Question Five

5. Let $\mathbf{F} = \nabla f$, where $f(x, y) = \sin(x - 2y)$. Find curves C_1 and C_2 that are not closed and satisfy

$$\int_{C_1} \mathbf{F}.\mathbf{ds} = 0, \quad \int_{C_2} \mathbf{F}.\mathbf{ds} = 1.$$

$\vec{F} = \nabla f$

$$\int_{C_{A\to B}} \vec{F} = \int_{\binom{C}{AB}} (\nabla f) = f(A) - f(B)$$

$f(B) = 0$

$B = (0,0)$

$A = (x,y)$

$C \; A \to B$

$$\int_C \vec{F} = f(x,y) = \sin(x-2y)$$

$y = 0$, $x = 2\pi$ $\quad B = (2\pi, 0)$

$x = \frac{\pi}{2}$, $y = 0$ $\quad B = (\frac{\pi}{2}, 0)$

$\vec{C_1} : [0,1] \to \mathbb{R}^2$

$\vec{C_1}(t) := (2\pi t, 0) \Rightarrow \int_{C_1} \vec{F} = 0$

$\vec{C_2} : [0,1] \to \mathbb{R}^2 \quad \vec{C_2}(t) := (\frac{\pi}{2} t, 0) \Rightarrow \int_{C_2} \vec{F} = 1$

Question Six

6. Determine whether or not $\mathbf{F}$ is a conservative vector field. If it is, find a function f such that $\mathbf{F} = \nabla f$.

(a) $\mathbf{F}(x, y) = y^2 e^{xy}\mathbf{i} + (1 + xy)e^{xy}\mathbf{j}$, for all $(x, y) \in \mathbb{R}^2$.

(b) $\mathbf{F}(x, y) = (ye^x + \sin y)\mathbf{i} + (e^x + x\cos y)\mathbf{j}$, for all $(x, y) \in \mathbb{R}^2$.

(c) $\mathbf{F}(x, y) = (2xy + y^{-2})\mathbf{i} + (x^2 - 2xy^{-3})\mathbf{j}$, for all $(x, y) \in \mathbb{R}^2$ and $y > 0$.

Note: All regions given above are simply conected!

$$\vec{F} = (F_1, F_2)$$

$$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x} \implies \vec{F} \text{ is cons.}$$

a) $F_1(x,y) = y^2 e^{xy}$ $\quad F_2(x,y) = (1+xy)(e^{xy})$

$$\frac{\partial F_1}{\partial y}(x,y) = 2ye^{xy} + y^2 x e^{xy} \qquad \frac{\partial F_2}{\partial x}(x,y) = ye^{xy} + (1+xy)ye^{xy}$$

$$= e^{xy}(y + y + xy^2)$$

$\therefore$ cons.

$$f : D \to \mathbb{R}$$

$$\vec{F} = \vec{\nabla} f$$

$$(F_1, F_2) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$$

$$\frac{\partial f}{\partial x} = (y^2 e^{xy})$$

$$(y^2)\int e^{xy}$$

$$y^2\left(\frac{e^{xy}}{y}\right)$$

$$f(x,y) = (ye^{xy}) + c(y)$$

$$\frac{\partial f}{\partial y} = (e^{xy} + xye^{xy}) + c'(y)$$

$$= F_2(x,y) = (1+xy)e^{xy}$$

$$c'(y) = 0$$

$$c'(y) = K$$

$$f(x,y) = ye^{xy} + K$$

$c'(y) = k$

$$f(x,y) = ye^{xy} + k$$

6. Determine whether or not **F** is a conservative vector field. If it is, find a function f such that $\mathbf{F} = \nabla f$.

   (a) $\mathbf{F}(x,y) = y^2e^{xy}\mathbf{i} + (1+xy)e^{xy}\mathbf{j}$, for all $(x,y) \in \mathbb{R}^2$.

   (b) $\mathbf{F}(x,y) = (ye^x + \sin y)\mathbf{i} + (e^x + x\cos y)\mathbf{j}$, for all $(x,y) \in \mathbb{R}^2$.

