3. Use spherical coordinates to find the volume of the solid that lies above the cone $z = \sqrt{x^2 + y^2}$ and below the sphere $x^2 + y^2 + z^2 = z$.

$$z = \sqrt{x^2+y^2}$$

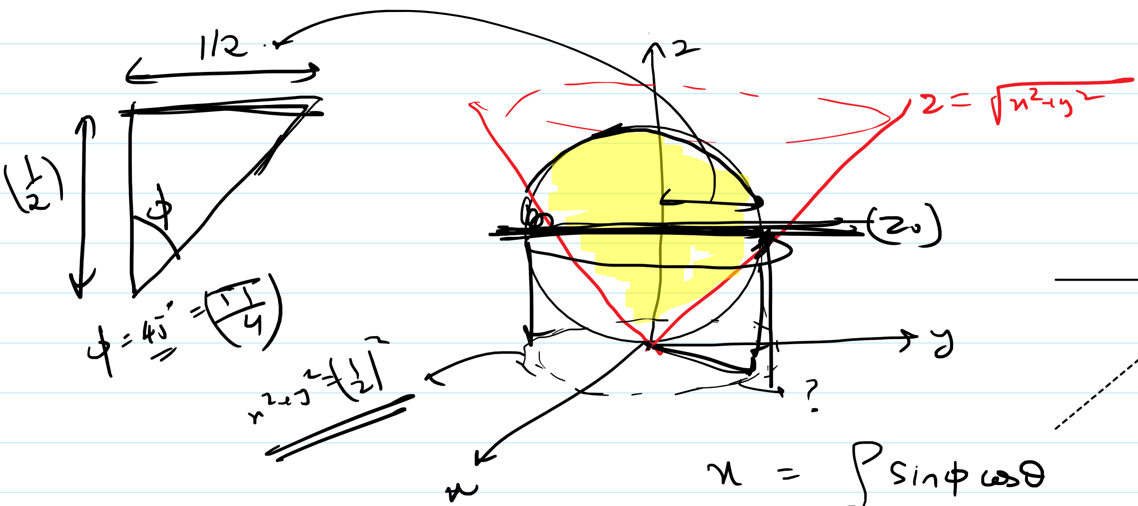
$$x^2+y^2+z^2 = z$$

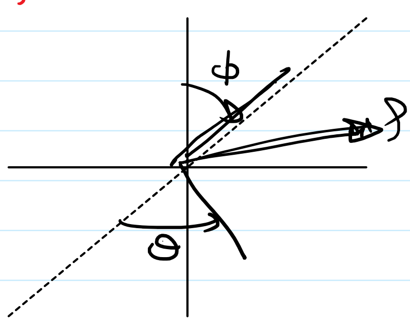
$$x^2+y^2+z^2-z = 0$$

$$x^2+y^2+z^2-2\times\frac{1}{2}z+\left(\frac{1}{2}\right)^2 = \left|\frac{1}{2}\right|^2$$

$$\left(x^2+y^2+\left(z-\frac{1}{2}\right)^2\right) = \left(\frac{1}{2}\right)^2$$

$$C\ \left(0,0,\frac{1}{2}\right) \qquad R\left(\frac{1}{2}\right)$$





$$x = \rho \sin\phi\cos\theta$$
$$y = \rho \sin\phi\sin\theta$$
$$z = \rho \cos\phi$$

$$|J(\rho,\theta,\phi)| = \left(\rho^2\sin\phi\right)$$

$$z_0^2 = x^2+y^2$$
$$x^2+y^2+z_0^2 = z_0$$
$$2z_0^2 = z_0$$
$$z_0 = \frac{1}{2}$$

