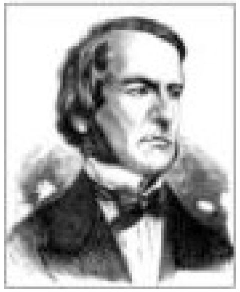
Algebra with Only 0s and 1s!

<u>Real-valued variable</u>

Variable may take any real number (e.g. $x = 3.1234567$)

<u>Boolean variable</u>

A variable assigned either 0 or 1 (e.g. $x = 0$ or $x = 1$)



George Boole
1815-1864

0 and 1 are also called "False" and "True" or "No" and "Yes"

<u>Real Operators</u>

+
-
$\times$
$\div$
...

<u>Boolean Operators</u>

`NOT`
`AND`
`OR`

Functions

Describing a real-valued function…

Words

f is a function of TWO real variables s.t. the output is the sum of their squares

Table

x	y	$f(x,y)$
0	0	0
1	2	5
1.2	1	2.44
1	1	2

Formula

$f(x) = x^2+y^2$

That table seems to be missing a “few” entries!

Boolean Functions

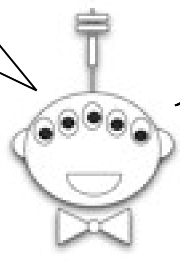
A Boolean function just takes Boolean arguments and gives a Boolean result.

Example: f(True, True) = True

The most common Boolean functions take 1 or 2 arguments.

There are exactly four 1-argument Boolean functions!

There are 16 two-argument functions, but only 5 are commonly used!



Boolean Functions

Describing a Boolean function (inputs and outputs: 0 and 1)

Words

f is a function of TWO binary (Boolean) variables s.t. the output is *1* if and only if exactly one of the two inputs is *1*

Table

x	y	$f(x,y)$
0	*0*	*0*
0	*1*	*1*
1	*0*	*1*
1	*1*	*0*

Formula

?

A table works fine now! It’s called a “truth table.”

NOT, AND, OR

x	NOT x
0	1
1	0

Also written
$\overline{x}$

x	y	x AND y
0	0	0
0	1	0
1	0	0
1	1	1

Also written
xy

x	y	x OR y
0	0	0
0	1	1
1	0	1
1	1	1

Also written
$x+y$

Playing with Functions…

Describe these functions in English:

$x\bar{x}$ xx $x+\bar{x}$ $(xy + \bar{x}\bar{y})$

Worksheet!

Playing with Functions…

How about Boolean formulae ("formulas") for:

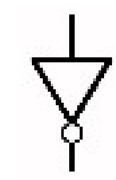
- A function of two variables x,y that evaluates to 1 iff x and y are not equal
- A function of two variables x,y that evaluates to 1 iff $x \geq y$

Digital Logic Gates

x	NOT *x*
0	1
1	0

Also written
$\overline{x}$

x



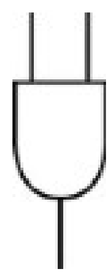
$\overline{x}$

NOT is often shown as just the small circle on another gate.

x	*y*	*x* AND *y*
0	0	0
0	1	0
1	0	0
1	1	1

Also written
xy

x *y*

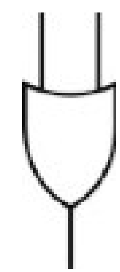


x AND *y*

x	*y*	*x* OR *y*
0	0	0
0	1	1
1	0	1
1	1	1

Also written
x+*y*

x *y*

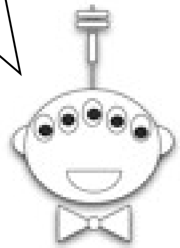


x OR *y*

XOR

x	y	x XOR y
0	0	0
0	1	1
1	0	1
1	1	0

Python uses ~, &, |, and ^ to represent NOT, AND, OR, and XOR, respectively.



Finding the Formula!

