1 What I am assuming

Fundamental theorem (infinite version)
open subgroup $\Rightarrow$ closed subgroup
closed subgroup of finite index $\Rightarrow$ open subgroup
Some basic properties of tensors
Some Category theory

2 Etale Algeras

Setting: k is our base field. $k_s = \bar{k}_{sep}$. L/k a finite separable extension. $G = Gal(k_s/k)$, this acts on $Hom_k(L, k_s)$ by post-composition.

Lemma 2.1. *Let X be discrete space. Then, $G \times X \to X$ is continuous iff the stabilizer is open in G.*

Lemma 2.2. *The above left action of G on $Hom_k(L, k_s)$ is continuous and transitive, hence $Hom_k(L, k_s)$ as a G-set is isomorphic to the left coset space of some open subgroup in G. For L Galois over k this coset space is in fact a quotient by an open normal subgroup.*

Proof. $U = Stab(\phi)$.
L is separable. Hence action is transitive. $G \simeq Hom_k(L, k_s)/U$ □

Theorem 2.3. *.The contravariant functor mapping a finite separable extension L/k to the finite G-set $Hom_k(L, k_s)$ gives an anti-equivalence between the category of finite separable extensions of k and the category of finite sets with continuous and transitive left G-action. Here Galois extensions give rise to G-sets isomorphic to some finite quotient of G.*

Proof. Essentially Surjective: Fixed field of the stabilizer.
Fully Faithful:
$L \to M$
$Hom_k(M, k_s) \to Hom_k(L, k_s)$
$\phi \mapsto f(\phi)$
$f(\phi)(L) \subset \phi(M)$ □

Theorem 2.4. *The functor mapping a finite étale k-algebra A to the finite set $Hom_k(A, k_s)$ gives an anti-equivalence between the category of finite étale k-algebras and the category of finite sets with continuous left G-action. Here separable field extensions give rise to sets with transitive G-action and Galois extensions to G-sets isomorphic to finite quotients of G.*

Proof. Let $A = \prod L_i$ $A \to k_s \leftrightarrow \cup L_i \to k_s$ $A \to B \leftrightarrow \cup L_i \to M_j$ □

Lemma 2.5. *A finite dimensional commutative algebra over a field F is isomorphic to a direct product of finite field extensions of F if and only if it is reduced.*

Proof. (product $\Rightarrow$ reduced) (reduced $\Rightarrow$ product) Take indecomposable algebra. No non-trivial idempotents.
$x^n = x^{2n}y^n$ □

Theorem 2.6. *Let A be a finite dimensional commutative k-algebra. Then the following are equivalent:*
1. A is étale.
2. $A \otimes \bar{k}$ is isomorphic to a finite product of copies of $\bar{k}$; 3. $A \otimes \bar{k}$ is reduced.

Proof. $2 \Leftrightarrow 3$.
$1 \to 2$. L/k, $L = k[x]/(f) \implies L \otimes \bar{k} \simeq \bar{k}[x]/(f)$.
$2 \to 1$. $A' := A/I, A \to \bar{k} \leftrightarrow A' \to \bar{k}$.
$Hom_k(A, k) \simeq Hom_{\bar{k}}(A \otimes \bar{k}, \bar{k})$ $dim_{\bar{k}}(A \otimes \bar{k}) = dim_k(A)$ □