Maximum Minimal Defensive Alliance in Graph

Ajinkya Gaikwad   Soumen Maity

Indian Institute of Science Education and Research, Pune, India

Introduction

Definition

A non-empty set $S \subseteq V$ is a defensive alliance in $G = (V, E)$ if $d_S(v) + 1 \geq d_{S^c}(v)$ for all $v \in S$.

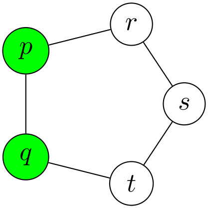


Figure 1: $\{p, q\}$ is a defensive alliance in G but $\{p\}$ is not a defensive alliance.

Motivation

In real life, an alliance is a collection of people, groups, or states such that the union is stronger than individual. The alliance can be either to achieve some common purpose, or to protect against attack. This motivates the definitions of defensive alliances in graphs.

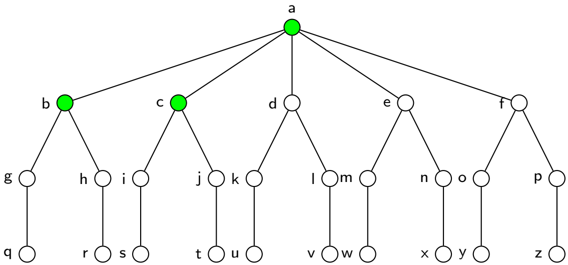
DEFENSIVE ALLIANCE
Input: An undirected graph $G = (V, E)$ and an integer $1 \leq \ell \leq |V(G)|$.
Question: Is there a defensive alliance $S \subseteq V(G)$ such that $|S| \leq \ell$?

It is known that the problems of finding a small defensive alliance is NP-complete [4].

Minimal Defensive Alliance

Definition

A defensive alliance S is *minimal* if no proper subset of S is a defensive alliance.



MAXIMUM MINIMAL DEFENSIVE ALLIANCE (MMDA)
Input: An undirected graph $G = (V, E)$ and an integer $k \geq 2$.
Question: Is there a minimal defensive alliance $S \subseteq V$ such that $|S| \geq k$?

Our Results

Known Results: Maximum Minimal Defensive Alliance is NP-complete, even for graphs of degree 3 or 4 [1].
Our Results:

- Maximum Minimal Defensive Alliance problem is polynomial time solvable on trees
- MMDA is W[1]-hard when parameterized by the treewidth of the graph.
- MMDA is fixed-parameter tractable (FPT) when parameterized by the neighbourhood diversity.
- Given a vertex $r \in V(G)$, deciding if G has a minimal defensive alliance containing vertex r is NP-complete.

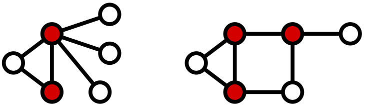
Parameterized Complexity

Main Idea: Instead of expressing the running time as a function $T(n)$ of n, we express it as a function $T(n, k)$ of input size n and some parameter k of the input.

What can be parameter k?

- the size k of the solution that we are looking for?
- the maximum degree of the input graph
- the neighbourhood diversity of the input graph
- the treewidth or clique width of the input graph

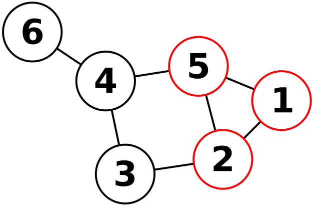
Parameterized Complexity: Vertex Cover



MINIMUM VERTEX COVER
Input: A graph G and an integer k.
Question: Is it possible to cover the edges with k vertices?

Complexity: NP-complete
Complete enumeration: $O(n^k)$
Parameterzied Complexity: $O(2^k n^2)$ algorithm exists; FPT

Parameterized Complexity: Clique



MAXIMUM CLIQUE
Input: A graph G and an integer k.
Question: Does G contain a clique of size k?

Complexity: NP-complete
Complete enumeration: $O(n^k)$
Parameterzied Complexity: $k^2 n^{O(k)}$; W[1]-hard (probably not FPT)

Some basic observations about MMDA

Lemma

If S is a minimal defensive alliance of size at least two in G, then S is connected and S cannot contain a vertex of degree one.

