ is said to be continuous at a point $z_0 \in \Omega$ if

$$\lim_{z \to z_0} f(z) = f(z_0)$$

Equivalently[a], f is continuous at z_0 if for all sequences $(z_n)_n$ $(z_n \in \Omega)$ such that $z_n \to z_0$ we have $f(z_n) \to f(z_0)$.

[a] The $\epsilon - \delta$ continuity definition $\Leftrightarrow$ the sequential definition

- f is said to be continuous if it is continuous at all $z_0 \in \Omega$.
- f is continuous iff u and v are continuous.

Differentiability

Definition (Complex Differentiability (CD))

Let $\Omega \subset \mathbb{C}$ be open. A function $\Omega \to \mathbb{C}$ is said to be complex-differentiable at $z_0 \in \mathbb{C}$ if the limit

$$\lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}$$

exists. If it does, we denote it by $f'(z_0)$.

- Note that h above is *complex*.
- Clearly this is stronger than differentiability of functions on $\mathbb{R}$ (Why?). As a result, we do **not** get an iff condition as in the case for continuity.
- f is said to be CD on Ω if it is CD on all $z \in \Omega$.
- Differentiability $\implies$ Continuity.

Holomorphicity

Definition (Holomorphicity)

Let $\Omega \subset \mathbb{C}$ be open. A function $\Omega \to \mathbb{C}$ is said to be holomorphic on Ω if it is complex differentiable at each $z_0 \in \Omega$ and the derivative f' is continuous on Ω. We denote $f \in C^1(\Omega)$.

- Can we drop the $C^1(\Omega)$ condition?
- f is called holomorphic **at a point** if it is holomorphic on an open disk containing that point.
- A function holomorphic on $\mathbb{C}$ is said to be entire.
- Holomorphic at a point $\implies$ CD at a point. Reverse?
- **Remark**: A function can be CD at a point and not holomorphic at the same point. Consider $f(z) = |z|^2$.

Properties

If $f : \Omega \to A$ and $g : \Omega \to B$ are holomorphic on Ω, then,

- $c_1 f + c_2 g$ is holomorphic on Ω, and $(c_1 f + c_2 g)' = c_1 f' + c_2 g'$.
- (fg) is holomorphic on Ω, and $(fg)' = f'g + g'f$.
- If $h : A \to \mathbb{C}$ is holomorphic on A, then $h \circ f(z) := h(f(z))$ is holomorphic on Ω, and $(h \circ f)'(z) = h'(f(z))f'(z)$.
- For $z_0 \in \Omega$ s.t. $g(z_0) \neq 0$, f/g is holomorphic at z_0, and,

$$\left(\frac{f}{g}\right)(z_0) = \frac{f'(z_0)g(z_0) - g'(z_0)f(z_0)}{g(z_0)^2}$$

Real Differentiability

For some $f : \Omega \to \mathbb{C}$ we will denote $F : \Omega_R \to \mathbb{R}^2$ the corresponding real function.Further, we let

$$F(x,y) = \big(u(x,y), v(x,y)\big)^T$$

Real Differentiability

$F : \Omega_R \to \mathbb{R}^2$ is differentiable at $(x,y) \in \Omega$ if there exists a 2×2 matrix $DF(x,y)$ such that

$$\lim_{h,k \to 0} \frac{|| \begin{pmatrix} u(x+h, y+k) \\ v(x+h, y+k) \end{pmatrix} - \begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix} - DF(x,y) \begin{pmatrix} h \\ k \end{pmatrix} ||}{|| \begin{pmatrix} h \\ k \end{pmatrix} ||} = 0$$

If so, we have

$$DF(x,y) = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}$$

Necessary Conditions for (complex) differentiability

Theorem (The CR Equations)

Let $f(z) = u(x, y) + \iota v(x, y)$ be defined on some open set Ω. Suppose that $f'(z_0)$ exists for some point $z_0 = x_0 + \iota y_0 \in \Omega$. Then the first order partial derivatives of u and v exist at that point (x_0, y_0) and satisfy the CR equations

$$u_x = v_y \ , \ v_x = -u_y$$

at that point.

i.e., $CD \implies CR$.

