Infinities and Beyond

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1 General results

Theorem 1 (Schröder-Bernstein (SB))**.** *If $\mathfrak{u}$ and $\mathfrak{v}$ are cardinal numbers such that $\mathfrak{u} \le \mathfrak{v}$ and $\mathfrak{v} \le \mathfrak{u}$, then $\mathfrak{u} = \mathfrak{v}$.*

Another way to phrase this is:

Theorem 1 (Schröder-Bernstein (SB))**.** *If U and V are sets such that there's an injection from U to V and an injection from V to U, then there is a bijection from U to V.*

Choice required: No.

Proof. By hypothesis, there exist one-to-one functions $f : U \to V$ and $g : V \to U$.
Define a function $\varphi : \mathcal{P}(U) \to \mathcal{P}(U)$ as follows:

$$\varphi(E) := U \setminus g[V \setminus f[E]] \tag{1}$$

Now, we claim that if $E \subset F \subset U$, then $\varphi(E) \subset \varphi(F)$.
Indeed, we have that $E \subset F \subset U \implies f[E] \subset f[F] \implies V \setminus f[F] \subset V \setminus f[E] \implies g[V \setminus f[F]] \subset g[V \setminus f[E]] \implies U \setminus g[V \setminus f[E]] \subset U \setminus g[V \setminus f[F]] \iff \varphi(E) \subset \varphi(F)$.

Thus, we have

$$E \subset F \subset U \implies \varphi(E) \subset \varphi(F) \tag{2}$$

Define $\mathcal{D} := \{E \in \mathcal{P}(U) : E \subset \varphi(E)\}$. Note that $\mathcal{D} \neq \varnothing$ as $\varnothing \in \mathcal{D}$.
Define $D := \displaystyle\bigcup_{E \in \mathcal{D}} E$.
Now, given any $E \in \mathcal{D}$, we have $E \subset D$. By (2), this gives us that $\varphi(E) \subset \varphi(D)$. Also, by definition of $\mathcal{D}$, we have that $E \subset \varphi(E)$.
Thus, $E \subset \varphi(D)$ for all $E \in \mathcal{D}$. It follows from the definition of D that $D \subset \varphi(D)$. Applying (2) again gives us $\varphi(D) \subset \varphi(\varphi(D))$ and hence, $\varphi(D) \in \mathcal{D}$. This now gives us that $\varphi(D) \subset D$.
The inclusions in both directions give us that $\varphi(D) = D$.

For the sake of clarity, we can now see that we have arrived at the following result:
There exist subsets $D \subset U$ and $R \subset V$ such that $f[D] = R$ and $g[V \setminus R] = U \setminus D$. (Let this D be the D defined as earlier and let $R := f[D]$.)
We can now simply define the following bijection $h : U \to V$ as

$$h(x) := \begin{cases} f(x) & \text{if } x \in D \\ g^{-1}(x) & \text{if } x \in U \setminus D \end{cases}$$

Note that h indeed is well-defined as we have defined the value of h for each x uniquely. The fact that it is well-defined for $x \in U \setminus D$ follows from the fact that $g[V \setminus R] = U \setminus D$ and thus, every $x \in U \setminus D$ does have a pre-image. This is unique by the hypothesis that g is one-to-one.
The fact that h is a bijection also follows from the properties of D and R. $\square$

Theorem 2 (Comparing cardinalities)**.** *Let U and V be sets. Then either $|U| \le |V|$ or $|V| \le |U|$.*

Choice required: Yes.

