MA 205 Tutorial Batch 3
Recap-5

Siddhant Midha

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Laurent Series

Theorem

Suppose f is analytic in the annulus $\mathcal{A}(p, r_1, r_2) := \{z : r_1 < |z - p| < r_2\}$. Let γ be any positively oriented simple closed contour around p lying in the annulus. Then,

$$f(z) = \sum_{n \in \mathbb{Z}} a_n (z - p)^n$$

holds for all $z \in \mathcal{A}(p, r_1, r_2)$. Where,

$$a_n := \frac{1}{2\pi\iota} \int_\gamma \frac{f(\eta)}{(\eta - p)^{n+1}} \; \forall n \in \mathbb{Z}$$

- The **principal part** of the Laurent Series of f at p is

$$\sum_{n \leq -1} a_n (z - p)^n$$

Question

Give the Laurent Series for,

1. $f(z) = 1/z^{10231}$ in $\{z : 0 < |z| < 1\}$
2. $f(z) = 1/z(z+1)$ in $\{z : 0 < |z| < 1\}$,$\{z : |z| > 1\}$
3. $f(z) = 1/(z^2+1)$ in $\{z : |z| < 1\}$,$\{z : |z - \iota| < 1\}$
4. $f(z) = e^{1/z}$

A consequence of the Laurent Series

Theorem

Suppose f is analytic in the annulus $\mathcal{A}(p, r_1, r_2) := \{z : r_1 < |z - p| < r_2\}$. Let a_n be the coefficients of its Laurent Series. Then,

1. f has a pole at $p \Leftrightarrow a_n = 0$ for all but finitely many $n < 0$, and $a_k \neq 0$ for some $k < 0$.
2. f has an essential singularity at $p \Leftrightarrow a_n \neq 0$ for **infinitely many** $n < 0$.

Thus,

1. The singularity is removable iff the principal part is zero.
2. The singularity is a pole iff the principal part has a finite (not zero) number of non-zero terms.
3. The singularity is essential iff the principal part has an infinite number of non-zero terms.

Residue Theorem

Definition (Residue at a point)

Suppose f is analytic in the annulus $\mathcal{A}(p, 0, r_2) := \{z : 0 < |z - p| < r_2\}$. Let

$$f(z) = \sum_{n \in \mathbb{Z}} a_n (z - p)^n$$

be the Laurent Series expansion of f in this annulus. Then, we define,

$$Res(f, p) := a_{-1}$$

As an immediate consequence, see that

$$\boxed{\int_C f(z)dz = 2\pi\iota \times Res(f, p)}$$

Residues of Meromorphic functions

Theorem

If f is analytic in $D^*(p, r)$ for some r and has a pole of order k at p, then,

$$Res(f, p) = \lim_{z \to p} \frac{1}{(k-1)!} \frac{d^{k-1}}{dz^{k-1}} \left((z-p)^k f(z) \right)$$

Questions

1. Find residue at 0 of $1/(z+z^2)$.
2. Find residue at 0 of $1/(z^2(1-z))$ using *both* methods.
3. $\int_{|z|=1} e^{1/z^2} dz$
4. $\int_{|z|=1} \frac{1}{\sin 1/z} dz$Uh? What? CRT? No!

The Cauchy Residue Theorem

This is a formal restatement of the earlier conclusion.

Theorem (CRT)

Let γ be a simple closed contour. Let f be analytic inside and on γ except for a finite number of isolated singularities p_k, then,

$$\int_\gamma f(z)dz = 2\pi\iota \sum_k Res(f, p_k)$$

Orders and Multiplicities

(Recall)

Theorem (The Order of a Pole)

Let f have a pole at $p \in \mathbb{C}$. Then, there exists some $k \in \mathbb{N}$ such that for some $r > 0$ we have,

$$f(z) = (z - p)^{-k} H(z)$$

where, H is holomorphic on $D(p, r)$ **and** $H(p) \neq 0$. We say that k is the order of the pole.

Orders and Multiplicities

And similarly, we define,

Definition (Multiplicity of a zero)

Given a function f, we say that $z_0 \in \mathbb{C}$ is a zero of multiplicity m of f, if

$$f^{(n)}(z_0) = 0 \ \forall n \leq m - 1 \ \textbf{and} \ f^{(m)}(z_0) \neq 0$$

We have that,

$$f(z) = (z - z_0)^m g(z)$$

for some holomorphic g which does not vanish in a neighbourhood of z_0.

The Argument Principle

This is a nice application of the CRT.

Theorem

Let f be a meromorphic function on and inside some closed contour γ, **and** has no poles or zeros on γ. Then,

$$\frac{1}{2\pi\iota}\int_{\gamma}\frac{f'(z)}{f(z)}dz = Z - P$$

where, Z is the number of zeros of f inside γ counted with multiplicites and P is the number of poles of f inside γ counted with order.

Jokes

1. Why did the mathematician name his dog Cauchy? –? Because he left a residue at each pole!
2. What is the contour integral over the Western Europe?

Zero! All the poles are in Eastern Europe.

`credit: MA 205 2021 Lectures`

And that is all

... for the recap's of MA 205!