markov lengths, mixed-state phases

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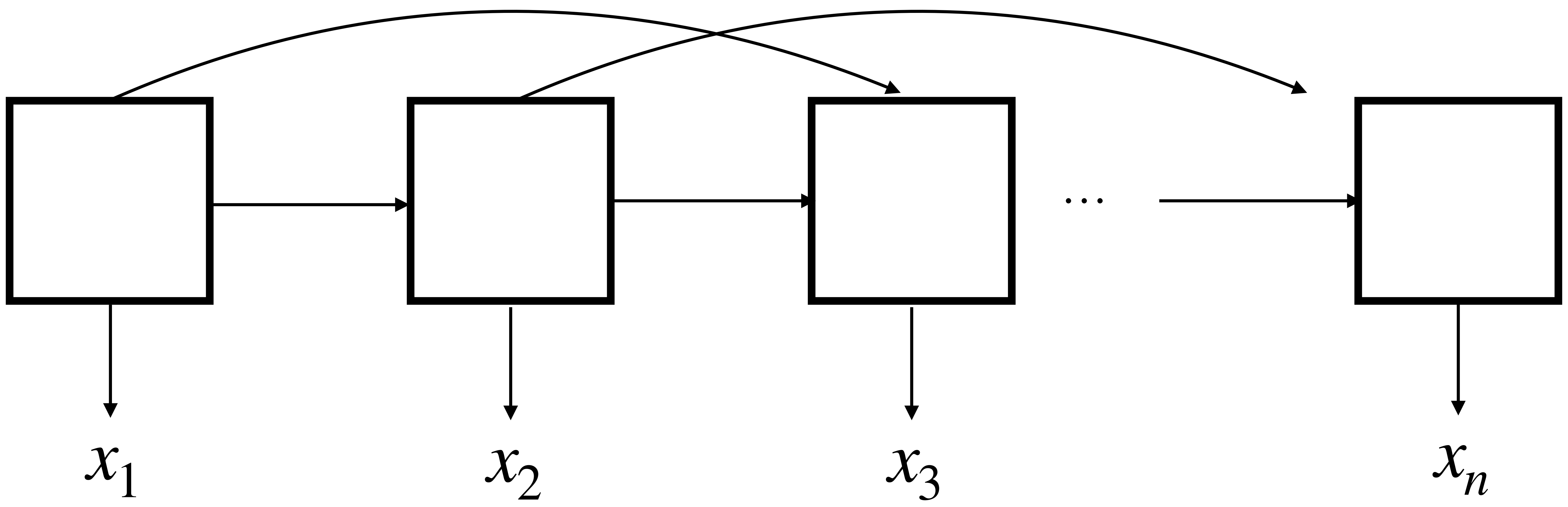
reviewing:

[1] Sang, Shengqi, and Timothy H. Hsieh. "Stability of mixed-state quantum phases via finite Markov length." *Physical Review Letters* 134.7 (2025): 070403.

[2] Negari, Amir-Reza, Tyler D. Ellison, and Timothy H. Hsieh. "Spacetime Markov length: a diagnostic for fault tolerance via mixed-state phases." *arXiv preprint arXiv:2412.00193* (2024).

markov length

$p(x_{1:N})$ $\qquad x_{1:N} = (x_1, x_2, \dots x_N)$ $\qquad x_i \in \{0,1\}$



$$p(x_{1:N}) = p(x_1)p(x_2)p(x_3)\dots$$

$$p(x_{1:N}) = p(x_1)p(x_2 \mid x_1)p(x_3 \mid x_2)\dots p(x_N \mid x_{N-1})$$

$$p(x_{1:N}) = p(x_1)p(x_2 \mid x_1)p(x_3 \mid x_1 x_2)\dots p(x_N : x_{1:N-1})$$

$$\mathscr{M} = \min_{M \geq 1}\{M : p(x_a \mid x_{a-M:a-1}) = p(x_a \mid x_{1:a-1}) \ \ \forall \ \ a \geq 2\}$$

markov length

$$p(x_{1:N}) = p(x_1)p(x_2 | x_1)p(x_3 | x_1 x_2)\dots$$

(1D) $$\mathcal{M} = \min_{M \geq 1}\{M : p(x_a | x_{a-M:a-1}) = p(x_a | x_{1:a-1}) \quad \forall \quad a \geq 2\}$$

$$p(x_{00} | \cup x_{ab}) = p(x_{00} | x_{01}, x_{10}, x_{11})$$

$$2^N \to 2^R N$$

$$R = O(\mathcal{M})$$

0	1	1	0
1	1	0	1
0	1	0	0
1	0	1	1
0	1	0	0

Probabilistic graphical models, graph inference, etc...

$$G = (V, E)$$

$$\forall v \in V \qquad \left(v \perp V - N(v) - \{v\}\right) \mid N(v)$$

entropies, information etc.

$$\mathbb{P}[X = x] = p_x \qquad H[X] = -\,\mathbb{E}_X \log p(X) \equiv -\sum_x p_x \log p_x$$

$$\mapsto S(\rho) = -\,\mathrm{tr}(\rho \log \rho)$$

MI $$\boxed{\begin{array}{c} I[X:Y] = S[X] + S[Y] - S[X \cup Y] \\ I[X:Y] = 0 \Leftrightarrow p_{XY} = p_X p_Y \end{array}}$$

CE $$S[X\,|\,Y] := \mathbb{E}_y[S(X\,|\,Y = y)] \equiv S[X \cup Y] - S[Y]$$

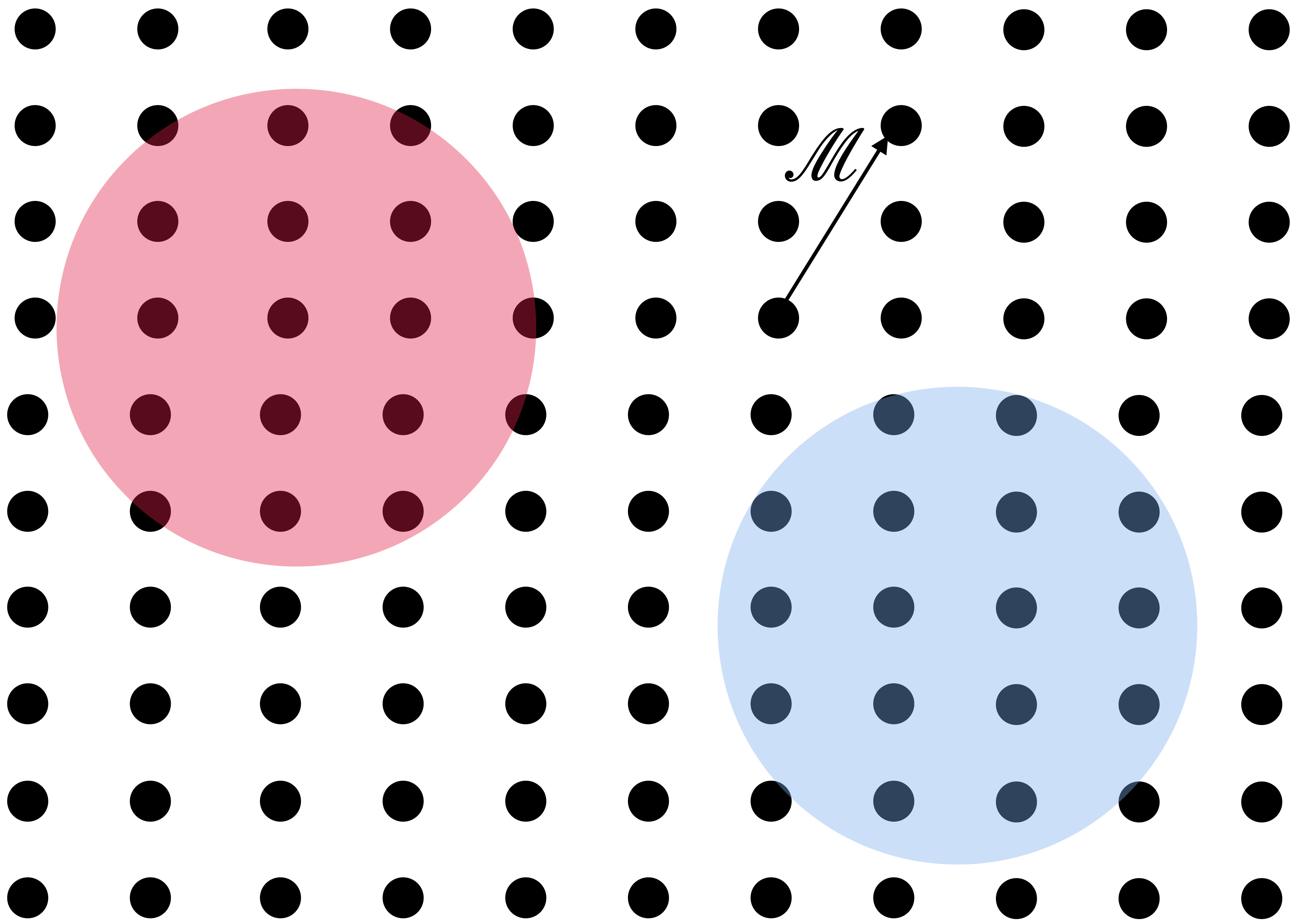
$$I[X:Y\,|\,Z] = I[X:YZ] - I[X:Z] = \mathbb{E} I(X:Y\,|\,Z = z)$$

