entanglement

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§1. Introduction

A bipartite pure state is entangled (with respect to that partition) iff it cannot be written as a pure state $|\psi\rangle = |\psi_A\rangle \otimes |\psi_B\rangle$. But, one might wish to quantify *how* entangled a state is compared to another. There are carefully designed ways to do this, the simplest of which we talk about now.

The notation of bipartite pure state entanglement is fully captured by the following theorem.

Theorem 1.0.1 (Schmidt Decomposition)**.** For any bipartite pure state $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$, there exists orthornormal bases $\{|\alpha_i\rangle\}$ and $\{|\beta_i\rangle\}$ of $\mathcal{H}_A$, $\mathcal{H}_B$ respectively, and positive numbers λ_i such that

$$|\psi\rangle = \sum_i \sqrt{\lambda_i} |\alpha_i\rangle \otimes |\beta_i\rangle \tag{1}$$

such that $\sum_i \lambda_i = 1$.

The values $\{\lambda_i\}$ comprise the Schmidt spectrum of $|\psi\rangle$, and characterize everything about the entanglement at that bipartition. Intuitively, if the distribution λ_i is 'well spread out,' then the entropy is high. This is quantified by the following entropic measures.

Definition 1.0.1 (Renyi Entropies)**.** For any distribution $p = \{p_i\}_i \in \mathcal{P}_n$, one defines for all $\alpha \geq 0$, $\alpha \neq 0$

$$H_\alpha(p) := \frac{1}{1-\alpha} \log \left(\sum_{i \in [n]} p_i^\alpha \right) \tag{2}$$

For $\alpha = 1$, we define

$$H_1(p) \equiv H(p) := -\sum_{i \in [n]} p_i \log p_i \tag{3}$$

The *α-entanglement entropy* of a state $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ is $H_\alpha(\lambda)$, where $\lambda \equiv \{\lambda_i\}$ is the Schmidt spectrum of ψ.

The notion of entanglement across a bipartition is naturally then extended to mixed states.

Definition 1.0.2 (Bipartite Separability)**.** A state $\rho \in D(\mathcal{H}_A \otimes \mathcal{H}_B)$ is said to be separable if

$$\rho = \sum_i p_i \rho_i^A \otimes \rho_i^B \tag{4}$$

The set of separable states in $D(\mathcal{H}_A \otimes \mathcal{H}_B)$ is denoted $\mathrm{Sep}(\mathcal{H}_A \otimes \mathcal{H}_B)$.

A bipartite state is said to be entanglement iff it is not separable. Also, the set $\mathrm{Sep}(\mathcal{H}_A \otimes \mathcal{H}_B)$ is convex, as expected. Moreover, separability implied 'classicality' of the correlations, as shown by the following lemma.

Lemma 1.0.1 (Prop. 6.6 of [1])**.** Every separable state $\rho \in \mathrm{Sep}(\mathcal{H}_A \otimes \mathcal{H}_B)$ there exist $\{p_i\}_i$ and $|\psi_{A(B)}\rangle_i \in \mathcal{H}_{A(B)}$ for $i \in [\mathrm{rank}(\rho)^2]$ such that

$$\rho = \sum_i p_i |\psi_A\rangle \langle\psi_A|_i \otimes |\psi_B\rangle \langle\psi_B|_i \tag{5}$$

§2. $N-$ particle entanglement

We now try to formalize a ladder of entanglement between the fully product state $|\psi\rangle = \otimes_{i=1}^n |\phi_i\rangle$, and a state which cannot be written as a product across any partition. This leads to the notion of $k-$producibility [2].

Definition 2.0.1 ($k-$producibility)**.** A pure state $|\psi\rangle \in \mathcal{H}^{\otimes N}$ is said to be k producible if it can be written as

$$|\psi\rangle = |\phi_1\rangle \otimes |\phi_2\rangle \otimes \cdots \otimes |\phi_M\rangle \tag{6}$$

where each ϕ_m is a state on maximally k particles.

Furthermore, we say that a state has genuine $k-$ party entanglement if it is $k-$producible but not $k-1$ producible. We extend this definition to mixed states by calling a mixed state $k-$producible if it can be written as a convex combination of pure $k-$producible states. We denote the convex set of $k-$producible states as Prod_k, and note that $\mathrm{Prod}_k \subset \mathrm{Prod}_{k+1}$. We thus denote $\mathrm{Ent}_k := \mathrm{Prod}_k/\mathrm{Prod}_{k-1}$. Then, the set of genuinely $N-$particle entangled states are Ent_N.

Lemma 2.0.1 (QFI and $k-$producibility [3])**.** For a $k-$producible state $\rho \in D(\mathcal{H}^N)$, the QFI $F[\rho, H]$ for $H = \sum_i h_i$ obeys

$$F[\rho, H] \leq \lfloor \frac{N}{k} \rfloor k^2 + \left(N - \lfloor \frac{N}{k} \rfloor k \right) \leq kN \tag{7}$$

Then, $k = O(1)$ implies that $F = O(N)$. Furthermore, $F = \Omega(N^2) \implies k = \Omega(N)$.

For pure states $|\psi\rangle \in \mathcal{H}^N$, one defines the geometric measure of entanglement E_G as

$$E_G(\psi) := 1 - \max_{|\phi\rangle \in \mathrm{Prod}_1} |\langle \phi|\psi\rangle|^2 \tag{8}$$

For mixed states, $\rho \in D(\mathcal{H}^N)$, one defines E_G through the convex roof construction

$$E_G(\rho) = \inf_{\rho = \{p_i, \psi_i\}} \sum_i p_i E_G(\psi_i) \tag{9}$$

Definition 2.0.2 (Geometric measure of entanglement)**.** For $\rho \in D(\mathcal{H}^N)$, one has that

$$E_G(\rho) = 1 - \max_{\sigma \in \mathrm{Prod}_1} F^2(\rho, \sigma) \tag{10}$$

Lemma 2.0.2. For any $\rho \in D(\mathcal{H}^N)$, one has

$$1 - \sqrt{1 - E_G(\rho)} \leq \min_{\sigma \in \mathrm{Prod}_1} ||\rho - \sigma||_1 \leq \sqrt{E_G(\rho)} \tag{11}$$

References

[1] John Watrous. *The theory of quantum information.* Cambridge university press, 2018.

[2] Otfried Gühne, Géza Tóth, and Hans J Briegel. Multipartite entanglement in spin chains. *New Journal of Physics*, 7(1):229, 2005.

[3] Philipp Hyllus, Wiesław Laskowski, Roland Krischek, Christian Schwemmer, Witlef Wieczorek, Harald Weinfurter, Luca Pezzé, and Augusto Smerzi. Fisher information and multiparticle entanglement. *Physical Review A*, 85(2), February 2012.