Proof. The idea will be to use Zorn's Lemma.
Let $\mathcal{F}$ be the set of all one-to-one functions f such that $\operatorname{dom} f \subset U$ and $\operatorname{rng} f \subset V$. Note that $\mathcal{F} \neq \varnothing$ as $\varnothing \in \mathcal{F}$.
We order $\mathcal{F}$ by inclusion. (Recall that every $f \in \mathcal{F}$ can regarded as a subset of $U \times V$.)
Let $\mathcal{C} \subset \mathcal{F}$ be a chain in $\mathcal{F}$. We show that $\mathcal{C}$ has an upper bound $u \in \mathcal{F}$.
Define $u = \bigcup_{f \in \mathcal{C}} f$.
One can straightaway observe that $\operatorname{dom} u = \bigcup_{f \in \mathcal{C}} \operatorname{dom} f \subset U$ and similarly, $\operatorname{rng} u \subset V$.
Now, we show that given any $x \in \operatorname{dom} u$, there a unique $y \in V$ such that $(x, y) \in u$.
Existence. This is easy, for if $x \in \operatorname{dom} u$, then $x \in \operatorname{dom} f$ for some $f \in \mathcal{C}$ and thus, $(x, f(x)) \in f \subset u$.
Uniqueness. Suppose (x, y_1) and (x, y_2) belong to u. We show that $y_1 = y_2$.
$(x, y_1) \in u \implies \exists f_1 \in \mathcal{C}[(x, y_1) \in f_1]$.
$(x, y_2) \in u \implies \exists f_1 \in \mathcal{C}[(x, y_2) \in f_2]$.
As $\mathcal{C}$ is a chain, we have that $f_1 \subset f_2$ or $f_2 \subset f_1$. WLOG, we assume that $f_1 \subset f_2$. Thus, $(x, y_1) \in f_2$.
However, f_2 is a function and thus, $y_1 = y_2$, as desired.
Thus, u is indeed a function.
Now we show that it is one-to-one as well. The argument is almost identical to what we gave for the uniqueness of y. We assume that (x_1, y) and (x_2, y) belong to u for some $y \in V$ and conclude that $x_1 = x_2$.
Thus, $u \in \mathcal{F}$. Now, it is easy to see that u is an upper bound of $\mathcal{C}$.

Thus, by Zorn's Lemma, we get that there exists a maximal element $m \in \mathcal{F}$.
Claim. Either $\operatorname{dom} m = U$ or $\operatorname{rng} m = V$.
Proof. Suppose not. Then $\operatorname{dom} m \neq U$ *and* $\operatorname{rng} m \neq V$. Thus, there exist $x \in U \setminus \operatorname{dom} m$ and $y \in V \setminus \operatorname{rng} m$.
Thus, $(x, y) \notin m$ giving us $m \subsetneq m \cup \{(x, y)\}$. However, $m \cup \{(x, y)\} \in \mathcal{F}$, contradicting the maximality of m.

If $\operatorname{dom} m = U$, then m is a one-to-one function from U to V giving us that $|U| \leq |V|$. Otherwise, m^{-1} is a one-to-one function from V to U giving us that $|V| \leq |U|$.

$\square$

Theorem 3 (Cantor)**.** *Let U be a set. Then $|U| < |\mathcal{P}(U)|$.*

Choice required: No.

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Proof. For $U = \varnothing$, the statement is true as $\mathcal{P}(\varnothing) = \{\varnothing\}$ is a nonempty set and there is no surjective function from an empty set to a nonempty set. On the other hand, $\varnothing : \varnothing \to \{\varnothing\}$ is an injection.

Now we suppose that $U \neq \varnothing$.
We first establish that $|U| \leq |\mathcal{P}(U)|$. Consider the map $i : U \to \mathcal{P}(U)$ defined as $x \overset{i}{\mapsto} \{x\}$. It is easy to see that this is an injection for $\{x\} = \{y\} \iff x = y$.

Now, we show that $|U| \neq |\mathcal{P}(U)|$. Suppose that there exists a bijection $h : U \to \mathcal{P}(U)$.
Define $S = \{x \in U : x \notin h(x)\}$.
By definition, we have that $S \subset U$ and thus, $S \in \mathcal{P}(U)$.
By assumption, h is a bijection and thus, there exists $x \in U$ such that $h(x) = S$.
Now, by the law of excluded middle, either $x \in S$ or $x \notin S$. We show that either leads to a contradiction.
Case 1. $x \in S$.
$x \in S \implies x \in h(x) \implies x \notin S$, where the first implication is by the definition of x and the second is by the definition of S.
Case 2. $x \notin S$.
$x \notin S \implies x \notin h(x) \implies x \in S$, where the first implication is by the definition of x and the second is by the definition of S.
Thus, we get that $x \in S \iff x \notin S$, a contradiction. $\square$

Theorem 4. *Every infinite set has a countably infinite subset.*
In other words, $|\mathbb{N}| \leq |A|$, if A is infinite.

Choice required: Yes.

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Proof. Let A be a any infinite set.
Claim. For any $n \in \mathbb{N}$, there exists a set $A_n \subset A$ such that $|A_n| = n$.
Proof. We prove this via induction. As $A \neq \varnothing$, there exists $A_1 \subset A$ such that $|A_1| = 1$.
Now, let $A_n \subset A$ be such that $|A_n| = n$. If $A \setminus A_n$ were empty, then we would get that A is finite. Thus, there exists $x \in A \setminus A_n$. Letting $A_{n+1} = A_n \cup \{x\}$, we have $A_{n+1} \subset A$ and $|A_{n+1}| = n + 1$.

