MA 109 D2 T1
Week One Extra Recap

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November 13, 2022

$\boxed{\text{Logistics!}}$

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A sequence which does not converge is said to diverge, or be non-convergent.

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then f_n converges, and the limit is $f_0 = a_0 = b_0$.

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$$\text{The sequence } a_n := (-1)^n \text{ does not converge. Proof?}$$

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Theorem (Monotone Convergence)

An upper bounded (lower bounded) real sequence a_n which is monotonically increasing (decreasing) converges. Further,

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  4. a_n and b_n are both convergent