   (c) $\mathbf{F}(x,y) = (2xy + y^{-2})\mathbf{i} + (x^2 - 2xy^{-3})\mathbf{j}$, for all $(x,y) \in \mathbb{R}^2$ and $y > 0$.

b)

$$F_1 = ye^x + \sin y \qquad F_2 = \underline{e^x + x\cos y}$$

$$\frac{\partial F_1}{\partial y} = e^x + \cos y = \frac{\partial F_2}{\partial x} = (e^x + \cos y)$$

$$\frac{\partial f}{\partial x} = F_1 = ye^x + \sin y$$

$$f(x,y) = ye^x + x\sin y + c(y)$$

$$\frac{\partial f}{\partial y} = e^x + x\cos y + c'(y) = e^x + x\cos y$$

$$c'(y) = 0$$

$$\therefore \quad f(x,y) = ye^x + x\sin y + k$$

6. Determine whether or not **F** is a conservative vector field. If it is, find a function f such that $\mathbf{F} = \nabla f$.

   (a) $\mathbf{F}(x,y) = y^2e^{xy}\mathbf{i} + (1+xy)e^{xy}\mathbf{j}$, for all $(x,y) \in \mathbb{R}^2$.

   (b) $\mathbf{F}(x,y) = (ye^x + \sin y)\mathbf{i} + (e^x + x\cos y)\mathbf{j}$, for all $(x,y) \in \mathbb{R}^2$.

   (c) $\mathbf{F}(x,y) = (2xy + y^{-2})\mathbf{i} + (x^2 - 2xy^{-3})\mathbf{j}$, for all $(x,y) \in \mathbb{R}^2$ and $y > 0$.

c)

$$F_1 = \underline{2xy} + \left(\frac{1}{y^2}\right) \qquad F_2 = x^2 - \left(\frac{2x}{y^3}\right)$$

$$\frac{\partial F_1}{\partial y} = 2x - \frac{2}{y^3} \qquad \frac{\partial F_2}{\partial x} = 2x - \frac{2}{y^3}$$

$$f(x,y) = (x^2y) + \left(\frac{x}{y^2}\right) + k$$

Question Seven

7. Let $\mathbf{F}$ be a vector field on $\mathbb{R}^2$. Find a function f such that $\mathbf{F} = \text{grad}\, f$ and using it evaluate $\int_{\mathbf{c}} \mathbf{F} \cdot \mathbf{ds}$, where $\mathbf{F}$ and $\mathbf{c}$ are given below:

(a) $\mathbf{F}(x,y,z) = (2xyz + \sin x)\mathbf{i} + x^2 z\mathbf{j} + x^2 y\mathbf{k}$ and $\mathbf{c}(t) = (\cos^5 t, \sin^3 t, t^4)$, $0 \le t \le \pi$.

(b) $\mathbf{F}(x,y) = (1+xy)e^{xy}\mathbf{i} + x^2 e^{xy}\mathbf{j}$ and $\mathbf{c}(t) = (\cos t, 2\sin t)$, $0 \le t \le \frac{\pi}{2}$.

(c) $\mathbf{F}(x,y,z) = yz\mathbf{i} + xz\mathbf{j} + (xy + 2z)\mathbf{k}$ and $\mathbf{c}$ is the line segment from $(1,0,-2)$ to $(4,6,3)$.

a) $\vec{F} = \nabla f)$

$\frac{\partial f}{\partial x} = (2xyz + \sin x)$ $\quad \frac{\partial f}{\partial y} = x^2 z$ $\quad \frac{\partial f}{\partial z} = x^2 y$

$f(x,y,z) = (x^2 z y) + k(x,z)$ — 1

$\frac{\partial f}{\partial x} = \left[2xzy + \frac{\partial k(x,z)}{\partial x}\right] = 2xyz + \sin x$

