MA 109 D2 T1
Practice Assignment

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Note to the student

1. This assignment is for the D2-T1 batch only.
2. This is a **practice** assignment, it carries no marks and is **not graded**.
3. It is **not** compulsary to submit this. The purpose is to give you more practice before the exams.
4. If you decide to submit, please do so in the third tutorial. It will be evaluated and given back by Tuesday 22/11/22.

A Proof Sample

Let us prove that the sequence $(a_n)_n$ defined as

$$a_n := \frac{5n}{n^2+3}$$

converges.
We claim[1] that a_n converges to 0. We show this through the $\epsilon - N$ definition.
Back of the envelope calculation: We need to show that

$$\mid \frac{5n}{n^2+3} - 0 \mid < \epsilon$$

We note,

$$\begin{aligned} \frac{5n}{n^2+3} &< \frac{5n}{n^2} \\ &< \frac{5}{n} \end{aligned}$$

So, if we want to ensure that $|\frac{5n}{n^2+3}| < \epsilon$, it suffices to ensure $\frac{5}{n} < \epsilon$. Hence, it suffices to take $n > 5/\epsilon$. Now note that choosing *any* $N_0 > 5/\epsilon$ would work. For concreteness, we take

$$N_0(\epsilon) = \lfloor 5/\epsilon \rfloor + 1$$

With this out of the way, we now give the proof.
Proof Let $\epsilon > 0$ be given. Choose $N_0(\epsilon) := \lfloor 5/\epsilon \rfloor + 1 \in \mathbb{N}$. Now, note that, for any $n \in \mathbb{N}$

$$\begin{aligned} \mid \frac{5n}{n^2+3} - 0 \mid = \mid \frac{5n}{n^2+3} \mid &< \frac{5n}{n^2} \\ &< \frac{5}{n} \end{aligned}$$

Thus, for all $n \in \mathbb{N}$ we have,

$$\mid \frac{5n}{n^2+3} - 0 \mid < \frac{5}{n} \quad (*)$$

[1] You do **not** have to justify how you made this claim.

Now let $n > N_0(\epsilon)$ be a natural number. Thus,

$$\begin{aligned}\frac{1}{n} &< \frac{1}{N_0(\epsilon)} \\ &< \frac{1}{\lfloor 5/\epsilon \rfloor + 1} \\ &< \frac{1}{5/\epsilon} \\ &< \frac{\epsilon}{5}\end{aligned}$$

Thus, we see that for $n > N_0(\epsilon)$ we have,

$$\frac{5}{n} < \epsilon \quad (**)$$

The inequalities $(*)$ and $(**)$ both hold for natural numbers n such that $n > N_0(\epsilon)$. Thus,

$$\mid \frac{5n}{n^2+3} - 0 \mid < \epsilon \text{ whenever } n > N_0(\epsilon)$$

Hence, we are done. ■

That is, for any $\epsilon > 0$, we were able to find an $N \in \mathbb{N}$ such that $n > N$ ensured that $|a_n - L| < \epsilon$. This is exactly what the definition demands – it throws an ϵ at you, and demands an $N_0(\epsilon)$ such that the standard conditions hold. This is why it is important to let ϵ be arbitrary in your proof. On the contrary, if you want to disprove convergence, then you need show that there exists *at least one* ϵ such that a suitable $N_0(\epsilon)$ *cannot* be found. Recall how we did this for $(-1)^n$.

The Assignment

1. Consider the sequence $(a_n)_n$ defined as,

$$a_n := \frac{3n^2 - 1}{10n + 5n^2}$$

   Determine if it converges or not. If it does, find the limit. Use nothing but the $\epsilon - N$ definition.

2. For any sequence $(a_n)_n$, show the convergence of a_n implies the convergence of $b_n := |a_n|$. Does the converse hold?

3. Consider $f : \mathbb{R} \to \mathbb{R}$ defined as,

$$f(x) := x^2 + 2$$

   Prove that f is continuous at $x = 2$ by using the $\epsilon - \delta$ definition.

4. Formulate the definition of a finite limit at (positive) infinity. Using this, prove or disprove that

$$\lim_{x \to \infty} \frac{\sin\left(x^2 + 3x + 120\right)}{x^2}$$

   exists.