$$(S[X] + S[YZ] - S[XYZ]) - (S[X] + S[Z] - S[XZ])$$

CMI $$\boxed{I[X:Y\,|\,Z] = S[XZ] + S[YZ] - S[XYZ] - S[Z]}$$

$$\mathrm{dist}(X, Y) > 2\mathscr{M} \implies p_{XYZ} = p_Z p_{X|Z} p_{Y|Z} \implies I[X:Y\,|\,Z] = 0$$

$$Z = \Lambda - (X \cup Y) \qquad \mathrm{CMI} = 0 \Leftrightarrow \mathrm{MC}\ X \to Z \to Y$$



pure state phases

“same long range correlations”

gapped local Hamiltonians $\rightarrow (H_0, |\psi_0\rangle)$, $(H_1, |\psi_1\rangle)$

$|\psi_0\rangle \ \cong \ |\psi_1\rangle$

$|\psi_0\rangle = U|\psi_1\rangle$

(uniform) gap

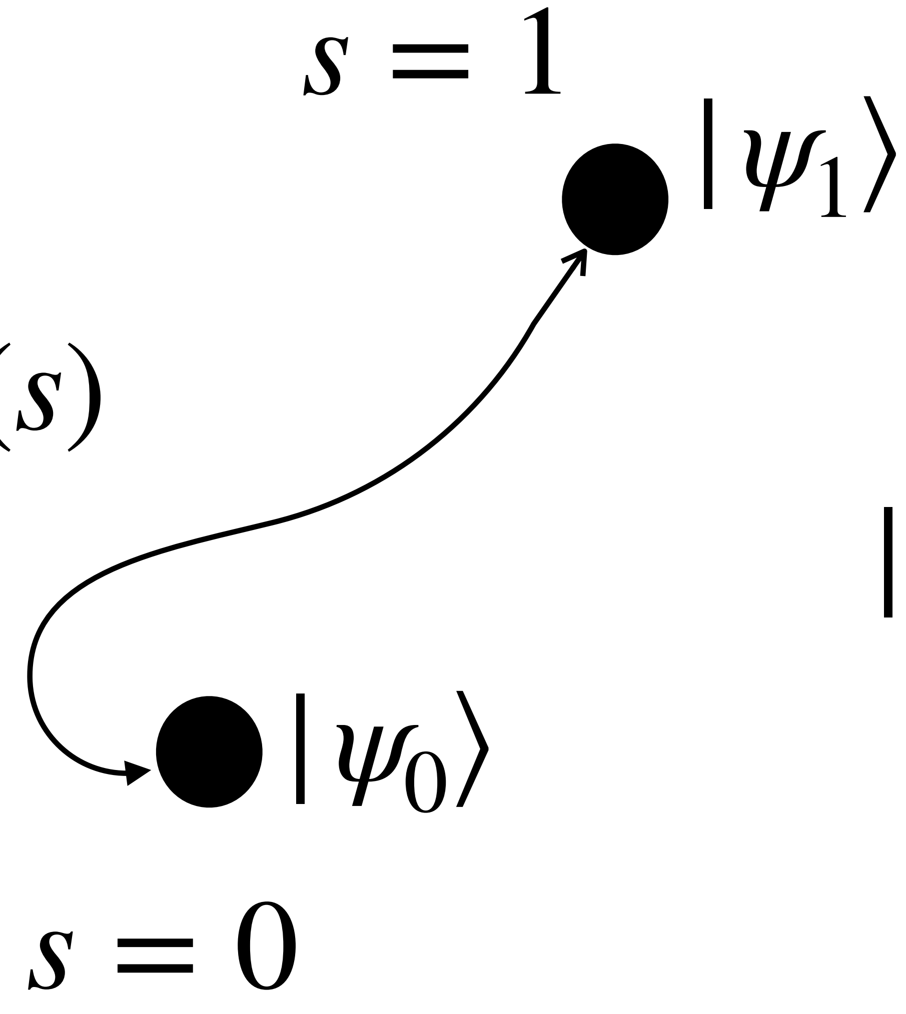
$\Delta(s) \geq \Delta = O(1)$

correlation length

smoothly varying $\langle O \rangle$

$H(s=0) = H_0$

$H(s=1) = H_1$

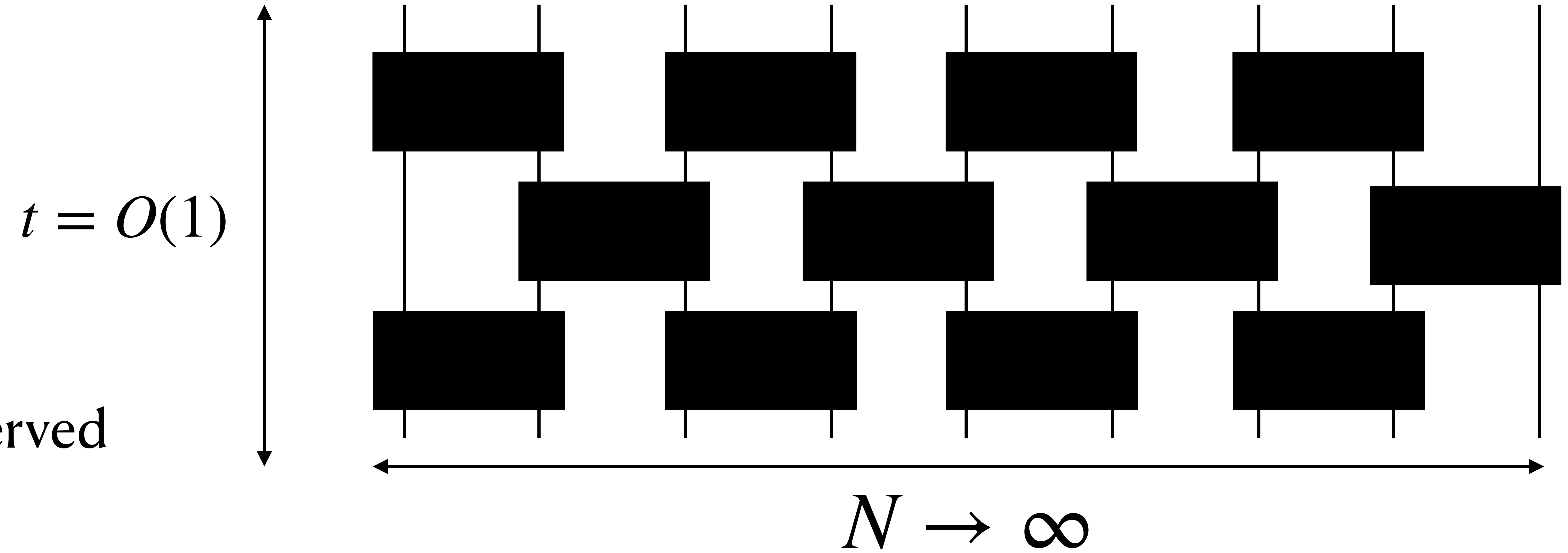


$|\psi_0\rangle \cong |\psi_s\rangle \forall s \in [0,1]$

$$U(s) = \mathcal{T} e^{-\imath \int_{t=0}^{s} H(t)dt}$$

$$|\psi_1\rangle = U(s=1)|\psi_0\rangle$$

Quasi-adiabatic Continuation (Hastings and Wen, 2005)



