SIDDHARTH SATYAM 180763

PROGRAMMING ASSIGNMENT 4

* We consider the unsteady state 1D diffusion equation in a slab with radioactive decay on a grid of N nodes.

* The governing equation →

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} - \lambda T \quad ; \quad 0 < x < L \; ; \; t > 0.$$

Initial condition → $t=0$, $T(x,0) = 0$.

Boundary condition → $x=0$, $T(0,t) = 1$ ; $x = L$, $T(L,t) = 0$.

* Discretization using FTCS scheme :→ (Implicit)

$$\frac{T_i^{n+1} - T_i^{n}}{\Delta t} = \alpha \frac{T_{i+1}^{n+1} - 2T_i^{n+1} + T_{i-1}^{n+1}}{(\Delta x)^2} - \lambda T_i^{n+1}$$

$$\Rightarrow \boxed{F_0 T_{i+1}^{n+1} - (2F_0 + \lambda \Delta t + 1) T_i^{n+1} + F_0 T_{i-1}^{n+1} = -T_i^{n}}$$

where, $F_0 = \frac{\alpha \Delta t}{(\Delta x)^2}$ → Grid Fourier number.

*
$$\begin{bmatrix} 1 & 0 & 0 & \cdots & \cdots & 0 \\ F_0 & -(2F_0+\lambda\Delta t+1) & F_0 & \cdots & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & \ddots & \ddots \\ 0 & 0 & \cdots & \cdots & 0 & 1 \end{bmatrix} \begin{bmatrix} T_1^{n+1} \\ T_2^{n+1} \\ \vdots \\ T_{i-1}^{n+1} \\ T_i^{n+1} \\ T_{i+1}^{n+1} \\ \vdots \\ T_N^{n+1} \end{bmatrix} = \begin{Bmatrix} 1 \\ -T_2^{n} \\ \vdots \\ -T_i^{n} \\ \vdots \\ 0 \end{Bmatrix}$$

↳ A tri diagonal matrix.

↳ Diagonally dominant (strongly)

$\left[ \because \; 2F_0 + \lambda \Delta t + 1 > 2F_0 \right.$
$\left. \text{and. } 1 > 0 \right]$

* We had to choose a time step such that the left wall temperature gradient reaches twice the steady state value in 20 iterations of time.

$\because$ Steady state solution $T(x) = \dfrac{\sinh\sqrt{\lambda}(1-x)}{\sinh\sqrt{\lambda}}$

$\Rightarrow \dfrac{\partial T}{\partial x} = -3\coth(3)$

$\Rightarrow \left(\dfrac{T_2^{20} - T_1^{20}}{\Delta x}\right) = 2 \times (-3\coth(3))$

$= \underline{-6.0298}$.

* So we choose the time step to be, for $N=101 \rightarrow \Delta t = 0.000535$

$N=201 \rightarrow \Delta t = 0.00054$

Such that at 20th iteration, $\dfrac{\partial T}{\partial x} \simeq -6.029$.

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ANALYTICAL SOLUTION $\rightarrow$

$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2} - \lambda T$$

* We assume, $T(x,t) = \varphi(x,t)\, e^{-\lambda t}$

$$\Rightarrow \cancel{-\lambda \varphi e^{-\lambda t}} + \frac{\partial \varphi}{\partial t} \cancel{e^{-\lambda t}} = \cancel{e^{-\lambda t}} \frac{\partial^2 \varphi}{\partial x^2} \cancel{-\lambda \varphi e^{-\lambda t}}$$

$$\Rightarrow \frac{\partial \varphi}{\partial t} = \frac{\partial^2 \varphi}{\partial t^2}; \quad 0 < x < 1;\ t > 0.$$

Initial cond$^n$ $\rightarrow$ $t=0$, $\varphi(x,0) = 0$

Boundary cond$^n$ $\rightarrow$ $x=0$, $\varphi(0,t) = e^{\lambda t}$; $x=1$, $\varphi(1,t) = 0$

★ Now, we obtain the solution using Duhamel's theorem

Hence, we obtain,

$$\varphi_H(x,t) = (1-x) + \sum_{n=1}^{\infty} C_n \sin(n\pi x)\, e^{-n^2\pi^2 t}$$

→ we sum upto 100 terms

$$C_n = \frac{\int_0^1 (x-1)\sin(n\pi x)\,dx}{\int_0^1 \sin^2(n\pi x)\,dx} = \frac{\frac{\pi n - \sin n\pi}{n^2\pi^2}}{\frac{1}{2} - \frac{\sin 2\pi n}{4\pi n}} = \frac{\frac{\pi n}{n^2\pi^2}}{\frac{1}{2}} = \frac{2}{n\pi}$$

($\sin n\pi = 0$ as $n \in \mathbb{Z}$)

$$\varphi(x,t) = \int_0^t e^{\lambda\tau} \frac{\partial}{\partial t}\varphi_H(x,t)\Big|_{x,t-\tau} d\tau$$

$$= \int_0^t e^{\lambda t} \sum_1^{100} C_n \sin(n\pi x)\cdot(-n^2\pi^2)\, e^{-n^2\pi^2(t-\tau)} d\tau$$

$$\Rightarrow T(x,t) = \int_0^t e^{-\lambda(t-\tau)} \sum_1^{100} C_n \sin(n\pi x)\,(-n^2\pi^2)\, e^{-n^2\pi^2(t-\tau)} d\tau$$

Taking $\Sigma$ out of the integral,

$$T(x,t) = \sum_1^{100} \int_0^t C_n \sin(n\pi x)\,(-n^2\pi^2)\, e^{(t-\tau)(-\lambda - n^2\pi^2)} d\tau$$

$$= \sum_1^{100} \int_0^t B_n\, e^{(t-\tau)(-\lambda - n^2\pi^2)} d\tau$$

$$= \sum_1^{100} \frac{B_n}{\lambda + n^2\pi^2}\left(1 - e^{-t(\lambda + n^2\pi^2)}\right)$$

$$= \sum_1^{100} D_n \left(1 - e^{-(\lambda + n^2\pi^2)t}\right)$$

★ The initial guess for Gauss siedel has been taken vector <double> x = array of zeros.