A Summary of Sorts

Siddhant Midha

02-03-2022

Table of Contents

Double Integrals on Rectangles

Let $R := [a, b] \times [c, d] \subset \mathbb{R}^2$. Let $f : R \to \mathbb{R}$ be a bounded function. Some definitions,

- $$m(f) := inf\{f(x, y) \mid (x, y) \in R\}$$
  $$M(f) := sup\{f(x, y) \mid (x, y) \in R\}$$
- Consider a partition P of R defined by,
  $$P := \{(x_i, y_j) \mid i = 0, 1 \ldots n \text{ and } j = 0, 1 \ldots m\}$$
- Further define $R_{ij} := [x_{i-1}, x_i] \times [y_{j-1}, y_j]$
  $$m_{ij}(f) := inf\{f(x, y) \mid (x, y) \in R_{ij}\}$$
  $$M_{ij}(f) := sup\{f(x, y) \mid (x, y) \in R_{ij}\}$$

Double Integrals on Rectangles

Recall $R_{ij} := [x_{i-1}, x_i] \times [y_{j-1}, y_j]$ and define $\Delta_{ij} := \text{Area}(R_{ij})$

- $$L(f, P) := \sum_{i,j} m_{ij}(f)\Delta_{ij}$$
  $$U(f, P) := \sum_{i,j} M_{ij}(f)\Delta_{ij}$$
- $$L(f) := sup\{L(f, P) \mid R \text{ is a partition of } R\}$$
  $$U(f) := inf\{U(f, P) \mid R \text{ is a partition of } R\}$$

Definition (Darboux Integrability)

A **bounded** function $f : R \to \mathbb{R}$ is said to be (Darboux) double integrable if

$$L(f) = U(f)$$

Double Integrals on Rectangles

Riemann's Condition

Let $f : [a, b] \times [c, d] \to \mathbb{R}$ be a **bounded** function. Then, f is integrable iff for every $\epsilon > 0$, there is a partition P_ϵ of $[a, b] \times [c, d]$ such that,

$$U(f, P_\epsilon) - L(f, P_\epsilon) < \epsilon$$

Recall that we can prove a handful of things using this condition.

- f is integrable $\implies$ $|f|$ is integrable
- f is bounded and monotonic in each variable $\implies$ f is integrable

Double Integrals on Rectangles

Given a rectangle R and a partition P, we define a tagged partition $P^t := (P, t)$ which is equipped with a set of tags $t_{ij} = (x^*_{ij}, y^*_{ij})$. Recall, the definition of the Riemann Sum,

$$S(f, P^t) := \sum_{i,j} f(t_{ij})\Delta_{ij}$$

Definition (Riemann Integrability)

A **bounded** function $f : R \to \mathbb{R}$ is said to be Riemann Integrable if $\exists S \in \mathbb{R}$ s.t.

$$\forall \epsilon > 0 \exists \delta > 0 : |S(f, P^t) - S| < \epsilon$$

for **every** tagged partition P^t satisfying $||P|| < \delta$.

Fact: The two definitions of integrability are equivalent.

Table of Contents

Fubini's Theorem

Theorem

Let $R := [a, b] \times [c, d]$ be a rectangle and let f be integrable on R.

- *If for each $x \in [a, b]$, the Riemann integral $\int_c^d f(x, y)dy$ exists, then the iterated integral $\int_a^b (\int_c^d f(x, y)dy)dx$ exists and is equal to $\iint_R f$*
- *If for each $y \in [c, d]$, the Riemann integral $\int_a^b f(x, y)dx$ exists, then the iterated integral $\int_c^d (\int_a^b f(x, y)dx)dy$ exists and is equal to $\iint_R f$*

Further we have the following result.

Proposition

Let R be a rectangle and let f be a continuous function on R. Then f is integrable on R and the iterated integrals exist and are equal to the integral of f.

