

# Christoffel Darboux analysis

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joint work with

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- E. Saff (Vanderbilt U.)
- N. Stylianopoulos (Cyprus)

## Complex orthogonal polynomials

$\mu \geq 0$  positive Borel measure, rapidly decreasing on  $\mathbb{C}$ , of infinite support,  
so that the *orthogonal polynomials*  $P_n(z)$ ,  $n \geq 0$ , are well defined by

$$P_n(z) = \gamma_n z^n + O(z^{n-1}), \quad \gamma_n > 0,$$

and

$$\langle P_n, P_k \rangle_{2,\mu} = \delta_{nk}.$$

**Real moment data (observables) and complex orthogonal polynomials are interchangeable via elementary matrix operations.**

## Christoffel-Darboux kernel

$$K_N(z, w) = \sum_{k=0}^{N-1} P_k(z) \overline{P_k(w)},$$

is the reproducing kernel in the space  $\mathbb{C}_N[z]$  of polynomials of degree less than  $N$ :

$$\langle h, K_N(\cdot, w) \rangle_{2,\mu} = h(w), \quad \deg h < N.$$

E. B. Christoffel, *Über die Gaußsche Quadratur und eine Verallgemeinerung derselben*, J. Reine Angew. Math. 55 (1858), 61-82.

G. Darboux, *Mémoire sur l'approximation des fonctions de très-grands nombres, et sur une classe étendue de développements en série*, Liouville J. (3) 4 (1878), 5-56; 377-416.

## Christoffel function

$$|h(w)| \leq \|h\| \|K_N(\cdot, w)\|$$

with optimal solution  $K_N(\cdot, w)$ :

$$\max \frac{|h(w)|}{\|h\|} = \|K_N(\cdot, w)\|,$$

or equivalently

$$\Lambda_N(\mu, w) := \min \frac{\|h\|_{2,\mu}^2}{|h(w)|^2} = \frac{1}{K_N(w, w)}$$

where  $\deg h < N$  and  $h(w) \neq 0$ .

**The asymptotics of Christoffel's function  $\Lambda_N(\mu, w)$  were and remain central for many problems of mathematical analysis.**

## Moment indeterminateness (see Seminar 2 at IISC)

Note that the orthogonal polynomials  $P_n(z)$  depend only on the moments of the underlying measure  $\mu$ :

$$c_{kn} = \int_{\mathbb{C}} z^k \bar{z}^n d\mu(z), \quad k, n \geq 0.$$

Solving the moment problem (i.e. recovery of  $\mu$  from  $(c_{kn})_{k,n=0}^\infty$ ) encounters a natural and difficult obstacle: *is the measure  $\mu$  unique?*

M. Riesz (1923), R. Nevanlinna (1924): *If  $\text{supp}\mu \subset \mathbb{R}$ , then the moments determine the measure if and only if*

$$\lim_{N \rightarrow \infty} \Lambda_N(\mu, z) = 0,$$

*for at least one  $z \in \mathbb{C} \setminus \mathbb{R}$  (and hence for all).*

## Maximal point masses

Note that always  $\Lambda_{N+1}(\mu, z) \leq \Lambda_N(\mu, z)$ , so

$$\Lambda(\mu, z) = \lim_{N \rightarrow \infty} \Lambda_N(\mu, z)$$

exists.

In case  $x \in \mathbb{R}$ ,

$$\mu(\{x\}) \leq \Lambda(\mu, x) = \min \int |h(y)|^2 d\mu(y), \quad h(x) = 1,$$

and the upper bound is attained (by extremal solutions of the moment problem).

## The unit circle

$w(t) \geq 0$  integrable weight on  $[-\pi, \pi]$ . With geometric mean

$$G(w) = \exp\left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \ln w(t) dt\right),$$

if  $w$  satisfies Szegö's condition

$$\int_{-\pi}^{\pi} \ln w(t) dt > -\infty,$$

and  $G(w) = 0$  otherwise.

