Hi

MA 106 D1-T3
Tutorial-6

Siddhant Midha

04-05-2022

Question 1

Nilpotent Matrices

A matrix $A \in \mathbb{C}^{n\times n}$ for some $n \in \mathbb{N}$ is said to be *nilpotent* if $\exists k \in \mathbb{N}$ such that

$$(A^k = O)$$

$A \to$ nilpotent

did not specify any rest on k

Question 1

Show that if A is nilpotent then $I - A$ is invertible. (Hint: Illegal Expansion)

$(1-a)^{-1} = 1 + a + \cdots$

$a^{l} = 0$

$\forall\, l \geq k+1$

Question 1

Show that if A is nilpotent then $I - A$ is invertible. (Hint: Illegal Expansion)

- Let k be the degree of nilpotency.

Question 1

$$(1 + a + a^2 \cdots a^n) = \frac{(a^n - 1)}{a - 1}$$

Show that if A is nilpotent then $I - A$ is invertible. (Hint: Illegal Expansion)

- Let k be the degree of nilpotency.
- Let
$$M := I + A + \ldots A^{k-1}$$

$$(MA) = A + A^2 + \cdots A^k = A + \cdots A^{k-1}$$
$$= M - I$$

$$M(I - A) = M - MA = M - (M - I) = I$$

Question 1

If A is nilpotent, what are its eigenvalues and characteristic polynomial?

Question 1

If A is nilpotent, what are its eigenvalues and characteristic polynomial?

- Let

$$Av = \lambda v$$

$$A^2 v = \lambda A v = \lambda^2 v$$

$$\vdots$$

$$A^k v = \lambda^k v = 0 \quad \text{as } A^k = 0$$

$$\therefore \lambda = 0$$

So, all e-vals are zero!

$$\therefore p_A(x) = x^n$$

Question 1

If A is nilpotent, what are its eigenvalues and characteristic polynomial?

- Let

$$Av = \lambda v$$

- (Assume $k \neq 0$)

Question 1

If A is nilpotent, what are its eigenvalues and characteristic polynomial?

- Let

$$Av = \lambda v$$

- (Assume $k \neq 0$) Let $B = A/\lambda$. $\lambda \neq 0$

B is also nilpotent.

Question 1

If A is nilpotent, what are its eigenvalues and characteristic polynomial?

- Let

$$Av = \lambda v$$

- (Assume $k \neq 0$) Let $B = A/\lambda$. We have,

$$(I - B)\underbrace{(I + B + \ldots B^{k-1}}_{M'}) = I$$

$$\times(\lambda^{n})$$

Question 1

If A is nilpotent, what are its eigenvalues and characteristic polynomial?

- Let

$$Av = \lambda v$$

- (Assume $k \neq 0$) Let $B = A/\lambda$. We have,

$$(I - B)(I + B + \dots B^{k-1}) = I$$

multiply by λ^k

$$(\lambda I - A)(\lambda^{k-1} I + \dots A^{k-1}) = \lambda^k I$$

Question 1

If A is nilpotent, what are its eigenvalues and characteristic polynomial?

- Let

$$Av = \lambda v$$

- (Assume $k \neq 0$) Let $B = A/\lambda$. We have,

$$(I - B)(I + B + \dots B^{k-1}) = I$$

multiply by λ^k

$$\underline{(\lambda I - A)}(\lambda^{k-1} I + \dots A^{k-1}) = \lambda^k I$$

Take determinant.

$p_A(\lambda)$

$\lambda^k \times \lambda^k \dots \lambda^k$

Question 1

If A is nilpotent, what are its eigenvalues and characteristic polynomial?

- Let
$$Av = \lambda v$$

- (Assume $k \neq 0$) Let $B = A/\lambda$. We have,
$$(I - B)(I + B + \dots B^{k-1}) = I$$
multiply by λ^k
$$(\lambda I - A)(\lambda^{k-1} I + \dots A^{k-1}) = \lambda^k I$$
Take determinant. p_A divides x^{nk}.

Question 1

If A is nilpotent, what are its eigenvalues and characteristic polynomial?

- Let
$$Av = \lambda v$$

- (Assume $k \neq 0$) Let $B = A/\lambda$. We have,
$$(I - B)(I + B + \ldots B^{k-1}) = I$$
multiply by λ^k
$$(\lambda I - A)(\lambda^{k-1} I + \ldots A^{k-1}) = \lambda^k I$$
Take determinant. p_A divides x^{nk}. But, $deg(p_A) = n$.

