MA 109 D2 T1
Practice Assignment Solutions

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1. Consider the sequence $(a_n)_n$ defined as,

$$a_n := \frac{3n^2 - 1}{10n + 5n^2}$$

Determine if it converges or not. If it does, find the limit. Use nothing but the $\epsilon - N$ definition.

Solution First, we guess that the limit is $3/5$ and we figure out the $N_0(\epsilon)$ needed. Note that, we need,

$$\begin{aligned} |\frac{3n^2 - 1}{10n + 5n^2} - \frac{3}{5}| &< \epsilon \\ |\frac{30n + 5}{5(10n + 5n^2)}| &< \epsilon \\ |\frac{6n + 1}{10n + 5n^2}| &< \epsilon \end{aligned}$$

Noting that

$$\frac{6n + 1}{10n + 5n^2} < \frac{6n + 12}{10n + 5n^2} = \frac{6(n + 2)}{5n(n + 2)} = \frac{6}{5n}$$

Thus it suffices to have $(6/5n) < \epsilon$, that is, $n > (6/5\epsilon)$. Hence, we can choose

$$N_0(\epsilon) := \lfloor \frac{6}{5\epsilon} \rfloor + 1$$

Now we give the proof.

Proof Let $\epsilon > 0$ be given. We claim that the sequence converges to $3/5$. Note that for any $n \in \mathbb{N}$ we have,

$$|\frac{3n^2 - 1}{10n + 5n^2} - \frac{3}{5}| = \frac{6n + 1}{10n + 5n^2}$$

Further, see that,

$$\frac{6n + 1}{10n + 5n^2} < \frac{6n + 12}{10n + 5n^2} = \frac{6(n + 2)}{5n(n + 2)} = \frac{6}{5n}$$

also holds for all $n \in \mathbb{N}$. Thus we have that,

$$|\frac{3n^2 - 1}{10n + 5n^2} - \frac{3}{5}| < \frac{6}{5n} \quad \forall n \in \mathbb{N} \quad (*)$$

Now choose

$$N_0(\epsilon) := \lfloor \frac{6}{5\epsilon} \rfloor + 1$$

Let $n \in \mathbb{N}$ be such that $n > N_0(\epsilon)$. Then we have,

$$\begin{aligned} \frac{1}{n} &< \frac{1}{N_0(\epsilon)} \\ &< \frac{1}{\lfloor \frac{6}{5\epsilon} \rfloor + 1} \\ &< \frac{1}{\frac{6}{5\epsilon}} = \frac{5\epsilon}{6} \end{aligned}$$

This implies that

$$\frac{6}{5n} < \epsilon \text{ whenever } n > N_0(\epsilon) \quad (**)$$

Thus, by $(*)$ and $(**)$ we have that,

$$\left| \frac{3n^2-1}{10n+5n^2} - \frac{3}{5} \right| < \epsilon \text{ whenever } n > N_0(\epsilon)$$

Hence, we are done. □

2. For any sequence $(a_n)_n$, show the convergence of a_n implies the convergence of $b_n := |a_n|$. Does the converse hold?

   Solution

   - For the first part, we have to show that the convergence of a_n implies the convergence of $b_n := |a_n|$.
     Proof Let a_n be convergent to $L \in \mathbb{R}$. Thus, we have that for any $\epsilon_1 > 0$ there exists $N_0(\epsilon_1) \in \mathbb{N}$ such that

     $$n > N_0(\epsilon_1) \implies |a_n - L| < \epsilon_1$$

     Now, let $\epsilon > 0$ be given. Let $\epsilon_1 := \epsilon$, and choose $N(\epsilon)$ to be $N_0(\epsilon_1)$ as given by the convergence of a_n. Now note that,

     $$||a_n| - |L|| \leq |a_n - L| \quad \forall n \in \mathbb{N} \quad (*)$$

     holds by the triangle inequality. Now, let $n > N(\epsilon)$ be a natural number. See that,

     $$|a_n - L| < \epsilon$$

     is guaranteed by the convergence of a_n. Combining this and $(*)$, we get that

     $$||a_n| - |L|| < \epsilon \text{ whenever } n > N(\epsilon)$$

     Hence, we are done. □

   - For the second part, we claim that the converse does not hold. For the same, consider the sequence

     $$a_n := (-1)^n \quad \forall n \in \mathbb{N}$$

     Note that $b_n := |a_n|$ is just the constant sequence 1 and hence it converges. But a_n does not converge to any real number. [1]

3. Consider $f : \mathbb{R} \to \mathbb{R}$ defined as,

   $$f(x) := x^2 + 2$$

   Prove that f is continuous at $x = 2$ by using the $\epsilon - \delta$ definition.

   Solution The point to realize here is that continuity is a *local property*. That is, if a δ works for some ϵ in the $\epsilon - \delta$ definition, then any $\delta' < \delta$ also works for that ϵ. With that in mind, we choose δ to be less than one. Now,

   $$|f(x) - f(2)| = |x^2 - 4| = |x+2||x-2|$$

   If $x \in (2-\delta, 2+\delta)$, then $1 < x < 3$ (Why?). Thus, $3 < x+2 < 5$. Thus, $x \in (2-\delta, 2+\delta) \implies |x+2| < 5$. Hence,

   $$|f(x) - f(2)| < 5|x-2| = 5\delta$$

   It is clear that choosing $\delta(\epsilon) := \epsilon/5$ should work, as long as we make sure it is less than one (that is what we started with!). Now we give the proof.
   Proof Let $\epsilon > 0$ be given. Choose

   $$\delta(\epsilon) := \frac{1}{2}\min\{1, \epsilon/5\}$$

   Let $x \in (2-\delta, 2+\delta)$. Note that this implies that $1 < x < 3$, and hence $|x+2| < 5$. Now, see that,

   $$|f(x) - f(2)| = |x^2 - 4| = |x+2||x-2| < 5 \cdot \delta$$

   Since $\delta = 0.5\min\{1, \epsilon/5\}$, we have that $5\delta = \min\{5/2, \epsilon/2\}$ which is always less than ϵ (How?). Hence,

   $$|f(x) - f(2)| < \epsilon \text{ whenever } |x-2| < \delta(\epsilon)$$

   We are done. □

[1] Ideally you should proceed to give a proof for that, but we shall leave it here as we discussed the proof already.

4. Formulate the definition of a finite limit at (positive) infinity. Using this, prove or disprove that

$$\lim_{x\to\infty} \frac{\sin\left(x^2+3x+120\right)}{x^2}$$

exists.

Solution

We say that $\lim_{x\to\infty} f(x) = L$ if for any $\epsilon > 0$ there exists $M > 0$ such that

$$|f(x) - L| < \epsilon \text{ whenever } x > M$$

We claim that the given limit exists and is zero.
Proof Let $\epsilon > 0$ be given. Choose $M(\epsilon) := 1/\sqrt{\epsilon}$. Let $x > M(\epsilon)$ and note that,

$$\begin{aligned} \mid \frac{\sin\left(x^2+3x+120\right)}{x^2} - 0 \mid &\leq \mid \frac{1}{x^2} \mid \\ &< \frac{1}{(M(\epsilon))^2} \\ &< \epsilon \end{aligned}$$

Where the first inequality followed from the fact that $|\sin| \leq 1$ and the second inequality followed from the fact that $x > M(\epsilon) > 0$. Hence,

$$\frac{\sin\left(x^2+3x+120\right)}{x^2} - 0 \mid < \epsilon \text{ whenever } x > M(\epsilon)$$

Thus, we are done. □