Problem 1

let $A := \begin{bmatrix} a_1 & a_2 & \cdots & a_n \\ b_1 & b_2 & \cdots & b_n \end{bmatrix} \in \mathbb{R}^{2\times n}$ is a gramian.

i) $\det(AA^T)$ — ①

ii) $\boxed{P = A^T A} = (p_{ij})$

$\boxed{\alpha_{ij} := \begin{vmatrix} p_{ii} & p_{ij} \\ p_{ji} & p_{jj} \end{vmatrix}}$ ← principal minor

$\sum_{1 \le i < j \le n} \alpha_{ij}$ — ②

$S := \{v_1, v_2 \ldots v_n\}$

$G = [g_{ij}] \qquad g_{ij} = \langle v_i, v_j \rangle$

$A^T = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \\ \vdots & \vdots \\ a_n & b_n \end{bmatrix} = [c_1 \ \ c_2]$

$\rightarrow c_1, \rightarrow c_2$

$AA^T = \begin{bmatrix} c_1^T \\ c_2^T \end{bmatrix} [c_1 \ \ c_2]$

$A = \begin{bmatrix} c_1^T \\ c_2^T \end{bmatrix}$

$= \begin{bmatrix} c_1^T c_1 & \cdots \\ & \cdots \end{bmatrix}$

$\{c_1, c_2\}$

$A^T A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \\ \vdots & \vdots \\ a_n & b_n \end{bmatrix} \begin{bmatrix} a_1 & a_2 & \cdots \\ b_1 & b_2 & \cdots \end{bmatrix} = [p_{ij}]$

$= \begin{bmatrix} a_1^2 + b_1^2 & a_1 a_2 + b_1 b_2 & \cdots \\ & & \\ & & \end{bmatrix}$

$$\boxed{p_{ij} = a_i a_j + b_i b_j}$$

$$= \begin{bmatrix} b_{11} & b_{12} & \cdots \\ b_{21} & b_{22} & \cdots \\ & & \end{bmatrix}$$

Principal minors — how many? — $\binom{n}{2}$

$$\alpha_{ij} := \begin{vmatrix} p_{ii} & p_{ij} \\ p_{ji} & p_{jj} \end{vmatrix}$$

$n-1$
$+$
$n-2$
$+$
$\vdots$
1
$\shortparallel$
$\frac{n(n-1)}{2}$

$$\sum_{1 \le i < j \le n} \alpha_{ij}$$

$$\alpha_{12} = \begin{vmatrix} b_{11} & p_{12} \\ p_{21} & p_{22} \end{vmatrix}$$

$$\boxed{p_{ij} = a_i a_j + b_i b_j}$$

$$\alpha_{12} = \begin{vmatrix} a_1^2 + b_1^2 & a_1 a_2 + b_1 b_2 \\ a_2 a_1 + b_2 b_1 & a_2^2 + b_2^2 \end{vmatrix}$$

$$= (a_1^2 + b_1^2)(a_2^2 + b_2^2) - \underline{(a_1 a_2 + b_1 b_2)^2}$$

$$= a_1^2 b_2^2 + b_1^2 a_2^2 - 2 a_1 a_2 b_1 b_2$$

$$= (a_1 b_2 - b_1 a_2)^2$$

$$= (a_1 b_2 - b_1 a_2)^2$$

$$= \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}^2$$

Sum $\rightarrow \sum_{1 \le i < j \le n} \begin{vmatrix} a_i & a_j \\ b_i & b_j \end{vmatrix}^2$ — (2)

$$\det(AA^T) = \det \begin{bmatrix} c_1^T c_1 & c_1^T c_2 \\ c_2^T c_1 & c_2^T c_2 \end{bmatrix}$$

$$= \left(\sum_i a_i^2\right)\left(\sum_i b_i^2\right) - \left(\sum_i a_i b_i\right)^2 \quad — (1)$$

$$c_1 = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \quad c_2 = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

$$(1) = (2)$$

The Cauchy-Binet formula

Statement [edit]

