MA 205: Complex Analysis Endsem TSC

Siddhant Midha

11th September 2022

Welcome to the (first? second!) TSC for Complex Analysis 2022!

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- Finally, if you notice some mistake, do let us know.

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- Preliminaries
- 2 Functions, Continuity, Differentiability
- 3 Power Series
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- 5 Singularities and Residues

Definition (Open Disks)

For any $z\in\mathbb{C}$, and for any r>0 define the open disk, denoted B(z,r) as

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A subset $S \subseteq \mathbb{C}$ is said to be open if for all $z \in S$ there exists an r > 0 such that $B(z, r) \subset S$.

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Recall: $z \in \mathbb{C}$ is a limit point of $\Omega \subset \mathbb{C}$ if there exists a sequence $z_n \in \Omega$, $z_n \neq z$, such that $z_n \to z$.

Connectedness

Definition (Connected)

A subset $S \subseteq \mathbb{C}$ is said to be connected if given any 2 points $x, y \in S$, there exists a continuous path joining them. i.e, a continuous function $f: [0,1] \to S$ such that f(0) = x and f(1) = y.

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Definition (Domain)

A open and connected subset of \mathbb{C} is called a domain.

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Sequences and Convergence

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A sequence in \mathbb{C} is a function $f : \mathbb{N} \cup \{0\} \to \mathbb{C}$. We denote $z_n = f(n)$.

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A sequence z_n is said to be converging to some $z \in \mathbb{C}$ if $\forall \epsilon > 0$, $\exists N_{\epsilon} \in \mathbb{N}$ s.t.

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Theorem

If $z_n = x_n + \iota y_n$ is a sequence in \mathbb{C} , then

$$z_n \rightarrow z = x + \iota y \Leftrightarrow x_n \rightarrow x \text{ and } y_n \rightarrow y$$

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Continuity

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A function $f:\Omega\subset\mathbb{C}\to\mathbb{C}$ is said to be continuous at a point $z_0\in\Omega$ if

$$\lim_{z\to z_0}f(z)=f(z_0)$$

Equivalently^a, f is continuous at z_0 if for all sequences $(z_n)_n$ $(z_n \in \Omega)$ such that $z_n \to z_0$ we have $f(z_n) \to f(z_0)$.

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 a The $\epsilon-\delta$ continuity definition \Leftrightarrow the sequential definition

- f is said to be continuous if it is continuous at all $z_0 \in \Omega$.
- f is continuous iff u and v are continuous.

Definition (Complex Differentiability (CD))

Let $\Omega \subset \mathbb{C}$ be open. A function $\Omega \to \mathbb{C}$ is said to be complex-differentiable at $z_0 \in \mathbb{C}$ if the limit

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exists. If it does, we denote it by $f'(z_0)$.

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- ullet Differentiability \Longrightarrow Continuity.



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Let $\Omega \subset \mathbb{C}$ be open. A function $\Omega \to \mathbb{C}$ is said to be holomorphic on Ω if it is complex differentiable at each $z_0 \in \Omega$ and the derivative f' is continuous on Ω . We denote $f \in C^1(\Omega)$.

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- A function holomorphic on $\mathbb C$ is said to be entire.
- Holomorphic at a point \implies CD at a point. Reverse?
- **Remark**: A function can be CD at a point and not holomorphic at the same point. Consider $f(z) = |z|^2$.

If $f: \Omega \to A$ and $g: \Omega \to B$ are holomorphic on Ω , then,

• $c_1f + c_2g$ is holomorphic on Ω , and $(c_1f + c_2g)' = c_1f' + c_2g'$.

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- If $h: A \to \mathbb{C}$ is holomorphic on A, then $h \circ f(z) := h(f(z))$ is holomorphic on Ω , and $(h \circ f)'(z) = h'(f(z))f'(z)$.