Szegö (1914)  $\Lambda(w(t)dt, 0) = G(w)$ .

Formulated equivalently as an extremum problem, with  $z = e^{it}$ :

$$\lim_{n \rightarrow \infty} \min_{A_j} \frac{1}{2\pi} \int_{-\pi}^{\pi} |z^n + A_1 z^{n-1} + \dots + A_n|^2 w(t) dt = G(w).$$

## Density of complex polynomials in Lebesgue space

$\mu \geq 0$  positive Borel measure on  $[-\pi, \pi]$  with Lebesgue decomposition

$$\mu = w(t)dt + \mu_{sc} + \mu_d.$$

where  $w(t)$  has bounded variation.

Kolmogorov (1941), Krein (1945) *The system  $1, z, z^2, \dots$  is dense in  $L^p(\mu)$ ,  $p \geq 1$ , if and only if*

$$\int_{-\pi}^{\pi} |\ln w(t)| dt = \infty.$$

Hint: By Szegő's Theorem, if  $G(w) > 0$ , then  $e^{-it}$  cannot be approximated by  $1, e^{it}, e^{i2t}, \dots$ .

## Accelerated convergence of Fourier series

Let  $\mu \geq 0$  be a positive measure supported on a compact set  $I \subset \mathbb{R}$ . For a continuous function  $f \in C(I)$  we set

$$S_N(\alpha, f, x) = \sum_{k=0}^{N-1} \langle f, P_k \rangle P_k(x).$$

Then

$$\sup_{\|f\|_{\infty, I} \leq 1} |S_N(\alpha, f, x)|^2 \leq \|f\|_{2, \alpha}^2 K_N(\alpha; x, x) \leq \alpha(I) K_N(\alpha; x, x).$$

Consequently

$$|f(x) - S_N(\alpha, f, x)| = |f(x) - Q(x) - S_N(\alpha, f - Q, x)| \leq \\ \inf_{\deg Q < N} \|f - Q\|_{\infty, I} (1 + \alpha(I) \sqrt{K_N(\alpha; x, x)}).$$

Lebesgue (1905) was the first to use this scheme for studying the convergence of Fourier type developments.

## The unit disk

$\mu = dA$  on  $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$

$$K_N(z, w) \rightarrow K(z, w) = \frac{1}{\pi(1 - z\bar{w})^2}, \quad z, w \in \mathbb{D},$$

the Bergman kernel, and

$$K_N(z, z) = \frac{N+1}{\pi} \frac{1}{|z|^2 - 1} |z|^{2N+2} (1 + O(1/N)), \quad |z| > 1.$$

measures how fast a polynomial of degree less than  $N$  lifts outside the disk, keeping its square norm average on  $\mathbb{D}$  bounded.

Classical asymptotics extended to more general domains by Carleman (1922) and Suetin (1969).

# 1. Shape reconstruction in 2D

$G = \cup G_j$  an *archipelago*, i.e. a finite union of simply connected domains, with real analytic boundary  $\Gamma = \partial G$  and  $\mu = \chi_G d\text{Area}$ .

"Observables" are finitely many moments

$$a_{mn} = \int z^m \bar{z}^n \, d\mu(z)$$

such as derived from geometric tomography.

They determine the complex orthogonal polynomials  $P_n(z)$ ,  $0 \leq n \leq N$ , and the CD-kernel

$$K_n(z, w) = \sum_{j=0}^{n-1} P_j(z) \overline{P_j(w)}.$$

# Christoffel function as defining function of the archipelago

Normalized Christoffel function

$$\gamma_n(z) = [K_n(z, z)]^{-1/2}$$

satisfies:

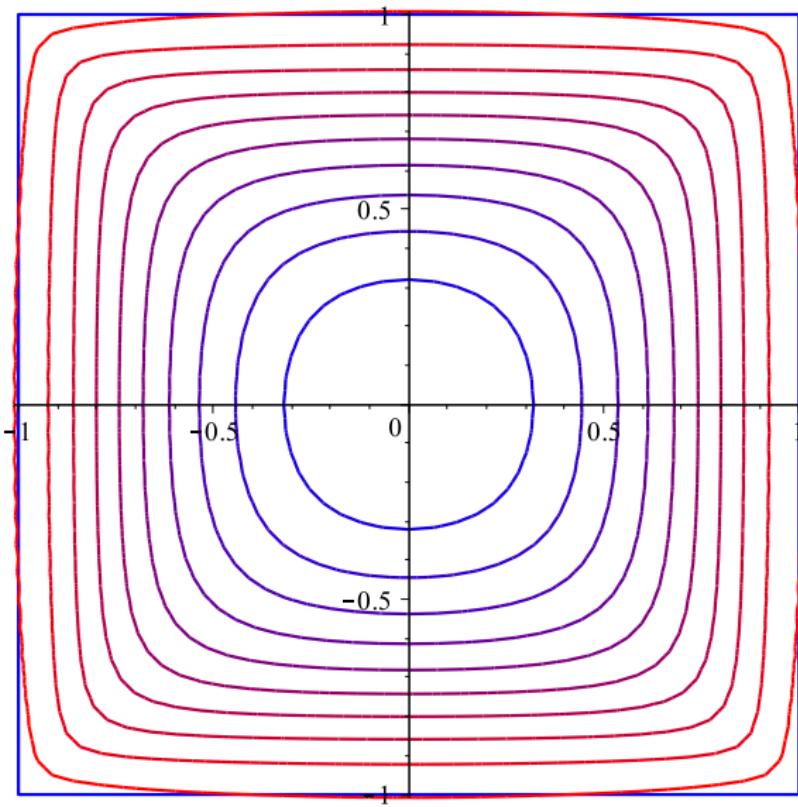
$$\sqrt{\pi} \operatorname{dist}(z, \Gamma) \leq \gamma_n(z) \leq C \operatorname{dist}(z, \Gamma)$$

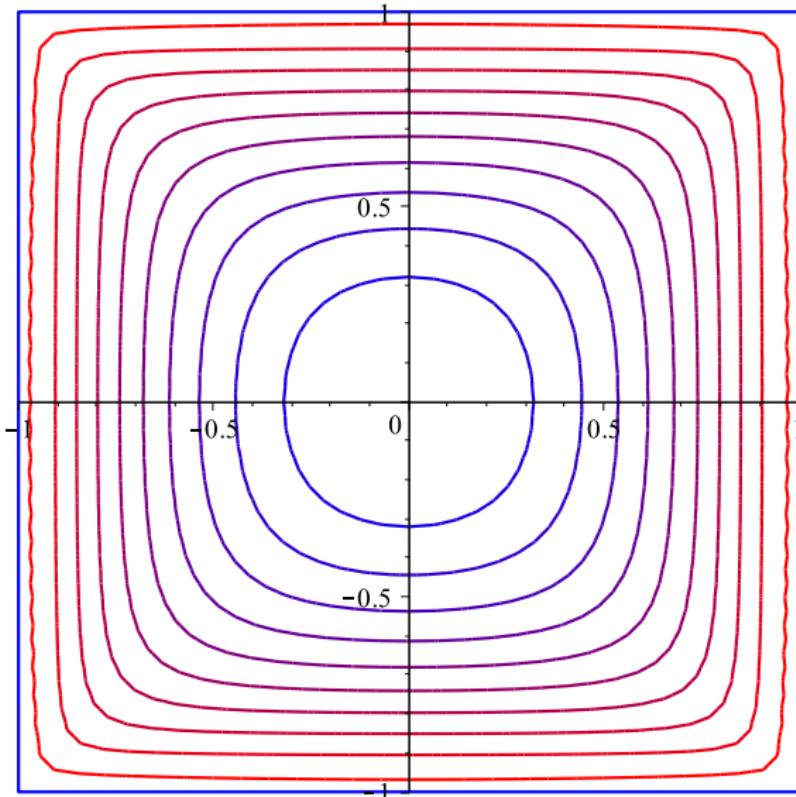
for  $z \in G$ , close to  $\Gamma$ . Moreover:

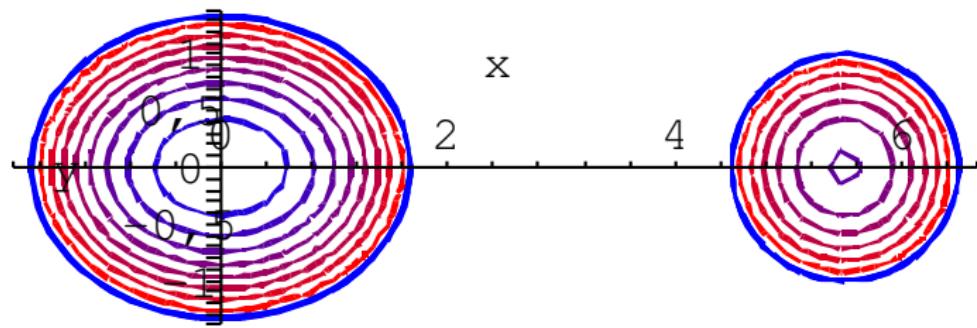
$$\gamma_n(z) = O\left(\frac{1}{n}\right), \quad z \in \Gamma.$$

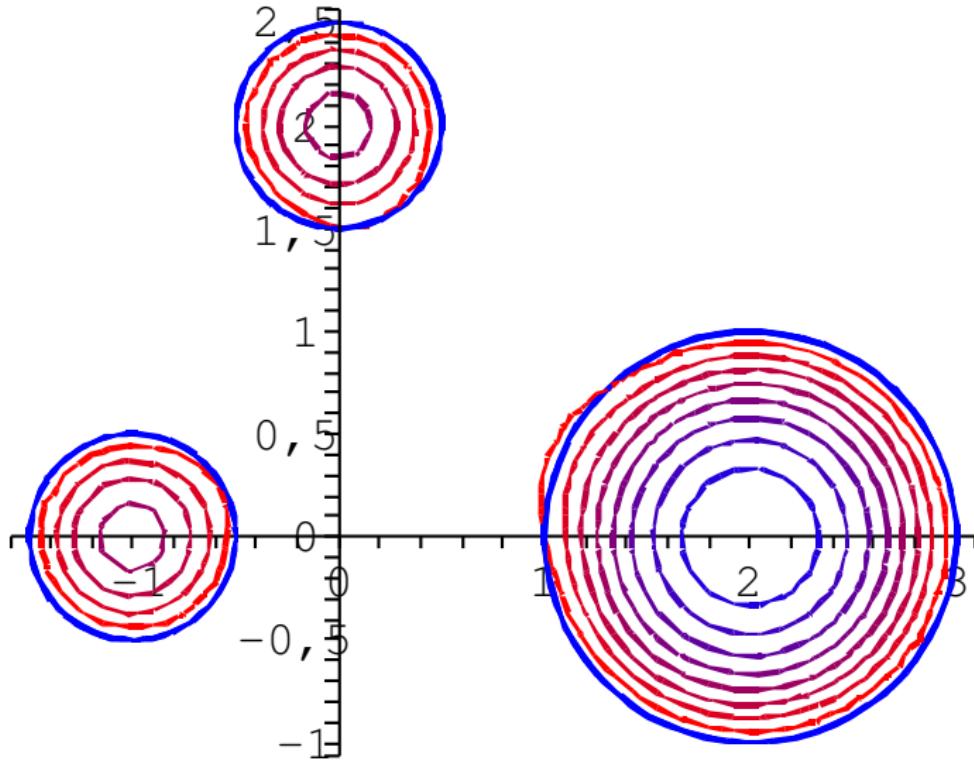
Outside  $\overline{G}$  this function decreases exponentially to zero. On analytic boundaries one has sharp estimates.

