MA 205: $\mathbb{C}$omplex Analysis
Endsem TSC

Siddhant Midha

11th September 2022

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- Finally, if you notice some mistake, do let us know.

Table of Contents

1. Preliminaries
2. Functions, Continuity, Differentiability
3. Power Series
4. Integrals and all that
5. Singularities and Residues

Open and Closed Sets

Definition (Open Disks)

For any $z \in \mathbb{C}$, and for any $r > 0$ define the open disk, denoted $B(z, r)$ as

$$B(z, r) := \{z_1 | d(z, z_1) < r\}$$

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Recall: $z \in \mathbb{C}$ is a limit point of $\Omega \subset \mathbb{C}$ if there exists a sequence $z_n \in \Omega$, $z_n \neq z$, such that $z_n \to z$.

Connectedness

Definition (Connected)

A subset $S \subseteq \mathbb{C}$ is said to be connected if given any 2 points $x, y \in S$, there exists a continuous path joining them. i.e, a continuous function $f : [0, 1] \to S$ such that $f(0) = x$ and $f(1) = y$.

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Definition (Domain)

A open and connected subset of $\mathbb{C}$ is called a domain.

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Theorem

If $z_n = x_n + \iota y_n$ is a sequence in $\mathbb{C}$, then

$$z_n \to z = x + \iota y \Leftrightarrow x_n \to x \text{ and } y_n \to y$$

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$$\lim_{z \to z_0} f(z) = f(z_0)$$

Equivalently[a], f is continuous at z_0 if for all sequences $(z_n)_n$ $(z_n \in \Omega)$ such that $z_n \to z_0$ we have $f(z_n) \to f(z_0)$.

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- f is said to be continuous if it is continuous at all $z_0 \in \Omega$.
- f is continuous iff u and v are continuous.

Differentiability

Definition (Complex Differentiability (CD))

Let $\Omega \subset \mathbb{C}$ be open. A function $\Omega \to \mathbb{C}$ is said to be complex-differentiable at $z_0 \in \mathbb{C}$ if the limit

$$\lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}$$

exists. If it does, we denote it by $f'(z_0)$.

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- Differentiability $\implies$ Continuity.

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- **Remark**: A function can be CD at a point and not holomorphic at the same point. Consider $f(z) = |z|^2$.

Properties

If $f : \Omega \to A$ and $g : \Omega \to B$ are holomorphic on Ω, then,

- $c_1 f + c_2 g$ is holomorphic on Ω, and $(c_1 f + c_2 g)' = c_1 f' + c_2 g'$.

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- If $h : A \to \mathbb{C}$ is holomorphic on A, then $h \circ f(z) := h(f(z))$ is holomorphic on Ω, and $(h \circ f)'(z) = h'(f(z))f'(z)$.

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- If $h : A \to \mathbb{C}$ is holomorphic on A, then $h \circ f(z) := h(f(z))$ is holomorphic on Ω, and $(h \circ f)'(z) = h'(f(z))f'(z)$.
- For $z_0 \in \Omega$ s.t. $g(z_0) \neq 0$, f/g is holomorphic at z_0, and,

$$\left(\frac{f}{g}\right)(z_0) = \frac{f'(z_0)g(z_0) - g'(z_0)f(z_0)}{g(z_0)^2}$$

Real Differentiability

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Real Differentiability

$F : \Omega_R \to \mathbb{R}^2$ is differentiable at $(x,y) \in \Omega$ if there exists a 2×2 matrix $DF(x,y)$ such that

$$\lim_{h,k \to 0} \frac{|| \begin{pmatrix} u(x+h, y+k) \\ v(x+h, y+k) \end{pmatrix} - \begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix} - DF(x,y) \begin{pmatrix} h \\ k \end{pmatrix} ||}{|| \begin{pmatrix} h \\ k \end{pmatrix} ||} = 0$$

If so, we have

$$DF(x,y) = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}$$

Necessary Conditions for (complex) differentiability

Theorem (The CR Equations)

Let $f(z) = u(x,y) + \iota v(x,y)$ be defined on some open set Ω.
Suppose that $f'(z_0)$ exists for some point $z_0 = x_0 + \iota y_0 \in \Omega$. Then the first order partial derivatives of u and v exist at that point (x_0, y_0) and satisfy the CR equations

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Let f be defined on some open set Ω be differentiable at some $z = (x + \iota y) \in \Omega$. Then, the real counterpart F will be differentiable at $(x,y) \in \Omega_R$.