$\frac{\partial k}{\partial x} = \sin x$

$k(x,z) = (-\cos x) + c(z)$ — 2

$f(x,y,z) = x^2 yz + (-\cos x) + c(z)$

$\frac{\partial f}{\partial z} = [x^2 y + c'(z)] = x^2 y$

$c'(z) = 0$

$f(x,y,z) = x^2 yz + (-\cos x) + k$

C from A to B

$A = (1,0,0)$

$B = (-1,0,\pi^4)$

$\int_c \vec{F} = f(B) - f(A)$

$= [(-\cos(-1)) + k] - [-\cos(1)]$

$= 0$

7. Let $\mathbf{F}$ be a vector field on $\mathbb{R}^2$. Find a function f such that $\mathbf{F} = \text{grad}\, f$ and using it evaluate $\int_{\mathbf{c}} \mathbf{F} \cdot \mathbf{ds}$, where $\mathbf{F}$ and $\mathbf{c}$ are given below:

(a) $\mathbf{F}(x,y,z) = (2xyz + \sin x)\mathbf{i} + x^2 z\mathbf{j} + x^2 y\mathbf{k}$ and $\mathbf{c}(t) = (\cos^5 t, \sin^3 t, t^4)$, $0 \le t \le \pi$.

(b) $\mathbf{F}(x,y) = (1+xy)e^{xy}\mathbf{i} + x^2 e^{xy}\mathbf{j}$ and $\mathbf{c}(t) = (\cos t, 2\sin t)$, $0 \le t \le \frac{\pi}{2}$.

(c) $\mathbf{F}(x,y,z) = yz\mathbf{i} + xz\mathbf{j} + (xy + 2z)\mathbf{k}$ and $\mathbf{c}$ is the line segment from $(1,0,-2)$ to $(4,6,3)$.

b) $f(x,y) = (xe^{xy} + k)$

$A = (1,0)$

$B = (0,2)$

$f(B) - f(A) = (0 + k) - (1e^{1\times 0} + k)$

$$f(B) - f(A) = (0 + k) - (1e^{1\times 0} + k)$$
$$= \underline{-1}$$

7. Let $\mathbf{F}$ be a vector field on $\mathbb{R}^2$. Find a function f such that $\mathbf{F} = \operatorname{grad} f$ and using it evaluate $\int_{\mathbf{c}} \mathbf{F} \cdot \mathbf{ds}$, where $\mathbf{F}$ and $\mathbf{c}$ are given below:

(a) $\mathbf{F}(x, y, z) = (2xyz + \sin x)\mathbf{i} + x^2z\mathbf{j} + x^2y\mathbf{k}$ and $\mathbf{c}(t) = (\cos^5 t, \sin^3 t, t^4)$, $0 \le t \le \pi$.

(b) $\mathbf{F}(x, y) = (1 + xy)e^{xy}\mathbf{i} + x^2e^{xy}\mathbf{j}$ and $\mathbf{c}(t) = (\cos t, 2\sin t)$, $0 \le t \le \frac{\pi}{2}$.

(c) $\mathbf{F}(x, y, z) = yz\mathbf{i} + xz\mathbf{j} + (xy + 2z)\mathbf{k}$ and $\mathbf{c}$ is the line segment from $\underset{A}{\underline{(1, 0, -2)}}$ to $\underset{B}{\underline{(4, 6, 3)}}$.

c)

$$f(x,y,z) = (xyz + z^2 + c)$$

$$f(B) - f(A)$$
$$= ((4\times 6\times 3) + 3^2 + c) - (0 + (-2)^2 + c)$$
$$= (72) + 9 - 4$$
$$= 72 + 5$$
$$= \underline{\underline{77}}$$

Question Eight

$(\vec{v} : \mathbb{R}^3 \to \mathbb{R}^3)$

8. For $\mathbf{v} = (2xy + z^3)\mathbf{i} + x^2\mathbf{j} + 3xz^2\mathbf{k}$, show that $\nabla\phi = \mathbf{v}$ for some ϕ and hence calculate $\oint_C \mathbf{v} \cdot d\underline{s}$ where C is any arbitrary smooth closed curve.