Let A be an $m\times n$ matrix and B an $n\times m$ matrix. Write $[n]$ for the set {1, ..., n}, and $\binom{[n]}{m}$ for the set of m-combinations of $[n]$ (i.e., subsets of size m; there are $\binom{n}{m}$ of them). For $S \in \binom{[n]}{m}$, write $A_{[m],S}$ for the $m\times m$ matrix whose columns are the columns of A at indices from S, and $B_{S,[m]}$ for the $m\times m$ matrix whose rows are the rows of B at indices from S. The Cauchy–Binet formula then states

$$\det(AB) = \sum_{S\in\binom{[n]}{m}} \det(A_{[m],S})\det(B_{S,[m]}).$$

Example: Taking m = 2 and n = 3, and matrices $A = \begin{pmatrix} 1 & 1 & 2 \\ 3 & 1 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 1 \\ 3 & 1 \\ 0 & 2 \end{pmatrix}$, the Cauchy–Binet formula gives the determinant

$$\det(AB) = \begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix} \cdot \begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ 1 & -1 \end{vmatrix} \cdot \begin{vmatrix} 3 & 1 \\ 0 & 2 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ 3 & -1 \end{vmatrix} \cdot \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix}.$$

Indeed $AB = \begin{pmatrix} 4 & 6 \\ \end{pmatrix}$ and its determinant is -28 which equals

$$\det(AB) = \begin{vmatrix} 3 & 1 \end{vmatrix} \cdot \begin{vmatrix} 3 & 1 \end{vmatrix} + \begin{vmatrix} 1 & -1 \end{vmatrix} \cdot \begin{vmatrix} 0 & 2 \end{vmatrix} + \begin{vmatrix} 3 & -1 \end{vmatrix} \cdot \begin{vmatrix} 0 & 2 \end{vmatrix} .$$

Indeed $AB = \begin{pmatrix} 4 & 6 \\ 6 & 2 \end{pmatrix}$, and its determinant is -28 which equals $-2 \times -2 + -3 \times 6 + -7 \times 2$ from the right hand side of the formula.

this hold for $3 \times n$ as well.

any $k \times n$ with $k < n$

Problem 2

$$\det\left(\begin{bmatrix} \cos\alpha & 1 & & & & \\ 1 & 2\cos\alpha & 1 & & & \\ & 1 & 2\cos\alpha & 1 & & \\ & & 1 & \ddots & 1 & \\ & & & 1 & 2\cos\alpha & 1 \\ & & & & 1 & 2\cos\alpha \end{bmatrix}_{n\times n}\right) = \cos(n\alpha)$$

hint: Induction (induction on n)

Base case : Trivial

Assume it holds for $n=k \in \mathbb{N}/\{1\}$

let's see for $n=k+1$,

$$D_{k+1} = \begin{vmatrix} \cos\alpha & 1 & & & \\ 1 & 2\cos\alpha & 1 & & \\ & 1 & \ddots & & \\ & & 1 & 2\cos\alpha & 1 \\ & & & 1 & 2\cos\alpha \end{vmatrix}$$

$$D_{k+1} = (2\cos\alpha)\underbrace{\begin{vmatrix} \cos\alpha & & & \\ 1 & 2\cos\alpha & & \\ & & \ddots & \\ & & 1 & 2\cos\alpha \end{vmatrix}}_{D_k} - 1 \begin{vmatrix} \overbrace{\begin{matrix} \cos\alpha & 1 & & \\ 1 & 2\cos\alpha & 1 & \\ & 1 & \ddots & 1 \\ & & 1 & 2\cos\alpha \end{matrix}}^{D_{k-1}} & \\ & 1 \end{vmatrix}$$

$k-1$

expand along this column

$$= (2\cos\alpha)\cos k\alpha - \cos((k-1)\alpha)$$

$$= \left[\cos(k+1)\alpha + \cos(k-1)\alpha\right] - \cos(k-1)\alpha$$

$$= \cos(k+1)\alpha$$

☐

Problem 3

$x, y \in \mathbb{C}^n \quad A \in \mathbb{C}^{n \times m} \quad \boxed{A^{\dagger}} \text{ or } A^{*} = (\bar{A})^T$

adjoint

$\langle x, y \rangle := x^T \bar{y} = y^{\dagger} x$

Show that

$E = \|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2 = 4\langle x, y\rangle$

