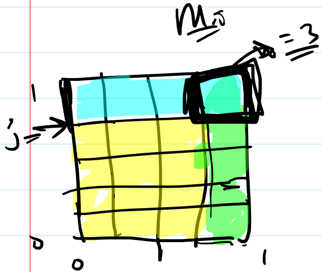
Q1 (a)

$R \subseteq \mathbb{R}^2$ (m.g)

(a) Let $R := [0,1] \times [0,1]$ and $f(x,y) := [x] + [y] + 1$ for all $(x,y) \in R$, where $[u]$ is the greatest integer less than equal to u, for any $u \in \mathbb{R}$. Using the definition of integration over rectangles, show that f is integrable over R. Also, find its value.



$$f(x,y) = \begin{cases} 1 & x \in [0,1) \text{ and } y \in [0,1) \\ 2 & x=1\ y \in [0,1) \text{ or } y=1\ x \in [0,1) \\ 3 & x=1\ y=1 \end{cases}$$

$m_{ij} = 1 \ \forall i,j$

M_{ij}

$$P_n = \left\{ \left(\frac{i}{n}, \frac{j}{n}\right) \,\middle|\, i,j = 0, \dots n \right\}$$

$$\alpha(f, P_n) = (1) \Rightarrow \alpha(1) = 1 \qquad \iint_R f = 1$$

$$U(f, P_n) = \sum_{i=0}^{n-2} \sum_{j=0}^{n-2} (1)\left(\frac{1}{n}\right)^2 + \left[\sum_{j=n-1}^{n-1} \sum_{i=0}^{n-1} (2)\left(\frac{1}{n}\right)^2\right] \times 2$$

$$= \frac{(n-1)^2}{n^2} + \frac{2}{n^2} \times 1 \times n \times 2 - \frac{1}{n^2} \left(\frac{1}{n}\right)^2$$

$$= \frac{n^2 - 2n + 1}{n^2} + \frac{4}{n} - \frac{1}{n^2}$$

$$= 1 - \frac{2}{n} + \frac{1}{n^2} + \frac{4}{n} - \frac{1}{n^2}$$

$$= 1 + \frac{2}{n}$$

$$U(f, P_n) - \alpha(f, P_n) = \left(\frac{2}{n}\right)$$

$\frac{2}{n} < \varepsilon$

$n > \left(\frac{2}{\varepsilon}\right)$ $\quad n = \left\lfloor \frac{2}{\varepsilon} \right\rfloor + 1$

given $\varepsilon > 0$, choose $n = \left\lfloor \frac{2}{\varepsilon} \right\rfloor + 1$

$$|U(f, P_n) - \alpha(f, P_n)| < \varepsilon$$

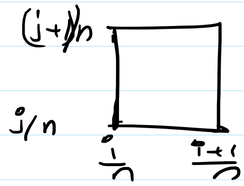
P_n

By RC, we are done.

Q1 (b)

(b) Let $R := [0,1]\times[0,1]$ and $f(x,y) := (x+y)^2$ for all $(x,y)\in R$. Show that f is integrable over R and find its value using Riemann sum.

$$P = \left\{ \left(\frac{i}{n}, \frac{j}{n}\right) \,\middle|\, i,j = 0,1,\dots,n \right\}$$



$$m_{ij} = \left(\frac{i+j}{n}\right)^2$$

$$M_{ij} = \left(\frac{(i+1)+(j+1)}{n}\right)^2$$

$$U(f,P_n) - L(f,P_n) = \sum_i \sum_j (M_{ij} - m_{ij})\,\Delta_{ij}$$

$$= \left(\frac{1}{n^2}\right) \sum_i \sum_j \left(\frac{1+1}{n}\right)\left(\frac{\cancel{2i+1+2j+1}}{n}\right)$$

$$= \left(\frac{2\times 2}{n^4}\right) \sum_i \sum_j \left(\frac{i+j+1}{1}\right)$$

$$= \frac{4}{n^2} \sum_i \sum_j (i+j+1)$$

$$= \frac{4}{n^2}\left((n^2) + \left[\left(\frac{n(n-1)}{2}\right) n \times 2\right]\right)$$

