Problem 1

11. List all possibilities for the reduced row echelon matrices of order 4×4 having exactly one pivot. Count the number of free parameters (degrees of freedom) in each case. For example one of the possibilty is $\begin{bmatrix} 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ wherein there are 2 degrees of freedom. Repeat for 0, 2, 3 and 4 pivots.

- zero pivots $\Rightarrow$ $A = [0]_{4\times 4}$

- one pivot

$x \to$ any real number

$$A = \begin{bmatrix} 1 & x & x & x \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{DoF} = 3 \quad \text{or} \quad A = \begin{bmatrix} 0 & 1 & x & x \\ & 0 & & \end{bmatrix} \text{DOF} = 2$$

$$\text{or} \quad A = \begin{bmatrix} 0 & 0 & 1 & x \\ & 0 & & \end{bmatrix} \quad \text{or} \quad A = \begin{bmatrix} 0 & 0 & 0 & 1 \\ & 0 & & \end{bmatrix} \text{DOF} = 0$$

DOF = 1

4 possibilities

- two pivots

$$\begin{bmatrix} 1 & 0 & x & x \\ 0 & 1 & x & x \\ 0 & 0 & & \\ 0 & 0 & 0 & \end{bmatrix} \text{4 DOF} \quad \begin{bmatrix} 1 & x & 0 & x \\ 0 & 0 & 1 & x \\ & & & \\ & & & \end{bmatrix} \text{3 DOF} \quad \begin{bmatrix} 1 & x & x & 0 \\ 0 & 0 & 0 & 1 \\ & & & \\ & & & \end{bmatrix} \text{2 DOF}$$

$$\begin{bmatrix} 0 & 1 & 0 & x \\ & & 1 & x \\ & & & \\ & & & \end{bmatrix} \text{2 DOF} \quad \begin{bmatrix} 0 & 1 & x & 0 \\ & & 0 & 1 \\ & & & \\ & & & \end{bmatrix} \text{1 DOF}$$

$$\begin{bmatrix} & & 1 & 0 \\ & & & 1 \\ & & & \\ & & & \end{bmatrix} \text{0 DOF}$$

6 possibilities

- three pivots

$$\begin{bmatrix} 1 & & & x \\ & 1 & & x \\ & & 1 & x \\ & & & \end{bmatrix} \text{3 DOF} \quad \begin{bmatrix} 1 & & x & \\ & 1 & x & \\ & & & 1 \\ & & & \end{bmatrix} \text{2 DOF}$$

$$\begin{bmatrix} 1 & x & & \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & & \end{bmatrix}$$

$$\begin{bmatrix} 1 & x & & \\ & & 1 & \\ & & & 1 \\ & & & \end{bmatrix}_{1\,DOF} \quad \begin{bmatrix} 0 & 1 & & \\ & & 1 & \\ & & & 1 \\ & & & \end{bmatrix}_{0\,DOF}$$ 4 possibilities

· four pivots $\quad I_{4\times 4}$ (0 DOF) $\quad$ 1 possibility

poss → $(1, 4, 6, 4, 1)$

$({}^4C_0, {}^4C_1, {}^4C_2, {}^4C_3, {}^4C_4)$

${}^nC_{(\#\text{ pivots})}$

Problem 2

2. Find whether the following sets of vectors are linearly dependent or independent:

(i) $[1, -1, 1], [1, 1, -1], [-1, 1, 1], [0, 1, 0]$.

(ii) $[1, 9, 9, 8], [2, 0, 0, 3], [2, 0, 0, 8]$.

if want to see when $v_1, v_2 \cdots v_n$ are in/dep

$[c_1 v_1 + c_2 v_2 \cdots]$

$$= \boxed{\underbrace{[v_1 \ v_2 \cdots v_n]}_{A} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = 0}$$

$Ax = 0$

does this have a non zero solution?

(i) dependent

why? $\cdot$ # num of vecs $= 4 > \dim(\mathbb{R}^3)$

By fundamental lemma, we are done.

