The Thomae Function

$T : [0,1] \to \mathbb{R}$

$$T(x) := \begin{cases} \frac{1}{q} & x = \frac{p}{q} \quad p, q \in \mathbb{N} \\ 0 & \text{o/w} \end{cases}$$

Let $r \in [0,1] \cap \mathbb{Q}$.

Let (x_n) be a sequence of irrationals such that (why can we find this?) $x_n \to r$

But, $f(x_n) = 0 \quad \forall n \in \mathbb{N} \Rightarrow f(x_n)$ does NOT converge to $f(r)$

$\therefore$ discontinuous at rationals

Let $i \in [0,1] \cap \mathbb{Q}^c$

Let $S_\varepsilon := \{x \mid f(x) \geq \varepsilon,\ x \in [0,1]\}$

Let $|S| = L < \infty$

how? Let $d := \min\{|x - i| \mid x \in S_\varepsilon\}$ choose $\delta := \frac{d}{2}$

Since S is finite, we can choose $\delta > 0$ such that $N_\delta(i) \cap S_\varepsilon = \emptyset$

where, $N_\delta(i) := (i - \delta, i + \delta)$

Let $x \in N_\delta(i)$
- $x \in \mathbb{Q} \Rightarrow f(x) < \varepsilon$ (why?)
- $x \notin \mathbb{Q} \Rightarrow f(x) = 0 < \varepsilon$

$\Rightarrow |f(x) - f(i)| < \varepsilon$

$\therefore 0 < |x - i| < \delta \Rightarrow |f(x) - f(i)| < \varepsilon$

$\therefore$ continuous at irrationals

Thus, T is continuous at irrationals & discontinuous at rationals

Now, let, $P = \{0 = x_0 < x_1 < \cdots < x_n = 1\}$ s.t. $\|P\| < \left(\frac{\varepsilon}{L}\right)$

Clearly $L(f, P) = 0$ (why?)

$$U(f, P) = \sum_i M_i \Delta_i$$

$$= \sum_{i : [x_i, x_{i+1}] \cap S_\varepsilon = \emptyset} M_i \Delta_i + \sum_{i : [x_i, x_{i+1}] \cap S_\varepsilon \neq \emptyset} M_i \Delta_i$$

$$= (S_1 + S_2)$$

$\underline{S_1}$ $\quad M_i < \varepsilon$!

$$\Rightarrow S_1 < \varepsilon \sum_{[\cdots]} \Delta_i \leq \varepsilon$$

$\underline{S_2}$: Again, atmost $2L$ intervals will cover the points of summation.

further, $M_i \leq 1$

$$\Rightarrow \quad S_2 \leq 1 \times (2L) \times \frac{\varepsilon}{L}$$

$$\Rightarrow \quad S_2 \leq 2\varepsilon$$

$\therefore \quad U(f,P) \leq 3\varepsilon$

$\Rightarrow \quad U(f,P) - L(f,P) \leq 3\varepsilon$

Done? Yes. Why? Justify!

T is integrable on $[0,1]$.