Some questions

Can you find a f such that,

1. iterated integrals exist, are not equal, f is not integrable
2. iterated integrals exist, are not equal, f is integrable
3. f is double integrable but 1-D integrals may not exist
4. both iterated integrals exist and are equal, f is integrable
5. both iterated integrals exist and are equal, f is not integrable

Table of Contents

Measure Zero and Content Zero

Definition: Covering

Let $A, Q_1, Q_2 \dots \subset \mathbb{R}^n$. We say that the collection of sets $\{Q_i\}$ *covers* A if,

$$A \subseteq \bigcup_i Q_i$$

Further, we call $\{Q_i\}_i$ a covering.

Definition: Measure and Content Zero

A set $A \subset \mathbb{R}^n$ has measure zero **in** $\mathbb{R}^n$ if $\forall \epsilon > 0$ there is a covering $Q_1, Q_2 \dots$ of A such that

$$\sum_i Volume(Q_i) < \epsilon$$

Note that the covering is atmost countable (it is either finite or countably infinite). If the covering is finite, we say that A has *content zero* in $\mathbb{R}^n$.

An Important Result

Theorem

Let R be a rectangle in $\mathbb{R}^2$ and let $f : R \to \mathbb{R}$ be a **bounded** *function. Let D be the set of points at which f fails to be continuous. Then, f is integrable over R if and only if D has measure zero.*

Now while we won't elaborate here, recall how we used the concept of integration over rectangles coupled with this theorem to extend ourselves to integrating over more general regions. Then we had,

Theorem

Let $D \subset \mathbb{R}^2$ be a bounded set whose boundary ∂D is given by the finitely many continouos closed curves then any **bounded** *and* **continuous** *function $f : D \to \mathbb{R}^2$ is integrable over D.*

Table of Contents

Elementary Regions

We defined the following regions,

1. Type 1 A region D is type-1, if there exists continuous functions h_1, h_2 such that

$$D = \{(x, y) \mid a \leq x \leq b \text{ and } h_1(x) \leq y \leq h_2(x)\}$$

2. Type 2 A region D is type-1, if there exists continuous functions f_1, f_2 such that

$$D = \{(x, y) \mid c \leq y \leq d \text{ and } f_1(y) \leq x \leq f_2(y)\}$$

3. Type 3 A region is of type 3 if it can be written as a union of type 1 and type 2 regions.

The Jacobian

Let $A, B \subset \mathbb{R}^2$ be open in $\mathbb{R}^2$. Let $h : A \to B$ be C^1 diffeomorphism of bounded open sets. Let

$$h(u, v) = (h_1(u, v), h_2(u, v))$$

We define the Jacobian of this transformation $J[h](u, v)$ as,

$$J[h](u, v) := \begin{pmatrix} \frac{\partial h_1}{\partial u} & \frac{\partial h_1}{\partial v} \\ \frac{\partial h_2}{\partial u} & \frac{\partial h_2}{\partial v} \end{pmatrix}$$

We deal with the determinant of this matrix as we shall see. And with some abuse of notation, we call the determinant the Jacobian.

Change of Variables

Theorem

Let $h : A \to B$ be a C^1 diffeomorphism of bounded open sets in $\mathbb{R}^2$. Let $f : B \to \mathbb{R}$ be a continuous function. Then f is integrable over B iff the function $(f \circ h)|detJ(h)|$ is integrable over A. Further,

$$\int_B f = \int_A (f \circ h)|detJ(h)|$$

Note that a C^1 diffeomorphism is a h bijective, differentiable map and $J(h)$ is continuous and invertible on A.

Some familiar Transformations

1. We recall the affine transformation,

$$x = au + bv + t_1$$

$$y = cu + dv + t_2$$

the Jacobian is $ad - bc$

2. We recall the polar transformation,

$$x = r\cos\theta$$

$$y = r\sin\theta$$

the Jacobian is r.

