1\. (a). given $\lim_{x\to 5} \frac{f(x)}{x^3} = 5$. ① [A]

Then, $\lim_{x\to 5} \frac{f(x)}{x} = \lim_{x\to 5}\left(\frac{f(x)}{x^3}\cdot x^2\right)$

$= \lim_{x\to 5}\frac{f(x)}{x^3}\cdot \lim_{x\to 5} x^2$

$= 5\cdot 25 = \boxed{125}$

(b) Let, $\alpha, \beta \in \mathbb{R}$ with $\beta > 0$.

$$f(x) = \begin{cases} x^\alpha \sin\frac{1}{x^\beta} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

f is diff. on $\mathbb{R}\setminus\{0\}$.

f is cont. at $0 \Leftrightarrow \alpha > 0$.

Now, $\lim_{h\to 0}\frac{f(h)-f(0)}{h} = \lim_{h\to 0} h^{\alpha-1}\sin\frac{1}{h^\beta}$

$= 0 \Leftrightarrow \alpha - 1 > 0$

$\Leftrightarrow \alpha > 1$.

$\therefore f$ is diff. on $\mathbb{R} \Leftrightarrow \boxed{\alpha > 1}$

$$\therefore f'(x) = \begin{cases} \alpha x^{\alpha-1}\sin\frac{1}{x^\beta} - \beta\cdot x^{\alpha-\beta-1}\cos\frac{1}{x^\beta} & x \neq 0 \\ 0 & x = 0. \end{cases}$$

f' is cont. on $\mathbb{R} \Leftrightarrow \alpha - 1 > 0$ & $\alpha - \beta - 1 > 0$

$\Leftrightarrow \boxed{\alpha > \beta + 1}$.

(c). $f:[0,1]\to\mathbb{R}$ diff. on $[0,1]$. [A] (2)

(3)

& $f\left(\frac{1}{n}\right)=0$.

$\because f$ is cont. at $0 \Rightarrow f(0)=\lim_{n\to\infty} f\left(\frac{1}{n}\right)=0$.

Now, since f is diff. at 0

$$f'(0)=\lim_{n\to\infty}\frac{f\left(\frac{1}{n}\right)-f(0)}{\frac{1}{n}}=\lim_{n\to\infty}\frac{0}{\frac{1}{n}}=\boxed{0}.$$

(d)

$$x^{109}-e^{-x}-2022=0.$$

Let $f(x)=x^{109}-e^{-x}-2022$.

Observe that f is continuous on $\mathbb{R}$.

Also, $f(x)\longrightarrow+\infty$ as $x\to+\infty$

and $f(x)\longrightarrow-\infty$ as $x\longrightarrow-\infty$.

$\therefore$ By IVP, $\exists\, x_0\in\mathbb{R}$ s.t. $f(x_0)=0$.

Let $x_1\neq x_2\in\mathbb{R}$ s.t. $f(x_1)=f(x_2)=0$.

Then, by Rolle's theorem, $\exists\, c\in(x_1,x_2)$ such that

$$f'(c)=0$$

i.e., $109\cdot c^{108}+e^{-c}=0$ — which is not possible.

Hence, [A] ③

exact number of real roots of $x^{109} - e^{-x} - 2022 = 0$ is

$\boxed{1}$.

(e):

$f : \mathbb{R} \to \mathbb{R}$ continuous on $\mathbb{R}$ with $f(0) = 1$.

Also $f(x) = f(x^2) \quad \forall x \in \mathbb{R}$.

Then, $f\left(\frac{1}{\sqrt{2}}\right) = f\left(\frac{1}{2}\right) = f\left(\frac{1}{2^2}\right) = f\left(\frac{1}{2^4}\right) = \cdots$

$= f\left(\frac{1}{2^n}\right) = \cdots$

$\therefore f\left(\frac{1}{\sqrt{2}}\right) = f\left(\frac{1}{2^n}\right) \quad \forall n \in \mathbb{N}$.

$\therefore f\left(\frac{1}{\sqrt{2}}\right) = \lim_{n\to\infty} f\left(\frac{1}{2^n}\right)$

$= f\left(\lim_{n\to\infty} \frac{1}{2^n}\right) \quad (\because f \text{ is cont on } \mathbb{R})$

$= f(0)$

$\boxed{= 1}$

Q.

