MA 205 Tutorial Batch 3
Recap-4

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Singularities

Definition (Singularities)

Given a function f, a point $z \in \mathbb{C}$ is said to be a singularity of f if

- f is not defined at z. Or,
- f is not holomorphic at z.

Read differently: Something goes wrong at a singlularity. Also called a singular point. Examples,

1. $f(z) := 1/z$ has a singularity at 0.
2. $f(z) := z^2/z$ has a singularity at 0.

Isolated Singularities

Notation: $D^*(z,r) := D(z,r)/\{z\}$. The 'punctured' disk.

Definition (Isolated Singularities)

Given a function f, a point $z \in \mathbb{C}$ is said to be an isolated singularity, if $\exists r > 0$ s.t. f is holomorphic on $D^*(z,r)$.

A singularity which is not isolated is called a non-isolated singularity.
Examples,

1. $f(z) := 1/z$ has a isolated singularity at 0.
2. $f(z) := (z-1)/(z(z-2)(z-2.00001))$ has isolated singularities at $0, 2, 2.00001$.
3. $f(z) := \tan(1/z)$ has a non-isolated singularity at $z = 0$. Why?
4. $f(z) := \bar{z}$ has *a* non-isolated singularity. Where?

Classifying Isolated Singularities - I

[#1] Removable Singularities

An isolated singularity $z \in \mathbb{C}$ of f is said to be removable if there exists a holomorphic function $\tilde{f} : D(z, r) \to \mathbb{C}$ for some $r > 0$ **such that** $\tilde{f}(z) = f(z) \forall z \in D^*(z, r)$.

Also, **fact**: A function has a removable singularity at a point p iff $\lim_{z \to p} f(z)$ exists.

Theorem (RRST)

Suppose f is bounded an holomorphic on $D^*(p, r)$. Then, p is a removable singularity for f.

Proof? Start with $g(z) := (z - p)^2 f(z)$. Explicitly construct the desired $\tilde{f}$.

Classifying Isolated Singularities - II

[#2] Poles

An isolated singularity $z \in \mathbb{C}$ of f is said to be a *pole* if $\lim_{z\to p} |f(z)| = \infty$.

Theorem (The Order of a Pole)

Let f have a pole at $p \in \mathbb{C}$. Then, there exists some $k \in \mathbb{N}$ such that for some $r > 0$ we have,

$$f(z) = (z-p)^{-k} H(z)$$

where, H is holomorphic on $D(p, r)$ **and** $H(p) \neq 0$. We say that k is the order of the pole.

A pole of order one is called a *simple pole*.

Classifying Isolated Singularities - III

[#3] Essential Singularities

An isolated singularity of f which is neither removable, nor a pole is called an *essential singularity*.

Theorem (Casorati-Weirstrass)

Suppose f has an essential singularity at p. Then, for any $r > 0$, f takes values arbitrarily close to every complex number on the disk $D^*(p, r)$.

Also, we have the following stronger result.

Theorem (Great Picard's Theorem)

Suppose f has an essential singularity at p. Then, for any $r > 0$, f takes on **all possible** complex values, **infinitely often**, with at most a single exception in $D^*(p, r)$.

Also recall the **Little Picard Theorem**: An entire function can omit at most one complex number.

Some Questions

Classify the singularities:

1. $e^{1/z}$ Essential
2. $\sin(1/z)$ Essential
3. $z \cos 1/z$ Essential
4. $z^4/(z^3 + z)$ Two poles, one removable

Laurent Series

Theorem

Suppose f is analytic in the annulus $\mathcal{A}(p, r_1, r_2) := \{z : r_1 < |z - p| < r_2\}$. Let γ be any positively oriented simple closed contour around p lying in the annulus. Then,

$$f(z) = \sum_{n \in \mathbb{Z}} a_n (z - p)^n$$

holds for all $z \in \mathcal{A}(p, r_1, r_2)$. Where,

$$a_n := \frac{1}{2\pi\iota} \int_\gamma \frac{f(\eta)}{(\eta - p)^{n+1}} \; \forall n \in \mathbb{Z}$$

Question

Where can we apply the Laurent Series mechanism, if

1. $f(z) = 1/z$
2. $f(z) = z$. What happens to the a_{-n}'s $(n > 0)$?
3. $f(z) = 1/((z-1)(z-2))$

Theorem

Suppose f is analytic in the annulus $\mathcal{A}(p, r_1, r_2) := \{z : r_1 < |z - p| < r_2\}$. Let a_n be the coefficients of its Laurent Series. Then,

1. f has a pole at $p \Leftrightarrow a_n = 0$ for all but finitely many $n < 0$, and $a_k \neq 0$ for some $k < 0$.
2. f has an essential singularity at $p \Leftrightarrow a_n \neq 0$ for **infinitely many** $n < 0$.

A Joke

"A man of the Polish persuasion got on a Boeing 767 for a routine flight back to the Motherland. He was getting comfortable in coach when a stewardess screamed out, 'The pilot and co-pilot are dead! Is there anyone left that can fly this aircraft?' The Polish man said, 'I was a pilot back in the war. Let me have a go at the controls.' So he bravely sauntered up to cockpit. When he opened the door, he was awestruck by the array of lights, dials, screens and switches in front of him, and he froze up. The stewardess shook him and said, 'Aren't you going to sit down and take the reins?' He said in a quavering voice, 'I'm just a simple Pole in a complex plane!'"

`source:` https://asimplepoleinacomplexplane.blogspot.com