Question Three

3. For a simple closed curve given in polar coordinates show using Green's theorem that the area enclosed is given by

$$\boxed{A = \frac{1}{2}\oint_C r^2 d\theta.}$$

Use this to compute the area enclosed by the following curves:

(i) The cardioid: $r = a(1-\cos\theta)$, $0 \le \theta \le 2\pi$; .

$$\left[\frac{1}{2}\int x\,dy - y\,dx\right]$$

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

$$\left(\frac{\partial F_2}{\partial y} - \frac{\partial F_1}{\partial x}\right) = 1$$

$$\int x\,dy \quad , \quad -\int y\,dx$$

$$\frac{1}{2}\int_{\theta=0}^{2\pi} (a(1-\cos(\theta)))^2 d\theta$$

$$\frac{a^2}{2}\int_0^{2\pi} (1+\cos^2\theta - 2\cos\theta)\,d\theta$$

$$\frac{a^2}{2}\int_0^{2\pi} 1 + \frac{1+\cos(2\theta)}{2} - 2\cos\theta\,d\theta$$

$$\int_0^{2\pi} \cos\theta\,d\theta = 0 = \int_0^{2\pi} \cos(2\theta)\,d\theta$$

$$\left(\frac{a^2}{2}\right) \times \left(1+\frac{1}{2}\right)2\pi = \left[\frac{3}{4}a^2\right] \times 2\pi = \left(\frac{3}{2}\pi a^2\right)$$

Question Four

The region under one arch of the cycloid

$$\mathbf{r} = a(t - \sin t)\mathbf{i} + a(1 - \cos t)\mathbf{j},\ 0 \le t \le 2\pi.$$

$$-\oint y\,dx$$

$$y = a(1-\cos t)$$

$$dx = a(1-\cos t)\,dt$$

$$-\int_0^{\pi/2} (a(1-\cos t))(a(1-\cos t))\,dt$$

$$= -a^2 \int_0^{\pi/2} (1-\cos t)^2\,dt$$

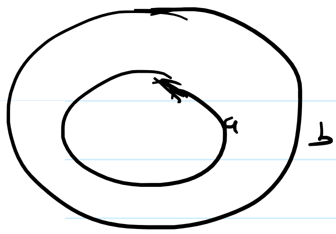
Question Five

Let $D = \{(x,y) \in \mathbb{R}^2 \mid a^2 \le x^2+y^2 \le b^2\}$, where $0 < a < b$. Evaluate

$$\int_{\partial D} xe^{-y^2}dx + [-x^2ye^{-y^2} + 1/(x^2+y^2)]dy,$$

where ∂D is positively oriented.

$[\underline{\underline{\partial D}}]$



$$\vec{c}(\theta) = \begin{cases} (b\cos\theta,\ b\sin\theta) & \theta \in [0, 2\pi) \\ (a\cos\theta,\ -a\sin\theta) & \theta \in [2\pi, 4\pi] \end{cases}$$

$$\overbrace{\int_{\partial D}\left(xe^{-y^2}dx + -x^2ye^{-y^2}dy\right)}^{\underline{0+0}} + \underline{\underline{\int_{\partial D}\left(\frac{1}{x^2+y^2}\right)dy}}$$

$$\left[\frac{1}{2}\vec{\nabla}\left(\underline{x^2e^{-y^2}}\right)\right]$$

$$\downarrow$$

$$\underline{0}$$

$\underline{\partial D}$

$$\vec{c}(\theta) = \begin{cases} (b\cos\theta,\ b\sin\theta) & \theta \in [0, 2\pi) \quad c_1 \\ (a\cos\theta,\ -a\sin\theta) & \theta \in [2\pi, 4\pi] \end{cases}$$

$$\oint_{\partial D} \frac{1}{x^2+y^2}\,dy \qquad \left(0,\ \frac{1}{x^2+y^2}\right)$$

$$\int_{c_1} + \int_{c_2}$$

$$\int_0^{2\pi} \frac{(b\cos\theta)}{b^2}\,d\theta + \int_{2\pi}^{4\pi} \frac{(-a\cos\theta)}{a^2}\,d\theta$$

$$= \underline{0 + 0} = 0$$

Ans : $0 + 0 = 0$

Question Eight

Let C be any counterclockwise closed curve in the plane and let $\mathbf{n}$ be the outward unit normal to the curve C. Compute $\oint_C \nabla(x^2 - y^2) \cdot \mathbf{n} ds$.