Reversibility comes for free

“global structure” preserved

mixed state phases

finite time

(quasi-)local *Liouvillian* evolution

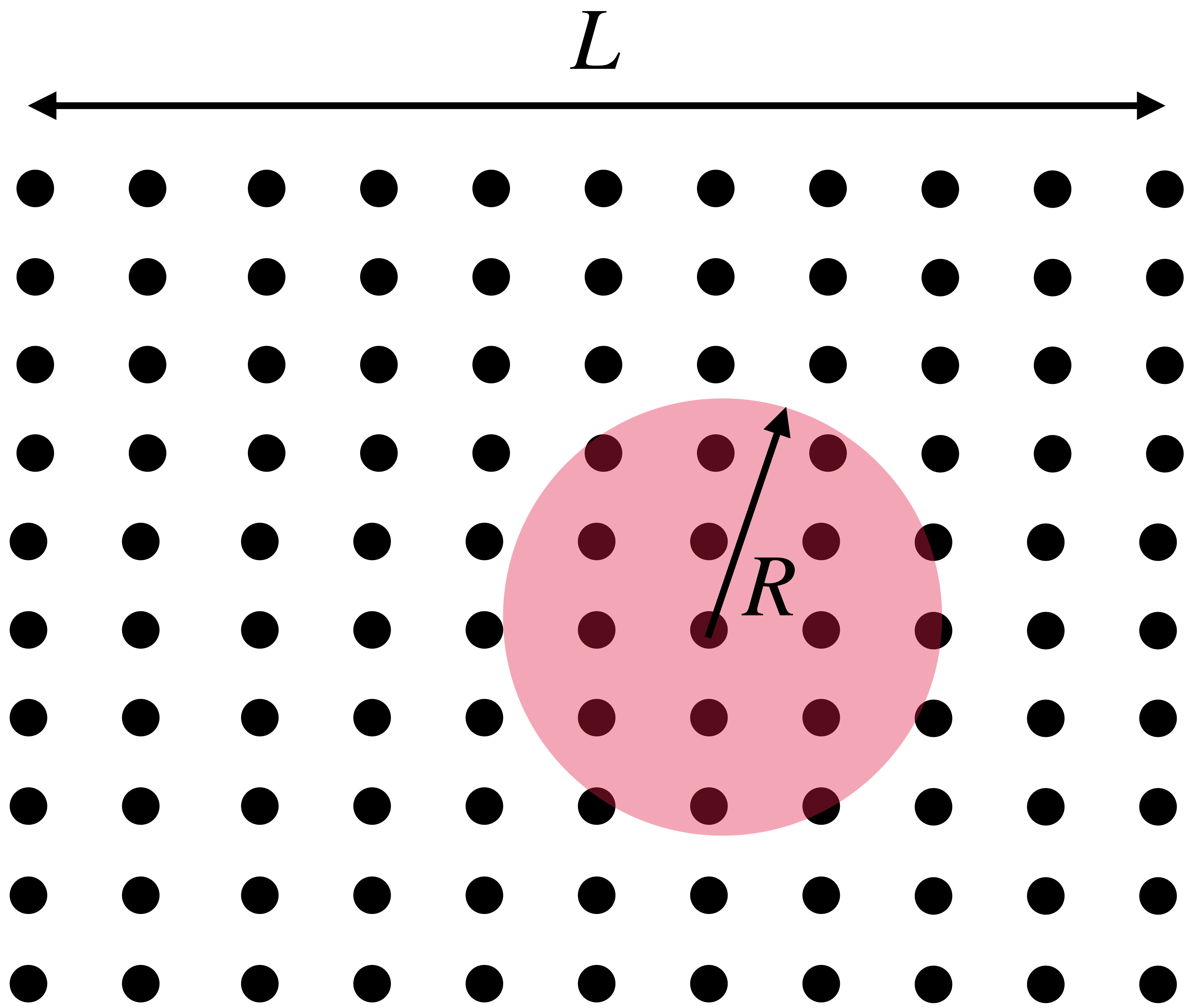
$$\rho_0 \cong \rho_1 \qquad \mathcal{T} e^{\int_0^1 \mathcal{L}_{0\to1}(s)ds}\rho_0 \approx \rho_1$$

And...

$$\mathcal{T} e^{\int_{s=0}^1 \mathcal{L}_{1\to0}(s)ds}\rho_1 \approx \rho_0$$

Reversibility is non-trivial!

$$\mathcal{L}(s) = \sum_x \mathcal{L}_x(s) \qquad x \in \Lambda$$



$$\mathrm{supp}[\mathcal{L}_x(s)] = \mathrm{polylog}(L)$$

$$\|\mathcal{L}_x(s)\| = \mathrm{polylog}(L)$$

$$A \approx B \Leftrightarrow \|A - B\|_1 < \varepsilon$$

cmi in quantum many-body systems

$\rho \equiv \rho_{ABC}$ $\qquad$ $S(A) = -\operatorname{tr}(\rho_A \log \rho_A)$

$$I_\rho(A : C | B) = I_\rho(A : BC) - I_\rho(A : B)$$

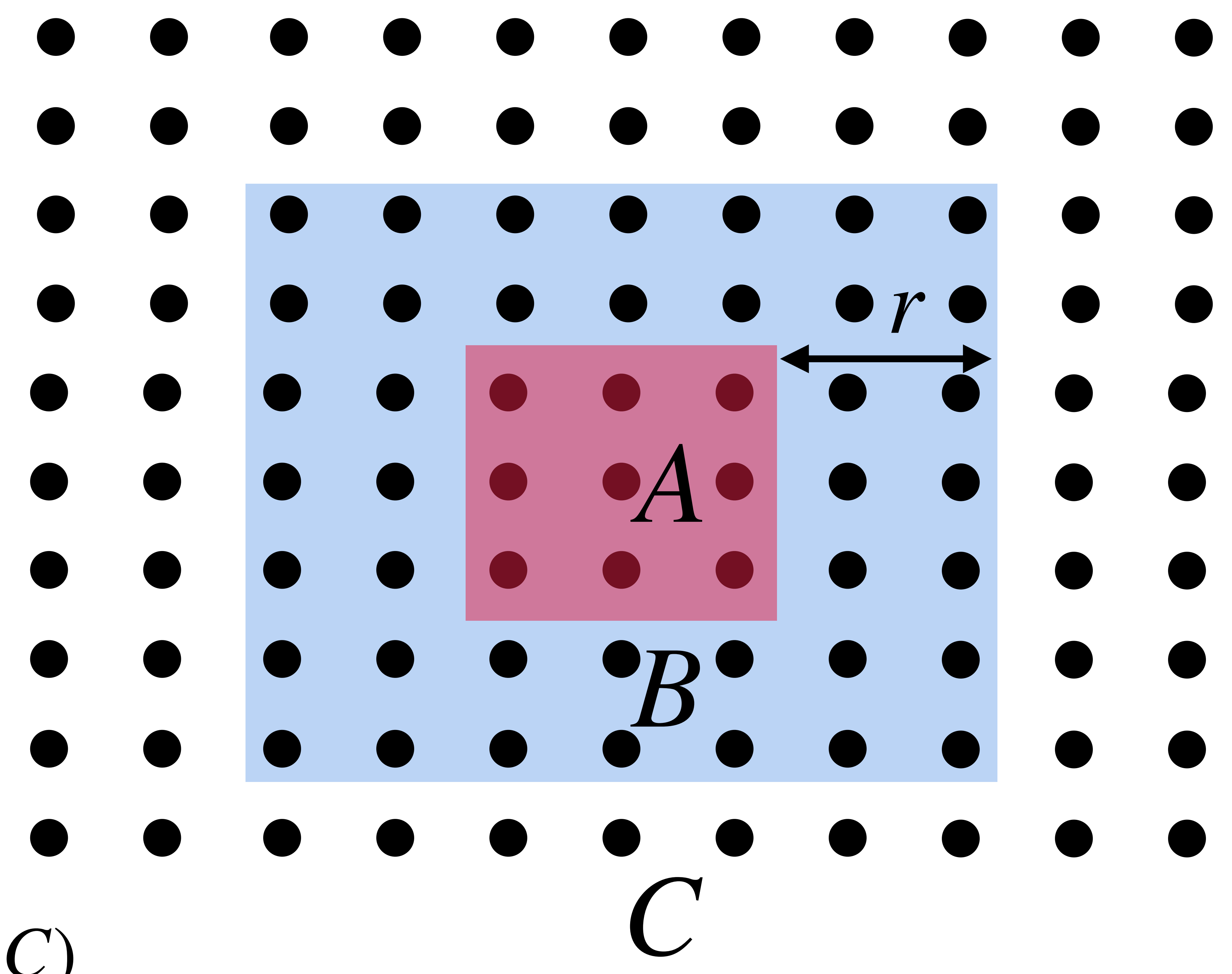
$$I_\rho(A : C | B) = S(AB) + S(BC) - S(B) - S(ABC) \geq 0 \text{ by SSA}$$

