MA 106 D1-T3
Tutorial-5

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27-04-2022

Some definitions

- Normal $A \in \mathbb{C}^{n \times n}$

$$AA^* = A^*A$$

Some definitions

- Normal

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- Unitary

$$AA^* = A^*A = I$$

Some definitions

- Normal

$$AA^* = A^*A$$

$$A^* = A^\dagger = (\bar{A})^T = \overline{(A^T)}$$

- Unitary

$$AA^* = A^*A = I$$

- Hermitian / self adjoint

$$A = A^*$$

Some definitions

- Normal

$$AA^* = A^*A$$

- Unitary

$$AA^* = A^*A = I$$

- Hermitian

$$A = A^*$$

- Orthogonal

$O \in \mathbb{R}^{n \times n}$

$$OO^T = O^TO = I$$

Some definitions

- Normal

$$AA^* = A^*A$$

- Unitary

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- Hermitian

$$A = A^*$$

$AP = DP$

$A =$

$A = P^{-1}DP$

- Orthogonal

$$OO^T = O^TO = I$$

- Unitarily (resp. orthogonally) diagonalizable: Diagnolizable with a unitary (resp. orthogonal) P.

Spectral Theorem [1]

Theorem $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$

Let $A \in \mathbb{K}^{n \times n}$

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- *If $\mathbb{K} = \mathbb{C}$, then A is normal $\boxed{\text{iff}}$ it is unitarily diagonalizable.*

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Let $A \in \mathbb{K}^{n\times n}$.

- *If $\mathbb{K} = \mathbb{C}$, then A is normal iff it is unitarily diagonalizable.*
- *If $\mathbb{K} = \mathbb{R}$, then, A is unitarily diagonalizable implies that A is normal.*

But not the other way around!

Spectral Theorem [1]

Theorem

Let $A \in \mathbb{K}^{n\times n}$.

- *If* $\mathbb{K} = \mathbb{C}$, *then* A *is normal iff it is unitarily diagonalizable.*
- *If* $\mathbb{K} = \mathbb{R}$, *then,* A *is unitarily diagonalizable implies that* A *is normal.*

Why was one implication knocked off in the real case?

Spectral Theorem [1]

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- *If $\mathbb{K} = \mathbb{R}$, then, A is unitarily diagonalizable implies that A is normal.*

Why was one implication knocked off in the real case?Recall the Issue pointed out in the previous recap slides.

Spectral Theorem [1]

Take

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Take

$$\left(A := \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix} \in \mathbb{R}^{2\times 2} \right)$$

$$p_A(x) = x^2 + 4$$

Spectral Theorem [1]

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$$A := \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$$

$$\implies A^* = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}$$

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Thus $AA^* = A^*A$.Normal.

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Take

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$$\implies A^* = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}$$

Thus $AA^* = A^*A$.Normal.Not unitarily diagonalizable (over $\mathbb{R}$).

Spectral Theorem [2]

Theorem

Let $A \in \mathbb{K}^{n \times n}$.

Spectral Theorem [2]

if $A = A^*$ then $AA^* = A^*A$

herm $\Rightarrow$ normal

Theorem

Let $A \in \mathbb{K}^{n\times n}$.

- *If* $\mathbb{K} = \mathbb{C}$, *then* A *is hermitian iff it is unitarily diagonalizable and has all eigenvalues real.*

Spectral Theorem [2]

Theorem

Let $A \in \mathbb{K}^{n \times n}$.

- *If* $\mathbb{K} = \mathbb{C}$, *then* A *is hermitian iff it is unitarily diagonalizable and has all eigenvalues real.*
- *If* $\mathbb{K} = \mathbb{R}$, *then* A *is symmetric iff* A *is orthogonally diagonalizable and has all eigenvalues real.*

Question 1

We have, the quadratic form as

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$$\underbrace{\begin{bmatrix} x & y & z \end{bmatrix}}A\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$\alpha^T A \alpha$

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$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

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Thus we have

$$a_{11}x^2 + a_{22}y^2 + a_{33}z^2 + (a_{12} + a_{21})xy + (a_{13} + a_{31})xz + (a_{23} + a_{32})yz$$

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We choose our A to be symmetric in what follows.

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We choose our A to be symmetric in what follows.Why is that?