Now, let $\{A_n\}_{n\in\mathbb{N}}$ be any such family of subsets of A as described above. The existence of such a family is given by axiom of choice.
For each $n \in \mathbb{N}$, define

$$B_n = A_{2^n} \setminus \left(\bigcup_{k=0}^{n-1} A_{2^k}\right).$$

Given $n < m$, we have that if $x \in B_n$, then $x \in A_{2^n}$ but then $x \notin B_m$. Thus the family $\{B_n\}_{n\in\mathbb{N}}$ is a pairwise disjoint family of subsets of A, and for each $n \in \mathbb{N}$ we have

$$|B_n| \geq 2^n - \sum_{k=0}^{n-1} 2^k = 2^n - (2^n - 1) = 1.$$

Thus, each B_n is nonempty.
Applying the axiom of choice to $\{B_n\}_{n\in\mathbb{N}}$ gives a choice function $f : \mathbb{N} \to \bigcup_{n\in\mathbb{N}} B_n \subset A$ such that $f(n) \in B_n$ for each $n \in \mathbb{N}$.
As the sets are pairwise disjoint, we have it that f is one-to-one.
Thus, $f[\mathbb{N}]$ is a countably infinite subset of A. □

Theorem 5. *Any subset of a countable set is countable.*

Choice required: No.

Proof. Let A be a countable set and let $B \subset A$. If B is finite, then there is nothing to prove. Now, suppose that B is infinite. Then, A cannot be finite and thus, is countably infinite. Let g be a bijection from $\mathbb{N}$ to A. Let $a_n := g(n)$.
We now define a bijection $f : \mathbb{N} \to B$ as follows:
$f(1) = a_{n_1}$ where n_1 is the smallest $n \in \mathbb{N}$ such that $a_n \in B$; $f(k+1) = a_{n_{k+1}}$ where n_{k+1} is the smallest $n \in \mathbb{N}$ such that $a_n \in B \setminus \{f(1), \ldots, f(k)\}$.

We now show that f is a bijection.
One-to-one. Let $n, m \in \mathbb{N}$ with $n \neq m$. WLOG, $n < m$.
Then, $f(m) \in B \setminus \{f(1), \ldots, f(n), \ldots, f(m-1)\}$ and thus $f(m) \neq f(n)$.
Onto. Let $x \in B$. Then, $x = a_m$ for some $m \in \mathbb{N}$.
Define $S = \{n \in \mathbb{N} : n < m, a_n \in B\}$. Then we have $f(|S| + 1) = x$. □

Theorem 6. *If A is any nonvoid countable set, then there exists a surjective function $f : \mathbb{N} \to A$.*

Choice required: No.

Proof. Since A is countable, there exists a one-to-one function $g : A \to \mathbb{N}$. Fix some $a \in A$. Define $f : \mathbb{N} \to A$ as

$$f(n) := \begin{cases} g^{-1}(n) & \text{if } n \in \operatorname{rng} g \\ a & \text{if } n \notin \operatorname{rng} g \end{cases}$$

Given any $x \in A$, we have that $f(g(x)) = x$. Thus, f is surjective. □

Theorem 7. *If A and B are two nonvoid sets and if there is a mapping f from A onto B, then $|B| \leq |A|$, that is, there is a one-to-one map from B to A.*

Choice required: Yes.

Proof. Let g be a choice function function for the family $\{f^{-1}(b)\}_{b\in B}$. Then g is a one-to-one mapping from B to A. This follows from the fact that $b_1 \neq b_2 \implies f^{-1}(b_1) \cap f^{-1}(b_2) = \varnothing$. $\square$

Theorem 8. *The union of any countable family of countable sets is a countable set, i.e., if $\{A_i\}_{i\in I}$ is a family of sets such that I is a countable and each A_i is countable, then $A = \bigcup_{i\in I} A_i$ is countable.*

Choice required: Yes.