The Minterm Expansion Principle

Consider this function…

Words	Truth Table			Formula
	x	y	x XOR y	
A function of TWO binary inputs x,y where the output is 1 iff $x \neq y$	0	0	0	
	0	1	1	$\overline{x}y$
	1	0	1	$x\overline{y}$
	1	1	0	

$$f(x,y) = \overline{x}y + x\overline{y}$$

From Formula to Circuit!

<u>Words</u>

f is a function of TWO binary (Boolean) variables s.t. the output is *1* if and only if exactly one of the two inputs is *1*

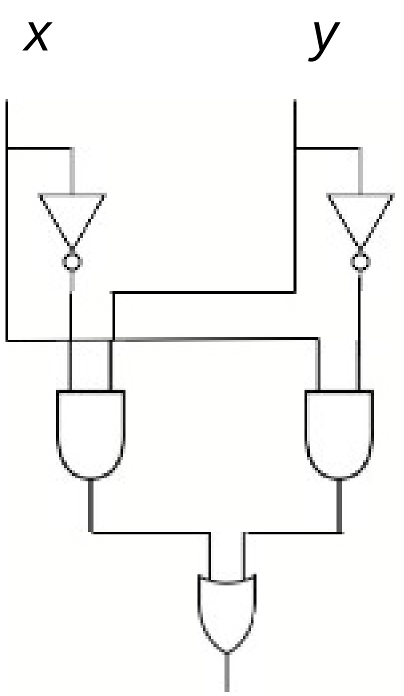
<u>Table</u>

x	*y*	*f(x,y)*
0	*0*	*0*
0	*1*	*1*
1	*0*	*1*
1	*1*	*0*

<u>Formula</u>

$\overline{x}y + x\overline{y}$

<u>Circuit</u>



From Formula to Circuit!

<u>Words</u>

f is a function of TWO binary (Boolean) variables s.t. the output is *1* if and only if exactly one of the two inputs is *1*

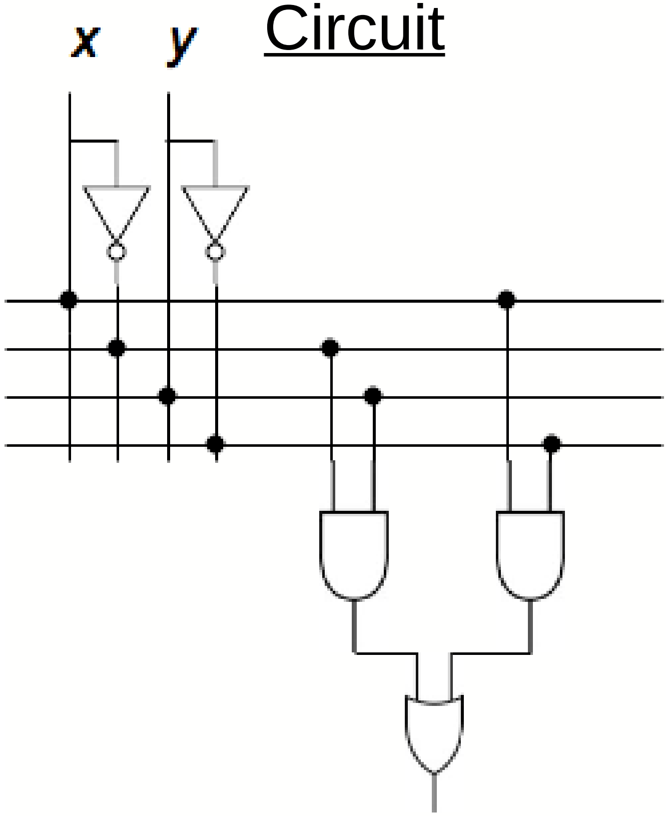
<u>Table</u>

x	y	$f(x,y)$
0	0	0
0	1	1
1	0	1
1	1	0

<u>Formula</u>

$\overline{x}y + x\overline{y}$

<u>Circuit</u>



You Try It!

