17.6 DEL APPLIED TO VECTOR POINT FUNCTIONS *(Divergence & Curl)*

1. Divergence: Let $\vec{f}(x,y,z) = f_1(x,y,z)\hat{\imath} + f_2(x,y,z)\hat{\jmath} + f_3(x,y,z)\hat{k}$ be a continuously differentiable vector point function. Divergence $\vec{f}(x,y,z)$ of is a scalar which is denoted by $\nabla \cdot \vec{f}$ and is defined as

$$\nabla \cdot \vec{f} = \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot \left(f_1\,\hat{\imath} + f_2\,\hat{\jmath} + f_3\hat{k}\,\right) = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}.$$

t is also denoted by $div\,\vec{f}$.

Physical interpretation of Divergence

Consider the case of fluid flow.

Let $\vec{v} = v_x\hat{\imath} + v_y\hat{\jmath} + v_z\hat{k}$ be the velocity of the fluid at a point $P(x,y,z)$. Consider a small parallelopiped with edges $\delta x, \delta y$ and δz parallel to the x, y and z axis

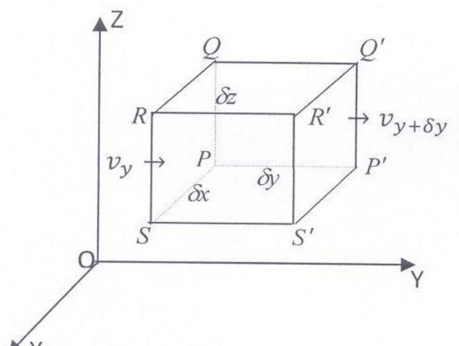

Fig. 7.9

respectively in the mass of fluid, with one of its corner at point P.

So, the mass of fluid flowing in through the face $PQRS$ per unit time $= v_y\ \delta z\ \delta x$

and the mass of fluid flowing out of the face $P'Q'R'S'$ per unit time

$$= v_{y+\delta y}\ \delta z\ \delta x = \left(v_y + \frac{\partial v_y}{\partial y}\ \delta y\right)\ \delta z\ \delta x$$

$\therefore$ The net decrease in fluid mass in the parallelopiped corresponding to flow along y-axis

$$= \left(v_y + \frac{\partial v_y}{\partial y}\ \delta y\right)\ \delta z\ \delta x - v_y\ \delta z\ \delta x = \frac{\partial v_y}{\partial y}\ \delta x\ \delta y\ \delta z$$

Similarly, the net decrease in fluid mass in the parallelopiped corresponding to the flow along x-axis and z-axis is $\frac{\partial v_x}{\partial x}\ \delta x\ \delta y\ \delta z$ and $\frac{\partial v_z}{\partial z}\ \delta x\ \delta y\ \delta z$ respectively.

So, total decrease in mass of fluid mass in the parallelopiped per unit time

$$= \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right)\ \delta x\ \delta y\ \delta z$$

Thus, the rate of loss of fluid per unit volume $= \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$

$$= \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot \left(v_x\hat{\imath} + v_y\hat{\jmath} + v_z\hat{k}\right)$$

$$= \nabla \cdot \vec{v} = div\ \vec{v}$$

Hence, $div\ \vec{v}$ gives the rate at which fluid is originating or diminishing at a point per unit volume.

If the fluid is incompressible, there can be no loss or gain in the volume element *i.e.* $div\ \vec{v} = 0$.

Observations:

(i) if $\vec{v}$ represent the electric flux, then $div\ \vec{v}$ is the amount of flux which diverges per unit volume in unit time.

(ii) if $\vec{v}$ represent the heat flux, then $div\ \vec{v}$ is the rate at which the heat is issuing from a point per unit volume.

(iii) If the flux entering any element of space is the same as that leaving it *i.e.*$div\ \vec{v} = 0$ everywhere, then such a vector point function is called **Solenoidal.**

Example 29: If $\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$ then show that $div\,\vec{r} = 3$.

Solution: $div\ \vec{r} = \nabla \cdot \vec{r} = \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot \left(x\hat{\imath} + y\hat{\jmath} + z\hat{k}\right)$

$$= \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 1 + 1 + 1 = 3$$

Example 30: Evaluate $div\,\vec{f}$ where $\vec{f} = 2x^2z\,\hat{\imath} - xy^2z\,\hat{\jmath} + 3y^2x\,\hat{k}$ at (1, 1, 1).

Solution: $div\,\vec{f} = \left(\hat{\imath}\frac{\partial\emptyset}{\partial x} + \hat{\jmath}\frac{\partial\emptyset}{\partial y} + \hat{k}\frac{\partial\emptyset}{\partial z}\right) \cdot \left(2x^2z\,\hat{\imath} - xy^2z\,\hat{\jmath} + 3y^2x\,\hat{k}\right)$

$$= \frac{\partial}{\partial x}(2x^2 z) + \frac{\partial}{\partial y}(-xy^2 z) + \frac{\partial}{\partial z}(3y^2 x)$$

$$= 4xz - 2xyz + 0$$

At (1, 1, 1), $div\,\vec{f} = 4(1)(1) - 2(1)(1)(1) = 2$

Example 31: Determine the constant a so that the vector $\vec{f} = (x + 3y)\hat{\imath} + (y - 2z)\hat{\jmath} + (x + az)\,\hat{k}$ is solenoidal.

Solution: Given that the vector $\vec{f}$ is solenoidal, so $div\,\vec{f} = 0$

$$\left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot \left((x + 3y)\hat{\imath} + (y - 2z)\hat{\jmath} + (x + az)\,\hat{k}\right) = 0$$

$$\frac{\partial}{\partial x}(x + 3y) + \frac{\partial}{\partial y}(y - 2z) + \frac{\partial}{\partial z}(x + az) = 0$$

$$1 + 1 + a = 0 \quad => \quad a = -2$$

2. Curl: Let $\vec{f}(x, y, z) = f_1(x, y, z)\hat{\imath} + f_2(x, y, z)\hat{\jmath} + f_3(x, y, z)\hat{k}$ be a continuously differentiable vector point function. Curl of $\vec{f}(x, y, z)$ of is a vector which is denoted by $\nabla \times \vec{f}$ and is defined as

$$\nabla \times \vec{f} = \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \times \vec{f} = \hat{\imath} \times \frac{\partial \vec{f}}{\partial x} + \hat{\jmath} \times \frac{\partial \vec{f}}{\partial y} + \hat{k} \times \frac{\partial \vec{f}}{\partial z}$$

Also in component form, curl of $\vec{f}(x, y, z)$ is

$$\nabla \times \vec{f} = \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \times \left(f_1\,\hat{\imath} + f_2\,\hat{\jmath} + f_3\hat{k}\right)$$

$$= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix} = \hat{\imath}\left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right) + \hat{\jmath}\left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right) + \hat{k}\left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)$$

Physical Interpretation of Curl

Consider the motion of a rigid body rotating with angular velocity $\vec{\omega}$ about an axis OA, where O is a fixed point in the body. Let $\vec{r}$ be the position vector of any point P of the body. The point P describing a circle whose center is M and radius is PM = $r \sin\theta$ where θ is the angle between $\vec{\omega}$ and $\vec{r}$, then the velocity of P is $\omega r \sin\theta$. This velocity is normal to the plane POM *i.e.* normal to the plane of $\vec{\omega}$ and $\vec{r}$.

So, if $\vec{v}$ is the linear velocity of P, then $\vec{v} = \omega\, r \sin\theta\, \hat{n} = \vec{\omega} \times \vec{r}$

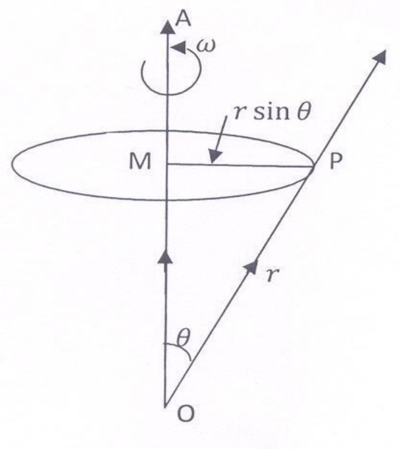

Fig. 7.10

$(\hat{n}\ being\ normal\ to\ \vec{\omega}\ and\ \vec{r})$

Now, if $\vec{\omega} = \omega_1\,\hat{\imath} + \omega_2\,\hat{\jmath} + \omega_3\hat{k}$ and $\vec{r} = x\,\hat{\imath} + y\,\hat{\jmath} + z\,\hat{k}$, then

$$Curl\ \vec{v} = \vec{\omega} \times \vec{r} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \omega_1 & \omega_2 & \omega_3 \\ x & y & z \end{vmatrix} = \hat{\imath}(\omega_2 z - \omega_3 y) + \hat{\jmath}(\omega_3 x - \omega_1 z) + \hat{k}(\omega_1 y - \omega_2 x)$$

And

$$Curl\ \vec{v} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (\omega_2 z - \omega_3 y) & (\omega_3 x - \omega_1 z) & (\omega_1 y - \omega_2 x) \end{vmatrix}$$

$$= \hat{\imath}(\omega_1 + \omega_1) + \hat{\jmath}(\omega_2 + \omega_2) + \hat{k}(\omega_3 + \omega_3)$$
$$= 2\omega_1\,\hat{\imath} + 2\omega_2\,\hat{\jmath} + 2\omega_3\,\hat{k} = 2\,\vec{\omega}$$

Hence $\vec{\omega} = \frac{1}{2} Curl\ \vec{v}$

Thus the angular velocity of rotation at any point is equal to half the curl of the velocity vector.

Observations:

(i) The curl of a vector point function gives the measure of the angular velocity at a point.

(ii) If the curl of a vector point function becomes zero i.e. $\nabla \times \vec{f} = 0$, then $\vec{f}$ is called an irrotational vector.

Example 32: If $\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$ then show that $curl\ \vec{r} = \vec{0}$.

Solution: $curl\ \vec{r} = \nabla \times \vec{r} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = \hat{\imath}\left(\frac{\partial z}{\partial y} - \frac{\partial y}{\partial z}\right) + \hat{\jmath}\left(\frac{\partial x}{\partial z} - \frac{\partial z}{\partial x}\right) + \hat{k}\left(\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y}\right)$

$$= \hat{\imath}(0-0) + \hat{\jmath}(0-0) + \hat{k}(0-0) = \vec{0}.$$

Example 33: Find a so that the vector $\vec{f} = (axy - z^3)\,\hat{\imath} + (a-2)x^2\,\hat{\jmath} + (1-a)xz^2\,\hat{k}$ is irrotational.

Solution: Given that $\vec{f}$ is irrotational, therefore $curl\ \vec{f} = \vec{0}$ … (1)

But $curl\ \vec{f} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (axy - z^3) & (a-2)x^2 & (1-a)xz^2 \end{vmatrix}$

$$= \hat{\imath}(0-0) + \hat{\jmath}\{-3z^2 - (1-a)z^2\} + \hat{k}\{(a-2)2x - ax\}$$

$$= 0\,\hat{\imath} + (-4+a)z^2\,\hat{\jmath} + (-4+a)x\,\hat{k}$$

Using (1), $0\,\hat{\imath} + (-4+a)z^2\,\hat{\jmath} + (-4+a)x\,\hat{k} = 0\,\hat{\imath} + 0\,\hat{\jmath} + 0\,\hat{k}$

Comparing the corresponding components both sides,

$$-4 + a = 0 \quad => \quad a = 4.$$

Example 34: If $\vec{f} = (xy^2)\,\hat{\imath} + 2x^2yz\,\hat{\jmath} - 3yz^2\,\hat{k}$, find the $curl\,\vec{f}$ at the point (1, -1, 1).

Solution: $curl\ \vec{f} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy^2 & 2x^2yz & -3yz^2 \end{vmatrix}$

$$= \hat{\imath}(-3z^2 - 2x^2y) + \hat{\jmath}(0-0) + \hat{k}(4xyz - 2xy)$$

At (1, -1, 1),

$curl\ \vec{f} = \hat{\imath}(-3(1)^2 - 2(1)^2(-1)) + \hat{\jmath}(0-0) + \hat{k}(4(1)(-1)(1) - 2(1)(-1))$

$= -\hat{\imath} - 2\,\hat{k}.$

17.7 DEL APPLIED TO THE PRODUCT OF POINT FUNCTIONS

Let $\phi,\ \psi$ are two scalar point functions and $\vec{f}, \vec{g}$ are two vector point functions, then

1. $\nabla(\phi\,\psi) = \phi\,\nabla\psi + \psi\,\nabla\phi$
2. $\nabla\cdot(\phi\,\vec{f}) = (\nabla\phi)\cdot\vec{f} + \phi\,(\nabla\cdot\vec{f})$
3. $\nabla\times(\phi\,\vec{f}) = (\nabla\phi)\times\vec{f} + \phi\,(\nabla\times\vec{f})$
4. $\nabla(\vec{f}\cdot\vec{g}) = (\vec{f}\cdot\nabla)\vec{g} + (\vec{g}\cdot\nabla)\vec{f} + \vec{f}\times(\nabla\times\vec{g}) + \vec{g}\times(\nabla\times\vec{f})$ [KUK 2007]
5. $\nabla\cdot(\vec{f}\times\vec{g}) = \vec{g}\cdot(\nabla\times\vec{f}) - \vec{f}\cdot(\nabla\times\vec{g})$
6. $\nabla\times(\vec{f}\times\vec{g}) = \vec{f}(\nabla\cdot\vec{g}) - \vec{g}(\nabla\cdot\vec{f}) + (\vec{g}\cdot\nabla)\vec{f} - (\vec{f}\cdot\nabla)\vec{g}$ [KUK 2011]

Proof 1: Consider $\nabla(\phi\,\psi) = \sum\hat{\imath}\,\frac{\partial}{\partial x}(\phi\,\psi) = \sum\hat{\imath}\,\left(\phi\frac{\partial\psi}{\partial x} + \psi\frac{\partial\phi}{\partial x}\right) = \sum\hat{\imath}\,\left(\phi\frac{\partial\psi}{\partial x}\right) + \sum\hat{\imath}\,\left(\psi\frac{\partial\phi}{\partial x}\right)$

$$= \phi\sum\hat{\imath}\,\frac{\partial\psi}{\partial x} + \psi\sum\hat{\imath}\,\frac{\partial\phi}{\partial x} = \phi\,\nabla\psi + \psi\,\nabla\phi$$

Proof 2: Consider $\nabla\cdot(\phi\,\vec{f}) = \sum\hat{\imath}\cdot\frac{\partial}{\partial x}(\phi\,\vec{f}) = \sum\hat{\imath}\cdot\left(\frac{\partial\phi}{\partial x}\vec{f} + \phi\frac{\partial\vec{f}}{\partial x}\right) = \sum\hat{\imath}\cdot\left(\frac{\partial\phi}{\partial x}\vec{f}\right) + \sum\hat{\imath}\cdot\left(\phi\frac{\partial\vec{f}}{\partial x}\right)$

$$= \left(\sum\hat{\imath}\frac{\partial\phi}{\partial x}\right)\cdot\vec{f} + \phi\left(\sum\hat{\imath}\cdot\frac{\partial\vec{f}}{\partial x}\right) = (\nabla\phi)\cdot\vec{f} + \phi\,(\nabla\cdot\vec{f})$$

Proof 3: Consider $\nabla\times(\phi\,\vec{f}) = \sum\hat{\imath}\times\frac{\partial}{\partial x}(\phi\,\vec{f}) = \sum\hat{\imath}\times\left(\frac{\partial\phi}{\partial x}\vec{f} + \phi\frac{\partial\vec{f}}{\partial x}\right) = \sum\hat{\imath}\times\left(\frac{\partial\phi}{\partial x}\vec{f}\right) + \sum\hat{\imath}\times\left(\phi\frac{\partial\vec{f}}{\partial x}\right)$

$$= \left(\sum\hat{\imath}\frac{\partial\phi}{\partial x}\right)\times\vec{f} + \phi\left(\sum\hat{\imath}\times\frac{\partial\vec{f}}{\partial x}\right) = (\nabla\phi)\times\vec{f} + \phi\,(\nabla\times\vec{f})$$

Proof 4: Consider $\nabla(\vec{f}\cdot\vec{g}) = \sum\hat{\imath}\,\frac{\partial}{\partial x}(\vec{f}\cdot\vec{g}) = \sum\hat{\imath}\,\left(\frac{\partial\vec{f}}{\partial x}\cdot\vec{g} + \vec{f}\cdot\frac{\partial\vec{g}}{\partial x}\right) = \left(\sum\hat{\imath}\,\frac{\partial\vec{f}}{\partial x}\right)\cdot\vec{g} + \sum\hat{\imath}\,\left(\vec{f}\cdot\frac{\partial\vec{g}}{\partial x}\right)$

… (1)

But, $\vec{g} \times \left(\hat{\imath} \times \frac{\partial \vec{f}}{\partial x}\right) = \left(\vec{g} \cdot \frac{\partial \vec{f}}{\partial x}\right)\hat{\imath} - (\vec{g} \cdot \hat{\imath})\frac{\partial \vec{f}}{\partial x}$

or $\left(\vec{g} \cdot \frac{\partial \vec{f}}{\partial x}\right)\hat{\imath} = \vec{g} \times \left(\hat{\imath} \times \frac{\partial \vec{f}}{\partial x}\right) + (\vec{g} \cdot \hat{\imath})\frac{\partial \vec{f}}{\partial x}$

So, $\sum\left(\vec{g} \cdot \frac{\partial \vec{f}}{\partial x}\right)\hat{\imath} = \vec{g} \times \sum\left(\hat{\imath} \times \frac{\partial \vec{f}}{\partial x}\right) + \sum(\vec{g} \cdot \hat{\imath})\frac{\partial \vec{f}}{\partial x} = \vec{g} \times \left(\nabla \times \vec{f}\right) + (\vec{g} \cdot \nabla)\vec{f}$ … (2)

Interchanging $\vec{f}$ and $\vec{g}$ in (2)

$$\sum\left(\vec{f} \cdot \frac{\partial \vec{g}}{\partial x}\right)\hat{\imath} = \vec{f} \times \sum\left(\hat{\imath} \times \frac{\partial \vec{g}}{\partial x}\right) + \sum(\vec{f} \cdot \hat{\imath})\frac{\partial \vec{g}}{\partial x} = \vec{f} \times (\nabla \times \vec{g}) + \left(\vec{f} \cdot \nabla\right)\vec{g} \quad \ldots (3)$$

Using (2) and (3) in (1), we get

$$\nabla\left(\vec{f} \cdot \vec{g}\right) = \left(\vec{f} \cdot \nabla\right)\vec{g} + (\vec{g} \cdot \nabla)\vec{f} + \vec{f} \times (\nabla \times \vec{g}) + \vec{g} \times \left(\nabla \times \vec{f}\right)$$

Proof 5: Consider $\nabla \cdot \left(\vec{f} \times \vec{g}\right) = \sum \hat{\imath} \cdot \frac{\partial}{\partial x}\left(\vec{f} \times \vec{g}\right) = \sum \hat{\imath} \cdot \left(\frac{\partial \vec{f}}{\partial x} \times \vec{g} + \vec{f} \times \frac{\partial \vec{g}}{\partial x}\right)$

$$= \sum \hat{\imath} \cdot \left(\frac{\partial \vec{f}}{\partial x} \times \vec{g}\right) + \sum \hat{\imath} \cdot \left(\vec{f} \times \frac{\partial \vec{g}}{\partial x}\right) = \vec{g} \cdot \sum\left(\hat{\imath} \times \frac{\partial \vec{f}}{\partial x}\right) - \vec{f} \cdot \sum\left(\hat{\imath} \times \frac{\partial \vec{g}}{\partial x}\right)$$

$$= \vec{g} \cdot \left(\nabla \times \vec{f}\right) - \vec{f} \cdot (\nabla \times \vec{g})$$

[using $\vec{a} \cdot \left(\vec{b} \times \vec{c}\right) = \vec{c} \cdot \left(\vec{a} \times \vec{b}\right) = -\vec{b} \cdot (\vec{a} \times \vec{c})$]