(ii) $v_1 = \begin{bmatrix} 1 \\ 9 \\ 9 \\ 8 \end{bmatrix}$ $v_2 = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 3 \end{bmatrix}$ $v_3 = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 8 \end{bmatrix}$

a) v_2 and v_3 are independent by observation.

b) Can you generate v_1 using v_2 and v_3? No! second & third element of v_2 & v_3 are zero. of v_1 are not.

c) conclude independent.

Problem 3

3. Find the ranks of the following matrices:

(i) $\begin{bmatrix} 8 & -4 \\ -2 & 1 \\ 6 & -3 \end{bmatrix}$, (ii) $\begin{bmatrix} m & n \\ n & m \\ p & p \end{bmatrix}$ $(m^2 \neq n^2)$, (iii) $\begin{bmatrix} 0 & 8 & -1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \\ 0 & 4 & 5 \end{bmatrix}$.

i) $A = \begin{bmatrix} 8 & -4 \\ -2 & 1 \\ 6 & -3 \end{bmatrix} \begin{matrix} \to r_1 \\ \to r_2 \\ \to r_3 \end{matrix}$, columns c_1, c_2

See that $c_1 = -2c_2$
hence, $\dim(\operatorname{span}\{\underline{c_1, c_2}\}) = 1$
$\underline{c_1,c_2}$ = column rank = rank.

a) $\text{rank}(A) \leq 2$

b) By observation, $r_1 = -4r_2$
$r_1 = +\frac{4}{3} r_3$

thus, by ERO's we can get $\begin{bmatrix} - & - \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\therefore$ rank $= 1$

ii) $A = \begin{bmatrix} m & n \\ n & m \\ p & p \end{bmatrix}$ $m^2 \neq n^2$

① $\text{rank}(A) \leq 2$ — ①

② By the theorem on determinantal rank, and the fact that $\det(\,\,) \neq 0$ $\boxed{\text{rank} = 2}$

Lemma

Let $A \in \mathbb{R}^{n \times n}$. A $k \times k$ submatrix of A has non zero determinant iff $Rank(A) \geq k$. $\longrightarrow$ $\text{Rank}(A) \geq 2$ — ②

iii) $A = \begin{bmatrix} 0 & 8 & -1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \\ 0 & 4 & 5 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 8 & -1 \\ 0 & 0 & 3 \\ 0 & 4 & 5 \end{bmatrix} \begin{matrix} R_1 \\ R_2 \\ R_3 \\ R_4 \end{matrix}$

$\downarrow R_2 \to R_2 - 2R_4$

$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & -11 \\ 0 & 0 & 3 \\ 0 & 4 & 5 \end{bmatrix}$

$\downarrow R_2 \leftrightarrow R_4$

$0 \quad 4 \quad 5$

$$\Big\downarrow R_2 \leftrightarrow R_4$$

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 4 & 5 \\ 0 & 0 & 3 \\ 0 & 0 & -11 \end{bmatrix}$$

$$\Big\downarrow R_4 \to R_4 + \frac{11}{3} R_3$$

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 4 & 5 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

<u>Rank = 3</u>

(look up the rref function in MATLAB)

TLDR:

i) Observe that c1 = kc2 for some scalar k. Thus dimension of span({c1,c2}) is 1. Rank is 1.
j) We find a (the highlighted) 2 x 2 determinant with non zero rank. We remark that rank can be atmost 2. We conclude it is 2.
k) Standard REF operation, with the fact that rank is preserved under EROs.

Problem 4

. For $a < b$, consider the system of equations:

$$\begin{aligned} x &+ y &+ z &= 1 \\ ax &+ by &+ 2z &= 3 \\ a^2x &+ b^2y &+ 4z &= 9. \end{aligned}$$

Find the pairs (a, b) for which the system has infinitely many solutions.

$$\underbrace{\begin{pmatrix} 1 & 1 & 1 \\ a & b & 2 \\ a^2 & b^2 & 4 \end{pmatrix}}_{A}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 9 \end{pmatrix}$$

$Ax = b$

$\det(A) = (b-a)(2-a)(2-b) = 0$

But $b - a \neq 0$

So, either $a = 2$ (1) or $b = 2$ (2)

(1)

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 2 & b & 2 & 3 \\ 4 & b^2 & 4 & 9 \end{array}\right]$$

