New universality classes for minimal spanning trees

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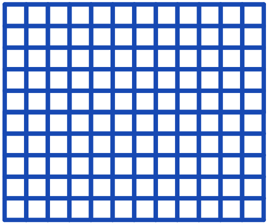
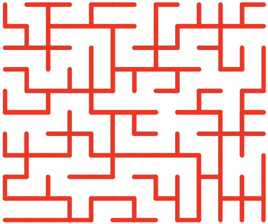
Based on joint work with Shankar Bhamidi

$G = (V, E)$ finite connected graph.
A spanning tree of G is a tree that is a subgraph of G with vertex set V.



G



A spanning tree of G

Minimal spanning tree (MST)

$G = (V, E)$ finite connected graph.

$b : E \to [0, \infty)$ a weight function.

An MST of G minimizes $\sum_{e \in T} b(e)$ over all spanning trees T of G.

MST is unique if $b(e) \neq b(e')$ whenever $e \neq e'$.

Many different settings:

- Underlying graph deterministic/random. Edge weights are i.i.d.
- Underlying graph is a random geometric graph. Edge weights are functions of the edge lengths.

Vast literature on random MSTs.

Not many results on the global geometry of MSTs.

Expected distance scaling in the MST

Disordered networks

$G = (V, E)$ finite connected graph.

Assign costs $\exp(\beta\varepsilon_e)$ to the edges $e \in E$, where ε_e, $e \in E$, are pairwise distinct.

For large enough β, for any $u, v \in V$, the path in G connecting u and v with the minimum total cost is the path that minimizes

$$\max_{e \in P} \varepsilon_e$$

among all paths P connecting u and v in G.

This is also the path connecting u and v in the MST of G constructed using the weights $\exp(\beta\varepsilon_e)$, $e \in E$.

Disordered networks

- Scaling of distances in the MST was studied through simulations in the early 2000s in the statistical physics literature.
  [Braunstein et al.; Physical Review Letters (2003), Braunstein et al.; International Journal of Bifurcation and Chaos (2007), Chen et al.; Physical review letters (2006), Wu et al.; Physical review letters (2006)].
- Suppose G is a scale-free random network on n vertices with degree exponent $\tau > 3$. Predicted behavior of scaling of distance between typical points:

$$l_{opt} \sim \begin{cases} n^{1/3}, & \text{if } \tau > 4, \\ n^{(\tau-3)/(\tau-1)}, & \text{if } \tau \in (3,4)\,. \end{cases}$$

- Different behavior compared to uniform spanning trees.

Known results for models exhibiting mean-field behavior

Diameter bound for the MST of the complete graph

- Addario-Berry, Broutin, Reed (RSA, 2009): $\mathrm{diam}(M_n^{\mathrm{comp}}) = O_P(n^{1/3})$.
- Nachmias, Peres (AOP, 2008):

  $\mathrm{diam}($ *largest component of critical Erdős-Rényi* $) = \Theta_P(n^{1/3})$

  $\implies \mathrm{diam}(M_n^{\mathrm{comp}}) = \Omega_P(n^{1/3})$.

Full scaling limit of the MST of the complete graph

Addario-Berry, Broutin, Goldschmidt, Miermont (AOP, 2017):

- $n^{-\frac{1}{3}} \cdot M_n^{\mathrm{comp}} \xrightarrow{\mathrm{d}} \mathcal{M}$ w.r.t. GHP topology.
- $\mathcal{M}$ almost surely compact, binary, and has Minkowski dimension 3.

Full scaling limit of the MST of random 3-regular graphs

Addario-Berry, Sen (PTRF, 2021):

- $n^{-\frac{1}{3}} \cdot M_n^{(3)} \xrightarrow{\mathrm{d}} 6^{1/3} \cdot \mathcal{M}$ w.r.t. GHP topology.
- The mass measure on $\mathcal{M}$ is non-atomic almost surely.

The heavy-tailed regime: $\tau \in (3, 4)$

The random graph model $\mathcal{G}_n(\boldsymbol{w})$

- n vertices labeled $1, \ldots, n$.
- Weights $w_1^{(n)} \geq w_2^{(n)} \geq \cdots \geq w_n^{(n)} > 0$.
- Place edges independently between i and j with respective probabilities

$$1 - \exp\Big(- w_i^{(n)} w_j^{(n)} / \sum_k w_k^{(n)} \Big) \quad \text{or} \quad \Big(w_i^{(n)} w_j^{(n)} / \sum_k w_k^{(n)} \Big) \wedge 1 \,.$$

- This is referred to as the rank-1 IRG.
- It is also the random graph associated with the multiplicative coalescent at a fixed time. This makes the scaling limit of the MST for this model the candidate for the universal MST scaling limit in the heavy-tailed regime under some general assumptions.

Assumptions

Fix $\tau \in (3,4)$. Let

$$\alpha = \frac{1}{\tau - 1} \quad \text{and} \quad \eta = \frac{\tau - 3}{\tau - 1}\,.$$

$\left(w_1^{(n)}, w_2^{(n)}, \ldots, w_n^{(n)}\right)$, $n \geq 1$, satisfy the following:

- Supercriticality: $\liminf_{n\to\infty} \dfrac{\sum_{j=1}^n (w_j^{(n)})^2}{\sum_{j=1}^n w_j^{(n)}} > 1\,.$
- Aldous-Limic condition: For each $i \geq 1$, there exists $\theta_i^* > 0$ such that

$$\lim_{n\to\infty} \frac{w_i^{(n)}}{n^\alpha} \cdot \left(\frac{n}{\sum_{j=1}^n (w_j^{(n)})^2}\right)^{1/2} = \theta_i^*.$$

- There exist constants $A_1, A_2 \in (0,\infty)$ such that for all $n \geq 2$ and $1 \leq i \leq n/2$,

$$A_1 \left(\frac{n}{i}\right)^\alpha \leq w_i^{(n)} \leq A_2 \left(\frac{n}{i}\right)^\alpha.$$

- For all $n \geq 2$, $w_n^{(n)} \geq A_1 \left(\log n\right)^{\frac{3}{2}} \cdot n^{-\frac{\eta}{4}}$.

