# MA 205 Tutorial Batch 3
# Recap-3

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# Theorem of the day

**Theorem (Cauchy Integral Theorem)**

*Let $\Omega$ be a bounded domain in $\mathbb{C}$, with piecewise smooth boundary $\partial\Omega$ and $f \in C^1(\bar{\Omega})$ is holomorphic on $\Omega$. Then,*

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Note that, in these theorems we are dealing with $\partial\Omega$ being traversed anticlockwise. Also, we do *not* need $f \in C^1(\bar{\Omega})$, as holomorphicity of $f$ guarantees holomorphicity and thus continuity of $f'$.

# Another way ...

Another way to state the CIT, which can avoid possible mistakes.

**Theorem (CIT - Aliter)**

*If $f : \Omega \to \mathbb{C}$ is holomorphic, and $\Omega$ is a* **simply connected** *domain, then for every closed piecewise smooth curve $\gamma$ within $\Omega$ we have,*

$$\int_\gamma f(z)dz = 0$$

# Consequences

- **Holo $\implies$ Analytic**. If $f$ is holomorphic at a point $z_0 \in \mathbb{C}$, then we have that $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$ for all $z : |z - z_0| < r$ for some small $r$.

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Hence we have the statement,

$$\textbf{Holomorphic} \Leftrightarrow \textbf{Analytic}$$

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  3. $f^{(n)}(z_0) = 0 \forall n$ for some $z_0 \in \Omega \implies f \equiv 0$ on $\Omega$.
- **Identity Principle**: If $f, g$ holomorphic agree on a 'suitable' set of points, then $f \equiv g$ on $\Omega$.

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With this and properties like the CIF, one can surely ponder about the rigidity of holomorphic functions.