$\downarrow$ $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 4R_1$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & b-2 & 0 & 1 \\ 0 & b^2-4 & 0 & 5 \end{array}\right]$$

$\downarrow$ $R_3 \to R_3 - (b+2)R_2$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & b-2 & 0 & 1 \\ 0 & 0 & 0 & 3-b \end{array}\right]$$

] → demand "consistency", i.e, $\text{Rank}(A) = \text{Rank}([A|b])$

$b = 3$

(2) $b = 2 \Rightarrow a = 3$ X

(show)

$\boxed{\text{Only sol is } (2, 3)}$

Problem 5

Show that the row space of a matrix does not change by row operations. Show that the dimension of the column space is unchanged by row operations.

If c_i are independent, so are Ec_i for any invertible matrix E

(seen in class)

let $\{r_i\}$ be the rows of A

let $\{r_i'\}$ be the rows of EA where E is the matrix of some ERO.

(1) $\text{span}(\{r_i'\}) \subseteq \text{span}(\{r_i\})$

let $A = \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_m \end{bmatrix}$ and $EA = \begin{bmatrix} r_1 \\ r_2 + \beta r_3 \\ \vdots \\ r_m \end{bmatrix}$

$$c_1 r_1 + c_2 (r_2 + \beta r_3) + \dots c_m r_m$$

$$= [c_1 r_1 + (c_2) r_2 + r_3(c_3 + \beta) + \dots c_m r_m] \in \text{Span}\{r_i\}$$

$EA = \begin{bmatrix} r_1 \\ c r_2 \\ \vdots \end{bmatrix}$ $\quad c_1 r_1 + c c_2 r_2 + \dots c_m r_m \in \text{Span}\{r_i\}$

$EA = \begin{bmatrix} r_2 \\ r_1 \\ \vdots \end{bmatrix}$ $\quad c_1 r_2 + c_2 r_1 + \dots \in \text{Span}\{r_i\}$

$\Downarrow$

$$\text{span}\{r_i'\} \subseteq \text{span}\{r_i\} \quad - \ast$$

now, perform the inverse ERO

$$E^{-1}A' = A$$

By $\ast$, $\text{Span}\{r_i\} \subseteq \text{span}\{r_i'\} \quad - \#$

By $\ast$ & $\#$,

$$\text{Span}\{r_i\} = \text{Span}\{r_i'\}$$

(That is, we show that upon ONE ERO, the span remains same. After that it follows by induction)

Problem 6 (i)

$A\underline{x} = \underline{b}$

Consider the system,

$$\underbrace{\begin{bmatrix} 1 & 0 & 2 & -1 & +1 & -2 & -1 \\ 2 & 2 & 6 & 0 & 4 & 2 & 4 \\ 1 & -1 & 1 & -2 & 0 & -5 & -4 \\ 2 & 2 & 6 & 0 & 4 & 2 & 4 \end{bmatrix}}_{A} \left| \underbrace{\begin{matrix} 1 \\ 10 \\ -3 \\ 10 \end{matrix}}_{b} \right.$$

Perform EROs. Use REF to answer:

What is Rank(A)? Nullity(A)? Is the system solvable?

$$R_4 \to R_4 - R_2$$

$$\left[\begin{array}{ccccccc|c} 1 & 0 & 2 & -1 & +1 & -2 & -1 & 1 \\ 2 & 2 & 6 & 0 & 4 & 2 & 4 & 10 \\ 1 & -1 & 1 & -2 & 0 & -5 & -4 & -3 \\ & & & 0 & & & & 0 \end{array}\right]$$

$$R_3 \to R_3 - R_1$$
$$R_2 \to R_2 - 2R_1$$

$$\left[\begin{array}{ccccccc|c} 1 & 0 & 2 & -1 & +1 & -2 & -1 & 1 \\ 0 & 2 & 2 & 2 & 2 & 6 & 6 & 8 \\ 0 & -1 & -1 & -1 & -1 & -3 & -3 & -4 \\ & & & 0 & & & & 0 \end{array}\right]$$

$$R_3 \to R_3 + 2R_2 \cdot \frac{1}{2}$$

$$\left[\begin{array}{ccccccc|c} 1 & 0 & - & - & & & & 1 \\ 0 & 2 & - & - & - & - & - & 8 \\ & & & 0 & & & & 0 \\ & & & & & & & \end{array}\right]$$

$$\underbrace{\left[\quad 0 \quad \right.}_{A'} \left| 0 \right]$$

$$\text{rank}(A) = \text{rank}(A') = 2$$

$$N(A) = (7) - 2 = 5$$

Soloution?

