MA 205 Tutorial Batch 3
Recap-3

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Theorem of the day

Theorem (Cauchy Integral Theorem)

Let Ω *be a bounded domain in* $\mathbb{C}$*, with piecewise smooth boundary* $\partial\Omega$ *and* $f \in C^1(\bar{\Omega})$ *is holomorphic on* Ω*. Then,*

$$\int_{\partial\Omega} f(z)dz = 0$$

Theorem (Cauchy Integral Formula)

Let Ω *be a bounded domain in* $\mathbb{C}$ *with piecewise smooth boundary* $\partial\Omega$*, and* $f \in C^1(\bar{\Omega})$ *is holomorphic on* Ω*. Then for all* $z \in \Omega$*, we have,*

$$f(z) = \frac{1}{2\pi\iota} \int_{\partial\Omega} \frac{f(\eta)}{\eta - z} d\eta$$

Note that, in these theorems we are dealing with $\partial\Omega$ being traversed anticlockwise. Also, we do *not* need $f \in C^1(\bar{\Omega})$, as holomorphicity of f guarantees holomorphicity and thus continuity of f'.

Another way ...

Another way to state the CIT, which can avoid possible mistakes.

Theorem (CIT - Aliter)

If $f : \Omega \to \mathbb{C}$ is holomorphic, and Ω is a **simply connected** *domain, then for every closed piecewise smooth curve γ within Ω we have,*

$$\int_\gamma f(z)dz = 0$$

Consequences

- **Holo $\implies$ Analytic**. If f is holomorphic at a point $z_0 \in \mathbb{C}$, then we have that $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$ for all $z : |z - z_0| < r$ for some small r. Where,

$$a_n = \frac{1}{2\pi\iota} \int_{D(z_0,r)} \frac{f(\eta)}{(\eta - z_0)^{n+1}d\eta}$$

Note that any r s.t. $D(z_0, r)$ is contained in the region of holomorphicity gives the same a_n.
Also, we had previously seen that power series are holomorphic in their region of convergence. Thus, Analytic $\implies$ Holomorphic. Hence we have the statement,

$$\textbf{Holomorphic} \Leftrightarrow \textbf{Analytic}$$

Consequences

- $f^{(n)}(z) = \frac{n!}{2\pi\iota} \int_{\partial\Omega} \frac{f(\eta)}{(\eta - z)^{n+1}} d\eta$ for all $z \in \Omega$. Particularly,

  $$|f^{(n)}(z_0)| \leq \frac{n! M}{R^n} \textbf{ Cauchy's Estimate}$$

  if f is holomorphic in $\{z : |z - z_0| < R\}$ and $|f| < M$ there.
- **Louiville's Theorem**: A bounded above entire function is a constant. Simple proof?
- **FTC**: A non-constant complex polynomial has atleast one root in $\mathbb{C}$.
- **Morera's Theorem**: For some domain Ω, if $f : \Omega \to \mathbb{C}$ is continuous and $\int_\gamma f(z)dz = 0$ for all $\gamma = \partial R$ for $R \subset \Omega$ being a rectangle, then f is holomorphic on Ω. Converse of Cauchy much?

Zeros of Analytical Functions

Theorem

Suppose $f : \Omega \to \mathbb{C}$ is holomorphic, where Ω is a domain. Further suppose $f(z_0) = 0$. Then, we have

1. *$f \equiv 0$ on Ω, or,*
2. *$\exists m \in \mathbb{N}$ and a holomorphic function $g : \Omega \to \mathbb{C}$ such that $g(z_0) \neq 0$ and $f(z) = (z - z_0)^m g(z)$ in $D(r, z_0)$ for some small r.*

This shows some interesting things:

- **Isolated Zeros**: Zeros of a non constant analytic function on a domain Ω are *isolated*.
- **Vanishing Behaviour**
  1. Vanishes at a sequence of points with a limit point in $\Omega \implies f \equiv 0$ on Ω.
  2. Vanishes on an open subset $A \subset \Omega \implies f \equiv 0$ on Ω.
  3. $f^{(n)}(z_0) = 0 \forall n$ for some $z_0 \in \Omega \implies f \equiv 0$ on Ω.
- **Identity Principle**: If f, g holomorphic agree on a 'suitable' set of points, then $f \equiv g$ on Ω.

The Niceness of Holomorphicity

Let $f = u + \iota v$ be holomorphic on a simply connected domain. Then,

- **Holomorphicity of integral** For any z_0 in the domain if we define $F(z) := \int_{z_0}^{z} f(\eta) d\eta$, then F is holomorphic on the domain and $F' = f$.
- **Derivatives are Holomorphic** Using the CIF, we see that $f', f'', ...$ all are holomorphic in the domain.
- **Partial derivatives are nice** Since $f', f''...$ all are holomorphic, partial derivatives of u, v of all orders are continuous.

With this and properties like the CIF, one can surely ponder about the rigidity of holomorphic functions.