$$x^2+y^2 = \left(\frac{1}{2}\right)^2$$

$$\left(0 \le \phi \le \pi/4\right) \quad — ①$$

$$\left(0 \le \theta < 2\pi\right) \quad — ②$$

$$(\rho^2) = \rho\cos\phi$$

$$\rho = \cos\phi$$

$$0 \le \rho \le \cos\phi$$

$$\int_0^{2\pi}\int_0^{\pi/4}\int_0^{\cos\phi} (1)(\rho^2 \sin\phi)\, d\rho\, d\phi\, d\theta$$

$$(2\pi)\left[\int_0^{\pi/4}\left(\int_0^{\cos\phi} \rho^2 \sin\phi\, d\rho\right) d\phi\right]$$

$$(2\pi)\left[\int_0^{\pi/4} (\sin\phi) \times \left(\frac{\rho^3}{3}\right)_0^{\cos\phi} d\phi\right]$$

$$= \frac{2\pi}{3}\int_0^{\pi/4} \sin\phi\, (\cos\phi)^3\, d\phi$$

$$t = \cos\phi$$

$$= -\frac{2\pi}{3}\int_1^{1/\sqrt{2}} t^3\, dt$$

$$= -\frac{2\pi}{3}\left(\frac{\left(\frac{1}{\sqrt{2}}\right)^4 - 1^4}{4}\right)$$

$$= \frac{\pi}{6}\left(1 - \frac{1}{4}\right)$$

$$= \frac{\pi}{6} \times \frac{3}{4} = \frac{\pi}{8}$$

4. Use cylindrical coordinates to evaluate $\int\int\int_W (x^2+y^2)\,dz\,dy\,dx$, where

$W = \{(x,y,z) \in \mathbb{R}^3 \mid \boxed{-2 \le x \le 2,} \quad -\sqrt{4-x^2} \le y \le \sqrt{4-x^2}, \quad \sqrt{x^2+y^2} \le z \le 2\}.$

$\boxed{r \le z \le 2}$

$0 \le r \le 2$

$$\left\{\begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{array}\right.$$

$$|J(r,\theta,z)| = r$$

$$\int_0^{2\pi}\int_0^{2}\int_r^{2} (r^2)\times r \, dz\, dr\, d\theta$$

$$(2\pi)\times\int_0^2 (2-r)r^3\,dr$$

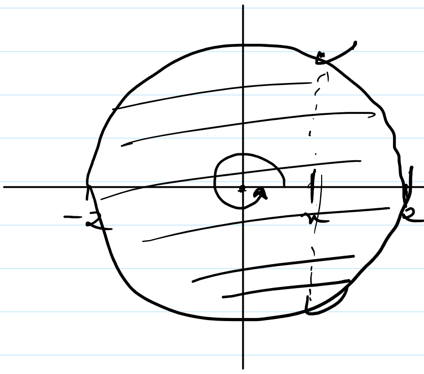
$$= 2\pi \times \left[2\int_0^2 r^3 dr - \int_0^2 r^4 dr\right]$$

$$= 2\pi\left[2\times\left(\frac{r^4}{4}\right)_0^2 - \left(\frac{r^5}{5}\right)_0^2\right]$$

