Graph polynomials from Lie algebras

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* Based on a joint work with Kartik Singh.

Some background from Lie algebras:

Lie algebra: A Lie algebra $\mathfrak{g}$ is a v.sp / $\mathbb{C}$ with the Lie product $[\ ]: \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ st

① $[x\ y] = -[y\ x]$ $\forall\ x, y \in \mathfrak{g}$ (skew-symm)

② $[x\ [y\ z]] = [[x\ y]\ z] + [y\ [x\ z]]$ (Jacobi identity)
$\forall\ x, y, z \in \mathfrak{g}$

Ideal of $\mathfrak{g}$: A subspace I of $\mathfrak{g}$ is said to be an ideal if $[x\ y] \in I$, $\forall\ x \in I$ and $y \in \mathfrak{g}$.

Quotient algebra: If $I \trianglelefteq \mathfrak{g}$, then $\mathfrak{g}/I$ is naturally a Lie algebra with the Lie product $[x+I, y+I] := [x\ y]+I$
$\forall\ x, y \in \mathfrak{g}$.

Free Lie algebra: Let X be a set and let $\mathbb{C}\langle X\rangle$ be the free associative algebra generated by X.

$\mathbb{C}\langle X\rangle$ is a Lie algebra with the Lie bracket $\boxed{[a\ b] := ab - ba}$ for all $a, b \in \mathbb{C}\langle X\rangle$. We usually denote $\boxed{[\mathbb{C}\langle X\rangle]}$ to emphasize the Lie algebra structure.

$$X \longrightarrow [\mathbb{C}\langle X\rangle]$$

$$\mathcal{FL}(X) = \text{the Lie-subalgebra generated by } X \text{ in } [\mathbb{C}\langle X\rangle]$$

Free Lie alg generated by X

<u>Remark</u>: $[\mathbb{C}\langle X\rangle]$ is in general a very big Lie algebra containing X. For example $\mathcal{FL}(X)$ only contains the Lie monomials formed from X, but $\mathbb{C}\langle X\rangle$ contains all possible monomials.

Partially Commutative Lie alg:

Let G be a $\boxed{\text{finite simple connected graph}}$ with the vertex set I and the edge set E.

Alphabets: $\{e_i \mid i \in I\}$

relations: $I(G) = \langle [e_i\, e_j] \mid \{i, j\} \notin E \rangle$

$$\boxed{\mathcal{L}_G = \frac{\mathcal{FL}(I)}{I(G)}}$$

the partially commutative Lie algebra associated with the graph G

Grading on $\mathcal{L}_G$: Let $\{\alpha_1, \ldots, \alpha_n\}$ be the standard basis of $\mathbb{Z}^n$ and let $Q_+ := \mathbb{Z}_+ - \mathrm{span}\{\alpha_1, \ldots, \alpha_n\}$.

$$\boxed{gr(e_i) = \alpha_i \quad \text{for } 1 \le i \le n}$$

gives us a grading on $\mathcal{L}_G$

$$\boxed{\mathcal{L}_G = \bigoplus_{\alpha \in Q_+} \mathfrak{g}_\alpha}$$

Note: $\boxed{\text{Ex}}$

$$\mathfrak{g}_\alpha = \mathrm{Span}_{\mathbb{C}} \left\{ [e_{i_1}[e_{i_2} \cdots [e_{i_{k-1}}, e_{i_k}]] \,\middle|\, \sum_{r=1}^{k} \alpha_{i_r} = \alpha \right\}$$

right normed Lie words

If $\alpha = \sum_{i=1}^{n} k_i \alpha_i$ then $k_i = \#\, e_i$ appears in the right normed Lie word $[e_{i_1}[e_{i_2} \cdots [e_{i_{k-1}} e_{i_k}] \ldots]$

$$\Delta_+ = \{ \alpha \in Q_+ \mid \mathfrak{g}_\alpha \neq 0 \}$$ ← the set of roots of $\mathfrak{g}_G$

$$m_\alpha = \dim \mathfrak{g}_\alpha \quad \text{for } \alpha \in \Delta_+$$ ← root multiplicity

Denominator identity: Let $\{x_i\}_{i=1}^n$ be n-commuting variables.