$$\vec{v} = v_1 \hat{i} + v_2 \hat{j} + v_3 \hat{k}$$

$$\left\{ \begin{aligned} \frac{\partial v_1}{\partial y} &= \frac{\partial v_2}{\partial x} \\ \frac{\partial v_1}{\partial z} &= \frac{\partial v_3}{\partial x} \\ \frac{\partial v_2}{\partial z} &= \frac{\partial v_3}{\partial y} \end{aligned} \right. \implies \exists \text{ such a } \phi$$

$$\vec{v} = \vec{\nabla}\phi$$

$$\frac{\partial \phi}{\partial x} = v_1 = 2xy + z^3$$

$$\phi(x,z,y) = y \int 2x + z^3 \cdot 1$$

$$= yx^2 + z^3 x + c(y,z)$$

$$\frac{\partial \phi}{\partial y} = \left(x^2 + 0 + \frac{\partial c}{\partial y}\right) = v_2 = x^2$$

$$\frac{\partial c}{\partial y} = 0$$

$$c \equiv k(z)$$

$$\phi(x,y,z) = yx^2 + z^3 x + k(z)$$

$$\frac{\partial \phi}{\partial z} = 0 + (3z^2)x + k'(z) = v_3 = 3xz^2$$

$$k'(z) = 0$$

$$k'(z) = \alpha \in \mathbb{R}$$

$$\phi(x,y,z) = (yx^2 + z^3 x + \alpha)$$

$$\oint_C \vec{v} = 0$$

Question Nine

9. Let $\boxed{S = \mathbb{R}^2 \setminus \{(0,0)\}}$. Let

$$\mathbf{F}(x,y) = -\frac{y}{x^2+y^2}\mathbf{i} + \frac{x}{x^2+y^2}\mathbf{j} := F_1(x,y)\mathbf{i} + F_2(x,y)\mathbf{j}.$$

(a) Show that $\left(\frac{\partial}{\partial y}F_1(x,y) = \frac{\partial}{\partial x}F_2(x,y)\right)$ on S.

(b) Compute $\int_C \mathbf{F}.\mathbf{ds}$ where C is the circle: $x^2+y^2=1$.

(c) Is $\mathbf{F}$ a conservative field on S? No!

a)

$$F_1(x,y) = \frac{-y}{x^2+y^2} \qquad F_2(x,y) = \frac{x}{x^2+y^2}$$

$$\frac{\partial F_1}{\partial y} = -\left[\frac{1(x^2+y^2) - (2y)(y)}{(x^2+y^2)^2}\right]$$

$$= -\left(\frac{x^2-y^2}{(x^2+y^2)^2}\right)$$

$$= \left(\frac{y^2-x^2}{(x^2+y^2)^2}\right)$$

$$\frac{\partial F_2}{\partial x} = \frac{1(x^2+y^2) - (2x)x}{(x^2+y^2)^2}$$

$$= \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x} \quad \text{on } S$$

$$(S = \mathbb{R}^2 / \{(0,0)\})$$

9. Let $S = \mathbb{R}^2 \setminus \{(0,0)\}$. Let

$$\mathbf{F}(x,y) = -\frac{y}{x^2+y^2}\mathbf{i} + \frac{x}{x^2+y^2}\mathbf{j} := F_1(x,y)\mathbf{i} + F_2(x,y)\mathbf{j}.$$

(a) Show that $\frac{\partial}{\partial y}F_1(x,y) = \frac{\partial}{\partial x}F_2(x,y)$ on S.

(b) Compute $\int_C \mathbf{F}.\mathbf{ds}$ where C is the circle: $x^2+y^2=1$.

$$\vec{c}(t) = (\cos(t), \sin(t)) \quad t \in [0, 2\pi]$$

$$\int_C \vec{F} = \int_0^{2\pi} \vec{F}(\vec{c}(t)) \cdot \vec{c}'(t)\, dt$$

$$= \int^{2\pi} (-\sin t, +\cos t) \cdot (-\sin t, \cos t)\, dt$$

$$= \int_0^{2\pi} (-\sin t, +\cos t) \cdot (-\sin t, \cos t)\, dt$$

$$= \int_0^{2\pi} (\sin^2 t + \cos^2 t)\, dt$$

$$= 1 \cdot \int_0^{2\pi} dt = \underline{2\pi} \neq \underline{0}$$

Question Ten

10. A radial force field is one which can be expressed as $\mathbf{F}(x,y,z) = f(r)\mathbf{r}$ where $\mathbf{r} = (\mathbf{x},\mathbf{y},\mathbf{z})$ is the position vector and $r = \|\mathbf{r}\|$. Show that, if f is continuous, $\mathbf{F}$ is conservative in $\mathbb{R}^3$.
(Hint. Try to guess what the potential function could be, provided f is continuous.)

