# Distance Measures for Quantum Information

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Note that these are *static* measures of distance between probability distributions.

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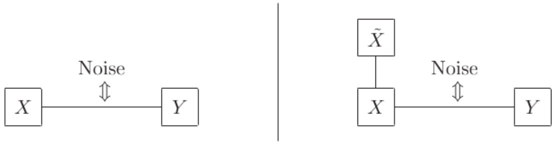


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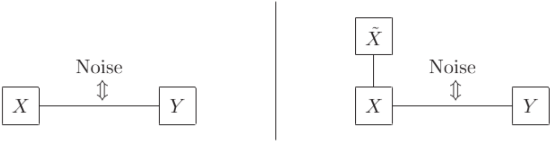


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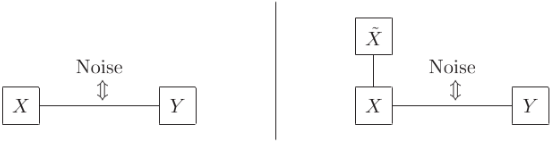


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Proof. Key step, $\rho - \sigma \rightarrow Q - S$.

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$$D(\rho_A, \sigma_A) \leq D(\rho_{AB}, \sigma_{AB})$$

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3. $A(\rho, \sigma) := \arccos F(\rho, \sigma)$ is a metric on $Dens(\mathcal{H})$

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The left side of the inequality can be proved using POVMs and simple mathematical manipulation.

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If we talk about the channel alone, we can minimize over all states [1]

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Illustrations

- Depolarizing Channel
- Phase Damping Channel
- Gate Fidelities

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# Defining a quantum information source - Attempt (1)

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This is called the *ensemble average fidelity*. Provided $\bar{F} \approx 1$ one can be confident that the source is being preserved by the channel.

# Defining a quantum information source - Attempt (2)

This notion is inspired from the idea of converting dynamic distance to correlation between a system and its copy. The correlation here translates to entanglement – A channel which preserves information well is one which preserves entanglement well.

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We define *entanglement fidelity*,

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\begin{aligned}
F_e(\rho, \mathcal{E}) &:= F(|RQ\rangle, |RQ'\rangle) \\
&= \langle RQ| \left[ (I_R \otimes \mathcal{E})(|RQ\rangle \langle RQ|) \right] |RQ\rangle
\end{aligned}
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where $\rho = Tr_R(|RQ\rangle \langle RQ|)$. It can be shown that only the choice of $\rho$ and $\mathcal{E}$ affect the EF, not the choice of purification.

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Thus, $F_e \leq \bar{F}$!

# Concluding Remarks

Thus, any quantum channel $\mathcal{E}$ which does a good job of preserving the entanglement between a source described by a density operator and a reference system will automatically do a good job of preserving an ensemble source described by probabilities $p_j$ and states $\rho_j$ such that $\rho = \sum_j p_j \rho_j$. In this sense the notion of a quantum source based on entanglement fidelity is a more stringent notion than the ensemble definition.