$$= 2^6\pi\left[\frac{1}{4} - \frac{1}{5}\right]$$

$$= \frac{2^6}{20}\pi$$

$$= \boxed{\frac{16}{5}\pi}$$

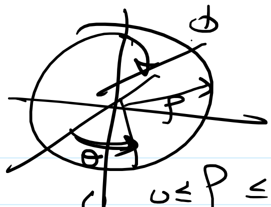


r ✓
θ ✓
z ✓

5. Describe the solid whose volume is given by the integral

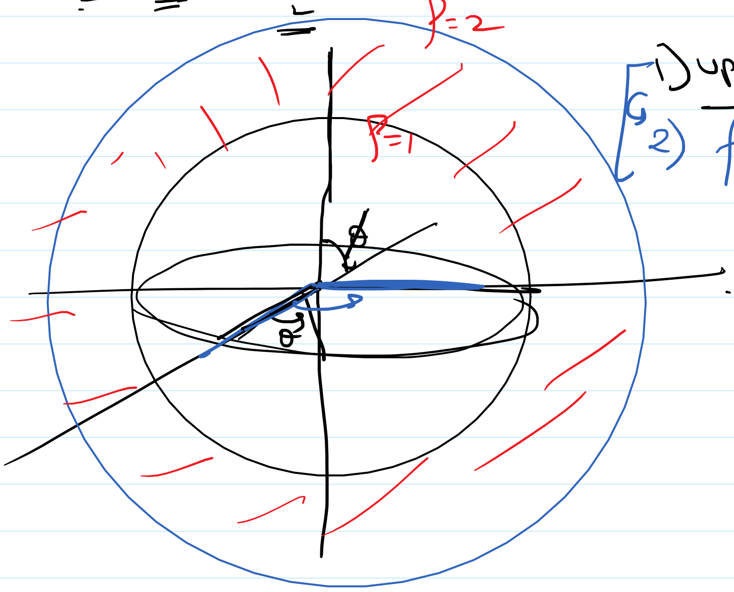
$$\int_0^{\pi/2}\int_0^{\pi/2}\int_1^2 \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta,$$

and evaluate the integral.



$0 \le \rho \le R$
$0 \le \phi \le \pi$
$0 \le \theta \le 2\pi$

$1 \le \rho \le 2$
$0 \le \phi \le \frac{\pi}{2}$
$0 \le \theta \le \frac{\pi}{2}$



1) upper half
2) first octant

5. Describe the solid whose volume is given by the integral

$$\int_0^{\pi/2}\int_0^{\pi/2}\int_1^2 \boxed{\rho^2 \sin\phi} \, d\rho \, d\phi \, d\theta,$$

and evaluate the integral.

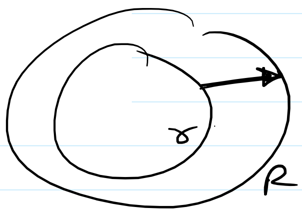
$\boxed{f := 1}$

$$\left(\int_1^2 \rho^2 \, d\rho\right)\left(\int_0^{\pi/2} \sin\phi \, d\phi\right)\left(\int_0^{\pi/2} d\theta\right)$$

$$\left(\frac{\rho^3}{3}\right)_1^2 \times 1 \times \frac{\pi}{2}$$

$$\frac{\pi}{2} \times \frac{(8-1)}{3} = \left(\frac{7\pi}{6}\right)$$

6. Find $\iiint_F \dfrac{1}{(x^2+y^2+z^2)^{n/2}}dV$, where F is the region bounded by the spheres with center the origin and radii r and R, $0 < r < R$.



$$\int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi}\int_{\rho=r}^{R} \frac{1}{(\rho^2)^{n/2}} (\rho^2 \sin\phi)\, d\rho\, d\phi\, d\theta$$

$$\left(\int_0^{2\pi} d\theta\right)\left(\int_0^{\pi} \sin\phi\, d\phi\right)\left(\int_r^R \rho^{2-n}\, d\rho\right)$$

$$(2\pi) \times 2 \times I(\rho) = \boxed{4\pi I(\rho)}$$

$$I(\rho) = \int_r^R \rho^{2-n}\, d\rho$$

$$\checkmark\ I(\rho) = \begin{cases} \dfrac{R^{3-n} - r^{3-n}}{(3-n)} & n \neq 3 \\ \log_e\left(\dfrac{R}{r}\right) & n = 3 \end{cases}$$

4. Calculate the line integral of

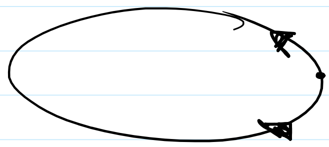
$$\mathbf{F}(x,y) = (x^2+y^2)\mathbf{i} + (x-y)\mathbf{j}$$

once around the ellipse $b^2x^2 + a^2y^2 = a^2b^2$ in the counter clockwise direction.