Question 1

If A is nilpotent, what are its eigenvalues and characteristic polynomial?

- Let

$$Av = \lambda v$$

- (Assume $k \neq 0$) Let $B = A/\lambda$. We have,

$$(I - B)(I + B + \ldots B^{k-1}) = I$$

multiply by λ^k

$$(\lambda I - A)(\lambda^{k-1} I + \ldots A^{k-1}) = \lambda^k I$$

Take determinant. p_A divides x^{nk}. But, $deg(p_A) = n$. Conclude. that $\boxed{p_A(x) = x^n}$.

Question 1

Let

$$A := \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$GM(0) = \text{nullity}(A) = 4 - 3 = 1$$

Question 1

Let

$$A := \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$v = \begin{bmatrix} x & y & z & w \end{bmatrix}^T$$

Question 1

Let

$$A := \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$v = \begin{bmatrix} x & y & z & w \end{bmatrix}^T$$

$$Av = \begin{bmatrix} y \\ z \\ w \\ 0 \end{bmatrix} \begin{matrix} \to 0 \\ \to 0 \\ \to 0 \\ \end{matrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

one free variable $\Rightarrow$ dim = 1

Question 1

Let

$$A := \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$v = \begin{bmatrix} x & y & z & w \end{bmatrix}^T$$

$$Av = \begin{bmatrix} y \\ z \\ w \\ 0 \end{bmatrix}$$

Dimension $= 1$.

Question 1

Let

$$A := \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Question 1

Let

$$A := \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$v = \begin{bmatrix} x & y & z & w \end{bmatrix}^T$$

Question 1

Let

$$A := \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$v = \begin{bmatrix} x & y & z & w \end{bmatrix}^T$$

$$Av = \begin{bmatrix} y \\ 0 \\ w \\ 0 \end{bmatrix}$$

Question 1

Let

$$A := \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

this is corrected
there was a typo
in the sheet

$$v = \begin{bmatrix} x & y & z & w \end{bmatrix}^T$$

$$Av = \begin{bmatrix} y \\ 0 \\ w \\ 0 \end{bmatrix}$$

Dimension $= 2$.

Question 1

Let

$$A := \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Question 1

Let

$$A := \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$v = \begin{bmatrix} x & y & z & w \end{bmatrix}^T$$

Question 1

Let

$$A := \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$v = \begin{bmatrix} x & y & z & w \end{bmatrix}^T$$

$$Av = \begin{bmatrix} y \\ z \\ 0 \\ 0 \end{bmatrix}$$

Question 1

Let

$$A := \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$v = \begin{bmatrix} x & y & z & w \end{bmatrix}^T$$

$$Av = \begin{bmatrix} y \\ z \\ 0 \\ 0 \end{bmatrix}$$

Dimension $= 2$.

Question 1

$$\overset{A}{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}, \overset{B}{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}}$$

AB

Product of two commuting nilpotents is nilpotent (why?). Show that this fails if matrices do not commute.

let ow of nilpot of $A = k_1$

. - - - of $B = k_2$

then, if A, B commute, $k_2 \le k_1$

then $(AB)^{k_2} = 0$

$A^{k_2} B^{k_2} = 0$

Question 1

Product of commuting nilpotents is nilpotent (why?). Show that this fails if matrices do not commute.

Let

$$A := \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad B := \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$$

$(AB)_{11} = 1$ $\qquad$ $(BA)_{11} = 0$

$\Rightarrow$ not commuting.

Check if AB is nilpotent.

Additional notes on Q1

let A be nilpotent and diag.

$A = P^{-1}DP = \underline{O}$

would be O!

- The only nilpotent matrix which is diagonalizable is the null matrix. (Why?)
- Any matrix with a non-zero eigenvalue cannot be nilpotent.
- Show that all complex eigen values zero $\implies$ nilpotency. (hint: show that $A^n = O$, where n is the size of the matrix)
- If the degree of nilpotency is defined as the smallest k such that $A^k = 0$, hence conclude that this degreee cannot exceed n.

