Trajectory level quantum dynamics in open quantum systems: the MIPT and the BTC phenomenon

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Last Updated: December 7, 2023

Abstract

This report summarizes my study of trajectory-level open quantum system dynamics. It details the continuous time crystal phase and the measurement-induced phase transition exhibited in monitored many-body systems. This document contains notes, reviews, and some simulations I performed. Original research resulting as a part of this study will be outlined in a paper.

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§§1.1. Introduction: MIPT in RQCs

The nature of entanglement in quantum many-body systems has been crucial for understanding many dynamical and phase-transition phenomenon in quantum systems. One important class of such systems are monitored quantum systems, which are being continuously observed by (or dissipated into) the environment. Different behaviours have been observed in integrable and non-integrable (chaotic) quantum systems as the strength of measurement is varied. This is manifested in the scaling of entanglement in a quantum system with the size N of the system. Notably, in the *area law* phase, the entanglement of a state is $\mathcal{O}(1)$, and in the *volume law* phase, the entanglement of a state is $\mathcal{O}(N)$.

In [CTDL19], the authors observed the absence of a stable volume-law phase under the presence of any monitoring, i.e., the dynamics saturate to an area-law behaviour for any arbitrarily small amount of measurement. In stark contrast, [LCF18], outline how in the setting of brick-wall random quantum circuits perturbed by measurements, (a chaotic system) there is a finite measurement strength p_c uptill when the volume-law survives. We take this as the starting point of studying the so called *measurement induced quantum phase-transition* (MIPT), with random unitary circuits being our first playground.

The model under study is shown in Fig. 1. This is a brick-wall circuit, which consists of random unitaries U interdispersed by measurements with probability p. Open boundary conditions are considered. With a lattice consisting of an even number of qubits, the evolution is formally described for even and odd time steps as follows. For t even, define the unitary

$$\mathcal{U}_e := \Pi_i U_{2i,2i+1} \tag{1.1}$$

and for t odd, define,

$$\mathcal{U}_o := \Pi_i U_{2i-1,2i} \tag{1.2}$$

with the individual two-qubit unitaries U being drawn from a random ensemble. This could be the from Haar measure, or from the set of random Clifford gates. Now, let K_i denote the probabilistic measurement of qubit i, which with probability $1-p$ does nothing, and with probability p measures the qubit i in the computational Z basis. Thus, we define the net evolution operator at time step t as,

$$\mathcal{E}_t := (\Pi_i K_i)\mathcal{U}_\alpha,\ \alpha \in \{e, u\} \tag{1.3}$$

Note that this is equivalent to the way [LCF18] performs rank-1 measurements. We evolve the system with such dynamics, and ask the question about the entanglement dynamics of the evolved quantum trajectories $\{|\Psi(t)\rangle\}$ for times $t \in \{1, 2, \ldots, 2N\}$ where N is the size of the system. The results of the entanglement dynamics are shown in Fig. 2. For $p = 0$, we observe a linear increase in the entanglement entropy up untill a saturation value, and note that the saturated entropy is volume law – it scales linearly with N. As p is increased, this

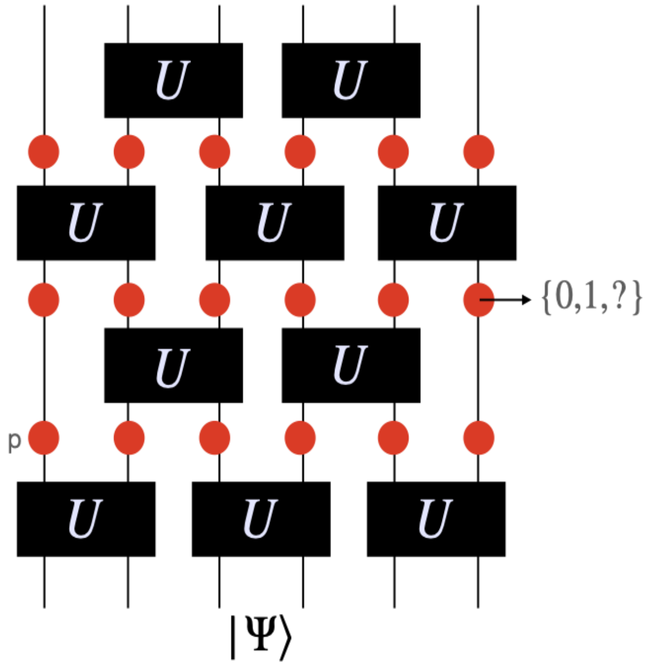


Figure 1: The typical playground for the MIPT: The *brickwall* random quantum circuit, consisting of random unitaries U sampled independently from, say, the Haar measure, and intermittent projective measurements (red dots) which occur with a probability p. The output from a single red dot is formally is in $\{0, 1, ?\}$, with it being ? with probability $1 - p$ when a measurement does not happen. Also, $p(0) + p(1) = p$.