Check using $\boxed{(F_1, F_2) \cdot \hat{n}}$

Calculate using FTC

$$\oint_{C = \partial D} \nabla(x^2 - y^2) \cdot \hat{n}\, ds$$

$$= \iint_D \vec{\nabla} \cdot (\vec{\nabla}(x^2 - y^2))\, dx dy$$

$$= \iint_D (x^2)_{xx} + (-y^2)_{yy}\, dx dy$$

$$= \iint_D 2 - 2\, dx dy = 0$$

Question Nine

Let D be a region in $\mathbb{R}^2$ with boundary ∂D satisfying the hypothesis stated in the 'Green's theorem'. Let $\phi : \mathbb{R}^2 \to \mathbb{R}$ be a C^2 function.

(i) Show that $\nabla^2\phi = \operatorname{div}(\operatorname{grad}\phi)$, where the operator ∇^2 is defined by

$$\nabla^2\phi(x,y) = \frac{\partial^2\phi}{\partial^2 x}(x,y) + \frac{\partial^2\phi}{\partial^2 y}(x,y).$$

The operator ∇^2 is called 'Laplace operator'.

(ii) Show that the Green's Identity holds:

$$\iint_D \nabla^2\phi \, d(x,y) = \oint_{\partial D} \left(\frac{\partial\phi}{\partial \mathbf{n}}\right) ds, \longrightarrow (*)$$

where $\mathbf{n}$ is the outward unit normal to the curve ∂D.

(Hint. Use the divergence form of Green's theorem for the vector field $\mathbf{F} = \operatorname{grad}\phi$)

(iii) Using the above identity, compute

$$\left(\oint_C \frac{\partial\phi}{\partial \mathbf{n}} ds\right)$$

for $\boxed{\phi = e^x \sin y,}$ and D the triangle with vertices $(0,0), (4,2), (0,2)$.

i) $\phi(x,y)$

$$\vec{\nabla}\phi = \phi_x \hat{i} + \phi_y \hat{j}$$

$$\vec{\nabla}\cdot(F_1, F_2) = (F_{1x} + F_{2y})$$

$$\vec{\nabla}\cdot(\vec{\nabla}\phi) = (\phi_x)_x + (\phi_y)_y = \phi_{xx} + \phi_{yy}$$

$$\vec{\nabla}^2(\cdot) \equiv \vec{\nabla}\cdot\vec{\nabla}(\cdot)$$

2)

$$\int_{\partial D} \vec{F}\cdot\hat{n}\, ds = \iint_D \vec{\nabla}\cdot\vec{F}\, dx dy$$

$$\vec{F} = \vec{\nabla}\phi$$

$$\int_{\partial D} (\vec{\nabla}\phi)\cdot\hat{n}\, ds = \iint \vec{\nabla}\cdot(\vec{\nabla}\phi)\, dx dy$$

$$\iint_{\partial D} \frac{\partial\phi}{\partial\hat{n}}\, ds = \iint \nabla^2\phi\, dx dy$$

$$\left(\frac{\partial\phi}{\partial\hat{n}}\right) = \text{directional derivative along } \hat{n}$$

$$= \left[(\vec{\nabla}\phi)\cdot\hat{n}\right]$$

3)

$$\phi(x,y) = e^x \sin y$$

$$\phi_x = e^x \sin y \qquad \phi_y = +e^x \cos y$$

$\phi_x = e^x \sin y$ $\qquad\qquad$ $\phi_y = +e^x \cos y$

$\phi_{xx} = e^x \sin y$ $\qquad\qquad$ $\phi_{yy} = -e^x \sin y$

$$\nabla^2 \phi = \phi_{xx} + \phi_{yy} = 0$$

LHS of (ϕ) is zero

Hence, RHS $= 0$

$$\oint_C \frac{\partial \phi}{\partial \hat{n}} \, ds = 0$$

Question Ten

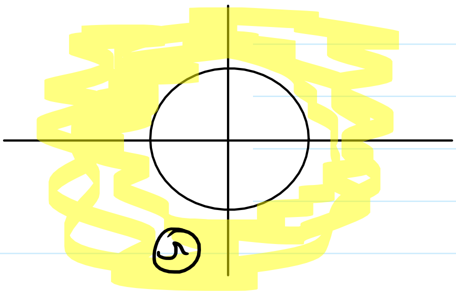
10. Let us consider the region $\Omega = \{(x, y) \mid x^2 + y^2 > 1\}$ and the vector field be defined on Ω. Evaluate the following line integrals where the loops are traced in the counter clockwise sense

(i)
$$\oint_C \frac{y\,dx - x\,dy}{x^2 + y^2}$$
where C is any simple closed curve in Ω enclosing the origin.