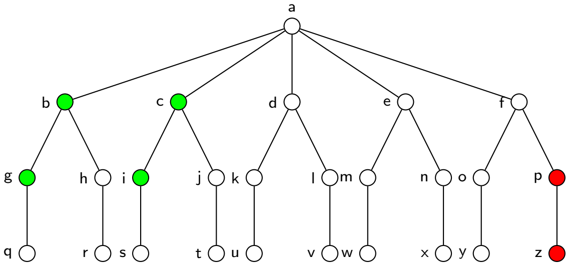


Figure 2: $\{b, g, c, i\}$ is not a minimal defensive alliance; $\{p, z\}$ is also not a minimal defensive alliance.

Some basic observations about MMDA

Lemma

If a non-empty set $S \subseteq V(G)$ is connected and each $v \in S$ is marginally protected, then S is a globally minimal defensive alliance in G.

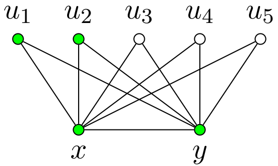


Figure 3: $S = \{x, y, u_1, u_2\}$ is a minimal defensive alliance in G.

Polynomial Time Algorithm on Trees

Lemma

Let $T = (V, E)$ be a tree with n vertices, with $n \geq 2$. A set $S \subseteq V$ is a minimal defensive alliance of size at least 2 in T if and only if S is connected and every vertex of S is marginally protected.

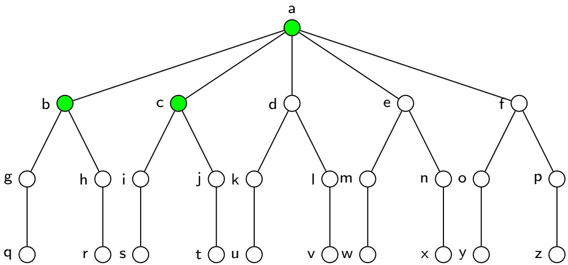
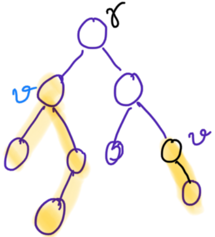


Figure 4: Tree T.

Polynomial Time Algorithm on Trees

We define different possible states of a vertex v as follows:

- 0: vertex v is not in the solution.
- 1_c: vertex v is in the solution and it is marginally protected by its children only; the parent of v is not in the solution.
- 1_p : vertex v is in the solution and it is marginally protected by its parent and children; the parent of v is in the solution.



$A_v(s)$= the size of the largest minimal defensive alliance of the subtree rooted at v when the state of v is s; if no minimal defensive alliance exists, we put $A_v(s) = -\infty$. Final goal is to compute $\max\left\{A_r(0), A_r(1_c)\right\}$ where r is the root of T.

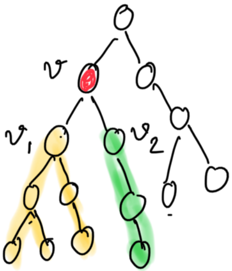
Poly Time Algorithm on Trees

Leaf node: For a leaf node v, we have $A_v(0) = 0$ and $A_v(1_c) = A_v(1_p) = -\infty$.

Non-leaf node: Let v be a non-leaf node with the set of children $\mathcal{C}_v = \{v_1, v_2, \ldots, v_d\}$.

Case 1 Let the state of v be 0. That is, v is not included in the solution. As any minimal defensive alliance must be connected and v is not in the solution, we try all its children and pick the best:

$$A_v(0) = \max_{1 \le i \le d} \Big\{ A_{v_i}(0), A_{v_i}(1_c) \Big\}.$$



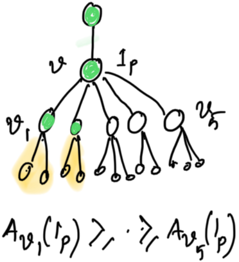
Polynomial Time Algorithm on Trees

Case 2: Let the state of v be 1_p or 1_c. Let $(v_1, v_2, \ldots, v_d)$ be a descending ordering of $\mathcal{C}_v$ according to values $A_{v_i}(1_p)$, that is,

$$A_{v_1}(1_p) \geq \ldots \geq A_{v_d}(1_p).$$

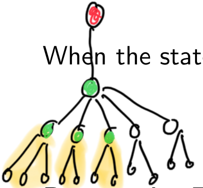
Let $\mathcal{C}_{v,\lceil \frac{d-2}{2} \rceil} = \left\{ v_1, v_2, \ldots, v_{\lceil \frac{d-2}{2} \rceil} \right\}$ and $\mathcal{C}_{v,\lceil \frac{d}{2} \rceil} = \left\{ v_1, v_2, \ldots, v_{\lceil \frac{d}{2} \rceil} \right\}$. When the state of v is 1_p, we have

$$A_v(1_p) = 1 + \sum_{x \in \mathcal{C}_{v,\lceil \frac{d-2}{2} \rceil}} A_x(1_p).$$