Proof 6: Consider $\nabla \times \left(\vec{f} \times \vec{g}\right) = \sum \hat{\imath} \times \frac{\partial}{\partial x}\left(\vec{f} \times \vec{g}\right) = \sum \hat{\imath} \times \left(\frac{\partial \vec{f}}{\partial x} \times \vec{g} + \vec{f} \times \frac{\partial \vec{g}}{\partial x}\right)$

$$= \sum \hat{\imath} \times \left(\frac{\partial \vec{f}}{\partial x} \times \vec{g}\right) + \sum \hat{\imath} \times \left(\vec{f} \times \frac{\partial \vec{g}}{\partial x}\right)$$

$$= \sum\left[(\hat{\imath} \cdot \vec{g})\frac{\partial \vec{f}}{\partial x} - \left(\hat{\imath} \cdot \frac{\partial \vec{f}}{\partial x}\right)\vec{g}\right] + \sum\left[\left(\hat{\imath} \cdot \frac{\partial \vec{g}}{\partial x}\right)\vec{f} - \left(\hat{\imath} \cdot \vec{f}\right)\frac{\partial \vec{g}}{\partial x}\right]$$

[using $\vec{a} \times \left(\vec{b} \times \vec{c}\right) = (\vec{a} \cdot \vec{c})\vec{b} - \left(\vec{a} \cdot \vec{b}\right)\vec{c}$]

$$= \sum(\vec{g} \cdot \hat{\imath})\frac{\partial \vec{f}}{\partial x} - \vec{g}\left(\sum \hat{\imath} \cdot \frac{\partial \vec{f}}{\partial x}\right) + \vec{f}\left(\sum \hat{\imath} \cdot \frac{\partial \vec{g}}{\partial x}\right) - \sum\left(\vec{f} \cdot \hat{\imath}\right)\frac{\partial \vec{g}}{\partial x}$$

$$= (\vec{g} \cdot \nabla)\vec{f} - \vec{g}\left(\nabla \cdot \vec{f}\right) + \vec{f}(\nabla \cdot \vec{g}) - \left(\vec{f} \cdot \nabla\right)\vec{g}$$

$$= \vec{f}(\nabla \cdot \vec{g}) - \vec{g}\left(\nabla \cdot \vec{f}\right) + (\vec{g} \cdot \nabla)\vec{f} - \left(\vec{f} \cdot \nabla\right)\vec{g}$$

17.8 DEL APPLIED TWICE TO POINT FUNCTIONS

Let $\emptyset$ be a scalar point function and $\vec{f}$ be a vector point function, then $\nabla\emptyset$ and $\nabla \times \vec{f}$ being the vector point functions, we can find their divergence and curl; whereas $\nabla \cdot \vec{f}$ being the scalar point function, we can find its gradient only. Thus we have following formulae:

1. $div\ \ grad\ \emptyset\ = \nabla \cdot (\nabla\emptyset\ ) = \nabla^2\emptyset\ = \frac{\partial^2\emptyset}{\partial x^2} + \frac{\partial^2\emptyset}{\partial y^2} + \frac{\partial^2\emptyset}{\partial z^2}$
2. $curl\ \ grad\ \emptyset = \nabla \times \nabla\emptyset = \vec{0}$
3. $div\ curl\ \vec{f} = \nabla \cdot \left(\nabla \times \vec{f}\right) = 0$
4. $curl\ curl\ \vec{f} = \nabla \times \left(\nabla \times \vec{f}\right) = \nabla\left(\nabla \cdot \vec{f}\right) - \nabla^2\vec{f} = grad\ div\ \vec{f} - \nabla^2\vec{f}$ [KUK 2006]
5. $grad\ div\ \vec{f} = \nabla\left(\nabla \cdot \vec{f}\right) = curl\ curl\ \vec{f} + \nabla^2\vec{f} = \nabla \times \left(\nabla \times \vec{f}\right) + \nabla^2\vec{f}$

Proof 1: $\nabla^2\emptyset = \nabla\cdot(\nabla\emptyset\ ) = \nabla\cdot\left(\hat{\imath}\frac{\partial\emptyset}{\partial x}+\hat{\jmath}\frac{\partial\emptyset}{\partial y}+\hat{k}\frac{\partial\emptyset}{\partial z}\right)$

$$=\frac{\partial}{\partial x}\left(\frac{\partial\emptyset}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{\partial\emptyset}{\partial y}\right)+\frac{\partial}{\partial z}\left(\frac{\partial\emptyset}{\partial z}\right)=\frac{\partial^2\emptyset}{\partial x^2}+\frac{\partial^2\emptyset}{\partial y^2}+\frac{\partial^2\emptyset}{\partial z^2}$$

Here ∇^2 is called the Laplacian Operator and $\nabla^2\emptyset = 0$ is called the Laplace's Equation.

Proof 2: $curl\ \ grad\ \emptyset = \nabla\times\nabla\emptyset = \nabla\times\left(\hat{\imath}\frac{\partial\emptyset}{\partial x}+\hat{\jmath}\frac{\partial\emptyset}{\partial y}+\hat{k}\frac{\partial\emptyset}{\partial z}\right)$

$$=\begin{vmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ \frac{\partial\emptyset}{\partial x} & \frac{\partial\emptyset}{\partial y} & \frac{\partial\emptyset}{\partial z}\end{vmatrix}=\sum\hat{\imath}\left(\frac{\partial^2\emptyset}{\partial y\partial z}-\frac{\partial^2\emptyset}{\partial z\partial y}\right)=\vec{0}$$

Proof 3: $div\ curl\ \vec{f} = \nabla\cdot(\nabla\times\vec{f}) = \left(\sum\hat{\imath}\frac{\partial}{\partial x}\right)\cdot\left(\hat{\imath}\times\frac{\partial\vec{f}}{\partial x}+\hat{\jmath}\times\frac{\partial\vec{f}}{\partial y}+\hat{k}\times\frac{\partial\vec{f}}{\partial z}\right)$

$$=\sum\hat{\imath}\cdot\left(\hat{\imath}\times\frac{\partial^2\vec{f}}{\partial x^2}+\hat{\jmath}\times\frac{\partial^2\vec{f}}{\partial x\partial y}+\hat{k}\times\frac{\partial^2\vec{f}}{\partial x\partial z}\right)$$

$$=\sum\left(\hat{\imath}\times\hat{\imath}\cdot\frac{\partial^2\vec{f}}{\partial x^2}+\hat{\imath}\times\hat{\jmath}\cdot\frac{\partial^2\vec{f}}{\partial x\partial y}+\hat{\imath}\times\hat{k}\cdot\frac{\partial^2\vec{f}}{\partial x\partial z}\right)$$

$$=\sum\left(\hat{k}\cdot\frac{\partial^2\vec{f}}{\partial x\partial y}-\hat{\jmath}\cdot\frac{\partial^2\vec{f}}{\partial x\partial z}\right)=0$$

Proof 4: $curl\ curl\ \vec{f} = \nabla\times(\nabla\times\vec{f}) = \left(\sum\hat{\imath}\frac{\partial}{\partial x}\right)\times\left(\hat{\imath}\times\frac{\partial\vec{f}}{\partial x}+\hat{\jmath}\times\frac{\partial\vec{f}}{\partial y}+\hat{k}\times\frac{\partial\vec{f}}{\partial z}\right)$

$$=\sum\hat{\imath}\times\left(\hat{\imath}\times\frac{\partial^2\vec{f}}{\partial x^2}+\hat{\jmath}\times\frac{\partial^2\vec{f}}{\partial x\partial y}+\hat{k}\times\frac{\partial^2\vec{f}}{\partial x\partial z}\right)$$

$$=\sum\hat{\imath}\times\left(\hat{\imath}\times\frac{\partial^2\vec{f}}{\partial x^2}\right)+\sum\hat{\imath}\times\left(\hat{\jmath}\times\frac{\partial^2\vec{f}}{\partial x\partial y}\right)+\sum\hat{\imath}\times\left(\hat{k}\times\frac{\partial^2\vec{f}}{\partial x\partial z}\right)$$

$$=\sum\left\{\left(\hat{\imath}\cdot\frac{\partial^2\vec{f}}{\partial x^2}\right)\hat{\imath}-(\hat{\imath}\cdot\hat{\imath})\frac{\partial^2\vec{f}}{\partial x^2}\right\}+\sum\left\{\left(\hat{\imath}\cdot\frac{\partial^2\vec{f}}{\partial x\partial y}\right)\hat{\jmath}-(\hat{\imath}\cdot\hat{\jmath})\frac{\partial^2\vec{f}}{\partial x\partial y}\right\}+\sum\left\{\left(\hat{\imath}\cdot\frac{\partial^2\vec{f}}{\partial x\partial z}\right)\hat{k}-(\hat{\imath}\cdot\hat{k})\frac{\partial^2\vec{f}}{\partial x\partial z}\right\}$$

$$=\sum\left\{\left(\hat{\imath}\cdot\frac{\partial^2\vec{f}}{\partial x^2}\right)\hat{\imath}+\left(\hat{\imath}\cdot\frac{\partial^2\vec{f}}{\partial x\partial y}\right)\hat{\jmath}+\left(\hat{\imath}\cdot\frac{\partial^2\vec{f}}{\partial x\partial z}\right)\hat{k}\right\}-\sum\frac{\partial^2\vec{f}}{\partial x^2}$$

$$=\sum\hat{\imath}\ \frac{\partial}{\partial x}\left(\hat{\imath}\cdot\frac{\partial\vec{f}}{\partial x}+\hat{\jmath}\cdot\frac{\partial\vec{f}}{\partial y}+\hat{k}\cdot\frac{\partial\vec{f}}{\partial z}\right)-\sum\frac{\partial^2\vec{f}}{\partial x^2}=\nabla(\nabla\cdot\vec{f})-\nabla^2\vec{f}$$

Proof 5: To get this formula we are to re-arrange the terms in last proof.

Example 35: Show that $\nabla^2(r^n) = n(n+1)r^{n-2}$.

Solution: We know that $r^2 = x^2+y^2+z^2$

On differentiation w. r. t. x, $\quad\frac{\partial r}{\partial x}=\frac{x}{r}$

Similarly $\quad\frac{\partial r}{\partial y}=\frac{y}{r}$ and $\frac{\partial r}{\partial z}=\frac{z}{r}$

Now $\quad\nabla^2(r^n)=\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)(r^n)=\frac{\partial}{\partial x}\left(\frac{\partial r^n}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{\partial r^n}{\partial y}\right)+\frac{\partial}{\partial z}\left(\frac{\partial r^n}{\partial z}\right)\quad\ldots(1)$

And $\quad\frac{\partial}{\partial x}\left(\frac{\partial r^n}{\partial x}\right)=\frac{\partial}{\partial x}\left(n\,r^{n-1}\frac{\partial r}{\partial x}\right)=\frac{\partial}{\partial x}\left(n\,r^{n-1}\frac{x}{r}\right)=n\ \frac{\partial}{\partial x}(r^{n-2}x)$

$$= n\left(r^{n-2} + x\,(n-2)r^{n-3}\frac{\partial r}{\partial x}\right) = n(r^{n-2} + x^2(n-2)r^{n-4})$$

Similarly $\quad \dfrac{\partial}{\partial y}\left(\dfrac{\partial r^n}{\partial y}\right) = n(r^{n-2} + y^2(n-2)r^{n-4})$

$$\frac{\partial}{\partial z}\left(\frac{\partial r^n}{\partial z}\right) = n(r^{n-2} + z^2(n-2)r^{n-4})$$

Using all these values in (1),

$$\nabla^2(r^n) = n(r^{n-2} + x^2(n-2)r^{n-4})) + n(r^{n-2} + (n-2)r^{n-4}) + n(r^{n-2} + z^2(n-2)r^{n-4})$$

$$= 3n\,r^{n-2} + n(n-2)r^{n-4}(x^2 + y^2 + z^2)$$

$$= 3n\,r^{n-2} + n(n-2)r^{n-2} = n(n+1)r^{n-2}$$

Example 36: Show that *(*i*) $\nabla^2 f(r) = f''(r) + \frac{2}{r}f'(r)$ (*ii*) $\nabla\cdot(\phi\,\nabla\psi - \psi\,\nabla\phi) = \phi\,\nabla^2\psi - \psi\,\nabla^2\phi$

***[KUK 2008]**

Solution: (*i*) $\nabla^2 f(r) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) f(r) = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x} f(r)\right) + \frac{\partial}{\partial y}\left(\frac{\partial}{\partial y} f(r)\right) + \frac{\partial}{\partial z}\left(\frac{\partial}{\partial z} f(r)\right)$ …(1)

And $\quad \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x} f(r)\right) = \frac{\partial}{\partial x}\left(f'(r)\ \frac{\partial r}{\partial x}\right) = \frac{\partial}{\partial x}\left(f'(r)\ \frac{x}{r}\right) = f'(r)\frac{\partial}{\partial x}\left(\frac{x}{r}\right) + \frac{x}{r}\ f''(r)\frac{\partial r}{\partial x}$

$$= f'(r)\left(\frac{1}{r} - \frac{x}{r^2}\frac{\partial r}{\partial x}\right) + \frac{x}{r}\ f''(r)\frac{\partial r}{\partial x}$$

$$= \frac{f'(r)}{r} - x^2\frac{f'(r)}{r^3} + x^2\frac{f''(r)}{r^2}$$

Similarly $\quad \dfrac{\partial}{\partial y}\left(\dfrac{\partial}{\partial y} f(r)\right) = \dfrac{f'(r)}{r} - y^2\dfrac{f'(r)}{r^3} + y^2\dfrac{f''(r)}{r^2}$

$$\frac{\partial}{\partial z}\left(\frac{\partial}{\partial z} f(r)\right) = \frac{f'(r)}{r} - z^2\frac{f'(r)}{r^3} + z^2\frac{f''(r)}{r^2}$$

Using all these values in (1)

$$\nabla^2 f(r) = \left(\frac{f'(r)}{r} - x^2\frac{f'(r)}{r^3} + x^2\frac{f''(r)}{r^2}\right) + \left(\frac{f'(r)}{r} - y^2\frac{f'(r)}{r^3} + y^2\frac{f''(r)}{r^2}\right)$$

$$+ \left(\frac{f'(r)}{r} - z^2\frac{f'(r)}{r^3} + z^2\frac{f''(r)}{r^2}\right)$$

$$= \frac{3f'(r)}{r} - (x^2 + y^2 + z^2)\frac{f'(r)}{r^3} + \frac{f''(r)}{r^2}(x^2 + y^2 + z^2)$$

$$= \frac{3f'(r)}{r} - \frac{f'(r)}{r} + f''(r)$$

$$= \frac{2f'(r)}{r} + f''(r)$$

Hence $\quad \nabla^2 f(r) = f''(r) + \frac{2}{r}f'(r)$.

(*ii*) Consider $\quad \nabla\cdot(\phi\,\nabla\psi - \psi\,\nabla\phi) = \nabla\cdot(\phi\,\nabla\psi) - \nabla\cdot(\psi\,\nabla\phi)$

$$= \{\nabla\phi\cdot\nabla\psi + \phi(\nabla\cdot\nabla\psi)\} - \{\nabla\psi\cdot\nabla\phi + \psi(\nabla\cdot\nabla\phi)\}$$

$$= \nabla\phi\cdot\nabla\psi + \phi\,\nabla^2\psi - \nabla\phi\cdot\nabla\psi - \psi\,\nabla^2\phi$$

$$= \phi\,\nabla^2\psi - \psi\,\nabla^2\phi$$

Hence $\quad \nabla\cdot(\phi\,\nabla\psi-\psi\,\nabla\phi)=\phi\,\nabla^2\psi-\psi\,\nabla^2\phi.$

Example 37: Find the value of n for which the vector $r^n\,\vec{r}$ is solenoidal, where $\vec{r}=x\hat{\imath}+y\hat{\jmath}+z\hat{k}$.

Solution: Consider $div(r^n\,\vec{r})=\nabla\cdot(r^n\,\vec{r})=(\nabla r^n)\cdot\vec{r}+r^n(\nabla\cdot\vec{r})$ … (1)

But $\quad \nabla r^n=\left(\hat{\imath}\frac{\partial}{\partial x}+\hat{\jmath}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)r^n=n\,r^{n-1}\left(\hat{\imath}\frac{\partial r}{\partial x}+\hat{\jmath}\frac{\partial r}{\partial y}+\hat{k}\frac{\partial r}{\partial z}\right)$

$=n\,r^{n-1}\left(\frac{x}{r}\hat{\imath}+\frac{y}{r}\hat{\jmath}+\frac{z}{r}\hat{k}\right)=n\,r^{n-2}\,\vec{r}$ … (2)

And $\quad \nabla\cdot\vec{r}=3$ … (3)

Therefore, $\quad div(r^n\,\vec{r})=(n\,r^{n-2}\,\vec{r})\cdot\vec{r}+(3)r^n=n\,r^{n-2}\,(\vec{r}\cdot\vec{r})+3r^n$

$=n\,r^{n-2}\,(r^2)+3r^n=(n+3)r^n$ … (4)

As given the vector $r^n\,\vec{r}$ is solenoidal, so $div(r^n\,\vec{r})=0$

So using (4), $\quad (n+3)r^n=0 \quad$ implies that $\quad n=-3 \quad$ (since $r\neq 0$)

Example 38: If $\vec{a}$ and $\vec{b}$ are irrotational, prove that $\vec{a}\times\vec{b}$ is solenoidal.

Solution: Given $\vec{a}$ and $\vec{b}$ are irrotational, so $\nabla\times\vec{a}=\vec{0}=\nabla\times\vec{b}$ … (1)

Consider $Div\left(\vec{a}\times\vec{b}\right)=\vec{b}\cdot(\nabla\times\vec{a})-\vec{a}\cdot\left(\nabla\times\vec{b}\right)=\vec{b}\cdot\vec{0}-\vec{a}\cdot\vec{0}=0 \quad$ [using (1)]

Thus $\vec{a}\times\vec{b}$ is solenoidal.

Example 39: Show that the vector field $\vec{f}=\left(z^2+2x+3y\right)\hat{\imath}+(3x+2y+z)\hat{\jmath}+(y+2zx)\hat{k}$ is irrotational but not solenoidal. Also obtain a scalar function $\emptyset$ such that $\nabla\emptyset=\vec{f}$.

Solution: Consider $Curl\,\vec{f}=\nabla\times\vec{f}=\begin{vmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ z^2+2x+3y & 3x+2y+z & y+2zx\end{vmatrix}$

$=\hat{\imath}(1-1)-\hat{\jmath}(2z-2z)+\hat{k}(3-3)=\vec{0}$

So $\vec{f}$ is irrotational vector field.

Also consider $\quad div\,\vec{f}=\nabla\cdot\vec{f}=\left(\hat{\imath}\frac{\partial}{\partial x}+\hat{\jmath}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\cdot\vec{f}=2+2+2x=2(x+2)\neq 0$

So $\vec{f}$ is not solenoidal vector field.

Now $d\emptyset=\frac{\partial\emptyset}{\partial x}\,dx+\frac{\partial\emptyset}{\partial y}\,dy+\frac{\partial\emptyset}{\partial z}\,dz=\left(\hat{\imath}\frac{\partial\emptyset}{\partial x}+\hat{\jmath}\frac{\partial\emptyset}{\partial y}+\hat{k}\frac{\partial\emptyset}{\partial z}\right)\cdot\left(dx\,\hat{\imath}+dy\,\hat{\jmath}+dz\,\hat{k}\right)$

$=grad\,\emptyset\,\cdot d\vec{r}=\vec{f}\cdot d\vec{r}$ (as $\vec{f}=grad\,\emptyset$)

$=\left((z^2+2x+3y)\hat{\imath}+(3x+2y+z)\hat{\jmath}+(y+2zx)\hat{k}\right)\cdot\left(dx\,\hat{\imath}+dy\,\hat{\jmath}+dz\,\hat{k}\right)$

$=(z^2+2x+3y)dx+(3x+2y+z)dy+(y+2zx)dz$

$=(z^2dx+2zx\,dz)+(3y\,dx+3x\,dy)+(z\,dy+y\,dz)+2x\,dx+2y\,dy$

$=d(xz^2)+3d(xy)+d(yz)+d(x^2)+d(y^2)$

Integrating both sides, we get

$$\emptyset = xz^2 + 3xy + yz + x^2 + y^2 + c$$

Example 40: If $\vec{V}_1$ and $\vec{V}_2$ be the vectors joining the fixed points (x_1, y_1, z_1) and (x_2, y_2, z_2) respectively to a variable point (x, y, z), prove that ***[KUK 2010]**

(*i*) $div\left(\vec{V}_1 \times \vec{V}_2\right) = 0$ *(*ii*) $curl\left(\vec{V}_1 \times \vec{V}_2\right) = 2(\vec{V}_1 - \vec{V}_2)$ (*iii*) $grad\left(\vec{V}_1 \cdot \vec{V}_2\right) = \vec{V}_1 + \vec{V}_2$.