$C(x,y) = \|x+y\|^2 - \|x-y\|^2$

$= \langle x+y, x+y\rangle - \langle x-y, x-y\rangle \qquad \|v\| = \sqrt{\langle v, v\rangle}$

$= (\langle x,x\rangle + \langle y,y\rangle + \langle x,y\rangle + \langle y,x\rangle) - (\langle x,x\rangle + \langle y,y\rangle - \langle x,y\rangle - \langle y,x\rangle)$

$= 2(\langle x,y\rangle + \langle y,x\rangle)$

$E = C(x,y) + iC(x, iy)$

$= 2\left[(\langle x,y\rangle + \langle y,x\rangle) + i(\langle x,iy\rangle + \langle iy,x\rangle)\right]$

$= 2\left[(\langle x,y\rangle + \langle y,x\rangle) + i(-i\langle x,y\rangle + i\langle y,x\rangle)\right]$

$i^2 = -1$

$= 2\left[(\langle x,y\rangle + \cancel{\langle y,x\rangle}) + \langle x,y\rangle - \cancel{\langle y,x\rangle}\right]$

$= 4\langle x,y\rangle$

Problem 4

4. Orthogonalize the following ordered set of row-vectors in $\mathbb{R}^4$.

$$\{[1,1,0,0],\ [1,0,1,0],\ [1,0,0,1],\ [0,1,1,0],\ [0,1,0,1],\ [0,0,1,1]\}$$

$w_1 \quad w_2 \quad w_3 \quad w_4 \quad w_5 \quad w_6$

Do you get an orthogonal basis? Does $[-2,-1,1,2]$ belong to the linear span? Use ~~Bessel's inequality~~.

$$v_1 := \frac{w_1}{\|w_1\|} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\\0\\0\end{bmatrix} \qquad \langle w_4, v_1\rangle = \frac{1}{\sqrt{2}}$$

$$\tilde{v}_2 := w_2 - \langle v_1, w_2\rangle v_1$$

$$= \begin{bmatrix}1\\0\\1\\0\end{bmatrix} - \left(\frac{1}{\sqrt{2}}\right)\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\\0\\0\end{bmatrix} = \begin{bmatrix}1\\0\\1\\0\end{bmatrix} - \frac{1}{2}\begin{bmatrix}1\\1\\0\\0\end{bmatrix} = \begin{bmatrix}1/2\\-1/2\\1\\0\end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix}1\\-1\\2\\0\end{bmatrix}$$

$$v_2 := \frac{\tilde{v}_2}{\|\tilde{v}_2\|} = \frac{1}{\sqrt{6}}\begin{bmatrix}1\\-1\\2\\0\end{bmatrix}$$

$$\langle v_1, v_2\rangle = 0$$

$$\tilde{v}_3 := w_3 - \langle v_1, w_3\rangle v_1 - \langle v_2, w_3\rangle v_2$$

$$= \begin{bmatrix}1\\0\\0\\1\end{bmatrix} - \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\\0\\0\end{bmatrix} - \frac{1}{\sqrt{6}}\frac{1}{\sqrt{6}}\begin{bmatrix}1\\-1\\2\\0\end{bmatrix}$$

$$= \begin{bmatrix}1\\0\\0\\1\end{bmatrix} - \begin{bmatrix}1/2\\1/2\\0\\0\end{bmatrix} - \begin{bmatrix}1/6\\-1/6\\1/3\\0\end{bmatrix} \qquad \frac{1}{2}+\frac{1}{6} = \frac{3+1}{6} = \frac{2}{3}$$

$$\frac{1}{6}-\frac{1}{2} = \frac{1-3}{6} = \frac{-1}{3}$$

$$= \begin{bmatrix}1-2/3\\0-\frac{1}{2}+\frac{1}{6}\\-1/3\\1\end{bmatrix} = \begin{bmatrix}1/3\\-1/3\\-1/3\\1\end{bmatrix} = \frac{1}{3}\begin{bmatrix}1\\-1\\-1\\3\end{bmatrix}$$