(a) Let $x_n = (-1)^n$, then $x_n^2 + \frac{x_n}{n}$ converges but x_n do not converge.

(b) Let $x_n = (-1)^n$, $\{x_n^2\}$ converges but $\{x_n\}$ do not converge.

[A] (4) (5)

(c) $x_n^2 \longrightarrow 0$. Let $\epsilon > 0$. Then $\exists\, N \in \mathbb{N}$ s.t.

$$|x_n^2 - 0| < \epsilon^2 \quad \forall n \geq N.$$

$$\Rightarrow |x_n| < \epsilon \quad \forall n \geq N$$

$$\Rightarrow x_n \longrightarrow 0.$$

(d). $\left\{x_n + \frac{x_n}{n}\right\}_n$ converges

$x_n + \frac{x_n}{n} = x_n\left(1+\frac{1}{n}\right)$ — converges.

Also, $\left\{1+\frac{1}{n}\right\}_n$ — converges.

$\therefore \dfrac{x_n\left(1+\frac{1}{n}\right)}{1+\frac{1}{n}} = x_n$ — converges.

[1]

3. Let $\{a_n\}_n$ be a bounded sequence in $\mathbb{R}$.

For each $n \in \mathbb{N}$, we define $x_n = \sup\{a_k : k > n\}$.

Let, $n_1 < n_2$. Then,

$$\{a_k : k > n_1\} \supseteq \{a_k : k > n_2\}$$

$$\therefore \sup\{a_k : k > n_1\} \geq \sup\{a_k : k > n_2\}$$

i.e, $x_{n_1} \geq x_{n_2}$.

That is $\{x_n\}$ is a decreasing seq. —— [1]

$\because$ $\{a_n\}_n$ is bounded, so, $\{x_n\}_n$ is bounded below. —— [1]

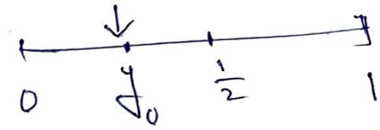
Hence, $\{x_n\}_n$ converges. —— [1]

7. Let $f:[0,1]\to\mathbb{R}$ cont. on $[0,1]$ & diff. on $(0,1)$. [A] 6

Given, $f(0)=f(1)=0$ and $\exists\, y_0 \neq \frac{1}{2}$ such that

$f(y_0)=1$.

Case 1: Let $0<y_0<\frac{1}{2}$:

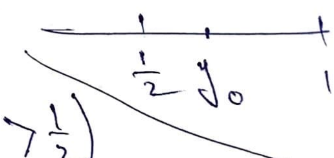


Apply MVT to f on $[0,y_0]$.

Then,
$$\frac{f(y_0)-f(0)}{y_0-0}=f'(\delta) \quad \text{for some } 0<\delta<y_0 \qquad \text{— [1]}$$

i.e.,
$$\frac{1}{y_0}=f'(\delta).$$

$$\therefore\ |f'(\delta)|=\frac{1}{|y_0|}>2. \qquad \text{— [1]}$$



Case 2:

$\frac{1}{2}<y_0<1$: $\left(\because\ 1-y_0>\frac{1}{2}\right)$

Apply MVT to f on $[y_0,1]$.

$$\therefore\ \frac{f(1)-f(y_0)}{1-y_0}=f'(\delta) \quad \text{for some } y_0<\delta<1. \qquad \text{— [1]}$$

i.e.,
$$\frac{-1}{1-y_0}=f'(\delta).$$

$$\therefore\ |f'(\delta)|=\frac{1}{1-y_0}>2 \qquad \text{— [1]}$$

1. (a) 27 ⑦ [B]

(b) $\alpha > 1$, $\alpha > 2\beta + 1$.

(c) $f'(0) = 1$.

(d) 2.

(e) 2

2.
(a)
~~(b)~~
~~(c)~~
(d)

3.
$\{x_n\}_n$ is increasing [1]

$\{x_n\}$ is bounded / bounded above [1]

Hence convergent. [1].