F is a cumulative distribution function with support in $[0, \infty)$ such that for some $\beta_F > 4/\eta$, and $c_F \in (0, \infty)$,

$$\limsup_{x \downarrow 0} \left(x^{-\beta_F} F(x) \right) < \infty, \lim_{x \to \infty} x^{\tau - 1} [1 - F(x)] = c_F, \text{ and } \int_0^\infty x^2 F(dx) > \int_0^\infty x F(dx).$$

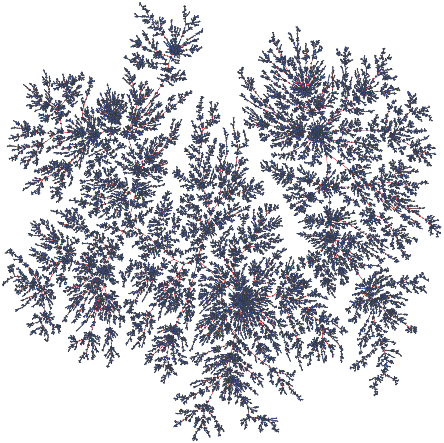
- $W_1, \ldots, W_n$ i.i.d. with cdf F, and let $w_1^{(n)} \geq \ldots \geq w_n^{(n)}$ be the corresponding ordered values.
- Let $w_i^{(n)} = [1 - F]^{-1} \big(i/(n+1) \big)$ for $1 \leq i \leq n$.

Result

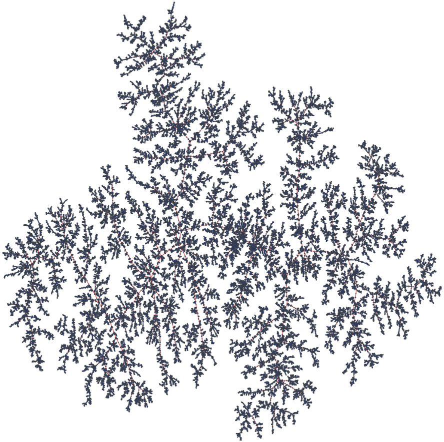
- Consider $\mathcal{G}_n(\boldsymbol{w})$ where the weight sequence satisfies the stated assumptions.
- Assign i.i.d. Uniform$[0, 1]$ weights to the edges.
- Let M_n be the MST of the component of the vertex 1.

Bhamidi and S. (2021+)

- There exists a random compact metric space $\mathcal{M}^{\boldsymbol{\theta}^*}$ whose law depends only on $\boldsymbol{\theta}^* := (\theta_i^*;\, i \geq 1)$ such that $n^{-\eta} \cdot M_n \xrightarrow{\mathrm{d}} \mathcal{M}^{\boldsymbol{\theta}^*}$ w.r.t. GH topology.
- A.s., every point in $\mathcal{M}^{\boldsymbol{\theta}^*}$ either has degree one (leaf), or two, or infinity (hub).
- A.s., both the set of leaves and the set of hubs are dense in $\mathcal{M}^{\boldsymbol{\theta}^*}$.
- A.s., the Minkowski dimension of $\mathcal{M}^{\boldsymbol{\theta}^*}$ equals $\frac{1}{\eta} = \frac{\tau-1}{\tau-3}$.



$w_i = 3(n/i)^{\alpha}$, where $\tau = 3.05$ and $n = 80000$.



$w_i = 3(n/i)^{\alpha}$, where $\tau = 3.95$ and $n = 80000$

Proof ingredients

- Consider bond percolation on $\mathcal{G}_n(\boldsymbol{w})$ with edge retention probability $c_n(1+\lambda/n^\eta)$ for $\lambda > 0$.
- Can prove a concentration inequality for the supremum of the centered breadth-first walk using a Klein-Rio bound. Building on this, can prove tail bounds on various quantitities associated with the component of the vertex 1. These bounds can in particular be used to show that the rest of the graph is "subcritcal."
- Need a diameter bound outside the component of the vertex 1. To do this, stochastically upper bound the component of a vertex by a three-layer branching process, and then obtain tail bounds on the heights of such branching processes. A key ingredient here is a technique recently developed by Addario-Berry (PTRF, 2019).

Proof ingredients (contd.)

- Diameter bound+critical scaling limit+some additional work would yield the existence of the scaling limit $\mathcal{M}^{\boldsymbol{\theta}^*}$.
- The claims about the degrees of points in $\mathcal{M}^{\boldsymbol{\theta}^*}$ can be proved using properties of inhomogeneous continuum random trees.
- Proof of the Minkowski bound uses discrete approximation.

Other standard heavy-tailed models

The MST of the giant component of the following models should be in the same universality class:

- Supercritcal simple random graph or configuration model with degree exponent $\tau \in (3, 4)$.
- A sequence of dense graphs converging to an L^p graphon where $p \in (2, 3)$.

Thank You.