Theorem

Let f be defined on some open set Ω be differentiable at some $z = (x + \iota y) \in \Omega$. Then, the real counterpart F will be differentiable at $(x, y) \in \Omega_R$.

i.e., $CD \implies RD$.

Remark

We have seen that CR and RD are both necessary conditions for CD. But, none of them implies CD. Consider,

- $f(z) = \bar{z}$. RD, not CD.
- [2020 Quiz] Consider,

$$f(z) := \begin{cases} \frac{\bar{z}^2}{z} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases}$$

  CR equations are satisfied at zero, but it is not CD at zero.
- We shall see that together they are sufficient to show CD.

Necessary and Sufficient Condition

Theorem

Let $f(z) = u(x, y) + \iota v(x, y)$ be defined on some open set Ω and let $F : \Omega_R \to \mathbb{R}^2$ be the corresponding real function. For some $z_0 = x_0 + \iota y_0 \in \Omega$, if

1. F is differentiable at (x_0, y_0).
2. The $DF(x_0, y_0)$ is of the following form
$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$
(equivalently, the CR equations are satisfied at (x_0, y_0))

Then, we have that $f'(z_0)$ exists and equals $a + \iota b$. Further, the converse holds.

Another Sufficient Condition

Theorem

Let $f(z) = u(x, y) + \iota v(x, y)$ be defined on some open set Ω. For some $z_0 = x_0 + \iota y_0 \in \Omega$, if

1. the partial derivatives of u and v exist in some neighbourhood of (x_0, y_0) and are continuous at (x_0, y_0), and
2. the CR equations are satisfied at (x_0, y_0)

Then, we have that $f'(z_0)$ exists.

This turns out to be easier to check.

Harmonic Functions

Definition (Harmonic Function)

A function $g : \Omega_R \subset \mathbb{R}^2 \to \mathbb{R}$ is said to be harmonic if it has continuous partial derivatives of the first and second order, and satisfies

$$\triangle g(x, y) = g_{xx}(x, y) + g_{yy}(x, y) = 0 \ \forall (x, y) \in \Omega_R$$

Theorem

If a function $f(z) = u(x, y) + \iota v(x, y)$ is CD in a domain Ω, then u and v are harmonic in D_R.

Summarizing ...

- CD $\implies$ RD.
- RD $\not\Longrightarrow$ CD. CD is a special case of RD.
- $CD \implies CR$.
- $CR \not\Longrightarrow CD$.
- $(CR + RD) \iff CD$

Table of Contents

Series

Definition (Series)

A *series* is an expression of the form $\sum_n z_n$, for $z_n \in \mathbb{C}$.

1. A *series* $\sum_n z_n$ is said to converge to L if the *sequence* $s_n := \sum_{i=0}^n z_n$ converges to L.
2. Absolute Convergence: A series $\sum_n z_n$ is said to converge absolutely if $\sum_n |z_n|$ converges.
3. **Fact**: Absolute Convergence $\implies$ Convergence.

Definition (Power Series)

A *power series* is an expression of the form $\sum_n a_n(z - z_0)^n$, for $a_n, z_0 \in \mathbb{C}$.

The word 'expression' signifies that the series/power series may or may not be meaningful (read convergent).

The Convergence Theorem

Convergence of Power Series

Given the power series,

$$P = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

such that $a_n \in \mathbb{C} \forall n$, $z_0 \in \mathbb{C}$, we have that *only one* of the following is true

1. P converges only at $z = z_0$.
2. P converges at all $z \in \mathbb{C}$.
3. There exists $R \in \mathbb{R}$, $\infty > R > 0$, such that P converges for all $z : |z - z_0| < R$ and diverges for all $z : |z - z_0| > R$.[a]

Usually, we allow for $R = 0, \infty$ for convenience.

[a] no comments on the boundary!