Question 2

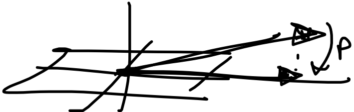
Projection Matrices

A square matrix P is said to be a projection if

$$P^2 = P$$

P would project on some subspace V of $\mathbb{R}^3$

$\mathbb{R}^3$,



Question 2

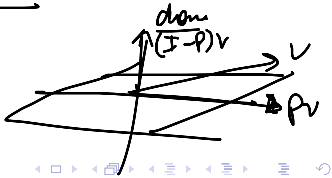
$\boxed{\text{look up "orthogonal" projection}}$

Show that if P is a projection so is $I - P$.

$$
\begin{aligned}
(I-P)^2 &= (I-P)(I-P) \\
&= I - P - P + P^2 \\
&= I - P - P + P \\
&= (I - P)
\end{aligned}
$$

$v \in \mathbb{R}^3$

$$v = (Pv) + (v - Pv) = Pv + \boxed{(I-P)v}$$



Question 2

Eigenvalues of P?

Question 2

Eigenvalues of P?

$$Pv = \lambda v$$

Question 2

Eigenvalues of P?

$$Pv = \lambda v$$

$$P(\lambda v) = \lambda P v = \lambda^2 v$$

$$\boxed{P^2 v} = \lambda^2 v = Pv = \lambda v$$

$$P^2 v = Pv = \lambda v$$

$$(\lambda - \lambda^2) v = 0$$

$$\Downarrow$$

$$\lambda = \lambda^2$$

$$\lambda = 0 \text{ or } 1$$

(why?) — $v \neq 0$ By defn

Question 2

Eigenvalues of P?

$$Pv = \lambda v$$

$$P^2 v = \lambda^2 v = Pv = \lambda v$$

Question 2

If P is invertible then it is the identity.

Question 2

If P is invertible then it is the identity.

Let M be the inverse.

Question 2

If P is invertible then it is the identity.

Let M be the inverse.

$$PM = I$$

Question 2

If P is invertible then it is the identity.

Let M be the inverse.

$$PM = I$$

$$PPM = P$$

Question 2

If P is invertible then it is the identity.

Let M be the inverse.

$$PM = I$$

$$PPM = P$$

$$PM = P = I$$

Question 2

Suppose $det(P) = 0$, and let $v_1 \dots v_k$ be a basis for nullspace of P.

If we extend this to a basis $v_1, \dots v_k, v_{k+1} \dots v_n$ (of $\mathbb{R}^n$),

- Prove/Disprove the linear independence of $\boxed{Pv_{k+1}, \dots Pv_n}$.
- Deduce about diagonalizablity of P.

Let there exist c_i not all zero such that

Question 2

Suppose $det(P) = 0$, and let $v_1 \dots v_k$ be a basis for nullspace of P.

If we extend this to a basis $v_1, \dots v_k, v_{k+1} \dots v_n$ (of $\mathbb{R}^n$),

- Prove/Disprove the linear independence of $Pv_{k+1}, \dots Pv_n$.
- Deduce about diagonalizablity of P.

Let there exist c_i not all zero such that

$$c_{k+1}Pv_{k+1} + \cdots + c_nPv_n = 0$$

Question 2

Suppose $det(P) = 0$, and let $v_1 \dots v_k$ be a basis for nullspace of P.
If we extend this to a basis $v_1, \dots v_k, v_{k+1} \dots v_n$ (of $\mathbb{R}^n$),

- Prove/Disprove the linear independence of $Pv_{k+1}, \dots Pv_n$.
- Deduce about diagonalizablity of P.

Let there exist c_i not all zero such that

$$c_{k+1}Pv_{k+1} + \dots + c_nPv_n = 0$$

$$P(\underbrace{c_{k+1}v_{k+1} + \dots + c_nv_n}_{\text{is in the nullspace}}) = 0$$

Question 2

Suppose $det(P) = 0$, and let $v_1 \dots v_k$ be a basis for nullspace of P.

If we extend this to a basis $v_1, \dots v_k, v_{k+1} \dots v_n$ (of $\mathbb{R}^n$),

- Prove/Disprove the linear independence of $Pv_{k+1}, \dots Pv_n$.
- Deduce about diagonalizablity of P.

Let there exist c_i not all zero such that

$$c_{k+1}Pv_{k+1} + \cdots + c_n Pv_n = 0$$

$$P(c_{k+1}v_{k+1} + \cdots + c_n v_n) = 0$$

Thus,

Question 2

Suppose $det(P) = 0$, and let $v_1 \dots v_k$ be a basis for nullspace of P.
If we extend this to a basis $\boxed{v_1, \dots v_k, v_{k+1} \dots v_n}$ (of $\mathbb{R}^n$),

- Prove/Disprove the linear independence of $Pv_{k+1}, \dots Pv_n$.
- Deduce about diagonalizablity of P.

