MA 109 D2 T1
Week One Recap

Siddhant Midha
https://siddhant-midha.github.io/

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Sets

Definition (Set)

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Some notation.

- $\mathbb{N}$: The set of natural numbers.
- $\mathbb{Z}$: The set of real numbers.
- If a set S contains some element a, we write $a \in S$.
- To refer to all the elements in the set S, we use $\forall s \in S$.
- 'There exists s in S': $\exists s \in S$.
- $\mathbb{Q}$: The set of rational numbers (numbers of the form p/q for $p, q \in \mathbb{Z}$).
- $\mathbb{R}$: The set of real numbers.

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The cardinality of a finite set S, denoted as $|S|$, is defined as

1. $|S| = 0$ if $S = \emptyset$.
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Can we talk about cardinality of infinite sets?

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Similarly, the Greatest Lower Bound (GLB) is defined. More commonly, we refer to LUB as the supremum, and the GLB as the infimum.

$\mathbb{Q}$ and $\mathbb{R}$

- Consider, $\{\frac{1}{1}, \frac{1}{1+\frac{1}{1+1}}, \frac{1}{1+\frac{1}{1+\frac{1}{1+1}}}, \ldots\} \cap (0, \frac{\sqrt{5}-1}{2}) \subset \mathbb{Q}$. Upper bounded? Supremum exists?

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A sequence which does not converge is said to diverge, or be non-convergent.

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then f_n converges, and the limit is $f_0 = a_0 = b_0$.

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We use monotonic and eventually monotonic synonymously.

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Where we know that the supremum exists due to the completeness of $\mathbb{R}$. Does the converse of the MCT hold? No. Take $a_n := (-1)^n/n$.