$I_\rho(A : C|B) \approx 0 \implies$ correlations b/w A and C mediated by B

$$\boxed{\begin{array}{c} I_\rho(A : C | B) \leq \text{poly}(|A|, |C|)\ e^{-r/\xi} \\ r = \text{dist}(A, C) \end{array}}$$

Markov length $\simeq \xi$

ρ is ξ – FML $\qquad$ $\xi = O(1)$



$\psi = |\psi\rangle\langle\psi| \implies I_\psi(A : C | B) = S(C) + S(A) - S(AC) = I_\psi(A : C)$

cmi and recovery

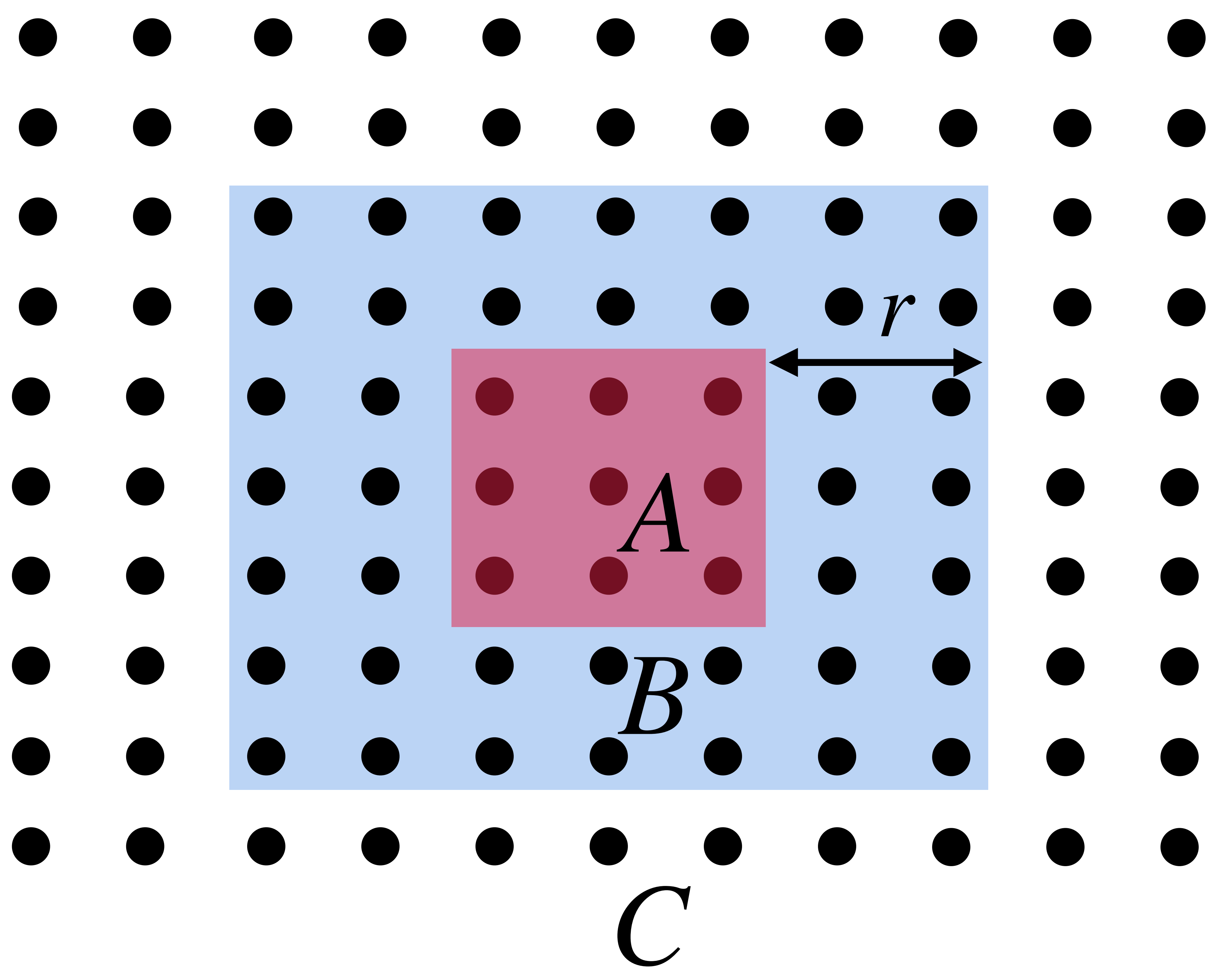
$$I_\rho(A:C|B) = I_\rho(A:BC) - I_\rho(A:B)$$

$$\rho'_{ABC} = \mathscr{E}(\rho_{ABC}) \qquad \text{supp}(\mathscr{E}) = A$$

Lemma 1

$$\exists \mathscr{R},\ \text{supp}(\mathscr{R}) = A \cup B$$

$$\|\mathscr{R} \circ \mathscr{E}(\rho_{ABC}) - \rho_{ABC}\|_1^2 \leq 2\log 2 \cdot I_\rho(A:C|B)$$



$$D(\rho||\sigma) - D(\mathscr{E}(\rho)||\mathscr{E}(\sigma)) \geq 0$$

$$D(\rho||\sigma) - D(\mathscr{E}(\rho)||\mathscr{E}(\sigma)) \geq -2\log F(\rho, (\mathscr{R}_{\mathscr{E},\rho} \circ \mathscr{E})[\rho])$$

$$\rho = \rho_{ABC},\ \sigma = \rho_{AB} \otimes \rho_C \qquad 1 - F(\rho,\sigma) \geq \frac{1}{4}\|\rho - \sigma\|_1^2$$

$$\text{RHS} \leq I_\rho(A:C|B) \qquad \text{LHS} \geq \frac{1}{2\log 2}\|\mathscr{R} \circ \mathscr{E}[\rho] - \rho\|_1^2$$

∎

main result of [1]

Lemma 2

local $\mathscr{L}_{0\to1}(t)$ $\quad \mathscr{G}_s = \mathscr{T} e^{\int_0^s \mathscr{L}_{0\to1}(t)dt}$

$$\rho_s = \mathscr{G}_s \rho_0$$

ρ_s is ξ – FML for $\xi = O(1)\ \forall s \in [0,1]$

$\exists$quasilocal-$\tilde{\mathscr{G}}$ $\quad \|\rho_0 - \tilde{\mathscr{G}}(\rho_1)\|_1 \leq \varepsilon$

