Dynamics of the Mapping Class Group on the $PSL_2\mathbb{C}$ character-variety

Subhojoy Gupta (IISc)

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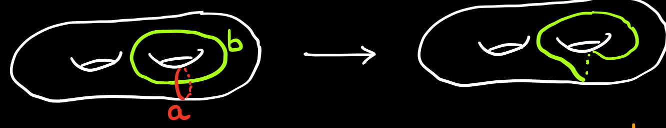
jt. work with Indranil Biswas, Mahan Mj & Junho Whang

S : a closed oriented surface of genus $g \geq 2$

Mapping Class Group $\quad MCG(S) = \mathrm{Homeo}^+(S) / \mathrm{Homeo}_0(S)$

orientation preserving homeomorphisms, upto homotopy e.g.



Dehn twist along a

$a \mapsto a$

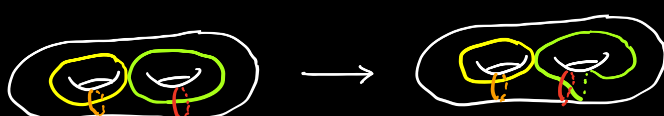
$b \mapsto ba$

$\varphi \in MCG(S)$ induces an automorphism

$$\varphi_* : \pi_1 S \to \pi_1 S$$

<u>Fact</u> $\quad MCG(S) \cong Out(\pi_1 S) = Aut(\pi_1 S)/Inn(\pi_1 S)$

inner
automorphism
$g \mapsto aga^{-1}$



"change of marking"

$PSL_2\mathbb{C}$ character-variety

$PSL_2\mathbb{C} = SL_2\mathbb{C}/\{\pm I\}$

$$\mathcal{X}(S) = \mathrm{Hom}(\pi_1 S, PSL_2\mathbb{C}) /\!\!/ PSL_2\mathbb{C}$$

representations
$\rho : \pi_1 S \to PSL_2\mathbb{C}$
up to conjugation
$g \cdot \rho = g \cdot \rho \cdot g^{-1}$

this quotient is Hausdorff for the subset of irreducible (simple) representations

"GIT quotient"
$\rho_1 \sim \rho_2$ iff
closures of their conjugation-orbits intersect.

Each equivalence class has a semisimple (or reductive) representation

$$\dim_{\mathbb{C}} \mathcal{X}(S) = 6g-6$$

- 2 connected components: those that lift to $SL_2\mathbb{C}$ and those that don't

$$\begin{tikzcd} & SL_2\mathbb{C} \arrow[d] \\ \pi_1 S \arrow[ur, dashed] \arrow[r, "\rho"] & PSL_2\mathbb{C} \end{tikzcd}$$

Why "character" variety ?

α : simple closed curve

$$\chi_\alpha : \mathcal{X}(S) \longrightarrow \mathbb{C}$$

$$\rho \longmapsto tr^2(\rho(\alpha))$$

trace function

is a well-defined function.

These generate the coordinate ring $\mathbb{C}[\mathcal{X}(S)]$.

