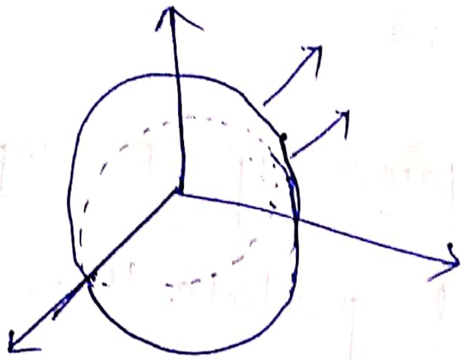
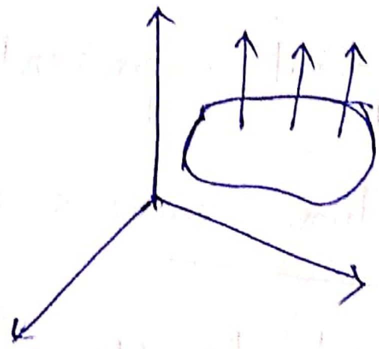
Surface Integrals.

Def :- If S has a unique normal at each point on the surface and direction of normal depends on points of Surface S, the Surface S is called <u>Smooth</u> Surface.

ex.



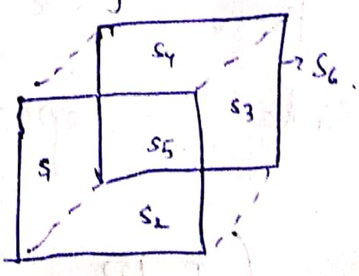
Sphere.



. plane Surface.

def:- If S is not smooth but can be devided into finite no. of smooth portions. Such surfaces are called piecewise smooth surface.

ex a cube. which 6 piecewise smooth surface.



Def Let S be the surface, & $F(R)$ be the continous vector function then the integral of $F(R)$ over the surface S is given by

$\int_S F \cdot dS$ or $\int_S F \cdot N \, ds$ where N is the unit outward normal at p to S.

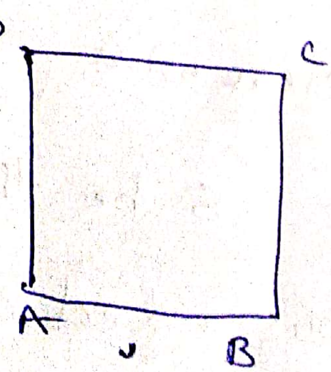
and is also called as normal surface integral of F (R) over S.

physical representation :-

Flux across a surface :- If F represents the velocity of a fluid particle then the total flux (rate of flow) across a closed surface S is the integral $\int_S F \cdot ds$.

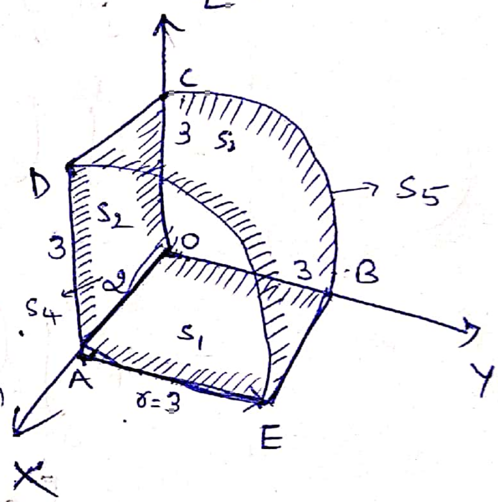
Note Line integral :- If the curve is a square then we need to integrate for each side seperately and then add the values

$$\int_C F \cdot dR = \int_{AB} F \cdot dR + \int_{BC} F \cdot dR + \int_{CD} F \cdot dR + \int_{DA} F \cdot dR.$$



ex Evaluate $\int_S F\cdot N\,ds$ where $F = 2x^2y\,i - y^2 j + 4xz^2 k$ and S is the closed surface of the region in first octant bounded by the cylinder $y^2+z^2=9$ and planes $x=0$, $x=2$, & $y=0$ & $z=0$.

Sol the given Surface is smooth but piecewise and it has 5 Surfaces.



S_1 – rectangle OAEB in xy

S_2 – rectangle OADC. in xz

S_3 – circular Quadrant OBC in YZ

S_4 – Circular Quadrant AED in

S_5 – Curved Surface BCDE of Cylinder.

