

Transformation of R.V. and Multivariate Gaussian

Kavya Gupta

April 13, 2024

Contents

1	Introduction	2
2	Transformation of Random Variable	2
3	Multi-variate Gaussian Disribution	2
3.1	Definition	2
3.2	A is diagonal	3
3.3	A is non-singular square matrix	3

1 Introduction

In this article, we will study about the following topics of statistics:

- Transformation of random variables
- Multivariate Gaussian random variable

2 Transformation of Random Variable

Given any continuous r.v. X with PDF $P_X(x)$ and given any function $g(X)$ (defined on range of X) we intend to find PDF associated with the r.v. $Y = g(X)$.

For simplicity, let's assume $g(\cdot)$ is monotonic increasing.

Then by probability mass conservation,

$$P(a < X < b) = P(g(a) < Y < g(b)) = \int_{g(a)}^{g(b)} Q(y) dy$$

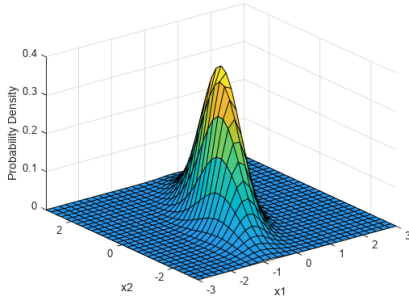
Using $y = g(x)$, we get the below relation upon simplification,

$$Q(y) = P(g^{-1}(y)) \frac{dg^{-1}(y)}{dy}$$

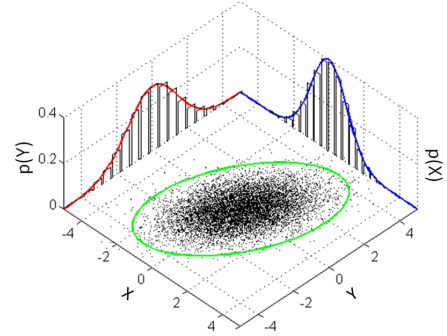
To handle monotonically decreasing $g(\cdot)$ as well¹,

$$Q(y) = \begin{cases} +P(g^{-1}(y)) \frac{dg^{-1}(y)}{dy}, & \text{for } g(\cdot) \text{ monotonically increasing.} \\ -P(g^{-1}(y)) \frac{dg^{-1}(y)}{dy}, & \text{for } g(\cdot) \text{ monotonically decreasing.} \end{cases} \quad (1)$$

For more information, refer [1]



(a) Example 1



(b) Example 2

3 Multi-variate Gaussian Distribution

3.1 Definition

Let X be a vector of random variables of dimension D .

A r.v. X has a joint PDF as multi-variate Gaussian distribution \exists finite i.i.d. standard Gaussian

¹we could have used modulus operator but I wanted things to look more complicated

r.v. W_1, W_2, \dots, W_N with $N > D$ such that

$$A = XW + \mu$$

Refer fig[1a] and fig[1b] for visual examples. This has many applications in machine learning, refer [2] and [3].

3.2 A is diagonal

In this case, the X_i are independent. The standard deviation of distribution of X_i is A_{ii} .

3.3 A is non-singular square matrix

Let's take $\mu = 0$ for simplicity.

Similar to univariate case, where scaling was determined by $|\frac{dg^{-1}(y)}{dy}|$ the scaling for multi-variate case is determined by determinant of matrix of derivatives, Jacobian matrix.

Also, $W = A^{-1}X$. We intend to find the volume of the parallelepiped formed due to this transformation. **Claim** The volume of parallelepiped described by column vectors of matrix A^{-1} is given by $\det(A^{-1})$

$$P(X) = \frac{1}{(2\pi)^{\frac{D}{2}}} \cdot \frac{1}{\sqrt{\det(C)}} \cdot \exp(0.5 \cdot XT \cdot C^{-1} \cdot X) \quad (2)$$

Sample Values of bivariate normal distribution		
x	y	f(x,y)
0	0	1.6
0	1	0.096
$\sqrt{2}$	$\sqrt{2}$	0.02

References

- [1] <https://www.example.com>.
- [2] <https://www.example.com>.
- [3] Donald E. Knuth. *The T_EX Book*. Addison-Wesley Professional, 1986.