saturation value decreases, and for $p = 0.2$, the saturation value becomes $\mathcal{O}(1)$, i.e., indepedent of N. Thus, we establish a the existence of a transition from volume-law to area-law. Also, as we vary the strength of the measurements p, we notice that the time taken for the entanglement to saturate to it's steady values reduces from $\sim 2N$ in the $p = 0$ case to $\mathcal{O}(1/p)$ in the $p > p_c$ case.

Now, how do we find the critical point p_c? We can do this in two ways. One, by noting the variation of the entropy on N for different values of p. The critical value will be when the plot of $\mathcal{E}$ v.s. N is a horizontal line. Another way is to plot the long-term entropy as a function of p for different values of N, which should lead to a collapse in the plots at $p \approx p_c$. We do this for a **single** trajectory in Fig.3, and the plots (a) and (b) indicate that $p \approx p_c$. We further aim to quantify this by employing the scaling ansatz

$$S_A(p, N) = L^\gamma F((p - p_c)N^{1/\nu}) \tag{1.4}$$

where $F(\cdot)$ is the scaling function. We can do this by plotting S_A/N^γ as a function of $(p - p_c)N^{1/\nu}$. We fix $p_c = 0.15$, and vary the critical exponents γ, ν to get a fit, and plot the results in Fig.3(c) for the fit $\nu = 1.84$ and $\gamma = 0.3$. Of course, this is a single trajectory so the scaling is not perfect. We average over fifty trajectories and plot the results in Fig.4, which confirm the critical point to be $p_c = 0.15$, agreeing with that from [LCF18]. One can now proceed to get more information about the scaling function [LCF18] by demanding that

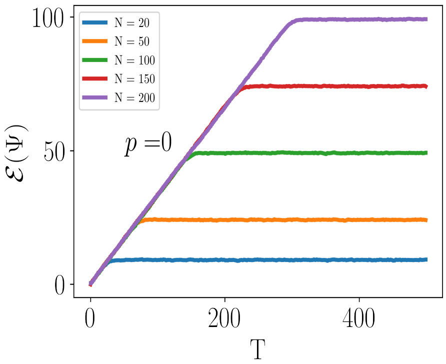
$$S_A = \mathcal{O}(L_A) \text{ for } p < p_c \text{ and } S_A = \mathcal{O}(1) \text{ for } p > p_c \tag{1.5}$$

we get that $F(x) \sim |x|^{(1-\gamma)\nu}$ for $x \to -\infty$ and $F(x) \sim x^{-\gamma\nu}$ for $x \to \infty$, thereby implying

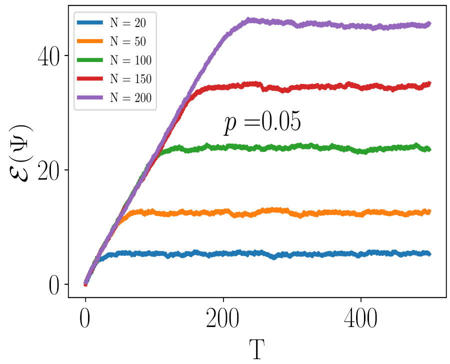
$$\lim_{N\to\infty} N^{-1} S_A(p, N) \sim |p - p_c|^{(1-\gamma)\nu} \textbf{ for } p < p_c \tag{1.6}$$

$$\lim_{N\to\infty} S_A(p, N) \sim |p - p_c|^{-\gamma\nu} \textbf{ for } p > p_c \tag{1.7}$$

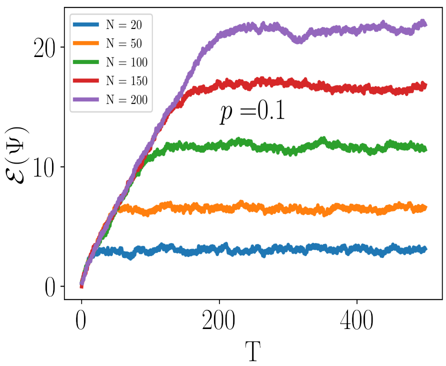
In comparison with Ising models, the entropy per unit qubit serves as a density much like the magnetization which is the magnetic moment per unit site. The entropy density thus serves as an order parameter for the weak measurement phase, vanishing continuously at the critical point with exponent $(1 - \gamma)\nu \sim 4/3$.