Now we find for all surfaces and all

for S_1 $\int_{S_1} F\cdot N\,ds = \int (2x^2y\,i - y^2 j + 4xz^2 k)\cdot\underline{(-k)}\,ds.$

$[\because z=0$ in xy plane$]$

$= -4\int xz^2\cdot ds = 0$

Similarly S_2 · $\int_{S_2} F\cdot N\,ds = 0$ & $\int_{S_3} F\cdot N\,ds = 0$

now for S_4, AED the outward normal is i

So $\int_{S_4} (2x^2 y\, i - y^2 j + 4xz^2 k) \cdot i\, ds$

$= \int_{S_4} 2x^2 y\, ds = \int_0^3 \int_0^{\sqrt{9-z^2}} 8y\, dy\, dz.$ $(\because x = 2)$

$= 8 \int_0^3 \int_0^{\sqrt{9-z^2}} y\, dy\, dz. = \cancel{8 \int_0^3 \int_0^{\sqrt{9-z^2}}}$

$= 8 \int_0^3 \frac{(\sqrt{9-z^2})^2}{2} \, dz.$

$= 4 \int_0^3 (9 - z^2)\, dz$

$= 4 \left[9z - \frac{z^3}{3} \right]_0^3$

$= 4 \left[27 - \frac{27}{3} \right]$

$= 4 \left[2 \cdot \frac{27}{3} \right] = 8 \cdot \frac{\cancel{27}^{9}}{\cancel{3}}$

$= 72$

Now for S_5 we have normal is ∇f

eqn is $y^2 + z^2 = 9$

$\nabla (y^2 + z^2) = 2y\, j + 2z\, k$

$N = \frac{2y\, j + 2z\, k}{\sqrt{4y^2 + 4z^2}}$

$$= \frac{yj + zk}{3} \qquad (\because y^2 + z^2 = 9).$$

and we have $|N \cdot k| = \frac{z}{3}$

so $$ds = \frac{dx\,dy}{\frac{z}{3}} \qquad \left(\because ds = \frac{dx\,dy}{|N\cdot k|}\right)$$

projection on xy plane

$$= \int_{S_5} F\cdot N\, ds = \int_0^2 \int_0^3 (\cancel{2y^2} x\, i - y^2 j + 4xz^2 k)\cdot\left(\frac{yj + zk}{\cancel{3}}\right) \frac{dx\,dy}{\frac{z}{\cancel{3}}}$$

$$= \int_{x=0}^{2} \int_0^3 \left(-\frac{y^3}{z} + \frac{4xz^3}{z}\right) dx\,dy$$

$$= \int_{x=0}^{2} \int_{y=0}^{3} \left(\frac{-y^3}{z} + 4xz^2\right) dx\,dy.$$

using parametric form $y = 3\sin\theta$, $z = 3\cos\theta$

$dy = 3\cos\theta\, d\theta$.

$$\int_{x=0}^{2} \int_0^{\pi/2} \left[\frac{-27\sin^3\theta}{3\cos\theta} + 4x(9\cos^2\theta)\right] 3\cos\theta \cdot d\theta \cdot dx.$$

$$= \int_0^2 \int_0^{\pi/2} (-27\sin^3\theta + 4x \cdot 27\cos^3\theta)\, d\theta\, dx$$

$$= \int_{x=0}^{2} \int_0^{\pi/2} (-27\sin^3\theta + 108\, x\cos^3\theta)\, d\theta\, dx$$

$$= \int_0^2 \left(-27\cdot\frac{2}{3} + 108x\cdot\frac{2}{3}\right) dx$$

$$= \frac{2}{3}\left[-27\cdot x + 108\cdot\frac{x^2}{2}\right]_0^2 = \frac{2}{3}\left[-27(2) + 54(2)^2\right] = (-36 + 144)$$

$= 108.$

Total Value

$= 0 + 0 + 0 + 72 + 108$

$= 180$

Green's theorem in a plane.

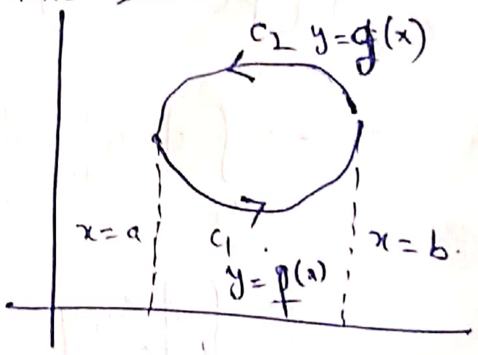
St: Let S be a closed ~~curv~~ region in xy-plane enclosed by a curve c. Let P & Q be continous and differentiable function of x & y in S.

then
$$\oint_c P\,dx + \phi\,dy = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy.$$