Follow up theorem

Convergence of Power Series

The *radius of convergence* of a power series as defined before is given as

$$R = \frac{1}{\limsup |a_n|^{1/n}}$$

Herein, we allow for $R = 0, \infty$ by letting $1/0 = \infty, 1/\infty = 0$.

What is the limsup?

Definition

limsup For a **real** sequence x_n, define,

$$s_n := \sup\{x_n, x_{n+1} \dots\} \forall n$$

Define $\limsup x_n := \lim s_n$.

Points to be noted,

- s_n is a **non-increasing** sequence. (Why?)
- Because of that, the limit of s_n always exists (can be $\pm\infty$). (Why?)
- Thus, the lim-sup always exists. The limit might not.
- When the limit exists, the limsup is equal to the limit.

Power series are holomorphic

Theorem

1. The power series $\sum_n a_n(z-z_0)^n$ defines a holomorphic function $f : D(z_0, R) \to \mathbb{C}$, $f(z) := \sum_n a_n(z-z_0)^n$ where R is the RoC.
2. The derivative of f is given by term by term differentiation of the power series. Further, it has the same RoC as of the power series defining f.
3. Thus, power series are infinitely differentiable in their disc of convergence.

This leads to the statement

$$\textbf{Analytic} \implies \textbf{Holomorphic}$$

Questions

Questions?

1. [2021 Tutorial] RoC of

$$\sum_{n:n \text{ is prime}} z^n$$

   Answer: 1

2. [2021 Tutorial] RoC of

$$\sum_n (2^n + \iota)(z - \iota)^n$$

   Answer: $1/2$

3. [2020 Endsem] RoC of

$$\sum_n \frac{(-1)^n z^{n^2}}{n!}$$

   Answer: 1

Checking Convergence

Monotone Convergence Theorem (MCT)

For a **real** sequence x_n, we have that if x_n is monotone and bounded, then it converges.

Let $\sum_n z_n$ be a complex series. Note the following,

1. **Necessary Condition for Convergence** If $\sum_n z_n$ converges, then $|z_n| \to 0$ as $n \to \infty$. (Recall the tutorial question about $\sum nz^n$)
2. **Necessary & Sufficient Condition for Convergence** Note that $|z_n| \geq 0$ (thus $s_n := \sum_{k=0}^{n} |z_k|$ is monotonic increasing), we have that $\sum_n |z_n|$ converges iff $s_n = \sum_{k=0}^{n} |z_k|$ is bounded above. (follows from the MCT, recall the tutorial question about $\sum z^n/n^2$)

Table of Contents

Integration Along Curves

Definition (Curve)

A curve in $\mathbb{C}$ is an infinitely differentiable (smooth) map $\gamma : [a, b] \to \mathbb{C}$.

- We have, that

$$\int_\gamma f(z)dz := \int_a^b f(\gamma(t))\gamma'(t)dt$$

- $|\int_\gamma f(z)dz| \leq \max_{z \in \text{Image}(\gamma)} |f(z)| \cdot \text{length}(\gamma)$
- If f is holomorphic on an open set containing Image(γ), then,

$$\int_\gamma f'(z)dz = f(\gamma(b)) - f(\gamma(a))$$

Primitives etc.

Definition (Primitives)

A holomorphic function $\Omega \to \mathbb{C}$ is said to admit a primitive F in Ω if $F'(z) = f(z)$ for all $z \in \Omega$.

Theorem

If γ is a closed curve in an open set $\Omega \in \mathbb{C}$, and $f : \Omega \to \mathbb{C}$ has a primitive in Ω, then,

$$\int_{\gamma} f(z)dz = 0$$

Follows that $f(z) := 1/z$ does not have a primitive in $\mathbb{C}^*$.

Theorem

If f is holomorphic in a domain (thus, connected), and $f' \equiv 0$ in that region, then f is a constant.