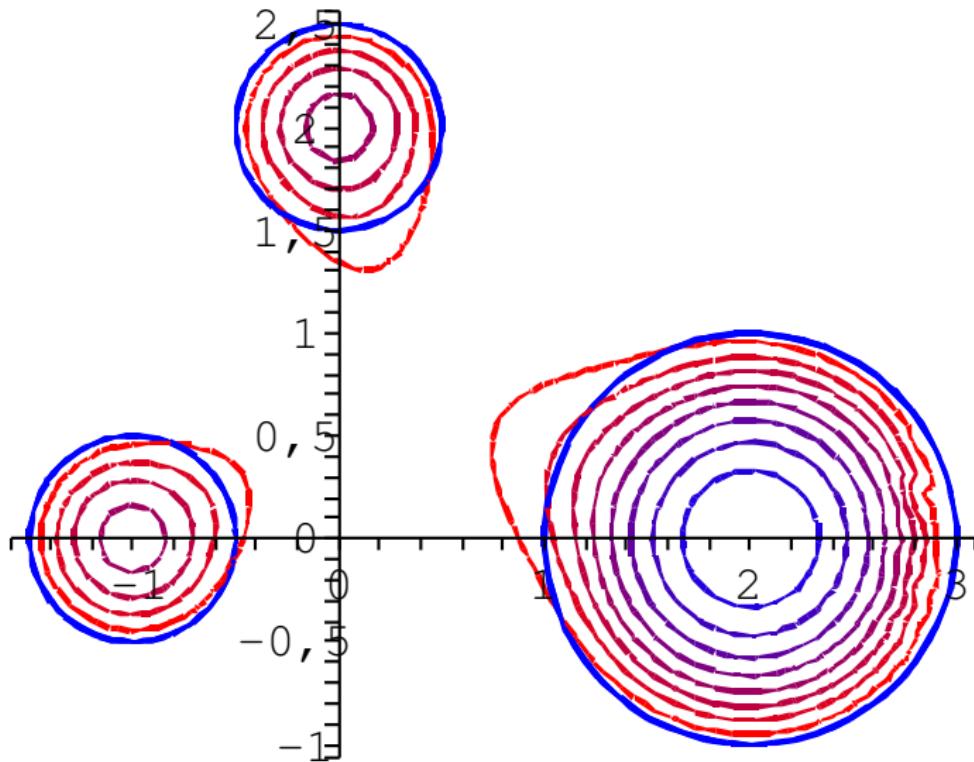
# Reconstruction experiments











## 2. Spectral analysis of dynamical systems

**Nonlinear** dynamics

$$x_{k+1} = T(x_k)$$

$$T : X \rightarrow X$$

**Linear** operator

$$\textcolor{red}{U}f = f \circ T$$

$$U : \mathcal{H} \rightarrow \mathcal{H}$$

Measure-preserving invertible setting:  $\mathcal{H} = L_2(\nu) \Rightarrow \textcolor{red}{U}$  unitary  $\Rightarrow \sigma(U) \subset \mathbb{T}$

**Goal:** Understand spectrum of  $\textcolor{red}{U}$  from data

**Spectral theorem**

$$1. \quad U = \int_{\mathbb{T}} z \, d\textcolor{red}{E}(z)$$

$$2. \quad \mu_f(\cdot) := \langle \textcolor{red}{E}(\cdot)f, f \rangle_{L_2(\nu)} \text{ is a positive measure on } \mathbb{T}$$

**Fact:** If  $\overline{\text{span}\{f, Uf, U^{-1}f, U^2f, U^{-2}f, \dots\}} = \mathcal{H}$ , then  $\mu_f$  determines  $\textcolor{red}{U}$

# Reconstructing $\mu_f$ from moments

$$\mu = \mu_{\text{at}} + \mu_{\text{ac}} + \mu_{\text{sc}}$$

↓      ↓      ↓

Point spectrum  
(Eigenvalues)