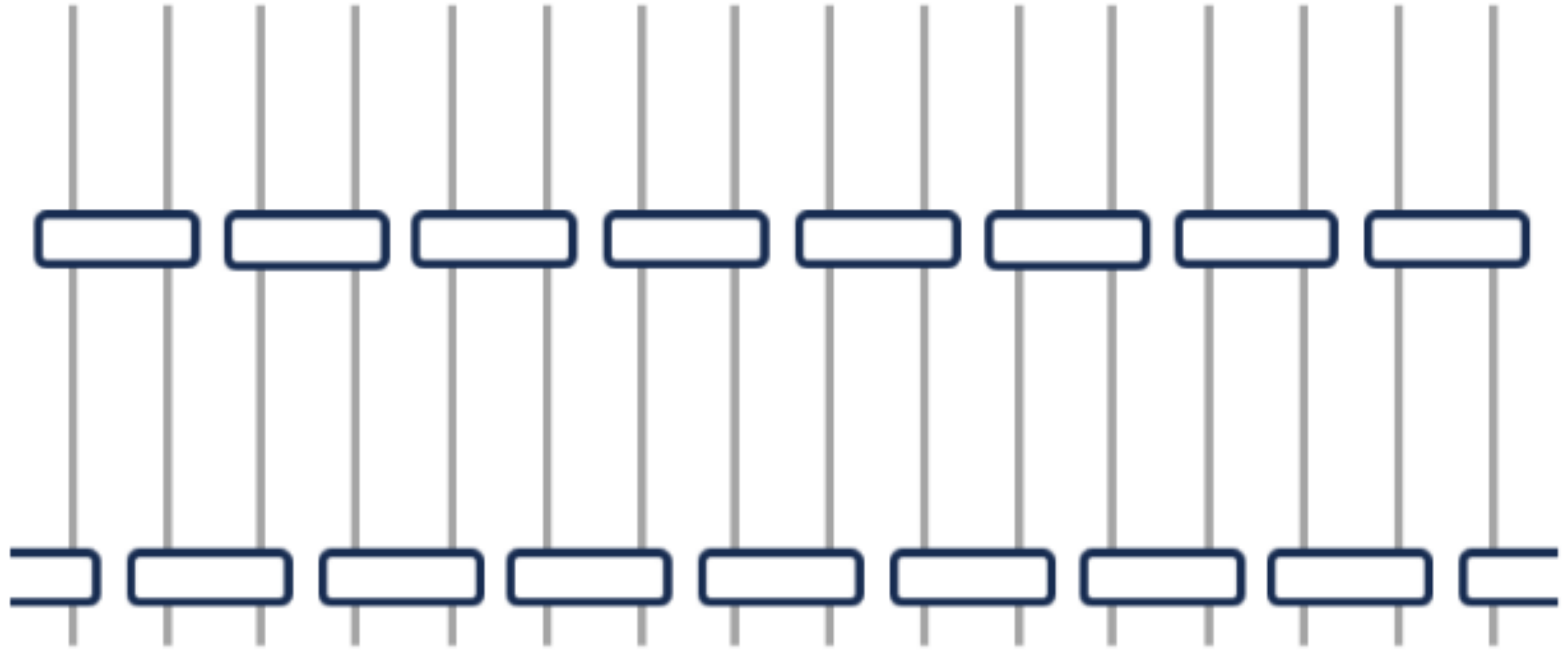
$\forall \varepsilon > 0$

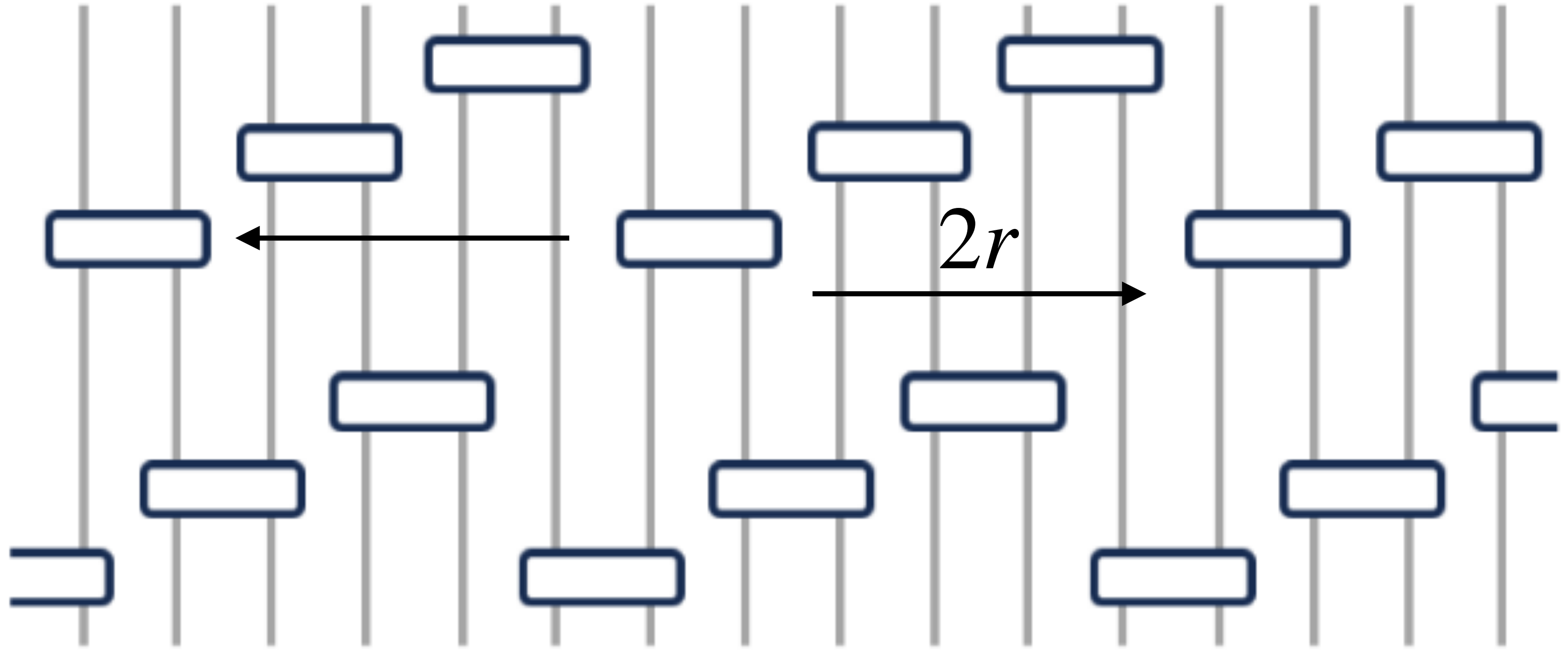
proof by picture

$\mathcal{G}$

depth overhead $O(r^D)$

$\mathcal{G}$ (reorganized)

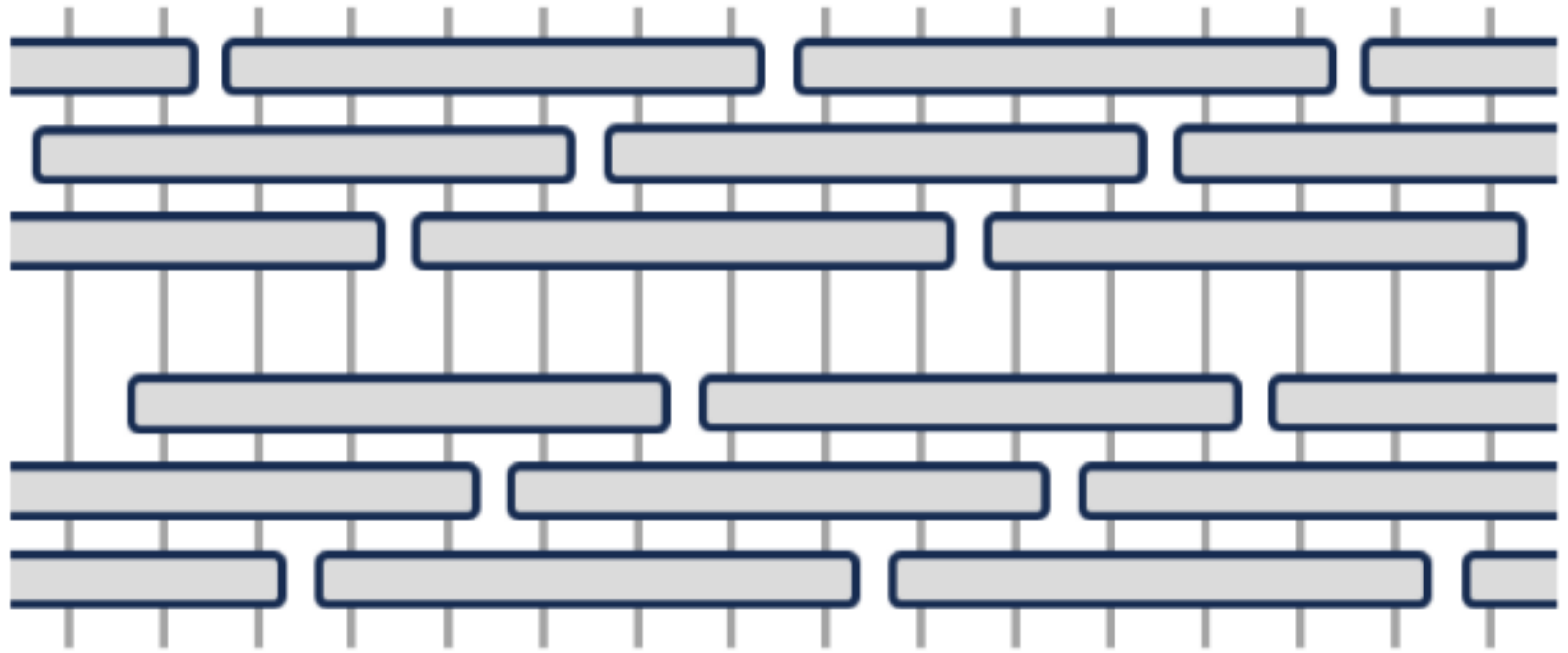




replace with Petz recovery

$\widetilde{\mathcal{G}}$

recovery only depends on local RDMs



$$\tilde{\mathscr{L}}_{1\to 0}(\tau) = \sum_x \tilde{\mathscr{L}}_x(\tau)$$

$$|\,\mathrm{supp}(\tilde{\mathscr{L}}_x(\tau))\,| = O(\mathrm{polylog}(L))$$

$$\varepsilon = \|\mathscr{R} \circ \mathscr{E}(\rho_{ABC}) - \rho_{ABC}\|_1 \leq \sum_{l,x} \varepsilon_{lx}$$

layer l, site x

$$\varepsilon_{lx} = \mathrm{poly}(L) e^{-r/2\xi}$$

$$\implies r \geq \xi \log\left(\frac{\mathrm{poly}(L)}{\varepsilon t}\right)$$

ξ controls locality of time-reversal

toric code example

$$\hat{H} = -\left(\sum_v A_v + \sum_p B_p\right)$$

$$A_v = \prod_{i \in v} Z_i \qquad B_p = \prod_{i \in p} X_p \qquad [A_v, B_p] = 0$$

