MA 205 Autumn 2022
Tutorial Sheet 5

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1. Construct a meromorphic function on $\mathbb{C}$ with infinitely many poles.

Sol.
First, argue that $\sin z = 0$ only at the points familiar to us on the $\mathbb{R}$eal Axis. Define $f : \mathbb{C}/\{n\pi | n \in \mathbb{Z}\} \to \mathbb{C}$ as,

$$f(z) := \frac{1}{\sin z}$$

and see that this function satisfies the requirement.

$\blacksquare$

2. Find Laurent expansions for $f(z) = \dfrac{2(z-1)}{z^2 - 2z - 3}$ on the regions:

1. $|z| < 1$,
2. $1 < |z| < 3$,
3. $3 < |z|$.

Sol.
See that,

$$f(z) = \frac{2(z-1)}{(z+1)(z-3)} = \frac{1}{z-3} + \frac{1}{z+1}$$

1. Write,

$$\begin{aligned} f(z) &= \frac{1}{1-(-z)} - \frac{1}{3}\frac{1}{1-(z/3)} \\ &= \sum_n (-z)^n - \frac{1}{3}\sum_n (z/3)^n \\ &= \sum_n z^n \left((-1)^n - (1/3)^{n+1}\right) \\ &= \sum_n a_n z^n \end{aligned}$$

   where,

$$a_n = \begin{cases} 0 & \text{if } n < 0 \\ \left((-1)^n - (1/3)^{n+1}\right) & \text{if } n \geq 0 \end{cases}$$

2. Write,

$$\begin{aligned} f(z) &= \frac{1}{z}\frac{1}{1+1/z} - \frac{1}{3}\frac{1}{1-(z/3)} \\ &= \frac{1}{z}\sum_n (-1/z)^n - \frac{1}{3}\sum_n (z/3)^n \\ &= \sum_n b_n z^n \end{aligned}$$

where,

$$b_n = \begin{cases} (-1)^{-n+1} & \text{if } n < 0 \\ -(\frac{1}{3})^{n+1} & \text{if } n \geq 0 \end{cases}$$

3. Write,

$$\begin{aligned} f(z) &= \frac{1}{z}\frac{1}{1+1/z} + \frac{1}{z}\frac{1}{1-(3/z)} \\ &= \frac{1}{z}\sum_n (-1/z)^n + \frac{1}{z}\sum_n (3/z)^n \\ &= \sum_n \frac{3^n + (-1)^n}{z^{n+1}} \\ &= \sum_n c_n z^n \end{aligned}$$

where,

$$c_n = \begin{cases} 3^{-n-1} + (-1)^{-n-1} & \text{if } n < 0 \\ 0 & \text{if } n \geq 0 \end{cases}$$

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3. Evaluate

$$\int_0^{2\pi} \frac{\cos^2(3x)}{5 - 4\cos(2x)} dx$$

Sol.
Using The Euler's Formula, cos(3x) and cos(2x) can be rewritten as

$$\frac{e^{3ix} + e^{-3ix}}{2}$$

and

$$\frac{e^{2ix} + e^{-2ix}}{2}$$

Further, taking e^{ix} as z, the integral changes to

$$\frac{1}{8i} \times \int_{|z|=1} \frac{-(z^6+1)^2}{2z^5(1-(\frac{5z^2}{2} - z^4))} dz$$

This function has poles 0 (order 5), $\pm\sqrt{2}$, $\pm\frac{1}{\sqrt{2}}$ (order 1). The integral is hence the sum of the residues at $z = 0$, $z = \pm\frac{1}{\sqrt{2}}$. The residues at $\pm\frac{1}{\sqrt{2}}$ can be evaluated easily by finding

$$\left.\frac{-(z^6+1)^2(z-\sqrt{0.5})}{2z^5(1-(\frac{5z^2}{2} - z^4))}\right|_{z=\sqrt{0.5}}$$

and

$$\left.\frac{-(z^6+1)^2(z+\sqrt{0.5})}{2z^5(1-(\frac{5z^2}{2} - z^4))}\right|_{z=-\sqrt{0.5}}$$

Each giving a residue of -27/8. The residue at 0 could be found using the Laurent expansion of the term inside the integral and finding the coefficient of z_{-1}. Since we have a division by z^5 we will need to find coefficient of z^4 of the remaining expression. Furthermore, a multiplication with $(1+z^6)^2$, has contribution only from 1 to the coefficient of z^4. The required expansion is

$$\frac{1}{(1-(\frac{5z^2}{2} - z^4))} = 1 + (\frac{5z^2}{2} - z^4) + (\frac{5z^2}{2} - z^4)^2 \ldots$$

From this we get the coefficient of z^4 as -1+25/4 = 21/4. The integral is thus,

$$\frac{2\pi i}{8i} \times (21/4 - 27/4) = -\frac{3\pi}{8}$$

<u>Aliter:</u> Write

$$\cos^2(3x) = \frac{1+\cos(6x)}{2} = \frac{1}{2} + \frac{1}{2}\text{Re}(e^{6\iota x})$$

This will make the calculation of the residues a bit easier.

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4. Evaluate

$$\int\limits_{|z-2|=4} \frac{2z^3+z^2+4}{z^4+4z^2}\, dz$$

Sol.
Note that the integrand has poles at $z = 0$ and $z = \pm 2\iota$ of orders 2 and 1 respectively. Thus, we break the integral into three parts about each of the poles.

