A Summary of Sorts

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1. Some brief topology
2. Conservativeness
3. Green's Theorem
4. Surfaces
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Openness

Let $a \in \mathbb{R}^n$. Given $\epsilon > 0$ define,

$$B_a(\epsilon) := \{x \mid x \in \mathbb{R}^n, |x - a| < \epsilon\}$$

We call this an *epsilon ball* centered around a.

Open

$A \subseteq \mathbb{R}^n$ is open if

$$\forall x \in A \exists \epsilon > 0 \text{ s.t. } B_a(\epsilon) \subset A$$

Thus, an open set is one which has all points as interior points.

Closedness

Closed

$A \subseteq \mathbb{R}^n$ is closed if A^c is open.

We can also define closedness in terms of limit points, refer to the note I shared on teams.

Regions

Let $A \subseteq \mathbb{R}^n$.

1. A is connected if it **cannot** be written as a disjoint union of two non empty subsets A_1 and A_2 *such that* $A_1 = A \cap U_1$ and $A_2 = A \cap U_2$ such that U_1, U_2 are open in $\mathbb{R}^n$.
2. A is path connected if any two points in A can be joined by a path which lies inside A.
3. A is simply connected if A is connected and any simple closed curve lying in A can be contracted to a point in A.

Some (/non) implications

- Path connected $\implies$ connected
- Connected $\not\Longrightarrow$ path connected (Example?)
- Open + Connected $\Leftrightarrow$ Path connected [1]

[1] We are talking about $\mathbb{R}^n$ here

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What was this again?

Definition

A vector field $\mathbf{F}$ is conservative if it is a gradient of some scalar function.

$$\mathbf{F} = \nabla f$$

FTC for vector fields

Let $f : D \subset \mathbb{R}^n$ be a differentiable function and let ∇f be continuous on a smooth part $\mathbf{c}$. Then,

$$\int_{\mathbf{c}} \nabla f = \int_a^b \nabla f(\mathbf{c}(t))\mathbf{c}'(t)dt = f(\mathbf{c}(b)) - f(\mathbf{c}(a))$$

Some (/non) implications

- Conservative $\implies$ Path Independent
- Not path independent $\implies$ not Conservative
- Path independent $\not\Longleftarrow$ Conservative
- Not conservative $\not\Longleftarrow$ Not path independent
- Path independent + Domain is Path connected $\implies$ Conservative

Some more implications

Let $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j}$ be a C^1 field on an open $D \subset \mathbb{R}^n$. We take $n = 2$. Similar results hold for $n = 3$.

1. Recall the necessary condition.

$$\mathbf{F} = \nabla f \implies \frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}$$

2. Recall the sufficient condition. Let D be simply connected. Then,

$$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x} \implies \mathbf{F} \text{ is conservative}$$

Note the direction of the implications above.

Question

Is

$$\mathbf{F} = (\frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2})$$

a conservative field?

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Question

Is

$$\mathbf{F} = (\frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2})$$

a conservative field? Answer? The question is not well framed. Why? This illustrates that a vector field is not *just* a tuple of two functions.

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Orientation

Definition

The *positive orientation* of a curve C in $\mathbb{R}^2$ is given by the vector field $\mathbf{k} \times \mathbf{n}$. Where $\mathbf{n}$ is the unit normal vector pointing outward along the curve.

So, given a path in $\mathbb{R}^2$, we have the notion of *orientation* of the region enclosed by the path.
Further recall that, for a region with holes, the positive orientation is such that the outer boundary is oriented counter-clockwise, and the inner clockwise.

Green's Theorem

Theorem

1. *Let D be a bounded region in $\mathbb{R}^2$ with a positively oriented boundary ∂D consiting of a* **finite number** *of* **non-intersecting simple** *closed* **piecewise continuously differentiable** *curves.*
2. *Let Ω be an open set in $\mathbb{R}^2$ such that $D \cup \partial D \subset \Omega$.*
3. *Let $F_1 : \Omega \to \mathbb{R}$, $F_2 : \Omega \to \mathbb{R}$ be C^1 functions. Consider the vector field*

$$\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j}$$

Then, (finally!)

$$\int_{\partial D} F_1 dx + F_2 dy = \iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) d(x, y)$$

Variants

D is our region, $\mathbf{n}$ is the outward normal, $\mathbf{k}$ is the vector normal to the plane.

1. Using Div
$$\int_{\partial D} \mathbf{F} \cdot \mathbf{n} ds = \iint_D \nabla \cdot \mathbf{F} d(x, y)$$
2. Using Curl
$$\int_{\partial D} \mathbf{F} \cdot d\mathbf{s} = \iint_D (\nabla \times \mathbf{F}) \cdot \mathbf{k} d(x, y)$$

(2) is a special case of - ?