Let there exist c_i not all zero such that

$$c_{k+1}Pv_{k+1} + \cdots + c_nPv_n = 0$$

$$P(c_{k+1}v_{k+1} + \cdots + c_nv_n) = 0$$

Thus,

$$(c_{k+1}v_{k+1} + \cdots + c_nv_n) = a_1v_1 + \cdots + a_kv_k$$

But!

$\{v_i\}$ are lin indp $\Rightarrow$ contradiction

Question 2

Suppose $det(P) = 0$, and let $v_1 \dots v_k$ be a basis for nullspace of P.

If we extend this to a basis $v_1, \dots v_k, v_{k+1} \dots v_n$ (of $\mathbb{R}^n$),

- Prove/Disprove the linear independence of $Pv_{k+1}, \dots Pv_n$.
- Deduce about diagonalizablity of P.

$\boxed{P\ DR}$

Let there exist c_i not all zero such that

$$c_{k+1}Pv_{k+1} + \dots + c_nPv_n = 0$$

$$P(c_{k+1}v_{k+1} + \dots + c_nv_n) = 0$$

Thus,

$$(c_{k+1}v_{k+1} + \dots + c_nv_n) = a_1v_1 + \dots + a_kv_k$$

But ...

P projection on some subspace $V \subseteq \mathbb{R}^n$?

$w = Pw + (I-P)w$

$\lambda = 1$ $\qquad$ $\lambda = 0$

Question 2

Suppose $det(P) = 0$, and let $v_1 \dots v_k$ be a basis for nullspace of P.

If we extend this to a basis $\boxed{v_1, \dots v_k}$, $v_{k+1} \dots v_n$ (of $\mathbb{R}^n$),

$\lambda = 0$

- Prove/Disprove the linear independence of $\boxed{Pv_{k+1}, \dots Pv_n}$.
- Deduce about diagonalizablity of P.

$\lambda = 1$

Let there exist c_i not all zero such that

Claim Pv_{k+1} is an e-vec of P w/ $\lambda = 1$.

$P(Pv_{k+1}) = P^2 v_{k+1} = Pv_{k+1}$

$$c_{k+1}Pv_{k+1} + \dots + c_nPv_n = 0$$

$$P(c_{k+1}v_{k+1} + \dots + c_nv_n) = 0$$

If we show $Pv_{k+1}, Pv_{k+2} \dots Pv_n$ are e-vecs of P w/ $\lambda = 1$ $\Downarrow$ diag

Thus,

$$(c_{k+1}v_{k+1} + \dots + c_nv_n) = a_1v_1 + \dots + a_kv_k$$

But ...

Thus, diagonalizable! $\to$ $\{v_1, \dots v_k\}$ basis for $\lambda = 0$

$\{Pv_{k+1}, \dots Pv_n\}$ basis for $\lambda = 1$

and e-vec w/ diff e-val are indep
∴ diagonalizable.

Question 3

$H_0 n = n - nn^T n = 0$

If $v \perp n$

$H_0 v = (I - nn^T)v$

Let $n \in \mathbb{R}^n$, $||n|| = 1$. Define

$$H := I - 2nn^T \quad \leftarrow \text{refl.}$$

$$H_0 := I - nn^T \quad \leftarrow \text{proj}$$

$nn^T nn^T = nn^T$

nn^T is proj $\Rightarrow$ $I - nn^T$ is proj

- Given H and H_0 (and discarding ~~the tut-0 knowledge~~), which one is a reflection and which one is a projection? algebraically
- Find their eigenvalues.
- Are they diagonalizable?
- Is H_0 idempotent? (geometric+algebraic way)

Question 3

- For a projection, $P^2 = P$.

Question 3

- For a projection, $P^2 = P$.
- For a reflection, $P^2 = I$.

