MA 106 D1-T3
Recap-1

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  - Multiply the i^{th} row by $c \neq 0$.

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- No. of pivots $\leq$ min(no. of rows, no. of columns)
- Not unique.
- Can be *reduced*.

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Theorem

Let A be a $n \times n$ square matrix. There exist ERM's E_i such that

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Definition

Let $A \in \mathbb{R}^{n \times n}$. It is said to be invertible if there exists $B \in \mathbb{R}^{n \times n}$ such that

$$AB = BA = I$$

where I is the $n \times n$ identity matrix. Such a B is unique and is denoted as A^{-1}

Gauss Jordan

Theorem

A square matrix A is invertible **iff** it is a product of ERM's.

Vector Subspaces

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Questions:

- Is $\mathbb{Q}^2$ a subspace of $\mathbb{R}^2$?
- Is R^2 a subspace of $\mathbb{R}^3$?

To do: Categorize the subspaces of $\mathbb{R}^2$ and $\mathbb{R}^3$

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- Redundancy?
- Linear Independence

Fundamental Lemma

Lemma

Let V be a vector subspace of $\mathbb{R}^n$ generated by k vectors. Any set of l vectors such that $l \geq k+1$ is linearly dependent.

Basis and Dimension

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Definition

Let V be a vector subspace of $\mathbb{R}^n$. A subset S of V is called a basis of V if

- S is linearly independent.
- $Span(S) = V$

Theorem

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1. V has a basis.
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(Revisiting the) Question: Show that $v_1, v_2, v_3 \in \mathbb{R}^3$ being linearly independent is equivalent to $det(\begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}) \neq 0$

Rank and all that

Definitions

Let $A \in \mathbb{R}^{m \times n}$. In terms of its columns and rows, A can be written as

$$\begin{bmatrix} c_1 & c_2 & \dots & c_n \end{bmatrix} \text{ or } \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_m \end{bmatrix}$$

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Kronecker-Capelli Theorem

Theorem

A system of linear equations

$$Ax = b$$

has a solution iff

$$Rank(A) = Rank([A|b])$$

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Rank Nullity Theorem

Given $A \in \mathbb{R}^{m \times n}$.

$$R(A) + N(A) = n$$

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Fact:

$$det\ rank(A) = rank(A)$$