$$\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \right)$$

"What are we integrating over?" → a geometric curve

$$\vec{C}(\theta) = (a\cos\theta, b\sin\theta)$$
$$\theta \in [0, 2\pi]$$



$$\int_C \vec{F} = \int_{\theta=0}^{2\pi} \underbrace{\vec{F}(\vec{C}(\theta)) \cdot \vec{C}'(\theta)}\, d\theta$$

$$\vec{F}(\vec{C}(\theta)) = \left((a^2\cos^2\theta + b^2\sin^2\theta), (a\cos\theta - b\sin\theta) \right)$$

$$\vec{C}'(\theta) = (-a\sin\theta, b\cos\theta)$$

$$\int_{\theta=0}^{2\pi} \left[\left(-a^3\sin\theta\cos^2\theta - ab^2\sin^3\theta \right) + \left(ba\cos^2\theta - b^2\sin\theta\cos\theta \right) \right] d\theta$$

$$\int_0^{2\pi} ba\cos^2\theta \, d\theta$$

$$= \int_0^{2\pi} ba\left(\frac{1+\cos 2\theta}{2} \right) d\theta$$

$$\int_0^{2\pi} \cos(2\theta)\, d\theta = 0$$

$$= \int_0^{2\pi} \frac{ba}{2}\, d\theta$$

$$= \boxed{\pi ba}$$

6. Calculate

$$\oint_C y\,dx + z\,dy + x\,dz$$

where C is the intersection of two surfaces $z = xy$ and $x^2 + y^2 = 1$ traversed once in a direction that appears counter clockwise when viewed from high above the xy-plane.

t

$(\cos(t), \sin(t))$

$f : \mathbb{R}^3 \to \mathbb{R}^2$ $\quad f(x,y,z) = (y, z, x)$

$\vec{r}(t) = (\cos(t), \sin(t), \cos(t)\sin(t))$
$t \in [0, 2\pi]$

$(\cos t, -\sin t)$

$$\int_0^{2\pi} (\sin t, \cos t \sin t, \cos t) \cdot (-\sin t, \cos t, \cos^2 t - \sin^2 t)\, dt$$

$$\int_0^{2\pi} \left(-\sin^2 t + \cos^2 t \sin t + \cos^3 t - \cos t \sin^2 t\right) dt$$

$$= \int_0^{2\pi} -\sin^2 t\, dt$$

$$= -\int_0^{2\pi} \left(\frac{1 - \cos 2t}{2}\right) dt$$

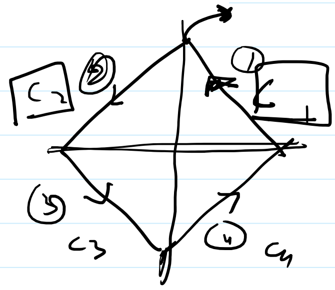
$$= (-\pi)$$

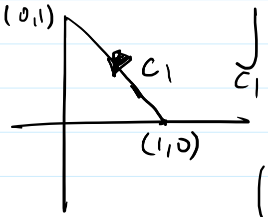
11. Compute the line integral

$$\oint_C \left(\frac{dx+dy}{|x|+|y|}\right)$$

where C is the square with vertices $(1,0),(0,1),(-1,0)$ and $(0,-1)$ traversed once in the counter clockwise direction.