Existence of Primitives

Theorem

Given a continuous function g on a domain Ω, we have that

$$g \text{ admits a primitive on } \Omega \Leftrightarrow \int_\gamma g(z)dz = 0 \;\forall \text{ closed } \gamma \subset \Omega$$

Given a holomorphic function in an open domain Ω, can we claim the existence of a primitive?

Theorem

Given a function $f : \Omega \to \mathbb{C}$ for a open and simply connected region Ω, then f has a primitive on Ω.

- Does this mean that we cannot have a primitive on a non-simply connected domain? No.
- Is the converse true? That is, if a function admits a primitive in a domain, does it have to be holomorphic in that domain? Yes!

The Cauchy Theorem(s)

Cauchy Integral Theorem

Let Ω be a bounded domain in $\mathbb{C}$, with piecewise smooth boundary $\partial\Omega$ and $f \in C^1(\bar{\Omega})$ is holomorphic on Ω. Then,

$$\int_{\partial\Omega} f(z)dz = 0$$

Cauchy Integral Formula

Let Ω be a bounded domain in $\mathbb{C}$ with piecewise smooth boundary $\partial\Omega$, and $f \in C^1(\bar{\Omega})$ is holomorphic on Ω. Then for all $z \in \Omega$, we have,

$$f(z) = \frac{1}{2\pi\iota} \int_{\partial\Omega} \frac{f(\eta)}{\eta - z} d\eta$$

Note that, in these theorems we are dealing with $\partial\Omega$ being traversed anticlockwise. Also, we do *not* need $f \in C^1(\bar{\Omega})$, as holomorphicity of f guarantees holomorphicity and thus continuity of f'.

Another way

Another way to state the CIT, which can avoid possible mistakes.

CIT - Aliter

If $f : \Omega \to \mathbb{C}$ is holomorphic, and Ω is a **simply connected** domain, then for every closed piecewise smooth curve γ within Ω we have,

$$\int_\gamma f(z)dz = 0$$

Questions

Questions?

1. [2020 Quiz]

$$\int_{|z|=1} \frac{e^z \sin(z) - z}{z^2 \cos(z)} dz$$

   Answer: 0

2. [2020 Quiz]

$$\int_{|z|=5} \frac{z}{(z-3)^2(z-1)} dz$$

   Answer: 0

3. [Slides]

$$\int_{|z|=5} \frac{e^z}{z^2(z-1)} dz$$

   Answer: $2\pi\iota(e-2)$.

Connecting the dots

Theorem (Morera's Theorem)

Given a continuous function g on a domain Ω, we have that if $\int_\gamma g(z)dz = 0$ for all $\gamma = \partial R$, whenever $R \subset \Omega$ is a rectangle, then, g is holomorphic on Ω.

Theorem

Given a function g which is complex differentiable at each point in a domain Ω, we have that $\int_\gamma f(z)dz = 0$ whenever $\gamma = \partial R$ and $R \subset \Omega$ is a rectangle.

Combining these two, we have,

Theorem (Goursat's Theorem)

If a function is complex differentiable at each point in an open set, then it is continuously differentiable in that set. Thus, it is holomorphic on that set.

- **Strong Regularity** If f is holomorphic at some z_0, then the derivatives of all orders are holomorphic at that point. Further, we have

$$f^{(n)}(z_0) = \frac{n!}{2\pi\iota} \int_{D(z_0,r)} \frac{f(\eta)}{((\eta - -z_0)^{n+1}} d\eta$$

for some small r.

Consequences of Cauchy's Theorem(s)

- **Holomorphic $\implies$ Analytic**. If f is holomorphic at a point $z_0 \in \mathbb{C}$, then we have that $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$ for all $z : |z - z_0| < r$ for some small r. Where,

$$a_n = \frac{1}{2\pi\iota} \int_{D(z_0,r)} \frac{f(\eta)}{(\eta - z_0)^{n+1} d\eta}$$

  Note that any r s.t. $D(z_0, r)$ is contained in the region of holomorphicity gives the same a_n. This means that an entire function has RoC $= \infty$ when expanded about any point as a power series. Also, we had previously seen that power series are holomorphic in their region of convergence. Thus, Analytic $\implies$ Holomorphic. Hence we have the statement,

  Holomorphic $\Leftrightarrow$ Analytic

Questions

Questions?

