SOS Report

Game Theory

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Introduction to Game Theory

Definition

"*The study of strategically interdependent behavior. It is a mathematical framework used to study strategic interactions between rational decision-makers*"

Strategic interdependence: What I do affects your outcomes, and what you do affects my outcomes.

- It is a theoretical framework to conceive social situations among competing players.
- The intention of game theory is to produce optimal decision-making of independent and competing actors in a strategic setting.
- Using game theory, real-world scenarios can be laid out and their outcomes predicted.
- Scenarios include the prisoner's dilemma and the dictator game among many others.
- Different types of game theory include cooperative/non-cooperative, zero-sum/non-zero-sum, and simultaneous/sequential.

Useful Terms in Game Theory

A few terms commonly used terms in the study of game theory are:

- **Game**: Any set of circumstances that has a result dependent on the actions of two or more decision-makers (players)
- **Player**: A strategic decision-maker within the context of the game
- **Strategy**: A complete plan of action a player will take given the set of circumstances that might arise within the game
- **Payoff**: The outcome associated with a specific combination of strategies chosen by the players in a game. It represents the utility or value that each player receives
- **Information set**: A collection of all possible actions and outcomes that a player can perceive at a particular point in a game
- **Equilibrium**: The stable state in a game where both players have made their decisions, have no incentive to change their strategy unilaterally, and an outcome is reached.

Combinatorial Games

Definition

*"Combinatorial games are **2-player** games which consist of:*

- ➢ *A set of possible positions (the states of the game)*
- ➢ *A **move rule** indicating for each position what the players can move to*
- ➢ *A **win rule** indicating a set of terminal positions where the game ends. Each terminal position has an associated outcome"*

To play one of these games, we choose a starting position and designate a player to move first. Then, the players alternate making moves until a terminal position is reached and the game ends. These games have **no element of randomness** to determine actions. These games also require each player to have perfect information.

Perfect Information: All players have complete knowledge of the game state and available moves.

Notation used:

The game is played between **L** and **R**.
Outcomes: L wins and R loses (+−), L loses and R wins (−+), or it is a draw (00)

Game Trees

Each branch node models a choice point for one of the players and every terminal node indicates an outcome. To make a game tree starting at position α with L moving first, we begin by making a root node containing an L (since L is first) and an α (since this is the starting position). If L can move to positions α_1 , ... , α_k, then we join k new nodes to the root node; each one of the new nodes contains one of these α_i positions. If any of these positions is terminal, then we put the appropriate outcome (either +−, −+, or 00) in this node. For the other positions, it will be R's turn to play, so all of these nodes will contain an R. Continue this process until it is complete.

W-L-D Game Trees

A tree with a distinguished root node (the starting position) in which each terminal node contains an outcome (+−, −+, or 00) and each branch node contains an L or an R.

Trivial Game Trees: W-L-D game tree consisting of just a single terminal node, i.e. no moves.

Strategy: Set of decisions telling the moves to make at each node where that player has a choice

- Winning strategy: a strategy that guarantees a win for a player who follows it
- Drawing strategy: a strategy that guarantees a player doesn't lose

Zermelo's Theorem

Theorem

"Every W-L-D game tree is one of **+-** *,* **-+** *or* **00**

- ➢ **+-** ⇒ *L has a winning strategy*
- ➢ **-+** ⇒ *R has a winning strategy*
- ➢ **00** ⇒ *both players have drawing strategy "*

Proof

Zermelo's theorem can be proved using the principle of mathematical induction. The proof of Zermelo's theorem is omitted here.

Strategy

Symmetry and **Strategy Stealing** can help us to determine which player has a winning strategy without having to analyze the game tree. Although these techniques do not apply to all games, they are powerful when they work.

Normal Play Games

Definition:

"A combinatorial game where the win rule dictates that the winner is the last player to move."

- ★ Normal Play Games cannot end in a draw.

Positions and their types

Normal-play games are simple to describe. They consist of a set of positions together with a rule that dictates, for each position, to which positions L can move and to which positions R can move.