Table of Contents

1. Here we start with integrating functions of the form $f : B \to \mathbb{R}$ where B is a cubiod by dividing it into smaller ones B_{ijk}.
2. Similar definitions for Darboux and Riemann Integrability are used here, which again, are equivaltent.
3. Fubini's Theorem
4. We again extend to integrating over non-cuboidal regions as we did in $\mathbb{R}^2$.
5. Elementary Regions are of the form,

$$E = \{(x, y, z) \mid \gamma_1(x, y) \leq z \leq \gamma_2(x, y), (x, y) \in D\}$$

   where γ_1, γ_2 are continuous functions on $D \subset \mathbb{R}^2$ and D is an elementary region in $\mathbb{R}^2$.
6. Thus integrals 'look like':

$$\iiint_E f dV = \int_a^b \int_{h_1(x)}^{h_2(x)} \int_{\gamma_1(x,y)}^{\gamma_2(x,y)} f(x, y, z) dzdydx$$

And we had the familiar transformations,

1. Recall the spherical transformation

$$x = \rho \cos\theta \sin\phi$$

$$y = \rho \sin\theta \sin\phi$$

$$z = \rho \cos\phi$$

   with the jacobian $\rho^2 \sin\phi$

2. Recall the cylindrical transformation

$$x = \rho \cos\theta$$

$$y = \rho \cos\theta$$

$$z = z$$

   with the jacobian ρ

Table of Contents

Openness

Let $a \in \mathbb{R}^n$. Given $\epsilon > 0$ define,

$$B_a(\epsilon) := \{x \mid x \in \mathbb{R}^n, |x - a| < \epsilon\}$$

We call this an *epsilon ball* centered around a.

Open

$A \subseteq \mathbb{R}^n$ is open if

$$\forall x \in A \exists \epsilon > 0 \text{ s.t. } B_a(\epsilon) \subset A$$

Thus, an open set is one which has all points as interior points.

Closedness

Closed

$A \subseteq \mathbb{R}^n$ is closed if A^c is open.

We can also define closedness in terms of limit points, refer to the note I shared on teams.

Regions

Let $A \subseteq \mathbb{R}^n$.

1. A is connected if it **cannot** be written as a disjoint union of two non empty subsets A_1 and A_2 *such that* $A_1 = A \cap U_1$ and $A_2 = A \cap U_2$ such that U_1, U_2 are open in $\mathbb{R}^n$.
2. A is path connected if any two points in A can be joined by a path which lies inside A.
3. A is simply connected if A is connected and any simple closed curve lying in A can be contracted to a point in A.

Some (/non) implications

- Path connected $\implies$ connected
- Connected $\centernot\implies$ path connected (Example?)
- Open + Connected $\Leftrightarrow$ Path connected [1]

[1]We are talking about $\mathbb{R}^n$ here

Table of Contents

Fields

We dealt with vector fields and scalar fields on $\mathbb{R}^n$ for $n = 2, 3$.

Definition

Let $D \subseteq \mathbb{R}^n$.

1. A scalar field is a map $f : D \to \mathbb{R}$.
2. A vector field is a map $\mathbf{F} : D \to \mathbb{R}^n$.

The Del Operator

Let us work with $\mathbb{R}^3$. We have the *del operator*,

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

Actions of the operator:

1. On scalar fields - Given a differentiable scalar field f, ∇f is the *gradient* of the field:

$$\nabla f := \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$$

2. On vector fields: Given a differentiable vector field $\mathbf{F} = (F_1, F_2, F_3)$, we can
   1. Produce a scalar field:

$$\nabla \cdot \mathbf{F} := \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

   2. Produce another vector field:

$$\nabla \times \mathbf{F}$$

Table of Contents

So Conservative!

Definition

A vector field $\mathbf{F}$ is conservative if it is a gradient of some scalar function.

$$\mathbf{F} = \nabla f$$

FTC for vector fields

Let $f : D \subset \mathbb{R}^n$ be a differentiable function and let ∇f be continuous on a smooth part $\mathbf{c}$. Then,

$$\int_{\mathbf{c}} \nabla f = \int_a^b \nabla f(\mathbf{c}(t)) \cdot \mathbf{c}'(t) dt = f(\mathbf{c}(b)) - f(\mathbf{c}(a))$$

Some (/non) implications

- Conservative $\implies$ Path Independent
- Not path independent $\implies$ not Conservative
- Path independent $\not\Longleftarrow$ Conservative
- Not conservative $\not\Longleftarrow$ Not path independent
- Path independent + Domain is Path connected $\implies$ Conservative

Some more implications

Let $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j}$ be a C^1 field on an open $D \subset \mathbb{R}^n$. We take $n = 2$. Similar results hold for $n = 3$.