$$v_3 := \frac{1}{\sqrt{12}}\begin{bmatrix}1\\-1\\-1\\3\end{bmatrix}$$

$$\tilde{v}_4 = \begin{bmatrix}0\\1\\1\\0\end{bmatrix} - \frac{1}{\sqrt{2}}\frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\9\end{bmatrix} - \frac{1}{\sqrt{6}}\cdot\frac{1}{\sqrt{6}}\begin{bmatrix}1\\-1\\2\\0\end{bmatrix} - \left(\frac{-2}{\sqrt{12}}\right)\frac{1}{\sqrt{12}}\begin{bmatrix}1\\-1\\-1\\3\end{bmatrix}$$

$$= \begin{bmatrix}0\\1\\1\\0\end{bmatrix} - \frac{1}{6}\begin{bmatrix}1\\-1\\2\\0\end{bmatrix} + \frac{2}{12}\frac{1}{6}\begin{bmatrix}1\\-1\\-1\\3\end{bmatrix} - \frac{1}{\sqrt{2}}\frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\9\end{bmatrix}$$

$$= \begin{bmatrix}0 - \frac{1}{6} + \frac{1}{6}\\ 1 + \frac{1}{6} - \frac{1}{6}\\ 1 - \frac{1}{3} - \frac{1}{6}\\ 1/2\end{bmatrix} - \frac{1}{\sqrt{2}}\frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\9\end{bmatrix}$$

$$\frac{1}{2}\begin{bmatrix}0\\2\\1\\1\end{bmatrix} - \frac{1}{\sqrt{2}}\frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\9\end{bmatrix} = \begin{bmatrix}0 - 1/2\\ 1 - 1/2\\ \frac{1}{2} - 0\\ \frac{1}{2}\end{bmatrix}$$

$$= \begin{bmatrix}-1/2\\1/2\\1/2\\1/2\end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix}-1\\1\\1\\1\end{bmatrix}$$

$$v_4 = \frac{1}{2}\begin{bmatrix}-1\\1\\1\\1\end{bmatrix}$$

$$\begin{cases} v_5 = \vec{0} \\ v_6 = \vec{0}\end{cases}$$

$\{v_1, v_2 \ldots v_6\}$

Problem 5

Orthonormalize the ordered set in $\mathbb{C}^5$:

$$\{[1, \imath, 0, 0, 0], [0, 1, \imath, 0, 0], [0, 0, 1, \imath, 0], [0, 0, 0, 1, \imath]\}$$

$w_1 \quad w_2 \quad w_3 \quad w_4$

relative to the unitary inner product $\langle \mathbf{v}, \mathbf{w}\rangle = \mathbf{w}^*\mathbf{v} = \sum_{j=1}^{5} v_j \bar{w}_j$.

$$v_1 := \frac{w_1}{\|w_1\|} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ i\\ 0\\ 0\\ 0\end{pmatrix}$$

$$\tilde{v}_2 := w_2 - \langle w_2, v_1\rangle v_1$$

$$\langle w_2, v_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & -i & 0 & 0 & 0\end{pmatrix}\begin{pmatrix}0\\ 1\\ i\\ 0\\ 0\end{pmatrix} = \frac{1}{\sqrt{2}}(-i)$$

$$= \begin{pmatrix}0\\ 1\\ i\\ 0\\ 0\end{pmatrix} - \left(\frac{-i}{\sqrt{2}}\right)\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ i\\ 0\\ 0\\ 0\end{pmatrix}$$

$$= \begin{pmatrix}0\\ 1\\ i\\ 0\\ 0\end{pmatrix} + \frac{1}{2}\begin{pmatrix}i\\ -1\\ 0\\ 0\\ 0\end{pmatrix} = \begin{pmatrix}i/2\\ 1/2\\ i\\ 0\\ 0\end{pmatrix} = \frac{1}{2}\begin{pmatrix}i\\ 1\\ 2i\\ 0\\ 0\end{pmatrix}$$

$$v_2 := \frac{\tilde{v}_2}{\|\tilde{v}_2\|} = \frac{1}{\sqrt{6}}\begin{pmatrix}i\\ 1\\ 2i\\ 0\\ 0\end{pmatrix}$$