We have

multivariate independence poly

$$\boxed{\sum_{S \subseteq \Omega} (-1)^{|S|} \prod_{i \in S} x_i = \prod_{\alpha \in \Delta_+} (1 - x^\alpha)^{\dim \mathfrak{g}_\alpha}}$$

where

$$\Omega = \{ S \subseteq V \mid S \text{ is } \boxed{\text{indpt}} \text{ subset of } V \}$$

there is no edge between vertices of S

By def, ϕ and singeltons are part of Ω.

Example: 

P_4.

$$\Omega = \{ \phi, \{1\}, \{2\}, \{3\}, \{4\}, \{1,3\}, \{2,4\} \}$$

$$\boxed{\text{SUM SIDE} = 1 - x_1 - x_2 - x_3 + x_1 x_3 + x_2 x_4}$$

PRODUCT SIDE can be calculated for this example. But in general, it is not that easy to compute.

The universal enveloping algebra of $\mathcal{L}_G$

A pair $(\mathcal{U}(\mathcal{L}_G), \iota)$ is said to be the universal enveloping alg of $\mathcal{L}_G$ if given (A, f) where A is an asso. alg and $f : \mathcal{L}_G \to [A]$ is a Lie alg hom.,

$\exists$ $\bar{f} : \mathcal{U}(\mathcal{L}_G) \to A$, asso. alg homom.

St the following diagram commute:

$$\begin{tikzcd}
\mathcal{U}(\mathcal{L}_G) \arrow[dr, "\bar{f}"] & \\
\mathcal{L}_G \arrow[u, "\iota"] \arrow[r, "f"'] \arrow[ur, phantom, "2"] & {[A]}
\end{tikzcd}$$

← asso. alg

← ass. alg hom

Lie alg hom. → ι

Lie alg hom

↑ asso. alg treated as a Lie alg with the Lie bracket $[ab] = ab - ba$

Ex: $\mathcal{U}(\mathcal{L}_G) = \mathbb{C}\langle I \rangle \big/ \langle e_i e_j = e_j e_i \mid \{i,j\} \notin E \rangle$

$gr(e_i) = \alpha_i$ gives grading on $\mathcal{U}(\mathcal{L}_G)$

The Hilbert series of $\mathcal{U}(\mathcal{L}_G)$:

Write

$$\mathcal{U}(\mathcal{L}_G) = \bigoplus_{\alpha \in Q_+} \mathcal{U}(\mathcal{L}_G)_\alpha .$$

where $\mathcal{U}(\mathcal{L}_G)_\alpha = \mathrm{Span}_{\mathbb{C}} \left\{ e_{i_1} \cdots e_{i_k} \;\middle|\; \sum_{r=1}^{k} \alpha_{i_r} = \alpha \right\}$

The Hilbert series of $\mathcal{U}(\mathcal{L}_G)$ defined to be

$$H_G(\underline{x}) := \sum_{\alpha \in Q_+} \dim \mathcal{U}(\mathcal{L}_G)_\alpha \, x^\alpha$$

Theorem: [Bourbaki]

$$\boxed{H_G(\underline{x}) = \frac{1}{\prod_{\alpha \in \Delta_+} (1 - x^\alpha)^{\dim \mathfrak{g}_\alpha}}}$$

Pf: follows from PBW theorem.

Theorem: [Cartier-Foata, 1969] [Xavier Viennot, 89']

$$\boxed{\mathcal{H}_G(\underline{x}) = \frac{1}{\sum_{S \subseteq \Omega} (-1)^{|S|} \left(\prod_{i \in S} x_i \right)}}$$

Remark: Proof of this theorem is obtained by defining a very explicit involution on $\mathcal{U}(\mathcal{L}_G)$. See Xavier Viennot lecture series on Heaps theory.