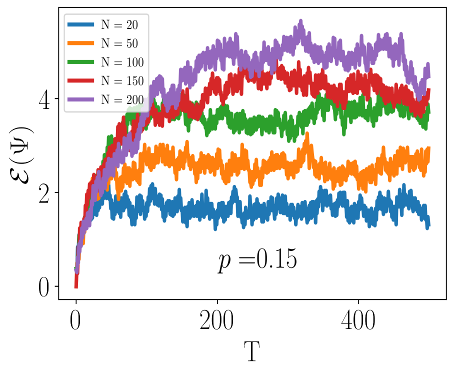
(a) $p = 0$



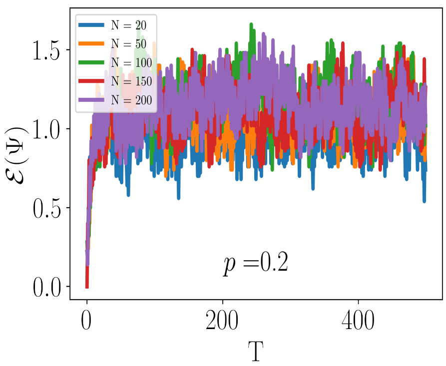
(b) $p = 0.05$



(c) $p = 0.1$

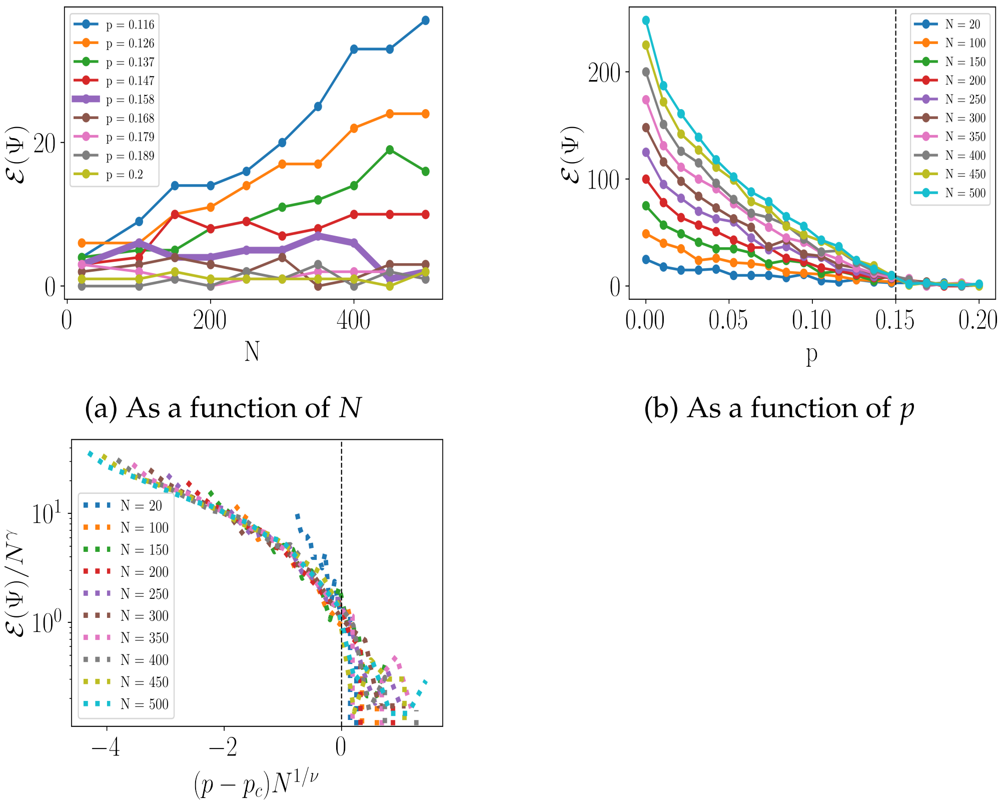


(d) $p = 0.15$



(e) $p = 0.2$

Figure 2: Entanglement dynamics of brickwall circuits with single-qubit projective measurements in the computational basis happening probability p. The critical measurement strength is identified as $p_c \approx 0.15$. Averaged over 50 trajectories.

