Problem One

$A, B \in \mathbb{R}^{n\times n}$, $n \in \mathbb{N}$

Show that

$A+iB$ is invertible $\Rightarrow \det\begin{pmatrix} A & B \\ -B & A \end{pmatrix} > 0$

$i := \sqrt{-1}$

Block Row operations

given some block matrix

$$O = \begin{pmatrix} O_1 & O_2 \\ O_3 & O_4 \end{pmatrix} \qquad O_i \in \mathbb{R}^{n\times n}\ \forall i$$

$$EO = \begin{pmatrix} O_1 - aO_3 & O_2 - aO_4 \\ O_3 & O_4 \end{pmatrix}$$

$$E = \begin{pmatrix} I & -aI \\ 0 & I \end{pmatrix}$$

now, let $O := \begin{pmatrix} A & B \\ -B & A \end{pmatrix}$

$$E := \begin{pmatrix} I & -iI \\ 0 & I \end{pmatrix}$$

what is EO

$$EO = \begin{pmatrix} A+iB & B-iA \\ -B & A \end{pmatrix}$$

$$= \begin{pmatrix} A+iB & -i(A+iB) \\ -B & A \end{pmatrix}$$

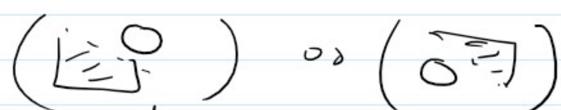
fact: $\det(M) = \det(M^T)$

using this fact, we say that column operations do not change the $\det(\cdot)$

$$C_2 \to C_2 + iC_1$$

$$\tilde{E}(EO) = \begin{pmatrix} A+iB & 0 \\ -B & A-iB \end{pmatrix}$$

Claim 1 The determinant of any triangular matrix is the product of its diagonal entries.



lower upper

Proof: use induction.

Claim 2 $\det\begin{pmatrix} I_k & O \\ O & A_n \end{pmatrix} = \det(A)$

$A \in \mathbb{R}^{n\times n}$ $k\times k$ . . . . . ?

$$\det\begin{pmatrix} I_k & 0 \\ 0 & A_n \end{pmatrix} = \det(A)$$

$A \in \mathbb{R}^{n\times n}$
$I \in \mathbb{R}^{k\times k} \quad \forall k \in \mathbb{N}$

Proof

$k = 1$

$$\begin{pmatrix} 1 & \\ & \boxed{A} \end{pmatrix}$$

let it be true for $k = \ell \in \mathbb{N}$

see that for $k = \ell + 1$ we have

$$\begin{pmatrix} 1 & \\ & \begin{matrix} I_\ell & \\ & A \end{matrix} \end{pmatrix} \quad \det A$$

$$\begin{pmatrix} P & \overset{n\times n}{0} \\ R & Q \end{pmatrix} = \underbrace{\underbrace{\begin{pmatrix} P & 0 \\ 0 & I \end{pmatrix}}_{\det = \det(P)} \underbrace{\begin{pmatrix} I & 0 \\ R & I \end{pmatrix}}_{\det = 1}}_{\begin{pmatrix} P & 0 \\ R & I \end{pmatrix}} \underbrace{\begin{pmatrix} I & 0 \\ 0 & Q \end{pmatrix}}_{\det = \det(Q)}$$

$$\underbrace{\begin{pmatrix} P & 0 \\ R & I \end{pmatrix}\begin{pmatrix} I & 0 \\ 0 & Q \end{pmatrix}}_{\begin{pmatrix} P & 0 \\ R & Q \end{pmatrix}}$$

$$\det\left[\begin{pmatrix} A+iB & 0 \\ -B & A-iB \end{pmatrix}\right] = \underbrace{\det(A+iB)}_{z}\ \underbrace{\det(A-iB)}_{\bar{z}}$$

$$= |z|^2$$

using the fact that $A+iB$ is inv,
$z \neq 0$

$\therefore\ |z|^2 > 0$

$$\det\begin{pmatrix} A+iB & 0 \\ -B & A-iB \end{pmatrix} > 0$$

Problem Two

Let $A := \begin{bmatrix} 2 & 0 & 6 & 0 & 4 \\ 5 & 3 & 2 & 2 & 7 \\ 2 & 5 & 7 & 5 & 5 \\ 2 & 0 & 9 & 2 & 7 \\ 7 & 8 & 4 & 2 & 1 \end{bmatrix} \in \mathbb{R}^{5\times5}$

Show that $\det(A)$ is divisible by 17.