$$\tilde{v}_3 := w_3 - \langle w_3, v_1\rangle v_1 - \langle w_3, v_2\rangle v_2$$

$$\langle w_3, v_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & -i & 0 & 0 & 0\end{pmatrix}\begin{pmatrix}0\\ 0\\ 1\\ i\\ 0\end{pmatrix} = 0$$

$$\langle w_3, v_2\rangle = \frac{1}{\sqrt{6}}\begin{pmatrix}-i & 1 & -2i & 0 & 0\end{pmatrix}\begin{pmatrix}0\\ 0\\ 1\\ i\\ 0\end{pmatrix} = \frac{-2i}{\sqrt{6}}$$

$$\tilde{v}_3 = \begin{pmatrix}0\\ 0\\ 1\\ i\\ 0\end{pmatrix} - \left(\frac{-2i}{\sqrt{6}}\right)\frac{1}{\sqrt{6}}\begin{pmatrix}i\\ 1\\ 2i\\ 0\\ 0\end{pmatrix}$$

$$= \begin{pmatrix}-1/3\\ i/3\\ 1-2/3\\ i\\ 0\end{pmatrix} = \frac{1}{3}\begin{pmatrix}-1\\ i\\ 1\\ 3i\\ 0\end{pmatrix}$$

$$\begin{pmatrix} \vdots \\ i \\ 0 \end{pmatrix} \qquad \begin{pmatrix} \vdots \\ 3i \\ 0 \end{pmatrix}$$

$$v_3 := \frac{\tilde{v}_3}{\|\tilde{v}_3\|} = \frac{1}{\sqrt{12}} \begin{pmatrix} -1 \\ i \\ 1 \\ 3i \\ 0 \end{pmatrix}$$

$$\tilde{v}_4 := w_4 - \langle w_4, v_1 \rangle v_1 - \langle w_4, v_2 \rangle v_2 - \langle w_4, v_3 \rangle v_3$$

$$\langle w_4, v_1 \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ i \end{pmatrix} = 0$$

$$\langle w_4, v_2 \rangle = \frac{1}{\sqrt{6}} \begin{pmatrix} -i & 1 & -2i & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ i \end{pmatrix} = 0$$

$$\langle w_4, v_3 \rangle = \frac{1}{\sqrt{12}} \begin{pmatrix} -1 & -i & 1 & -3i & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ i \end{pmatrix} = \left(\frac{-3i}{\sqrt{12}} \right)$$

$$\Rightarrow \tilde{v}_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ i \end{pmatrix} - \left(\frac{-3i}{\sqrt{12}} \right) \times \frac{1}{\sqrt{12}} \begin{pmatrix} -1 \\ i \\ 1 \\ 3i \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} -i/4 \\ -1/4 \\ +i/4 \\ 1/4 \\ i \end{pmatrix} = \frac{1}{4} \begin{pmatrix} -i \\ -1 \\ i \\ 1 \\ 4i \end{pmatrix}$$

$$v_4 := \frac{\tilde{v}_4}{\|\tilde{v}_4\|} = \frac{1}{\sqrt{20}} \begin{pmatrix} -i \\ -1 \\ i \\ 1 \\ 4i \end{pmatrix}$$

$$\left\{ \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \frac{1}{\sqrt{6}}\begin{pmatrix} i \\ 1 \\ 2i \\ 0 \\ 0 \end{pmatrix}, \quad \frac{1}{\sqrt{12}}\begin{pmatrix} -1 \\ i \\ 1 \\ 3i \\ 0 \end{pmatrix}, \quad \frac{1}{\sqrt{20}}\begin{pmatrix} -i \\ -1 \\ i \\ 1 \\ 4i \end{pmatrix} \right\}$$

is the o-normal set desired.

Problem 6

$A \in \mathbb{C}^{n \times n}$

Show that rows are orthonormal $\Rightarrow$ so are columns.

