MA 109 D2 T1
Week Seven Recap

Siddhant Midha
https://siddhant-midha.github.io/

December 13, 2022

Tangent Lines and Planes

- Let $F : D \subset \mathbb{R}^2 \to \mathbb{R}$ have partial derivatives at an interior point $P = (x_0, y_0)$ and let $\nabla F(x_0, y_0) \neq (0, 0)^T$. Consider the curve, $C := \{(x, y) \mid F(x, y) = 0\}$ and assume that P lies on C. Then the equation of the tangent line to C at P is given by,

$$F_x(x_0, y_0)(x - x_0) + F_y(x_0, y_0)(y - y_0) = 0$$

  Special Case: $F(x, y) = y - f(x)$

- Let $F : D \subset \mathbb{R}^3 \to \mathbb{R}$ have partial derivatives at an interior point $P = (x_0, y_0, z_0)$ and let $\nabla F(x_0, y_0, z_0) \neq (0, 0, 0)^T$. Consider the surface, $S := \{(x, y, z) \mid F(x, y, z) = 0\}$ and assume that P lies on S. Then the equation of the tangent plane to S at P is given by,

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0$$

  Special Case: $F(x, y, z) = z - f(x, y)$

Normal Lines and Tangent Curves

- Note that for all points on the tangent plane, $\nabla F(P) \cdot (x - x_0, y - y_0, z - z_0)^T = 0$. Thus we have the normal line,

$$x = x_0 + F_x(P)t \ ; \ y = y_0 + F_y(P)t \ ; \ z = z_0 + F_z(P)t \quad ; \ t \in \mathbb{R}$$

- **Tangent Curves**: Given a smooth parametrized curve $C = \{(x(t), y(t), z(t) \mid t \in [a, b])\}$, we define for all $t_0 \in [a, b]$ the tangent vector to C at t_0 as $(x'(t_0), y'(t_0), z'(t_0))$ (assume $\neq (0, 0, 0)$).
- Consider a C lying on the surface defined by $F(x, y, z) = 0$. Then, for any $t_0 \in [a, b]$, we have,

$$\boxed{\nabla F(x(t_0), y(t_0), z(t_0)) \cdot (x'(t_0), y'(t_0), z'(t_0)) = 0}$$

Extrema & Saddle Points

Recall the concept of **local extrema**.

Proposition (Necessary Conditions for Local Extrema)

Let $D \subset \mathbb{R}^2$, and let (x_0, y_0) be an interior point of D. Suppose $f : D \to \mathbb{R}$ has a local extremum at (x_0, y_0). If u is a unit vector and $(\nabla_u f)(x_0, y_0)$ exists, then $(\nabla_u f)(x_0, y_0) = 0$.

The converse is false. $f(x, y) = xy$
Also, recall **critical points** (do: derivative $\to$ gradient!)

Definition (Saddle Points)

For a 'nice' function $f : D \subset \mathbb{R}^2 \to \mathbb{R}$, and some interior point $P \in D$, we call P to be a saddle point of f if $\nabla F(P) = (0, 0)^T$ but P is **not** a local extrema.

Hessian Test

Proposition (The Hessian Test)

Let $D \subset \mathbb{R}^2$ and let (x_0, y_0) be an interior point of D. Suppose $f : D \to \mathbb{R}$ is such that the first-order and second-order partial derivatives of f exist and are continuous in a neighbourhood of (x_0, y_0), and $(\nabla f)(x_0, y_0) = (0, 0)$. Consider the discriminant (or the Hessian)

$$(\Delta f)(x_0, y_0) := f_{xx}(x_0, y_0) f_{yy}(x_0, y_0) - f_{xy}(x_0, y_0)^2$$

of f at (x_0, y_0). We have,

1. If $(\Delta f)(x_0, y_0) > 0$ and $f_{xx}(x_0, y_0) < 0$, then f has a local maximum at (x_0, y_0).
2. If $(\Delta f)(x_0, y_0) > 0$ and $f_{xx}(x_0, y_0) > 0$, then f has a local minimum at (x_0, y_0).
3. If $(\Delta f)(x_0, y_0) < 0$, then f has a saddle point at (x_0, y_0).

The discrimimant test is inconclusive if $(\Delta f)(x_0, y_0) = 0$.

Constrained Extrema

Setting: Let $f, g : D \subset \mathbb{R}^2 \to \mathbb{R}$ be nice functions. We aim to find extrema of f on D constrained to the fact that $g = 0$.

Proposition (LMT)

Let $D \subset \mathbb{R}^2$, and let (x_0, y_0) be an interior point of D. Suppose $f, g : D \to \mathbb{R}$ have continuous partial derivatives in a neighbourhood of (x_0, y_0). Let $C := \{(x, y) \in D : g(x, y) = 0\}$.Suppose,

1. $g(x_0, y_0) = 0$
2. $(\nabla g)(x_0, y_0) \neq (0, 0)$
3. f, when restricted to C, has a local extremum at (x_0, y_0).

Then, there is $\lambda_0 \in \mathbb{R}$ such that

$$(\nabla f)(x_0, y_0) = \lambda_0 (\nabla g)(x_0, y_0).$$

The real number λ_0 is called a Lagrange multiplier.

Lagrange Multiplier's: The Procedure

1. Establish C: the set where g vanishes.
2. Establish that f attains its bounds on C (use the **EVT**)
3. Solve the system of equations

$$\nabla f = \lambda \nabla g$$

$$g = 0$$

4. Use the solutions from the above system of equations to check whether $\nabla g \neq 0$ on those points
5. Conclude that f attains extrema on those points.
6. Note that this procedure only tells us that these points are the extrema – does not comment on whether they are minima or maxima.