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Proof. Let $\{A_i\}_{i\in I}$ be as in the theorem. WLOG, we assume that I is nonvoid and so is A_i for each $i \in I$.
Applying Theorem 6 to obtain a surjection $g : \mathbb{N} \to I$.
Now, note that for each $i \in I$, there exists a surjective function $f_i : \mathbb{N} \to A_i$.
Using the axiom of choice, we can fix one such surjection for each $i \in I$.
Now, we define $h : \mathbb{N} \times \mathbb{N} \to A$ by $h(m, n) = f_{g(m)}(n)$. Then h is a surjective function. By Theorem 7, we get that $|A| \leq |\mathbb{N} \times \mathbb{N}| = |\mathbb{N}|$, where the last equality follows from Theorem 9. $\square$

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2 Cardinalities of specific sets

Theorem 9. $|\mathbb{N} \times \mathbb{N}| = |\mathbb{N}|$

Choice required: No.

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Proof. $(m, n) \mapsto 2^{m-1}(2n - 1)$ is a bijection from $\mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$. $\square$

Theorem 10. $|\mathbb{Z}| = |\mathbb{Q}| = |\mathbb{N}|$

Choice required: No.

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Proof. $\mathbb{Z}$ and $\mathbb{Q}$ can both be written as a countable union of countable sets and thus, are countable. One can also avoid choice and appeal to SB by choosing suitable functions. $\square$

Theorem 11. $2^{\aleph_0} = \mathfrak{c}$.

Choice required: No.

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Proof. Let $A = \{0, 1\}^{\mathbb{N}}$. Then, $|A| = 2^{\aleph_0}$. Let $B = [0, 1)$. Then $|B| = \mathfrak{c}$. Thus, it suffices to show that $|A| = |B|$. We shall construct injections from A to B and vice-versa and then appeal to SB.

$A \to B$.
Define $f : A \to B$ as $f(\varphi) = \displaystyle\sum_{n=1}^{\infty} \frac{\varphi(n)}{3^n}$.
This can be thought of as mapping an infinite sequence of 0 and 1 to the corresponding ternary number. As we don't have sequences with infinitely many trailing 2s, it follows that f is one-to-one.

$B \to A$.
Given any $x \in B$, it has a unique binary representation if we don't allow trailing 1s. Said formally, there is a unique representation of the form:

$$x = \sum_{n=1}^{\infty} \frac{x_n}{2^n}$$

where each x_n is 0 or 1 and $x_n = 0$ for infinitely many $n \in \mathbb{N}$.
Define $g : B \to A$ by $g(x) = \varphi_x$ where $\varphi_x : \mathbb{N} \to \{0, 1\}$ is defined as $\varphi_x(n) = x_n$.
Thus, g is a one-to-one mapping from B to A.

By SB, we are done. $\square$

Theorem 12. $|\mathbb{R}^{\mathbb{N}}| = |\mathbb{R}|$ *or* $\mathfrak{c}^{\aleph_0} = \mathfrak{c}$.

Choice required: No.

Proof. By Theorem 11, there exists a bijection $f : \mathbb{R} \to \{0,1\}^{\mathbb{N}}$. Given any $r \in \mathbb{R}$, let $f_r := f(r)$. That is, f_r is a function from $\mathbb{N}$ to $\{0,1\}$ for each $r \in \mathbb{R}$.

Now, given any sequence $(x_n)_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$, we get a sequence of functions $(f_{x_n})_{n\in\mathbb{N}} \in (\{0,1\}^{\mathbb{N}})^{\mathbb{N}}$.
This sequence corresponds to a function $g : \mathbb{N} \times \mathbb{N} \to \{0,1\}$ defined as $g(m,n) = f_{x_m}(n)$.

It is easy to see the this correspondence is one-to-one. Thus, we get that

$$|\mathbb{R}^{\mathbb{N}}| = |\{0,1\}^{\mathbb{N}\times\mathbb{N}}| = |\{0,1\}^{\mathbb{N}}| = |\mathbb{R}|.$$

Note that we have used Theorem 9, that is $|\mathbb{N} \times \mathbb{N}| = |\mathbb{N}|$. □

Theorem 13. *Let* $\mathfrak{a}$ *be an infinite cardinal number. Then* $\mathfrak{a}^{\mathfrak{a}} = 2^{\mathfrak{a}}$.

Choice required: Yes.

