MA 109 D2 T1
Week Five Recap

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$\mathbb{R} \to \mathbb{R}^2$

- **Sequences**: A sequence in $\mathbb{R}^2$ is of the form $((x_n, y_n))$
- Note,

$$|x_n - x_0|, |y_n - y_0| \leq ||(x_n, y_n) - (x_0, y_0)|| \leq |x_n - x_0| + |y_n - y_0|$$

- That is,

$$(x_n, y_n) \to (x_0, y_0) \Leftrightarrow x_n \to x_0 \text{ and } y_n \to y_0$$

- **Functions**: We will deal with $f : D \subset \mathbb{R}^2 \to \mathbb{R}$.
  1. $\{(x, y, f(x, y)) \mid (x, y) \in D\}$ is the graph of f
  2. For some fixed $c \in \mathbb{R}$, $\{(x, y, c) \mid (x, y) \in D \text{ and } f(x, y) = c\}$ is a contour line.
  3. For some fixed $c \in \mathbb{R}$, $\{(x, y) \mid (x, y) \in D \text{ and } f(x, y) = c\}$ is a level curve.
  4. **Limits and Continuity**: Again, two definitions for both. Again equivalent. Similar relations between limits and continuity.
  5. But note that, we will only deal with *interior points* here.

Partial Derivatives

'Partial' Derivatives – Rate of change along the x and y axes.

Definition (Partial Derivatives)

Let $f : D \subset \mathbb{R}^2 \to \mathbb{R}$, and let $(x_0, y_0) \in D$ be an interior point of D. Then, we say that,

1. f has a partial derivative w.r.t. x at (x_0, y_0) if

$$\lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}$$

   exists. We denote it as $f_x(x_0, y_0)$.

2. f has a partial derivative w.r.t. y at (x_0, y_0) if

$$\lim_{h \to 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h}$$

   exists. We denote it as $f_y(x_0, y_0)$.

Gradient

Definition (Gradient)

Let $f : D \subset \mathbb{R}^2 \to \mathbb{R}$, and let $(x_0, y_0) \in D$ be an interior point of D. Suppose both partial derivatives of f exist at (x_0, y_0), then we define the gradient of f at (x_0, y_0)

$$\nabla f(x_0, y_0) := (f_x(x_0, y_0), f_y(x_0, y_0))^T \in \mathbb{R}^2$$

Definition (Directional Derivatives)

Let $f : D \subset \mathbb{R}^2 \to \mathbb{R}$, and let $(x_0, y_0) \in D$ be an interior point of D. For some $u = (u_1, u_2)^T \in \mathbb{R}^2$, we say that f has a directional derivative along u at (x_0, y_0) if

$$\lim_{t \to 0} \frac{f(x_0 + tu_1, y_0 + tu_2) - f(x_0, y_0)}{t}$$

exists, and denote it by $D_u f(x_0, y_0)$.

'Differentiability'?

In the 1D case, we saw that, Differentiability implies Continuity. Well, should we take the definitions of directional/partial derivatives as the definition of 2D Differentiability?

Consider, $f : \mathbb{R}^2 \to \mathbb{R}$ such that $f(x, y) := 0 \forall (x, y) : x \neq 0$ and $y \neq 0$, $f(x, 0) := 1000 \forall x$ and $f(0, y) := 1000 \forall y$.

Does the 'derivative' exist at $(0, 0)$? Continuity? Well?

Thus, the directional derivative is not a *strong enough* definition! Can have functions for which DD's exist at a point, but is not continuous at that point.

Tangent Lines

- 1D: $y = f(x_0) + f'(x_0)(x - x_0)$
- 2D $z = f(x_0, y_0) + \partial_x(x_0, y_0)(x - x_0) + \partial_y(x_0, y_0)(y - y_0)$

1D Definition:

$$\lim_{h \to 0} \frac{|f(x_0 + h) - f(x_0) - f'(x_0)h|}{|h|} = 0$$

What about 2D?

Differentiability

Definition (Differentiability)

A function $f : D \to \mathbb{R}$ is said to be differentiable at an interior point (x_0, y_0) if, $\partial_x f$ and $\partial_y f$ exist at that point and,

$$\lim_{(h,k)\to 0} \frac{|f(x_0 + h, y_0 + k) - f(x_0, y_0) - \partial_x(x_0, y_0)h - \partial_y(x_0, y_0)k|}{||(h, k)||} = 0$$

Differentiability at a point *implies* the following,

- Existence of partial derivatives at that point.
- Existence of ALL directional derivatives at that point.
- The fact that $D_u f(x_0, y_0) = \nabla f(x_0, y_0) \cdot u$ for every unit vector u.
- Continuity of f at that point.

Alternate Condition

Proposition

Let $f : D \to \mathbb{R}$ and let (x_0, y_0) be an interior point of D. If the partial derivatives of f exist in some neighbourhood of (x_0, y_0) and are continuous at (x_0, y_0) then f is differentiable at (x_0, y_0).

Example

$$f(x,y) := xy\frac{x^2-y^2}{x^2+y^2} \forall (x,y) \neq (0,0); \quad f(0,0) := 0$$

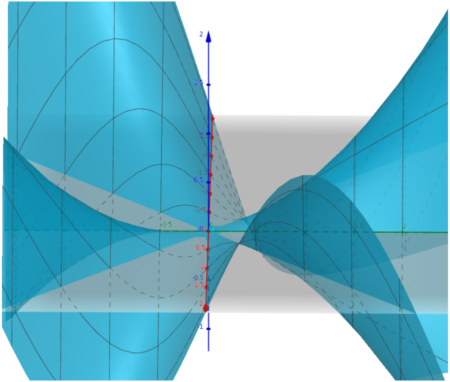

Figure: The graph of f