1. Recall the necessary condition.

$$\mathbf{F} = \nabla f \implies \frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}$$

2. Recall the sufficient condition. Let D be simply connected. Then,

$$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x} \implies \mathbf{F} \text{ is conservative}$$

Note the direction of the implications above.

Question

Is

$$\mathbf{F} = (\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2})$$

a conservative field?

Question

Is

$$\mathbf{F} = (\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2})$$

a conservative field? Answer?

Question

Is

$$\mathbf{F} = (\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2})$$

a conservative field? Answer? The question is not well framed. Why?

Question

Is

$$\mathbf{F} = (\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2})$$

a conservative field? Answer? The question is not well framed. Why? This illustrates that a vector field is not *just* a tuple of two functions.

Table of Contents

Orientation

Definition

The *positive orientation* of a curve C in $\mathbb{R}^2$ is given by the vector field $\mathbf{k} \times \mathbf{n}$. Where $\mathbf{n}$ is the unit normal vector pointing outward along the curve.

So, given a path in $\mathbb{R}^2$, we have the notion of *orientation* of the region enclosed by the path.
Further recall that, for a region with holes, the positive orientation is such that the outer boundary is oriented counter-clockwise, and the inner clockwise.

Green's Theorem

Theorem

1. *Let D be a bounded region in $\mathbb{R}^2$ with a positively oriented boundary ∂D consiting of a* **finite number** *of* **non-intersecting simple** *closed* **piecewise continuously differentiable** *curves.*
2. *Let Ω be an open set in $\mathbb{R}^2$ such that $D \cup \partial D \subset \Omega$.*
3. *Let $F_1 : \Omega \to \mathbb{R}$, $F_2 : \Omega \to \mathbb{R}$ be C^1 functions. Consider the vector field*

$$\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j}$$

Then, (finally!)

$$\int_{\partial D} F_1 dx + F_2 dy = \iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) d(x, y)$$

Variants

D is our region, $\mathbf{n}$ is the outward normal, $\mathbf{k}$ is the vector normal to the plane.

1. Using Div

$$\int_{\partial D} \mathbf{F} \cdot \mathbf{n} ds = \iint_D \nabla \cdot \mathbf{F} d(x,y)$$

2. Using Curl

$$\int_{\partial D} \mathbf{F} \cdot d\mathbf{s} = \iint_D (\nabla \times \mathbf{F}) \cdot \mathbf{k} d(x,y)$$

(2) is a special case of - ?

Applications

- Recall how area is defined. Now, we can apply Green's Theorem and see that,

$$A(D) = \frac{1}{2}\int_{\partial D} xdy - ydx = \int_{\partial D} xdy = -\int_{\partial D} ydx$$

- In polar coordinates,

$$A(D) = \frac{1}{2}\int r^2 d\theta$$

Table of Contents

Definitions

Definition

Let $D \subseteq \mathbb{R}^2$ be path connected. A *parametrised surface* is a **continuous** function $\varphi : D \to \mathbb{R}^3$.

Now, recall the definitions of $\varphi_u(u, v)$ and $\varphi_v(u, v)$. At some point (u, v) these two give us the *tangent plane* at that point.

Tangent plane

At some point (x_0, y_0, z_0), the equation of the tangent plane is given by

$$(\varphi_u \times \varphi_v)(u_0, v_0) \cdot (x - x_0, y - y_0, z - z_0) = 0$$

Area of a Surface

$$Area(\varphi) := \iint_D ||\varphi_u \times \varphi_v|| d(u, v)$$

With the notation $dS = ||\varphi_u \times \varphi_v|| d(u, v)$ we have,

$$Area(\varphi) = \iint_D dS$$

Note that D is the domain of the surface parametrisation.