When the state of v is 1_c, we have

$$A_v(1_c) = 1 + \sum_{x \in \mathcal{C}_{v,\lceil \frac{d}{2} \rceil}} A_x(1_p).$$



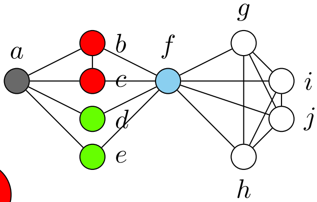
Root node: For the root node r, the state 1_p is not a valid state for the root node as root node has no parent. When the state of v is 1_c, we have $A_r(1_c) = 1 + \sum_{x \in \mathcal{C}_{v,\lfloor \frac{d}{2} \rfloor}} A_x(1_p)$. The answer we seek is $\max\{A_r(0), A_r(1_c)\}$.

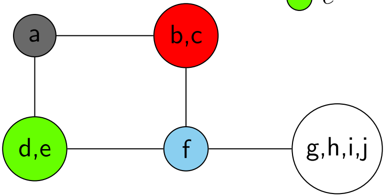
FPT algorithm parameterized by Neighbourhood Diversity

We say two vertices u and v have the same *type* if and only if $N(u) \setminus \{v\} = N(v) \setminus \{u\}$.

Definition

The neighbourhood diversity of a graph $G = (V, E)$, denoted by $\texttt{nd}(G)$, is the least integer w for which we can partition the set $V(G)$ of vertices into w classes, such that all vertices in each class have the same type [2].





FPT Algorithm Parameterized by Neighbourhood Diversity

Maximize $\sum_{C_i \in I_1 \cup I_2} x_i$

Subject to

$x_i = 1$ for all $i \ : \ C_i \in I_1$;

$x_i \in \{2, \ldots, |C_i|\}$ for all $i \ : \ C_i \in I_2$

$$1 + \sum_{C_i \in N_H(C_j) \cap (I_1 \cup I_2)} 2x_i \geq \sum_{C_i \in N_H(C_j)} n_i, \quad \text{for all } C_j \in \mathcal{I},$$

$$\sum_{C_i \in N_H[C_j] \cap (I_1 \cup I_2)} 2x_i \geq \sum_{C_i \in N_H[C_j]} n_i, \quad \text{for all } C_j \in \mathcal{C},$$

for $j = 1$ to k;

$$1 + \sum_{C_i \in N_H(C_j) \cap (I_1 \cup I_2)} 2x'_i < \sum_{C_i \in N_H(C_j)} n_i, \forall\ \mathbf{x}'_i \in R_j; C_j \text{ is an independent class}$$

$$\sum_{C_i \in N_H[C_j] \cap (I_1 \cup I_2)} 2x'_i < \sum_{C_i \in N_H[C_j]} n_i, \forall\ \mathbf{x}'_i \in R_j; C_j \text{ is a clique class}$$

Theorem

The MAXIMUM MINIMAL DEFENSIVE ALLIANCE *problem is fixed-parameter tractable when parameterized by the neighbourhood diversity.*

Rooted Minimal Defensive Alliance

Rooted Minimal Defensive Alliance
Input: An undirected graph $G = (V, E)$, a vertex $r \in V$.
Question: Does there exist a minimal defensive alliance S, such that $r \in S$?

Theorem

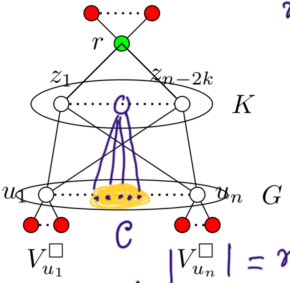
The Rooted Minimal Defensive Alliance problem is NP-complete.