$$\text{Rank}([A|b]) = 2 = \text{Rank}(A)$$

$\therefore$ Soloutable.

Theorem

A system of linear equations

$$Ax = b$$

has a solution iff

$$Rank(A) = Rank([A|b])$$

Problem 6 (ii)

Consider the system,

$$\underbrace{\left[\begin{array}{ccccccc|c} 1 & 0 & 2 & -1 & +1 & -2 & -1 & 1 \\ 2 & 2 & 6 & 0 & 4 & 2 & 4 & 10 \\ 1 & -1 & 1 & -2 & 0 & -5 & -4 & -3 \\ 2 & 2 & 6 & 0 & 4 & 2 & 4 & 10 \end{array}\right]}_{A \qquad\qquad\qquad\qquad b}$$

Perform EROs. Use REF to answer:

Find a k x k submatrix of A with non zero determinant.

Definition

$A \in \mathbb{R}^{m\times n}$ is said to have determinental rank k (denoted $det\ rank(A)$ if

1. There exists some $k \times k$ submatrix of A with non zero determinant.
2. *All* $(k+1) \times (k+1)$ submatrices of A have zero determinant.

$$det\ rank(A) = rank(A)$$

Firstly note that since the rank is 2, we will not bother about k greater than 2, and k = 1 is trivial.
A matrix has non zero determinant => rows are independent, columns are independent.

Let pseudoREF be the composition of all the REF steps except for row exchanges.
1) Apply pseudoREF. Let there be k pivotal columns. (note the modified notion of pivotal here)
2) Take the n × k matrix (call it B) consulting of those columns as its columns.
3) Apply pseudoREF on the transpose of this matrix B (or equivalently apply pseudoCEF on B) to find independent rows of B. (Note that independent columns of REF are in correspondence with the independent columns of the original matrix).
4) Find the submatrix as the collection of those rows of B.

note that $\boxed{\text{first two colums}}$ (c_1 and c_2) of REF(A) were independent thus $k = 2$.

Thus, let $B = \begin{bmatrix} 1 & 0 \\ 2 & 2 \\ +1 & -1 \\ 2 & 2 \end{bmatrix} \to (4 \times 2) = n \times k$ matrix

$B = [c_1, c_2]$

$B^T = \begin{bmatrix} 1 & 2 & +1 & 2 \\ 0 & 2 & -1 & 2 \end{bmatrix}$

$$B' = \begin{bmatrix} 1 & 2 & +1 & 2 \\ 0 & 2 & -1 & 2 \end{bmatrix}$$

independent columns of B^T

$\Updownarrow$

ind. rows of B

$$\therefore \quad K \times K \text{ submatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 2 \end{bmatrix}$$

Problem 6 (iii)