Question 3

- For a projection, $P^2 = P$.
- For a reflection, $P^2 = I$.
- Already established that eigenvalues of H_0 are 0 and 1.

dim of eigenspace

1

$n-1$

$Hn=0$

Question 3

- For a projection, $P^2 = P$.
- For a reflection, $P^2 = I$.
- Already established that eigenvalues of H_0 are 0 and 1.
- Note, $H = 2H_0 - I$. Thus, $\boxed{\lambda_H = 2\lambda_{H_0} - 1.}$

$\lambda_H \to \{-1, 1\}$ $\quad$ $\lambda_{H_0} \to \{0, 1\}$

$H_0 v = \lambda v$

$Hv = (2H_0 - I)v$

$= (2\lambda - 1)v$

$Hw = \lambda w$

$\Downarrow$

$H_0 w =$

Question 3

- For a projection, $P^2 = P$.
- For a reflection, $P^2 = I$.
- Already established that eigenvalues of H_0 are 0 and 1.
- Note, $H = 2H_0 - I$. Thus, $\lambda_H = 2\lambda_{H_0} - 1$.
- Yes!

Question 3

- For a projection, $P^2 = P$.
- For a reflection, $P^2 = I$.
- Already established that eigenvalues of H_0 are 0 and 1.
- Note, $H = 2H_0 - I$. Thus, $\lambda_H = 2\lambda_{H_0} - 1$.
- Yes!
- Yes!

Question 4

Let

$$A_1 := \begin{bmatrix} 2 & 10^5 & 10^9 \\ 0 & 1 & \pi \\ 0 & 0 & 3 \end{bmatrix}, A_2 := \begin{bmatrix} 3 & 0 & 0 \\ e & 2 & 0 \\ 10^8 & 10^{10} & 1 \end{bmatrix}$$

Are they similar?

$\underline{\underline{\{1, 2, 3\}}}$

Question 4

Let

$$A_1 := \begin{bmatrix} 2 & 10^5 & 10^9 \\ 0 & 1 & \pi \\ 0 & 0 & 3 \end{bmatrix}, A_2 := \begin{bmatrix} 3 & 0 & 0 \\ e & 2 & 0 \\ 10^8 & 10^{10} & 1 \end{bmatrix}$$

Are they similar? Note that they are diagonalizable.

Question 4

Let

$$A_1 := \begin{bmatrix} 2 & 10^5 & 10^9 \\ 0 & 1 & \pi \\ 0 & 0 & 3 \end{bmatrix}, A_2 := \begin{bmatrix} 3 & 0 & 0 \\ e & 2 & 0 \\ 10^8 & 10^{10} & 1 \end{bmatrix}$$

Are they similar? Note that they are diagonalizable. Why?

Question 4

Let

$$A_1 := \begin{bmatrix} 2 & 10^5 & 10^9 \\ 0 & 1 & \pi \\ 0 & 0 & 3 \end{bmatrix}, A_2 := \begin{bmatrix} 3 & 0 & 0 \\ e & 2 & 0 \\ 10^8 & 10^{10} & 1 \end{bmatrix}$$

Are they similar? Note that they are diagonalizable. Why?
Distinct Eigenvalues!

Question 4

Let

$$A_1 := \begin{bmatrix} 2 & 10^5 & 10^9 \\ 0 & 1 & \pi \\ 0 & 0 & 3 \end{bmatrix}, A_2 := \begin{bmatrix} 3 & 0 & 0 \\ \mathrm{e} & 2 & 0 \\ 10^8 & 10^{10} & 1 \end{bmatrix}$$

Are they similar? Note that they are diagonalizable. Why? Distinct Eigenvalues! Now let $P^{-1}A_1P = diag(2, 1, 3)$

Question 4

Let

$$A_1 := \begin{bmatrix} 2 & 10^5 & 10^9 \\ 0 & 1 & \pi \\ 0 & 0 & 3 \end{bmatrix}, A_2 := \begin{bmatrix} 3 & 0 & 0 \\ e & 2 & 0 \\ 10^8 & 10^{10} & 1 \end{bmatrix}$$

Are they similar? Note that they are diagonalizable. Why? Distinct Eigenvalues! Now let $P^{-1}A_1P = diag(2,1,3)$ and $Q^{-1}A_2Q = diag(3,2,1)$.

looking for some Z s.t. $Z^{-1}A_1Z = A_2$

need some R s.t. $R^{-1}\,\mathrm{diag}(3,2,1)\,R = \mathrm{diag}(2,1,3)$

Question 4

Let

$$A_1 := \begin{bmatrix} 2 & 10^5 & 10^9 \\ 0 & 1 & \pi \\ 0 & 0 & 3 \end{bmatrix}, A_2 := \begin{bmatrix} 3 & 0 & 0 \\ e & 2 & 0 \\ 10^8 & 10^{10} & 1 \end{bmatrix}$$