Solution: Here, $\vec{V}_1 = (x - x_1)\hat{\imath} + (y - y_1)\hat{\jmath} + (z - z_1)\hat{k}$ and $\vec{V}_2 = (x - x_2)\hat{\imath} + (y - y_2)\hat{\jmath} + (z - z_2)\hat{k}$

$$(i)\ \vec{V}_1 \times \vec{V}_2 = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ x - x_1 & y - y_1 & z - z_1 \\ x - x_2 & y - y_2 & z - z_2 \end{vmatrix}$$

$$= [(y - y_1)(z - z_2) - (y - y_2)(z - z_1)]\hat{\imath} + [(x - x_2)(z - z_1) - (x - x_1)(z - z_2)]\hat{\jmath}$$
$$+ [(x - x_1)(y - y_2) - (x - x_2)(y - y_1)]\hat{k}$$

So $div\left(\vec{V}_1 \times \vec{V}_2\right) = \frac{\partial}{\partial x}[(y - y_1)(z - z_2) - (y - y_2)(z - z_1)]$
$$+\frac{\partial}{\partial y}[(x - x_2)(z - z_1) - (x - x_1)(z - z_2)]$$
$$+\frac{\partial}{\partial z}[(x - x_1)(y - y_2) - (x - x_2)(y - y_1)] = 0$$

(ii) $curl\left(\vec{V}_1 \times \vec{V}_2\right)$=

$$\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (y - y_1)(z - z_2) - (y - y_2)(z - z_1) & (x - x_2)(z - z_1) - (x - x_1)(z - z_2) & (x - x_1)(y - y_2) - (x - x_2)(y - y_1) \end{vmatrix}$$

$$= [(x - x_1) - (x - x_2) - (x - x_2) + (x - x_1)]\hat{\imath}$$
$$+[(y - y_1) - (y - y_2) - (y - y_2) + (y - y_1)]\hat{\jmath}$$
$$+[(z - z_1) - (z - z_2) - (z - z_2) + (z - z_1)]\hat{k}$$
$$= 2\left[(x - x_1)\hat{\imath} + (y - y_1)\hat{\jmath} + (z - z_1)\hat{k}\right] - 2[(x - x_2)\hat{\imath} + (y - y_2)\hat{\jmath} + (z - z_2)\hat{k}]$$

$$= 2(\vec{V}_1 - \vec{V}_2)$$

(iii) $\vec{V}_1 \cdot \vec{V}_2 = (x - x_1)(x - x_2) + (y - y_1)(y - y_2) + (z - z_1)(z - z_2)$

So $grad\left(\vec{V}_1 \cdot \vec{V}_2\right) = \hat{\imath}\frac{\partial}{\partial x}[(x - x_1)(x - x_2)] + \hat{\jmath}\frac{\partial}{\partial y}[(y - y_1)(y - y_2)] + \hat{k}\frac{\partial}{\partial z}[(z - z_1)(z - z_2)]$

$$= \hat{\imath}[(x - x_1) + (x - x_2)] + \hat{\jmath}[(y - y_1) + (y - y_2)] + \hat{k}[(z - z_1) + (z - z_2)]$$

$$= \left[(x - x_1)\hat{\imath} + (y - y_1)\hat{\jmath} + (z - z_1)\hat{k}\right] + \left[(x - x_2)\hat{\imath} + (y - y_2)\hat{\jmath} + (z - z_2)\hat{k}\right]$$

$$= \vec{V}_1 + \vec{V}_2$$

Example 41: If $\vec{a}$ is a constant vector and $\vec{r} = x\,\hat{\imath} + y\,\hat{\jmath} + z\,\hat{k}$, prove that **[KUK 2009]**

(*i*) $grad\,(\vec{a}\cdot\vec{r}) = \vec{a}$ (*ii*) $div\,(\vec{a}\times\vec{r}) = 0$ (*iii*)) $curl\,(\vec{a}\times\vec{r}) = 2\vec{a}$ (*iv*) $curl\,[(\vec{a}\cdot\vec{r})\vec{r}] = \vec{a}\times\vec{r}$

Solution: Let $\vec{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\,\hat{k}$ is the constant vector.

(i) $\vec{a}\cdot\vec{r} = \left(a_1\hat{\imath} + a_2\,\hat{\jmath} + a_3\,\hat{k}\right)\cdot\left(x\,\hat{\imath} + y\,\hat{\jmath} + z\,\hat{k}\right) = (a_1x + a_2y + a_3z)$

So $grad\,(\vec{a}\cdot\vec{r}) = \hat{\imath}\frac{\partial}{\partial x}[a_1x + a_2y + a_3z] + \hat{\jmath}\frac{\partial}{\partial y}[a_1x + a_2y + a_3z] + \hat{k}\frac{\partial}{\partial z}[a_1x + a_2y + a_3z]$

$= a_1\hat{\imath} + a_2\,\hat{\jmath} + a_3\,\hat{k} = \vec{a}$

(ii) $\vec{a}\times\vec{r} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a_1 & a_2 & a_3 \\ x & y & z \end{vmatrix} = \hat{\imath}(a_2z - a_3y) + \hat{\jmath}(a_3x - a_1z) + \hat{k}(a_1y - a_2x)$

$$div\,(\vec{a}\times\vec{r}) = \frac{\partial}{\partial x}(a_2z - a_3y) + \frac{\partial}{\partial y}(a_3x - a_1z) + \frac{\partial}{\partial z}(a_1y - a_2x) = 0$$

(iii) $curl\,(\vec{a}\times\vec{r}) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (a_2z - a_3y) & (a_3x - a_1z) & (a_1y - a_2x) \end{vmatrix}$

$= \hat{\imath}(a_1 + a_1) + \hat{\jmath}(a_2 + a_2) + \hat{k}(a_3 + a_3) = 2\left(a_1\hat{\imath} + a_2\hat{\jmath} + a_3\,\hat{k}\right) = 2\vec{a}$

(iv) $(\vec{a}\cdot\vec{r})\vec{r} = (a_1x + a_2y + a_3z)x\hat{\imath} + (a_1x + a_2y + a_3z)y\hat{\jmath} + (a_1x + a_2y + a_3z)z\,\hat{k}$

$Curl\,[(\vec{a}\cdot\vec{r})\vec{r}] = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (a_1x + a_2y + a_3z)x & (a_1x + a_2y + a_3z)y & (a_1x + a_2y + a_3z)z \end{vmatrix}$

$= \hat{\imath}(a_2z - a_3y) + \hat{\jmath}(a_3x - a_1z) + \hat{k}(a_1y - a_2x) = \vec{a}\times\vec{r}$ [using part (ii)]

Example 42: Find $\vec{f}\times(\nabla\times\vec{g})$ at the point (1, -1, 2),

if $\vec{f} = xz^2\,\hat{\imath} + 2y\,\hat{\jmath} - 3xz\,\hat{k}$, $\vec{g} = 3xz\,\hat{\imath} + 2yz\,\hat{\jmath} - z^2\,\hat{k}$.

Solution: $\nabla\times\vec{g} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3xz & 2yz & -z^2 \end{vmatrix} = \hat{\imath}(0 - 2y) + \hat{\jmath}(3x - 0) + \hat{k}(0 - 0) = -2y\,\hat{\imath} + 3x\,\hat{\jmath} + 0\,\hat{k}$

Now $\vec{f}\times(\nabla\times\vec{g}) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ xz^2 & 2y & -3xz \\ -2y & 3x & 0 \end{vmatrix} = (9x^2z)\hat{\imath} + (6xyz)\,\hat{\jmath} + (3x^2z^2 + 4y^2)\,\hat{k}$

At (1, -1, 2), $\vec{f}\times(\nabla\times\vec{g}) = (9(1)^2(2))\,\hat{\imath} + (6(1)(-1)(2))\,\hat{\jmath} + (3(1)^2(2)^2 + 4(-1)^2)\,\hat{k}$

$= 18\,\hat{\imath} - 12\,\hat{\jmath} + 16\,\hat{k}$

Example 43: If $\vec{f} = yz^2\,\hat{\imath} - 3xz^2\,\hat{\jmath} + 2xyz\,\hat{k}$, $\vec{g} = 3x\,\hat{\imath} + 4z\,\hat{\jmath} - xy\,\hat{k}$ and $\emptyset = xyz$; find

(i) $\vec{f}\times\nabla\emptyset$ (ii) $\left(\vec{f}\times\nabla\right)\emptyset$ (iii) $\left(\nabla\times\vec{f}\right)\times\vec{g}$ (iv) $\vec{g}\cdot\left(\nabla\times\vec{f}\right)$

Solution: $\nabla\emptyset = \hat{\imath}\frac{\partial\emptyset}{\partial x} + \hat{\jmath}\frac{\partial\emptyset}{\partial y} + \hat{k}\frac{\partial\emptyset}{\partial z} = yz\,\hat{\imath} + zx\,\hat{\jmath} + xy\,\hat{k}$

(i) $\vec{f} \times \nabla\emptyset = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ yz^2 & -3xz^2 & 2xyz \\ yz & xz & xy \end{vmatrix} = -5x^2yz^2\,\hat{\imath} + xy^2z^2\,\hat{\jmath} + 4xyz^3\,\hat{k}$

(ii) $\vec{f} \times \nabla = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ yz^2 & -3xz^2 & 2xyz \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{vmatrix}$

$$= \hat{\imath}\left\{-3xz^2\frac{\partial}{\partial z} - 2xyz\frac{\partial}{\partial y}\right\} + \hat{\jmath}\left\{2xyz\frac{\partial}{\partial x} - yz^2\frac{\partial}{\partial z}\right\} + \hat{k}\left\{yz^2\frac{\partial}{\partial y} + 3xz^2\frac{\partial}{\partial x}\right\}$$

Now $\left(\vec{f} \times \nabla\right)\emptyset = \hat{\imath}\left\{-3xz^2\frac{\partial\emptyset}{\partial z} - 2xyz\frac{\partial\emptyset}{\partial y}\right\} + \hat{\jmath}\left\{2xyz\frac{\partial\emptyset}{\partial x} - yz^2\frac{\partial\emptyset}{\partial z}\right\} + \hat{k}\left\{yz^2\frac{\partial\emptyset}{\partial y} + 3xz^2\frac{\partial\emptyset}{\partial x}\right\}$

$$= \hat{\imath}\{-3xz^2(yx) - 2xyz(xz)\} + \hat{\jmath}\{2xyz(yz) - yz^2(xy)\} + \hat{k}\{yz^2(xz) + 3xz^2(yz)\}$$

$$= -5x^2yz^2\,\hat{\imath} + xy^2z^2\hat{\jmath} + 4xyz^3\,\hat{k}$$

(iii) $\nabla \times \vec{f} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz^2 & -3xz^2 & 2xyz \end{vmatrix} = (2xz + 6xz)\hat{\imath} + (2yz - 2yz)\hat{\jmath} + (-3z^2 - z^2)\,\hat{k}$

$$= 8xz\,\hat{\imath} + 0\,\hat{\jmath} - 4z^2\,\hat{k}$$

Now $\left(\nabla \times \vec{f}\right) \times \vec{g} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 8xz & 0 & -4z^2 \\ 3x & 4z & -xy \end{vmatrix} = (0 + 16z^3)\hat{\imath} + (-12xz^2 + 8x^2yz)\hat{\jmath} + (32xz^2 - 0)\,\hat{k}$

$$= 16z^3\,\hat{\imath} + (-12xz^2 + 8x^2yz)\hat{\jmath} + 32xz^2\,\hat{k}$$

$$\vec{g} \cdot \left(\nabla \times \vec{f}\right) = \left(3x\,\hat{\imath} + 4z\,\hat{\jmath} - xy\,\hat{k}\right) \cdot \left(8xz\,\hat{\imath} + 0\,\hat{\jmath} - 4z^2\,\hat{k}\right) = 24x^2z + 4xyz^2$$

Example 44: Find the directional derivative of $\nabla \cdot (\nabla\emptyset)$ at the point (1, -2, 1) in the direction of the normal to the surface $xy^2z = 3x + z^2$ where $\emptyset = 2x^3y^2z^4$. [Raipur 2005]

Solution: Given $\emptyset = 2x^3y^2z^4$

So $\nabla\emptyset = \hat{\imath}\,\frac{\partial\emptyset}{\partial x} + \hat{\jmath}\,\frac{\partial\emptyset}{\partial y} + \hat{k}\,\frac{\partial\emptyset}{\partial z} = \hat{\imath}\,(6x^2y^2z^4) + \hat{\jmath}\,(4x^3yz^4) + \hat{k}(8x^3y^2z^3)$

And $f = \nabla \cdot (\nabla\emptyset) = \frac{\partial}{\partial x}(6x^2y^2z^4) + \frac{\partial}{\partial y}(4x^3yz^4) + \frac{\partial}{\partial z}(8x^3y^2z^3)$

$$= 12xy^2z^4 + 4x^3z^4 + 24\,x^3y^2z^2$$

Consider $grad(f) = \nabla f = \hat{\imath}\,\frac{\partial f}{\partial x} + \hat{\jmath}\,\frac{\partial f}{\partial y} + \hat{k}\,\frac{\partial f}{\partial z}$

$$= \hat{\imath}\,(12y^2z^4 + 12x^2z^4 + 72x^3y^2z^2) + \hat{\jmath}\,(24xyz^4 + 48x^3yz^2)$$
$$+ \hat{k}(48xy^2z^3 + 16x^3z^3 + 48x^3y^2z)$$

At point (1, -2, 1)

$grad(f) = \hat{\imath}\,(48 + 12 + 288) + \hat{\jmath}\,(-48 - 96) + \hat{k}(192 + 16 + 192)$

$$= 348\,\hat{\imath} - 144\,\hat{\jmath} + 400\,\hat{k}$$

Now consider the surface $g = xy^2z - 3x - z^2 = 0$

So, $\nabla g = \hat{\imath}\ \frac{\partial g}{\partial x} + \hat{\jmath}\ \frac{\partial g}{\partial y} + \hat{k}\ \frac{\partial g}{\partial z} = \hat{\imath}\,(y^2 z - 3) + \hat{\jmath}\,(2xyz) + \hat{k}(xy^2 - 2z)$

The vector normal to the surface (1) at point (1, -2, 1) is given by

$$\vec{n} = \hat{\imath} - 4\hat{\jmath} + 2\hat{k}$$

And $\quad |\vec{n}| = \sqrt{1+16+4} = \sqrt{21}$

So the direction derivative of $f = \nabla \cdot (\nabla \emptyset)$ at the point (1, -2, 1) in the direction of the normal to the surface (1) is $grad(f)\,.\frac{\vec{n}}{|\vec{n}|}$

$$= \left(348\,\hat{\imath} - 144\,\hat{\jmath}\ + 400\,\hat{k}\right).\frac{1}{\sqrt{21}}\left(\hat{\imath} - 4\hat{\jmath} + 2\hat{k}\right)$$

$$= \frac{1}{\sqrt{21}}(348 + 576 + 800)$$

$$= \mathbf{1724/\sqrt{21}}$$

Example 45: If *r* is the distance of a point (*x*, *y*, *z*) from the origin, prove that

$curl\left(\hat{k} \times grad\ \frac{1}{r}\right) + grad\ \left(\hat{k} \cdot grad\ \frac{1}{r}\right) = \vec{0}$ where $\hat{k}$ is the unit vector in the direction of OZ.

Solution: Here $\vec{r} = x\,\hat{\imath} + y\,\hat{\jmath} + z\,\hat{k}$ so that $r = \sqrt{x^2+y^2+z^2}$

Now $\quad grad\ \frac{1}{r} = \left(\hat{\imath}\ \frac{\partial}{\partial x} + \hat{\jmath}\ \frac{\partial}{\partial y} + \hat{k}\ \frac{\partial}{\partial z}\right)\frac{1}{r} = \left(\hat{\imath}\ \frac{\partial}{\partial x} + \hat{\jmath}\ \frac{\partial}{\partial y} + \hat{k}\ \frac{\partial}{\partial z}\right)(x^2+y^2+z^2)^{-\frac{1}{2}}$

$$= -\frac{1}{2}(x^2+y^2+z^2)^{-\frac{3}{2}}(2x\,\hat{\imath} + 2y\,\hat{\jmath} + 2z\,\hat{k}\ ) = -\frac{1}{r^3}\vec{r}$$

$$grad\ \left(\hat{k} \cdot grad\ \frac{1}{r}\right) = grad\ \left(-\frac{1}{r^3}z\right) = -grad\ \left(\frac{z}{(x^2+y^2+z^2)^{3/2}}\right)$$

$$= \frac{3zx}{(x^2+y^2+z^2)^{5/2}}\hat{\imath} + \frac{3zy}{(x^2+y^2+z^2)^{5/2}}\hat{\jmath} + \frac{x^2+y^2-2z^2}{(x^2+y^2+z^2)^{5/2}}\hat{k} \qquad \ldots (1)$$

And $\quad curl\left(\hat{k} \times grad\ \frac{1}{r}\right) = curl\left(\hat{k} \times -\frac{1}{r^3}\vec{r}\right) = curl\left(\frac{1}{r^3}(y\,\hat{\imath} - x\,\hat{\jmath})\right)$

$$= -\frac{3zx}{(x^2+y^2+z^2)^{5/2}}\hat{\imath} - \frac{3zy}{(x^2+y^2+z^2)^{5/2}}\hat{\jmath} - \frac{x^2+y^2-2z^2}{(x^2+y^2+z^2)^{5/2}}\hat{k} \qquad \ldots (2)$$

Adding (1) and (2), $\quad curl\ \left(\hat{k} \times grad\ \frac{1}{r}\right) + grad\ \left(\hat{k} \cdot grad\ \frac{1}{r}\right) = \vec{0}$.

Example 46: Prove that $\vec{a} \cdot \nabla\left(\vec{b} \cdot \nabla\frac{1}{r}\right) = \frac{3(\vec{a}\cdot\vec{r})(\vec{b}\cdot\vec{r})}{r^5} - \frac{\vec{a}\cdot\vec{b}}{r^3}$, where $\vec{a}$ and $\vec{b}$ are constant vectors.

Solution: From example 45, $\nabla\frac{1}{r} = -\frac{1}{r^3}\vec{r} = -(x^2+y^2+z^2)^{-\frac{3}{2}}(x\,\hat{\imath} + y\,\hat{\jmath} + z\,\hat{k}\ )$

Let the constant vectors are $\vec{a} = a_1\,\hat{\imath} + a_2\,\hat{\jmath} + a_3\,\hat{k}$ and $\vec{b} = b_1\,\hat{\imath} + b_2\,\hat{\jmath} + b_3\,\hat{k}$

So $\nabla\left(\vec{b} \cdot \nabla\frac{1}{r}\right) = \nabla\left(-(x^2+y^2+z^2)^{-\frac{3}{2}}(b_1x + b_2y\ + b_3z\ )\right)$

$$= -(x^2+y^2+z^2)^{-\frac{3}{2}}\nabla(b_1x + b_2y\ + b_3z\ ) - (b_1x + b_2y\ + b_3z\ )\nabla(x^2+y^2+z^2)^{-\frac{3}{2}}$$

$$= -(x^2+y^2+z^2)^{-\frac{3}{2}}\left(b_1\,\hat{\imath} + b_2\,\hat{\jmath} + b_3\,\hat{k}\right) + 3(b_1x + b_2y\ + b_3z\ )(x^2+y^2+z^2)^{-\frac{5}{2}}(x\,\hat{\imath} + y\,\hat{\jmath} + z\,\hat{k}\ )$$

$$= -\frac{\vec{b}}{r^3} + \frac{3\left(\vec{b}\cdot\vec{r}\right)\vec{r}}{r^5}$$

Now $\vec{a}\cdot\nabla\left(\vec{b}\cdot\nabla\frac{1}{r}\right)=\vec{a}\cdot\left(-\frac{\vec{b}}{r^3}+\frac{3(\vec{b}\cdot\vec{r})\vec{r}}{r^5}\right)=\frac{3(\vec{a}\cdot\vec{r})(\vec{b}\cdot\vec{r})}{r^5}-\frac{\vec{a}\cdot\vec{b}}{r^3}$

Hence Proved.

