MA 109 D2 T1
Week Two Recap

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The Idea

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We defined and spent time understanding convergence of sequences. Now, we will apply those properties to talk about functions over $\mathbb{R}$.

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Further, we have that, both the definitions are **equivalent**! Can you give a proof?

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Proposition

Let $x_0 \in \mathbb{R}$, and let $A \subset \mathbb{R}$ such that $N_r(x_0) \subset A$ for some $r > 0$. The function $f : A \to \mathbb{R}$ is continuous at x_0 iff $\lim_{x\to x_0} f(x)$ exists and is equal to $f(x_0)$. That is,

$$\boxed{\text{Continuity at } x_0 \Leftrightarrow \lim_{x\to x_0} f(x) = f(x_0)}$$

Questions

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$$1_{\mathbb{Q}} := \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

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Also attempt the following: Just like we made precise the handwavy definition of a sequence 'going to infinity', give a definition using the $\epsilon - \delta$ way for a function 'going to infinity' at some $x \in \mathbb{R}$.

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- Thus, if a function is not continuous at a point, it cannot be differentiable a that point.

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- Differentiability $\implies$ Continuity? Yes
- Thus, if a function is not continuous at a point, it cannot be differentiable a that point.
- Continuity $\implies$ Differentiability?

Differentiability

Definition (Differentiability)

Let $U \subset \mathbb{R}$ be an **open** interval, and let $f : U \to \mathbb{R}$ be a function. Let $c \in U$ and let $\epsilon_0 > 0$ be such that $N_c(\epsilon_0) \subset U$[a]. Define $d(\cdot; f, c) : (-\epsilon_0, \epsilon_0)/\{0\} \to \mathbb{R}$ as,

$$d(h; f, c) := \frac{f(c+h) - f(c)}{h}$$

Then, we say that f is differentiable at c if $\lim_{h \to 0} d(h; f, c)$ exists. If it does, we denote it by $f'(c)$.

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