PH 534: Quantum Information and Computing Tutorial

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Notations

- States: $|\psi\rangle$
- $|\psi^{\pm}\rangle = \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle)$, $|\phi^{\pm}\rangle = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle)$, $|\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$
- Density operators: $\rho, \sigma ...$
- Hilbert Space: $\mathcal{H}$
- Set of linear operators from $\mathcal{H}_1$ to $\mathcal{H}_2$: $L(\mathcal{H}_1, \mathcal{H}_2)$
- Set of density matrices on $\mathcal{H}$: $D(\mathcal{H}) \subset L(\mathcal{H})$
- Set of linear operators from $L(\mathcal{H}_1)$ to $L(\mathcal{H}_2)$: $T(\mathcal{H}_1, \mathcal{H}_2)$
- Set of quantum channels from $L(\mathcal{H}_1)$ to $L(\mathcal{H}_2)$: $C(\mathcal{H}_1, \mathcal{H}_2) \subset T(\mathcal{H}_1, \mathcal{H}_2)$

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$$Tr_B(\rho_{AB}) := \sum_j (I_A \otimes \langle j|_B)\rho_{AB}(I_A \otimes |j\rangle_B)$$

Question

Write the matrix representation, and calculate the reduced single-qubit density matrices ρ_A and ρ_B.

1. $\rho_{AB} = \frac{1}{3}|00\rangle\langle 00| + \frac{2}{3}|\psi^+\rangle\langle\psi^+|$, where $|\psi^\pm\rangle = \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle)$.
2. $\rho_{AB} = (1-p)\mathbb{I}/4 + p\,|\psi^-\rangle\langle\psi^-|$, where $\mathbb{I}$ is the 4×4 identity matrix.
3. $\rho_{AB} = p\,|\psi^-\rangle\langle\psi^-| + (1-p)|\psi^+\rangle\langle\psi^+$.

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Theorem 2.7 of QCQI

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where $\lambda_i \geq 0$ and $\sum_i \lambda_i^2 = 1$.

Question Consider the state, $|\psi\rangle = \frac{1}{\sqrt{3}}(|00\rangle + |01\rangle + |10\rangle)$. Find its Schmidt Decomposition.

Purification

- Suppose
$$\rho_A = \sum_i p_i \left| i_A \right\rangle \left\langle i_A \right|$$
- Introduce R with $\mathcal{H}_R = \mathcal{H}_A$ (can we do better?) and a orthonormal basis $\left| i_R \right\rangle$.

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$$\rho_A = Tr_R(\left|\psi_{AR}\right\rangle \left\langle\psi_{AR}\right|)$$

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Question Compute the purification for the state

$$\rho = \frac{1}{2}(\left|0\right\rangle \left\langle 0\right| + \left|+\right\rangle \left\langle +\right|)$$

Further, apply the unitary $U := \mathbb{1} \otimes \sigma_y$ to the purified state, and calculate the reduced state to show whether or not such unitaries create equivalent purifications.

Quantum Operations: Beyond Unitaries

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  1. $Tr[\mathcal{E}(\rho)] \in [0,1] \forall \rho$ ($Tr[\mathcal{E}(\rho)]$ is the probability that ρ undergoes the transformation $\mathcal{E}$).
  2. Convex linearity

$$\mathcal{E}(\sum_i p_i \rho_i) = \sum_i p_i \mathcal{E}(\rho_i)$$

     for all density matrices ρ_i and probabilities p_i s.t. $\sum_i p_i = 1$
  3. $\mathcal{E}$ is completely positive. Not only does $\mathcal{E}$ preserve positivity, $(I \otimes \mathcal{E})$ also preserves positivity for I being the identity on an aribtrarily dimensional system's hilbert space.

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Formally,

$$\mathcal{E} : L(\mathcal{H}_1) \rightarrow L(\mathcal{H}_2) \in C(\mathcal{H}_1, \mathcal{H}_2)$$

Simple example

For a single qubit state ρ, a measurement in the computational basis can be described by the operations $\mathcal{E}_0(\rho) \equiv |0\rangle\langle 0|\rho|0\rangle\langle 0|$ and $\mathcal{E}_1(\rho) \equiv |1\rangle\langle 1|\rho|1\rangle\langle 1|$

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That is, if no measurement is happening, the map $\mathcal{E}$ would be a *completely positive trace preserving (CPTP)* map.

The operator sum representation

Theorem 8.1 of QCQI

The map $\mathcal{E}$ satisfies the axioms for a valid quantum operation iff there exists a set of operators $\{E_i\}$ such that

$$\mathcal{E}(\rho) = \sum_i E_i \rho E_i^*$$

for all valid density matrices ρ and $0 \preccurlyeq \sum_i E_i^* E_i \preccurlyeq I$

Note: $A \preccurlyeq B$ if $B - A$ is PSD.

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Note: $A \preccurlyeq B$ if $B - A$ is PSD. So if we are just dealing with CPTP maps, then these E_i satisfy $\sum_i E_i^* E_i = I$, and are called the *kraus operators.*

Some maffs

1. **The Vectorization Map**: vec : $L(\mathcal{H}_2, \mathcal{H}_1) \to \mathcal{H}_1 \otimes \mathcal{H}_2$, defined as

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2. **The Choi Representation** $\mathcal{J} : T(\mathcal{H}_1, \mathcal{H}_2) \to L(\mathcal{H}_2 \otimes \mathcal{H}_1)$
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If $\mathcal{H}_1 = \mathcal{H}_2$,
$$J(\mathcal{E}) = \sum_{i,j} \mathcal{E}(|i\rangle \langle j|) \otimes |i\rangle \langle j|$$

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The Choi Theorem

If $\mathcal{E} : \mathcal{H} \to \mathcal{H}$ is a quantum channel with kraus operators $\{A_i\}$, then,

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2. Suppose,

$$\mathcal{J}(\mathcal{E}) = \frac{2}{3} |\phi^-\rangle \langle \phi^-| + \frac{4}{3} |\psi^+\rangle \langle \psi^+|$$

Calculate the Kraus operators for $\mathcal{E}$.