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i.e., $CD \implies RD$.

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  CR equations are satisfied at zero, but it is not CD at zero.
- We shall see that together they are sufficient to show CD.

Necessary and Sufficient Condition

Theorem

Let $f(z) = u(x, y) + \iota v(x, y)$ be defined on some open set Ω and let $F : \Omega_R \to \mathbb{R}^2$ be the corresponding real function. For some $z_0 = x_0 + \iota y_0 \in \Omega$, if

1. F is differentiable at (x_0, y_0).
2. The $DF(x_0, y_0)$ is of the following form
$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$
(equivalently, the CR equations are satisfied at (x_0, y_0))

Then, we have that $f'(z_0)$ exists and equals $a + \iota b$. Further, the converse holds.

Another Sufficient Condition

Theorem

Let $f(z) = u(x, y) + \iota v(x, y)$ be defined on some open set Ω. For some $z_0 = x_0 + \iota y_0 \in \Omega$, if

1. the partial derivatives of u and v exist in some neighbourhood of (x_0, y_0) and are continuous at (x_0, y_0), and
2. the CR equations are satisfied at (x_0, y_0)

Then, we have that $f'(z_0)$ exists.

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Then, we have that $f'(z_0)$ exists.

This turns out to be easier to check.

Harmonic Functions

Definition (Harmonic Function)

A function $g : \Omega_R \subset \mathbb{R}^2 \to \mathbb{R}$ is said to be harmonic if it has continuous partial derivatives of the first and second order, and satisfies

$$\triangle g(x, y) = g_{xx}(x, y) + g_{yy}(x, y) = 0 \ \forall (x, y) \in \Omega_R$$

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Theorem

If a function $f(z) = u(x, y) + \iota v(x, y)$ is CD in a domain Ω, then u and v are harmonic in D_R.

Summarizing ...

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Series

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The word 'expression' signifies that the series/power series may or may not be meaningful (read convergent).

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Usually, we allow for $R = 0, \infty$ for convenience.

[a] no comments on the boundary!

Follow up theorem

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The *radius of convergence* of a power series as defined before is given as

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Herein, we allow for $R = 0, \infty$ by letting $1/0 = \infty, 1/\infty = 0$.

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- Thus, the lim-sup always exists. The limit might not.
- When the limit exists, the limsup is equal to the limit.

Power series are holomorphic

Theorem

1. The power series $\sum_n a_n(z-z_0)^n$ defines a holomorphic function $f : D(z_0, R) \to \mathbb{C}$, $f(z) := \sum_n a_n(z-z_0)^n$ where R is the RoC.

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This leads to the statement

$$\textbf{Analytic} \implies \textbf{Holomorphic}$$

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For a **real** sequence x_n, we have that if x_n is monotone and bounded, then it converges.

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Let $\sum_n z_n$ be a complex series. Note the following,

1. **Necessary Condition for Convergence** If $\sum_n z_n$ converges, then $|z_n| \to 0$ as $n \to \infty$. (Recall the tutorial question about $\sum nz^n$)
2. **Necessary & Sufficient Condition for Convergence** Note that $|z_n| \geq 0$ (thus $s_n := \sum_{k=0}^{n} |z_k|$ is monotonic increasing), we have that $\sum_n |z_n|$ converges iff $s_n = \sum_{k=0}^{n} |z_k|$ is bounded above. (follows from the MCT, recall the tutorial question about $\sum z^n/n^2$)

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- If f is holomorphic on an open set containing Image(γ), then,
$$\int_\gamma f'(z)dz = f(\gamma(b)) - f(\gamma(a))$$

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Theorem

If f is holomorphic in a domain (thus, connected), and $f' \equiv 0$ in that region, then f is a constant.

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Cauchy Integral Theorem

Let Ω be a bounded domain in $\mathbb{C}$, with piecewise smooth boundary $\partial\Omega$ and $f \in C^1(\bar{\Omega})$ is holomorphic on Ω. Then,

$$\int_{\partial\Omega} f(z)dz = 0$$

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$$f(z) = \frac{1}{2\pi\iota} \int_{\partial\Omega} \frac{f(\eta)}{\eta - z} d\eta$$

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Another way

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CIT - Aliter

If $f : \Omega \to \mathbb{C}$ is holomorphic, and Ω is a **simply connected** domain, then for every closed piecewise smooth curve γ within Ω we have,

$$\int_\gamma f(z)dz = 0$$

Questions

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$$\int_{|z|=1} \frac{e^z \sin(z) - z}{z^2 \cos(z)} dz$$

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   Answer: $2\pi\iota(e-2)$.