ASSIGNMENT 4

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1. If $\vec{f}=(x+y+1)\hat{\imath}+\hat{\jmath}-(x+y)\hat{k}$, show that $\vec{f}\cdot curl\,\vec{f}=0$.
2. Evaluate (a) $div\left[3x^2\hat{\imath}+5xy^2\hat{\jmath}+xyz^3\hat{k}\right]$ at the point (1, 2, 3).

   (b) $curl\,[e^{xyz}(\hat{\imath}+\hat{\jmath}+\hat{k})]$.
3. Find the value of 'a' if the vector $(ax^2y+yz)\hat{\imath}+(xy^2-xz^2)\hat{\jmath}+(2xyz-2x^2y^2)\hat{k}$ has zero divergence. Find the curl of above vector which has zero divergence .
4. If $\vec{v}=(\vec{r})/\sqrt{x^2+y^2+z^2}$, show that $\nabla\cdot\vec{v}=2/\sqrt{x^2+y^2+z^2}$ and $\nabla\times\vec{v}=\vec{0}$.
5. If $u=x^2+y^2+z^2$ and $\vec{v}=x\hat{\imath}+y\hat{\jmath}+z\hat{k}$, show that $div\,(u\,\vec{v})=5u$.
6. Show that each of following vectors are solenoidal:

   (a) $(x+3y)\hat{\imath}+(y-3z)\hat{\jmath}+(x-2z)\hat{k}$ (b) $3y^4z^2\hat{\imath}+4x^3z^2\hat{\jmath}+3x^2y^2\hat{k}$ (c) $\nabla u\times\nabla v$
7. If $\vec{r}=x\hat{\imath}+y\hat{\jmath}+z\hat{k}$ and $r=|\vec{r}|\neq 0$, show that

   (a) $\nabla(1/r^2)=-2\vec{r}/r^4$; $\nabla\cdot(\vec{r}/r^2)=1/r^2$ (b) $\nabla\cdot(r^n\vec{r})=(n+3)r^n$; $\nabla\times(r^n\vec{r})=\vec{0}$.

   (c) $\nabla\left[\nabla\cdot\frac{\vec{r}}{r}\right]=-\frac{2}{r^3}\vec{r}$
8. Prove that (a) $\nabla\vec{a}^2=2(\vec{a}\cdot\nabla)\vec{a}+2\vec{a}\times(\nabla\times\vec{a})$, where $\vec{a}$ is the constant vector.

   (b) $\nabla\times(\vec{r}\times\vec{u})=\vec{r}(\nabla\cdot\vec{u})-2\vec{u}-(\vec{r}\cdot\nabla)\vec{u}$.
9. (a) If $\emptyset=(x^2+y^2+z^2)^{-n}$, find the $div\ grad\ \emptyset$ and determine n if $div\ grad\ \emptyset=0$.

   (b) Show that $grad\ r^n=n(n+1)r^{n-2}$, where $r^2=x^2+y^2+z^2$.
10. In electromagnetic theory, we have $\nabla\cdot\vec{D}=\rho,\ \nabla\cdot\vec{H}=0,\ \nabla\times\vec{D}=-\frac{1}{c}\frac{\partial\vec{H}}{\partial t},\nabla\times\vec{H}=\frac{1}{c}\left(\rho\vec{V}+\frac{\partial\vec{D}}{\partial t}\right)$.

    Prove that $\nabla^2\vec{H}-\frac{1}{c^2}\frac{\partial^2\vec{H}}{\partial t^2}=-\frac{1}{c}\nabla\times\left(\rho\vec{V}\right)$ and $\nabla^2\vec{D}-\frac{1}{c^2}\frac{\partial^2\vec{D}}{\partial t^2}=\nabla\rho+\frac{1}{c^2}\frac{\partial}{\partial t}\left(\rho\vec{V}\right)$.
11. If $u=x^2yz,\ v=xy-3z^2$, find (i) $\nabla(\nabla u\cdot\nabla v)$ (ii) $\nabla\cdot(\nabla u\times\nabla v)$.
12. For a solenoidal vector $\vec{f}$, show that $curl\ curl\ curl\ curl\ \vec{f}=\nabla^4\vec{f}$.
13. Calculate (i) $curl(grad\ f)$, given $f(x,y,z)=x^2+y^2-z$ [BPTU 2006]

    (ii) $curl(curl\ \vec{a})$, given $\vec{a}=x^2y\,\hat{\imath}+y^2z\,\hat{\jmath}+z^2y\,\hat{k}$
14. Show that each of the following vectors are solenoidal:

    (i) $(-x^2+yz)\,\hat{\imath}+(4y-z^2x)\,\hat{\jmath}+(2xz-4z)\,\hat{k}$

    (ii) $3y^4z^2\,\hat{\imath}+4x^3z^2\,\hat{\jmath}+3x^2y^2\,\hat{k}$

    (iii) $\nabla\phi\times\nabla\psi$

INTEGRAL VECTOR CALCULUS

17.9 INTEGRATION OF VECTORS

If two vector functions $\vec{f}(t)$ and $\vec{g}(t)$ be such that $\frac{d}{dt}(\vec{g}(t)) = \vec{f}(t)$, then $\vec{g}(t)$ is called an integral of $\vec{f}(t)$ with respect to a scalar variable *t* and we can write $\int \vec{f}(t)dt = \vec{g}(t)$.

Indefinite Integral

If $\vec{c}$ be an arbitrary constant vector and $\vec{f}(t) = \frac{d}{dt}(\vec{g}(t)) = \frac{d}{dt}(\vec{g}(t) + \vec{c})$, then $\int \vec{f}(t)dt = \vec{g}(t) + \vec{c}$. This is called the indefinite integral of $\vec{f}(t)$.

Definite Integral

If $\frac{d}{dt}(\vec{g}(t)) = \vec{f}(t)$ for all values of t in the interval (a, b), then the definite integral of $\vec{f}(t)$ between a and b is defined and denoted by $\int_a^b \vec{f}(t)dt = [\vec{g}(t)]_a^b = \vec{g}(b) - \vec{g}(a)$.

Example 47: If $\boldsymbol{\frac{d^2\vec{f}}{dt^2} = 6t\,\hat{\imath} - 12t^2\,\hat{\jmath} + 4\cos t\;\hat{k}}$, find $\boldsymbol{\vec{f}}$, given that $\boldsymbol{\frac{d\vec{f}}{dt} = -\,\hat{\imath} - 3\,\hat{k}}$ and $\boldsymbol{\vec{f} = 2\,\hat{\imath} + \hat{\jmath}}$ at $\boldsymbol{t = 0}$.

Solution: Given that $\frac{d^2\vec{f}}{dt^2} = 6t\,\hat{\imath} - 12t^2\,\hat{\jmath} + 4\cos t\;\hat{k}$ … (1)

Integrating (1) with respect to t,

$$\int \frac{d^2\vec{f}}{dt^2}dt = \int\left(6t\,\hat{\imath} - 12t^2\,\hat{\jmath} + 4\cos t\;\hat{k}\right)dt$$

Implying $\frac{d\vec{f}}{dt} = 3t^2\,\hat{\imath} - 4t^3\,\hat{\jmath} + 4\sin t\;\hat{k} + \vec{c}_1$ … (2)

Now, integrating (2) with respect to t,

$$\int \frac{d\vec{f}}{dt}dt = \int\left(3t^2\,\hat{\imath} - 4t^3\,\hat{\jmath} + 4\sin t\;\hat{k} + \vec{c}_1\right)dt$$

Implying $\vec{f} = t^3\,\hat{\imath} - t^4\,\hat{\jmath} - 4\cos t\;\hat{k} + \vec{c}_1 t + \vec{c}_2$ … (3)

Also we are given that at $t = 0$, $\frac{d\vec{f}}{dt} = -\,\hat{\imath} - 3\,\hat{k}$ … (4)

$\vec{f} = 2\,\hat{\imath} + \hat{\jmath}$ … (5)

Using (2) and (4), $\vec{c}_1 = -\,\hat{\imath} - 3\,\hat{k}$

Using (3) and (5), $-4\;\hat{k} + \vec{c}_2 = 2\,\hat{\imath} + \hat{\jmath}$ => $\vec{c}_2 = 2\,\hat{\imath} + \hat{\jmath} + 4\;\hat{k}$

Putting the values of the constant vectors $\vec{c}_1$ & $\vec{c}_2$ in (3), we get

$$\vec{f} = t^3\,\hat{\imath} - t^4\,\hat{\jmath} - 4\cos t\;\hat{k} + \left(-\,\hat{\imath} - 3\,\hat{k}\right)t + \left(2\,\hat{\imath} + \hat{\jmath} + 4\;\hat{k}\right)$$

$$= (t^3 - t + 2)\,\hat{\imath} + (1 - t^4)\,\hat{\jmath} - (4\cos t + 3\,t + 4)\,\hat{k}$$

ASSIGNMENT 5

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1. For given $\vec{f}(t) = (5t^2 - 3t)\,\hat{\imath} + 6t^3\,\hat{\jmath} - 7t\,\hat{k}$, evaluate $\int_2^4 \vec{f}(t)dt$.
2. Given $\vec{r}(t) = 3t^2\,\hat{\imath} + t\,\hat{\jmath} - t^3\,\hat{k}$, evaluate $\int_0^1\left(\vec{r} \times \frac{d^2\vec{r}}{dt^2}\right)dt$.

3. If $\vec{r}(t) = \begin{cases} 2\,\hat{\imath} - \hat{\jmath} + 2\,\hat{k}, & when\ t = 1 \\ 3\,\hat{\imath} - 2\hat{\jmath} + 4\,\hat{k}, & when\ t = 2 \end{cases}$, show that $\int_1^2 \left(\vec{r} \cdot \frac{d\vec{r}}{dt}\right) dt = 10.$
4. The acceleration of a particle at any time $t \geq 0$ is given by $12\cos 2t\ \hat{\imath} - 8\sin 2t\ \hat{\jmath} + 16t\ \hat{k}$, the displacement and velocity are initially zero. Find the velocity and displacement at any time.

17.10 LINE INTEGRAL

Let $\vec{r}(u) = x(u)\hat{i} + y(u)\hat{j} + z(u)\hat{k}$ defines a curve C joining points P_1 and P_2 where $\vec{r}(u)$ is the position vector of (x, y, z) and the value of u at P_1 and P_2 is u_1 and u_2, respectively.

Now if $A(x, y, z) = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$ be vector function of defined position and continuous along C, then the integral of the tangential component of A along C from P_1 to P_2 written as $\int_{P_1}^{P_2} \vec{A}.d\vec{r}$ is known as the line integral. Also in terms of Cartesian components, we have

$$\int_{P_1}^{P_2} \vec{A}.d\vec{r} = \int_{P_1}^{P_2} (A_1\hat{i} + A_2\hat{j} + A_3\hat{k}).(dx\hat{i} + dy\hat{j} + dz\hat{k})$$

$$= \int_{P_1}^{P_2} (A_1\ dx + A_2\ dy + A_3\ dz) = \int_C (A_1\ dx + A_2\ dy + A_3\ dz)$$

If $\vec{A}$ (vector function of the position) represents the force $\vec{F}$ on a particle moving along C, then the line integral represents the work done by the force $\vec{F}$. If C is a simple closed curve, then the integral around C is generally written as

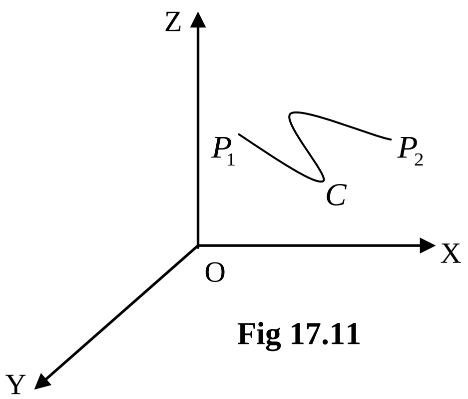

Fig 17.11

$$\oint \vec{A}.d\vec{r} = \oint (A_1\ dx + A_2\ dy + A_3\ dz)$$

In Fluid Mechanics and Aerodynamics, the above integral is called circulation of $\vec{A}$ about C, where $\vec{A}$ represents the velocity of the fluid.

Let $A = grad\,\phi$, then we have

$$\int_P^Q \vec{A}.d\vec{r} = \int_P^Q (\nabla\phi).d\vec{r} = \int_P^Q \left(\hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}\right).(\hat{i}\ dx + \hat{j}\ dy + \hat{k}\ dz)$$

$$= \int_P^Q \left(\frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy + \frac{\partial \phi}{\partial z} dz \right)$$

$$= \int_P^Q d\phi = [\phi]_P^Q = \phi_Q - \phi_P \qquad \ldots (1)$$

We see that the integral $\int_P^Q \vec{A}.d\vec{r}$ depends on the value of ϕ at the end P and Q and not on the particular path. In case ϕ is single valued and the integral is taken round a closed curve, the terminal points P and Q coincide and $\phi_B = \phi_P$.

[Because function ϕ is uniform]

The integration along a closed curve is denoted by the sign of circle in the mid of the integral sign *i.e.* for a uniform function, we have

$$\oint_C (\nabla \phi).d\vec{r} = 0 \qquad \ldots (2)$$

The converse of the above result is also true i.e. if there exists a vector $\vec{A}$ and its integral round every closed curve in the region under consideration vanishes, then there exist a point function ϕ such that $\vec{A} = grad\phi$.

To prove this consider any closed curve *ABCD* such that the integral round it is zero, so integral along *ABC* must be equal to that along *ADC.* Similarly, the integral along *ABC* must be equal to that along any curve joining *A* to *C, i.e.* independent of the path from *A* to *C* with *A* be a fixed point and *C* a variable point. Then due to the fact that line integral is independent of the path chosen, the value of the line integral from *A* to *C* must be a scalar point function, say ϕ *i.e.* $\int_A^C \vec{A}.d\vec{r} = \phi$

Now if $d\phi$ is the increment in ϕ due to a small displacement $\overrightarrow{dr}$ of $\vec{r}$, then we have $d\phi = \vec{A}.d\vec{r}$

But we already know that $d\phi = \nabla\phi.d\vec{r}$, so $\vec{A}.d\vec{r} = (\nabla\phi).d\vec{r}$,

$\Rightarrow \quad (\vec{A} - \nabla\phi).d\vec{r} = 0,$ which is true for all $d\vec{r}$ and hence $\vec{A} = \nabla\phi$.

The vector $\vec{A}$ is called a **potential vector** (or gradient vector), and in cartesian component; the condition that $\vec{A}.d\vec{r} = A_1\,dx + A_2\,dy + A_3\,dz$ be a perfect differential can be thrown easily into the form

$$\frac{\partial A_2}{\partial z} - \frac{\partial A_3}{\partial y} = 0, \quad \frac{\partial A_3}{\partial x} - \frac{\partial A_1}{\partial z} = 0, \quad \frac{\partial A_1}{\partial y} - \frac{\partial A_2}{\partial x} = 0 \qquad \ldots (3)$$

Circulation: If $\vec{A}$ represents the velocity of a fluid particle, then the line integral $\int_C \vec{A}.d\vec{r}$ is called the circulation of $\vec{A}$ along *C.*

The vector point function $\vec{A}$, is said to be irrotational in a region, if its circulation along every closed curve in the region is zero *i.e.* $\int_C \vec{A}.d\vec{r} = 0$

Theorem: *The necessary and sufficient condition for a vector point function $\vec{A}$ to be irrotaional in a simply connected region is the curl $\vec{A} = 0$ at every point of the region.*

Work: If $\vec{A}$ represents the force acting on a particle moving along an arc PQ then the work done during the small displacement $d\vec{r}$ is equal to $\vec{A}.d\vec{r}$. Therefore, the total work done by $\vec{A}$ during the displacement from P to Q is given by the line integral $\int_P^Q \vec{A}.d\vec{r}$.

Example 48: Evaluate the line integral $\int[(x^2 + xy)dx + (x^2 + y^2)dy]$, where C is the square formed by the lines $y = \pm 1$ and $x = \pm 1$.

Solution: Curve C is square in the xy plane where $z = 0$

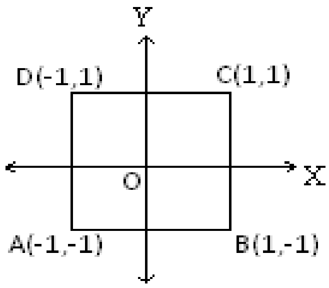

Fig. 7.12

$\therefore \quad \vec{r} = x\hat{\imath} + y\hat{\jmath}$ in xy plane

$$d\vec{r} = dx\hat{\imath} + dy\hat{\jmath} \qquad \ldots (1)$$

Now $\int_C \vec{F}.d\vec{r} = \int[(x^2 + xy)dx + (x^2 + y^2)dy]$

Path of the integration is shown in figure 17.12, it consists of lines AB, BC, CD and DA. As curve C is a square, then

On AB, $y = -1 \quad \Rightarrow \quad dy = 0$ and x varies from -1 to 1

On BC, $x = 1 \quad \Rightarrow \quad dx = 0$ and y varies from -1 to 1

On CD, $y = 1 \quad \Rightarrow \quad dy = 0$ and x varies from 1 to -1

On DA, $x = -1 \quad \Rightarrow \quad dx = 0$ and y varies from 1 to -1

$$\int_C \vec{F}.d\vec{r} = \int_{AB}(x^2 - x)dx + \int_{BC}(1 + y^2)dy + \int_{CD}(x^2 + x)dx + \int_{DA}(1 + y^2)dy$$

$$= \int_{-1}^{1}(x^2 - x)dx + \int_{-1}^{1}(1 + y^2)dy + \int_{1}^{-1}(x^2 + x)dx + \int_{1}^{-1}(1 + y^2)dy$$

$$= \left[\frac{x^3}{3} - \frac{x^2}{2}\right]_{-1}^{1} + \int_{-1}^{1}(1 + y^2)dy + \left[\frac{x^3}{3} + \frac{x^2}{2}\right]_{1}^{-1} - \int_{-1}^{1}(1 + y^2)dy$$

$$= \left(\frac{1}{3} - \frac{1}{2} + \frac{1}{3} + \frac{1}{2}\right) + \left(-\frac{1}{3} + \frac{1}{2} - \frac{1}{3} - \frac{1}{2}\right) = 0$$

Example 49: If $\vec{F} = 3xy\,\hat{\imath} - y^2\hat{\jmath}$, evaluate $\int_C \vec{F}.d\vec{r}$ where C is the arc of the parabola $y = 2x^2$ from $(0, 0)$ to $(1, 2)$.

Solution: Because the integration is performed in the xy-plane ($z = 0$), we take

$\vec{r} = x\hat{\imath} + y\hat{\jmath}$ so that $d\vec{r} = dx\,\hat{\imath} + dy\,\hat{\jmath}$

$\therefore \quad \vec{F}.d\vec{r} = (3xy\,\hat{\imath} - y^2\hat{\jmath}).(dx\,\hat{\imath} + dy\,\hat{\jmath}) = 3xy\,dx - y^2dy$

On the curve *C:* $y = 2x^2$ from $(0, 0)$ to $(1, 2)$

$$\vec{F}.d\vec{r} = 3x(2x^2)dx - (2x^2)^2 4x\,dx = (6x^3 - 16x^5)dx$$

Also x varies from 0 to 1.

$$\int_C \vec{F}.d\vec{r} = \int_0^1 (6x^3 - 16x^5)dx = \left[\frac{6x^4}{4} - \frac{16x^6}{6}\right]_0^1 = \frac{3}{2} - \frac{8}{3} = -\frac{7}{6}$$

Note: If the curve is traversed in the opposite direction, that is from $(1,2)$ to $(0,0)$, the value of the integral would be $\frac{7}{6}$.