1. [2018 Midsem] If f is a holomorphic function on an open set containing the closed unit disk, and,

$$\int_{|z|=1} f(z)\bar{z}^j dz = 0$$

holds true for all $j = 0, 1, 2 \ldots,$ then show that $f \equiv 0$.

- $f^{(n)}(z) = \frac{n!}{2\pi\iota}\int_{\partial\Omega}\frac{f(\eta)}{(\eta - z)^{n+1}}d\eta$ for all $z \in \Omega$. Particularly,

$$|f^{(n)}(z_0)| \leq \frac{n! M_R}{R^n}$$ **Cauchy's Estimate**

  if f is holomorphic on a open set containing $D(z_0, R)$ and $M_R = \max\{|f(z)| : |z - z_0| = R\}$
- **Liouville's Theorem**: A bounded above entire function is a constant.
- **FTA**: A non-constant complex polynomial has atleast one root in $\mathbb{C}$.
- **Mean Value Property**: If f is holomorphic on Ω and $D(z_0, r) \subset \Omega$, then,

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{\iota t})dt$$

Questions?

1. [2018 Midsem] If f is an entire function such that

$$|f(z)| \leq 1 + \sqrt{|z|} \; \forall z \in \mathbb{C}$$

show that f is constant.

Zeros of Analytic Functions

Theorem

Suppose $f : \Omega \to \mathbb{C}$ is holomorphic, where Ω is a domain. Further suppose $f(z_0) = 0$. Then, we have

1. $f \equiv 0$ on Ω, or,
2. $\exists m \in \mathbb{N}$ and a holomorphic function $g : \Omega \to \mathbb{C}$ such that $g(z_0) \neq 0$ and $f(z) = (z - z_0)^m g(z)$ in $D(r, z_0)$ for some small r.

- **Isolated Zeros**: Zeros of a non constant analytic function on a domain Ω are *isolated*. Formally, the set of zeros do not have a limit point.
- **Vanishing Behaviour**
  1. Vanishes at a sequence of points with a limit point in $\Omega \implies f \equiv 0$ on Ω.
  2. Vanishes on an open subset $A \subset \Omega \implies f \equiv 0$ on Ω.
  3. $f^{(n)}(z_0) = 0 \forall n$ for some $z_0 \in \Omega \implies f \equiv 0$ on Ω.
- **Identity Principle**: If f, g holomorphic agree on a 'suitable' set of points, then $f \equiv g$ on Ω.

Questions?

1. []
   1. Let f be an entire function such that $\exp(f)$ is a constant. Show that f is constant.
   2. Suppose f and g are entire, and the following holds true for all $z \in \mathbb{C}$,

   $$\exp(f(z)) + \exp(g(z)) = 1$$

   Show that f, g are constants.
2. [2018 Midsem] Suppose that an entire function g that satisfies

   $$g(1-z) = 1 - g(z)\ \forall z \in \mathbb{C}$$

   Then,
   1. Can g be a constant?
   2. Assume that g is non-constant, then show that $g(\mathbb{C}) = \mathbb{C}$.

Table of Contents

Singularities

Definition (Singularities)

Given a function f, a point $z \in \mathbb{C}$ is said to be a singularity of f if

- f is not defined at z. Or,
- f is not holomorphic at z.

Read differently: Something goes wrong at a singlularity. Also called a singular point. Examples,

1. $f(z) := 1/z$ has a singularity at 0.
2. $f(z) := z^2/z$ has a singularity at 0.

Isolated Singularities

Notation: $D^*(z, r) := D(z, r)/\{z\}$. The 'punctured' disk.

Definition (Isolated Singularities)

Given a function f, a point $z \in \mathbb{C}$ is said to be an isolated singularity, if $\exists r > 0$ s.t. f is holomorphic on $D^*(z, r)$.