The Minterm Expansion Principle

Consider this function…

Words

Truth Table

Formula

A function of TWO binary inputs *x*,*y* where the output is 1 iff $x \geq y$

Circuit

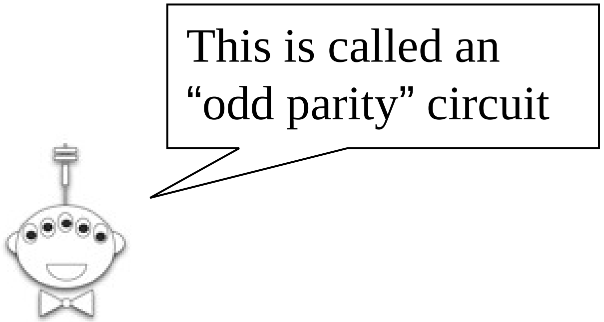
Try This One…

Consider this function…

<u>Words</u> | <u>Truth Table</u> | <u>Formula</u>

A function of THREE binary inputs *x,y,z* where the output is 1 iff the number of 1’s is odd

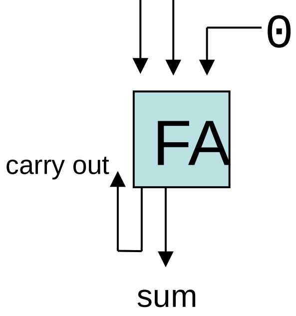
<u>Circuit</u>



A Circuit for Adding!

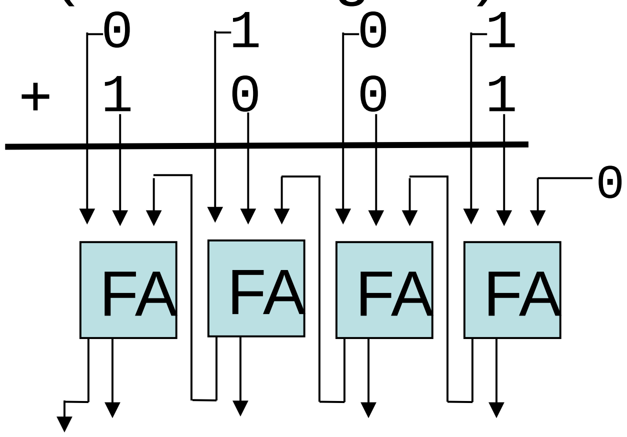
Base 2 Addition
(4-bit unsigned)

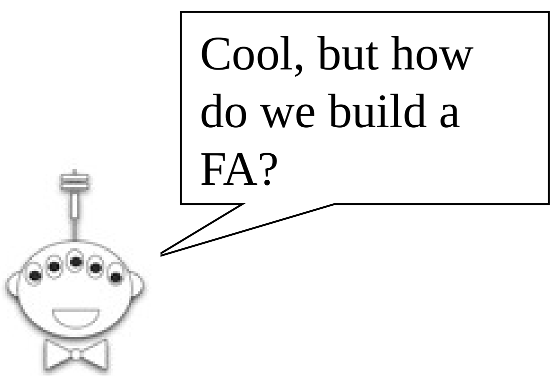
```
   0  1  0  1
+  1  0  0  1
```



A Circuit for Adding!

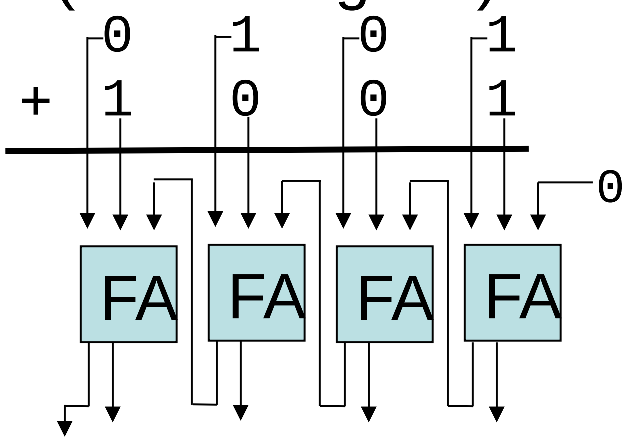
Base 2 Addition
(4-bit unsigned)