Hint Look at the numbers formed by the rows. → div by 17

if $\tilde{A}$ is formed by performing

$$C_1 \to 10^4 C_1 + 10^3 C_2 + 10^2 C_3 + 10 C_4 + C_5 \checkmark$$

then, $\det(\tilde{A}) = 10^4 \det(A)$

the first column of $\tilde{A}$ is div by 17

now, let B be formed from $\tilde{A}$ by

$$C_1 \to \left(\frac{C_1}{17}\right)$$

then $\det(B) = \frac{1}{17} \det(\tilde{A})$

$\det(\tilde{A}) = 17 \det(B)$

$$17 \det(B) = 10^4 \det(A)$$

Claim if a matrix D has integer entr, then $\det(D)$ is an integer.

Entries of both A & B are integers.

$\det(A), \det(B) \in \mathbb{Z}$

$$\to \left[\ \underline{\underline{\det(A)}} = \frac{17}{10^4} \underline{\underline{\det(B)}} \right.$$

can I now say that $\det(B) = 10^4 \times z$, $z \in \mathbb{Z}$

ans =

-5.6000e+05

$\det(A) = \frac{17}{10^4} \times z \times 10^4 = \underline{17z}$

$\therefore$ det(A) is a multiple of 17.

Problem Three

Show that, a necessary condition for $\begin{pmatrix} x^2+ax+b=0 \\ x^2+px+q=0 \end{pmatrix}$ to have a common root is that

$$\det\left(\underbrace{\begin{bmatrix} 1 & a & b & 0 \\ 0 & 1 & a & b \\ 1 & p & q & 0 \\ 0 & 1 & p & q \end{bmatrix}}_{A}\right) = 0$$

common root $\Rightarrow$

Let α be the common root

$$C_4 \to C_4 + \alpha^3 C_1 + \alpha^2 C_2 + \alpha C_3$$

$$\begin{pmatrix} \cdot & \cdot & \cdot & \alpha^3+\alpha^2 a+\alpha b \\ \cdot & \cdot & \cdot & \alpha^2+\alpha a+b \\ \cdot & \cdot & \cdot & \alpha^3+\alpha^2 p+\alpha q \\ \cdot & \cdot & \cdot & \alpha^2+\alpha p+q \end{pmatrix}$$

0 ?
$=$

$$\alpha^2 + a\alpha + b = 0 \quad -①$$
$$\alpha^2 + p\alpha + q = 0 \quad -②$$

$$0\alpha^3 + \alpha^2 + a\alpha + b = 0$$

① $\times$ α

$$\alpha^3 + a\alpha^2 + b\alpha + 0 = 0$$

$$0\alpha^3 + \alpha^2 + p\alpha + q = 0$$

$$\alpha^3 + p\alpha^2 + q\alpha + 0 = 0$$

$$\begin{pmatrix} & & \\ & A & \\ & & \end{pmatrix} \underbrace{\begin{pmatrix} \alpha^3 \\ \alpha^2 \\ \alpha^1 \\ \alpha^0 \end{pmatrix}}_{\neq 0} = 0$$

$Ax = 0$ has non trivial sol^n

$\Rightarrow \underline{\det(A) = 0}$

3) Show that a Necessary Condition for

$$\begin{cases} x^2 + ax + b = 0 \\ x^2 + px + q = 0 \end{cases}$$ to have a Common root is that

$$\begin{vmatrix} 1 & a & b & 0 \\ 0 & 1 & a & b \\ 1 & p & q & 0 \\ 0 & 1 & p & q \end{vmatrix} = 0$$