Proof.

$$2^{\mathfrak{a}} \leq \mathfrak{a}^{\mathfrak{a}} \leq (2^{\mathfrak{a}})^{\mathfrak{a}} = 2^{\mathfrak{a}\cdot\mathfrak{a}} = 2^{\mathfrak{a}}.$$

□

Remark. Choice was used to conclude that $\mathfrak{a} \cdot \mathfrak{a} = \mathfrak{a}$. However, there are cardinalities for which this is true even without choice. For them, the theorem holds even without choice.
In fact, $\mathfrak{a} \cdot \mathfrak{a} = \mathfrak{a}$ for all cardinalities implies AC.

Theorem 14. *Let* S *be the set of continuous functions from* $\mathbb{R}$ *to* $\mathbb{R}$.
$|S| = \mathfrak{c}$.

Choice required: No.

Proof. First, we show that $|S| \geq |\mathbb{R}| = \mathfrak{c}$.
Note that given any $r \in \mathbb{R}$, the constant function $x \mapsto r$ belongs to S. It is easy to see that this gives an injection $\mathbb{R} \hookrightarrow S$.

Now, we show that $|S| \leq |\mathbb{R}^{\mathbb{N}}| = |\mathbb{R}| = \mathfrak{c}$, where the equality $|\mathbb{R}^{\mathbb{N}}| = \mathfrak{c}$ follows from Theorem 12.
We know that $|\mathbb{Q}| = |\mathbb{N}|$. Let $q : \mathbb{N} \to \mathbb{Q}$ be a bijection.
Given any $f \in S$, define the following sequence $(x_n) \in \mathbb{R}^{\mathbb{N}}$

$$x_n = f(q(n)).$$

Now, note that if two continuous functions agree at all rational points, then they must be equal. ($\because \mathbb{Q}$ is dense in $\mathbb{R}$.)
Thus, the above mapping $f \mapsto (x_n)$ is an injection $S \hookrightarrow \mathbb{R}^{\mathbb{N}}$.

By SB, we conclude that $|S| = \mathfrak{c}$. □

Theorem 15. *Let* S *be the set of discontinuous functions from* $\mathbb{R}$ *to* $\mathbb{R}$.
$|S| = 2^{\mathfrak{c}}$.

Choice required: No.

Proof. $S \subset \mathbb{R}^{\mathbb{R}}$ and thus, $|S| \leq |\mathbb{R}^{\mathbb{R}}| = 2^{\mathfrak{c}}$. (Theorem 13.)

Now, we show that $|S| \geq 2^{\mathfrak{c}}$.
We create a injection from $\mathcal{P}(\mathbb{R}) \setminus \{\varnothing, \mathbb{R}\}$ to S.
Let $A \in \mathcal{P}(\mathbb{R}) \setminus \{\varnothing, \mathbb{R}\}$. Define $\varphi(A) = \chi_A$, the indicator function of $A \subset \mathbb{R}$.
It is easy to see that χ_A is discontinuous. This follows from the fact that $A = \chi_A^{-1}(\{1\})$ and $\mathbb{R} \setminus A = \chi_A^{-1}(\{0\})$ would have to be open subsets of $\mathbb{R}$, if χ_A were continuous but $\mathbb{R}$ is connected, so this is not possible. ($\because A \notin \{\varnothing, \mathbb{R}\}$.)

As $|\mathcal{P}(\mathbb{R}) \setminus \{\varnothing, \mathbb{R}\}| = |\mathcal{P}(\mathbb{R})| = 2^{\mathfrak{c}}$, the result follows from SB. $\square$

Theorem 16. *Let S be the set of continuous functions from $\mathbb{Q}$ to $\mathbb{Q}$.*
$|S| = \mathfrak{c}$.

Choice required: No.

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Proof. First, we show that $|S| \leq \mathfrak{c}$.
Note that $S \subset \mathbb{Q}^{\mathbb{Q}}$ and thus $|S| \leq |\mathbb{Q}^{\mathbb{Q}}| = \aleph_0^{\aleph_0} = 2^{\aleph_0} = \mathfrak{c}$. (Theorems 10, 13, and 11.)

Now, we show that $|S| \geq |\mathbb{N}^{\mathbb{N}}| = |\mathbb{R}| = \mathfrak{c}$.
Let $f \in \mathbb{N}^{\mathbb{N}}$ be given. Using this, we create a function $\varphi_f : \mathbb{Q} \to \mathbb{Q}$ as follows:
$\varphi_f(x) = f(1)$ for all $x < 1$, $\varphi_f(n) = f(n)$ for all $n \in \mathbb{N}$,
for $x \in \mathbb{Q} \setminus \mathbb{N}$ and $x > 1$, let $p = \lfloor x \rfloor$ and define $\varphi_f(x) = (x - p)(f(p+1) - f(p)) + f(p)$.