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- For $z_0 \in \Omega$ s.t. $g(z_0) \neq 0$, f/g is holomorphic at z_0 , and,

$$\left(\frac{f}{g}\right)(z_0) = \frac{f'(z_0)g(z_0) - g'(z_0)f(z_0)}{g(z_0)^2}$$

Real Differentiability

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Real Differentiability

 $F:\Omega_R\to\mathbb{R}^2$ is differentiable at $(x,y)\in\Omega$ if there exists a 2×2 matrix DF(x,y) such that

$$\lim_{h,k\to 0} \frac{\left|\left| \begin{pmatrix} u(x+h,y+k) \\ v(x+h,y+k) \end{pmatrix} - \begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix} - DF(x,y) \begin{pmatrix} h \\ k \end{pmatrix} \right|\right|}{\left|\left| \begin{pmatrix} h \\ k \end{pmatrix} \right|\right|} = 0$$

If so, we have

$$DF(x,y) = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}$$

Theorem (The CR Equations)

Let $f(z) = u(x,y) + \iota v(x,y)$ be defined on some open set Ω . Suppose that $f'(z_0)$ exists for some point $z_0 = x_0 + \iota y_0 \in \Omega$. Then the first order partial derivatives of u and v exist at that point (x_0, y_0) and satisfy the CR equations

$$u_x = v_y$$
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Theorem

Let f be defined on some open set Ω be differentiable at some $z = (x + \iota y) \in \Omega$. Then, the real counterpart F will be differentiable at $(x, y) \in \Omega_R$.

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- [2020 Quiz] Consider,

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We shall see that together they are sufficient to show CD.

Necessary and Sufficient Condition

Theorem

Let $f(z) = u(x,y) + \iota v(x,y)$ be defined on some open set Ω and let $F: \Omega_R \to \mathbb{R}^2$ be the corresponding real function. For some $z_0 = x_0 + \iota y_0 \in \Omega$, if

- **1** F is differentiable at (x_0, y_0) .
- 2 The $DF(x_0, y_0)$ is of the following form

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

(equivalently, the CR equations are satisfied at (x_0, y_0))

Then, we have that $f'(z_0)$ exists and equals $a + \iota b$. Further, the converse holds.

Another Sufficient Condition

Theorem

Let $f(z) = u(x, y) + \iota v(x, y)$ be defined on some open set Ω . For some $z_0 = x_0 + \iota y_0 \in \Omega$, if

- the partial derivatives of u and v exist in some neighbourhood of (x_0, y_0) and are continuous at (x_0, y_0) , and
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Then, we have that $f'(z_0)$ exists.

This turns out to be easier to check.

Harmonic Functions

Definition (Harmonic Function)

A function $g:\Omega_R\subset\mathbb{R}^2\to\mathbb{R}$ is said to be harmonic if it has continuous partial derivatives of the first and second order, and satisfies

$$\triangle g(x,y) = g_{xx}(x,y) + g_{yy}(x,y) = 0 \ \forall (x,y) \in \Omega_R$$

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Theorem

If a function $f(z) = u(x, y) + \iota v(x, y)$ is CD in a domain Ω , then u and v are harmonic in D_R .

Summarizing ...

 \bullet CD \Longrightarrow RD.

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- \bullet CD \Longrightarrow RD.
- ullet RD \longmapsto CD. CD is a special case of RD.

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- $(CR + RD) \iff CD$

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- Fact: Absolute Convergence ⇒ Convergence.

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A *series* is an expression of the form $\sum_n z_n$, for $z_n \in \mathbb{C}$.

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The word 'expression' signifies that the series/power series may or may not be meaningful (read convergent).



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- **③** There exists $R \in \mathbb{R}$, ∞ > R > 0, such that P converges for all $z : |z z_0| < R$ and diverges for all $z : |z z_0| > R$.

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Usually, we allow for $R = 0, \infty$ for convenience.

ano comments on the boundary!

Follow up theorem

Convergence of Power Series

The radius of convergence of a power series as defined before is given as

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Herein, we allow for $R=0,\infty$ by letting $1/0=\infty,1/\infty=0$.

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- Thus, the lim-sup always exists. The limit might not.
- When the limit exists, the limsup is equal to the limit.