Hint: Look up 4 linear equations in y_1, y_2, y_3, y_4 with non trivial solution

$c^3, c^2, c, 1$

Problem Four

Find the values of β for which Cramer's rule is applicable. For the remaining value(s) of β, find the number of solutions.

$$\begin{aligned} x + 2y + 3z &= 20 \\ x + 3y + z &= 13 \\ x + 6y + \beta z &= \beta. \end{aligned}$$

Start with performing ERO's on $[A|b]$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 20 \\ 1 & 3 & 1 & 13 \\ 1 & 6 & \beta & \beta \end{array}\right]$$

$$\Big\downarrow \begin{array}{l} R_2 \to R_2 - R_1 \\ R_2 \to R_3 - R_1 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 20 \\ 0 & 1 & -2 & -7 \\ 0 & 4 & \beta-3 & \beta-20 \end{array}\right]$$

$$\Big\downarrow R_3 \to R_3 - 4R_2$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 20 \\ 0 & 1 & -2 & -7 \\ 0 & 0 & \underline{\beta+5} & \beta+8 \end{array}\right] \quad \beta - \cancel{20} + 28$$

$\det(A) = \underline{\beta+5}$ $\Rightarrow$ for $\beta \neq -5$ Cramer's Rule is applicable

if $\beta = -5$, $\left.\begin{array}{l} \text{Rank}(A) = 2 \\ \text{Rank}([A|b]) = 3 \end{array}\right]$ no sol^n

Cramer's rule is applicable <=> det(A) is non zero, hence one solution
Cramer's rule not applicable => beta = -5, and no solution
Thus we either have one solution or no solution

Problem Five

Find whether the following set of vectors is linearly dependent or independent:

$$\{\underset{v_1}{a\mathbf{i} + b\mathbf{j} + c\mathbf{k}},\ \underset{v_2}{b\mathbf{i} + c\mathbf{j} + a\mathbf{k}},\ \underset{v_3}{c\mathbf{i} + a\mathbf{j} + b\mathbf{k}}\}.$$

$$V = [v_1,\ v_2\ v_3] = \begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}$$

$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \xrightarrow{C_3 \to C_3 + C_2 + C_1} \begin{vmatrix} a & b & a+b+c \\ b & c & a+b+c \\ c & a & a+b+c \end{vmatrix}$$

$$(a+b+c) \begin{vmatrix} a & b & 1 \\ b & c & 1 \\ c & a & 1 \end{vmatrix}$$

$$\Big\downarrow \begin{array}{l} R_1 \to R_1 - R_3 \\ R_2 \to R_2 - R_3 \end{array}$$

$$(a+b+c) \begin{vmatrix} a-c & b-a & 0 \\ b-c & c-a & 0 \\ c & a & 1 \end{vmatrix}$$

Expand along R_3

$$(a+b+c)\left(\overbrace{(c-a)(a-c)}^{-(a-c)^2} - (b-a)(b-c)\right)$$

$$(a+b+c)\left(-a^2 - c^2 + 2ac - (b^2 + ac - ab - bc)\right)$$

$$(a+b+c)\left(-a^2 - b^2 - c^2 + ab + bc + ca\right)$$

$$\downarrow$$

$$-\frac{1}{2}(a+b+c)\left((a-b)^2 + (b-c)^2 + (c-a)^2\right)$$

$\det(V) = 0$ iff $a+b+c=0$ or $a=b=c$

$\Updownarrow$

v_1, v_2, v_3 are lin dep.

o/w, independent.

Problem Six

Invert the matrix $H = \begin{bmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{bmatrix}$.