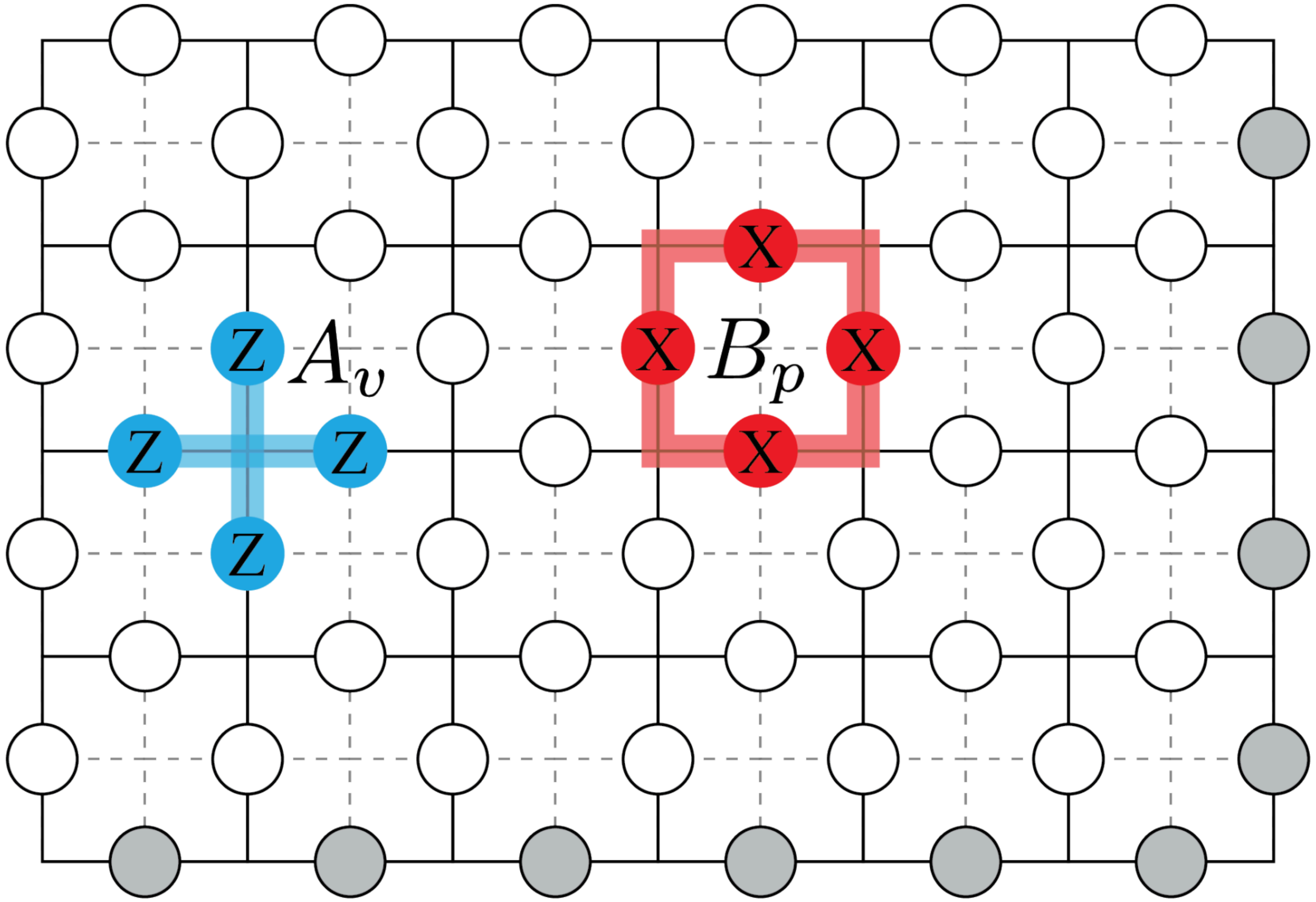
$$\mathcal{E}_p[\rho] = (1-p)\rho + pZ\rho Z$$

$$\rho_0 = |\psi\rangle\langle\psi|$$

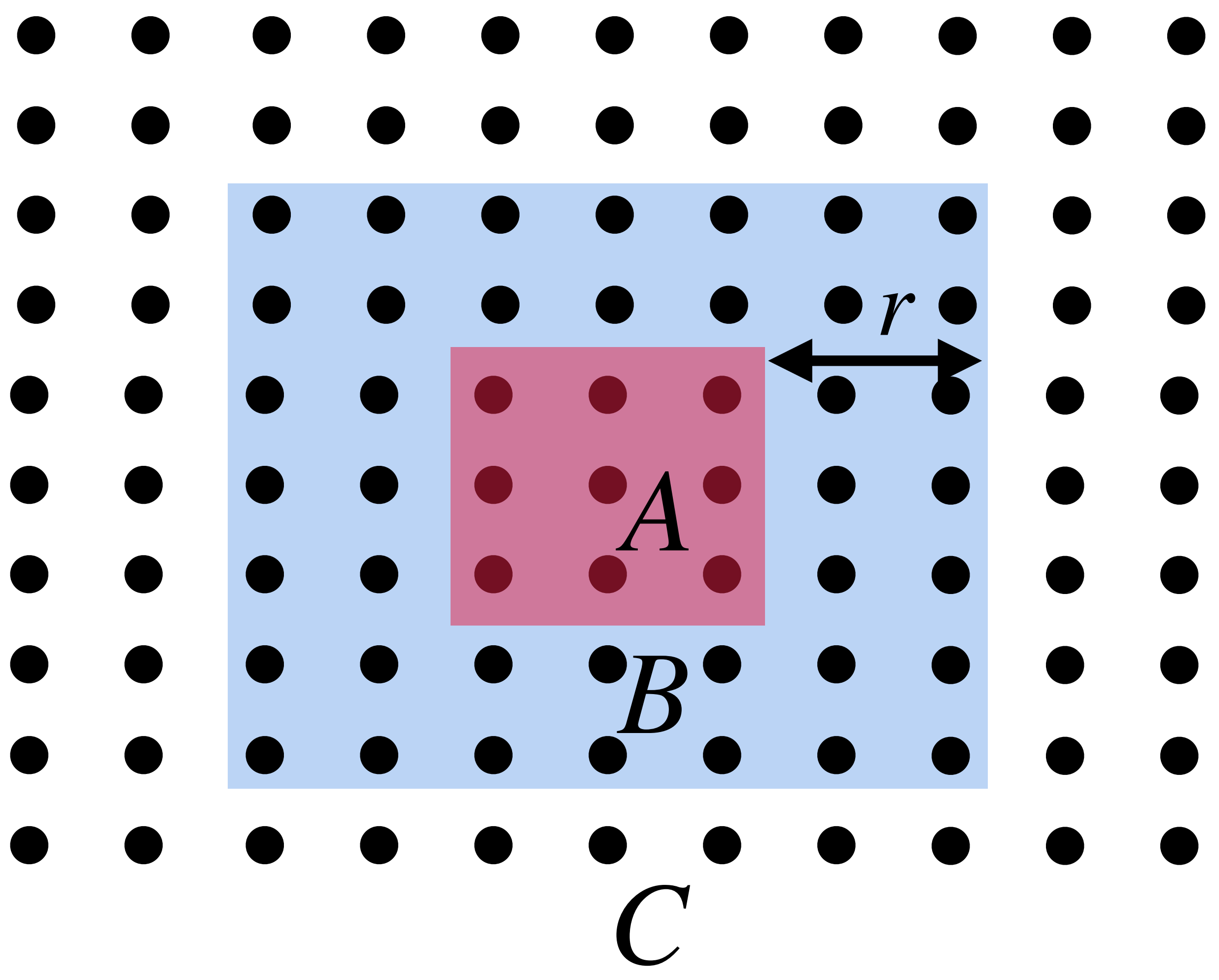
$$\rho_p := \mathcal{E}_p^{\otimes N}[\rho_0]$$

