Problem 1

Suppose P is a n × n non-singular matrix. A and B are n × n matrices. Show that, if B = $P^{-1}AP$ then A and B have the same characteristic equation. (similar)

$$p_B(x) = \det(xI - B)$$

$$= \det(xI - P^{-1}AP)$$

$$= \det(xP^{-1}P - P^{-1}AP)$$

$$= \det(P^{-1}(xI - A)P)$$

$$= \det(P^{-1})\det(xI - A)\det(P)$$

$$= \det(xI - A) = p_A(x) \;!$$

Done

Problem 2

A and B are n × n matrices.

1. Show that I_n - AB is invertible iff I_n - BA is invertible

2. Do AB and BA have the same eigenvalues?

$$(1-x)^{-1} = 1 + x + x^2 + x^3 + x^4 + \dots$$
$x \in \mathbb{R}$

$$(I - AB)^{-1} = I + AB + ABAB + \dots$$
$$= I + A(B + BAB + \dots)$$
$$= I + A(I + BA + BABA + \dots)B$$
$$= \boxed{I + A(I - BA)^{-1}B}$$
$$\times (I - AB) = I$$

assume that $(I - AB)$ is inv.

$\Rightarrow \exists C$ st

$$C(I - AB) = I$$
$$(C - CAB = I)$$
$B \times \qquad\qquad \times A$
$$CA - CABA = A$$
$$BCA - BCABA = BA$$
$$BCA(I - BA) = BA$$
$$BCA(I - BA) - BA = 0$$
$$+I \qquad\qquad +I$$
$$BCA(I - BA) + (I - BA) = I$$
$$(BCA + I)(I - BA) = I$$

$$(BCA+I)(I-BA) = I$$

$$\Longrightarrow (I-BA)^{-1} = BCA + I$$

$\therefore$ $(I-BA)$ is invertible iff $(I-AB)$ is. — (#1) $\boxed{B}$

Do AB and BA have the same eigenvalues?

let α be an e-val of AB

$\Downarrow$

$\det(\alpha I - AB) = 0$

$\alpha = 0$

$\det(-AB) = 0$ $\to (-1)^n \det(A)\det(B)$

$\Downarrow$

$\det(-BA) = 0$

$\Downarrow$

$\det(\alpha I - BA) = 0$

$\therefore \alpha$ is an e-val of BA

$\alpha \neq 0$

$\det(\alpha I - AB) = 0$

$\alpha^n \det(I - \frac{1}{\alpha} AB) = 0$

now, (#1) implies that $\det(I - \frac{1}{\alpha} BA)$ has to be 0!

If not, then $I - \frac{1}{\alpha} AB$ is not inv and $I - \frac{1}{\alpha} BA$ is

$\Downarrow$

$\det(I - \frac{1}{\alpha} BA) = 0$

$\Downarrow$

$\det(\alpha I - BA) = 0$

$\therefore \alpha$ is an e-val of BA.

$\boxed{B}$

4) if $\det(A) = \det(B) = 0$,
show that AB & BA have same e-vals, w/o using (#)

hint (#1) use RREF.

hint #(2) apply inv of RREF to B.

Problem 3

Show that
Nullity(A) = k ⇒ x^k divides the polynomial given by det(xI - A)

Recall from Recap,

A note on the characteristic polynomial

What do the coefficients of the characteristic polynomial look like?

Fact ex: Prove this by induction.

Given $A \in \mathbb{K}^{n\times n}$, let its characteristic polynomial be given as

$$p_A(x) = det(xI - A) = x^n + c_1 x^{n-1} + \ldots c_{n-1}x + c_n$$

x^k

Then,

$$c_k = (-1)^k \times (\text{sum of k} \times \text{k principal minors}) \forall k \in \{1, 2 \ldots n-1\}$$

$$N(A) = k \xRightarrow{\text{Rank Nullity thm.}} R(A) = n-k$$

By the thm on determinantal rank,
any submatrix of size $\ell > n-k$
will have $det(\cdot) = 0$

now, this means,

$$c_\ell = 0 \;\forall\, \ell \text{ s.t. } \boxed{\ell > n-k}$$
$$\Downarrow$$
$$k > n-\ell$$