Chromatic polynomials of graphs from Lie algebras:

$$\left(\sum_{S \subseteq \Omega} (-1)^{|S|} \left(\prod_{i \in S} x_i \right) \right)^{-q} = \sum_{\alpha \in Q_+} (-1)^{|\alpha|} \pi_\alpha^G(-q) x^\alpha$$

the Hilbert series of the algebra $\mathcal{U}(\mathcal{L}_G)^{\otimes q}$

α-Multi-coloring Chromatic polynomial of G

α-multi-coloring: Let $q \in \mathbb{N}$ and $\alpha = \sum_{i=1}^{n} k_i \alpha_i \in Q_+$

A α-multi-coloring is a map $f : I \to 2^{[q]}$ st

$2^{[q]}$: Power set of $\{1, \dots, q\}$

for any vertex $i \in I$, $|f(i)| = k_i$

and for any adjacent vertices i & $j \in I$ of G we have $f(i)$ & $f(j)$ are disjoint.

$$\boxed{\pi_\alpha^G(q) = \#\ \alpha\text{-multi-coloring of } G}$$

Ex: $$\boxed{\pi_\alpha^G(q) = \frac{\pi_{G(\alpha)}(q)}{\alpha!}}$$

$\pi_{G(\alpha)}$: the usual chromatic polynomial

where $\alpha! = \prod_{i=1}^{n} k_i!$

and $G(\alpha)$ is the graph obtained from G as follows: for each $i \in I$, take a clique of size k_i and connect all the edges of i^{th} clique and all the edges of j^{th} clique if $\{i, j\} \in E$.

Now using $\sum = \prod$ (denominator identity)

we can get an expression of $\pi_\alpha^G(-q)$ in terms of multiplicities of $\mathcal{L}_G$.

Using Möbius inversion formula, we get

$$\dim \mathfrak{g}_\alpha = \sum_{\ell \mid \alpha} \frac{\mu(\ell)}{\ell} \left| \pi^G_{\frac{\alpha}{\ell}}(q)[q] \right|$$

the co-eff of q in $\pi_\alpha^G(q)$

Examples: $G = K_n$ – Complete graph

For $\alpha = \sum_{i=1}^{n} k_i \alpha_i$, we have

$$\pi_\alpha^G(q) = \binom{q}{k_1}\binom{q-k_1}{k_2} \cdots \binom{q-(k_1+\cdots+k_{n-1})}{k_n}$$

$G = T_n$ – tree with n-vertices

$$\pi_\alpha^G(q) = \binom{q-k_1'}{k_1}\binom{q-k_2'}{k_2} \cdots \binom{q-k_{(n-1)}'}{k_{n-1}}\binom{q}{k_n}$$

where $I = \{1, \dots, n\}$ ordered such a way that the vertex i is a leaf of the subgraph of $\langle i, i+1, \dots, n \rangle$ and i' is the unique vertex in $\langle i, i+1, \dots, n \rangle$ that is adjacent to i.

Matching Polynomial of a graph:

$$\mu(G, x) = \sum_{k \geq 0} (-1)^k m_k(G)\, x^k$$

where $m_k(G) = \#$ matchings with k-edges

(a matching is a subset of E such that any two edges from it are not adjacent)

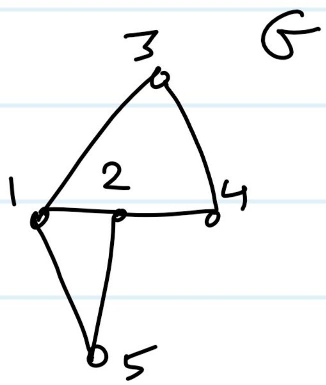
Godsil Constructed a rooted tree (T, r)

