MA 205 Autumn 2022
Tutorial Sheet 3

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1. Expand $\frac{1+z}{1+2z^2}$ into a power series around 0. Find the radius of convergence.

Sol.
We have

$$\begin{aligned} f(z) &= \frac{1+z}{1-(-2z^2)} \\ &= (1+z)[\sum_{n=0}^{\infty}(-2z^2)^n] \\ &= \sum_n (-2)^n z^{2n} + \sum_n (-2)^n z^{2n+1} \\ &= \sum_{k=0}^{\infty} a_k z^k \end{aligned}$$

where,

$$a_k = \begin{cases} (-2)^{(k-1)/2} & \text{when k is odd,} \\ (-2)^{k/2} & \text{when k is even.} \end{cases}$$

Clearly, we have $\limsup |a_k|^{1/k} = \sqrt{2}$. Thus, $R = 1/\sqrt{2}$. ■

2. Let γ be the boundary of the triangle $\{0 < y < 1-x;\ 0 \le x \le 1\}$ taken with the anticlockwise orientation. Evaluate

1. $\int_\gamma \mathrm{Re}(z)dz$,
2. $\int_\gamma z^2 dz$,

Sol.
Define three paths,

$$\begin{aligned} \gamma_1 &: [0,1] \to \mathbb{C}\ \gamma_1(t) := t \\ \gamma_2 &: [0,1] \to \mathbb{C}\ \gamma_2(t) := (1-t) + \iota t \\ \gamma_3 &: [0,1] \to \mathbb{C}\ \gamma_3(t) := \iota(1-t) \end{aligned}$$

Then, we have,

$$\int_\gamma f(z)dz = \int_{\gamma_1} f(z)dz + \int_{\gamma_2} f(z)dz + \int_{\gamma_3} f(z)dz$$

Now,

1. $f(z) = \text{Re}(z)$. Thus, we have

$$\int_\gamma f(z)dz = \frac{1}{2} + (\iota - 1)(\frac{1}{2}) + 0 = \frac{\iota}{2}$$

2. $f(z) = z^2$. Zero! Why? Or,

$$\int_\gamma f(z)dz = \frac{1}{3} - (\frac{1}{3} + \frac{\iota}{3}) + \frac{\iota}{3}$$

■

3. Compute $\displaystyle\int_{|z-1|=1} \frac{2z-1}{z^2-1}dz$.

Sol.
We have,

$$\int_{|z-1|=1} \frac{\frac{2z-1}{z+1}}{z-1}dz = 2\pi\iota \times \frac{2(1)-1}{1+1} = \pi\iota$$

as a direct application of the Cauchy Integration Formula (CIF). ■

4. Consider the parametrization of the unit circle: $\gamma = e^{\iota t}$, $0 \le t \le 2\pi$. Compute

1. $\displaystyle\int_\gamma \frac{e^z}{z}\, dz$,

2. $\displaystyle\int_\gamma \frac{z^3}{z^2+4}\, dz$.

Sol.

1. $2\pi\iota \times e^0 = 2\pi\iota$, again by the CIF.

2. $2\pi\iota \times 0 = 0$, similarly.

■

5. Let f be a holomorphic function on $\mathbb{C}$ (entire function) such that $f\left(\frac{1}{n}\right) = 0$ for $n \in \mathbb{N}$. Show that $f \equiv 0$.

Sol.
Use the theorem on `Slide 23, Lecture 2B` and conclude. ■

6. Let f be an entire function such that $|\text{Re}(f(z))| \le M$ for all $z \in \mathbb{C}$ and M is a constant. Conclude that f is constant. Hint: Consider e^f.

Sol.
Define $g : \mathbb{C} \to \mathbb{C}$ as $g(z) = e^{f(z)}$. Note that since $e^{(\cdot)}$ and f are entire, so is g. Now, we have,

$$\begin{aligned}|g(z)| &= |e^{\text{Re}(f(z))}e^{\iota \text{Im}(f(z))}| \\ &= |e^{\text{Re}(f(z))}| \\ &\le e^{|\text{Re}(f(z))|} \\ &\le e^M\end{aligned}$$

By Louiville's Theorem (`Slide 30, Lecture 2B`), we have that g is a constant, and thus f is a constant. ■

7. Let f be an entire function such that $f(x) = e^x$ for all $x \in \mathbb{R}$. Evaluate $f(\pi i)$.

Sol.
Define $g : \mathbb{C} \to \mathbb{C}$ as $g(z) := e^z - f(z)$. Note that g is entire, and,

$$g(x + 0\iota) = 0 \forall x \in \mathbb{R}$$

Conclude by Loiville's theorem (or other ways) that $g \equiv 0$. Thus $f(z) = e^z$. Subsequently, $f(\iota\pi) = e^{\iota\pi} = -1$.
■

8. Show that the image of the upper half space $\{z \in \mathbb{C} : \mathrm{Im}(z) \geq 0\}$ under the map e^{iz} is contained in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| \leq 1\}$.

Sol.
Let $z = x + \iota y$, such that $y \geq 0$. Now,

$$f(z) = e^{\iota z} = e^{\iota x - y} = e^{\iota x} e^{-y}$$

Thus, $|f(z)| = e^{-y} \leq 1$, as $y \geq 0$. ■