Are they similar? Note that they are diagonalizable. Why? Distinct Eigenvalues! Now let $P^{-1}A_1P = diag(2,1,3)$ and $Q^{-1}A_2Q = diag(3,2,1)$

Can find R such that, $\underline{diag(2,1,3) = R^{-1}diag(3,2,1)}R$.

$$R = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Question 4

Let

$$A_1 := \begin{bmatrix} 2 & 10^5 & 10^9 \\ 0 & 1 & \pi \\ 0 & 0 & 3 \end{bmatrix}, A_2 := \begin{bmatrix} 3 & 0 & 0 \\ e & 2 & 0 \\ 10^8 & 10^{10} & 1 \end{bmatrix}$$

Are they similar? Note that they are diagonalizable. Why?
Distinct Eigenvalues! Now let $P^{-1}A_1P = diag(2,1,3)$ and
$Q^{-1}A_2Q = diag(3,2,1)$
Can find R such that, $diag(2,1,3) = R^{-1}diag(3,2,1)R$. Thus,
$RP^{-1}A_1PR^{-1} = diag(3,2,1)$

$RP^{-1}A_1PR^{-1} = Q^{-1}A_2Q$

$\therefore$ Similar

Question 5

real

- Eigenvalues of a skew-symmetric matrix? $A = -A^T$
- Diagonalizability?

Few points:

Question 5

- Eigenvalues of a skew-symmetric matrix?
- Diagonalizability?

Few points:

- A is hermitian $\Leftrightarrow$ ιA is skew-hermitian

$i = \sqrt{-1}$

Question 5

- Eigenvalues of a skew-symmetric matrix?
- Diagonalizability?

Few points:

- A is hermitian $\Leftrightarrow$ ιA is skew-hermitian
- If λ_i are the eigenvalues of A, then $c\lambda_i$ are the eigenvalues of cA.

Question 5

- Eigenvalues of a skew-symmetric matrix?
- Diagonalizability?

Few points:

- A is hermitian $\Leftrightarrow$ ιA is skew-hermitian
- If λ_i are the eigenvalues of A, then $c\lambda_i$ are the eigenvalues of cA.
- If A is real skew symmetric, then ιA is hermitian.

$$(iA)^* = -i(A^*) = -i(A^T)$$
$$= -i(-A) = \underline{iA}$$

Question 5

- Eigenvalues of a skew-symmetric matrix?
- Diagonalizability?

Few points:

- A is hermitian $\Leftrightarrow \iota A$ is skew-hermitian
- If λ_i are the eigenvalues of A, then $c\lambda_i$ are the eigenvalues of cA.
- If A is real skew symmetric, then ιA is hermitian.

Now use the Spectral Theorem [2] And use the second point above.

Question 5

- Eigenvalues of a skew-symmetric matrix?
- Diagonalizability?

Few points:

- A is hermitian $\Leftrightarrow$ ιA is skew-hermitian
- If λ_i are the eigenvalues of A, then $c\lambda_i$ are the eigenvalues of cA.
- If A is real skew symmetric, then ιA is hermitian.

Now use the Spectral Theorem [2] And use the second point above. Thus the eigenvalues of our skew symmetric matrix (let's say S) are either purely imaginary or zero - (#).

Question 5

- Eigenvalues of a skew-symmetric matrix?
- Diagonalizability?

Few points:

- A is hermitian $\Leftrightarrow \iota A$ is skew-hermitian
- If λ_i are the eigenvalues of A, then $c\lambda_i$ are the eigenvalues of cA.
- If A is real skew symmetric, then ιA is hermitian.

Now use the Spectral Theorem [2] And use the second point above. Thus the eigenvalues of our skew symmetric matrix (let's say S) are either purely imaginary or zero - (#). Also see,

$$p_S(x) = det(xI - S) = det(xI - S^T) = det(xI + S) = (-1)^n det((-x)I - S)$$

$p_S(-x)$

$$\boxed{p_S(x) = (-1)^n p_S(-x)}$$

Question 5

- Eigenvalues of a skew-symmetric matrix?
- Diagonalizability?

Few points:

- A is hermitian $\Leftrightarrow \iota A$ is skew-hermitian
- If λ_i are the eigenvalues of A, then $c\lambda_i$ are the eigenvalues of cA.
- If A is real skew symmetric, then ιA is hermitian.