Example 50: A vector field is given by $\vec{F} = (\sin y)\hat{\imath} + x(1 + \cos y)\hat{\jmath}$. Evaluate the line integral over the circular path given by $x^2 + y^2 = a^2,\ z = 0$. [PTU 2003]

Solution: The parametric equations of the circular path are $x = a\cos t\,, y = a\sin t\,, z = 0$ where t varies from 0 to 2π. Since the particle moves in the xy-plane ($z = 0$), we can take $\vec{r} = x\,\hat{\imath} + y\,\hat{\jmath}$.

$$\oint_C \vec{F}.d\vec{r} = \oint_C [\sin y\,\hat{\imath} + x(1 + \cos y)\hat{\jmath}]\,.(dx\,\hat{\imath} + dy\,\hat{\jmath})$$

$$= \oint_C [\sin y\,dx + x(1 + \cos y)dy]$$

$$= \oint_C [(\sin y\,dx + x\cos y\,dy) + x\,dy] = \oint_C d(x\sin y) + \oint_C x\,dy$$

$$= \int_0^{2\pi} d[\text{a}\cos t\sin(\text{a}\sin t)]dt + \int_0^{2\pi} \text{a}\cos t\,.\,\text{a}\cos t\,dt$$

$$= [\text{a}\cos t\sin(\text{a}\sin t)]_0^{2\pi} + a^2\int_0^{2\pi}\cos^2 t\,dt$$

$$= \frac{a^2}{2}\int_0^{2\pi}(1 + \cos 2t)dt = \frac{a^2}{2}\left[t + \frac{\sin 2t}{2}\right]_0^{2\pi} = \frac{a^2}{2}(2\pi) = \pi a^2$$

Example 51: Compute the line integral $\int_C (y^2dx - x^2dy)$ about the triangle whose vertices are $(1,\ 0)$, $(0, 1)$ and $(-1,\ 0)$. [NIT Uttrakhand 2011]

Solution: Here the closed curve C is a triangle ABC.

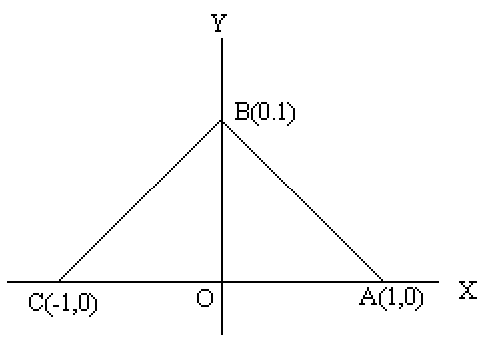

Fig. 7.13

On AB: Equation of line AB is

$$y - 0 = \frac{1-0}{0-1}(x - 1) \quad \Rightarrow \quad y = 1 - x$$

$\therefore \quad dy = -dx$ and x varies from 1 to 0.

On BC: Equation of line BC is

$$y - 1 = \frac{0-1}{-1-0}(x - 0) \quad \Rightarrow \quad y = 1 + x$$

$\therefore \quad dy = dx$ and x varies from 0 to -1.

On CA: $y = 0$. Therefore, $dy = 0$ and x varies from -1 to 1.

$$\int_C (y^2dx - x^2dy) = \int_{AB}(y^2dx - x^2dy) + \int_{BC}(y^2dx - x^2dy) + \int_{CA}(y^2dx - x^2dy)$$

$$= \int_1^0[(1-x)^2dx - x^2(-dx)] + \int_0^{-1}[(1+x)^2dx - x^2dx] + \int_{-1}^1 0\,dx$$

$$= \int_1^0 (2x^2 - 2x + 1)dx + \int_0^{-1} (2x + 1)dx + 0$$

$$= \left[\frac{2x^3}{3} - \frac{2x^2}{2} + x\right]_1^0 + \left[\frac{2x^2}{2} + x\right]_0^{-1} = \left(-\frac{2}{3} + 1 - 1\right) + (1 - 1) = -\frac{2}{3}$$

Example 52: If $\vec{F} = (3x^2 + 6y)\hat{\imath} - 14yz\hat{\jmath} + 20xz^2\hat{k}$, evaluate $\int_C \vec{F}.d\vec{r}$ where

(i) C is the line joining the point $(0,0,0)$ to $(1,1,1)$

(ii) C is given by $x = t, y = t^2, z = t^3$ from the point $(0,0,0)$ to $(1,1,1)$.

Solution: (i) Equation of line joining (0,0,0) to (1,1,1) is

$$\frac{x-0}{1-0} = \frac{y-0}{1-0} = \frac{z-0}{1-0} = t \text{ (say)}$$

$\therefore$ Parametric equations of the line C are $x = t, y = t, z = t; 0 \le t \le 1$

$$\therefore \quad \vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k} = t\hat{\imath} + t\hat{\jmath} + t\hat{k}$$

$$\Rightarrow \quad \frac{d\vec{r}}{dt} = \hat{\imath} + \hat{\jmath} + \hat{k}$$

Now $\vec{F} = (3x^2 + 6y)\hat{\imath} - 14yz\hat{\jmath} + 20xz^2\hat{k} = (3t^2 + 6t)\hat{\imath} - 14t^2\hat{\jmath} + 20t^3\hat{k}$

$$\int_C \vec{F}.d\vec{r} = \int_C \left(\vec{F}.\frac{d\vec{r}}{dt}\right) dt$$

$$= \int_C \left[(3t^2 + 6t)\hat{\imath} - 14t^2\hat{\jmath} + 20t^3\hat{k}\right].\left(\hat{\imath} + \hat{\jmath} + \hat{k}\right)dt$$

$$= \int_0^1 (3t^2 + 6t - 14t^2 + 20t^3)\, dt$$

$$= \left[\frac{3t^3}{3} + \frac{6t^2}{2} - \frac{14t^3}{3} + \frac{20t^4}{4}\right]_0^1 = \left(1 + 3 - \frac{14}{3} + 5\right) = \frac{13}{3}$$

(ii) Here the curve C is given by $x = t, y = t^2, z = t^3$ from the point (0,0,0) to (1,1,1)

$$\therefore \quad \vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k} = t\hat{\imath} + t^2\hat{\jmath} + t^3\hat{k}$$

$$\Rightarrow \quad \frac{d\vec{r}}{dt} = \hat{\imath} + 2t\,\hat{\jmath} + 3t^2\,\hat{k}$$

Now $\vec{F} = (3x^2 + 6y)\hat{\imath} - 14yz\hat{\jmath} + 20xz^2\hat{k} = 9t^2\hat{\imath} - 14t^5\hat{\jmath} + 20t^7\hat{k}$

$$\int_C \vec{F}.d\vec{r} = \int_C \left(\vec{F}.\frac{d\vec{r}}{dt}\right) dt$$

$$= \int_C \left[9t^2\hat{\imath} - 14t^5\hat{\jmath} + 20t^7\hat{k}\right].\left(\hat{\imath} + 2t\,\hat{\jmath} + 3t^2\,\hat{k}\right)dt$$

$$= \int_0^1 (9t^2 - 28t^6 + 60t^9)\, dt$$

$$= \left[\frac{9t^3}{3} - \frac{28t^7}{7} + \frac{60t^{10}}{10}\right]_0^1 = (3 - 4 + 6) = 5$$

Example 53: Find the circulation of $\vec{F}$ around the curve C where $\vec{F} = y\hat{\imath} + z\hat{\jmath} + x\hat{k}$ and C is the circle $x^2 + y^2 = 1,\ z = 0$.

Solution: Circulation of $\vec{F}$ along the curve C is $\oint_C \vec{F}.d\vec{r}$

Equation of circle is $x^2 + y^2 = 1,\ z = 0$

Its parameteric equations are $x = \cos\theta\,,\ y = \sin\theta\,,\ z = 0$

Now $\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k} = \cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath} + 0\hat{k}$ so that

$$d\vec{r} = \left(-\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath} + 0\hat{k}\right)d\theta$$

Also $\vec{F} = y\hat{\imath} + z\hat{\jmath} + x\hat{k} = \sin\theta\,\hat{\imath} + 0\hat{\jmath} + \cos\theta\,\hat{k}$

$$\therefore \quad \oint_C \vec{F}.d\vec{r} = \oint_C \left(\sin\theta\,\hat{\imath} + 0\hat{\jmath} + \cos\theta\,\hat{k}\right).\left(-\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath} + 0\hat{k}\right)d\theta$$

$$= \int_0^{2\pi} -\sin^2\theta\ d\theta \qquad \text{(since along the circle, } \theta \text{ varies from 0 to } 2\pi)$$

$$= -\int_0^{2\pi} \frac{1-\cos 2\theta}{2}\,d\theta = -\left[\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right]_0^{2\pi}$$

$$= -\left[\left(\frac{2\pi}{2} - 0\right) - (0 - 0)\right] = -\pi$$

Example 54: Find the work done in moving a particle once round a circle C in the xy plane, if the circle has its centre at the origin and radius 2 units and the force field is given as

$$\boldsymbol{\vec{F} = (2x - y + 2z)\hat{\imath} + (x + y - z)\hat{\jmath} + (3x - 2y - 5z)\hat{k}.}$$

Solution: Equation of a circle having centre $(0,0)$ with radius 2 in xy plane is $x^2 + y^2 = 4$. Parametric equations of this circle are $x = 2\cos t\,,\ y = 2\sin t\,,\ z = 0$.

Since integration is to be performed around a circle in xy plane,

$$\therefore \quad \vec{r} = x\hat{\imath} + y\hat{\jmath} = 2\cos t\,\hat{\imath} + 2\sin t\,\hat{\jmath} \qquad \Rightarrow \qquad \frac{d\vec{r}}{dt} = -2\sin t\,\hat{\imath} + 2\cos t\,\hat{\jmath}$$

Work done, $\int_C \left(\vec{F}.\frac{d\vec{r}}{dt}\right) dt$

$$= \int\left\{(2x - y + 2z)\hat{\imath} + (x + y - z)\hat{\jmath} + (3x - 2y - 5z)\hat{k}\right\}.\{-2\sin t\,\hat{\imath} + 2\cos t\,\hat{\jmath}\}\ dt$$

$$= \int\left\{(4\cos t - 2\sin t)\hat{\imath} + (2\cos t + 2\sin t)\hat{\jmath} + (6\cos t - 4\sin t)\hat{k}\right\}.\{-2\sin t\,\hat{\imath} + 2\cos t\,\hat{\jmath}\}\ dt$$

In moving round the circle, t varies from 0 to 2π

$$\therefore \quad \text{Work done} = \int_0^{2\pi}\{(4\cos t - 2\sin t)(-2\sin t) + (2\cos t + 2\sin t)(2\cos t)\}\,dt$$

$$= \int_0^{2\pi}[-8\cos t\sin t + 4\sin^2 t + 4\cos^2 t + 4\sin t\cos t]\,dt$$

$$= \int_0^{2\pi}[4 - 4\sin t\cos t]\,dt = \left[4t - 4\,\frac{\sin^2 t}{2}\right]_0^{2\pi}$$

$$= [(8\pi - 2\sin^2 2\pi) - (0 - 0)] = 8\pi$$

ASSIGNMENT 6

1. Using the line integral, compute the work done by the force $\vec{F} = (2y+3)\hat{\imath} + (xz)\hat{\jmath} + (yz - x)\hat{k}$, when it moves a particle from $(0,0,0)$ to $(2,1,1)$ along the curve $x = 2t^2, y = t, z = t^3$.
2. Find the work done in moving a particle in the force field $\vec{F} = 3x^2\hat{\imath} + (2xz - y)\hat{\jmath} + z\hat{k}$, along
   (a) the straight line from $(0,0,0)$ to $(2,1,3)$.
   (b) the curve defined by $x^2 = 4y, 3x^3 = 8z$ from $x = 0$ to $x = 2$.
3. If C is a simple closed curve in the xy plane not enclosing the origin, show that $\int_C \vec{F}.d\vec{r} = 0$, where $\vec{F} = \frac{y\hat{\imath} - x\hat{\jmath}}{x^2+y^2}$.
4. If $\vec{f} = (5xy - 6x^2)\,\hat{\imath} + (2y - 4x)\,\hat{\jmath}$, evaluate $\int_C \vec{f}.d\vec{r}$ along the curve C in xy-plane, $y = x^3$ from the point $(1,\ 1)$ to $(2,\ 8)$.
5. Evaluate $\int_C (xy + z^2)ds$ where C is the arc of the helix $x = \cos t\,, y = \sin t\,, z = t$ which joins the points $(1,\ 0,\ 0)\ and\ (-1,\ 0,\ \pi)$.
6. If $\vec{F} = 2y\hat{\imath} - z\hat{\jmath} + x\hat{k}$, evaluate $\int_C \vec{F} \times d\vec{r}$ along the curve $x = \cos t, y = \sin t, z = 2\cos t$ from $t = 0$ to $t = \frac{\pi}{2}$.

17.11 SURFACE INTEGRALS AND FLUX

An integral which is to be evaluated over a surface is called a surface integral. Suppose S is a surface of finite area. Divide the area S into n sub-areas $\delta S_1, \delta S_2, \dots\dots, \delta S_n$. In each area δS_i, choose an arbitrary point $P_i(x_i, y_i, z_i)$. Let φ define a scalar point function over the area S.

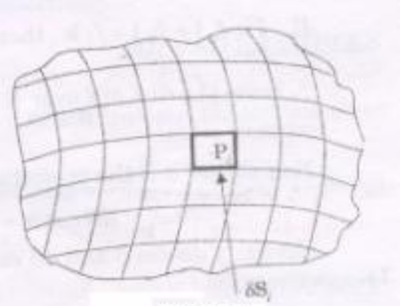

Fig. 7.14

Now from the sum $\sum_{i=1}^{n} \varphi(P_i)\ \delta S_i$, where $\varphi(P_i) = \varphi(x_i, y_i, z_i)$

Now let us take the limit of the sum as $n \to \infty$, each sub-area δS_i reduces to a point and the limit if it exists is called the surface integral of φ over S and is denoted by $\iint_\varphi\ dS$.

Note: If S is piecewise smooth then the function $\varphi(x,y,z)$ is continuous over S and then the limit exists and is independent of sub-divisions and choice of the point P_i.

Flux: Suppose S is a piecewise smooth surface so that the vector function $\vec{F}$ defined over S is continuous over S. Let P be any point of the surface S and suppose $\hat{n}$ is a unit vector at P in the direction of outward drawn normal to the surface S at P. Then the component of $\vec{F}$ along $\hat{n}$ is $\vec{F}.\hat{n}$ and the integral of $\vec{F}.\hat{n}$ over S is called the surface integral of $\vec{F}$ over S and is denoted by $\iint_S \vec{F}.\hat{n}\,dS$. It is also called flux of $\vec{F}$ over S.

Different Forms of Surface Integral

(i) Flux of $\vec{F}$ over $S = \iint_S \vec{F}.\hat{n}\,dS$ … (1)

Now let $\overrightarrow{dS}$ denote a vector (called vector area) whose magnitude is that of differential of surface area *i.e.*, dS and whose direction is that of $\hat{n}$. Then clearly

$$\overrightarrow{dS} = \hat{n} dS$$

From (1), flux of $\vec{F}$ over $S = \iint_S \vec{F}.\overrightarrow{dS}$ …. (2)

(ii) Suppose outward drawn normal to the surface S at P makes angles $\alpha,\ \beta,\ \gamma$ with the positive direction of axes and if l, m, n denote the direction cosines of this outward drawn normal, then

$$l = \cos\alpha\,,\quad m = \cos\beta\,,\quad n = \cos\gamma$$

Therefore, $\hat{n} = \cos\alpha\,\hat{\imath} + \cos\beta\,\hat{\jmath} + \cos\gamma\,\hat{k}$

If $\vec{F} = F_1\hat{\imath} + F_2\hat{\jmath} + F_3\hat{k}$ then $\vec{F}.\hat{n} = F_1\cos\alpha + F_2\cos\beta + F_3\cos\gamma$

$\therefore$ From (1), flux of $\vec{F}$ over $S = \iint_S (F_1\cos\alpha + F_2\cos\beta + F_3\cos\gamma)\, dS$ … (3)

Now $dS\cos\alpha$ is the projection of area dS on the yz plane, therefore $dS\cos\alpha = dydz$. Similarly $dS\cos\beta$ and $dS\cos\gamma$ are the projections of the area dS on the zx and xy plane respectively and therefore $dS\cos\beta = dzdx,\ dS\cos\gamma = dxdy$.

$\therefore$ From (3), $\iint_S \vec{F}.\hat{n}\, dS = \iint_S \vec{F}_1 dydz + \vec{F}_2 dzdx + \vec{F}_3 dxdy$ … (4)

Note: *In order to evaluate surface integral it is convenient to express them as double integrals by taking the projection of surface S on one of the coordinate planes. This will happen only if any line perpendicular to co-ordinate plane chosen meets the surface in one point and not more than one point. Surface S is divided into sub surfaces, if above requirement is not met, so that sub surfaces may satisfy the above requirement.*

(iii) Suppose surface S is such that any line perpendicular to xy plane does not meet S in more than one point. Let the equation of surface S be $Z = h(x, y)$.

Let R_1 denotes the orthogonal projection of S on the xy plane. Then projection of dS on the xy plane $= dS\cos\gamma$, where γ is the acute angle which the normal to the surface S makes with positive direction of Z-axis.

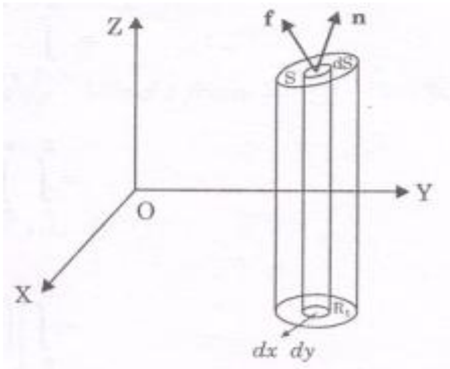

Fig. 7.15

$\therefore$ $dS\cos\gamma = dxdy$ … (5)

But $\cos\gamma = \frac{|\hat{n}.\hat{k}|}{|\hat{n}|} = |\hat{n}.\hat{k}|$

Therefore from (5), $dS = \frac{dx\ dy}{|\hat{n}.\hat{k}|}$ … (6)

Thus $\iint_S \vec{F}.\hat{n}\, dS = \iint_{R_1} \vec{F}.\hat{n}\frac{dx\ dy}{|\hat{n}.\hat{k}|}$ … (7)

Similarly we have, $\iint_S \vec{F}.\hat{n}\, dS = \iint_{R_2} \vec{F}.\hat{n}\frac{dy\ dz}{|\hat{n}.\hat{\imath}|}$ … (8)

$\iint_S \vec{F}.\hat{n}\, dS = \iint_{R_3} \vec{F}.\hat{n}\frac{dz\ dx}{|\hat{n}.\hat{\jmath}|}$ … (9)

where R_2, R_3 are the projections of S on zx and xy planes, respectively.

Example 55: Evaluate $\iint_S \vec{F}.\hat{n}\, dS$ where $\vec{F} = z\,\hat{\imath} + x\,\hat{\jmath} + 3y^2 z\,\hat{k}$ and S is the surface of the cylinder $x^2 + y^2 = 16$ included in the first octant between $z = 0$ and $z = 5$.