The key property of any position is to which other positions each player can move. We represent a position by: **{** all moves L can make **|** all moves R can make **}**. We generally reserve the equals symbol for use in position notation $\gamma = \{\alpha_1 , \ldots , \alpha_m \mid \beta_1 , \ldots , \beta_n\}$

Types of Positions

Every position in a normal-play game is one of the following types:

➢ **L** ⇒ L has a winning strategy whoever goes first
➢ **R** ⇒ R has a winning strategy whoever goes first
➢ **N** ⇒ Next player to play has a winning strategy
➢ **P** ⇒ Second (or Previous) player has a winning strategy

If $\gamma = \{\alpha_1, \ldots, \alpha_m \mid \beta_1, \ldots, \beta_n\}$, the type of γ is given by the following chart:

	some β_j *is type* R *or* P	*all of* $\beta_1, \ldots, \beta_n$ *are types* L *or* N
some α_i *is type* L *or* P	N	L
all of $\alpha_1, \ldots, \alpha_m$ *are types* R *or* N	R	P

Sums of Positions

If α and β are positions in normal-play games, then we define $\alpha + \beta$ to be a new position consisting of the **components** α and β.

Determinate Sums

➢ If β is type P, then α and $\alpha + \beta$ are the same type
➢ If α and β are both type L, then $\alpha+\beta$ is type L.
➢ Similarly, if α and β are both type R, then $\alpha + \beta$ is type R

Indeterminate Sums

None of the question marks in the following table can be truthfully replaced by one of our types.

+	L	R	N	P
L	L	?	?	L
R	?	R	?	R
N	?	?	?	N
P	L	R	N	P

Equivalence

Definition

"Two positions α and α' in (possibly different) normal-play games are equivalent if for every position β in any normal-play game, the two positions $\alpha + \beta$ and $\alpha' + \beta$ have the same type."

That is, two positions are equivalent if they behave the same under summation.
We write $\alpha \equiv \alpha'$ to indicate this.

Equivalence Relations

The term equivalence relation is used to describe any relation that is:

- **Reflexive**: $\alpha \equiv \alpha$
- **Symmetric:** $\alpha \equiv \beta$ implies $\beta \equiv \alpha$
- **Transitive:** $\alpha \equiv \beta$ and $\beta \equiv \gamma$ implies $\alpha \equiv \gamma$

Following are some key points regarding equivalence:

- If $\alpha \equiv \alpha'$, then α and α' have the same type
- If α, β, γ are positions in normal-play games, then:
  - $\alpha + \beta \equiv \beta + \alpha$ *(commutativity)*
  - $(\alpha + \beta) + \gamma \equiv \alpha + (\beta + \gamma)$ *(associativity)*
- If β is type P, then $\alpha + \beta \equiv \alpha$
- If α and α' are type P, then $\alpha \equiv \alpha'$
- If $\alpha + \beta$ and $\alpha' + \beta$ are both type P, then $\alpha \equiv \alpha'$

Impartial Games

Definition:

"These are normal-play games in which the available moves for L and R are always the same."

In an impartial game, the identity of the player does not affect the available moves. This means impartial games have no positions of type L or type R.

A position in an impartial game is:

- Type **N** $\Rightarrow$ if there exists a move to a position of type P
- Type **P** $\Rightarrow$ if there is no move to a position of type P

Nim

An impartial game in which a position consists of ℓ piles of stones of sizes $a_1, a_2, \ldots, a_\ell$. To make a move, a player removes one or more (up to all) stones from a chosen pile. The last player to take a stone wins.

Binary Expansion:

- For every nonnegative integer ℓ,

  $2^0 + 2^1 + 2^2 + \cdots + 2^\ell = 2^{\ell+1} - 1$

- Every nonnegative integer n has a unique binary expansion.

Types of Positions

For any nonnegative integer a let $*a$ denote a Nim position (also called a "*nimber")* consisting of a single pile of a stones. Then a position with piles of sizes $a_1, a_2, \ldots, a_\ell$ may be denoted as $*a_1 + *a_2 + \cdots + *a_\ell$

Now we will break each pile into subpiles based on the binary expansion of the size of the pile.