$$f(x,y) = \left[\frac{1}{\boxed{|x|+|y|}_{\to 1}} (1,1) \right]$$

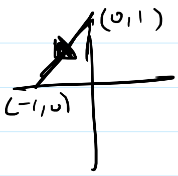




$$\int_{C_1} \underline{dx+dy}$$

$$\boxed{\nabla(x+y)} \quad (1,1)$$

$$\left(\int_C \nabla(x+y)\right) = (0+1)-(1+0) = 0$$



$\alpha_1 = 0$

$\alpha_2 = (-1+0)-(0+1) = -2$

$\alpha_3 = 0$

$\alpha_4 = +2$

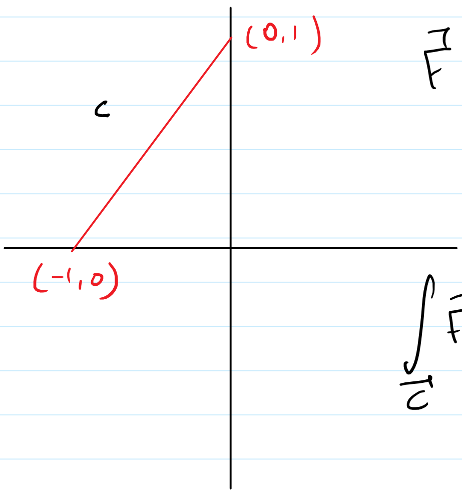
$$\alpha = \sum_i \alpha_i = \underline{\underline{0}}$$

$\vec{c}(t) = (1-t)(0,1) + t(-1,0)$

$= \underline{(-t,\ 1-t)}$ $\quad t \in [0,1]$

$\vec{c}(0) = (0,1)$

$\vec{c}(1) = (-1,0)$



$\vec{F} = (1,1) = \nabla(x+y)$

$$\int_{\underline{C}} \vec{F} = \int_{\underline{C}} \nabla \underbrace{(x+y)}_{f(x,y)}$$

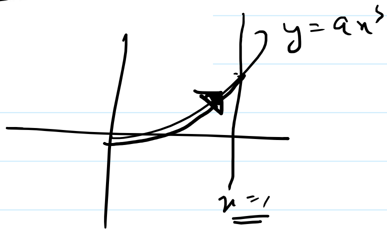
$= f(\vec{c}(1)) - f(\vec{c}(0))$

$= f(-1,0) - f(0,1)$

$= -1 - 1$

$= (-2)$

12. A force $F = xy\mathbf{i} + x^6y^2\mathbf{j}$ moves a particle from $(0,0)$ onto the line $x = 1$ along $y = ax^b$ where $a, b > 0$. If the work done is independent of b find the value of a.



$$\vec{c}(t) = (t, at^b) \qquad t \in [0,1]$$

$$\int_c \vec{F} = \int_{t=0}^{1} \vec{F}(\vec{c}(t)) \cdot \vec{c}\,'(t)\,dt$$

$$= \int_0^1 \left(at^{b+1}\hat{i} + t^6 a^2 t^{2b}\hat{j} \right) \cdot \left(1\hat{i} + abt^{b-1}\hat{j} \right) dt$$

$$= \int_0^1 \left[(at^{b+1}) + a^3 b\, t^{2b+6+b-1} \right] dt$$

$$I(a,b) = \int_0^1 \left[at^{b+1} + a^3 b\, t^{3b+5} \right] dt \qquad a, b > 0$$

$$= \left[\left(\frac{a}{b+2}\right) + \frac{a^3 b}{3b+6} \right]$$

$$I(a,b) = \left[\frac{3a + a^3 b}{3b+6} \right]$$

$$\left(\frac{\partial I(a,b)}{\partial b} \right) = 0 \qquad \forall\ a, b > 0$$

$$\left[\frac{(a^3)(3b+6) - (3)(3a + a^3 b)}{(3b+6)^2} \right] = 0$$

$$\cancel{3a^3b} + 6a^3 - 9a - \cancel{3a^3b} = 0$$

$$9a = 6a^3$$

$$a = 0 \quad \text{or} \quad 9 = 6a^2$$

$$a^2 = \frac{3}{2}$$

$$a = 0,\ +\sqrt{\frac{3}{2}},\ -\sqrt{\frac{3}{2}} \qquad a > 0$$

$\times \qquad\qquad \times$

$$\boxed{a = \sqrt{\frac{3}{2}}}$$