Solution: A vector normal to the surface S is given by

$$\vec{n} = \nabla(x^2 + y^2) = 2x\,\hat{\imath} + 2y\,\hat{\jmath} \qquad \ldots (1)$$

$\therefore$ $\hat{n} =$ unit vector normal to surface S at any point $(x, y, z) = \frac{2x\hat{\imath}+2y\hat{\jmath}}{\sqrt{4x^2+4y^2}}$ … (2)

$\because \quad x^2 + y^2 = 16,$ therefore $\hat{n} = \frac{2x\hat{\imath}+2y\hat{\jmath}}{\sqrt{4(x^2+y^2)}} = \frac{2x\hat{\imath}+2y\hat{\jmath}}{8} = \frac{x}{4}\hat{\imath} + \frac{y}{4}\hat{\jmath}$ … (3)

Now $\quad \iint_S \vec{F}.\hat{n}\, dS = \iint_R \vec{F}.\hat{n}\, \frac{dx\, dz}{|\hat{n}.\hat{\jmath}|}$

(projection on xy plane can't be taken as the surface s is peerpendicular to xy plane)

$$= \iint_R \left(\frac{xz}{4} + \frac{xy}{4}\right) \frac{dx\, dz}{\frac{y}{4}} = \iint_R \left(\frac{xz}{y} + x\right) dx\, dz, \quad \left(\text{since from (3),} \quad \hat{n}.\hat{\jmath} = \frac{y}{4}\right)$$

$$= \int_{z=0}^{5} \int_{x=0}^{4} \left(\frac{xz}{\sqrt{16-x^2}} + x\right) dx\, dz = \int_{z=0}^{5} \int_{x=0}^{4} \left(\frac{\frac{-z}{2}(-2x)}{\sqrt{16-x^2}} + x\right) dx\, dz$$

$$= \int_{z=0}^{5} \left[-\frac{z}{2}\frac{\sqrt{16-x^2}}{1/2} + \frac{x^2}{2}\right]_{x=0}^{4} dz = \int_{z=0}^{5} (4z + 8) dz = \left[\frac{4z^2}{2} + 8z\right]_{z=0}^{5} = 90$$

Example 56: Evaluate $\iint_S \vec{F}.\hat{n}\, dS$ where $\vec{F} = 6z\,\hat{\imath} - 4\,\hat{\jmath} + y\,\hat{k}$ and S is the portion of the plane $2x + 3y + 6z = 12$ in the first octant.

Solution: Vector normal to surface S is given by

$$\nabla(2x + 3y + 6z) = 2\hat{\imath} + 3\hat{\jmath} + 6\hat{k}$$

$\therefore \quad \hat{n}$ = unit vector normal to surface S at any point $(x, y, z) = \frac{2\hat{\imath}+3\hat{\jmath}+6\hat{k}}{\sqrt{4+9+36}} = \frac{2}{7}\hat{\imath} + \frac{3}{7}\hat{\jmath} + \frac{6}{7}\hat{k}$

Now $\quad \vec{F}.\hat{n} = \left(6z\,\hat{\imath} - 4\,\hat{\jmath} + y\,\hat{k}\right).\left(\frac{2}{7}\hat{\imath} + \frac{3}{7}\hat{\jmath} + \frac{6}{7}\hat{k}\right) = \frac{12}{7}z - \frac{12}{7} + \frac{6}{7}y$

Taking projection on xy plane

$$\iint_S \vec{F}.\hat{n}\, dS = \iint_R \vec{F}.\hat{n}\, \frac{dx\, dy}{|\hat{n}.\hat{k}|} = \iint_R \vec{F}.\hat{n}\, \frac{dx\, dy}{6/7} \qquad \ldots (1)$$

where R is the region of projection of S on xy plane. R is bounded by x-axis, y-axis and the line $2x + 3y = 12$, $z = 0$. In order to evaluate double integral in (1), y varies from 0 to 4 and x varies from 0 to $\frac{12-3y}{2}$. Therefore from (1)

$$\iint_S \vec{F}.\hat{n}\, dS = \int_{y=0}^{4} \int_{x=0}^{\frac{12-3y}{2}} [2z - 2 + y] dx\, dy, \qquad \text{[Find } z \text{ from } 2x + 3y + 6z = 12]$$

$$= \int_{y=0}^{4} \int_{x=0}^{\frac{12-3y}{2}} \left[2\left(2 - \frac{x}{3} - \frac{y}{2}\right) - 2 + y\right] dx\, dy$$

$$= \int_{y=0}^{4} \int_{x=0}^{\frac{12-3y}{2}} \left[2 - \frac{2x}{3}\right] dx\, dy = \int_0^4 \left[2x - \frac{x^2}{3}\right]_{x=0}^{\frac{12-3y}{2}} dy$$

$$= \int_0^4 \left[2 \times \frac{12-3y}{2} - \frac{1}{3}\left(\frac{12-3y}{2}\right)^2\right] dy$$

$$= \left[\frac{(12-3y)^2}{-6} + \frac{(12-3y)^3}{108}\right]_{y=0}^{4} = \frac{144}{6} - \frac{1728}{108} = 24 - 16 = 8$$

Example 57: $\iint_S \boldsymbol{\varphi\, \hat{n}\, dS}$ **where** $\boldsymbol{\varphi = \frac{3}{8}xyz}$ **and** $\boldsymbol{S}$ **is the surface of cylinder** $\boldsymbol{x^2 + y^2 = 16}$ **included in the first octant between** $\boldsymbol{z = 0}$ **to** $\boldsymbol{z = 5}$**.**

Solution: A vector normal to the surface S is given by

$$\vec{n} = \nabla(x^2 + y^2) = 2x\hat{\imath} + 2y\hat{\jmath}$$

$\therefore$ $\hat{n}$ = unit vector normal to surface S at any point $(x, y, z) = \frac{2x\hat{\imath}+2y\hat{\jmath}}{\sqrt{4x^2+4y^2}}$

$$\hat{n} = \frac{2x\hat{\imath}+2y\hat{\jmath}}{\sqrt{4(x^2+y^2)}} = \frac{2x\hat{\imath}+2y\hat{\jmath}}{8} = \frac{x}{4}\hat{\imath} + \frac{y}{4}\hat{\jmath} \qquad [\because \ x^2 + y^2 = 16\,]$$

Now $\iint_S \varphi\, \hat{n}\, dS = \iint_R \frac{3}{8}xyz \left(\frac{x}{4}\hat{\imath} + \frac{y}{4}\hat{\jmath}\right)\frac{dx\ dz}{|\hat{n}.\hat{\jmath}|}$ …(1)

where R is the region of projection of S on zx plane. Therefore, from (1)

$$\iint_S \varphi\, \hat{n}\, dS = \int_{z=0}^{5}\int_{x=0}^{4}\left(\frac{3}{32}x^2yz\,\hat{\imath} + \frac{3}{32}xy^2z\,\hat{\jmath}\right)\frac{dx\ dz}{y/4}, \qquad \text{where } \ y^2 = 16 - x^2$$

$$= \int_{z=0}^{5}\int_{x=0}^{4}\left(\frac{3}{8}x^2z\,\hat{\imath} + \frac{3}{8}xz\sqrt{(16-x^2)}\,\hat{\jmath}\right)dx\, dz$$

$$= \frac{3}{8}\int_{z=0}^{5}\left(\frac{x^3z}{3}\ \hat{\imath} - \frac{z}{2}\frac{(16-x^2)^{3/2}}{3/2}\ \hat{\jmath}\right)_{x=0}^{4} dZ$$

$$= \frac{3}{8}\int_{z=0}^{5}\left(\frac{64}{3}z\,\hat{\imath} + \frac{64}{3}z\,\hat{\jmath}\right)dz = 8\left[\frac{z^2}{2}\hat{\imath} + \frac{z^2}{2}\hat{\jmath}\right]_{z=0}^{5} = 100\,\hat{\imath} + 100\,\hat{\jmath}$$

Example 58: Evaluate $\int_S \boldsymbol{\vec{F}.dS}$ **where** $\boldsymbol{\vec{F} = x\hat{\imath} - (z^2 - zx)\hat{\jmath} - xy\hat{k}}$ **and** $\boldsymbol{S}$ **is the triangular surface with vertices** $\boldsymbol{(2,0,0)}$**,** $\boldsymbol{(0,2,0)}$ **and** $\boldsymbol{(0,0,4)}$**.**

Solution: The triangular surface S with vertices $(2,0,0)$, $(0,2,0)$, and $(0,0,4)$ is given by the equation

$$\frac{x}{2} + \frac{y}{2} + \frac{z}{4} = 1 \qquad \Rightarrow \qquad 2x + 2y + z = 4 \qquad \ldots (1)$$

Vector normal to surface S is given by $\quad \nabla(2x + 2y + z) = 2\hat{\imath} + 2\hat{\jmath} + \hat{k}$

$\therefore$ $\hat{n}$ = unit vector normal to surface S at any point $(x, y, z) = \frac{2\hat{\imath}+2\hat{\jmath}+\hat{k}}{\sqrt{4+4+1}} = \frac{2}{3}\hat{\imath} + \frac{2}{3}\hat{\jmath} + \frac{1}{3}\hat{k}$

Now $\vec{F}.\hat{n} = \left(x\hat{\imath} - (z^2 - zx)\hat{\jmath} - xy\hat{k}\right).\left(\frac{2}{3}\hat{\imath} + \frac{2}{3}\hat{\jmath} + \frac{1}{3}\hat{k}\right) = \frac{2}{3}x - \frac{2}{3}(z^2 - zx) - \frac{1}{3}xy$

Taking projection on xy plane

$$\iint_S \vec{F}.\hat{n}\, dS = \iint_R \vec{F}.\hat{n}\,\frac{dx\,dy}{|\hat{n}.\hat{k}|} = \iint_R \vec{F}.\hat{n}\,\frac{dx\,dy}{1/3} \qquad \ldots (2)$$

where R is the region of projection of S on xy plane. R is bounded by x-axis, y-axis and the line $2x+2y=4$ *i.e.*, $x+y=2$, $z=0$. In order to integrate double integral in (2), y varies from 0 to 2 and x varies from 0 to $2-y$. Therefore from (2)

$$\iint_S \vec{F}.\hat{n}\, dS = \int_{y=0}^{2}\int_{x=0}^{2-y}\left[\frac{2}{3}x - \frac{2}{3}(z^2 - zx) - \frac{1}{3}xy\right] dx\,dy \qquad \text{[Find } z \text{ from } 2x+2y+z=4]$$

$$= \int_{y=0}^{2}\int_{x=0}^{2-y}\left[\frac{2}{3}x - \frac{2}{3}\{(4-2x-2y)^2 - (4-2x-2y)x\} - \frac{1}{3}xy\right] dx\,dy$$

$$= \frac{1}{3}\int_{y=0}^{2}\int_{x=0}^{2-y}[2x - 2\{16+4x^2+4y^2-16x+8xy-16y-4x+2x^2+2xy\} - xy]dx\,dy$$

$$= \frac{1}{3}\int_{y=0}^{2}\int_{x=0}^{2-y}(-12x^2 - 8y^2 + 42x + 32y - 21xy - 32)dx\,dy$$

$$= \frac{1}{3}\int_{y=0}^{2}\left[-4x^3 - 8xy^2 + 21x^2 + 32xy - 21\frac{x^2y}{2} - 32x\right]_{x=0}^{2-y} dy$$

$$= \frac{1}{3}\int_{y=0}^{2}\left[-4(2-y)^3 - 8(2-y)y^2 + 21(2-y)^2 + 32(2-y)y - 21\frac{(2-y)^2y}{2} - 32(2-y)\right] dy$$

$$= \frac{1}{3}\int_{y=0}^{2}\left(\frac{3}{2}y^3 + 9y^2 + 18y - 12\right) dy = 38$$

Example 59: Evalutate $\int_S \vec{f}\cdot\hat{n}\,ds$ where $\vec{f} = 2x^2y\,\hat{\imath} - y^2\,\hat{\jmath} + 4xz^2\,\hat{k}$ and S is closed surface of the region in the first octant bounded by the cylinder $y^2+z^2=9$ and the planes $x=0, x=2$, $y=0$ and $z=0$.

Solution: The given closed surface S is piecewise smooth and is comprised of S_1 –the rectangular face OAEB in xy-plane; S_2 –the rectangular face OADC in xz-plane; S_3 – the circular quadrant ABC in yz-plane; S_4 – the circular quadrant AED and S_5 – the curved surface BCDE of the cylinder in the first octant (see Fig. 17.16).

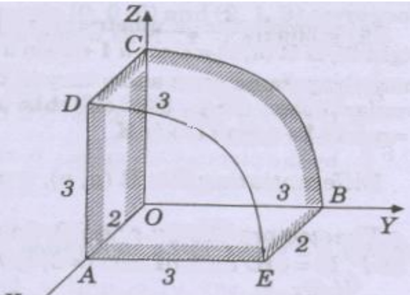

Fig. 7.16

$$\therefore \quad \int_S \vec{f}\cdot\hat{n}\,ds = \int_{S_1}\vec{f}\cdot\hat{n}\,ds + \int_{S_2}\vec{f}\cdot\hat{n}\,ds + \int_{S_3}\vec{f}\cdot\hat{n}\,ds + \int_{S_4}\vec{f}\cdot\hat{n}\,ds + \int_{S_5}\vec{f}\cdot\hat{n}\,ds \qquad \ldots (1)$$

Now $\int_{S_1}\vec{f}\cdot\hat{n}\,ds = \int_{S_1}(2x^2y\,\hat{\imath} - y^2\,\hat{\jmath} + 4xz^2\,\hat{k})\cdot(-\hat{k})\,ds = -4\int_{S_1} xz^2\,ds = 0$

$(as\ z = 0\ in\ xy - plane)$

Similarly, $\int_{S_2}\vec{f}\cdot\hat{n}\,ds = 0$ and $\int_{S_3}\vec{f}\cdot\hat{n}\,ds$

$$\int_{S_4}\vec{f}\cdot\hat{n}\,ds = \int_{S_4}(2x^2y\,\hat{\imath} - y^2\,\hat{\jmath} + 4xz^2\,\hat{k})\cdot\hat{\imath}\,ds = \int_{S_4} 2x^2y\,ds$$

$$= \int_0^3\int_0^{\sqrt{9-z^2}} 8y\,dydz = 4\int_0^3(9-z^2)dz = 72$$

To find $\hat{n}$ in S_5 , we note that $\nabla(y^2+z^2-9) = 2y\,\hat{\jmath} + 2z\hat{k}$,

Implying $\hat{n} = \frac{2y\,\hat{\jmath}+2z\hat{k}}{\sqrt{4(y^2+z^2)}} = \frac{y\,\hat{\jmath}+z\hat{k}}{3}$ and $|\hat{n}\cdot\hat{k}| = \frac{z}{3}$ so that $ds = dxdy/(z/3)$ $\qquad (as\ y^2+z^2=9)$

$$\int_{S_5} \vec{f} \cdot \hat{n}\, ds = \int_0^2 \int_0^3 \frac{(-y^3+4xz^3)}{3}\, dxdy/(z/3) = \int_0^2 \int_0^3 \left(\frac{-y^3}{z} + 4xz^2\right) dxdy$$

Now putting $y = 3\sin\theta\,,\ z = 3\cos\theta \quad \therefore dy = 3\cos\theta\, d\theta$

$$\int_0^2 \int_0^{\frac{\pi}{2}} \left[\frac{-27\sin^3\theta}{3\cos\,\theta} + 4x\,(9\cos^2\theta)\right] 3\cos\theta\ d\theta dx$$

ASSIGNMENT 7

1. If velocity vector is $\vec{F} = y\,\hat{\imath} + 2\,\hat{\jmath} + xz\,\hat{k}$ m/sec., show that the flux of water through the parabolic cylinder $y = x^2,\ \ 0 \le x \le 3,\ \ 0 \le z \le 2$ is $69\, m^3/sec.$
2. Evaluate $\iint_S \ \vec{F}.\hat{n}\, dS$ where $\vec{F} = (x+y^2)\,\hat{\imath} - 2x\,\hat{\jmath} + 2yz\,\hat{k}$ and S is the surface of the plane $2x + y + 2z = 6$ in the first octant.
3. If $\vec{F} = 4xz\,\hat{\imath} - y^2\,\hat{\jmath} + yz\,\hat{k}$; evaluate $\iint_S \ \vec{F}.\hat{n}\, dS$, where S is the surface of the cube bounded by $x = 0,\ x = 1,\ y = 0,\ y = 1,\ z = 0,\ z = 1.$
4. If $\vec{F} = 2y\,\hat{\imath} - 3\,\hat{\jmath} + x^2\,\hat{k}$ and S is the surface of the parabolic cylinder $y^2 = 8x$ in the first octant bounded by the planes $y = 4$ and $z = 6$, show that $\int_S \ \vec{F} \cdot \hat{n}\, ds = 132.$
5. Evaluate $\int_S \ \vec{F} \cdot \hat{n}\, ds$ where $\vec{F} = 6z\,\hat{\imath} - 4\,\hat{\jmath} + y\,\hat{k}$ and S is the portion of the plane $2x + 3y + 6z = 12$ in the first octant.

17.12 VOLUME INTEGRALS

Suppose V is the volume bounded by a surface S. Divide the volume V into sub-volumes δV_1, δV_2, …, δV_n. In each δV_i, choose an arbitrary point P_i whose coordinates are (x_i, y_i, z_i). Let φ be a single valued function defined over V. Form the sum $\sum \varphi\,(P_i)\,\delta V_i$, where $\varphi(P_i) = \varphi(x_i, y_i, z_i)$.

Now let us take the limit of the sum as $n \to \infty$, then the limit, if exists, is called the volume integral of φ over V and is denoted as $\iiint_V \ \varphi\ dV$.

Likewise if $\vec{F}$ is a vector point function defined in the given region of volume V then vector volume integral of $\vec{F}$ over V is $\iiint_V \ \vec{F}\ dV$.

Note: *Above volume integral becomes* $\iiint_V \ \varphi\ dx\ dy\ dz$ *if we subdivide the volume* V *into small cuboids by drawing lines parallel to three co-ordinate axes because in that case* $dV = dxdydz$.

Example 60: If $\vec{F} = (2x^2 - 3z)\hat{\imath} - 2xy\hat{\jmath} - 4x\hat{k}$, evualuate $\iiint_V \ \nabla \times \vec{F}\, dV$ where V is the region bounded by the co-ordinate planes and the plane $2x + 2y + z = 4$.

Solution: Consider

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x^2 - 3z & -2xy & -4x \end{vmatrix} = 0\,\hat{\imath} + \hat{\jmath} - 2y\,\hat{k}$$

Region bounded by $2x + 2y + z = 4$ and coordinate planes such that

$$2x \le 4,\ 2x + 2y \le 4,\ 2x + 2y + z \le 4$$

i.e. $\quad x \le 2, \quad y \le 2 - x, \quad z \le 4 - 2x - 2y$

$$\therefore \quad \iiint_V \nabla \times \vec{F}\, dV = \int_{x=0}^{2}\int_{y=0}^{2-x}\int_{z=0}^{4-2x-2y} \left(\hat{\jmath} - 2y\,\hat{k}\right) dz\, dy\, dx$$

$$= \int_{x=0}^{2}\int_{y=0}^{2-x} \left[z\,\hat{\jmath} - 2yz\,\hat{k}\right]_{z=0}^{z=4-2x-2y} dy\, dx$$

$$= \int_{x=0}^{2}\int_{y=0}^{2-x} \left\{(4 - 2x - 2y)\,\hat{\jmath} - 2y(4 - 2x - 2y)\,\hat{k}\right\} dy\, dx$$

$$= \int_{x=0}^{2} \left[(4y - 2xy - y^2)\hat{\jmath} - \left(4y^2 - 2xy^2 - \tfrac{4}{3}y^3\right)\hat{k}\right]_{y=0}^{y=2-x} dx$$

$$= \int_{x=0}^{2} \begin{bmatrix} \{8 - 4x - 2x(2 - x) - (2 - x)^2\}\hat{\jmath}\ - \\ \left(4(2 - x)^2 - 2x(2 - x)^2 - \tfrac{4}{3}(2 - x)^3\right)\hat{k} \end{bmatrix} dx$$

$$= \int_{x=0}^{2} \left[(4 - 4x + x^2)\hat{\jmath} - \left(-\tfrac{2}{3}x^3 + 4x^2 - 8x + \tfrac{16}{3}\right)\hat{k}\right] dx$$

$$= \left[4x - 2x^2 + \frac{x^3}{3}\right]_{x=0}^{x=2}\hat{\jmath} - \left[-\frac{1}{6}x^4 + \frac{4}{3}x^3 - 4x^2 + \frac{16}{3}x\right]_{x=0}^{x=2}\hat{k}$$

$$= \frac{8}{3}\hat{\jmath} - \frac{8}{3}\hat{k} = \frac{8}{3}(\hat{\jmath} - \hat{k})$$

ASSIGNMENT 8

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1. Evaluate $\iiint_V \varphi\, dV$ where $\varphi = 45x^2y$ and V is the region bounded by the planes $4x + 2y + z = 8,\ x = 0,\ y = 0,\ z = 0.$

2. If $\vec{F} = 2xz\,\hat{\imath} - x\,\hat{\jmath} + y^2\,\hat{k}$; evaluate $\iiint_V \vec{F}\, dV$ where V is the region bounded by the planes $x = 0,\ y = 0,\ x = 2,\ y = 6,\ z = x^2,\ z = 4.$

17.13 STOKE’S THEOREM (*Relation between Line and Surface Integral*)

Statement: Let S be a piecewise smooth open surface bounded by a piecewise smooth simple curve C . If $\vec{f}(x, y, z)$ be a continuous vector function which has continuous first partial derivative in a region of space which contains S, then $\oint_C \vec{f}.d\vec{r} = \iint_C curl\,\vec{f}.\hat{n}\,dS$, where $\hat{n}$ is the unit normal vector at any point of S and C is trasversed in positive direction.