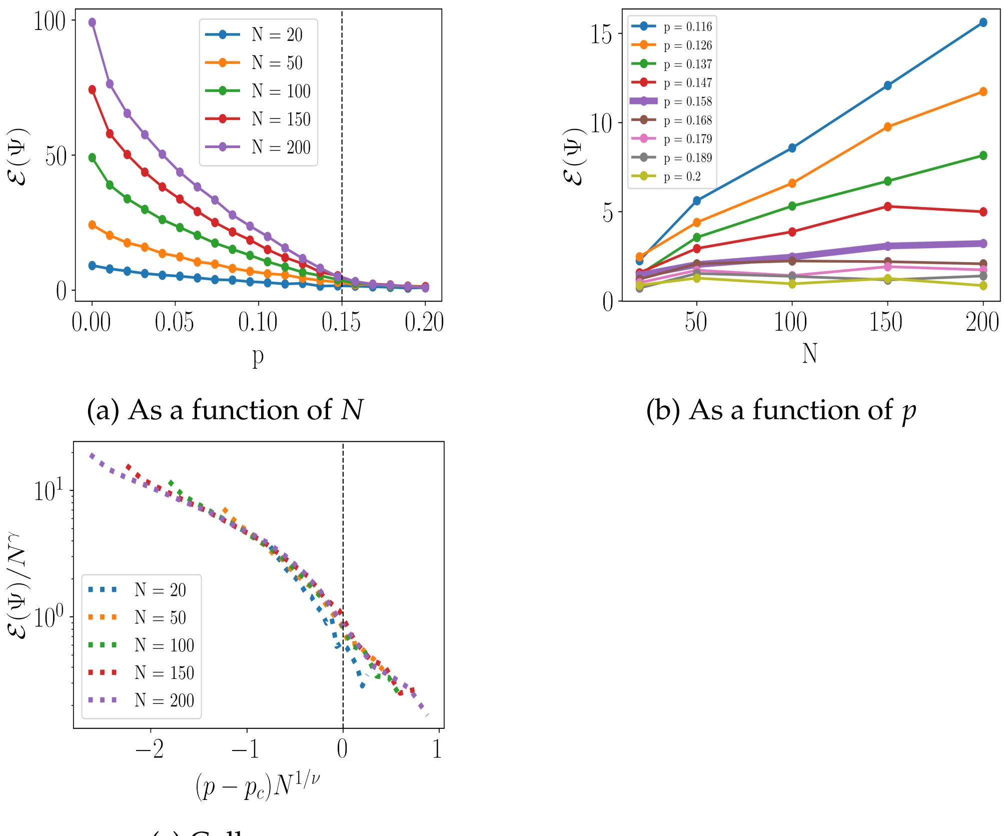


(a) As a function of N

(b) As a function of p

(c) Collapse

Figure 3: Long-time entanglement entropy of *single* trajectories of the brick-wall circuit. The critical value p_c can be intuitively seen to be around 0.15, but the finite N scaling is not very good.



(a) As a function of N

(b) As a function of p

(c) Collapse

Figure 4: Long-time entanglement entropy averaged over 50 trajectories of the brick-wall circuit. The critical point $p_c = 0.15$ is identified clearly owing to the clear finite-N scaling now.