(ii)
$$\oint_C \frac{y\,dx - x\,dy}{x^2 + y^2}$$
where C is any simple closed curve in Ω not enclosing the origin.

(iii) Let C be a smooth simple closed curve lying in Ω. Find
$$\oint_C \frac{\partial(\ln r)}{\partial y}dx - \frac{\partial(\ln r)}{\partial x}dy.$$

$$\left(\frac{y}{x^2+y^2} , \frac{-x}{x^2+y^2} \right)$$



(i)

r , $(0,0)$ $\quad r > 1$

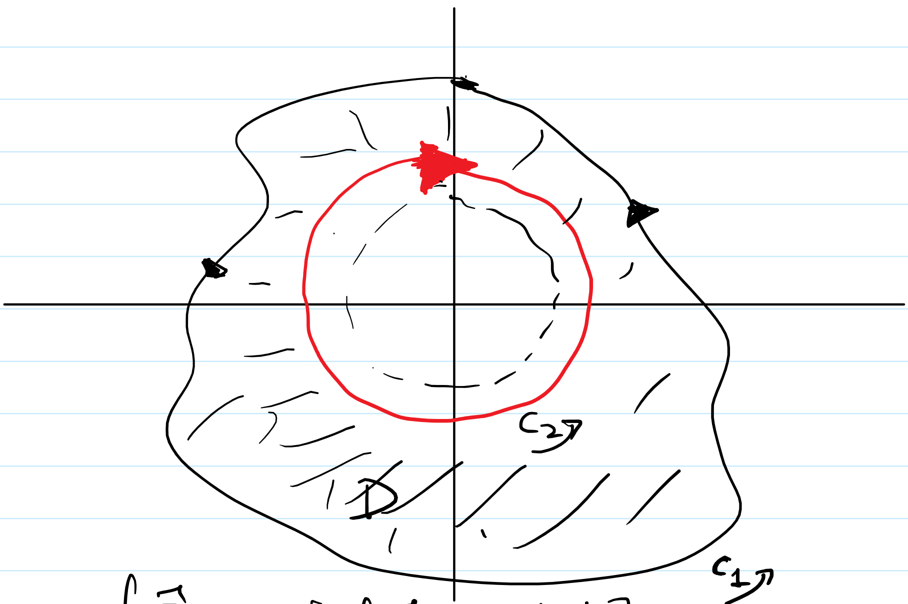
$\vec{c}(t) = (r\cos t, r\sin t) \quad t \in [0, 2\pi]$

$\vec{c}'(t) = (-r\sin t, r\cos t)$

$$\int_C \vec{F} = \int_0^{2\pi} \vec{F}(\vec{c}(t)) \cdot \vec{c}'(t)\,dt$$

$$= \frac{1}{r^2} \int_0^{2\pi} \underbrace{(-r\sin t)}_{c'} \times \underbrace{(r\sin t)}_{F_1} + \underbrace{(r\cos t)}_{c'} \times \underbrace{(-r\cos t)}_{\vec{F}_2}\,dt$$

$$= \int_0^{2\pi} -(\sin^2 t + \cos^2 t)\,dt = (-2\pi)$$



$$\int_{\partial D} \vec{F} = \left[\int_{C_1} \vec{F} - \int_{C_2} \vec{F} \right] = \iint_D \underbrace{(\vec{\nabla} \times \vec{F})}_{0} d(x,y)$$

$$\int \vec{F} = \left(\int \vec{F} \right) = (-2\pi)$$

$$\oint_{C_1} \vec{F} = \left(\int_{C_2} \vec{F}\right) = (-2\pi)$$

ii)

$$\left[\vec{\nabla} \times \vec{F} = 0\right]$$

$$\boxed{\int_{\partial D} \vec{F} = 0}$$

iii)

$$r = \sqrt{x^2+y^2}$$

$$\ln(r) = \frac{1}{2}\ln(x^2+y^2)$$

$$\frac{\partial(\ln(r))}{\partial x} = \frac{1}{2}\frac{2x}{(x^2+y^2)} = \frac{x}{x^2+y^2}$$

$$\frac{\partial(\ln(r))}{\partial y} = \left(\frac{y}{x^2+y^2}\right)$$

- C encloses the origin → (-2π)
- C does not enclose the origin → (0)