now, recall that,
the char poly
is of the form,
$$\Sigma\, c_\ell\, x^{n-\ell}$$

$\sum c_\ell x^{n-\ell}$

$\Rightarrow$ for $n-\ell < K$

coeff of $x^{n-\ell} = 0$

$\Rightarrow p_A(x) = x^K + (\cdot)\, x^{K+1} \ldots$

$\Rightarrow x^K$ divides $p_A(x)$

Problem 4

Find eigenvalues and eigenvectors of

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$$

$$\det(A - \lambda I) = \det \begin{pmatrix} 1-\lambda & 2 & 2 \\ 2 & 1-\lambda & 2 \\ 2 & 2 & 1-\lambda \end{pmatrix}$$

$$C_2 \to C_2 - C_1$$
$$C_3 \to C_3 - C_1$$

$$= \det \begin{pmatrix} 1-\lambda & 1+\lambda & 1+\lambda \\ 2 & -1-\lambda & 0 \\ 2 & 0 & -1-\lambda \end{pmatrix}$$

$$= (1+\lambda)(1+\lambda) \det \begin{pmatrix} 1-\lambda & 1 & 1 \\ 2 & -1 & 0 \\ 2 & 0 & -1 \end{pmatrix}$$

$$(1+\lambda)(1+\lambda)\left[2(0+1) - 1(\lambda-1-2)\right]$$

$$(1+\lambda)^2(2+3-\lambda) = -(\lambda-5)(\lambda+1)^2$$

$$\underline{\lambda = 5, -1, -1}$$

$\underline{\lambda = -1}$

$$A - \lambda I = (A + I)$$

$$= \begin{pmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{pmatrix}$$

$(A+I)x = 0$

$$\left[\begin{array}{ccc|c} 2 & 2 & 2 & 0 \\ 2 & 2 & 2 & 0 \\ 2 & 2 & 2 & 0 \end{array}\right]$$

$R_3 \to R_3 - R_1$

$R_2 \to R_2 - R_1$

$$\left[\begin{array}{ccc|c} 2 & 2 & 2 & 0 \\ & 0 & & \end{array}\right] \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$x_1 + x_2 + x_3 = 0$

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\} \to \lambda = -1$$

$$(A - 5I) = \begin{bmatrix} 1-5 & 2 & 2 \\ 2 & 1-5 & 2 \\ 2 & 2 & 1-5 \end{bmatrix}$$

$$= \begin{bmatrix} -4 & 2 & 2 \\ 2 & -4 & 2 \\ 2 & 2 & -4 \end{bmatrix}$$

$R_3 \to R_3 + \frac{1}{2} R_1$

$R_2 \to R_2 + \frac{1}{2} R_1$

$$\begin{bmatrix} -4 & 2 & 2 \\ 0 & -3 & 3 \\ 0 & 3 & -3 \end{bmatrix}$$

$R_3 \to R_3 + R_2$

$$R_3 \to R_3 + R_2$$

$$\begin{bmatrix} -4 & 2 & 2 \\ 0 & -3 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

$$x_2 = x_3$$

$$-2x_1 + x_2 + x_3 = 0$$

$$-2x_1 + 2x_2 = 0$$

$$x_1 = x_2 = x_3$$

$$\therefore \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\} \longrightarrow a = 5$$

e-val	lin indp. e-vector set	dim of e-space
-1	$\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}$	2
5	$\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$	1

(digress to Digression #2)

Problem 5

Find eigenvalues and eigenvectors of

$$A = \begin{bmatrix} 4 & -1 & -2 \\ 2 & 1 & -2 \\ 1 & -1 & 1 \end{bmatrix}$$

$$p_A(\lambda) = (\lambda^3 - 6\lambda^2 + 11\lambda - 6) = 0$$

$$p_A(\lambda) = \boxed{(\lambda-1)(\lambda-2)(\lambda-3)}$$

$$(A - I) = \begin{bmatrix} 3 & -1 & -2 \\ 2 & 0 & -2 \\ 1 & -1 & 0 \end{bmatrix} \quad \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$R_1 \to R_1 - 3R_3$