It is easy to show that $\varphi_f \in S$ and $f \neq g \implies \varphi_f \neq \varphi_g$ as φ_f agrees with f at all naturals.
(φ_f is the functions obtained by joining the points of the graph of f.)

The result now follows from SB. $\square$

Theorem 17. *Let S be the set of discontinuous functions from $\mathbb{Q}$ to $\mathbb{Q}$.*
$|S| = \mathfrak{c}$.

Choice required: No.

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Proof. First, we show that $|S| \leq \mathfrak{c}$.
Note that $S \subset \mathbb{Q}^{\mathbb{Q}}$ and thus $|S| \leq |\mathbb{Q}^{\mathbb{Q}}| = \aleph_0^{\aleph_0} = 2^{\aleph_0} = \mathfrak{c}$. (Theorems 10, 13, and 11.)

Now, we show that $|S| \geq |\mathbb{N}^{\mathbb{N}}| = |\mathbb{R}| = \mathfrak{c}$.
Let $f \in \mathbb{N}^{\mathbb{N}}$ be given. Using this, we create a function $\varphi_f : \mathbb{Q} \to \mathbb{Q}$ as follows:

$$\varphi_f(x) = \begin{cases} f(x) & \text{if } x \in \mathbb{N} \\ 0 & \text{if } x \notin \mathbb{N} \end{cases}$$

It is easy to show that $\varphi_f \in S$ and $f \neq g \implies \varphi_f \neq \varphi_g$ as φ_f agrees with f at all naturals.

The result now follows from SB. $\square$

Theorem 18. $|\mathbb{N}^{\mathbb{R}}| = |2^{\mathbb{R}}|$ *or* $\aleph_0^{\mathfrak{c}} = 2^{\mathfrak{c}}$.

Choice required: No.

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Proof.

$$|2^{\mathbb{R}}| \leq |\mathbb{N}^{\mathbb{R}}| \leq |\mathbb{R}^{\mathbb{R}}| = |2^{\mathbb{R}}|.$$

$\square$

Theorem 19. *Let $\mathfrak{a} \geq \aleph_0$. Then, $\mathfrak{a}! = 2^{\mathfrak{a}}$.*

Choice required: Yes.

Proof. Let A be a set with cardinality $\mathfrak{a}$ and let S be the set of all bijections from A to itself. By definition, we have $|S| = \mathfrak{a}!$.

Note that we have $|S| \leq |A^A| = 2^{\mathfrak{a}}$. (Theorem 13.)
Now we show that $|S| \geq |\mathcal{P}(A)| = 2^{\mathfrak{a}}$.
If A is infinite, then we have that $|A| = |A \times \{0, 1\}|$. (This uses choice.)
Thus, it suffices to show that there are as many bijections from $A \times \{0, 1\}$ as there are elements in $\mathcal{P}(A)$.
Let $B \in \mathcal{P}(A)$. Define the following function $f_B : A \times \{0, 1\} \to A \times \{0, 1\}$.

$$f_B\left((a, x)\right) = \begin{cases} (a, 0) & \text{if } a \notin B \text{ and } x = 0 \\ (a, 1) & \text{if } a \notin B \text{ and } x = 1 \\ (a, 0) & \text{if } a \in B \text{ and } x = 1 \\ (a, 1) & \text{if } a \in B \text{ and } x = 0 \end{cases}$$

That is, f_B fixes all elements of the form $(a, 0)$ and $(a, 1)$ if $a \notin B$ and swaps them otherwise.
It is clear that $B \mapsto f_B$ is an injection from $\mathcal{P}(A)$ to S and thus, we are done by SB. $\square$

3 Summary

1. $|2^{\mathbb{N}}| = |\mathbb{R}|$.
2. $|\mathbb{R}^{\mathbb{N}}| = |\mathbb{R}|$.
3. $|\mathbb{N}^{\mathbb{R}}| = |2^{\mathbb{R}}|$.
4. $|C(\mathbb{R}, \mathbb{R})| = |\mathbb{R}|$.
5. $|C(\mathbb{Q}, \mathbb{Q})| = |\mathbb{R}|$.
6. $|X^X| = 2^{|X|}$. (C)
7. $|X|! = 2^{|X|}$ if $|X| = \infty$. (C)