MA 106 D1-T3
Recap-2

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The Determinant Function

We shall consider functions of the form

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- (Skew) Symmetry

$$f(c_1, \ldots c_i \ldots c_j \ldots c_n) = (-)f(c_1, \ldots c_j \ldots c_i \ldots c_n)$$

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For any $n \times n$ matrix A,

$$det(A) = \sum_{j=1}^{n} a_{ij} A_{ij}$$

for all $i = 1, 2 \dots n$.

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$$det(G) \neq 0$$

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We saw in class, if A is non-singular then

$$A^{-1} = \frac{Adj(A)}{det(A)}$$