MA 205 Tutorial Batch 3
Recap-2

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Usually, we allow for $R = 0, \infty$ for convenience.

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Herein, we allow for $R = 0, \infty$ by letting $1/0 = \infty, 1/\infty = 0$.

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- Thus, <u>the lim-sup always exists. The limit might not.</u>
- When the limit exists, the limsup is equal to the limit.

Some properties [Can Ignore]

Let x_n be the (real) sequence of interest, and let $-\infty < L < \infty$ be its limsup. Let s_n be defined as before.

1. Since s_n is non-increasing, given any $\epsilon > 0$, $\exists n_0 \in \mathbb{N}$ s.t. $s_{n_0} < L + \epsilon$.
2. By definition of s_{n_0}, we have

$$x_n \leq s_n < L + \epsilon \forall n \geq n_0$$

3. Now, assume that $\exists n_0 \in \mathbb{N}$ s.t. $x_n \leq L - \epsilon \forall n \leq n_0$. This would imply $s_n \leq L - \epsilon \forall n \leq n_0$. Not possible (Why?).
4. Thus there are arbitrary large n s.t. $x_n > L - \epsilon$.

Armed with these points, one can go through `Slide-6,7` of `Lecture 1C` to understand the proof of the RoC better.

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- (Comparison test) If $x_n, y_n \in \mathbb{R}$, and $0 \leq x_n \leq y_n$, then
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- (Ratio Test) Suppose $x_n \neq 0 \forall n$. Further suppose $|x_{n+1}/x_n| \to L$ as $n \to \infty$. Then,
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