1. About $z = 0$: Use CIF and see that this would evaluate to zero.
2. About $z = 2\iota$: Use CIF and see that this would be $2\pi\iota \times 1$.
3. About $z = -2\iota$: Use CIF and see that this would be $2\pi\iota \times 1$.

Thus, the answer is $4\pi\iota$.

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5. Show

$$\int\limits_{-\infty}^{\infty} \frac{x}{(x^2+2x+2)(x^2+4)}\, dx = -\frac{\pi}{10}.$$

Sol.
Changing the variable of integration to z, we get

$$\int\limits_{-\infty}^{\infty} \frac{z}{(z^2+2z+2)(z^2+4)}\, dz$$

We shall take the standard contour for such problems, which is the union of

1. γ: Curve which goes from $-R$ to $+R$ on the real line
2. γ_R: Curve which goes along the semicircle $Re^{\iota t}$ as t goes from 0 to π

Argue that the integral that $\int_{\gamma_R} f(z)dz = 0$ as we let $R \to \infty$. The poles of the function are at -1 $\pm$i and $\pm$2i. To evaluate the integral around $\gamma \cup \gamma_R$ we have,

$$\int_\gamma f(z)dz + \int_{\gamma_R} f(z)dz = 2\pi\iota \times \sum_{z\in\{-1+\iota, 2\iota\}} \text{Res}(f,z)$$

Further, since the γ_R integral is zero, we have,

$$\int_\gamma f(z)dz = 2\pi\iota \times \sum_{z\in\{-1+\iota, 2\iota\}} \text{Res}(f,z)$$

We calculate the residues, as,

$$\left.\frac{z(z+1-i)}{(z^2+2z+2)(z^2+4)}\right|_{z=-1+i}$$

and

$$\left.\frac{z(z-2i)}{(z^2+2z+2)(z^2+4)}\right|_{z=2i}$$

Giving us the values $\frac{1+3i}{20}$ and -$\frac{1+2i}{20}$ respectively. Summing them up, we have,

$$2\pi i \times \frac{i}{20} = -\frac{\pi}{10}$$

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6. Compute using residue theory

$$\int_{-\infty}^{\infty} \frac{\cos x}{(1+x^2)^2}\, dx.$$

Sol.

Consder the function $f : \mathbb{C}/\{\pm\iota\} \to \mathbb{C}$ defined as

$$f(z) := \frac{e^{\iota z}}{(z^2+1)^2}$$

If we compute the integral of this function along the real line, and take the real part of it, we would get our desired integral. Define

1. γ: Curve which goes from $-R$ to $+R$ on the real line
2. γ_R: Curve which goes along the semicircle $Re^{\iota t}$ as t goes from 0 to π

We shall thus integrate over the curve stitched together by joining γ and γ_R. Now, note that,

$$f(Re^{\iota t}) = \frac{e^{\iota R\cos t} e^{-R\sin t}}{(R^2 e^{2\iota t}+1)^2}$$

That is, we can easily see that the function is bounded on the semicircle, as $t \in [0,\pi] \implies \sin t \geq 0$. In the limit $R \to \infty$, thus the integral along the semicircle vanishes, i.e., $\int_{\gamma_R} f(z)dz = 0$.
Now, we can apply residue theorem,

$$\int_{\gamma} f(z)dz + \int_{\gamma_R} f(z)dz = 2\pi\iota \times \mathrm{Res}(f, 2\iota)$$

Further, since the γ_R integral is zero, we have,

$$\int_{\gamma} f(z)dz = 2\pi\iota \times \mathrm{Res}(f, 2\iota)$$

Evaluating this, we have,

$$2\pi\iota \times \frac{d}{dz}\left(\frac{e^{\iota z}}{(z+\iota)^2}\right)|_{z=\iota}$$

This can be computed, and results in π/e.
Thus,

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{\cos x}{(1+x^2)^2}\, dx &= \mathrm{Re}(\int_{\gamma} f(z)dz) \\ &= \mathrm{Re}(\pi/e) \\ &= \pi/e\end{aligned}$$

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7. Let $a > 1$. Show by transforming into an integral over the unit circle, that

$$\int_0^{2\pi} \frac{1}{a^2+1-2a\cos\theta}\, d\theta = -\frac{2\pi}{1-a^2}.$$

Sol.

$$\begin{aligned}
\int_0^{2\pi} \frac{1}{a^2+1-a(e^{\iota\theta}+e^{-\iota\theta})} \frac{\iota e^{\iota\theta} d\theta}{\iota e^{\iota\theta}} &= \int_{|z|=1} \frac{1}{a^2+1-a(z+1/z)} \frac{dz}{z} \\
&= \frac{1}{\iota} \int_{|z|=1} \frac{1}{(a^2+1)z - a(z^2+1)} dz \\
&= \frac{1}{\iota} \int_{|z|=1} \frac{1}{(a-z)(az-1)} dz \\
&= \frac{1}{a\iota} \int_{|z|=1} \frac{1}{(a-z)(z-1/a)} dz \\
&= \frac{1}{a\iota} 2\pi\iota \times \frac{1}{a-1/a} \\
&= \frac{2\pi}{a^2-1}
\end{aligned}$$

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