$\left(\sum_0^{n-1}\right)$ $\frac{(n-1)(n-1+1)}{2}$ $\frac{n(n-1)}{2}$

$$= 4\left[\frac{1}{n^2} + \frac{n-1}{n^2}\right] = \left(\frac{4}{n}\right)$$

$$n = \left\lfloor \frac{4}{\varepsilon} \right\rfloor + 1$$

By the RC, f is int on R.

$$t_{ij} = \left(\frac{i}{n}, \frac{j}{n}\right)$$

$$\sum_i \sum_j \left(\frac{1}{n^2}\right)\left(\frac{i+j}{n}\right)^2 = \frac{1}{n^4}\left(\sum_i \sum_j i^2 + j^2 + 2ij\right)$$

$$= \frac{1}{n^4}\left[\left(\frac{n(n-1)(2n-1)}{6}\right) n \times 2 + 2\times\left(\frac{n(n-1)}{2}\right)^2\right]$$

$$\lim_{n\to\infty} S(f, P_n, t) = \left(\frac{4}{6}\right) + \left(\frac{1}{2}\right)$$

$$= \frac{2}{3} + \frac{1}{2} = \left(\frac{7}{6}\right)$$

$$\therefore \iint_R f = \left(\frac{7}{6}\right)$$

Q1 (c)

(c) Let $R := [a,b] \times [c,d]$ be a rectangle in $\mathbb{R}^2$ and let $f : R \to \mathbb{R}$ be integrable. Show that $|f|$ is also integrable over R.

$$P = \{(x_i, y_j)\}$$

$$R_{ij} = [x_i, x_{i+1}] \times [y_j, y_{j+1}]$$

$(u,v), (x,y) \in R_{ij}$

① $\left[|f(x,y)| - |f(u,v)| \le |f(x,y) - f(u,v)| \right.$ M_{ij}, m_{ij}

② $\begin{cases} \to f(x,y) \le M_{ij} \\ \to m_{ij} \le f(u,v) \Rightarrow -f(u,v) \le -m_{ij} \end{cases}$

$$\to |f(x,y) - f(u,v)| \le (M_{ij} - m_{ij})$$

$\Rightarrow \left[|f(x,y)| - |f(u,v)| \le M_{ij} - m_{ij} \right]$

Take sup over (x,y) and inf over (u,v) wrt $|f|$

$M_{|f|ij} = \sup\{|f(x,y)|\ (x,y) \in R_{ij}\}$

$m_{|f|ij} =$

$\Downarrow$

$$(M_{|f|ij} - m_{|f|ij}) \le (M_{ij} - m_{ij})$$

$$\sum_i \sum_j \Delta_{ij}$$

$\Downarrow$

$[U(|f|, P) - \alpha(|f|, P) \le U(f, P) - \alpha(f, P)]$ — ①

$\forall \varepsilon > 0, \quad \exists \ P_\varepsilon$

$[U(f, P_\varepsilon) - \alpha(f, P_\varepsilon) < \varepsilon$ — ②$]$

$$\Rightarrow U(|f|, P_\varepsilon) - \alpha(|f|, P_\varepsilon) < \varepsilon.$$

Done !

f is RI $\Rightarrow$ $|f|$ is RI

Q1 (d)

(d) Check the integrability of the function f over $[0,1] \times [0,1]$;

$$f(x,y) := \begin{cases} 1 & \text{if both } x \text{ and } y \text{ are rational numbers,} \\ -1 & \text{otherwise.} \end{cases}$$

What do you conclude about the integrability of $|f|$?