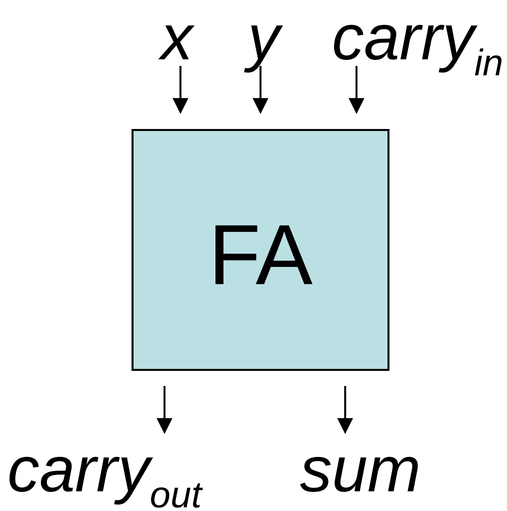




A Circuit for Adding!

Base 2 Addition
(4-bit unsigned)





x	*y*	$carry_{in}$	$carry_{out}$	*sum*
0	0	0	0	0
0	0	1	0	1
0	1	0	0	1
…				
…				
…				
…				
1	1	1	1	1

Properties of Boolean Functions

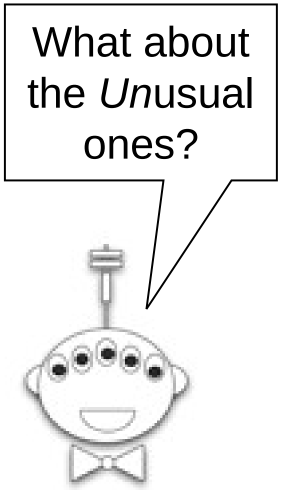
All the "usual" Boolean functions commute:

$$f(x, y) = f(y, x)$$

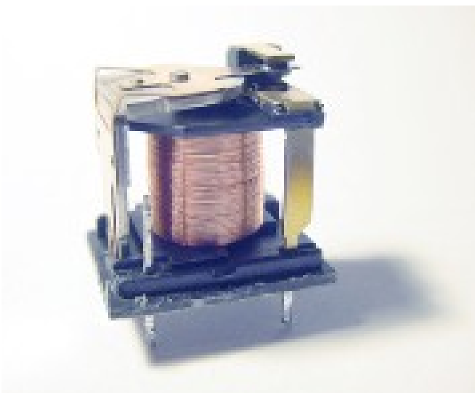
AND, OR, and XOR associate:

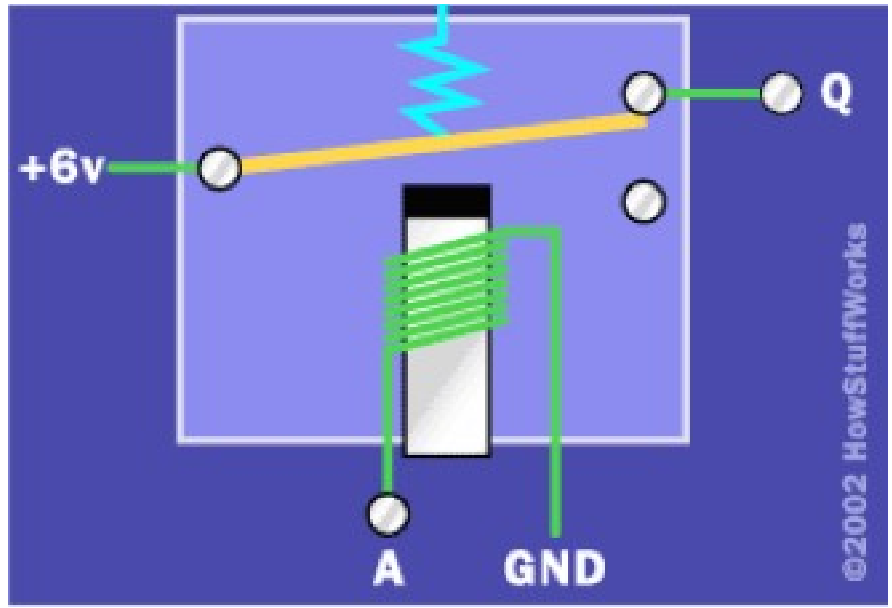
$$f(f(x, y), z) = f(x, f(y, z))$$

e.g., (x AND y) AND z = x AND (y AND z)