"Unitary" matrix

$$A = \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{bmatrix}$$

$$A^+ = \begin{bmatrix} r_1^+ & r_2^+ & \cdots & r_n^+ \end{bmatrix}$$

$$AA^+ = \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{bmatrix} \begin{bmatrix} r_1^+ & \cdots & r_n^+ \end{bmatrix}$$

$$= \begin{bmatrix} \underline{\underline{r_1 r_1^+}} & \cdots & r_1 r_n^+ \\ \vdots & & \vdots \\ r_n r_1^+ & \cdots & r_n r_n^+ \end{bmatrix}$$

$$p_{ij} = r_i r_j^+$$

$$\underline{\underline{\langle r_i, r_j \rangle}} = r_i r_j^+$$

<u>Claim</u> $AA^+ = I$

orthonormal $\Rightarrow$ $\langle r_i, r_j \rangle = \delta_{ij}$ $\begin{cases} 1 & \text{if } i = j \\ 0 & \text{o/w} \end{cases}$

But $(AA^+)_{ij} = \langle r_i, r_j \rangle = \delta_{ij}$

$$AA^+ = I$$

$$AA^{+} = I$$

$$A^{+} = A^{-1}$$

$$\Downarrow$$

$$A^{+}A = I$$

$$A = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

$$A^{+} = \begin{bmatrix} c_1^{+} & c_2^{+} & \cdots & c_n^{+} \end{bmatrix}$$

$$A^{+}A = \begin{bmatrix} c_1^{+}c_1 & c_1^{+}c_2 & \cdots & \\ & & \ddots & \\ c_n^{+}c_1 & & & c_n^{+}c_n \end{bmatrix} = I = [q_{ij}]$$

$$q_{ij} = \delta_{ij}$$

$$\Downarrow$$

$$\underline{c_i^{+}c_j = \delta_{ij}} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

$$\Downarrow$$

$$\langle c_i, c_j \rangle = \delta_{ij} \quad \therefore \{c_i\} \text{ is o-}\underline{\text{Normal}}.$$

Problem 7

$$v = \frac{1}{\sqrt{3}}\begin{bmatrix}1\\ i\\ -1\end{bmatrix} \quad w = \frac{1}{\sqrt{6}}\begin{bmatrix}1\\ -2i\\ -1\end{bmatrix} \quad \text{find } u \in \mathbb{C}^3 \text{ s.t.}$$

$\{u, v, w\}$ is orthonormal

take any vector $\tilde{v}$ which is lin indp w/ v, w.

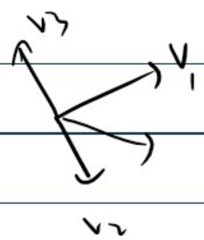
$$\tilde{v} = \begin{pmatrix}1\\0\\0\end{pmatrix}$$

$$v = \begin{pmatrix}1\\0\\0\end{pmatrix} - \left(\frac{1}{\sqrt{3}}\right)\frac{1}{\sqrt{3}}\begin{pmatrix}1\\ i\\ -1\end{pmatrix} - \frac{1}{\sqrt{6}}\frac{1}{\sqrt{6}}\begin{pmatrix}1\\ -2i\\ -1\end{pmatrix}$$

$$= \begin{pmatrix}1\\0\\0\end{pmatrix} - \frac{1}{3}\begin{pmatrix}1\\ i\\ -1\end{pmatrix} - \frac{1}{6}\begin{pmatrix}1\\ -2i\\ -1\end{pmatrix}$$

$$= \begin{pmatrix}1 - \frac{1}{2}\\ 0\\ \frac{1}{2}\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1\\0\\1\end{pmatrix}$$

$$v_n = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\1\end{pmatrix}$$



$$\left.\begin{array}{l} v_1 \cdot v_3 = 0 \\ v_2 \cdot v_3 = 0 \end{array}\right\}$$

let $u = \begin{pmatrix}a\\b\\c\end{pmatrix}$

$\langle v, u\rangle = 0$

$\langle w, u\rangle = 0$

$$a + ib - c = 0$$
$$a - 2ib - c = 0$$

$$\begin{bmatrix} 1 & +i & -1 \\ 1 & -2i & -1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$R_2 \to R_2 - R_1$$

$$\begin{bmatrix} 1 & i & -1 \\ 0 & -3i & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Downarrow$$

$$\underline{b = 0}$$

$$a - c = 0$$

$$a = c$$

$$\therefore \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$