Question 1(i)

Here,

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$0x^2 + 0y^2 + 0z^2 + 2(xy + yz + zx)$

Here,

$$A = \begin{array}{cc} & \begin{array}{ccc} x & y & z \end{array} \\ \begin{array}{c} x \\ y \\ z \end{array} & \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \end{array}$$

$\det(A - xI) = \det\begin{pmatrix} -x & 1 & 1 \\ 1 & -x & 1 \\ 1 & 1 & -x \end{pmatrix}$

$= (x+1)^2(x-2)$

$\boxed{-1, -1, 2}$

Question 1(i)

Here,

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

We get the eigenvalues as

Question 1(i)

$x^T A x = +ve$?

Here,

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

We get the eigenvalues as $-1, -1, 2$

Thus it is a hyperboloid w/ 2 sheets.

$\boxed{-,-,+}$

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- For $\lambda = -1$, we get the eigenvectors

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$$\frac{1}{\sqrt{2}}\begin{bmatrix}-1\\1\\0\end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix}-1\\0\\1\end{bmatrix}$$

Question 1(i)

- For $\lambda = \boxed{-1,}$ we get the eigenvectors

$$\frac{1}{\sqrt{2}}\begin{bmatrix}-1\\1\\0\end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix}-1\\0\\1\end{bmatrix}$$

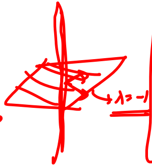
- For $\lambda = \boxed{2,}$ we get the eigenvector

Question 1(i)

want: principal axes.

$\langle v_1, v_3\rangle = 0$

$\langle v_2, v_3\rangle = 0$



- For $\lambda = -1$, we get the eigenvectors

$$\lambda=-1 \quad \boxed{\underset{v_1}{\frac{1}{\sqrt{2}}\begin{bmatrix}-1\\1\\0\end{bmatrix}}, \underset{v_2}{\frac{1}{\sqrt{2}}\begin{bmatrix}-1\\0\\1\end{bmatrix}}}$$

- For $\lambda = 2$, we get the eigenvector

$$\frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\end{bmatrix}$$

note the procedure:

$AO = OD$

$A = [ODO^T]$

let $\alpha = \begin{bmatrix}x\\y\\z\end{bmatrix}$

$\alpha^T A \alpha = 1$

$\alpha^T O D O^T \alpha = 1$

$\tilde{\alpha} := O^T\alpha$

$\Rightarrow \tilde{\alpha}^T D \tilde{\alpha} = 1$

$\tilde{\alpha}$ is now principal as $\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}$

To get the colums of O, perform GS-OP on $\lambda=-1$ space thus getting first two colums
third colum is by $\lambda = 2$ e-vector

to get back α, $\alpha = O\tilde{\alpha} \Rightarrow$ princ. axis is defined by the colums of O.

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- For $\lambda = -1$, we get the eigenvectors

$$\frac{1}{\sqrt{2}}\begin{bmatrix}-1\\1\\0\end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix}-1\\0\\1\end{bmatrix}$$

- For $\lambda = 2$, we get the eigenvector

$$\frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\end{bmatrix}$$

Have we found the transformation?

Question 1(i)

- For $\lambda = -1$, we get the eigenvectors

$$\underbrace{\frac{1}{\sqrt{2}}\begin{bmatrix}-1\\1\\0\end{bmatrix}}_{w_1}, \underbrace{\frac{1}{\sqrt{2}}\begin{bmatrix}-1\\0\\1\end{bmatrix}}_{w_2}$$

→ apply GS procedure here.

$\langle w_1, w_2\rangle \neq 0$ yet

- For $\lambda = 2$, we get the eigenvector

$$w_3 \quad \frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\end{bmatrix}$$

$\langle w_1, w_3\rangle = \langle w_2, w_3\rangle = 0$ as expected.

Have we found the transformation? No .

Question 1(ii)

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$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 3 \\ 0 & 3 & 4 \end{bmatrix}$$

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Thus we have a One sheeted hyperboloid.

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$$\frac{1}{\sqrt{4+2\sqrt{2}}}\begin{bmatrix} 0 \\ -\sqrt{2}-1 \\ 1 \end{bmatrix}$$

Question 1(ii)

- For $\lambda = 1$, we have

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \omega_1$$

- For $\lambda = 1 - 3\sqrt{2}$, we have

$$\frac{1}{\sqrt{4+2\sqrt{2}}} \begin{bmatrix} 0 \\ -\sqrt{2}-1 \\ 1 \end{bmatrix} \quad \omega_2$$

- For $\lambda = 1 + 3\sqrt{2}$, we have

$\langle \omega_i, \omega_j \rangle = \delta_{ij}$
as expected.

$$\frac{1}{\sqrt{4-2\sqrt{2}}} \begin{bmatrix} 0 \\ \sqrt{2}-1 \\ 1 \end{bmatrix} \quad \omega_3$$

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Have we found the transformation? Yes .