Theorem

1 The power series $\sum_n a_n(z-z_0)^n$ defines a holomorphic function $f:D(z_0,R)\to\mathbb{C}$, $f(z):=\sum_n a_n(z-z_0)^n$ where R is the RoC.

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This leads to the statement

Analytic \Longrightarrow Holomorphic

Questions?

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Let $\sum_{n} z_n$ be a complex series. Note the following,

- **1** Necessary Condition for Convergence If $\sum_n z_n$ converges, then $|z_n| \to 0$ as $n \to \infty$. (Recall the tutorial question about $\sum nz^n$)
- **Necessary & Sufficient Condition for Convergence** Note that $|z_n| \ge 0$ (thus $s_n := \sum_{k=0}^n |z_k|$ is monotonic increasing), we have that $\sum_n |z_n|$ converges iff $s_n = \sum_{k=0}^n |z_k|$ is bounded above. (follows from the MCT, recall the tutorial question about $\sum z^n/n^2$)

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- Preliminaries
- 2 Functions, Continuity, Differentiability
- 3 Power Series
- 4 Integrals and all that
- 5 Singularities and Residues

Definition (Curve)

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A holomorphic function $\Omega \to \mathbb{C}$ is said to admit a primitive F in Ω if F'(z) = f(z) for all $z \in \Omega$.

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Theorem,

If f is holomorphic in a domain (thus, connected), and $f' \equiv 0$ in that region, then f is a constant.

11th September 2022

Theorem

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Cauchy Integral Theorem

Let Ω be a bounded domain in \mathbb{C} , with piecewise smooth boundary $\partial\Omega$ and $f\in C^1(\bar{\Omega})$ is holomorphic on Ω . Then,

$$\int_{\partial\Omega}f(z)dz=0$$

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Note that, in these theorems we are dealing with $\partial\Omega$ being traversed anticlockwise.



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Note that, in these theorems we are dealing with $\partial\Omega$ being traversed anticlockwise. Also, we do *not* need $f\in C^1(\bar{\Omega})$, as holomorphicity of f guarantees holomorphicity and thus continuity of f'.

Another way

Another way to state the CIT, which can avoid possible mistakes.

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CIT - Aliter

If $f:\Omega\to\mathbb{C}$ is holomorphic, and Ω is a **simply connected** domain, then for every closed piecewise smooth curve γ within Ω we have,

$$\int_{\gamma} f(z)dz = 0$$

Questions?

Questions?

① [2020 Quiz]

$$\int_{|z|=1} \frac{e^z \sin(z) - z}{z^2 \cos(z)} dz$$

Questions?

1 [2020 Quiz]

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Answer: 0

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Slides

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Answer: $2\pi\iota(e-2)$.



Theorem (Morera's Theorem)

Given a continuous function g on a domain Ω , we have that if $\int_{\gamma} g(z)dz = 0$ for all $\gamma = \partial R$, whenever $R \subset \Omega$ is a rectangle, then, g is holomorphic on Ω .

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Theorem

Given a function g which is complex differentiable at each point in a domain Ω , we have that $\int_{\gamma} f(z)dz=0$ whenever $\gamma=\partial R$ and $R\subset\Omega$ is a rectangle.

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Theorem (Goursat's Theorem)

If a function is complex differentiable at each point in an open set, then it is continuously differentiable in that set. Thus, it is holomorphic on that set.

• Strong Regularity If f is holomorphic at some z_0 , then the derivatives of all orders are holomorphic at that point.

 Strong Regularity If f is holomorphic at some z₀, then the derivatives of all orders are holomorphic at that point. Further, we have

$$f^{(n)}(z_0) = \frac{n!}{2\pi\iota} \int_{D(z_0,r)} \frac{f(\eta)}{((\eta - z_0)^{n+1}} d\eta$$

for some small r.

• **Holomorphic** \Longrightarrow **Analytic**. If f is holomorphic at a point $z_0 \in \mathbb{C}$, then we have that $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ for all $z : |z - z_0| < r$ for some small r.