Examples

1. Fuchsian representations

$\rho : \pi_1 S \longrightarrow PSL_2\mathbb{R} \subset PSL_2\mathbb{C}$

discrete faithful representation

$PSL_2\mathbb{R} = Isom^+(\mathbb{H}^2)$

& $\mathbb{H}^2/\rho(\pi_1 S)$ is a hyperbolic surface

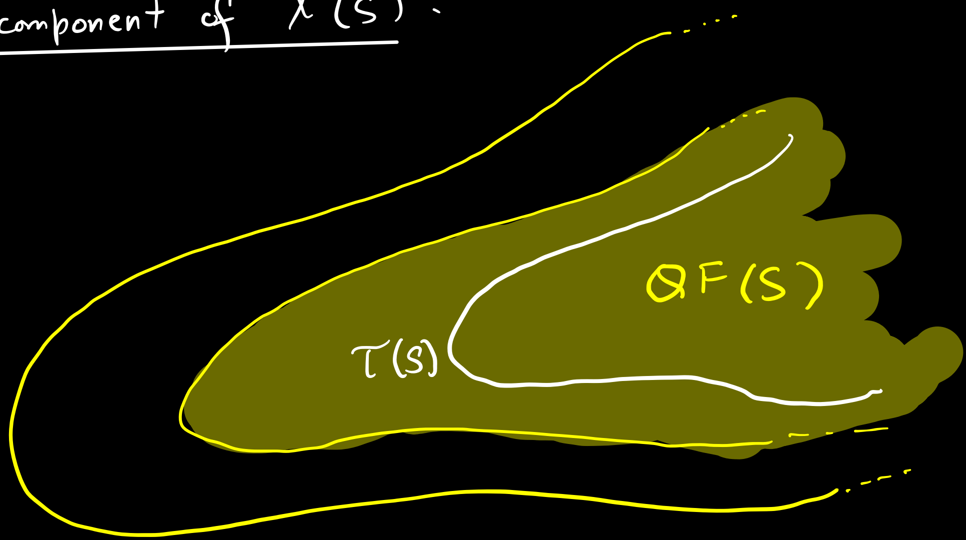
2. Quasi-Fuchsian representations

$\rho : \pi_1 S \to PSL_2\mathbb{C} = Isom^+(\mathbb{H}^3)$

- "perturbations" of Fuchsian reps
- discrete & faithful
- quotient $\mathbb{H}^3/\rho(\pi_1 S)$ is a

hyperbolic 3-manifold .

A component of $\mathcal{X}(S)$:



$QF(S)$: Quasi-Fuchsian reps
(open set)

$\mathcal{T}(S)$: Fuchsian reps
(Teichmüller space of S)

Mapping Class Group action on $\mathcal{X}(S)$:

$\varphi \in MCG(S)$

acting on

$\rho : \pi_1 S \to PSL_2\mathbb{C}$

$\rightsquigarrow$

$\rho \circ \varphi_*^{-1} : \pi_1 S \to PSL_2\mathbb{C}$

Q. What is the dynamics of this action?

What do the orbits look like?

Familiar example :

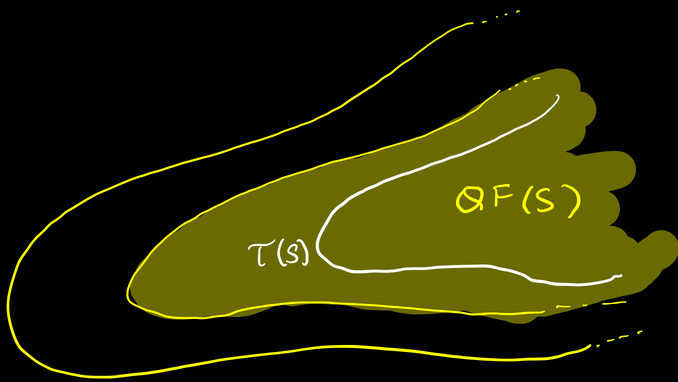
$MCG(S) \curvearrowright \mathcal{T}(S)$ Teichmüller space

is <u>properly discontinuous</u>

& the quotient $\mathcal{T}(S)/MCG(S)$

$\shortparallel$

$\mathcal{M}_g$ Riemann moduli space



FACT :

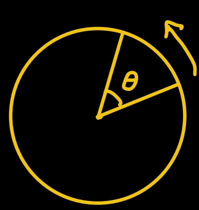
The action of MCG(S) is <u>properly discontinuous</u> on QF(S).

CONJECTURE (Goldman)

The action of MCG(S) outside QF(S) is ergodic.

ergodic = any MCG(S)-invariant subset has either full measure or zero measure.

e.g.



irrational rotation of the circle

$\mathcal{X}(S)$ has a measure coming from a symplectic volume form.

↳ symplectic form defined by Goldman Atiyah-Bott.

(Action of MCG(S) preserves this structure)

Ergodic $\Rightarrow$ almost every orbit is <u>dense</u>.

There's no example of an MCG(S)-orbit that is dense outside QF(S)!

Best result so far:

<u>Thm</u> (M. Lee)
MCG(S) does not act properly discontinuously on any open set larger than QF(S).

Also: Canary-Storm

"Motivation" for the conjecture :

Thm (Goldman)

The action of MCG(S) on the SU(2) character-variety is ergodic.

$$\mathrm{Hom}(\pi_1 S,\ SU(2)) / SU(2)$$

where $SU(2) = \left\{ \begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix} \,\middle|\, \alpha, \beta \in \mathbb{C},\ |\alpha|^2 + |\beta|^2 = 1 \right\}$

special unitary group

Proof uses the action of Dehn-twists & irrational rotations of the circle!

- generalized to any compact Lie group by Previte-Xia

Back to $PSL_2\mathbb{C}$ character-variety :

Q. When is a $MCG(S)$-orbit finite ?

When is it bounded ?

Thm (Biswas - G - Mj - Whang, closed surface case when $g \geq 2$)

The $MCG(S)$-orbit of $\rho : \pi_1 S \to PSL_2\mathbb{C}$ (semisimple) in $\chi(S)$ is

- finite iff the image of ρ is finite
- bounded iff ρ is conjugate to a rep'n into $PSU(2)$.