Balanced Position: For every power of 2, the total number of subpiles of that size is even

Following are some key points regarding balanced and unbalanced games:

★ The ability to move from an unbalanced position to a balanced position figures prominently in a winning Nim strategy

★ Every balanced Nim position is type P and every unbalanced Nim position is type N

★ In both cases, the balancing procedure provides a winning strategy

A Nim position with two piles $*a + *b$ will be type P if $a = b$ and type N otherwise

Equivalence in Nim

We call a position of the form $*a$ a "*nimber*"

Balanced positions are type P and all other positions are type N. This means $*0$ is type P and, since any two type P positions are equivalent, this tells us that every balanced position is equivalent to $*0$

Nim-sum: The Nim-sum of the nonnegative integers $a_1, \ldots, a_\ell$, denoted $a_1 \oplus \cdots \oplus a_\ell$, is the nonnegative integer b with the property that 2^j appears in the binary expansion of b if and only if this term appears an odd number of times in the expansions of $a_1, \ldots, a_\ell$

Following are some key points regarding balanced and unbalanced games:

- ★ If a_1 , ... , a_ℓ are nonnegative integers and $b = a_1 \oplus a_2 \cdots \oplus a_\ell$, then $*a_1 + *a_2 + \cdots + *a_\ell \equiv *b$
- ★ If $*a_1 + \cdots + *a_\ell$ is balanced, then b = 0, and the position is type P. Otherwise, $*a_1 + \cdots + *a_\ell$ is unbalanced, so $b \neq 0$ and the position (which is equivalent to $*b$) will be type N

The Sprague-Grundy Theorem and MEX Principle

Definition

*“For a set $S = \{a_1, a_2, \ldots , a_n\}$ of nonnegative integers, we define the **M**inimal **EX**cluded value, abbreviated MEX, of S to be the smallest nonnegative integer b which is not one of a_1 , ... , a_n “*

MEX Principle

Let $\alpha = \{\alpha_1 , \ldots , \alpha_k\}$ be a position in an impartial game. Suppose that $\alpha_i \equiv *a_i$ for every $1 \leq i \leq k$. Then $\alpha \equiv *b$ where b is the MEX of the set $\{a_1 , \ldots , a_k\}$

Sprague-Grundy Theorem

“Every position in an impartial game is equivalent to a nimber”

The Sprague-Grundy Theorem tells us that every position in each of these games is equivalent to some nimber, and the MEX Principle gives us a recipe for finding that equivalent nimber.

Hackenbush and Partizan Games

Hackenbush Game

Hackenbush is a normal-play game played between L and R. The game consists of a graph drawn with black and gray edges, some of which are attached to the ground. A move for L is to erase a bLack edge, while a move for R is to erase a gray edge. After an erasure, any part of the graph no longer connected to the ground floats away, out of the game. Since this game is normal-play, the last player to make a move wins.

To start, we define •0 to be the Hackenbush position with no edges (no moves available)

- ★ The position •0 is type P
- ★ A Hackenbush position α satisfies $\alpha \equiv$ •0 if and only if α is type P
- ★ For every Hackenbush position α, we have α + •0 $\equiv \alpha$
- ★ The position •0 in Hackenbush is like position $*0$ in Nim (the number 0 in the integers)

Negation: In Hackenbush, we can reverse the roles of the players in any position α by switching the colors of all the edges. This new position is denoted $-\alpha$ and is called the negative of α

If α and β are Hackenbush positions, then:

- $-(-\alpha) \equiv \alpha$
- $\alpha + (-\alpha) \equiv \bullet 0$
- $\beta + (-\alpha) \equiv \bullet 0$ implies $\alpha \equiv \beta$

Positions in Hackenbush

Integer Positions

We may view • as an operation that takes an integer and outputs a special Hackenbush position

For any integers m and n, we have:

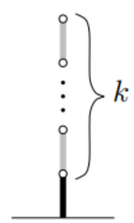
- $-(\bullet n) \equiv \bullet(-n)$,
- $(\bullet m) + (\bullet n) \equiv \bullet(m + n)$

The position $\bullet n$ will be:

- type L when $n > 0$
- type P when $n = 0$
- type R when $n < 0$

Fractional Positions

For every positive integer k, define $\bullet 1\ 2k$ to be the Hackenbush position shown in Figure:



For every positive integer k, we have $\bullet 1/2^k + \bullet\ 1/2^k \equiv \bullet\ 1/2^{k-1}$

Dyadic Numbers and Positions

"Any number that can be expressed as a fraction where the denominator is a power of 2 (and the numerator is an integer) is called a dyadic number"

★ Every dyadic number has a unique (finite) binary expansion

Birthdays of Dyadic Numbers

On day 0 the number 0 is born. If $a_1 < a_2 < \cdots < a_\ell$ are the numbers born on days 0, 1, ... , n, then on day $n + 1$ the following new numbers are born:

- ➢ the largest integer which is less than a_1
- ➢ the smallest integer which is greater than a_ℓ
- ➢ the number $(a_i + a_{i+1})/2$ for every $1 \le i \le \ell - 1$
- ★ Every open interval of real numbers (a, b), (a, ∞),$(-\infty, b)$, or $(-\infty, \infty)$, contains a unique oldest dyadic number

Dyadic Positions

For every dyadic number $q > 0$ with binary expansion $2^{d1} + 2^{d2} + \cdots + 2^{d\ell}$ (here $d_1 > d_2 > \cdots > d_\ell$ are integers which may be positive or negative), we define the position:

$\bullet q = \bullet 2^{d1} + \bullet 2^{d2} + \cdots + \bullet 2^{d\ell}$

- ➢ To get negative positions, we define $\bullet(-q) = -(\bullet q)$
- ➢ We call any position of the form $\bullet q$ a dyadic position
- ★ Let $a1, \ldots, an$ be numbers, each of which is either 0 or has the form $\pm 2k$ for some integer k. If $a1 + a2 + \cdots + an = 0$, then $\bullet a1 + \bullet a2 + \cdots + \bullet an \equiv \bullet 0$

If p, q are dyadic numbers, then

- ➢ $-(\bullet p) \equiv \bullet(-p)$
- ➢ $(\bullet p) + (\bullet q) \equiv \bullet(p + q)$

For a dyadic number q, $\bullet q$ is type:

- ➢ L *if* $q > 0$
- ➢ P *if* $q = 0$
- ➢ R *if* $q < 0$

The Simplicity Principle

"Consider a position in a partizan game given by $\gamma = \{\alpha_1, \ldots, \alpha_m \mid \beta_1, \ldots, \beta_n\}$ and suppose $\alpha_i \equiv \bullet a_i$ for $1 \le i \le m$, $\beta_j \equiv \bullet b_j$ for $1 \le j \le n$. If there do not exist a_i and b_j with $a_i \ge b_j$, then $\gamma \equiv \bullet c$ where c is the oldest number larger than all of $a_1, \ldots, a_m$ and smaller than all of $b_1, \ldots, b_n$"

The Simplicity Principle can be used (recursively) to show that many positions in other partizan games are equivalent to dyadic positions

- ★ If α, β are positions equivalent to $\bullet a, \bullet b$, then $\alpha + \beta \equiv \bullet a + \bullet b$
- ★ The Simplicity Principle does not apply to every position in a partizan game. For the position $\gamma = \{\alpha_1, \ldots, \alpha_m \mid \beta_1, \ldots, \beta_n\} \equiv \{\bullet a_1, \ldots, \bullet a_m \mid \bullet b_1, \ldots, \bullet b_n\}$ the Simplicity Principle is only applicable when $\max\{a_1, \ldots, a_m\} < \min\{b_1, \ldots, b_n\}$

Zero-Sum Matrix Games

A Zero-Sum Matrix Game is a game played between Rose and Colin according to the following rules:

- There is a fixed matrix A that is known to both players. Rose secretly chooses one of the rows and Colin secretly chooses one of the columns.
- Then, both players reveal their choices. If Rose chose row i and Colin chose column j, then the (i, j) entry of matrix A is the payoff of the game.
- This payoff indicates how much Colin must pay Rose.