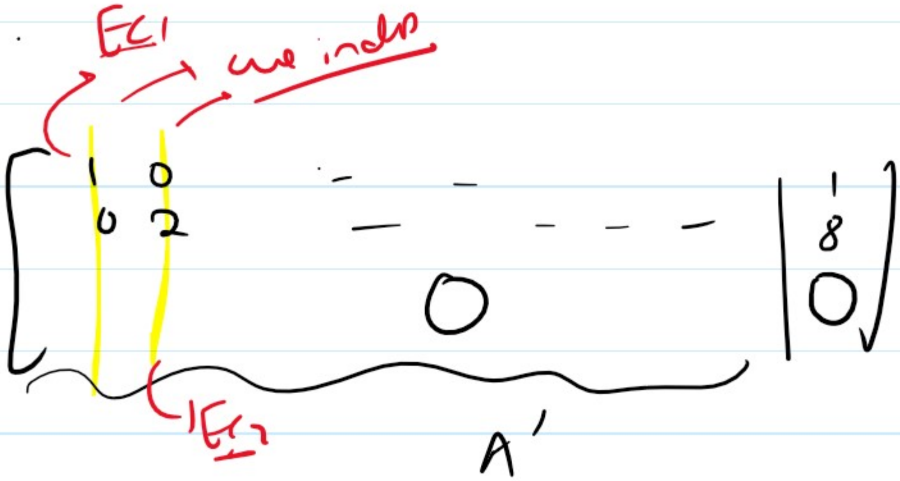
Consider the system,

$$\left[\begin{array}{ccccccc|c} 1 & 0 & 2 & -1 & +1 & -2 & -1 & 1 \\ 2 & 2 & 6 & 0 & 4 & 2 & 4 & 10 \\ 1 & -1 & 1 & -2 & 0 & -5 & -4 & -3 \\ 2 & 2 & 6 & 0 & 4 & 2 & 4 & 10 \end{array}\right]$$

c_1 c_2 $\underbrace{\qquad\qquad\qquad}_{A}$ $\underbrace{\quad}_{b}$

Perform EROs. Use REF to answer:

Find a basis for the column space of A.



Ec_1 and Ec_2 ind $\Rightarrow$ c_1 and c_2 ind.

We use the fact that if a certain set of vectors is independent, then upon transformation by an invertible matrix, they remain independent. Thus we first see that the first two columns of REF(A) are independent, and conclude that the first two columns of A are too (as EROs are just multiplication by invertible Ei's). Further, using that and the fact that the rank (and thus the dimension of the column space is 2) we are done.

Problem 6 (iv)

Consider the system,

$$\underbrace{\begin{bmatrix} 1 & 0 & 2 & -1 & +1 & -2 & -1 \\ 2 & 2 & 6 & 0 & 4 & 2 & 4 \\ 1 & -1 & 1 & -2 & 0 & -5 & -4 \\ 2 & 2 & 6 & 0 & 4 & 2 & 4 \end{bmatrix}}_{A} \left| \underbrace{\begin{matrix} 1 \\ 10 \\ -3 \\ 10 \end{matrix}}_{b} \right.$$

Perform EROs. Use REF to answer:

Find the complete set of solutions of Ax = b. Which are the free variables?

We have the REF of $[A|b]$ as,

$$\left[\underbrace{\begin{matrix} 1 & 0 & 2 & -1 & +1 & -2 & -1 \\ 0 & 2 & 2 & 2 & 2 & 6 & 6 \\ & & & 0 & & & \\ & & & 0 & & & \end{matrix}}_{A'} \left| \underbrace{\begin{matrix} 1 \\ 8 \\ 0 \\ 0 \end{matrix}}_{b'} \right.\right]$$

$$A' \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_7 \end{pmatrix}}_{x} = b'$$

Solve:

$$x_1 + 0x_2 + 2x_3 - x_4 + x_5 - 2x_6 - x_7 = 1$$
$$0x_1 + 2x_2 + 2x_3 + 2x_4 + 2x_5 + 6x_6 + 6x_7 = 8$$

$$x_1 = 1 - 2x_3 + x_4 - x_5 + 2x_6 + x_7$$
$$x_2 = 4 - x_3 - x_4 - x_5 - 3x_6 - 3x_7$$

$$x = \begin{pmatrix} 1 - 2x_3 + x_4 - x_5 + 2x_6 + x_7 \\ 4 - x_3 - x_4 - x_5 - 3x_6 - 3x_7 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{pmatrix}$$

$$x(x_3, x_4, x_5, x_6, x_7) = \underbrace{\begin{pmatrix} 1 \\ 4 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}}_{c} + x_3 \underbrace{\begin{pmatrix} -2 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}}_{\tilde{c}_1} + x_4 \underbrace{\begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}}_{\tilde{c}_2} + x_5 \underbrace{\begin{pmatrix} -1 \\ -1 \\ 0 \\ 0 \\ +1 \\ 0 \\ 0 \end{pmatrix}}_{\tilde{c}_3} + x_6 \underbrace{\begin{pmatrix} 2 \\ -3 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}}_{\tilde{c}_4} + x_7 \underbrace{\begin{pmatrix} 1 \\ -3 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}}_{\tilde{c}_5} \quad - (\#)$$

particular solution

Solution set, $S = \{ x(x_3, x_4, x_5, x_6, x_7) \mid x_3, x_4, x_5, x_6, x_7 \in \mathbb{R} \}$

