MA 109 D2 T1
Week One Recap

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Hello!

Again, a warm welcome to MA 109! Please note a few things,

- Material regarding the tutorials can be found at https://siddhant-midha.github.io/.
- Please feel free to raise your hand, and ask a doubt anytime.
- We will be meeting 20 minutes before the allotted tutorial time for the recap every week.
- The recap is not a substitute for the lectures.
- A feedback form can be found at the website. Please use this regularly.

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Sets

Definition (Set)

A set is an unordered collection of distinct objects.

Some notation.

- $\mathbb{N}$: The set of natural numbers.
- $\mathbb{Z}$: The set of real numbers.
- If a set S contains some element a, we write $a \in S$.
- To refer to all the elements in the set S, we use $\forall s \in S$.
- 'There exists s in S': $\exists s \in S$.
- $\mathbb{Q}$: The set of rational numbers (numbers of the form p/q for $p, q \in \mathbb{Z}$).
- $\mathbb{R}$: The set of real numbers.

Cardinality, etc.

Definition (Finite Set)

A set S is called finite if,

1. It has no elements (denoted $S = \emptyset$). Or,
2. There is a bijection $f : \{1, 2, \ldots n\} \to S$ for some $n \in \mathbb{N}$.

If a set is not finite, it is said to be infinite. This enables us to form a rigorous definition of cardinality.

Definition (Cardinality)

The cardinality of a finite set S, denoted as $|S|$, is defined as

1. $|S| = 0$ if $S = \emptyset$.
2. $|S| = n$ if a bijection $f : S \to \{1, 2, \ldots n\}$ exists.

Can we talk about cardinality of infinite sets?

Maxima, Minima, and all that

Let X be a set with an order. For instance, this can be $\mathbb{R}$, or $\mathbb{Q}$.

Definition (Maxima and Minima)

Let T be a subset of X. An element $e \in T$ is said to be,

- A *maximum* if $e \geq t$ for all $t \in T$.
- A *minimum* if $e \leq t$ for all $t \in T$.

Definition (Upper Bounded and Lower Bounded)

A subset T of X is said to be

- Upper bounded (in X) if there exists $x \in X$ such that
$$t \leq x \forall t \in T$$
- Lower bounded (in X) if there exists $x \in X$ such that
$$x \leq t \forall t \in T$$

- A set which is both upper bounded and lower bounded is said to be bounded.

LUB & GLB

We identify two special bounds.

Definition

For a subset T of X, an element $x \in X$ is said to be a Least Upper Bound (LUB) of T if,

- x is **an** upper bound of T.
- For any upper bound y of T, we have,

$$x \leq y$$

Similarly, the Greatest Lower Bound (GLB) is defined. More commonly, we refer to LUB as the supremum, and the GLB as the infimum.

$\mathbb{Q}$ and $\mathbb{R}$

- Consider, $\{\frac{1}{1}, \frac{1}{1+\frac{1}{1+1}}, \frac{1}{1+\frac{1}{1+\frac{1}{1+1}}}, \dots\} \cap (0, \frac{\sqrt{5}-1}{2}) \subset \mathbb{Q}$. Upper bounded? Supremum exists?
- If $X = \mathbb{Q}$, we find that not all upper bounded sets have a supremum (in $\mathbb{Q}$).
- $\mathbb{Q}$ has 'holes'.
- Cover up these gaps to obtain $\mathbb{R}$!
- $\mathbb{R}$ is complete: Every non-empty upper bounded (lower bounded) subset of $\mathbb{R}$ has a supremum (infimum) in $\mathbb{R}$.
- $\mathbb{Q}$ is **not** complete.

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Sequences

Definition (Sequences)

A *sequence* in a set X is a function $f : \mathbb{N} \to X$.

- Notation: We denote, $a_n \equiv f(n)$. We denote the entire sequence by $\{a_n\}_n$.
- Examples. Consider,
  1. $a_n := \frac{1}{n}$
  2. $b_n := (-1)^n$.
  3. $c_n := \sin n$.

Convergence of Sequences

Definition (Convergence)

A real sequence $\{a_n\}_n$ is said to converge to a real number L , if $\forall \epsilon > 0$, there exists $N_0 \in \mathbb{N}$ such that,

$$|a_n - L| < \epsilon \text{ whenever } n > N_0$$

A sequence which does not converge is said to diverge, or be non-convergent.

Properties

Proposition

Let a_n and b_n be real convergent sequences.

1. The sequences converge to unique real numbers. Denote them as a_0, and b_0 respectively.
2. a_n and b_n are bounded (that is, both lower and upper bounded).
3. The sequence $p_n := |a_{n+1} - a_n|$ converges to 0.
4. $c_n := a_n \pm b_n$ is convergent, and converges to $a_0 \pm b_0$.
5. $d_n := a_n \times b_n$ is convergent, and converges to $a_0 \times b_0$.
6. If $b_n \neq 0 \; \forall n$, then $e_n := a_n / b_n$ is convergent, and converges to a_0/b_0.
7. **Sandwich Property**: If $a_0 = b_0$ and there is a sequence f_n such that

$$a_n \leq f_n \leq b_n \; \forall n$$

then f_n converges, and the limit is $f_0 = a_0 = b_0$.

The MCT

We use monotonic and eventually monotonic synonymously.

Definition (Monotone sequence)

A sequence a_n is said to be monotonically increasing (decreasing) if there is $n_0 \in \mathbb{N}$ such that for all $n > n_0$ we have $a_{n+1} \geq a_n$ $(a_{n+1} \leq a_n)$.

If we replace $\geq$ by $>$ in the definition above, we get strict monotonicity.

Theorem (Monotone Convergence)

An upper bounded (lower bounded) real sequence a_n which is monotonically increasing (decreasing) converges. Further,

$$\lim_{n\to\infty} a_n = \sup\{a_n\} \ (\inf\{a_n\})$$

Where we know that the supremum exists due to the completeness of $\mathbb{R}$. Does the converse of the MCT hold? No. Take $a_n := (-1)^n/n$.