(Godsil path tree) associated to G st

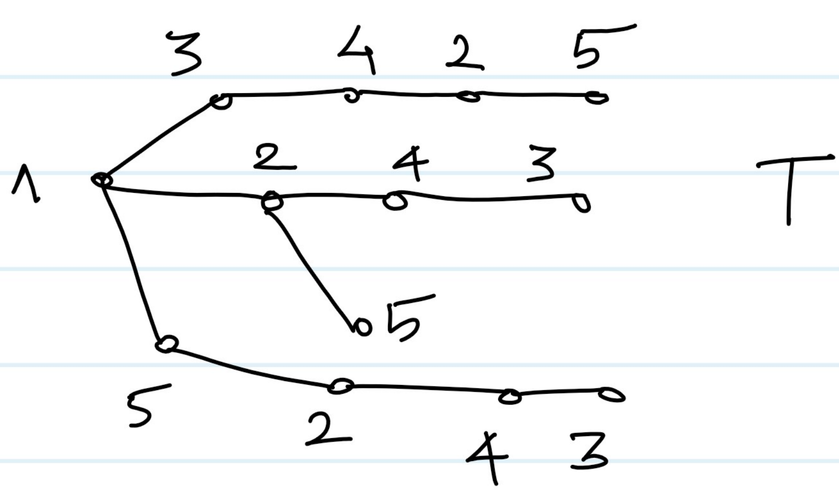
$$\boxed{\frac{\mu(G-u,x)}{\mu(G,x)} = \frac{\mu(T-r,x)}{\mu(T,x)}}$$

where $G-u$ is the graph obtained from G by removing the vertex u (and the edges adjacent to u)

Godsil tree: Example



$u = 1$



vertices of T: all paths strating from u

edges of T: the edges are the strict inclusions

<u>Remark</u>: $\mu(T, x)$ is given by the characteristic polynomial of the adjacent matrix of T

$$\boxed{\frac{\mu(T-r, x)}{\mu(G-u, x)} = \frac{\mu(T, x)}{\mu(G, x)}}$$

$\Rightarrow$ $\mu(G, x)$ divides $\mu(T, x)$.

Using this Godsil proved $\mu(G,x)$ has only real roots and he also obtained sharp bounds on the roots of $\mu(G, x)$

$G \rightsquigarrow L_G$ - Line graph of G

$$\mu(G, x) = I(L_G, x)$$

independence polynomial of L_G

Remark: L_G is claw-free graph

Chudnovsky and Seymour prove indep poly of claw-free free graphs have real rootedness property in 2007.

Real rootedness $\Rightarrow$ log-concavity of the co-eff $\Rightarrow$ unimodality of the co-eff.

Motivated from this F. Bencs (2018),

Constructed a stable-path tree T' asso. to G such that,

$$(*) \quad \boxed{\frac{I(G-u, x)}{I(G, x)} = \frac{I(T'-r, x)}{I(T', x)}}$$

Using this F. Bencs constructed many interesting families of graphs whose indp poly has real-rooted property.

(*) Can be generalized easily to multi-variate version.

We have,

$$(**)\quad \boxed{\frac{I(G-u,\underline{x})}{I(G,\underline{x})} = \frac{I(T'-r,\underline{x})}{I(T',\underline{x})}}$$

the LHS is the character of a highest weight rep of $\mathfrak{L}_G$

Question: $(**)$ Can it be viewed as the identity that follows from the isomorphism of reps of $\mathfrak{L}_G$?

Ans: Yes!

$\mathcal{M}(G)$ = Cartier-Foata monoid (or trace monoid) of G

$$\mathrm{span}_{\mathbb{C}}\,\mathcal{M}(G) = \mathcal{U}(\mathcal{L}_G)$$

$$\mathcal{M}_u(G) = \{\, w \in \mathcal{M}(G) \mid IA(w) = \{u\} \,\}$$

$IA(w)$ = the initial alphabets of w.

$$V_u = \mathrm{span}_{\mathbb{C}}\,\mathcal{M}_u(G) \leftarrow$$ the highest weight rep of $\mathcal{L}_G$

The Construction of Stable-free path T' asso. to G:

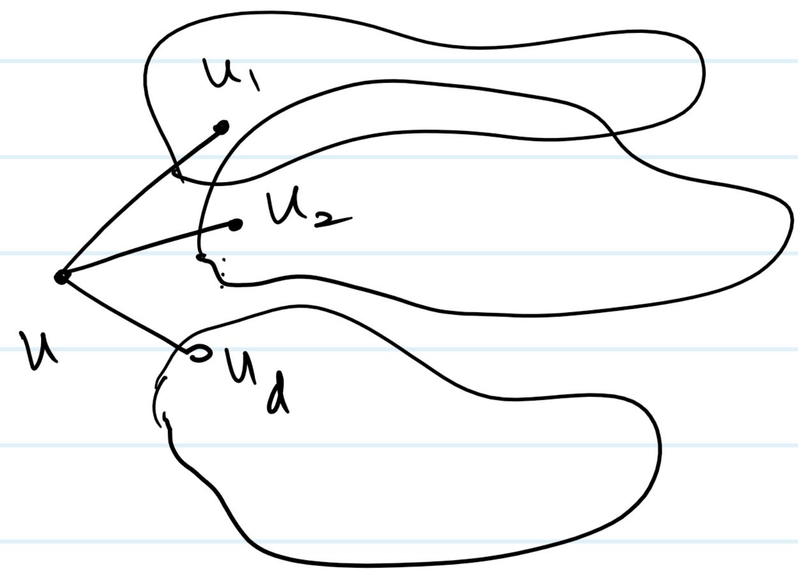
$I = \{1, \dots, n\}$ fix $u \in I$

nbd of u $\rightarrow$ $N_G(u) = \{u_1, \dots, u_d\}$

We recursively construct a rooted tree (T', r) and a surjective map $\ell_G : I(T') \to I$

If $d=0$, then G is single vertex, take $T' = G$

If $d \geq 1$

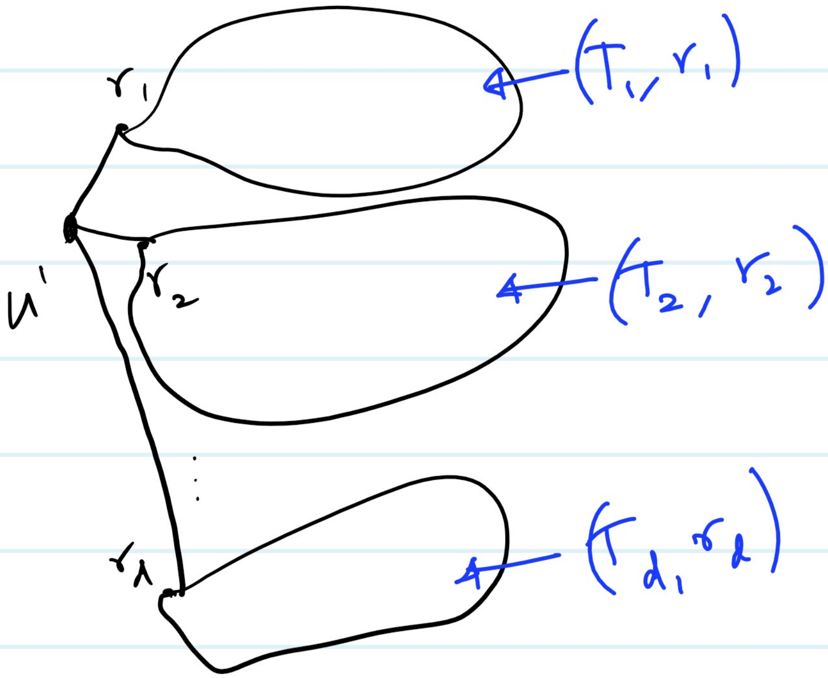


Write $N_G(u) = \{u_1 < \cdots < u_d\}$

for $u_i \in N_G(u)$, let G_i be the connected component of $G[I \setminus \{u, u_1, \ldots, u_{i-1}\}]$ containing u_i and we take the induced ordering on the vertices of G_i. By induction we have

$$(T_i, r_i) \text{ and } \ell_{G_i} : I(T_i) \to I(G_i)$$

st $\ell_G(r_i) = u_i$



Now take $\bigsqcup_{i=1}^{d} (T_i, r_i)$ and u' vertex and connect u' with the vertices r_i $(1 \le i \le d)$. ℓ_G - naturally designed.