$x \in \mathbb{Q}$ or $\mathbb{I}$ $\varepsilon > 0$

$x - \varepsilon$ x $x + \varepsilon$

$\mathbb{R}$

$S(f, P_n, t) \to S$

RI

$S(f,P) = -1$

$S(f,P) = +1$

t

rat. ir.

f is not int

(x_n, y_n)

(x, y)

$+1$ (x, y) -1

$x \in \mathbb{Q}$

$|f|$ is RI $\not\Rightarrow$ f is RI

Q3

3. Consider the function $f : [0,1] \times [0,1] \to \mathbb{R}$ defined as

$$f(x,y) = \begin{cases} 1 - 1/q & \text{if } x = p/q \quad \text{where } p, q \in \mathbb{N} \text{ are relatively prime and } y \text{ is rational,} \\ 1 & \text{otherwise.} \end{cases}$$

$M_{ij} = 1$

for any given $\varepsilon > 0$,

$S_\varepsilon := \{ x \mid 1 - f(x,y) > \varepsilon,\ x \in [0,1]\ y \in [0,1], y \in \mathbb{Q} \}$

$|S_\varepsilon|$ $\quad 1 - f(x,y) = 1 - \left(1 - \frac{1}{q}\right) = \left(\frac{1}{q}\right) > \varepsilon$

$\left(q < \frac{1}{\varepsilon} \right)$

Say that $|S_\varepsilon| = L$

Let $P = \{(x_i, y_j)\}$ such that $\|P\| < \left(\frac{\varepsilon}{L}\right)$

$U(f,P) - L(f,P)$

$= \sum_i \sum_j M_{ij} \Delta_{ij} - \sum_i \sum_j m_{ij} \Delta_{ij}$

$= \sum_i \sum_j (M_{ij} - m_{ij}) \Delta_{ij}$

$= \sum_{i : [x_i, x_{i+1}] \cap S_\varepsilon = \phi} \sum_j (M_{ij} - m_{ij}) \Delta_{ij} + \underbrace{\sum_{i : [x_i, x_{i+1}] \cap S_\varepsilon \neq \phi} \sum_j (M_{ij} - m_{ij}) \Delta_{ij}}_{S_2}$

$= S_1 + S_2$

S_1 $\quad (M_{ij} - m_{ij}) = (1 - m_{ij})$ $\quad$ why? $M_{ij} = 1 \ \forall i,j$

$\left(\underbrace{1 - f(x,y)}_{x \notin S_\varepsilon} < \varepsilon \right) \Rightarrow (1 - m_{ij}) < \varepsilon$

$S_1 = \sum_i \sum_j (1 - m_{ij}) \Delta_{ij} \leq \varepsilon \sum_i \sum_j \Delta_{ij} \leq \varepsilon \times (1)$

$\sum_i \sum_j \Delta_{ij} = 1$

$\therefore S_1 \leq \varepsilon$

S_2 $\quad$ $|S_\varepsilon| = L$

$\sum_{i : [x_i, x_{i+1}] \cap S_\varepsilon \neq \phi} \sum_j (M_{ij} - m_{ij}) \Delta_{ij}$

$\Delta_{ij} = (\Delta_i \Delta_j)$

$= \sum_i \sum_j (1 - m_{ij}) \Delta_i \Delta_j$ $\quad$ $\min = (k_i^?)$

$$= \sum_{(\;)} \sum_{j} (1 - m_{ij}) \Delta_i \Delta_j \qquad m_{iu} = (k_i^p)$$

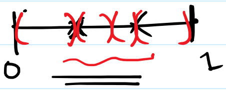
$[x_i, x_{i+1}] \cap S_\varepsilon \neq \phi$

$$= \sum_{(\;)} \sum_{j} (1 - k_i^p) \Delta_i \Delta_j$$

$$= \underbrace{\left(\sum_j \Delta_j \right)}_{\substack{\shortparallel \\ 1}} \left(\sum_{(\;)} (1 - k_i^p) \Delta_i^p \right) \qquad (S_\varepsilon)$$

$$\Rightarrow S_2 \leq \sum_{(\;)} (1 - kp) \Delta_i \qquad (2L)$$

[Can cover all the elements in S_ε, by atmost $(2L)$ intervals.] → Prove via induction



$$S_2 \leq \sum_{(\;)} (1 - k_i) \Delta_i \leq \sum_{(\;)} \Delta_i \leq (2L) \times \left(\frac{\varepsilon}{L}\right)$$