A singularity which is not isolated is called a non-isolated singularity.

Examples,

1. $f(z) := 1/z$ has a isolated singularity at 0.
2. $f(z) := (z - 1)/(z(z - 2)(z - 2.00001))$ has isolated singularities at $0, 2, 2.00001$.
3. $f(z) := \tan(1/z)$ has a non-isolated singularity at $z = 0$. Why?
4. $f(z) := \bar{z}$ has *a* non-isolated singularity. Where?

Classifying Isolated Singularities - I

[#1] Removable Singularities

An isolated singularity $z \in \mathbb{C}$ of f is said to be removable if there exists a holomorphic function $\tilde{f} : D(z, r) \to \mathbb{C}$ for some $r > 0$ **such that** $\tilde{f}(z) = f(z) \forall z \in D^*(z, r)$.

Also, **fact**: A function has a removable singularity at a point p iff $\lim_{z \to p} f(z)$ exists.

Theorem (RRST)

Suppose f is bounded an holomorphic on $D^*(p, r)$. Then, p is a removable singularity for f.

Proof? Start with $g(z) := (z - p)^2 f(z)$. Explicitly construct the desired $\tilde{f}$.

Classifying Isolated Singularities - II

[#2] Poles

An isolated singularity $z \in \mathbb{C}$ of f is said to be a *pole* if $\lim_{z \to p} |f(z)| = \infty$.

Theorem (The Order of a Pole)

Let f have a pole at $p \in \mathbb{C}$. Then, there exists some $k \in \mathbb{N}$ such that for some $r > 0$ we have,

$$f(z) = (z - p)^{-k} H(z)$$

where, H is holomorphic on $D(p, r)$ **and** $H(p) \neq 0$. We say that k is the order of the pole.

A pole of order one is called a *simple pole*.

Classifying Isolated Singularities - III

[#3] Essential Singularities

An isolated singularity of f which is neither removable, nor a pole is called an *essential singularity*.

Theorem (Casorati-Weirstrass)

Suppose f has an essential singularity at p. Then, for any $r > 0$, f takes values arbitrarily close to every complex number on the disk $D^*(p, r)$.

Also, we have the following stronger result.

Theorem (Great Picard's Theorem)

Suppose f has an essential singularity at p. Then, for any $r > 0$, f takes on **all possible** complex values, **infinitely often**, with at most a single exception in $D^*(p, r)$.

Also recall the **Little Picard Theorem**: An entire function can omit at most one complex number.

Some Questions

Questions?

Classify the singularities:

1. $e^{1/z}$ Answer: Essential at 0
2. $\sin(1/z)$ Answer: Essential at 0
3. [2020 Quiz]
$$\frac{(z+2)\cos 1/z}{z-3}$$
Answer: Simple pole at 3, essential at 0
4. $z^{100}\cos 1/z$ Answer: Essential at 0
5. $z^4/(z^3+z)$ Answer: Simple poles at $\pm\iota$, Removable at 0

Laurent Series

Definition (Meromorphic Function)

A function holomorphic on a domain except possibly for a set of poles, is said to be meromorphic on that domain.

Theorem (Laurent Series)

Suppose f is analytic in the annulus $\mathcal{A}(p, r_1, r_2) := \{z : r_1 < |z - p| < r_2\}$. Let γ be any positively oriented simple closed contour around p lying in the annulus. Then,

$$f(z) = \sum_{n \in \mathbb{Z}} a_n (z - p)^n$$

holds for all $z \in \mathcal{A}(p, r_1, r_2)$. Where,

$$a_n := \frac{1}{2\pi\iota} \int_\gamma \frac{f(\eta)}{(\eta - p)^{n+1}} \;\forall n \in \mathbb{Z}$$