Connecting the dots

Theorem (Morera's Theorem)

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Theorem (Goursat's Theorem)

If a function is complex differentiable at each point in an open set, then it is continuously differentiable in that set. Thus, it is holomorphic on that set.

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Holomorphic $\Leftrightarrow$ Analytic

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$$\int_{|z|=1} f(z)\bar{z}^j dz = 0$$

holds true for all $j = 0, 1, 2 \ldots,$ then show that $f \equiv 0$.

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- $f^{(n)}(z) = \frac{n!}{2\pi\iota} \int_{\partial\Omega} \frac{f(\eta)}{(\eta-z)^{n+1}} d\eta$ for all $z \in \Omega$.

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Suppose $f : \Omega \to \mathbb{C}$ is holomorphic, where Ω is a domain. Further suppose $f(z_0) = 0$.

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2. Assume that g is non-constant, then show that $g(\mathbb{C}) = \mathbb{C}$.

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2. $f(z) := (z-1)/(z(z-2)(z-2.00001))$ has isolated singularities at $0, 2, 2.00001$.
3. $f(z) := \tan(1/z)$ has a non-isolated singularity at $z = 0$. Why?
4. $f(z) := \bar{z}$ has *a* non-isolated singularity. Where?

Classifying Isolated Singularities - I

[#1] Removable Singularities

An isolated singularity $z \in \mathbb{C}$ of f is said to be removable if there exists a holomorphic function $\tilde{f} : D(z, r) \to \mathbb{C}$ for some $r > 0$ **such that** $\tilde{f}(z) = f(z) \forall z \in D^*(z, r)$.

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Proof? Start with $g(z) := (z - p)^2 f(z)$. Explicitly construct the desired $\tilde{f}$.

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A pole of order one is called a *simple pole*.

[#3] Essential Singularities

An isolated singularity of f which is neither removable, nor a pole is called an *essential singularity*.

Classifying Isolated Singularities - III

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Theorem (Casorati-Weirstrass)

Suppose f has an essential singularity at p. Then, for any $r > 0$, f takes values arbitrarily close to every complex number on the disk $D^*(p, r)$.

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Suppose f has an essential singularity at p. Then, for any $r > 0$, f takes on **all possible** complex values, **infinitely often**, with at most a single exception in $D^*(p, r)$.

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Also recall the **Little Picard Theorem**: An entire function can omit at most one complex number.

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Classify the singularities:

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$$\frac{(z+2)\cos 1/z}{z-3}$$

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Answer: Simple pole at 3, essential at 0
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Laurent Series

Definition (Meromorphic Function)

A function holomorphic on a domain except possibly for a set of poles, is said to be meromorphic on that domain.

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holds for all $z \in \mathcal{A}(p, r_1, r_2)$. Where,

$$a_n := \frac{1}{2\pi\iota} \int_\gamma \frac{f(\eta)}{(\eta - p)^{n+1}} \; \forall n \in \mathbb{Z}$$

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As an immediate consequence, see that

$$\boxed{\int_C f(z)dz = 2\pi\iota \times Res(f, p)}$$

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Questions

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5. $\int_{|z|=1} \frac{1}{\sin 1/z} dz$ Cannot apply the CRT!

The Cauchy Residue Theorem

This is a formal restatement of the earlier conclusion.

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Orders and Multiplicities

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$$f(z) = (z - z_0)^m g(z)$$

for some holomorphic g which does not vanish in a neighbourhood of z_0.

Orders and Multiplicities

And similarly, we define,

Definition (Multiplicity of a zero)

Given a function f, we say that $z_0 \in \mathbb{C}$ is a zero of multiplicity m of f, if

$$f^{(n)}(z_0) = 0 \ \forall n \leq m - 1 \text{ and } f^{(m)}(z_0) \neq 0$$

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where, Z is the number of zeros of f inside γ counted with multiplicites and P is the number of poles of f inside γ counted with order.

And that's a wrap!

Thank you, and all the best!