Such games are called zero-sum since whatever one player gains is what the other player loses.

Dominance

Dominance is the strategic concept of one strategy being superior to another for a player, regardless of the strategies chosen by their opponents.

Saddle point: a zero-sum matrix game A has a saddle point at (i, j) if the entry in row i and column j is the smallest in its row and the largest in its column

Theorem

"Let A be a zero-sum matrix game. If iterated deletion of dominated strategies reduces A to a 1×1 matrix consisting of the entry in position (i,j) of the original matrix, then A has saddle point at (i,j)"

Mixed strategies

For an arbitrary $m \times n$ zero-sum matrix game A, define a mixed strategy for Rose to be a row vector p = $[p_1\, p_2 \cdots p_m]$ with the property that $p_i \geq 0$ for $1 \leq i \leq m$ and $\sum m\ i{=}1\ p_i = 1$. When Rose plays p, she randomly chooses to play row i with probability pi . Analogously for Colin, a mixed strategy is a column vector q, for which $q_j \geq 0$ for $1 \leq j \leq n$ and $\sum n\ j{=}1\ q_j = 1$. When Colin plays q, he randomly chooses column j with probability q_j . (Again, we assume that Rose's and Colin's probabilistic choices are independent.)

von Neumann solutions and Minimax Theorem

Let p and q be mixed strategies for Rose and Colin in the zero-sum matrix game A.

- If p has a guarantee of r, then pAq $\geq r$
- If q has a guarantee of c, then pAq $\leq c$

➢ If p and q have guarantees of r and c, then $r \leq c$

Minimax Theorem

"Every zero-sum matrix game A has a unique number v, called the value of A, which is the maximum guarantee of a mixed strategy for Rose and the minimum guarantee of a mixed strategy for Colin"

Let A be a zero-sum matrix game with $ai,j = v$. Then A has a saddle point at(i, j)if and only if v together with the pure strategies of row i and column j form a von Neumann solution.

Equating the Opponent's Results

For a zero-sum matrix game A, we say that a mixed strategy p for Rose equates Colin's results if all entries of pA are equal. Similarly, we say that a mixed strategy q for Colin equates Rose's results if all entries of Aq are equal.

For every 2 × 2 zero-sum matrix game:

➢ If Rose has no dominant row, Colin has a mixed strategy equating her results
➢ If Colin has no dominant column, Rose has a mixed strategy equating his results

von Neumann Theorem

"For every zero-sum matrix game, the maximum guarantee of a mixed strategy for Rose is equal to the minimum guarantee of a mixed strategy for Colin"

General Games

The Prisoner's Dilemma

The Prisoner's Dilemma is a fundamental concept in game theory involving two rational players, each having the choice to either cooperate or betray the other. The dilemma lies in the fact that although mutual cooperation yields the highest joint payoff, each player can improve their individual payoff by betraying the other while the other cooperates. However, if both players betray, they end up with lower payoffs compared to mutual cooperation. This paradox illustrates the tension between individual rationality and collective interest, as the self-interested choices of both players lead to a suboptimal outcome for the group as a whole.

The Prisoner's Dilemma is a useful model for analyzing scenarios where cooperation is necessary for mutual benefit, but individual incentives might drive players towards non-cooperative behavior, thus shedding light on the challenges of achieving cooperation in various real-life situations, ranging from economics and politics to social interactions and evolutionary biology.

Utility in general games

Utilities refer to the numerical values that represent the preferences or benefits of each player for different outcomes or strategies. Utilities are used to quantify the players' subjective evaluations of the payoffs they receive from different choices they make during the game.

Matrix games and game trees

Just as every game tree can be turned into a matrix game, so every matrix game can be turned into a game tree. To introduce our matrix-to-tree process, we will again consider an example.