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$$A = \begin{bmatrix} -1 & 4 & -2 \\ 4 & -1 & 2 \\ -2 & 2 & 2 \end{bmatrix}$$

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$$-6, 3, 3$$

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- For $\underline{\lambda = 3}$, we have

$$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{5}}\begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} \to \underline{\text{appryus}}$$

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Have we found the transformation? No .

Question 2

$$\iiint_{\mathbb{R}^3} \exp\Big(-\underbrace{[\ldots\ -\ ]}_{x^T A x}\Big)\, dV.$$

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We find the eigenvalues to be $1, 1, 7$.

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We find the eigenvalues to be $\boxed{1, 1, 7}$. A sigh of relief - why?

~~what if?~~

$$\iiint_{\mathbb{R}^2} \exp\left(-\left[x^T A x\right]\right) dx$$

$$\left[1x^2 + 1y^2 + 7z^2\right]$$

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We can write A as

$$\cancel{A = Q^T D Q}$$

Better write

$$A = ODO^T$$

(keep in mind $AO = OD$)

Question 2

We again write the A matrix,

$$A = \begin{bmatrix} 2 & -2 & -1 \\ -2 & 5 & 2 \\ -1 & 2 & 2 \end{bmatrix}$$

We find the eigenvalues to be $1, 1, 7$. A sigh of relief - why? Now, how do we go about computing the integral? Do we *need* to compute the eigenbasis?
We can write A as

$$\cancel{A = O^T D O} \quad A = O D O^T$$

where $D = diag(1, 1, 7)$ and O is orthogonal.

Question 2

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$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \mapsto O^T \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix} = O^T \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\begin{bmatrix} \frac{\partial h_1}{\partial x} & \frac{\partial h_1}{\partial y} & \frac{\partial h_1}{\partial z} \\ \vdots & \vdots & \vdots \end{bmatrix} = J$$

$$\boxed{J = O^T}$$

$OO^T = I \Rightarrow \det(OO^T) = 1$

$\Downarrow$

$\det(O) = \pm 1 \Rightarrow |J| = 1$

(all orthogonal real matrices have $\det = \pm 1$)

Question 2

In effect, we perform the transformation

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \mapsto \bar{O} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

What is the Jacobian matrix for this transformation?

Question 2

Hence we are left with

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Hence we are left with

$$\iiint_{\mathbb{R}^3} \exp\{-X^2\} \exp\{-Y^2\} \exp\{-7Z^2\} dXdYdZ \; |\det(J)|$$

$\to \frac{1}{2}$

$$\left(\int_{-\infty}^{\infty} e^{-x^2} dx\right) \left(\int_{-\infty}^{\infty} e^{-y^2} dy\right) \left(\int_{-\infty}^{\infty} e^{-7z^2} dz\right)$$

$z' = \sqrt{7}\, z$

$$\frac{\int_{-\infty}^{\infty} e^{-z'^2} dz'}{\sqrt{7}} \qquad dz = \frac{1}{\sqrt{7}} dz'$$

$$= \sqrt{\pi} \times \sqrt{\pi} \times \frac{\sqrt{\pi}}{\sqrt{7}} = \frac{(\pi)^{3/2}}{\sqrt{7}}$$

as, $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$

Question 3

$$ax^2+by^2+cz^2+2hxy+2gxz+2fyz$$

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

We *again* have the A matrix, as

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What happens when this matrix loses rank?

$\Updownarrow$

non-zero nullity $\equiv$ atleast one e-val is 0

Question 3

$A = O^T D O$

As before, we convert the quadric to $\to \{\lambda_1 X^2 + \lambda_2 Y^2 + \lambda_3 Z^2\}$

$\det(\cdot) = 0 \Rightarrow$ atleast one of $\lambda_1, \lambda_2, \lambda_3$ is 0

WLOG, let $\lambda_3 = 0$. and let $\lambda_1, \lambda_2 \neq 0$ $\Rightarrow \lambda_1 X^2 + \lambda_2 Y^2 = \lambda_1 \left[X^2 + \frac{\lambda_2}{\lambda_1} Y^2\right]$

$= \lambda_1 \left[\left(X - \sqrt{\frac{-\lambda_2}{\lambda_1}} Y\right)\left(X + \sqrt{\frac{-\lambda_2}{\lambda_1}} Y\right)\right]$

done.

Can easily see other cases.