① (d I R)

General form: Differentiation under the integral sign [edit]

Theorem. Let $f(x,t)$ be a function such that both $f(x,t)$ and its partial derivative $f_x(x,t)$ are continuous in t and x in some region of the xt-plane, including $a(x) \le t \le b(x)$, $x_0 \le x \le x_1$. Also suppose that the functions $a(x)$ and $b(x)$ are both continuous and both have continuous derivatives for $x_0 \le x \le x_1$. Then, for $x_0 \le x \le x_1$,

$$\frac{d}{dx}\left(\int_{a(x)}^{b(x)} f(x,t)\,dt\right) = \boxed{f(x,b(x))\cdot\frac{d}{dx}b(x)} - f(x,a(x))\cdot\frac{d}{dx}a(x) + \int_{a(x)}^{b(x)}\frac{\partial}{\partial x}f(x,t)\,dt.$$

② (FTC)

First part [edit]

why

This part is sometimes referred to as the *first fundamental theorem of calculus*.[7]

Let f be a continuous real-valued function defined on a closed interval $[a,b]$. Let F be the function defined, for all x in $[a,b]$, by

$$\boxed{F(x) = \int_a^x f(t)\,dt.}$$

Then F is uniformly continuous on $[a,b]$ and differentiable on the open interval (a,b), and

$$F'(x) = f(x)$$

for all x in (a,b) so F is an antiderivative of f.

hint: def^n of conservative

$$\vec{F}(x,y,z) = f(r)\,\vec{r} = \vec{\nabla}(V)$$

$$\frac{\partial V}{\partial r}$$

$$b(x) < \sqrt{x^2+y^2+z^2}$$

$$V(x,y,z) = \left[\int_0^{\sqrt{x^2+y^2+z^2}} f(t)\,dt\right] = \int_0^r f(t)\,dt$$

$$\left(\frac{\partial V}{\partial x}\right) = f\left(\sqrt{x^2+y^2+z^2}\right) \times \frac{\partial}{\partial x}\left(\sqrt{x^2+y^2+z^2}\right)$$

$$= \left[f\left(\sqrt{x^2+y^2+z^2}\right) \times \frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+y^2+z^2}}\right]$$

$$r = \sqrt{x^2+y^2+z^2}$$

(unwanted)

$$\left[\frac{\partial V}{\partial x} = \left(\frac{f(r)}{r}\right)x\right]$$

$$(\vec{\nabla} V) = f(r)\,\hat{r} = \left[\frac{f(r)\,\vec{r}}{r}\right]$$

$$\sqrt{x^2+y^2+z^2} = \boxed{b_{y_2}(x)}$$

$$V(x,y,z) = \int_0^{b_{y_2}(x)} (t f(t))\,dt = \int_0^{b_{y_2}(x)} \underbrace{(t f(t))}_{g(t)}\,dt$$

$$\frac{\partial V}{\partial x} = \left[g(b_{y_2}(x)) \times \frac{d}{dx}b_{y_2}(x)\right]$$

$$= (b_{y_2}(x))\,f(b_{y_2}(x)) \times \frac{\partial}{\partial x}\left(\sqrt{x^2+y^2+z^2}\right)$$

$$= \cancel{\sqrt{x^2+y^2+z^2}}\,f\left(\sqrt{x^2+y^2+z^2}\right) \times \frac{x}{\cancel{\sqrt{x^2+y^2+z^2}}}$$

$$= x f(r)$$

$$\nabla V = x f(r)\hat{i} + y f(r)\hat{j} + z f(r)\hat{k}$$

$$= \left(f(r)\,\vec{r}\right) = \vec{F}(x, y, z)$$