§§1.2. Interesting works

[BCA20] When one works with the MIPT, one needs to average in two levels – one over the unitaries, and one over the possible measurement outcomes. The averaged entropy (say, half-system) can then be written as,

$$S_A = \langle \sum_{i_M} p_{i_m}(U) S_A(i_M, U) \rangle_U \tag{1.8}$$

where i_M is a possible measurement outcome (a *very* long bit vector, essentially), A is a subsystem, and U is a possible circuit realization. The authors of [BCA20] take into account averaging over the i_M by writing the joint system-ancilla state (note that you can think of the ancilla as the measurement probes) as

$$\rho_{ABM} = \sum_P i_M p_{i_M} |\Psi(U, i_M)\rangle \langle \Psi(U, i_M)| \otimes |i_M\rangle \langle i_M| \tag{1.9}$$

Now, if one wants to compute the entropy $S_A(U)$ averaged over the measurements but for a particular Haar realization U, it is nothing but the conditional entropy of A on M, given as

$$S_A(U) = S(A|M) = S_{AM} - S_M \tag{1.10}$$

These are easier to compute analytically than the entropies for a particular trajectory i_M. While S_M is just the classical Shannon entropy of p_{i_M}, $S(A|M)$ is still tricky. To go around this, the authors use a replica trick to compute this by introducing $S^{(n)}(A|M) = S^{(n)}_{AM} - S^{(n)}_{M}$ where

$$S^{(n)}_X := \frac{\log\langle \operatorname{Tr} \rho^n_X \rangle_U}{1-n} \tag{1.11}$$

This quantity converges to the conditional entropy $\langle S(A|M)\rangle_U$ for $n \to 1$.

Another quantity they deal with is the Fisher information, which talks about the learnability of the quantum system in the volume-law and area-law phases. Consider $|\Psi_0\rangle$ and $|\Psi_x\rangle := \delta U(x) |\Psi_0\rangle$ for x small and $\delta U(x) = \exp\{-\imath A x\} \approx 1$. If one looks at the probability distributions of the outcomes p_0 and p_x one can define the average (over U) KL divergence between them as,

$$D_{KL}(p_0, p_x) := \sum_{i_M} p_{0,i_M} \log \frac{p_{0,i_M}}{p_{x,i_M}}. \tag{1.12}$$

Further, we define the Fisher information as

$$\mathcal{F} = \partial_x^2 D_{KL}(p_0, p_x)|_{x=0} \tag{1.13}$$

This quantifies the amount of information that can be learned from the system. One expects a sharp transition in this quantity from the area-law to the volume-law phases, wherein the dissipation enables the environment to learn about the system in the former, and the highly non-local distribution in the latter prevents the environment from learning about the system qubits. A similar replica trick is used. Further, the replica quantities are mapped to some exactly solvable classical statistical mechanics models to enable analytic control over MIPT (to some extent).

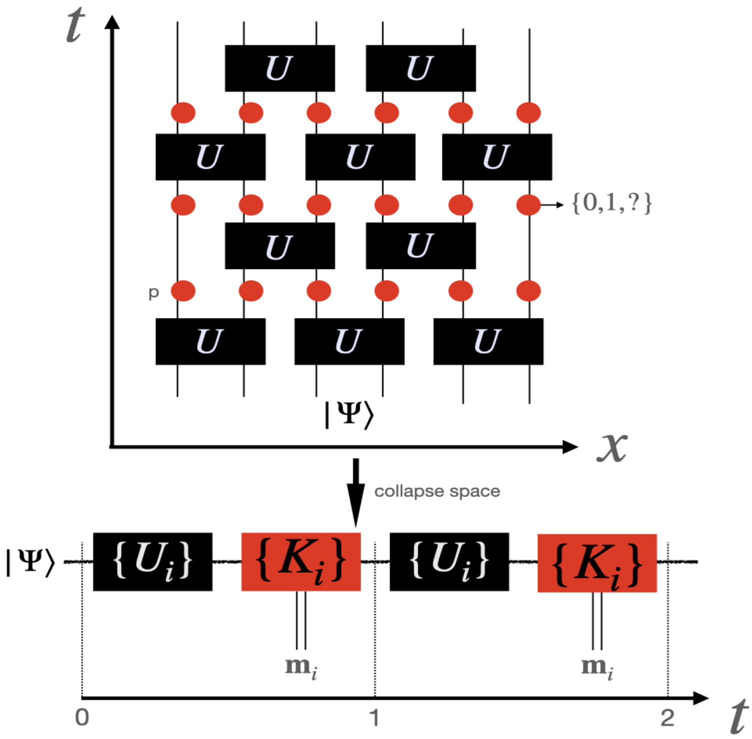


Figure 5: Alternate picture for the RQC-MIPT with a collapse of the spatial dimension. The measurement results at each time-step are $\mathbf{m}_i \in \{0, 1, ?\}^N$, where N is the number of qubits.

§2. Continuous time crystals

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