$(\Rightarrow$ proven$)$

~~Question 4~~ Q3 contd

~~Let $O \in \mathbb{R}^{3\times 3}$ be orthogonal.~~

note: if any one of $a_i, b_i, c_i = 0$ then we are done, so assume otherwise.

two cases

$\lambda_1, \lambda_2, \lambda_3$ of same sign

LHS $= 0$ iff $x = y = z = 0$

but, since $a_i, b_i, c_i \neq 0$, RHS will be $= 0$ for some x, y, z not all zero by (#).

Done.

Q3 contd (Thanks to Tanmojeet for pointing the "iff" out)

($\Leftarrow$) ie, factorized $\Rightarrow$ det(A) = 0

assume

$\lambda_1 x^2 + \lambda_2 y^2 + \lambda_3 z^2 = (a_1 x + b_1 y + c_1 z) \times (a_2 x + b_2 y + c_2 z)$

w/ $\lambda_1, \lambda_2, \lambda_3 \neq 0$

Claim: $\forall a, b, c \in \mathbb{C}$, all $\neq 0$, $\exists\, x, y, z \in \mathbb{R}$ not all zero s.t. $xa + by + cz = 0$ — (#)

(hint: use fundamental lemma)

(WLOG) let $\lambda_1, \lambda_2 > 0$, $\lambda_3 < 0$

So LHS $= 0$ at points where $z = \dfrac{\sqrt{\lambda_1 x^2 + \lambda_2 y^2}}{\sqrt{-\lambda_3}}$

$\Rightarrow$ locus is a cone

What does the locus of zeros of RHS look like? Split a_i, b_i, c_i into real and imaginary parts and see. Intersection of some planes? hence a different locus.

Done.

A Simple algebraic Soln also added at the end.

Question 4

Let $O \in \mathbb{R}^{3\times3}$ be orthogonal.Let $Ov = \lambda v$.

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Thus,

$$v^T \underbrace{\boxed{O^T O}}_{I} v = \lambda^2 v^T v$$

$v^T v = \lambda^2 v^T v \Rightarrow (\lambda^2 - 1)\underbrace{\boxed{v^T v}}_{\neq 0} = 0$

$\Downarrow$

$\underline{\lambda^2 = 1}$ $\quad \Rightarrow \underline{\underline{|\lambda| = 1}}$

for complex evalue,

$Ov = \lambda v$

take (*) $\Rightarrow v^* O^T = \bar{\lambda} v^*$

as $O^T = O^*$

(O is real)

$\Downarrow$

$v^* v = |\lambda|^2 v^* v$

$\Rightarrow \boxed{|\lambda| = 1}$

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Let $O \in \mathbb{R}^{3\times 3}$ be orthogonal.Let $Ov = \lambda v$. Then,

$$v^T O^T = \lambda v^T$$

Thus,

$$v^T O^T O v = \lambda^2 v^T v$$

Thus $|\lambda| = 1$.

Question 4

all roots of p_O may not be real
But, there will always be atleast one real root. Why?

Let $O \in \mathbb{R}^{3\times 3}$ be orthogonal. Let $Ov = \lambda v$. Then,

$$v^T O^T = \lambda v^T$$

Thus,

$$v^T O^T O v = \lambda^2 v^T v$$

$\boxed{\lambda = \pm 1}$

Thus $|\lambda| = 1$.
Are we done?

$\det(O) = \prod \lambda_i$ all λ_i may not be real.

Question 4

all real roots

assume none is +1

$\Rightarrow$ all are -1

$\Downarrow$

$\det(O) = (-1)^3 \neq +1$

a contradiction.

1 real root + 2 complex ones

$\lambda \times e^{i\phi} \times e^{-i\phi} = 1$

$\lambda = 1$

(as coeff of $P_O(x)$ are real)

$\Downarrow$

atleast one e-val is +1. $\boxtimes$

Question 4

Question 5

$U^* D U$

$$\underline{A^T A = AA^T = I} \xrightarrow{A \in \mathbb{R}^{n\times n}} \boxed{A^* A = AA^* = I}$$

A is normal (over $\mathbb{C}$)

We have

$\|v\| = 1$

$\rho, \sigma \in \mathbb{R}^n$

$$Av = \lambda v$$

$$A(\rho + \iota\sigma) = (\alpha + \iota\beta)(\rho + \iota\sigma)$$

$\alpha, \beta \in \mathbb{R}$

where, $\lambda \in \{\pm 1\}$

$$[(\alpha\rho) - \beta\sigma] + i[\beta\rho + \alpha\sigma]$$

$$\boxed{\begin{aligned} A\rho &= \alpha\rho - \beta\sigma \\ A\sigma &= \beta\rho + \alpha\sigma. \end{aligned}}$$