Question

An isometry maps a quantum system from a smaller to a larger Hilbert space. For instance, the isometry on a single qubit,

$$U|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |3\rangle)$$
$$U|1\rangle = \frac{1}{\sqrt{2}}(|1\rangle + |2\rangle),$$

maps the qubit from the Hilbert space basis $\{|0\rangle, |1\rangle\}$ to the four-dimensional basis $\{|0\rangle, |1\rangle, |2\rangle, |3\rangle\}$

1) Write the isometry U as a 4×2 matrix, and show that it $U^\dagger U = \mathbb{I}$, but $UU^\dagger \neq \mathbb{I}$.

2) Find a 4×4 unitary matrix, $\tilde{U}$, such that it gives U, when the last 2 columns are removed. (Hint: You have to show $\tilde{U}^\dagger \tilde{U} = \tilde{U}\tilde{U}^\dagger = \mathbb{I}$.

3) Alternatively, show that there exists a 4×2 matrix V, such that $U = \tilde{U} \cdot V$. Interpret, the role of V.

Appendix: Interpretations of the OpSum Representation

S+E $\rightarrow$ Kraus

- We have, the system coupled to an environment under unitary evolution as

$$U(\rho \otimes |e_0\rangle \langle e_0|)U^*$$

- Trace out the environment,

$$\sum_k (\mathbb{1} \otimes \langle e_k|)U(\rho \otimes |e_0\rangle \langle e_0|)U^*(\mathbb{1} \otimes |e_k\rangle)$$

- See that

$$(\rho \otimes |e_0\rangle \langle e_0|) = (\mathbb{1} \otimes |e_0\rangle)\rho(\mathbb{1} \otimes \langle e_0|)$$

- Define

$$E_k \equiv (\mathbb{1} \otimes \langle e_k|)U(\mathbb{1} \otimes |e_0\rangle)$$

and see the equivalence.

More physically ...

With the notation in the previous slide, define

$$\rho_k \equiv \frac{E_k \rho E_k^*}{tr(E_k \rho E_k^*)}$$

Thus, we can consider the act of applying $\mathcal{E}$ as applying U to $\rho \otimes |e_0\rangle \langle e_0|$ and then measuring the environment in the $|e_k\rangle$ basis.

That is equivalent to replacing ρ randomly by ρ_k with the probability $p(k) = tr(E_k \rho E_k^*)$, thus resulting in

$$\mathcal{E}(\rho) = \sum_k p(k)\rho_k = \sum E_k \rho E_k^*$$

as expected.

(More General) S+E → Kraus

Let P_m be a projective measurement on the S+E. Define

$$\mathcal{E}_m(\rho) = tr_E(P_m U(\rho \otimes \sigma) U^* P_m)$$

where $\sigma = \sum_i q_j |j\rangle\langle j|$. And let $|e_k\rangle$ be a basis for $\mathcal{H}^E$. Thus,

$$\mathcal{E}_m(\rho) = \sum_{jk} q_j (\mathbb{1} \otimes \langle e_k|) P_m U(\rho \otimes |j\rangle\langle j|) U^* P_m (\mathbb{1} \otimes |e_k\rangle)$$

Define

$$E^m_{jk} = \sqrt{q_j}(\mathbb{1} \otimes \langle e_k|) P_m U(\mathbb{1} \otimes |j\rangle)$$

and we get

$$\mathcal{E}_m(\rho) = \sum_{jk} E^m_{jk} \rho E^{m*}_{jk}$$

It is easy to see that $\sum_{jk} E^{m*}_{jk} E^m_{jk} \preccurlyeq \mathbb{1}$, not $= \mathbb{1}$. The state evolves to $\mathcal{E}_m(\rho)$ with probability $tr(\mathcal{E}_m(\rho))$.

Kraus $\to$ S+E

- We are given the set $\{\mathcal{E}_m\}$. We shall construct a S+E (+measurement) model.
- Let E^m_k be the kraus rep. for the operation $\mathcal{E}_m$.
- Introduce env E with an orthonormal basis $|m,k\rangle$ with indices in 1-1 correspondence.
- Let $|e_k\rangle$ be a basis for $\mathcal{H}^E$ and define

$$U|\psi\rangle \otimes |e_0\rangle \equiv \sum_{mk} E_{mk}|\psi\rangle |m,k\rangle$$

  As done earlier, we know this can be extended to a unitary on the composite system.

- Define

$$P_m \equiv \sum_m |m,k\rangle \langle m,k|$$

Kraus $\to$ S+E

Let $\rho = \sum p_i \left|i\right\rangle \left\langle i\right|$ be a state of our system.

Consider

$$\begin{aligned} U(\rho \otimes \left|e_0\right\rangle \left\langle e_0\right|)U^* &= \sum_j p_i U(\left|j\right\rangle \left\langle j\right| \otimes \left|e_0\right\rangle \left\langle e_0\right|)U^* \\ &= \sum_{mjk} p_j (E_{mk} \left|j\right\rangle \left\langle j\right| E^*_{mk} \otimes \left|m,k\right\rangle \left\langle m,k\right|) \end{aligned}$$

Now it is easy (and a bit annoying to write) to see that measuring P_m will result in $\mathcal{E}_m(\rho)$ with probability $tr(\mathcal{E}_m(\rho))$.

Kraus $\rightarrow$ S+E

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