Proof:- Let us devide the curve c into 2 curves c_1 & c_2 between $x=a$ & $x=b$

$c_1 \rightarrow y = f(x)$

$c_2 \rightarrow y = g(x)$



& we have $g(x) > f(x)$

or $f(x) < g(x)$

~~f(x)~~ now

$$\iint_S \frac{\partial P}{\partial y}\,dx\,dy = \int_{x=a}^{b} \int_{f(x)}^{g(x)} \frac{\partial P}{\partial y}\,dx\,dy.$$

$$= \int_{x=a}^{b} \left[P(x,y)\right]_{f(x)}^{g(x)} dx\,\cancel{dy}$$

$$= \int_{x=a}^{b} \left(P(x,g) - P(x,f)\right) dx.$$

$$= \int_a^b P(x, g) - \int_a^b P(x,f)\,dx$$

$$= -\int_{c_2} P(x,g) - \int_{c_1} P(x,f)\,dx = -\oint_c P(x,y)\,dx = -\oint_c P\,dx \quad \text{—(1)}$$

Similarly $\iint_S \frac{\partial Q}{\partial x} \, dx\,dy = \int_C Q\,dy$ — ②

now from ① we have

$$\iint_S \frac{\partial P}{\partial y} \, dx\,dy = -\int_C P\,dx \quad — ①$$

So by Subtracting ① from ② we get

$$\int_C Q\,dy + P\,dx = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

or

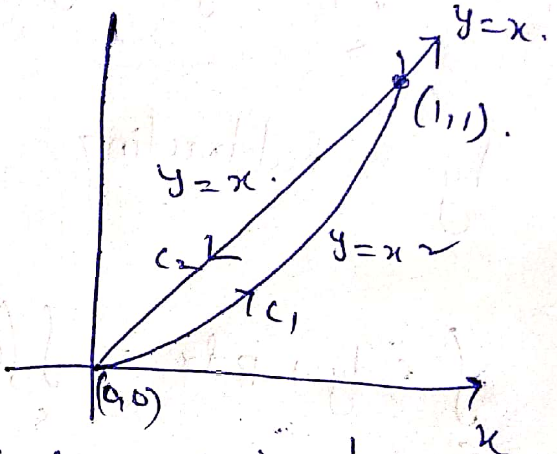
$$\int_C P\,dx + Q\,dy = \iint \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy.$$

Prob Verify Green's theorem for $\int_C (xy+y^2)\,dx + x^2\,dy$

where c is bounded by $y=x$ & $y=x^2$.

Sol here $\phi = xy+y^2$, $Q = x^2$.

$\left(\because \int \underline{P}\,dx + \underline{Q}\,dy \right)$.



now $\int \phi\,dx + Q\,dy =$

$$\int_{c_1} + \int_{c_2}$$

along c_1 we have $y=x^2$ & x varies from 0 to 1

$$\int_{c_1} = \int_{x=0}^{1} \left(x\cdot(x^2) + (x^2)^2 \right) dx + x^2\,d(x^2)$$

$$= \int_0^1 (x^3 + x^4 + 2x^3)\,dx$$

$$= \int_0^1 (3x^3 + x^4)\,dx$$

$$= \left[3\frac{x^4}{4} + \frac{x^5}{5} \right]_0^1$$

$$= \frac{3}{4} + \frac{1}{5} = \frac{19}{20}.$$

Similarly along C_2 we have $y=x$ & x varies

from 1 to 0

$$\int_{C_2} = \int_1^0 [x \cdot x + x^2]\, dx + x^2 \, dx$$

$$= \int_1^0 3x^2 \, dx = \cancel{3}\left[\frac{x^3}{\cancel{3}}\right]_1^0 = 0 - 1 = -1$$

Now $\int_C \phi\, dx + Q\, dy = \frac{19}{20} - 1 = \frac{-1}{20}.$

Now we verify $\iint_E \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\, dy$

$$= \iint_E \left[\frac{\partial}{\partial x}(x^2) - \frac{\partial}{\partial y}(xy + y^2)\right] dx\, dy$$

$$= \iint (2x - x - 2y)\, dx\, dy$$

$$= \int_{x=0}^{1} \int_{y=x^2}^{x} \cancel{2x -} x(x - 2y)\, dx\, dy$$

$$= \int_0^1 \left[xy - y^2\right]_{x^2}^{x} dx$$

$$= \int_0^1 [\cancel{x^2} - \cancel{x^2}] - [x^3 - x^4]\, dx$$

$$= \int_0^1 (x^4 - x^3)\, dx = \left[\frac{x^5}{5} - \frac{x^4}{4}\right]_0^1$$

$$= \left[\frac{1}{5} - \frac{1}{4}\right] = \frac{4-5}{20} = \frac{-1}{20}$$

Green's theorem is verified.