MA 205 Tutorial Batch 3
Recap-2

Siddhant Midha

Convergence

1. A sequence s_n is said to converge to s if $\forall \epsilon > 0 \exists N \in \mathbb{N}$ s.t. $n > N \implies |s_n - s| < \epsilon$.
2. A *series* $\sum_n z_n$ is said to converge if the *sequence* $s_n := \sum_{i=0}^{n} z_n$ converges.
3. Absolute Convergence: A series $\sum_n z_n$ is said to converge absolutely if $\sum_n |z_n|$ converges.
4. Fact: Absolute Convergence $\implies$ Convergence. Proof? Use the fact that $\mathbb{C}$ is complete.

Theorem of the Day

Convergence of Power Series

Given the power series,

$$P = \sum_{n=0}^{\infty} a_n (z - z_0)^2$$

such that $a_n \in \mathbb{C} \forall n$, $z_0 \in \mathbb{C}$, we have that *only one* of the following is true

1. P converges only at $z = z_0$.
2. P converges at all $z \in \mathbb{C}$.
3. There exists $R \in \mathbb{R}$, $\infty > R > 0$, such that P converges for all $z : |z - z_0| < R$ and diverges for all $z : |z - z_0| > R$.[a]

Usually, we allow for $R = 0, \infty$ for convenience.

[a] no comments on the boundary!

Follow up theorem

Before that, one should ask, why is it that we cannot comment on the boundary? Hint: Convergence does *not* imply absolute convergence.

Convergence of Power Series

The *radius of convergence* of a power series as defined before is given as

$$R = \frac{1}{\limsup |a_n|^{1/n}}$$

Herein, we allow for $R = 0, \infty$ by letting $1/0 = \infty, 1/\infty = 0$.

What is the limsup?

Definition

For a **real** sequence x_n, define,

$$s_n := \sup\{x_n, x_{n+1} \dots\} \forall n$$

Define, $\limsup x_n := \lim s_n$.

Points to be noted,

- s_n is a **non-increasing** sequence. (Why?)
- Because of that, the limit of s_n always exists (can be $\pm\infty$). (Why?)
- Thus, the lim-sup always exists. The limit might not.
- When the limit exists, the limsup is equal to the limit.

Some properties [Can Ignore]

Let x_n be the (real) sequence of interest, and let $-\infty < L < \infty$ be its limsup. Let s_n be defined as before.

1. Since s_n is non-increasing, given any $\epsilon > 0$, $\exists n_0 \in \mathbb{N}$ s.t. $s_{n_0} < L + \epsilon$.
2. By definition of s_{n_0}, we have

$$x_n \leq s_n < L + \epsilon \forall n \geq n_0$$

3. Now, assume that $\exists n_0 \in \mathbb{N}$ s.t. $x_n \leq L - \epsilon \forall n \leq n_0$. This would imply $s_n \leq L - \epsilon \forall n \leq n_0$. Not possible (Why?).
4. Thus there are arbitrary large n s.t. $x_n > L - \epsilon$.

Armed with these points, one can go through `Slide-6,7` of `Lecture 1C` to understand the proof of the RoC better.

Helpful Facts

Let $\sum_n x_n$ be a complex series. Note the following,

- If $\sum_n x_n$ converges, then $|x_n| \to 0$ as $n \to \infty$. Necessary Condition
- Note that $|x_n| \geq 0$ (thus $\sum_{k=0}^n |x_n|$ is monotonic increasing), we have that $\sum_n |x_n|$ converges iff $\sum_{k=0}^n |x_n|$ is bounded above. Necessary and Sufficient Condition
- (Comparison test) If $x_n, y_n \in \mathbb{R}$, and $0 \leq x_n \leq y_n$, then
  1. $\sum_n y_n$ converges $\implies$ $\sum_n x_n$ converges.
  2. $\sum_n x_n$ diverges $\implies$ $\sum_n y_n$ diverges.
- (Ratio Test) Suppose $x_n \neq 0 \forall n$. Further suppose $|x_{n+1}/x_n| \to L$ as $n \to \infty$. Then,
  1. $L > 1 \implies$ divergence.
  2. $L < 1 \implies$ absolute convergence $\implies$ convergence.
  3. $L = 1 \implies$ nada.
- (Root Test) Define $C := \limsup |x_n|^{1/n}$. Then,
  1. $C > 1 \implies$ divergence.
  2. $C < 1 \implies$ absolute convergence $\implies$ convergence.
  3. $C = 1 \implies$ nada.