Applications

- Recall how area is defined. Now, we can apply Green's Theorem and see that,

$$A(D) = \frac{1}{2}\int_{\partial D} xdy - ydx = \int_{\partial D} xdy = -\int_{\partial D} ydx$$

- In polar coordinates,

$$A(D) = \frac{1}{2}\int r^2 d\theta$$

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Definitions

Definition

Let $D \subseteq \mathbb{R}^2$ be path connected. A *parametrised surface* is a **continuous** function $\varphi : D \to \mathbb{R}^3$.

Now, recall the definitions of $\varphi_u(u, v)$ and $\varphi_v(u, v)$. At some point (u, v) these two give us the *tangent plane* at that point.

Tangent plane

At some point (x_0, y_0, z_0), the equation of the tangent plane is given by

$$(\varphi_u \times \varphi_v)(u_0, v_0) \cdot (x - x_0, y - y_0, z - z_0) = 0$$

Area of a Surface

$$Area(\varphi) := \iint_D ||\varphi_u \times \varphi_v|| d(u, v)$$

With the notation $dS = ||\varphi_u \times \varphi_v|| d(u, v)$ we have,

$$Area(\varphi) = \iint_D dS$$

Note that D is the domain of the surface parametrisation.

Oriented Surfaces

Definition

A surface S is said to be *orientable* if there exists a **continuous** vector field $\mathbf{F} : S \to \mathbb{R}^3$ such that for each point P in S, $\mathbf{F}(P)$ is a unit vector normal to the surface S at P.

An oriented parametrised surface φ comes equipped with a vector field of normal unit vectors

$$\mathbf{n} = \frac{\varphi_u \times \varphi_v}{||\varphi_u \times \varphi_v||}$$

Then, given any surface it is either orientation reversing or perserving. Also note that changing the orientation leads to changing the sign of a surface integral.

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Surface Integrals

Scalar Fields

Given a surface φ on the path connected set $E \subseteq \mathbb{R}^2$, its image S, and a bounded function $f : S \to \mathbb{R}$, we define,

$$\iint_S fdS := \iint_E f(x, y, z)||\varphi_u \times \varphi_v||d(u, v)$$

Surface Integrals

Vector Fields

Given a surface φ on the path connected set $E \subseteq \mathbb{R}^2$, its image S, and a bounded vector field such that its domain consists of S, we define,

$$\iint_S \mathbf{F} \cdot d\mathbf{S} := \iint_E \mathbf{F}(\varphi(u,v)) \cdot (\varphi_u \times \varphi_v) du dv$$

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Setting up

Recall the orientation of a surface. Now, given such an oriented surface S, an orientation is induced on the *boundary* ∂S. The direction of the same can be found by the right hand rule.
Further, if $\varphi : E \to \mathbb{R}^3$ (E is P.C.) is a smooth, orientation preserving parametrisation of S then the induced orientation of ∂S corresponds to the positive orientation of ∂E w.r.t. E.

Stokes Theorem

Theorem

1. *Let S be a bounded piecewise smooth oriented surface with non-empty boundary ∂S. Suppose S is closed in $\mathbb{R}^3$.*
2. *Let ∂S be a disjoint unionm of simple closed curves each of which is a piecewise non-singular parametrized curve with the induced orientation.*
3. *Let $\mathbf{F} = (F_1, F_2, F_3)$ be a C^1 vector field defined on an open set containing S. Then (finally again!),*

$$\int_{\partial S} \mathbf{F} \cdot d\mathbf{S} = \iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S}$$

Consequences

1. The surface integral of $\nabla \times \mathbf{F}$ over two different surfaces with the same orientation and the same boundary is the same.
2. Note that we required $\partial S \neq \emptyset$ before. If it is empty, then the surface integral of the curl is simply zero. But, this was proved separately, not given by the Stokes Theorem.
3. For computing line integrals, we can switch over to computing surface integrals with a *large* choice of surfaces to choose from.
4. If we have a curl free **smooth** vector field on open regions, we have path independence. Can we conclude that a suitable potential function must exist?
5. No. Moreover,

$$\text{curl free} + \text{domain is simply connected} \implies \text{conservative}$$

Sidenote

- $\text{curl}(\text{grad}) = 0$
- $\text{div}(\text{curl}) = 0$
- $\text{curl} = 0$ and domain is simply connected $\implies$ field is a grad field

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Divergence Theorem

Theorem

1. *Let W be a simple solid region of $\mathbb{R}^3$ whose boundary $S = \partial W$ is a closed surface and is positively oriented.*
2. *Let $\mathbf{F}$ be a smooth vector field on an open subset of $\mathbb{R}^3$ containing W. Then,*

$$\iint_{\partial W} \mathbf{F} \cdot d\mathbf{S} = \iiint_{W} \nabla \cdot \mathbf{F} dV$$

Consequences

- Divergence free fields enjoy Zero surface integrals over boundaries of simple solid regions.
- Break up the aforementioned boundary into two surfaces S_1, S_2 with the same boundary[2] . Then the surface integral of a div-free field is same across both.

[2] Please note the usage of boundary for surfaces and solid regions, and distinguish appropriately.

The end

Onto the last tutorial.