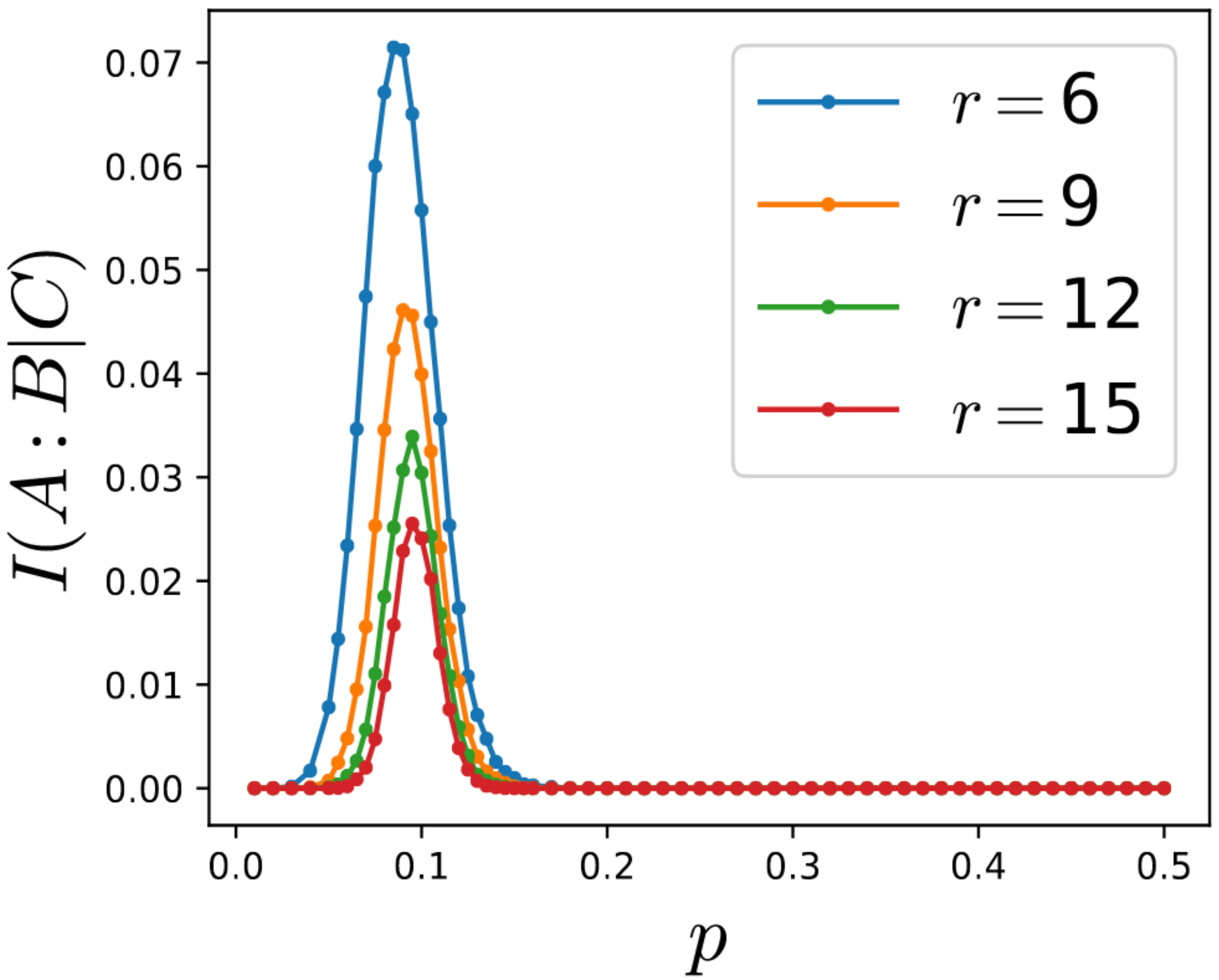
$\xi = O(1)$ $\xi = O(1)$

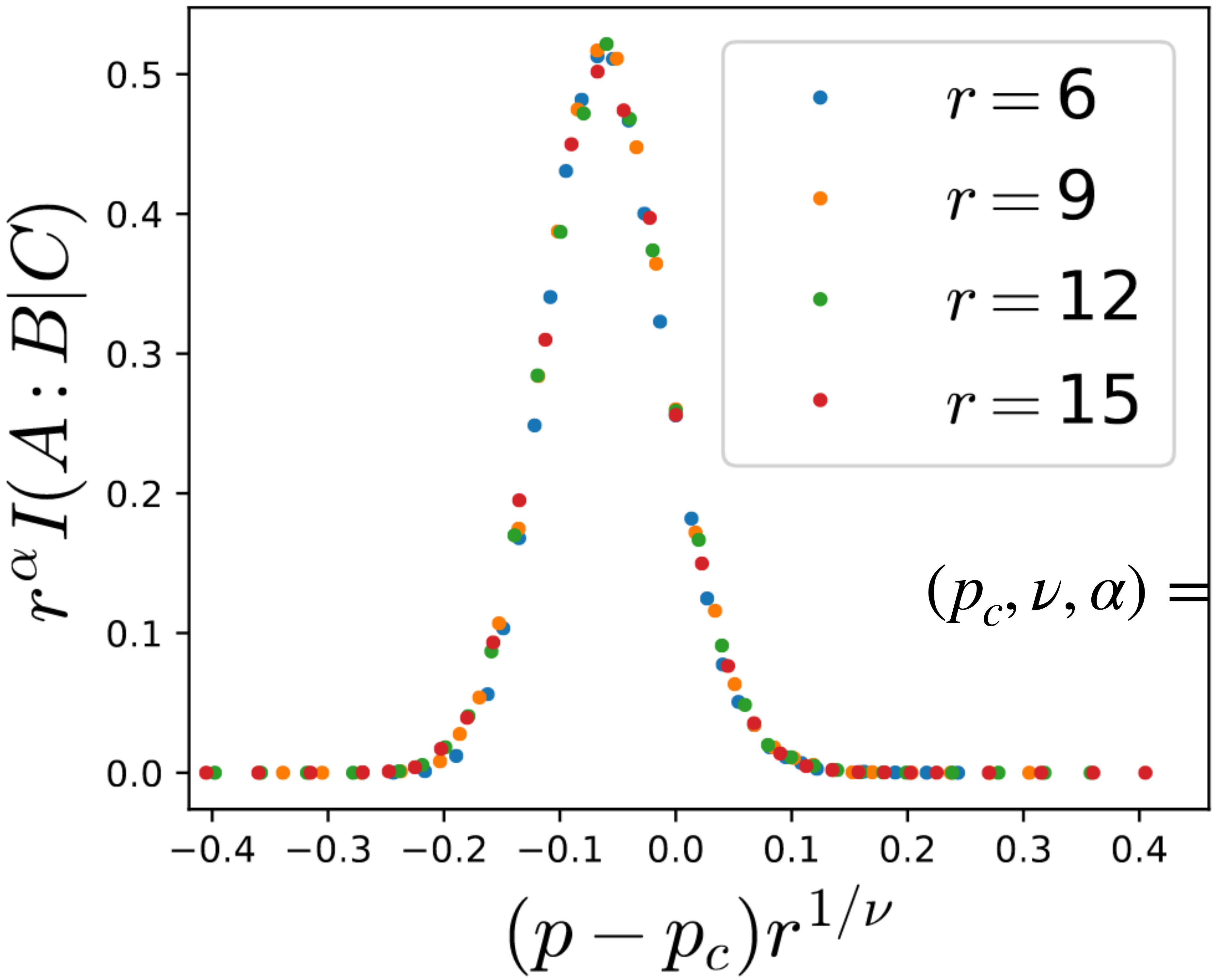
$p = 0$ $p = p_c$ $p = 0.5$



numerics

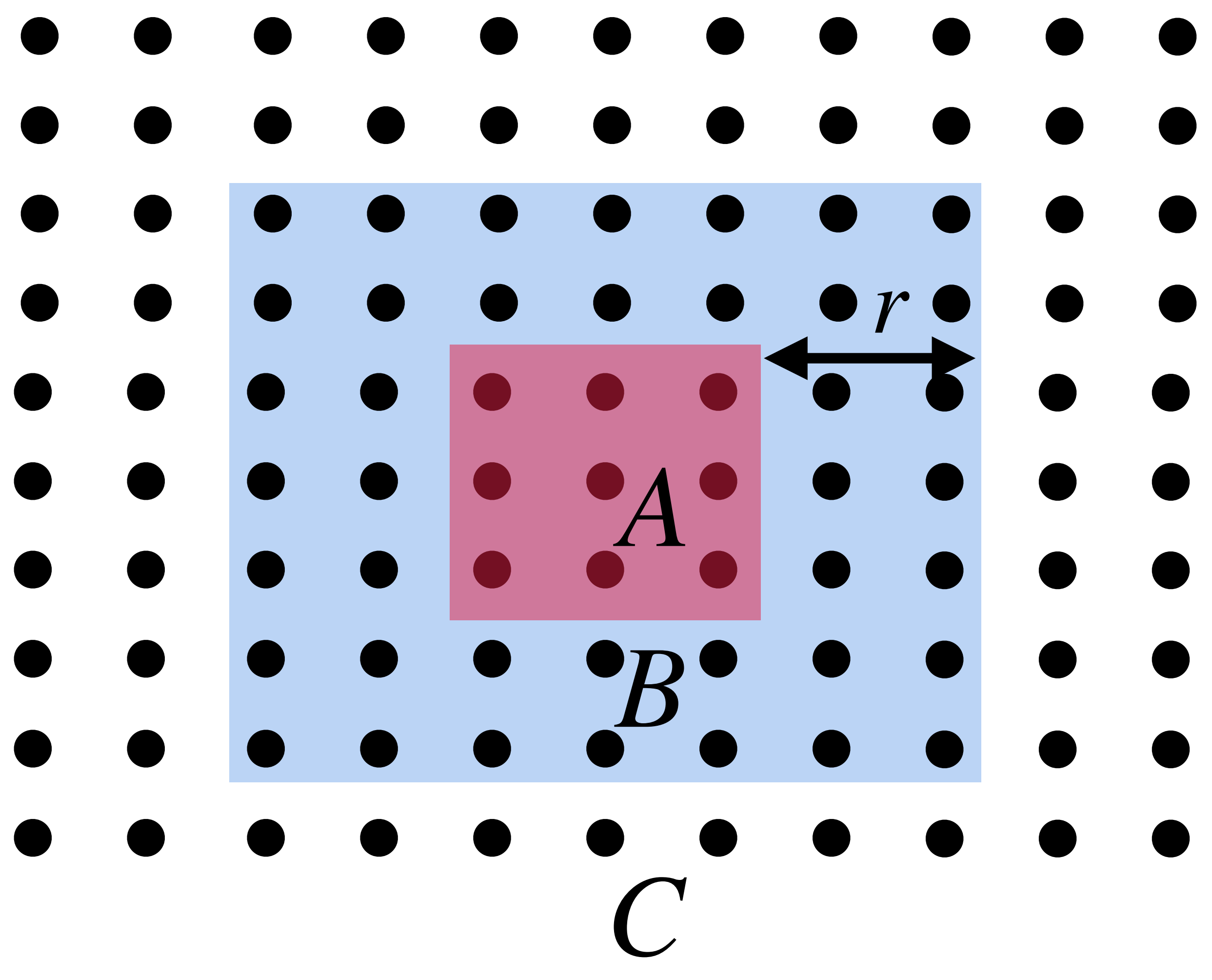


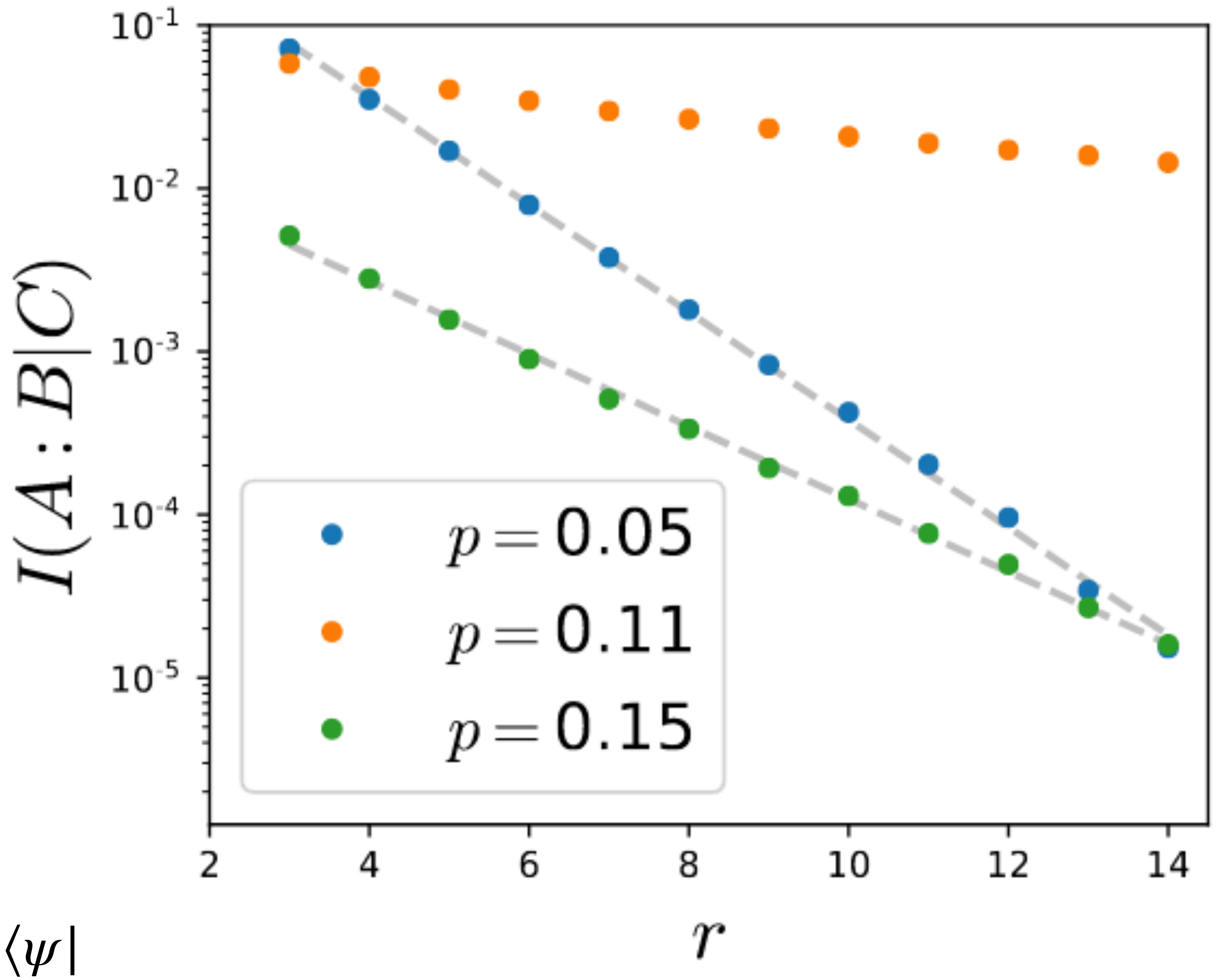




$$(p_c, \nu, \alpha) = (0.11, 1.8, 1.1)\,.$$

numerics





$\Longrightarrow$ can use this to decode back to the logical $\rho_0 = |\psi\rangle\langle\psi|$

everything remains quasilocal

summarising...

- mixed state phases are a thing, presumably
- phenomenology: markov length ξ stays finite within a phase, diverges at a transition
- markov length takes the role of a gap for mixed states
- $\rho_1 \cong \rho_2 \Leftrightarrow \exists$FML-$\xi$ $\rho_1 \to \rho_2 \Leftrightarrow \exists$FML-$\xi$ $\rho_2 \to \rho_1$
- furnishes quasi-local decoders as a byproduct