Proof: It is easy to see that Rooted Minimal Defensive Alliance is in NP. We prove it is NP-hard by giving a polynomial time reduction from Clique on regular graphs to Rooted Minimal Defensive Alliance. Let $I = (G, k)$ be an instance of Clique, where G is an s-regular graph. We construct an instance $I' = (G', r)$ of Rooted Minimal Defensive Alliance as follows:

Rooted Minimal Defensive Alliance

$\Rightarrow$ C is a clique in G then $S = C \cup K \cup \{r\}$ is a minimal DA

$|V_r^{\square}| = n - 2k$



$|V_{u_n}^{\square}| = n - s - 2$

G'

$\Leftarrow$ G' has a minimal DA S. $K \subseteq S$

Figure 5: Reduction from s-regular clique to rooted minimal defensive alliance containing r.

We claim that G has a k-clique if and only if G' admits a minimal defensive alliance containing r.

Treewidth

Treewidth: A measure of how "tree-like" the graph is. (Introduced by Robertson and Seymour [3]).

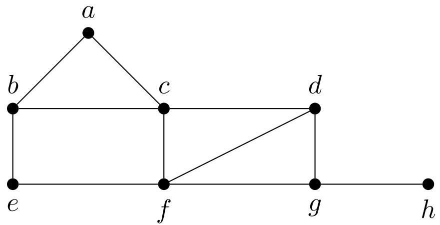
Tree decomposition: Vertices are arranged in a tree structure satisfying the following properties:

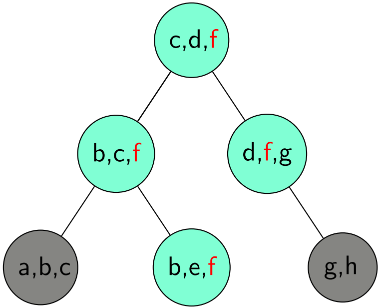
- If u and v are neighbors, then there is a bag containing both of them.
- For every vertex v, the bags containing v form a connected subtree.

Width of the decomposition: largest bag size -1.

treewidth: width of the best decomposition.

Fact: tw $= 1$ iff graph is a forest





W[1]-hardness Parameterized by Treewidth

Definition

Let A and B be two parameterized problems. A parameterized reduction from A to B is an algorithm that, given an instance of (x, k) of A, output an instance (x', k') of B such that

1. (x, k) is a yes-instance of A if and only if (x', k') is a yes-instance of B.
2. $k' \leq g(k)$ for some computable function g, and
3. the running time is $f(k) \times |x|^{O(1)}$ for some computable function f.

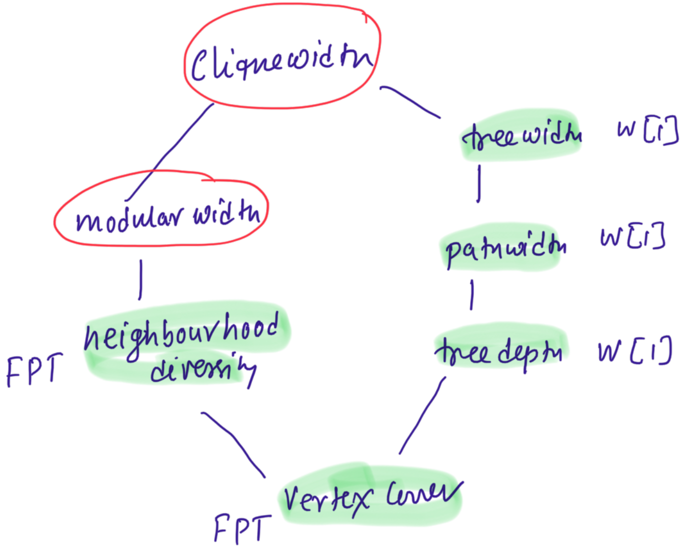
Theorem

The Maximum Minimal Defensive Alliance problem is W[1]-hard when parameterized by the treewidth of the graph.

pathwidth

treedepth

Open Problems



References

C. Bazgan, H. Fernau, and Z. Tuza.
Aspects of upper defensive alliances.
Discrete Applied Mathematics, 266:111 – 120, 2019.

M. Lampis.
Algorithmic meta-theorems for restrictions of treewidth.
Algorithmica, 64:19–37, 2012.

N. Robertson and P. Seymour.
Graph minors. iii. planar tree-width.
Journal of Combinatorial Theory, Series B, 36(1):49 – 64, 1984.

J. Sigarreta and J. Rodríguez.
On the global offensive alliance number of a graph.
Discrete Applied Mathematics, 157(2):219 – 226, 2009.

THANK YOU