MA 205 Tutorial Batch 3
Recap-1

Siddhant Midha

Complex Functions

We will deal with functions of the form

$$f : \Omega \subset \mathbb{C} \to \mathbb{C}$$

If we have $z = x + \iota y$, then $f(z)$ can be expressed as

$$f(z) = u(z) + \iota v(z)$$

Using the $z \leftrightarrow (x, y)$ association we can treat u and v as real valued functions

$$u, v : \Omega_R \to \mathbb{R}$$

where $\Omega_R \subset \mathbb{R}^2$ is the region corresponding to $\Omega \subset \mathbb{C}$.

Continuity

Definition

A function $f : \Omega \subset \mathbb{C} \to \mathbb{C}$ is said to be continuous at a point $z_0 \in \Omega$ if

$$\lim_{z \to z_0} f(z) = f(z_0)$$

Equivalently[a], f is continuous at z_0 if for all sequences $(z_n)_n$ $(z_n \in \Omega)$ such that $z_n \to z_0$ we have $f(z_n) \to f(z_0)$.

[a] The $\epsilon - \delta$ continuity definition $\Leftrightarrow$ the sequential definition

- f is said to be continuous if it is continuous at all $z_0 \in \Omega$.
- f is continuous iff u and v are continuous.

Examples

- $f(z) = c$, $c \in \mathbb{C}$ is continuous.
- $f(z) = z$ is continuous.
- By the sum and product rules, all polynomials are continuous.

Differentiability

Definition

Let $\Omega \subset \mathbb{C}$ be 'open'. A function $\Omega \to \mathbb{C}$ is said to be complex-differentiable at $z_0 \in \mathbb{C}$ if the limit

$$\lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}$$

exists. If it does, we denote it by $f'(z_0)$.

- Note that h above is *complex*.
- Clearly this is stronger than differentiability of functions on $\mathbb{R}$ (Why?). So, do we expect that we might not get an iff condition as in the case of continuity?
- f is said to be CD on Ω if it is CD on all $z \in \Omega$.
- Differentiability $\implies$ Continuity.

Holomorphicity

Definition

Let $\Omega \subset \mathbb{C}$ be 'open'. A function $\Omega \to \mathbb{C}$ is said to be holomorphic on Ω if it is complex differentiable at each $z_0 \in \Omega$ and the derivative f' is continuous on Ω. We denote $f \in C^1(\Omega)$.

- f is called holomorphic *on a point* if it is holomorphic on an open disk containing that point.
- A function holomorphic on $\mathbb{C}$ is said to be entire.
- Holomorphic on a point $\implies$ CD on a point. Reverse?

Recall ...

For some $f : \Omega \to \mathbb{C}$ we will denote $F : \Omega_R \to \mathbb{R}^2$ the corresponding real function. Further, we let

$$F(x, y) = \begin{pmatrix} u(x, y) \\ v(x, y) \end{pmatrix}$$

Real Differentiability

$F : \Omega_R \to \mathbb{R}^2$ is differentiable at $(x, y) \in \Omega$ if there exists a 2×2 matrix $DF(x, y)$ such that

$$\lim_{h,k \to 0} \frac{|| \begin{pmatrix} u(x+h, y+k) \\ v(x+h, y+k) \end{pmatrix} - \begin{pmatrix} u(x, y) \\ v(x, y) \end{pmatrix} - DF(x, y) \begin{pmatrix} h \\ k \end{pmatrix} ||}{|| \begin{pmatrix} h \\ k \end{pmatrix} ||} = 0$$

If so, we have

$$DF(x, y) = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}$$

Necessary Condition (for ?)

Theorem - CR Equations

Let $f(z) = u(x, y) + \iota v(x, y)$ be defined on some open set Ω. Further suppose that $f'(z_0)$ exists for some $z_0 = x_0 + \iota y_0 \in \Omega$. Then the first order partial derivatives of u and v exist at (x_0, y_0) and satisfy the CR equations

$$u_x = v_y \ , \ v_x = -u_y$$

i.e., $CD \implies CR$.

Sufficient Condition (for ?)

Theorem

Let $f(z) = u(x, y) + \iota v(x, y)$ be defined on some open set Ω. For some $z_0 = x_0 + \iota y_0 \in \Omega$, if

1. the partial derivatives of u and v exist in some neighbourhood of (x_0, y_0) and are continuous at (x_0, y_0), and
2. the CR equations are satisfied at (x_0, y_0)

Then, we have that $f'(z_0)$ exists.

Necessary and Sufficient Condition (for ?)

If we have a F with the DF matrix as

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

Then what can be say about the corresponding f? Some intuition?

Theorem

Let $f(z) = u(x,y) + \iota v(x,y)$ be defined on some open set Ω and let $F : \Omega_R \to \mathbb{R}^2$ be the corresponding real function. For some $z_0 = x_0 + \iota y_0 \in \Omega$, if

1. F is differentiable at (x_0, y_0).
2. The $DF(x_0, y_0)$ is of the following form

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

   (equivalently, the CR equations are satisfied at (x_0, y_0))

Then, we have that $f'(z_0)$ exists and equals $a + \iota b$. Further, the converse holds.

Summarizing ...

Why were the last two conditions different? Remember the differentiation theorem?

- CD $\implies$ RD.
- RD $\not\Longleftarrow$ CD. CD is a special case of RD.
- $CD \implies CR$.
- $CR \not\Longleftarrow CD$.
- $(CR + RD) \iff CD$