Check that:

1) $Ac = b$
2) $A\tilde{c}_i = 0 \quad \forall i \in \{1, 2, 3, 4, 5\}$
3) $\{\tilde{c}_i\}$ is linearly independent

x_3, x_4, x_5, x_6, x_7 are FREE variables.

x_3, x_4, x_5, x_6, x_7 are FREE variables.

x_1, x_2 are NOT.

Variables corresponding to NON-PIVOTAL rows are FREE.

Thus, if rank is r, there are $(\overset{\#\text{ colums}}{n} - r)$ free variables.

Way to find particular solution

Just zero out all the free (non-pivotal) variables in x, and then solve for $Ax = b$.

Another way to justify that $A\tilde{c}\_i = 0$ for all i is to recognize that some columns are non pivotal in the REF (note that order of columns is preserved while performing EROs) and hence can be generated by some other columns (if rank is not zero), so

for our case, let $A = [c_1 \; c_2 \; c_3 \cdots c_7]$

$\downarrow$ REF

$$A' = [\underbrace{c_1' \; c_2'}_{\text{pivotal}} \cdots c_7']$$

thus, $c_3' = ac_1' + bc_2'$

$0 = ac_1' + bc_2' - c_3'$

apply reverse ERO's,

$$0 = ac_1 + bc_2 - c_3$$

$$0 = [c_1 \; c_2 \cdots c_7]\begin{pmatrix} a \\ b \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$\therefore \begin{pmatrix} a \\ b \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ is a sol to $Ax = 0$

Check, $c_3 = 2c_1 + c_2$ in our case

so $a = 2$

$b = 1$

$\Rightarrow (2 \;\; 1 \; -1 \;\; 0 \; 0 \; 0 \; 0)^T$ is a sol^n of $Ax = 0$

compare this with the vector x_3 is multiplied to in (#)

Compare this with the vector
x_3 is multiplied to in (#)

Thus, each non pivotal column will lead to such a solution of Ax = 0, further justifying why

Nullity(A) = #columns - Rank(A)
Where Rank(A) is just #pivots in REF

Problem 6 (v)

Consider the system,

$$\underbrace{\left[\begin{array}{ccccccc|c} 1 & 0 & 2 & -1 & +1 & -2 & -1 & 1 \\ 2 & 2 & 6 & 0 & 4 & 2 & 4 & 10 \\ 1 & -1 & 1 & -2 & 0 & -5 & -4 & -3 \\ 2 & 2 & 6 & 0 & 4 & 2 & 4 & 10 \end{array}\right]}_{A \qquad\qquad\qquad\qquad b}$$

Perform EROs. Use REF to answer:

Using (iv) write a basis of the null space of A.

We saw that $\tilde{c}_1, \tilde{c}_2, \tilde{c}_3, \tilde{c}_4, \tilde{c}_5$ were lin. indp. in (iv).

Recall, in (i) we saw that $N(A) = 5$

well?

a Basis is $\{\tilde{c}_i\}$!

What if we did not do (iv)? we will just solve for $Ax = 0$ or $A_{aug} x_{aug} = 0$ then, right?

Removing the particular solution as in (iv), we will get exactly the same vector X. (convince yourself)

Thus, for the complete solution set of $Au = b$,

1) find particular solution $\rightarrow c$
2) find basis for null space $\rightarrow \{\tilde{c}_i\}$
3) Ans $= \{ c + \sum_i \lambda_i \tilde{c}_i \mid \lambda_i \in \mathbb{R} \}$

Now let A be a m x n matrix. With all of the above in mind, show that

- Ax = 0 has ONLY the zero solution iff REF(A) has n non-zero rows.
- (using the above point) if m < n Ax = 0 has a non-trivial solution.