Question

Questions? Give the Laurent Series for,

1. $f(z) = 1/z^{10231}$ in $\{z : 0 < |z| < 1\}$
2. $f(z) = 1/z(z+1)$ in $\{z : 0 < |z| < 1\}$,$\{z : |z| > 1\}$
3. $f(z) = 1/(z^2+1)$ in $\{z : |z| < 1\}$,$\{z : |z-\iota| < 1\}$
4. $f(z) = e^{1/z}$

A consequence of the Laurent Series

Theorem

Suppose f is analytic in the annulus $\mathcal{A}(p, r_1, r_2) := \{z : r_1 < |z - p| < r_2\}$. Let a_n be the coefficients of its Laurent Series. Then,

1. f has a pole at $p \Leftrightarrow a_n = 0$ for all but finitely many $n < 0$, and $a_k \neq 0$ for some $k < 0$.
2. f has an essential singularity at $p \Leftrightarrow a_n \neq 0$ for **infinitely many** $n < 0$.

Thus,

1. The singularity is removable iff the principal part is zero.
2. The singularity is a pole iff the principal part has a finite (not zero) number of non-zero terms.
3. The singularity is essential iff the principal part has an infinite number of non-zero terms.

Residue Theorem

Definition (Residue at a point)

Suppose f is analytic in the annulus $\mathcal{A}(p, 0, r_2) := \{z : 0 < |z - p| < r_2\}$. Let

$$f(z) = \sum_{n \in \mathbb{Z}} a_n (z-p)^n$$

be the Laurent Series expansion of f in this annulus. Then, we define,

$$Res(f, p) := a_{-1}$$

As an immediate consequence, see that

$$\boxed{\int_C f(z)dz = 2\pi\iota \times Res(f, p)}$$

Residues of Meromorphic functions

Theorem

If f is analytic in $D^*(p, r)$ for some r and has a pole of order k at p, then,

$$Res(f, p) = \lim_{z \to p} \frac{1}{(k-1)!} \frac{d^{k-1}}{dz^{k-1}} \left((z-p)^k f(z) \right)$$

Questions?

1. [2020 Quiz] Find the residue at $z = 0$ of
$$\frac{\sin z}{z^2(z - \frac{\pi}{2})}$$
2. Find residue at 0 of $1/(z + z^2)$.
3. Find residue at 0 of $1/(z^2(1 - z))$ using *both* methods.
4. $\int_{|z|=1} e^{1/z^2} dz$
5. $\int_{|z|=1} \frac{1}{\sin 1/z} dz$ Cannot apply the CRT!

The Cauchy Residue Theorem

This is a formal restatement of the earlier conclusion.

Theorem (CRT)

Let γ be a simple closed contour. Let f be analytic inside and on γ except for a finite number of isolated singularities p_k, then,

$$\int_\gamma f(z)dz = 2\pi\iota \sum_k Res(f, p_k)$$

(Recall)

Theorem (The Order of a Pole)

Let f have a pole at $p \in \mathbb{C}$. Then, there exists some $k \in \mathbb{N}$ such that for some $r > 0$ we have,

$$f(z) = (z-p)^{-k} H(z)$$

where, H is holomorphic on $D(p, r)$ **and** $H(p) \neq 0$. We say that k is the order of the pole.

And similarly, we define,

Definition (Multiplicity of a zero)

Given a function f, we say that $z_0 \in \mathbb{C}$ is a zero of multiplicity m of f, if

$$f^{(n)}(z_0) = 0 \ \forall n \leq m - 1 \textbf{ and } f^{(m)}(z_0) \neq 0$$

We have that,

$$f(z) = (z - z_0)^m g(z)$$

for some holomorphic g which does not vanish in a neighbourhood of z_0.

The Argument Principle

This is a nice application of the CRT.

Theorem

Let f be a meromorphic function on and inside some closed contour γ, **and** has no poles or zeros on γ. Then,

$$\frac{1}{2\pi\iota}\int_{\gamma}\frac{f'(z)}{f(z)}dz = Z - P$$

where, Z is the number of zeros of f inside γ counted with multiplicites and P is the number of poles of f inside γ counted with order.

And that's a wrap!

Thank you, and all the best!