Now use the Spectral Theorem [2] And use the second point above. Thus the eigenvalues of our skew symmetric matrix (let's say S) are either purely imaginary or zero - $(\#)$. Also see,

$$p_S(x) = det(xI - S) = det(xI - S^T) = det(xI + S) = (-1)^n det((-x)I - S)$$

$$= (-1)^n p_S(-x)$$

Question 5

So if n is,

Question 5

So if n is,

- Even, then $p_S(x) = p_S(-x)$. By $(\#)$ and the fact that coefficients of p_S are real, we conclude all purely imaginary eigenvalues.

Question 5

So if n is,

- Even, then $p_S(x) = p_S(-x)$. By (#) and the fact that coefficients of p_S are real, we conclude all purely imaginary eigenvalues.
- Odd, then $p_S(x) = -p_S(-x)$. atleast + ~~Thus, one eigenvalue is zero and all the others are purely imaginary~~.

Not diag unless $n=1$

Question 5 – Aliter

This is a direct way.

$A^* = (\bar{A}^T)$ real

$= A^T$

$= -A$

$$
\begin{aligned}
Av &= \lambda v \\
\boxed{v^* Av} &= \lambda v^* v \\
(A^* v)^* v &= \lambda v^* v \\
(-Av)^* v &= \lambda v^* v \\
(-\lambda v)^* v &= \lambda v^* v \\
-\bar{\lambda} &= \lambda
\end{aligned}
$$

$\Rightarrow \lambda$ is 0 or of the form ia, $a \neq 0$

Again note that $p_S(x) = (-1)^n p_S(-x)$ and coeffients of p_S are real. Conclude.

Question 5 – Aliter

What about diagonalizability over $\mathbb{C}$?

This is a direct way.

$$
\begin{aligned}
Av &= \lambda v \\
v^* A v &= \lambda v^* v \\
(A^* v)^* v &= \lambda v^* v \\
(-Av)^* v &= \lambda v^* v \\
(-\lambda v)^* v &= \lambda v^* v \\
-\bar{\lambda} &= \lambda
\end{aligned}
$$

Again note that $p_S(x) = (-1)^n p_S(-x)$ and coeffients of p_S are real. Conclude.

Hence what about diagonalizability? (over $\mathbb{R}$) If it were, then the "D" matrix would be all zeros! $\Rightarrow S = 0$

$\Rightarrow$ not diag if $S \neq 0$

Question 6

Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be a function such that $\boxed{f(0) = 0}$ and $||f(x) - f(y)|| = ||x - y||$. Is it true that $f(x) = Ax$ for some matrix A. If so, what kind of matrix?

Question 6

Claim: f is inner-product preserving.

Question 6

Claim: f is inner-product preserving. That is,

$$\langle f(x), f(y)\rangle = \langle x, y\rangle \forall x, y \in \mathbb{R}^3$$

Question 6

Claim: f is inner-product preserving. That is,

$$\langle f(x), f(y)\rangle = \langle x, y\rangle \forall x, y \in \mathbb{R}^3$$

Proof: First note that $||f(x)|| = ||x||$ take $y=0$ in

$||f(x)-f(y)|| = ||x-y||$

Question 6

Claim: f is inner-product preserving. That is,

$$\langle f(x), f(y)\rangle = \langle x, y\rangle \forall x, y \in \mathbb{R}^3$$

Proof: First note that $||f(x)|| = ||x||$. Now,

$$||f(x) - f(y)|| = ||x - y||$$

Question 6

Claim: f is inner-product preserving. That is,

$$\langle f(x), f(y)\rangle = \langle x, y\rangle \forall x, y \in \mathbb{R}^3$$

Proof: First note that $||f(x)|| = ||x||$. Now, 1

$$||f(x) - f(y)|| = ||x - y||$$

$$\langle (f(x) - f(y)), (f(x) - f(y))\rangle = \langle (x - y), (x - y)\rangle \quad — 2$$

$$\boxed{\cancel{\langle f(x), f(x)\rangle}} + \boxed{\cancel{\langle f_y, f_y\rangle}} - \langle f_x, f_y\rangle - \langle f_y, f_x\rangle$$

$$\langle f_x, f_y\rangle = \langle x, y\rangle$$

$$\boxed{\cancel{\langle x, x\rangle}} + \boxed{\cancel{\langle y, y\rangle}} - \langle x, y\rangle - \langle y, x\rangle$$

Question 6

Now let $x, y \in \mathbb{R}^3$ be arbitrary vectors and $a, b \in \mathbb{R}$ be arbitrary scalars.

