MA 205 Tutorial Batch 3
Recap-4

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Singularities

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4. $f(z) := \bar{z}$ has *a* non-isolated singularity. Where?

Classifying Isolated Singularities - I

[#1] Removable Singularities

An isolated singularity $z \in \mathbb{C}$ of f is said to be removable if there exists a holomorphic function $\tilde{f} : D(z, r) \to \mathbb{C}$ for some $r > 0$ **such that** $\tilde{f}(z) = f(z) \forall z \in D^*(z, r)$.

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Proof? Start with $g(z) := (z - p)^2 f(z)$. Explicitly construct the desired $\tilde{f}$.

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A pole of order one is called a *simple pole*.

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Suppose f has an essential singularity at p. Then, for any $r > 0$, f takes values arbitrarily close to every complex number on the disk $D^*(p, r)$.

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Suppose f has an essential singularity at p. Then, for any $r > 0$, f takes on **all possible** complex values, **infinitely often**, with at most a single exception in $D^*(p, r)$.

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Also recall the **Little Picard Theorem**: An entire function can omit at most one complex number.

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4. $z^4/(z^3+z)$

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3. $z\cos 1/z$ Essential
4. $z^4/(z^3+z)$ Two poles, one removable

Laurent Series

Theorem

Suppose f is analytic in the annulus $\mathcal{A}(p, r_1, r_2) := \{z : r_1 < |z - p| < r_2\}$.

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holds for all $z \in \mathcal{A}(p, r_1, r_2)$. Where,

$$a_n := \frac{1}{2\pi\iota} \int_\gamma \frac{f(\eta)}{(\eta - p)^{n+1}} \ \forall n \in \mathbb{Z}$$

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A Joke

"A man of the Polish persuasion got on a Boeing 767 for a routine flight back to the Motherland. He was getting comfortable in coach when a stewardess screamed out, 'The pilot and co-pilot are dead! Is there anyone left that can fly this aircraft?' The Polish man said, 'I was a pilot back in the war. Let me have a go at the controls.' So he bravely sauntered up to cockpit. When he opened the door, he was awestruck by the array of lights, dials, screens and switches in front of him, and he froze up. The stewardess shook him and said, 'Aren't you going to sit down and take the reins?' He said in a quavering voice, 'I'm just a simple Pole in a complex plane!'"
source: https://asimplepoleinacomplexplane.blogspot.com