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Note that any r s.t. $D(z_0,r)$ is contained in the region of holomorphicity gives the same a_n . This means that an entire function has $RoC = \infty$ when expanded about any point as a power series. Also, we had previously seen that power series are holomorphic in their region of convergence.

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Questions?

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● [2018 Midsem] If *f* is a holomorphic function on an open set containing the closed unit disk, and,

$$\int_{|z|=1} f(z)\bar{z}^j dz = 0$$

holds true for all $j = 0, 1, 2 \dots$, then show that $f \equiv 0$.

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Suppose $f:\Omega\to\mathbb{C}$ is holomorphic, where Ω is a domain. Further suppose $f(z_0)=0$. Then, we have $f(z_0)=0$ on $f(z_0)=0$ or $f(z_0)=0$

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- $f \equiv 0 \text{ on } \Omega, \text{ or, }$
- ② $\exists m \in \mathbb{N}$ and a holomorphic function $g: \Omega \to \mathbb{C}$ such that $g(z_0) \neq 0$ and $f(z) = (z z_0)^m g(z)$ in $D(r, z_0)$ for some small r.

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Zeros of Analytic Functions

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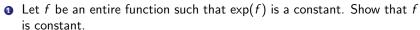
Suppose $f:\Omega\to\mathbb{C}$ is holomorphic, where Ω is a domain. Further suppose $f(z_0)=0$. Then, we have

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 - **3** $f^{(n)}(z_0) = 0 \forall n$ for some $z_0 \in \Omega \implies f \equiv 0$ on Ω .
 - **Identity Principle**: If f, g holomorphic agree on a 'suitable' set of points, then $f \equiv g$ on Ω .

Questions?

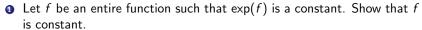
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Questions?





- Let f be an entire function such that $\exp(f)$ is a constant. Show that f is constant.
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Then,

- Can g be a constant?
- ② Assume that g is non-constant, then show that $g(\mathbb{C}) = \mathbb{C}$.



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- Preliminaries
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- 3 Power Series
- 4 Integrals and all that
- 5 Singularities and Residues

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[#1] Removable Singularities

An isolated singularity $z\in\mathbb{C}$ of f is said to be removable if there exists a holomorphic function $\tilde{f}:D(z,r)\to\mathbb{C}$ for some r>0 such that $\tilde{f}(z)=f(z)\forall z\in D^*(z,r).$

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Proof? Start with $g(z) := (z - p)^2 f(z)$. Explicitly construct the desired \tilde{f} .

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A pole of order one is called a *simple pole*.

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Also recall the **Little Picard Theorem**: An entire function can omit at most one complex number.

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- **5** $z^4/(z^3+z)$ Answer: Simple poles at $\pm \iota$, Removable at 0

Definition (Meromorphic Function)

A function holomorphic on a domain except possibly for a set of poles, is said to be meromorphic on that domain.

Theorem (Laurent Series)

Suppose f is analytic in the annulus $\mathcal{A}(p, r_1, r_2) := \{z : r_1 < |z - p| < r_2\}.$

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$$a_n := rac{1}{2\pi\iota} \int_{\gamma} rac{f(\eta)}{(\eta - p)^{n+1}} \ orall n \in \mathbb{Z}$$

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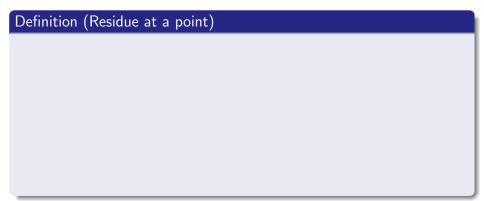
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As an immediate consequence, see that

$$\int_C f(z)dz = 2\pi\iota \times Res(f,p)$$

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Let γ be a simple closed contour. Let f be analytic inside and on γ except for a finite number of isolated singularities p_k , then,

$$\int_{\gamma} f(z)dz = 2\pi \iota \sum_{k} Res(f, p_{k})$$

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And that's a wrap!

Thank you, and all the best!