$[(1 - k_i) \leq 1]$

$$S_2 \leq 2\varepsilon$$

$$\left(U(f, P) - L(f, P) \right) = S_1 + S_2 \leq \varepsilon + 2\varepsilon \quad (\leq 3\varepsilon)$$

By the RC, f is RI on $[0,1] \times [0,1]$

3. Consider the function $f : [0, 1] \times [0, 1] \to \mathbb{R}$ defined as

$$f(x, y) = \begin{cases} 1 - 1/q & \text{if } x = p/q \quad \text{where} \quad p, q \in \mathbb{N} \quad \text{are relatively prime and } y \quad \text{is rational,} \\ 1 & \text{otherwise.} \end{cases}$$

given any y, $\phi^y(x) := \begin{cases} \left[\begin{array}{ll} 1 - \frac{1}{q} & \text{if } x = \frac{p}{q} \;\; p, q \in \mathbb{N} \\ 1 & \text{o/w} \end{array} \right] & y \in \mathbb{Q} \\ 1 & y \notin \mathbb{Q} \end{cases}$

$T : [0,1] \to \mathbb{R}$

Thomae

$$T(x) := \begin{cases} \frac{1}{q} & \text{if } x = \frac{p}{q} \quad p, q \in \mathbb{N} \\ 0 & \text{o/w} \end{cases}$$

$$① = \left[\phi^y(x) = \begin{cases} 1 - T(x) & y \in \mathbb{Q} \\ 1 & y \notin \mathbb{Q} \end{cases} \right.$$

$$A(y) = \int_0^1 \phi^y(x)\, dx = \begin{cases} 1 - \int_0^1 T(x) dx & y \in \mathbb{Q} \end{cases} \qquad \left[\int_0^1 T(x) dx = 0 \right.$$

$$\underline{\underline{A(y)}} = \int_0^1 \phi^y(x)\,dx = \begin{cases} 1 - \int_0^1 T(x)\,dx & y \in \mathbb{Q} \\ 1 & y \notin \mathbb{Q} \end{cases}$$

$$\int_0^1 T(x)\,dx = 0$$

$T \to$ cont at all irrationals, discont at all rationals

$$A(y) = 1$$

$$\int_0^1 A(y)\,dy = 1$$

Def $\quad S := \{ x \mid S(x) > \frac{\varepsilon}{2} \quad x \in [0,1] \}$

$$\phi^x(y) = \begin{cases} \underbrace{\begin{bmatrix} 1 - \frac{1}{q} & \text{if } y \in \mathbb{Q} \\ 1 & \text{if } y \notin \mathbb{Q} \end{bmatrix}}_{\tilde{D}(y)} & x = \frac{p}{q} \quad p, q \in \mathbb{N} \\ 1 & x \notin \mathbb{Q} \text{ or } x = 0 \end{cases}$$

$\tilde{D} : [0,1] \to \mathbb{R} \qquad \underline{\underline{\tilde{D}(y)}} = \begin{cases} \overline{\left(1 - \frac{1}{q}\right)} & y \in \mathbb{Q} \\ 1 & y \notin \mathbb{Q} \end{cases}$

$\tilde{D}$ is discont on all points in $[0,1]$

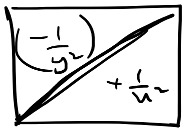
$$\phi^x(y) = \begin{cases} \tilde{D}(y) & x = p/q \\ 1 & x \notin \mathbb{Q} \text{ or } x = 0 \end{cases}$$

$\phi^y(x)$ <u>is int.</u> $\qquad$ $\phi^x(y)$ <u>is NOT</u>

Q4

4. Consider the function $f : [0,1] \times [0,1] \to \mathbb{R}$ defined by

$$f(x,y) = \begin{cases} \frac{1}{x^2} & \text{if } 0 < y < x < 1, \\ -\frac{1}{y^2} & \text{if } 0 < x < y < 1, \\ 0 & \text{otherwise} \end{cases}$$



Is f integrable over the rectangle? Do both iterated integrals exist? If they exist, do they have the same value?

