Let $(z_n)_n$ be a sequence in $\mathbb{C}$.
w/ $z_n \in \mathbb{C}\ \forall n \in \mathbb{N}$

1) $(z_n)_n$ is said to be convergent if
$\exists\ L \in \mathbb{C}$, s.t.
$\forall \varepsilon > 0$, $\exists\ N \in \mathbb{N}$, s.t.
$|z_n - L| < \varepsilon$ whenever $n > N$.

2) $(z_n)_n$ is said to be cauchy if
$\forall \varepsilon > 0\ \exists\ N \in \mathbb{N}$ s.t.
$\forall n, m > N$ we have $|z_n - z_m| < \varepsilon$.

fact : Convergent $\Rightarrow$ Cauchy

let $(z_n)_n$ be conv. $\longrightarrow L$

ie, $\forall \varepsilon_1 > 0 \quad \exists \quad N(\varepsilon_1)$ s.t. ....

Now, let $\varepsilon > 0$ be arbit.

let $\varepsilon_1 := \left(\frac{\varepsilon}{2}\right)$. By conv,

we have $\exists \quad N \equiv N(\varepsilon_1)$ s.t.

$\forall n > N$ we have $|z_n - L| < \frac{\varepsilon}{2}$

now let $n, m \in \mathbb{N}$ s.t.
$n, m > N$.

then,

$$|z_n - z_m| = |(z_n - L) + (L - z_m)|$$

$$\leq \text{(how)}$$

$$|z_n - L| + |z_m - L|$$

$$\frac{\varepsilon}{2} + \frac{\varepsilon}{2} \text{ (how)}$$

$\Rightarrow$ for arbit $\varepsilon > 0 \quad \exists \; N \in \mathbb{N}$
s.t.

$|z_n - z_m| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ whenever
$n, m > N$.

Thus, $\{z_n\}$ is Cauchy. ■

Fact: Cauchy $\Rightarrow$ Conv. IN $\mathbb{C}$. ("and in any "complete" space*, by defⁿ)

i.e., $\mathbb{C}$ is said to be "complete".

There can be "spaces"* where a Cauchy seq. doesn't converge.

eg. $(X, d) = ((0,1), |\cdot|)$ → usual $\mathbb{R}$ distance

Consider $z_n = \frac{1}{n}$, clearly does not converge in X. But one can show that it is Cauchy in X. (PTO)

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*: a space in this context is any set X with a distance notion d.
eg: $(X, d) = (\mathbb{R}, |\cdot|)$ or $(\mathbb{R}^n, \|\cdot\|)$

Pf: $\left(\frac{1}{n}\right)_n$ is cauchy.

$z_n = \frac{1}{n}$

let $\varepsilon > 0$.

$n \neq m \in \mathbb{N}$

$$|z_n - z_m| = \left|\frac{1}{n} - \frac{1}{m}\right| = \left|\frac{n-m}{nm}\right|$$

$$\wedge$$

$$\left|\frac{1}{mn}\right|$$

let $N \in \mathbb{N}$ s.t. $N > \frac{1}{\sqrt{\varepsilon}}$. $\Rightarrow \frac{1}{N^2} < \varepsilon$

now, if $n, m > N$

$$\Rightarrow \frac{1}{n}, \frac{1}{m} < \frac{1}{N}$$

$$\Rightarrow \frac{1}{nm} < \frac{1}{N^2} < \varepsilon.$$

$$\Rightarrow \left|\frac{1}{mn}\right| < \varepsilon$$

$$\Rightarrow |z_n - z_m| \leq \left|\frac{1}{mn}\right| < \varepsilon.$$

$\therefore$ Given $\varepsilon > 0$, we were able to find $N \in \mathbb{N}$ s.t. .....

$\therefore$ <u>Cauchy</u>

Now,

1) is $[0,1)$ complete ?

2) is $(0,1]$ complete ?

3) is $[0,1]$ complete ?