Oriented Surfaces

Definition

A surface S is said to be *orientable* if there exists a **continuous** vector field $\mathbf{F} : S \to \mathbb{R}^3$ such that for each point P in S, $\mathbf{F}(P)$ is a unit vector normal to the surface S at P.

An oriented parametrised surface φ comes equipped with a vector field of normal unit vectors

$$\mathbf{n} = \frac{\varphi_u \times \varphi_v}{||\varphi_u \times \varphi_v||}$$

Then, given any surface it is either orientation reversing or perserving. Also note that changing the orientation leads to changing the sign of a surface integral.

Table of Contents

Surface Integrals

Scalar Fields

Given a surface φ on the path connected set $E \subseteq \mathbb{R}^2$, its image S, and a bounded function $f : S \to \mathbb{R}$, we define,

$$\iint_S f dS := \iint_E f(x, y, z) ||\varphi_u \times \varphi_v|| d(u, v)$$

Surface Integrals

Vector Fields

Given a surface φ on the path connected set $E \subseteq \mathbb{R}^2$, its image S, and a bounded vector field such that its domain consists of S, we define,

$$\iint_S \mathbf{F} \cdot d\mathbf{S} := \iint_E \mathbf{F}(\varphi(u, v)) \cdot (\varphi_u \times \varphi_v) dudv$$

Table of Contents

Setting up

Recall the orientation of a surface. Now, given such an oriented surface S, an orientation is induced on the *boundary* ∂S. The direction of the same can be found by the right hand rule.
Further, if $\varphi : E \to \mathbb{R}^3$ (E is P.C.) is a smooth, orientation preserving parametrisation of S then the induced orientation of ∂S corresponds to the positive orientation of ∂E w.r.t. E.

Stokes Theorem

Theorem

1. *Let S be a bounded piecewise smooth oriented surface with non-empty boundary ∂S. Suppose S is closed in $\mathbb{R}^3$.*
2. *Let ∂S be a disjoint unionm of simple closed curves each of which is a piecewise non-singular parametrized curve with the induced orientation.*
3. *Let $\mathbf{F} = (F_1, F_2, F_3)$ be a C^1 vector field defined on an open set containing S. Then (finally again!),*

$$\int_{\partial S} \mathbf{F} \cdot d\mathbf{S} = \iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S}$$

Consequences

1. The surface integral of $\nabla \times \mathbf{F}$ over two different surfaces with the same orientation and the same boundary is the same.
2. Note that we required $\partial S \neq \emptyset$ before. If it is empty, then the surface integral of the curl is simply zero. But, this was proved separately, not given by the Stokes Theorem.
3. For computing line integrals, we can switch over to computing surface integrals with a *large* choice of surfaces to choose from.
4. If we have a "curl free" **smooth** vector field on open regions, we have path independence. Can we conclude that a suitable potential function must exist?
5. No. Moreover,

$$\text{curl free} + \text{domain is simply connected} \implies \text{conservative}$$

Sidenote

- curl(grad) $= 0$
- div(curl) $= 0$
- curl $= 0$ and domain is simply connected $\implies$ field is a grad field
- Does div $= 0$ and domain is simply connected imply that field is a curl field?

Table of Contents

Divergence Theorem

Theorem

1. *Let W be a simple solid region of $\mathbb{R}^3$ whose boundary $S = \partial W$ is a closed surface and is positively oriented.*
2. *Let $\mathbf{F}$ be a smooth vector field on an open subset of $\mathbb{R}^3$ containing W. Then,*

$$\iint_{\partial W} \mathbf{F} \cdot d\mathbf{S} = \iiint_W \nabla \cdot \mathbf{F} dV$$

Consequences

- Divergence free fields enjoy Zero surface integrals over boundaries of simple solid regions.
- Break up the aforementioned boundary into two surfaces S_1, S_2 with the same boundary[2] . Then the surface integral of a div-free field is same across both.

[2] Please note the usage of "boundary" for surfaces and solid regions, and distinguish appropriately.