Remarks

- We actually work with $SL_2\mathbb{C}$.
- We in fact handle surfaces with boundary & genus $g > 0$, and we show the only exceptions are the "special dihedral rep'ns" for $g = 1$.

- False for the four-holed sphere.

  (finite orbits classified by Lisovyy – Tykhyy)

  $\Updownarrow$

  algebraic solutions of the Painlevé VI equation

  (Boalch, Dubrovin, Hitchin, Mazzocco ...)

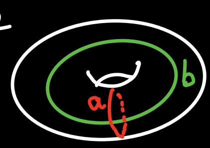
- False for the $PSL_n\mathbb{C}$ character-variety when $n \gg 0$.

  (Biswas – Koberda – Mj – Santharoubane)

$g = 1$

(Counter-) example: dihedral representation

T^2



$$\rho : \pi_1(T^2) \longrightarrow PSL_2\mathbb{C}$$

$$a \longmapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$b \longmapsto \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \qquad |\lambda| > 1$$

infinite image, but MCG-orbit is finite!

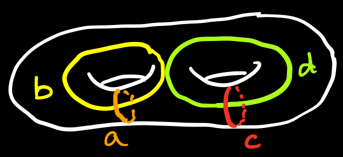
Dehn-twists around b:

$$\left.\begin{aligned} ab^k &\longmapsto \begin{pmatrix} 0 & \lambda^{-k} \\ -\lambda^k & 0 \end{pmatrix} \\ b &\longmapsto \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \end{aligned}\right\} \text{conjugate to the above rep'n!}$$

$$\begin{pmatrix} \lambda^{-k/2} & 0 \\ 0 & \lambda^{k/2} \end{pmatrix}$$

"special dihedral" : $\rho|_{S \setminus a}$ is diagonal .
$g=1$, $n>0$

Example $(g \geq 2)$

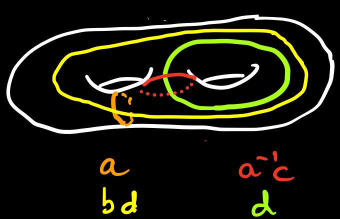


special dihedral

$\rho(a) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$ where $|\lambda| > 1$ $\quad \rho(c) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\rho(b) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \rho(d) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\downarrow$



$\rho(a') = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \quad \rho(c') = \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{pmatrix}$

$\rho(b) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \rho(d') = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Now Dehn twists around c' will produce an infinite orbit

Key ingredient of proof had number-theory motivations :

Thm (Patel - Shankar - Whang)

Let S have genus $g \geqslant 1$ and $n \geqslant 0$ boundary components.

Then if $\rho : \pi_1 S \to SL_2\mathbb{C}$ has finite order monodromy along every simple closed curve, then the image of ρ is finite.

(first step of the proof : $tr(\rho(\gamma))$ is an algebraic integer for γ a s.c.c.)

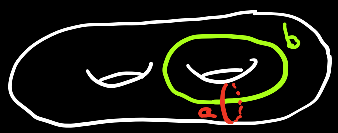
A flavour of the proof :

"Analysis of Dehn twists"

Suppose $\rho : \pi_1 S \to PSL_2\mathbb{C}$ has a finite MCG-orbit.

Want to show : $\rho(a)$ has finite order for any simple closed curve a.

Consider the case when a is non-separating.



Pick b that intersects a once.

Dehn twists about a

$\rightsquigarrow \{tr(\rho(a^k b)) \mid k \in \mathbb{Z}\}$

is a finite set

- If $\rho(a) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$ $\quad \lambda \in \mathbb{C}^*$

  & $\rho(b) = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}$

  then $tr(\rho(a^k b)) = \lambda^k b_1 + \lambda^{-k} b_4$

  Hence either λ is a root of unity $\rightsquigarrow$ done

  or $\rho(b) = \begin{pmatrix} 0 & b_2 \\ b_3 & 0 \end{pmatrix}$

  $\rightsquigarrow$ reduces to the "special dihedral" case, already handled.

- If $\rho(a) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$

  then $tr(\rho(a^k b)) = b_1 + b_4 + k t b_3$

  Hence either $t = 0 \rightsquigarrow$ done

  or $b_3 = 0 \rightsquigarrow \rho(b) = \begin{pmatrix} b_1 & b_2 \\ 0 & b_3 \end{pmatrix}$ & ρ is upper-triangular and hence diagonal

If a is separating, then induction on the complexity of surfaces, with "base cases" of $S_{1,1}$ and $S_{1,2}$ handled separately.

Thanks for listening!


τ(S)
∂F(S)