Question 6

Now let $x, y \in \mathbb{R}^3$ be arbitrary vectors and $a, b \in \mathbb{R}$ be arbitrary scalars. Consider,

$$\langle (f(ax+by) - af(x) - bf(y)), (f(ax+by) - af(x) - bf(y)) \rangle$$

$$\langle f(ax+by), f(ax+by) \rangle - 2a \langle f(ax+by), f(x) \rangle - 2b \langle f(ax+by), f(y) \rangle$$
$$+ a^2 \langle f(x), f(x) \rangle + b^2 \langle f(y), f(y) \rangle + 2ab \langle f(x), f(y) \rangle$$

$$a^2 \langle x, x \rangle + b^2 \langle y, y \rangle + 2ab \langle x, y \rangle - 2a(a\langle x, x \rangle + b \langle y, x \rangle) - 2b(a\langle x, y \rangle + b\langle y, y \rangle)$$
$$+ a^2 \langle x, x \rangle + b^2 \langle y, y \rangle + 2ab \langle x, y \rangle$$

Count them up! $\boxed{=0}$

Question 6

$\therefore \quad f(ax+by) = af(x)+bf(y) \quad \forall \, x, y \in \mathbb{R}$
$a, b \in \mathbb{R}$

Now let $f(x) = Ax$.

$\Downarrow$

$\exists A \in \mathbb{R}$ s.t. $f(x) = Ax$

Question 6

Now let $f(x) = Ax$. Using the fact that $||f(x)|| = ||x||$ we have

Question 6

Now let $f(x) = Ax$. Using the fact that $||f(x)|| = ||x||$ we have

$$\langle Ax, Ax\rangle = \langle x, x\rangle$$

Question 6

Now let $f(x) = Ax$. Using the fact that $||f(x)|| = ||x||$ we have

$$\langle Ax, Ax \rangle = \langle x, x \rangle$$

$$(Ax)^T(Ax) = x^T x \implies x^T A^T Ax - x^T x = 0$$

Question 6

Now let $f(x) = Ax$. Using the fact that $||f(x)|| = ||x||$ we have

$$\langle Ax, Ax \rangle = \langle x, x \rangle$$

$$(Ax)^T(Ax) = x^T x \implies x^T A^T Ax - x^T x = 0$$

$$x^T(A^T A - I)x = 0$$

Question 6

Now let $f(x) = Ax$. Using the fact that $||f(x)|| = ||x||$ we have

$$\langle Ax, Ax \rangle = \langle x, x \rangle$$

$$(Ax)^T(Ax) = x^T x \implies x^T A^T Ax - x^T x = 0$$

$$x^T(A^T A - I)x = 0$$

The previous equations holds *for all* $x \in \mathbb{R}^3$.

Question 6

Now let $f(x) = Ax$. Using the fact that $||f(x)|| = ||x||$ we have

$$\langle Ax, Ax \rangle = \langle x, x \rangle$$

$$(Ax)^T(Ax) = x^T x \implies x^T A^T Ax - x^T x = 0$$

$$x^T(A^T A - I)x = 0$$

The previous equations holds *for all* $x \in \mathbb{R}^3$. Symmetricity is crucial. Ways to tackle:

Question 6

Now let $f(x) = Ax$. Using the fact that $||f(x)|| = ||x||$ we have

$$\langle Ax, Ax \rangle = \langle x, x \rangle$$

$$(Ax)^T(Ax) = x^T x \implies x^T A^T A x - x^T x = 0$$

$$x^T(A^T A - I)x = 0$$

The previous equations holds *for all* $x \in \mathbb{R}^3$. Symmetricity is crucial. Ways to tackle:

- Quadratic form

Question 6

Now let $f(x) = Ax$. Using the fact that $||f(x)|| = ||x||$ we have

$$\langle Ax, Ax \rangle = \langle x, x \rangle$$

$$(Ax)^T(Ax) = x^T x \implies x^T A^T Ax - x^T x = 0$$

$$x^T(A^T A - I)x = 0$$

The previous equations holds *for all* $x \in \mathbb{R}^3$. Symmetricity is crucial. Ways to tackle:

- Quadratic form
- Spectral Theorem [2] part 2