Question 5

By the Spectral Theorem [1] we know that,

$$\langle(\rho + \iota\sigma), v\rangle = 0$$

We conclude that

$$\langle\rho, v\rangle = \langle\sigma, v\rangle = 0$$

Now let

scale ρ and σ appropriately

$$O := \begin{bmatrix} v & \rho & \sigma \end{bmatrix}$$

condition because $\|\rho\| = \|\sigma\|$ (if they were scaled differently, $A\rho = \alpha\rho - \beta\sigma$ and $A\sigma = \beta\rho + \alpha\sigma$ wouldn't hold)

note:

$A(\rho + i\sigma) = (\alpha + i\beta)(\rho + i\sigma)$

$\Downarrow$

$A(\rho - i\sigma) = (\alpha - i\beta)(\rho - i\sigma)$

$\therefore \langle \rho + i\sigma, \rho - i\sigma\rangle = 0$ (again by S.T. [1])

$\Downarrow$

$\langle\rho, \rho\rangle - \langle\sigma, \sigma\rangle + 2i\langle\rho, \sigma\rangle = 0$

$\therefore \langle\rho, \sigma\rangle = 0$

and $\|\rho\| = \|\sigma\|$ too.

Question 5

If $A = [c_1\ c_2\ c_3]$

$Ae_1 = c_1$

$Ae_2 = c_2$

$Ae_3 = c_3$

$$\boxed{O = [v\ \ \rho\ \ \sigma]}$$

$$AO = [Av\ \ A\rho\ \ A\sigma]$$

$$= [\cancel{A}v\ \ \alpha\rho - \beta\sigma\ \ \beta\rho + \alpha\sigma]$$

$$= [v\ \ \rho\ \ \sigma] \begin{bmatrix} 1 & 0 & 0 \\ 0 & \alpha & \beta \\ 0 & -\beta & +\alpha \end{bmatrix}$$

$\alpha\rho - \beta\sigma$

$$AO = O \begin{bmatrix} 1 & & \\ & \alpha & \beta \\ & -\beta & +\alpha \end{bmatrix}$$

$$O^T A O = \begin{bmatrix} 1 & & \\ & \alpha & \beta \\ & -\beta & \alpha \end{bmatrix}$$

$$\begin{bmatrix} \boxed{\beta} & & \\ & \boxed{\beta} & \\ & & \boxed{\beta} \end{bmatrix}$$

(here again $AO = DO$ → follow this convention)

Question 5

Numerical example:

let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix}$

$\det(A-\lambda I) = \det \begin{bmatrix} (1-\lambda) & & \\ & -\lambda & -1 \\ 1 & +1 & -\lambda \end{bmatrix}$

$p_A(\lambda) = (1-\lambda)(\lambda^2+1)$

$\lambda = 1, +i, -i$

$\Rightarrow$ Only one real eval

$\Rightarrow$ not real diagonalizable.

$\left\{ \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ i \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -i \\ 1 \end{pmatrix} \right\}$

$\lambda = 1 \quad \lambda = i \quad \lambda = -i$

$y = (ie_2 + e_3)$

$Ay = iy$

$A(ie_2+e_3) = i(ie_2+e_3) \Rightarrow Ae_3 = -e_2$

$Ae_2 = e_3$

$v = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$

now, $A[v \ \ e_2 \ \ e_3] = [v \ \ e_3 \ \ e_2]$

$= [v \ \ e_2 \ \ e_3] \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

$P := [v \ e_2 \ e_3]$

$\langle e_2, v \rangle \neq 0$ But, $\{v, e_2, e_3\}$ are lin indep.

Det, $\underbrace{P^{-1}AP}_{\text{I}} = \begin{bmatrix} \boxed{1} & 0 & 0 \\ 0 & \boxed{\begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix}} \\ 0 & & \end{bmatrix}$ ↳ "Block diagonal form" → Best we can do

Verify!

Let $\lambda_1 x^2 + \lambda_2 y^2 + \lambda_3 z^2 = (ax + by + cz)(px + qy + rz)$.
See that $aq + bp = ar + cp = br + cq = 0$,
$\lambda_1 = ap, \lambda_2 = bq, \lambda_3 = cr$.

$$arq + bpr = arq + cpq = bpr + cqp = 0$$

From the first equality, either $p = 0$ (we are done) or $br = cq$. If the latter, then the third equality reads $br(p + p) = 0$, thus $br = 0$, and thus $\lambda_2 \lambda_3 = 0$. Done.