NOT BOUNDED!

$$A(x) = \int_0^1 f(x,y)\, dy$$

$$= \int_0^x +\frac{1}{x^2}\, dy + \int_x^1 \left(-\frac{1}{y^2}\right) dy$$

$$= \left(\frac{1}{x}\right) + \left(\frac{1}{y}\right)_x^1$$

$$= (1)$$

$$\int_0^1 A(x)\, dx = 1$$

$$A(y) = \int_0^1 f(x,y)\, dx$$

$$= -1$$

$$\int_0^1 A(y)\, dy = -1.$$

Q6 (a)

$(:=) \to$ is defined as $\quad (\equiv) \quad \triangleq \quad (:=)$

6. (a) Let $R = [a,b] \times [c,d]$ and $\boxed{f(x,y) = \phi(x)\psi(y)}$ for all $(x,y) \in R$, where ϕ is continuous on $[a,b]$ and ψ is continuous on $[c,d]$. Show that

$$\int\int_R f(x,y)\,dx dy = \Big(\int_a^b \phi(x)\,dx\Big)\Big(\int_c^d \psi(y)\,dy\Big).$$

$P = \{(x_i, y_j)\} \qquad R_{ij}$

$t = \{t_{ij}\} \qquad t_{ij} = (t_i, t_j)$

$S(f,P,t) = \sum_i \sum_j f(t_{ij}) \Delta_{ij}$

$M := \max(S_x) = (b-a)M_\phi \qquad = \sum_i \sum_j \phi(t_i)\psi(t_j)\Delta_i \Delta_j$

$= \Big(\sum_i \phi(t_i)\Delta_i\Big)\Big(\sum_j \psi(t_j)\Delta_j\Big)$

let us be given $\varepsilon > 0$ $\qquad = S_x S_y$

given $\left[\frac{\varepsilon}{(b-a)M_\phi}\right] > 0 \qquad \exists \; \delta_1 > 0$

$\|P_1\| < \delta_1 \Rightarrow \left(|S_x - S_1| < \frac{\varepsilon}{(b-a)M_\phi}\right)$

given $\frac{\varepsilon}{|S_2|} > 0 \qquad \exists \; \delta_2 > 0$

$\|P_2\| < \delta_2 \Rightarrow \left(|S_y - S_2| < \varepsilon / |S_2|\right)$

when, $P = (P_1 \times P_2)$

$|S_x S_y - S_1 S_2| = |S_x S_y - S_x S_2 + S_x S_2 - S_1 S_2|$

$\leq |S_x||S_y - S_2| + |S_2||S_x - S_1|$

Take $\delta < \min(\delta_1, \delta_2)$

$\|P\| < \delta \Rightarrow |S_x S_y - S_1 S_2| \leq \frac{\varepsilon}{\max(|S_x|)}|S_x| + |S_2|\frac{\varepsilon}{|S_2|}$

$\boxed{|S_x S_y - S_1 S_2| \leq 2\varepsilon}$

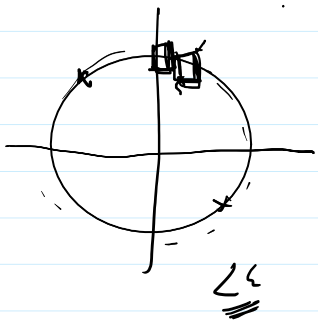
$S_1 \times S_2$

Q8

8. Consider the function f over $[-1,1]\times[-1,1]$:

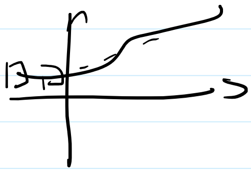
$$f(x,y)=\begin{cases} x+y & \text{if} \quad x^2+y^2\le 1, \\ 0 & \text{otherwise.}\end{cases}$$

Determine the set of points at which f is discontinuous. Is f integrable over $[-1,1]\times[-1,1]$?



$$D = \{(x,y) \mid x^2+y^2=1\} - \{(\tfrac{1}{\sqrt2}, \tfrac{-1}{\sqrt2}), (\tfrac{-1}{\sqrt2}, \tfrac{+1}{\sqrt3})\}$$

f is int!



$p(x)$