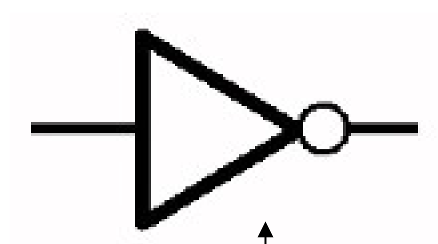
Implementing Gates with Relays





NOT Gate

A

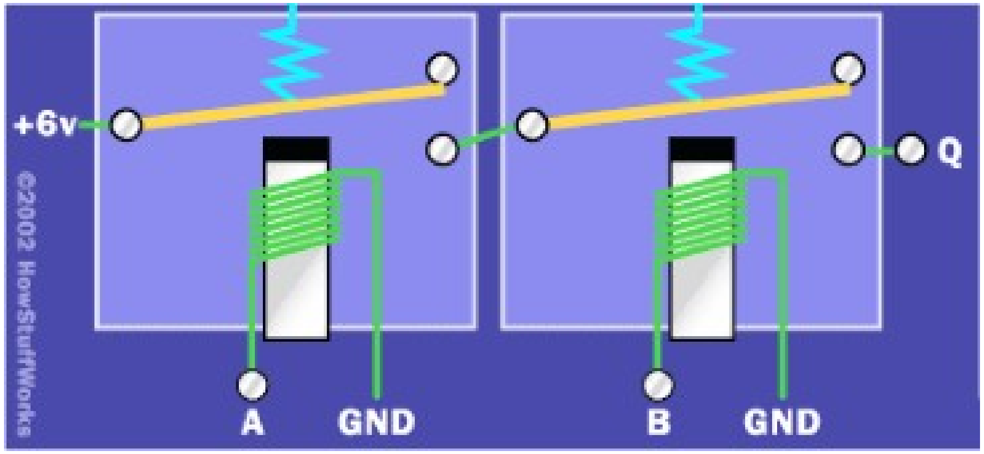


$Q = \overline{A}$

6v power

Figures courtesy of HowStuffWorks.com

And AND…



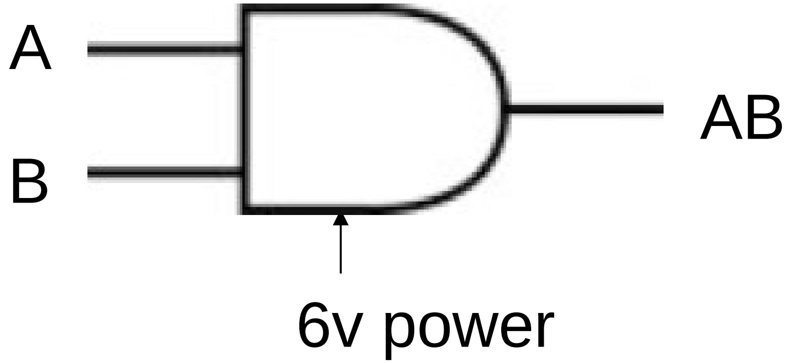
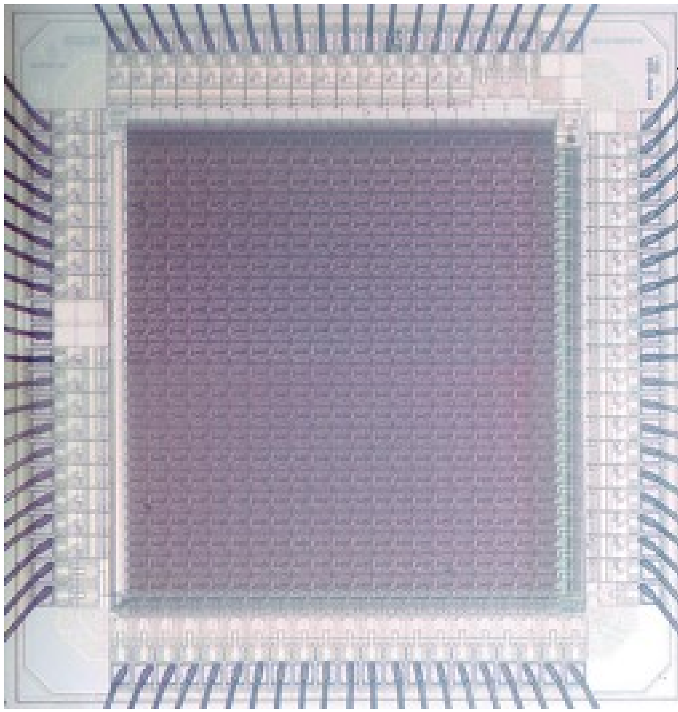


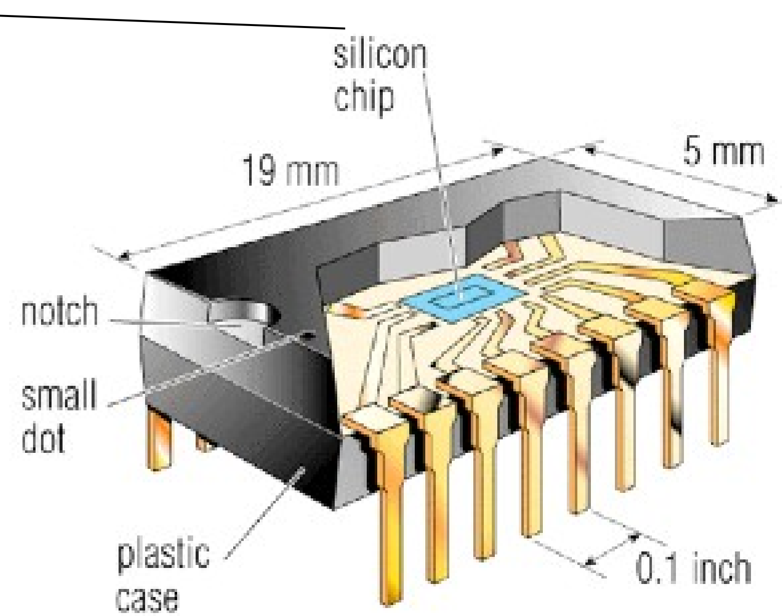
Figure courtesy of HowStuffWorks.com

Integrated Circuits



3mm

http://personalpages.manchester.ac.uk/staff/p.dudek/projects/scamp



http://www.helicon.co.uk

AND, OR, NOT Is a "Universal Set"

De Morgan's Laws:

$$\overline{x\,y} = \overline{x} + \overline{y}$$

$$\overline{x+y} = \overline{x}\;\overline{y}$$

Augustus De Morgan
1806-1871

Are there other universal sets of gates?

By Sophia Elizabeth De Morgan - Memoir of Augustus De Morgan, Public Domain
https://commons.wikimedia.org/w/index.php?curid=4722207