Direction of C is positive if an observer walking on the boundary of S in this direction with its head pointing in the direction of outward normal $\hat{n}$ to S has the surface on the left.

We may put the statement of Stoke’s theorem in words as under:

The line integral of the tangential component of a vector $\vec{f}$ taken around a simple closed curve C is equal to the surface integral of normal component of curl of $\vec{f}$ taken over S having C as its boundary.

Stoke's Theorem in Cartesian Form:

a) Cartesian Form of Stoke's Theorem in Plane (*or* Green's Theorem in Plane)

Choose system of coordinate axes such that the plane of the surface is in xy plane and normal to the surface S lies along the z-axis. Normal vector is constant in this case.

Suppose $\quad \vec{f} = f_1\,\hat{\imath} + f_2\,\hat{\jmath} + f_3\,\hat{k}$

$\therefore \quad \oint_C \vec{f}.d\vec{r} = \oint_C \vec{f}.\frac{d\vec{r}}{ds}\,ds = \oint_C \vec{f}.\vec{t}\,ds$ where $\vec{t} = \frac{d\vec{r}}{ds}$ is unit vector tangent to C.

$$\therefore \quad \oint_C \vec{f}.d\vec{r} = \oint_C \left(f_1\,\hat{\imath} + f_2\,\hat{\jmath} + f_3\,\hat{k}\right).\left(\hat{\imath}\,\frac{dx}{ds} + \hat{\jmath}\,\frac{dy}{ds} + \hat{k}\,\frac{dz}{ds}\right)ds$$

$$= \oint_C \left(f_1\,\frac{dx}{ds} + f_2\,\frac{dy}{ds} + f_3\,\frac{dz}{ds}\right)ds$$

But tangent at any point lies in the xy plane, so $\frac{dz}{ds} = 0$

$$\therefore \quad \oint_C \vec{f}.d\vec{r} = \oint_C \left(f_1\,\frac{dx}{ds} + f_2\,\frac{dy}{ds}\right)ds \qquad \ldots (1)$$

Now $\iint_S curl\,\vec{f}\,.\hat{n}\;dS = \iint_S curl\,\vec{f}\,.\hat{k}\;dS$ (Here normal is along Z-axis)

$$= \iint_S \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)dxdy \qquad \ldots (2)$$

Using (1) and (2), Stoke's theorem is

$$\oint_C (f_1dx + f_2dy) = \iint_S \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)dxdy$$

b) Cartesian Form of Stoke's Theorem in Space

Suppose $\vec{f} = f_1\,\hat{\imath} + f_2\,\hat{\jmath} + f_3\hat{k}$ and $\hat{n}$ is an outward drawn normal unit vector of S making angles α, β, γ with positive direction of axes.

$$\therefore \quad \hat{n} = \cos\alpha\;\hat{\imath} + \cos\beta\;\hat{\jmath} + \cos\gamma\;\hat{k}$$

Now, $\quad \nabla \times \vec{f} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$

$i.e. \quad curl\,\vec{f} = \left[\left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right)\hat{\imath} + \left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right)\hat{\jmath} + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)\hat{k}\right]$

$$\therefore \quad curl\,\vec{f}\,.\;\hat{n} = \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right)\cos\alpha + \left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right)\cos\beta + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)\cos\gamma \qquad \ldots (1)$$

Also $\quad \vec{f}.d\vec{r} = \left(f_1\,\hat{\imath} + f_2\,\hat{\jmath} + f_3\hat{k}\right).\left(dx\,\hat{\imath} + dy\,\hat{\jmath} + dz\,\hat{k}\right)$

or $\vec{f}.d\vec{r} = (f_1\,dx + f_2\,dy + f_3\,dz)$

Then Stoke's theorem is

$$\oint_C (f_1\,dx + f_2\,dy + f_3\,dz) = \iint_S \left[\left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right)\cos\alpha + \left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right)\cos\beta + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)\cos\gamma\right] dS$$

Example 61: Verify Stoke's theorem for $\vec{f} = (2x-y)\hat{\imath} - yz^2\hat{\jmath} - y^2z\hat{k}$ when S is the upper half of the surface of the sphere $x^2+y^2+z^2 = 1$ and C is its boundary.

Solution: The boundary C of the upper half of the sphere S is circle in the xy plane. Therefore, parametric equations of C are $x = \cos t,\ y = \sin t,\ z = 0$ when $0 \le t \le 2\pi$

Now, $\oint_C \vec{f}.d\vec{r} = \oint_C (f_1\,dx + f_2\,dy + f_3\,dz)$

$$= \oint_C (2x-y)dx - yz^2dy - y^2z\,dz = \oint_C (2\cos t - \sin t)(-\sin t)\,dt$$

$[\because\ x = \cos t,\ \therefore dx = -\sin t\ dt$ and other terms of integrand become zero as $z = 0]$

$$= \int_{t=0}^{2\pi}[(2\cos t)(-\sin t) + \sin^2 t]dt = \left[2\frac{\cos^2 t}{2}\right]_{t=0}^{2\pi} + \int_{t=0}^{2\pi}\sin^2 t\,dt$$

$$= (1-1) + 4\int_{t=0}^{\pi/2}\sin^2 t\,dt \qquad \text{(Property of definite integral)}$$

$$= 0 + 4.\frac{1}{2}.\frac{\pi}{2} = \pi \qquad \ldots(1)$$

Now, $$curl\,\vec{f} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x-y & -yz^2 & -y^2z \end{vmatrix} = (-2yz+2yz)\hat{\imath} + (0-0)\hat{\jmath} + (0+1)\hat{k} = \hat{k}$$

Now, $\iint curl\,\vec{f}\,.\,\hat{n}\ dS = \iint_S \hat{k}\,.\hat{n}\,dS$

$$= \iint_R \hat{k}\,.\hat{n}\,\frac{dx\,dy}{|\hat{n}.\hat{k}|} \qquad \text{[where } R \text{ is the projection of S on } xy-\text{plane]}$$

$$= \iint_R dx\,dy$$

Now projection of S on xy plane is circle $x^2 + y^2 = 1$.

$$= \int_{x=-1}^{1}\int_{y=-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dx\,dy = 4\int_{x=0}^{1}\int_{y=0}^{\sqrt{1-x^2}} dx\,dy \qquad \text{[By definite integral]}$$

$$= 4\int_{x=0}^{1}\sqrt{1-x^2}\,dx = 4\left[\frac{x\sqrt{1-x^2}}{2} + \frac{1}{2}\sin^{-1}x\right]_{x=0}^{1}$$

$$= 4\left[\frac{1}{2}\sin^{-1}1\right] = 4\times\frac{\pi}{4} = \pi \qquad \ldots(2)$$

From (1) and (2), Stoke's theorem is verified.

Example 62: Verify Stoke's theorem for the function $\vec{f} = (x^2+y^2)\,\hat{\imath} - 2xy\,\hat{\jmath}$ taken round the rectangle bounded by $x = \pm a,\ y = 0,\ y = b$. [KUK 2006]

Solution: Given $\vec{f} = (x^2 + y^2)\,\hat{\imath} - 2xy\,\hat{\jmath}$

Therefore, $\quad \vec{f}.d\vec{r} = [(x^2 + y^2)\,\hat{\imath} - 2xy\,\hat{\jmath}].[dx\,\hat{\imath} + dy\,\hat{\jmath}] = (x^2 + y^2)\,dx - 2xy\,dy$

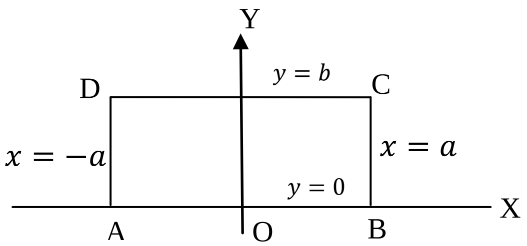

Fig. 17.17

$$\therefore \quad \oint_C \vec{f}.d\vec{r} = \oint_C (x^2 + y^2)\,dx - 2xy\,dy$$

$$= \int_{DA}(x^2 + y^2)\,dx - 2xy\,dy + \int_{AB}(x^2 + y^2)\,dx - 2xy\,dy$$

$$+ \int_{BC}(x^2 + y^2)\,dx - 2xy\,dy + \int_{CD}(x^2 + y^2)\,dx - 2xy\,dy \qquad \ldots (1)$$

On DA: $x = -a, \quad \therefore \quad dx = 0$

$$\therefore \quad \int_{DA}(x^2 + y^2)\,dx - 2xy\,dy = \int_{DA} -2(-a)y\,dy = \int_{y=b}^{0} 2ay\,dy = [ay^2]_b^0 = -ab^2$$

On AB: $y = 0, \quad \therefore \quad dy = 0$

$$\therefore \quad \int_{AB}(x^2 + y^2)\,dx - 2xy\,dy = \int_{AB} x^2\,dx = \int_{-a}^{a} x^2\,dx = \frac{2}{3}a^3$$

On BC: $x = a, \quad \therefore \quad dx = 0$

$$\therefore \quad \int_{BC}(x^2 + y^2)\,dx - 2xy\,dy = \int_{BC}(-2ay\,)dy = \int_{y=0}^{b}(-2ay)\,dy = -ab^2$$

On CD: $y = b, \quad \therefore \quad dy = 0$

$$\therefore \quad \int_{CD}(x^2 + y^2)\,dx - 2xy\,dy = \int_{CD}(x^2 + b^2)\,dx = \int_{x=a}^{-a}(x^2 + b^2)\,dx$$

$$= \left[\frac{x^3}{3} + b^2 x\right]_{x=a}^{-a} = -2\left(\frac{a^3}{3} + b^2 a\right)$$

Substituting these values in (1), we get

$$\oint_C \vec{f}.d\vec{r} = -ab^2 + \frac{2}{3}a^3 - ab^2 - 2\left(\frac{a^3}{3} + b^2 a\right) = -4ab^2 \qquad \ldots (2)$$

Now, $\quad curl\,\vec{f} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 + y^2 & -2xy & 0 \end{vmatrix} = 0\,\hat{\imath} + 0\,\hat{\jmath} + (-2y - 2y)\hat{k} = -4y\,\hat{k}$

Since Surface lies in xy-plane, therefore $\quad \hat{n} = \hat{k}$.

$\therefore \qquad \iint_S \; curl \; \vec{f} \,.\, \hat{n}\, dS = \iint_S \; \left(-4y\,\hat{k}\right).\hat{k}\, dS = \int_{y=0}^{b}\int_{x=-a}^{a} -4y\, dy\, dx = -4ab^2 \qquad \ldots (3)$

Hence from (2) and (3), theorem is verified.

Example 63: Evaluate by Stoke's theorem $\oint_C \; (yz\, dx + xz\, dy + xydz)$, where C is the curve $x^2 + y^2 = 1,\; z = y^2$.

Solution: $\oint_C \; (yz\, dx + xz\, dy + xydz) = \oint_C \; \left(yz\,\hat{\imath} + xz\,\hat{\jmath} + xy\,\hat{k}\right).\left(dx\,\hat{\imath} + dy\,\hat{\jmath} + dz\,\hat{k}\right)$

$$= \oint_C \; \vec{f}\,.\,d\vec{r}\,, \qquad \text{where } \vec{f} = yz\,\hat{\imath} + xz\,\hat{\jmath} + xy\,\hat{k}$$

Now, $\quad curl\,\vec{f} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & zx & xy \end{vmatrix} = (x - x)\hat{\imath} + (y - y)\,\hat{\jmath} + (z - z)\hat{k} = 0$

$\therefore \qquad \oint_C \; \vec{f}\,.\,d\vec{r} = \iint_S \; curl\,\vec{f}.\hat{n}\, dS = 0 \qquad \left[\because \; curl\,\vec{f} = 0\right]$

Example 64: Evaluate $\oint_C \; \vec{f}\,.\,d\vec{r}$ by Stoke's theorem, where $\vec{f} = y^2\,\hat{\imath} + x^2\,\hat{\jmath} - (x + z)\,\hat{k}$ and C is the boundary of triangle with vertices at $(0, 0, 0),\; (1, 0, 0),\; (1, 1, 0)$.

Solution: Here,

$$curl\,\vec{f} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & x^2 & -(x + z) \end{vmatrix} = 0\,\hat{\imath} + \;\hat{\jmath} + 2(x - y)\,\hat{k}$$

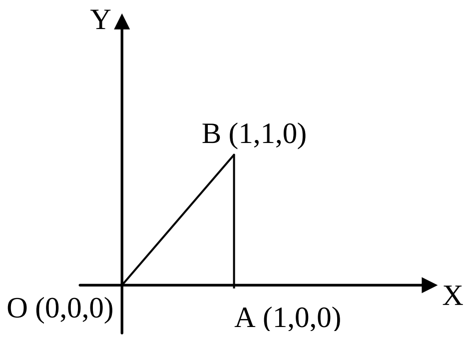

Fig. 17.18

Here triangle is in the xy plane as z co-ordinate of each vertex of the triangle is zero.

$\therefore \qquad \hat{n} = \hat{k}$

$\therefore \qquad curl\,\vec{f}\,.\hat{n} = \left[0\,\hat{\imath} + \;\hat{\jmath} + 2(x - y)\,\hat{k}\right].\hat{k} = 2(x - y)$

By Stoke's theorem, $\quad \oint_C \; \vec{f}\,.\,d\vec{r} = \iint_S \; curl\,\vec{f}.\hat{n}\, dS$

$$= \iint_S \; 2(x - y)\, dy\, dx$$

Note here the equation of OB is $y = x$, thus for S, x varies from 0 to 1 and y from 0 to x.

$\therefore \qquad \oint_C \vec{f}.d\vec{r} = \int_{x=0}^{1}\int_{y=0}^{x} 2(x-y)\,dy\,dx = \int_{x=0}^{1} 2\left(xy - \frac{y^2}{2}\right)_{y=0}^{x} dx$

$$= \int_{x=0}^{1} 2\left(x^2 - \frac{x^2}{2}\right)dx = \int_{x=0}^{1} x^2 dx = \frac{1}{3}$$

Note*: Green's Theorem in plane is special case of Stoke's Theorem:* *If R is the region in xy plane bounded by a closed curve C then this is a special case of Stoke's theorem. In this case $\hat{n} = \hat{k}$ and it is called vector form of Green's theorem in plane. Vector form of Green's theorem can be written as*

$$\iint_R \left(\nabla \times \vec{f}\right).\hat{k}\, dR = \oint_C \vec{f}.d\vec{r}$$

Example 65: Evaluate $\oint_C [(y - \sin x)dx + \cos x\, dy]$, where C is the triangle having vertices $(0,0)$, $\left(\frac{\pi}{2},0\right)$ and $\left(\frac{\pi}{2},1\right)$ (i) directly (ii) by using Green's theorem in plane [KUK 2011]

Solution:

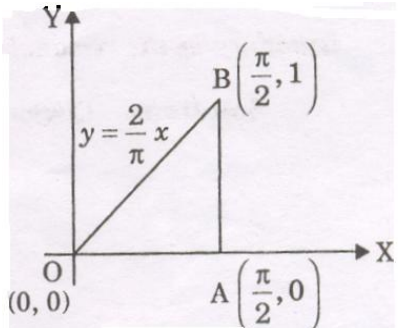

Fig. 7.19

(i) Here $\oint_C [(y - \sin x)dx + \cos x\, dy]$

$$= \oint_C [(y-\sin x)\hat{\imath} + \cos x\,\hat{\jmath}].(dx\,\hat{\imath} + dy\,\hat{\jmath})$$

$$= \oint_C \vec{f}.d\vec{r}$$

where $\vec{f} = [(y-\sin x)\hat{\imath} + \cos x\,\hat{\jmath}]$ and $d\vec{r} = dx\,\hat{\imath} + dy\,\hat{\jmath}$ and C is triangle OAB

Now $\oint_C \vec{f}.d\vec{r} = \oint_C [(y-\sin x)\,dx + \cos x\,dy]$

$$= \int_{OA}[(y-\sin x)\,dx + \cos x\,dy] + \int_{AB}[(y-\sin x)\,dx + \cos x\,dy]$$

$$+ \int_{BO}[(y-\sin x)\,dx + \cos x\,dy] \qquad \ldots (1)$$

On OA: $y = 0$, $\therefore \quad dy = 0$

$\therefore \qquad \int_{OA}[(y-\sin x)\,dx + \cos x\,dy] = \int_{OA}[-\sin x\,dx] = \int_{x=0}^{\pi/2} -\sin x\,dx = -1$

On AB: $x = \frac{\pi}{2}$, $\therefore \quad dx = 0$

$\therefore \qquad \int_{AB}[(y-\sin x)\,dx + \cos x\,dy] = 0$

On BO: $y = \frac{2}{\pi}x$, $\therefore \quad dy = \frac{2}{\pi}dx$

$\therefore \qquad \int_{BO}[(y-\sin x)\,dx + \cos x\,dy] = \int_{x=\pi/2}^{0}\left[\left(\frac{2}{\pi}x - \sin x\right)dx + \cos x\,\frac{2}{\pi}dx\right]$

$$= \left[\frac{2}{\pi}\frac{x^2}{2} + \cos x + \frac{2}{\pi}\sin x\right]_{x=\pi/2}^{0} = 1 - \left[\frac{\pi}{4} + \frac{2}{\pi}\right]$$

Substituting these values in (1), we get

$$\oint_C \vec{f}\,.\,d\vec{r} = -1 + 0 + 1 - \frac{\pi}{4} - \frac{2}{\pi} = -\left[\frac{\pi}{4} + \frac{2}{\pi}\right]$$

(ii) By Green's Theorem

$$\oint_C (f_1 dx + f_2 dy) = \iint_S \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right) dxdy$$

$$= \int_{x=0}^{\pi/2} \int_{y=0}^{2x/\pi} (-\sin x - 1) dy\, dx = \int_{x=0}^{\pi/2} (-\sin x - 1)[y]_{y=0}^{2x/\pi} dx$$

$$= \int_{x=0}^{\pi/2} -\frac{2}{\pi} x(\sin x + 1)\, dx = -\frac{2}{\pi}\int_{x=0}^{\pi/2} (x \sin x + x)\, dx$$

$$= -\frac{2}{\pi}\left[x(-\cos x) + \sin x + \frac{x^2}{2}\right]_{x=0}^{\pi/2} = -\frac{2}{\pi}\left[\left(-0 + 1 + \frac{\pi^2}{8}\right) - (0 + 0 + 0)\right]$$

$$= -\left(\frac{2}{\pi} + \frac{\pi}{4}\right)$$

Example 66: Verify Green's theorem in the plane for $\oint_C (xy + y^2)dx + x^2 dy$, where C is the closed curve of the region bounded by $y = x$ and $y = x^2$.