$R_2 \to R_2 - 2R_3$

$$\begin{bmatrix} 0 & 2 & -2 \\ 0 & 2 & -2 \\ +1 & -1 & 0 \end{bmatrix}$$

$R_1 \to R_1 + R_2$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & -2 \\ +1 & -1 & 0 \end{bmatrix} \begin{matrix} \Rightarrow x_2 = x_3 \\ \Rightarrow x_1 = +x_2 \end{matrix} \Bigg) \Rightarrow x_3 = x_3 = +x_1$$

$$\therefore \left\{ \begin{bmatrix} +1 \\ +1 \\ +1 \end{bmatrix} \right\}$$

$$(A-2I) = \begin{bmatrix} 2 & -1 & -2 \\ 2 & -1 & -2 \\ 1 & -1 & -1 \end{bmatrix}$$

$R_1 \to R_1 - R_2$

$R_2 \to R_2 - 2R_3$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & -1 & -1 \end{bmatrix} \begin{matrix} \\ \Rightarrow x_2 = 0 \\ \Rightarrow x_1 = x_3 \end{matrix}$$

$$\therefore \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$(A-3I) = \begin{bmatrix} 1 & -1 & -2 \\ 2 & -2 & -2 \\ 1 & -1 & -2 \end{bmatrix}$$

$R_2 \to R_2 - 2R_3$

$R_1 \to R_1 - R_3$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ 1 & -1 & -2 \end{bmatrix} \begin{matrix} \\ \Rightarrow x_3 = 0 \\ \Rightarrow x_1 = x_2 \end{matrix}$$

$$\therefore \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$$

e-value	lin indep e-vectorset	dim of e-space
1	$\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$	1
2	$\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$	1
3	$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$	1

Digression #1

Why Diagonalize?

given $A \in \mathbb{R}^{2\times 2}$

and given $A = P^{-1} D P$

where P is non sy, $D = \begin{pmatrix} \lambda_1 & \\ & \lambda_2 \end{pmatrix}$

$$e^{A} := \left[\sum_{i=0}^{\infty} \frac{A^i}{i!}\right] = P^{-1}\left(\boxed{\sum_i \frac{D^i}{i!}}\right) P$$

$$= P^{-1}\begin{pmatrix} e^{\lambda_1} & \\ & e^{\lambda_2} \end{pmatrix} P$$

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\left[\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = e^{\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} t}\begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix}\right]$$

Digression #2

What if we cannot diagonalize?
What can we do?

(continued from problem 4)

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$$

$$P = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}$$

$$AP = \begin{bmatrix} +1 & 0 & 5 \\ 0 & -1 & 5 \\ -1 & +1 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} -1 & & \\ & -1 & \\ & & 5 \end{bmatrix} \quad \begin{bmatrix} \ddots \end{bmatrix}$$

$$\underline{\underline{A = P D P^{-1}}}$$

what if we cannot diagonalize?
(can we do something)

$$(A - \lambda_1 I) v_1 = 0$$
$$(A - \lambda_2 I) v_2 = 0$$

But, i can find $\underline{w}$ s.t. "is a generalized eigenvector"
$$(A - \lambda_1 I) w = v_1$$

So can I get anything like a diagonal $\underline{\underline{\text{matrix}}}$?

hint: $\underline{P} = \begin{bmatrix} v_1 & w & v_2 \end{bmatrix}$

Q: how does this $\underline{\underline{\text{generalize}}}$?

AP

$$AP = P \begin{bmatrix} \boxed{\begin{matrix} \lambda_1 & 1 \\ & \lambda_1 \end{matrix}} & \\ & \boxed{\lambda_2} \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} \ddots & & \\ & & \\ & & \boxed{\lambda_2} \end{bmatrix}}$$

$\hookrightarrow J$

$$J = \begin{bmatrix} \boxed{\begin{matrix}\lambda_1 & 1 \\ & \lambda_1\end{matrix}} & & \\ & \boxed{\begin{matrix}\lambda_1 & 1 \\ & \lambda_1\end{matrix}} & \\ & & \boxed{\lambda_2} \end{bmatrix}$$

$$(x-\lambda_1)^4 (x-\lambda_2)$$