$A_{ij} = (-1)^{i+j} M_{ij}$

$M_{11} = \frac{1}{3} \cdot \frac{1}{5} - \frac{1}{4} \cdot \frac{1}{4} = \frac{1}{15} - \frac{1}{16} = \frac{1}{240}$

$M_{12} = \frac{1}{2} \cdot \frac{1}{5} - \frac{1}{4} \cdot \frac{1}{3} = \frac{1}{10} - \frac{1}{12} = \frac{1}{60}$

$M_{13} = \frac{1}{2} \cdot \frac{1}{4} - \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{8} - \frac{1}{9} = \frac{1}{72}$

$M_{21} = \frac{1}{2} \cdot \frac{1}{5} - \frac{1}{3} \cdot \frac{1}{4} = \frac{1}{10} - \frac{1}{12} = \frac{1}{60}$

$M_{22} = 1 \cdot \frac{1}{5} - \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{5} - \frac{1}{9} = \frac{4}{45}$

$M_{23} = 1 \cdot \frac{1}{4} - \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{4} - \frac{1}{6} = \frac{1}{12}$

$M_{31} = \frac{1}{2} \cdot \frac{1}{4} - \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{8} - \frac{1}{9} = \frac{1}{72}$

$M_{32} = 1 \cdot \frac{1}{4} - \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{4} - \frac{1}{6} = \frac{1}{12}$

$M_{33} = \frac{1}{3} \cdot 1 - \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{12}$

thus, the minor matrix, $\underline{M} = \begin{pmatrix} \frac{1}{240} & \frac{1}{60} & \frac{1}{72} \\ \frac{1}{60} & \frac{4}{45} & \frac{1}{12} \end{pmatrix}$

Thus, the minor matrix, $\underline{M} = \begin{pmatrix} \frac{1}{60} & \frac{4}{45} & \frac{1}{12} \\ \frac{1}{72} & \frac{1}{12} & \frac{1}{12} \end{pmatrix}$

Cofactor matrix, $C = \begin{array}{c} \quad \xrightarrow{i+j = \text{odd}} \\ \begin{pmatrix} \frac{1}{240} & -\frac{1}{60} & \frac{1}{72} \\ -\frac{1}{60} & \frac{4}{45} & -\frac{1}{12} \\ \frac{1}{72} & -\frac{1}{12} & \frac{1}{12} \end{pmatrix} \end{array}$

~~Adjoint~~

Adjugate, $\mathrm{Adj}(H) = C^T$

$$= \begin{pmatrix} \frac{1}{240} & -\frac{1}{60} & 1/72 \\ -1/60 & 4/45 & -1/12 \\ \frac{1}{72} & -\frac{1}{12} & 1/12 \end{pmatrix}$$

$$\det(H) = \frac{1}{2160}$$

$$\therefore \boxed{\underline{H}^{-1} = \frac{1}{\det(H)} \mathrm{Adj}(H) = \begin{pmatrix} 9 & -36 & 30 \\ -36 & 192 & -180 \\ 30 & -180 & 180 \end{pmatrix}}$$

Problem Seven

(Wronskian) Let $f_1, f_2, ..., f_n$ be functions over some interval (a, b). Their Wronskian is another function on (a, b) defined by a determinant involving the given functions and their derivatives upto the order $n-1$.

$$W_{f_1,f_2,...,f_n}(x) \overset{def}{=} \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix}.$$

Vector $\boxed{\text{SUB}}$ space.

· $\mathbb{R}^n$ is a vector space.

·

Prove that if $c_1f_1 + c_2f_2 + \cdots + c_nf_n = 0$ holds over the interval (a, b) for some constants $c_1, c_2, ..., c_n$ and $W_{f_1,f_2,...,f_n}(x_0) \neq 0$ at some x_0, then $c_1 = c_2 = \cdots = c_n = 0$. In other words, nonvanishing of $W_{f_1,f_2,...,f_n}$ at a single point establishes linear independence of $f_1, f_2, ..., f_n$ on (a, b).

$f \in C^\infty$

Caution: The converse is false. $W \equiv 0 \not\Rightarrow f_1, f_2, ..., f_n$ linearly dependent on (a, b). Though one can prove existence of a subinterval of (a, b) where linear dependence holds.

$\mathbb{C}^n, \mathbb{R}^n$

$$c_1 f_1 + c_2 f_2 \cdots c_n f_n = 0$$

$$c_1 f_1' + c_2 f_2' \cdots c_n f_n' = 0$$

$$\vdots$$

$$c_1 f_n^{n-1} \cdots = 0$$

$$\begin{pmatrix} & & \\ & W & \\ & & \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \end{pmatrix}$$