MA 109 D2 T1
Week One Extra Recap

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$\boxed{\text{Logistics!}}$

Convergence of Sequences

Definition (Convergence)

A real sequence $\{a_n\}_n$ is said to converge to a real number L , if $\forall \epsilon > 0$, there exists $N_0 \in \mathbb{N}$ such that,

$$|a_n - L| < \epsilon \text{ whenever } n > N_0$$

A sequence which does not converge is said to diverge, or be non-convergent.

Properties

Proposition

Let a_n and b_n be real convergent sequences.

1. The sequences converge to unique real numbers. Denote them as a_0, and b_0 respectively.
2. a_n and b_n are bounded (that is, both lower and upper bounded).
3. The sequence $p_n := |a_{n+1} - a_n|$ converges to 0.
4. $c_n := a_n \pm b_n$ is convergent, and converges to $a_0 \pm b_0$.
5. $d_n := a_n \times b_n$ is convergent, and converges to $a_0 \times b_0$.
6. If $b_n \neq 0 \ \forall n$, then $e_n := a_n / b_n$ is convergent, and converges to a_0 / b_0.
7. **Sandwich Property**: If $a_0 = b_0$ and there is a sequence f_n such that
$$a_n \leq f_n \leq b_n \ \forall n$$
then f_n converges, and the limit is $f_0 = a_0 = b_0$.

Implications

One has to be careful about the nature of implications and their use.

- If, $A \implies B$ holds, then B is a necessary condition for A.
- The implication $A \implies B$ is equivalent to $\neg B \implies \neg A$.
- Example 1: Convergence of $a_n \implies$ boundedness of $\{a_n\}$. Boundedness of $\{a_n\}$ is necessary for convergence. Further, unboundedness of $\{a_n\}$ implies a_n is **not** convergent!
- If $A \implies B$ holds, then A is a sufficient condition for A.
- The implication $A \implies B$ does not mean that $B \implies A$ too!
- Example 2: Boundedness is necessary for convergence, but it is not sufficient! For example,

$$\text{The sequence } a_n := (-1)^n \text{ does not converge. Proof?}$$

The MCT

Theorem (Monotone Convergence)

An upper bounded (lower bounded) real sequence a_n which is monotonically increasing (decreasing) converges. Further,

$$\lim_{n\to\infty} a_n = \sup\{a_n\} \ (\inf\{a_n\})$$

That is, for a monotonically increasing sequence,

$$\textbf{boundedness} \Leftrightarrow \textbf{convergence}$$

<u>Exercises</u>.

1. Show that
$$a_n := 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$
does not converge to any real number.
2. Show that
$$b_n := 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} \text{ does converge.}$$

Practice Questions

- We talk about sequences 'going to infinity'. How do we make this definition precise? Use the format of the $\epsilon - N$ definitions to think about this.
- An interesting equivalence. Show that, a real sequence a_n converges to $L \in \mathbb{R}$ if and only if it has the property that every neighbourhood $N_r(L)$ contains all but finitely many points of a_n. Note that we define neighbourhoods as

$$N_r(L) := \{x \in \mathbb{R} \mid |x - L| < r\}$$

- [Tutorial] Let a_n and b_n be real sequences. Do the following properties guarantee the convergence of $c_n := a_n b_n$?
  1. a_n is convergent
  2. a_n is convergent and b_n is bounded
  3. a_n converges to 0 and b_n is bounded
  4. a_n and b_n are both convergent