Solution:

$$\oint_C (xy + y^2)dx + x^2 dy = \int_{OBA} [(xy + y^2)dx + x^2 dy] + \int_{OA} [(xy + y^2)dx + x^2 dy] \quad \ldots (1)$$

Along curve OBA: $y = x^2 \quad \therefore \quad dy = 2x\, dx$

$$\therefore \quad \int_{OBA} [(xy + y^2)dx + x^2 dy] = \int_{x=0}^{1} [(x^3 + x^4)dx + 2x^3 dx] = \frac{19}{20}$$

Along curve AO: $y = x \quad \therefore \quad dy = dx$

$$\therefore \quad \int_{AO} [(xy + y^2)dx + x^2 dy] = \int_{AO} [(x^2 + x^2)dx + x^2 dx] = \int_{x=1}^{0} [3x^2]\, dx = -1$$

$\therefore$ from (1), $\quad \oint_C (xy + y^2)dx + x^2 dy = \frac{19}{20} - 1 = -\frac{1}{20} \quad \ldots (2)$

Here $f_1 = xy + y^2,\ f_2 = x^2$

$\Rightarrow \quad \frac{\partial f_1}{\partial y} = x + 2y,\ \frac{\partial f_2}{\partial x} = 2x$

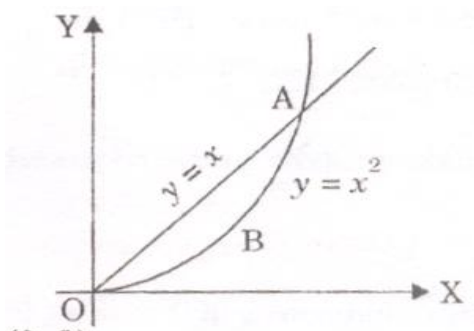

Fig. 7.20

By Green's theorem,

$$\oint_C (f_1\, dx + f_2\, dy) = \iint_S \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right) dy\, dx$$

$$= \iint_S (2x - x - 2y)\, dy\, dx$$

$$= \int_{x=0}^{1} \int_{y=x^2}^{y=x} (x - 2y) dy\, dx = \int_{x=0}^{1} [xy - y^2]_{y=x^2}^{y=x} dx$$

$$= \int_{x=0}^{1} (x^4 - x^3) dx = \left[\frac{x^5}{5} - \frac{x^4}{4}\right]_{x=0}^{x=1} = \frac{1}{5} - \frac{1}{4} = -\frac{1}{20} \quad \ldots (3)$$

Equation (2) and (3) verify the result.

ASSIGNMENT 9

1. Verify Green's theorem for $\int_C [(3x - 8y^2)dx + (4y - 6xy)dy]$ where C is the boundary of the region bounded by $x = 0,\ y = 0$ and $x + y = 1$. [KUK 2007]
2. Verify Green's theorem in plane for $\oint_C [(3x^2 - 8y^2)dx + (4y - 6xy)dy]$, where C is the boundary of the region defined by $y = \sqrt{x}$ and $y = x^2$. [KUK 2008]
3. Apply Green's theorem to evaluate $\int_C [(2x^2 - y^2)dx + (x^2 + y^2)dy]$, where C is the boundary of the area enclosed by x-axis and the upper half of the circle $x^2 + y^2 = 1$. [KUK 2010]
4. Evaluate the surface integral $\iint_S curl\, \vec{F} . \hat{n}\, dS$ by transforming it into a line integral, S being that part of the surface of the paraboloid $z = (1 - x^2 - y^2)$ for which $z \geq 0$ and $\vec{F} = y\hat{i} + z\,\hat{j} + x\hat{k}$. [KUK 2008]
5. Using Stoke's theorem, evaluate $\int_C [(x + y)dx + (2x - z)dy + (y + z)dz]$, where C is the boundary of the triangle with vertices $(2, 0, 0)$, $(0, 3, 0)$ and $(0, 0, 6)$. [KUK 2009]
6. Verify Stoke's theorem for a vector field defined by $\vec{f} = -y^3\,\hat{\imath} + x^3\,\hat{\jmath}$, in the region $x^2 + y^2 \leq 1,\ z = 0$.

17.14 GAUSS'S DIVERGENCE THEOREM (*Relation between Volume and Surface Integral*)

Statement: Suppose V is the volume bounded by a closed piecewise smooth surface S. Suppose $\vec{f}(x, y, z)$ is a vector function which is continuous and has continuous first partial derivatives in V.Then

$$\iiint_V \nabla . \vec{f}\ dV = \iint \ \vec{f} . \hat{n}\, dS$$

where $\hat{n}$ is the outward unit normal to the surface S.

In other words: The surface integral of the normal component of a vector $\vec{f}$ taken over a closed surface is equal to the integral of the divergence of $\vec{f}$ over the volume enclosed by the surface.

Divergence Theorem in Cartesian Coordinates

If $\vec{f} = f_1\hat{\imath} + f_2\hat{\jmath} + f_3\hat{k}$ then $div\, \vec{f} = \nabla . \vec{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$. Suppose $\alpha,\ \beta,\ \gamma$ are the angles made by the outward drawn unit normal with the positive direction of axes, then

$$\hat{n} = \cos\alpha\,\hat{\imath} + \cos\beta\,\hat{\jmath} + \cos\gamma\,\hat{k}$$

Now, $\vec{f} . \hat{n} = f_1\cos\alpha + f_2\cos\beta + f_3\cos\gamma$

Then divergence theorem is

$$\iiint_V \left(\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}\right) dx\, dy\, dz = \iint_S (f_1\cos\alpha + f_2\cos\beta + f_3\cos\gamma)\, dS$$

$$= \iint_S (f_1 dydz + f_2\, dzdx + f_3 dxdy)$$

$$[\because\ \cos\alpha\ dS = dydz\ etc.]$$

Proof: Let $\vec{f} = f_1\hat{\imath} + f_2\hat{\jmath} + f_3\hat{k}$ where $f_1,\ f_2$ and f_3 and their derivatives in any direction are finite and continuouos.

Suppose S is a closed surface such that it is possible to choose rectangular cartesian co-ordinate system such that any line drawn parallel to coordinate axes does not cut S in more than two points.

Let R be the orthogonal projection of S on the xy-plane. Any line parallel to z-axis through a point of R meets the boundary of S in two points. Let S_1 and S_2 be the lower and upper portions of S. Let the equations of these portions be

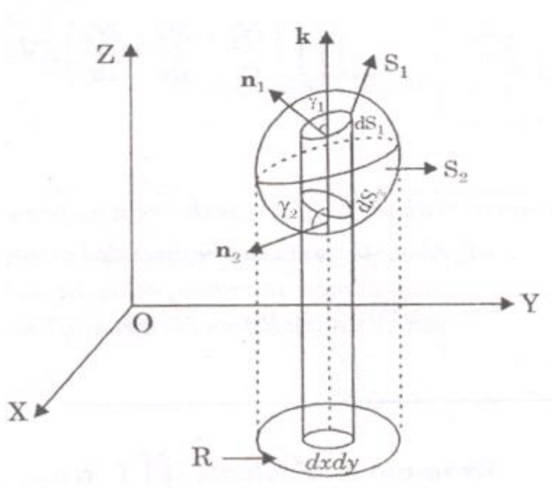

Fig. 7.21

$$z = \Phi_1(x,y) \text{ and } z = \Phi_2(x,y)$$

$$\text{where } \Phi_1(x,y) \geq \Phi_2(x,y)$$

Consider the volume integral

$$\iiint_V \frac{\partial f_3}{\partial z}\, dV = \iiint \frac{\partial f_3}{\partial z}\, dx\, dy\, dz = \iint \left[\int_{z=\Phi_2(x,y)}^{z=\Phi_1(x,y)} \left(\frac{\partial f_3}{\partial z} dz\right)\right] dx\, dy$$

$$\iiint_V \frac{\partial f_3}{\partial z}\, dV = \iint [f_3(x,y,z)]_{\Phi_2(x,y)}^{\Phi_1(x,y)} dx\, dy = \iint \{f_3(x,y,\Phi_1) - f_3(x,y,\Phi_2)\}\, dx\, dy \qquad \ldots(1)$$

Let $\hat{n}_1$ be the unit outward drawn vector making an acute angle γ_1 with $\hat{k}$ for the upper position S_1 as shown in the figure.

Now projection $dx\, dy$ of dS_1 on the xy plane is given as $dxdy = dS_1 \cos\gamma = dS_1\, \hat{k}.\hat{n}_1 = \hat{k}.\hat{n}_1 dS_1$

Now $\iint_R\ f_3(x,y,\Phi_1)\, dxdy = \iint_{S_1} f_3\, \hat{k}.\vec{n}_1 dS_1 \qquad \ldots(2)$

Similarly if $\hat{n}_2$ be the unit outward drawn normal to the lower surface S_2 making an angle γ_2 with $\hat{k}$. Obviously γ_2 is an obtuse angle

$$\therefore \qquad dxdy = dS_2 \cos(\pi - \gamma_2) = -dS_2 \cos\gamma_2 = -\hat{k}.\hat{n}_2\, dS_2$$

$$\therefore \qquad \iint_R\ f_3(x,y,\Phi_2)\, dx\, dy = -\iint_{S_2} f_3\, \hat{k}.\hat{n}_2 dS_2 \qquad \ldots(3)$$

From (1), (2) and (3), we have

$$\iiint_V \frac{\partial f_3}{\partial z}\, dV = \iint_{S_1} f_3\, \hat{k}.\hat{n}_1 dS_1 + \iint_{S_2} f_3\, \hat{k}.\hat{n}_2 dS_2 = \iint_S\ f_3\, \hat{k}.\hat{n}\, dS \qquad \ldots(4)$$

Similarly by projecting S on the other coordinate planes

$$\iiint_V \frac{\partial f_2}{\partial y}\, dV = \iint_S\ f_2\, \hat{\jmath}.\vec{n}\, dS \qquad \ldots(5)$$

And $\iiint_V \frac{\partial f_1}{\partial x}\, dV = \iint_S\ f_1\, \hat{\imath}.\vec{n}\, dS \qquad \ldots(6)$

Adding (4), (5) and (6), we get

$$\iiint_V \left(\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}\right) dV = \iint_S \left(f_1\,\hat{\imath} + f_2\,\hat{\jmath} + f_3\,\hat{k}\right) .\ \hat{n}\, dS = \iint_S\ \vec{f}\ .\ \hat{n}\, dS$$

Note: *With the help of this theorem we can express volume integral as surface integral or vice versa.*

Example 67: Evaluate $\iint_S \vec{f}.\hat{n}\, dS$ where $\vec{f} = 4xy\,\hat{\imath} + yz\,\hat{\jmath} - zx\,\hat{k}$ and S is the surface of the cube bounded by the planes $x = 0, x = 2, y = 0, y = 2, z = 0, z = 2$.

Solution: Here $\vec{f} = 4xy\,\hat{\imath} + yz\,\hat{\jmath} - zx\,\hat{k}$. By Gauss's divergence theorem

$$\iint_S \vec{f}\,.\,\hat{n}\,dS = \iiint_V \nabla.\vec{f}\,dV$$

$$= \iiint_V \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).\left(4xy\,\hat{\imath} + yz\,\hat{\jmath} - zx\,\hat{k}\right) dV$$

$$= \iiint_V (4y + z - x)\,dV = \int_{x=0}^{2}\int_{y=0}^{2}\int_{z=0}^{2}(4y + z - x)\,dz\,dy\,dx$$

$$= \int_{x=0}^{2}\int_{y=0}^{2}\left[4yz + \frac{z^2}{2} - xz\right]_{z=0}^{z=2} dy\,dx$$

$$= \int_{x=0}^{2}\left[\int_{y=0}^{2}[8y + 2 - 2x]\,dy\right] dx$$

$$= \int_{x=0}^{2}(4y^2 + 2y - 2xy)_{y=0}^{y=2} dx = \int_{x=0}^{2}(20 - 4x)\,dx$$

$$= [20x - 2x^2]_{x=0}^{x=2} = 32$$

Example 68: Use Gauss theorem to show that $\iint_S \left[(x^3 - yz)\,\hat{\imath} - 2x^2y\,\hat{\jmath} + 2\hat{k}\right].\hat{n}\,dS = \frac{a^5}{3}$

where S denotes the surface of the cube bounded by the planes, $x = 0$, $x = a$, $y = 0$, $y = a$, $z = 0$, $z = a$.

Solution: By Gauss's divergence theorem

$$\iint_S \left[(x^3 - yz)\,\hat{\imath} - 2x^2y\,\hat{\jmath} + 2\hat{k}\right].\hat{n}\,dS = \iiint_V \nabla.\left\{(x^3 - yz)\,\hat{\imath} - 2x^2y\,\hat{\jmath} + 2\hat{k}\right\} dV$$

$$= \int_0^a\int_0^a\int_0^a (3x^2 - 2x^2)\,dx\,dy\,dz$$

$$= \int_0^a\int_0^a\left[\int_0^a x^2 dx\right]dy\,dz = \int_0^a\int_0^a\left(\frac{x^3}{3}\right)_0^a dy\,dz$$

$$= \int_0^a\int_0^a\left(\frac{a^3}{3}\right)dy\,dz = \int_0^a\left(\frac{a^3}{3}y\right)_0^a dz = \int_0^a\frac{a^4}{3}dz = \frac{a^4}{3}(z)_0^a = \frac{a^5}{3}$$

Example 69: Evaluate $\iint_S \vec{f}.\hat{n}\,dS$ with the help of Gauss theorem for $\vec{f} = 6z\,\hat{\imath} + (2x + y)\hat{\jmath} - x\,\hat{k}$ taken over the region S bounded by the surface of the cylinder $x^2 + z^2 = 9$ included between $x = 0$, $y = 0$, $z = 0$ and $y = 8$.

Solution: $\vec{f} = 6z\,\hat{\imath} + (2x + y)\hat{\jmath} - x\,\hat{k}$

$$\nabla.\vec{f} = \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).\left\{6z\,\hat{\imath} + (2x + y)\hat{\jmath} - x\,\hat{k}\right\} = 1$$

By Gauss's divergence theorem,

$$\iint_S \vec{f}.\hat{n}\,dS = \iiint_V 1\,dx\,dy\,dz$$

$$= \int_{x=0}^{3}\int_{y=0}^{8}\int_{z=0}^{\sqrt{9-x^2}} dz\,dy\,dx = \int_{x=0}^{3}\int_{y=0}^{8}[z]_{z=0}^{z=\sqrt{9-x^2}}dy\,dx$$

$$= \int_{x=0}^{3}\int_{y=0}^{8}\sqrt{9-x^2}\,dy\,dx$$

$$= \int_{x=0}^{3}\left[\sqrt{9-x^2}\;\;y\right]_{y=0}^{8}dx = \int_{x=0}^{3} 8\sqrt{9-x^2}\,dx$$

$$= 8\left[\frac{x\sqrt{9-x^2}}{2}+\frac{9}{2}\sin^{-1}\frac{x}{3}\right]_{x=0}^{3} = 18\,\pi$$

Example 70: Evaluate $\iint_S (xdydz + ydzdx + zdxdy)$ where S is the surface of the sphere $x^2+y^2+z^2=a^2$. **[KUK 2011, 2009]**

Solution: By Gauss's divergence theorem

$$\iint_S (xdydz + ydzdx + zdxdy) = \iiint_V \left(\frac{\partial x}{\partial x}+\frac{\partial y}{\partial y}+\frac{\partial z}{\partial z}\right)dx\,dy\,dz = \iiint_V 3\,dx\,dy\,dz$$

$$= 3\iiint_V dx\,dy\,dz = 3\times \text{volume of the sphere } \mathrm{x}^2+\mathrm{y}^2+\mathrm{z}^2=\mathrm{a}^2$$

$$= 3\times\frac{4}{3}\pi \mathrm{a}^3 = 4\pi\mathrm{a}^3$$

Example 71: Show that $\iint_S \hat{n}\,dS = 0$ for any closed surface S.

Solution: Let C be any closed vector.

$\therefore$ $\quad C\iint_S \hat{n}\,dS = \iint_S C.\hat{n}\,dS = \iiint_V div\,C\,dV$

$\therefore$ $\quad C\iint_S \hat{n}\,dS = 0 \qquad [\because\ C \text{ is constant, therfore } div\,C = 0]$

$\Rightarrow$ $\quad \iint_S \hat{n}\,dS = 0$

Example 72: Prove that $\iiint_V \frac{1}{r^2}dV = \iint_S \frac{\vec{r}.\hat{n}}{r^2}dS$, where $\vec{r} = x\,\hat{\imath}+y\,\hat{\jmath}+z\,\hat{k}$ and $|\vec{r}| = r$.

Solution: $\iint_S \frac{\vec{r}.\hat{n}}{r^2}dS = \iint_S \frac{\vec{r}}{r^2}.\hat{n}\,dS = \iiint_V \left(\nabla.\frac{\vec{r}}{r^2}\right)dV$ … (1)

Now, $\quad \nabla.\left(\frac{\vec{r}}{r^2}\right) = \frac{1}{r^2}(\nabla.\vec{r}) + \vec{r}.\nabla\left(\frac{1}{r^2}\right)$ … (2)

Also, $\quad \vec{r} = x\,\hat{\imath}+y\,\hat{\jmath}+z\,\hat{k} \quad \Rightarrow \quad r^2 = x^2+y^2+z^2$

$\therefore \quad 2r\frac{\partial r}{\partial x} = 2x; \quad 2r\frac{\partial r}{\partial y} = 2y; \quad 2r\frac{\partial r}{\partial z} = 2z$

$\Rightarrow \quad \frac{\partial r}{\partial x} = \frac{x}{r}; \quad \frac{\partial r}{\partial y} = \frac{y}{r}; \quad \frac{\partial r}{\partial z} = \frac{z}{r}$ … (3)

Now, $\quad \nabla\left(\frac{1}{r^2}\right) = \left(\hat{\imath}\frac{\partial}{\partial x}+\hat{\jmath}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\left(\frac{1}{r^2}\right) = \frac{-2}{r^3}\left(\hat{\imath}\frac{\partial r}{\partial x}+\hat{\jmath}\frac{\partial r}{\partial y}+\hat{k}\frac{\partial r}{\partial z}\right)$ [using (3)]

$$= \frac{-2}{r^4}\left(x\,\hat{\imath}+y\,\hat{\jmath}+z\,\hat{k}\right) = \frac{-2}{r^4}\vec{r}$$

Also, $\nabla.\vec{r} = 1 + 1 + 1 = 3$

Substituting these values in (2), we have

$$\nabla.\left(\frac{\vec{r}}{r^2}\right) = \frac{1}{r^2}.3 + \vec{r}.\left(-\frac{2}{r^4}\right)\vec{r} = \frac{3}{r^2} - \frac{2}{r^4}r^2 = \frac{1}{r^2} \qquad [\because \vec{r}.\vec{r} = r^2]$$

$\therefore$ From (1), $\iint_S \frac{\vec{r}.\hat{n}}{r^2} dS = \iiint_V \frac{1}{r^2} dV$

ASSIGNMENT 10

1. Find $\iint_S \vec{F}.\hat{n}\, dS$ where $\vec{F} = (2x+z)\hat{i} - (xz+y)\hat{j} + (y^2+2z)\hat{k}$ and S is the surface of the sphere having centre $(3, -1, 2)$ and radius 3 units. [KUK 2006]
2. Use Divergence theorem to evaluate $\iint_S \vec{F}.d\vec{S}$ where $\vec{F} = x^3\hat{i} + y^3\hat{j} + z^3\hat{k}$ and S is the outer surface of the sphere $x^2 + y^2 + z^2 = 1$. [KUK 2007]
3. Verify Divergence theorem for $\vec{F} = (x^2 - yz)\hat{i} + (y^2 - zx)\hat{j} + (z^2 - xy)\hat{k}$ taken over rectangular parallelopiped $0 \le x \le a,\ 0 \le y \le b,\ 0 \le z \le c.$ [KUK 2010]