Compare and contrast matrix games and game trees

Nash Equilibrium

Nash equilibrium is a concept that represents a stable state of a game where no player has an incentive to unilaterally change their strategy, given the strategies chosen by other players. In other words, in a Nash equilibrium, each player's strategy is the best response to the strategies chosen by the other players. It is a point of strategic balance where no player can improve their payoff by changing their strategy alone.

Formally, in a game with players (P1, P2, ..., Pn), strategies (S1, S2, ..., Sn), and payoff functions (u1, u2, ..., un), a Nash equilibrium is a combination of strategies (s1*, s2*, ..., sn*) such that for each player i:

ui(s1*, s2*, ..., si*, ..., sn*) ≥ ui(s1*, s2*, ..., si, ..., sn*)

Applications of Nash Equilibrium

Economics: Nash equilibrium is widely used in economic modeling to analyze various market structures, pricing strategies, and oligopolistic competition. It helps to understand how firms interact strategically and reach stable outcomes in different economic scenarios.

Business Strategy: In strategic decision-making, Nash equilibrium helps analyze competitive dynamics and pricing strategies among competing firms to find stable and optimal outcomes.

Network Theory: Nash equilibrium is used to analyze and optimize the behavior of participants in networks, such as communication networks, transportation networks, and social networks.

Game Design: In the field of game design, Nash equilibrium helps create balanced and strategic gameplay by analyzing the interactions between players and ensuring stable outcomes.

The single most important theorem in classical game theory pairs best responses to produce what's called a Nash equilibrium. For a matrix game A, a pair of mixed strategies p for Rose and q for Colin form a Nash equilibrium if p is a best response to q and q is a best response to p.

- ★ Every matrix game has a Nash equilibrium

n-Player Games & Preferences and Society

In an n-player game, each player has a set of possible strategies from which they can choose, and the players receive payoffs based on the joint actions of all players. The utility or payoff for each player depends not only on their own strategy but also on the strategies chosen by all other players. This makes the analysis of n-player games more challenging than two-player games.

Matrix games and coalitions

Matrix games are a specific form of game theory where the payoffs for each player are presented in a matrix format. Each player has a finite set of strategies, and the outcome or payoff for each combination of strategies is shown in the matrix. The players' goal is to select strategies that maximize their payoffs.

Coalitions refer to groups of players who form alliances to achieve common goals. Studying matrix games and coalitions involves analyzing the strategic interactions between players and understanding how they can collaborate or compete to achieve favorable outcomes. Cooperative game theory is often employed to study coalitions and how players can work together to improve their collective outcomes.

Shapley value and its application

The Shapley value is a concept in cooperative game theory used to fairly allocate the total worth or value of a cooperative game among its players. It provides a unique and fair way to distribute the payoff among players based on their contributions to all possible coalitions.

The Shapley value is calculated by considering all possible orders in which players could join a coalition and determining the marginal contributions of each player to the coalition. The Shapley value provides an equitable distribution that satisfies properties such as efficiency, symmetry, and additivity.

Applications

Applications of the Shapley value include cost-sharing problems, fair resource allocation in collaborative projects, and sharing dividends among shareholders in a company.

Fair division and Stable marriages

Fair division is a branch of game theory that deals with the fair allocation of goods or resources among multiple players, each with their own preferences. Various methods, such as dividing items sequentially, envy-free algorithms, and the Knaster procedure, are used to ensure equitable outcomes in fair division problems.

Stable marriages, on the other hand, is a specific application of matching theory, where individuals with preferences for potential partners seek mutually acceptable matches. The goal is to find a stable matching, where no two individuals would prefer to be with each other over their current partners.

Arrow's Impossibility Theorem

Arrow's Impossibility Theorem, formulated by economist Kenneth Arrow, addresses the challenges of aggregating individual preferences into a collective or societal preference. It states that there is no perfect voting system that can satisfy all desirable properties simultaneously, such as transitivity, non-dictatorship, and Pareto efficiency. This theorem has significant implications in social choice theory and voting systems, highlighting the difficulties of achieving a fair and consistent collective decision-making process.