AC spectrum  
 $d\mu_{\text{ac}} = \rho(\theta)d\theta$

SC spectrum

## Point spectrum

$$\lim_{N \rightarrow \infty} \frac{1}{K_N(e^{i2\pi\theta}, e^{i2\pi\theta})} = \mu_{\text{at}}(\{e^{i2\pi\theta}\}) \quad \text{for all } \theta \in [0, 1]$$

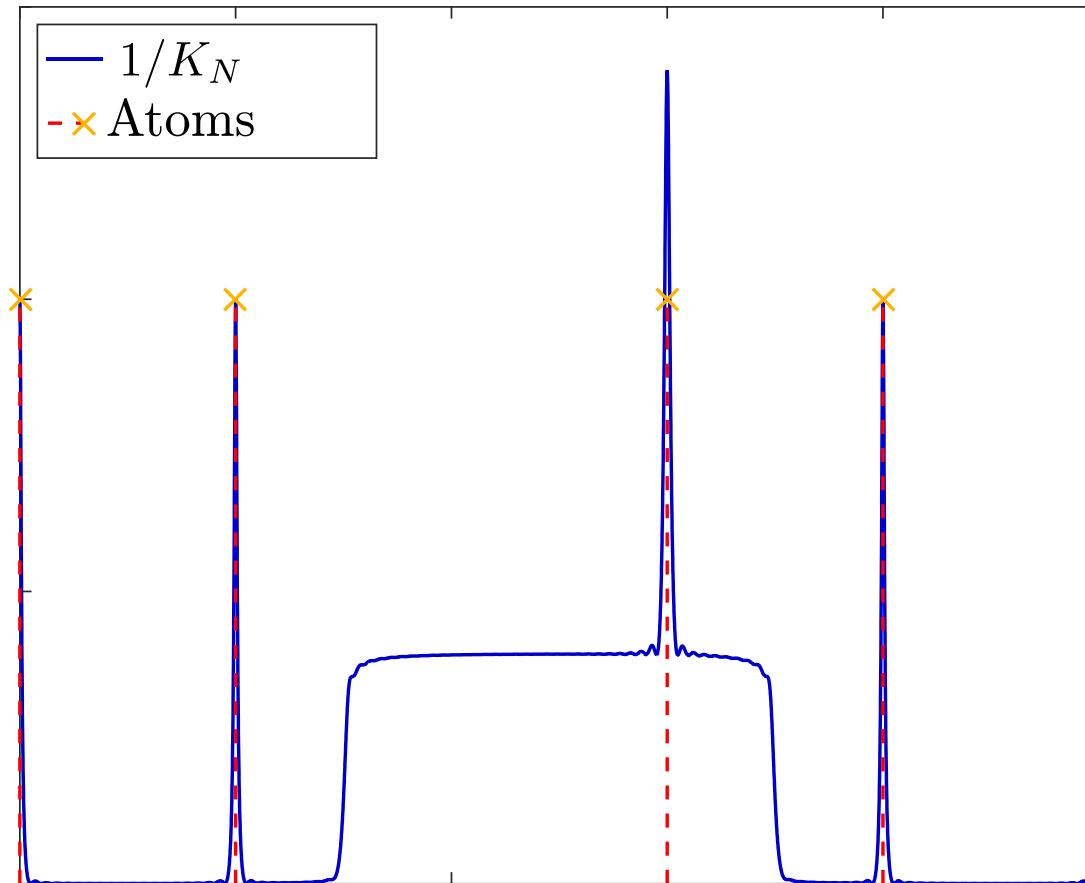
## AC spectrum [Mate, Nevai, Totik, 91']

$$\lim_{N \rightarrow \infty} \frac{N}{K_N(e^{i2\pi\theta}, e^{i2\pi\theta})} = \rho(\theta) \quad \text{for almost all } \theta \in [0, 1]$$

# Reconstructing $\mu_f$ from moments - Example

$$\mu = 4I_{[0.3,0.7]} d\theta + 0.1(\delta_0 + \delta_{0.2} + \delta_{0.6} + \delta_{0.8})$$

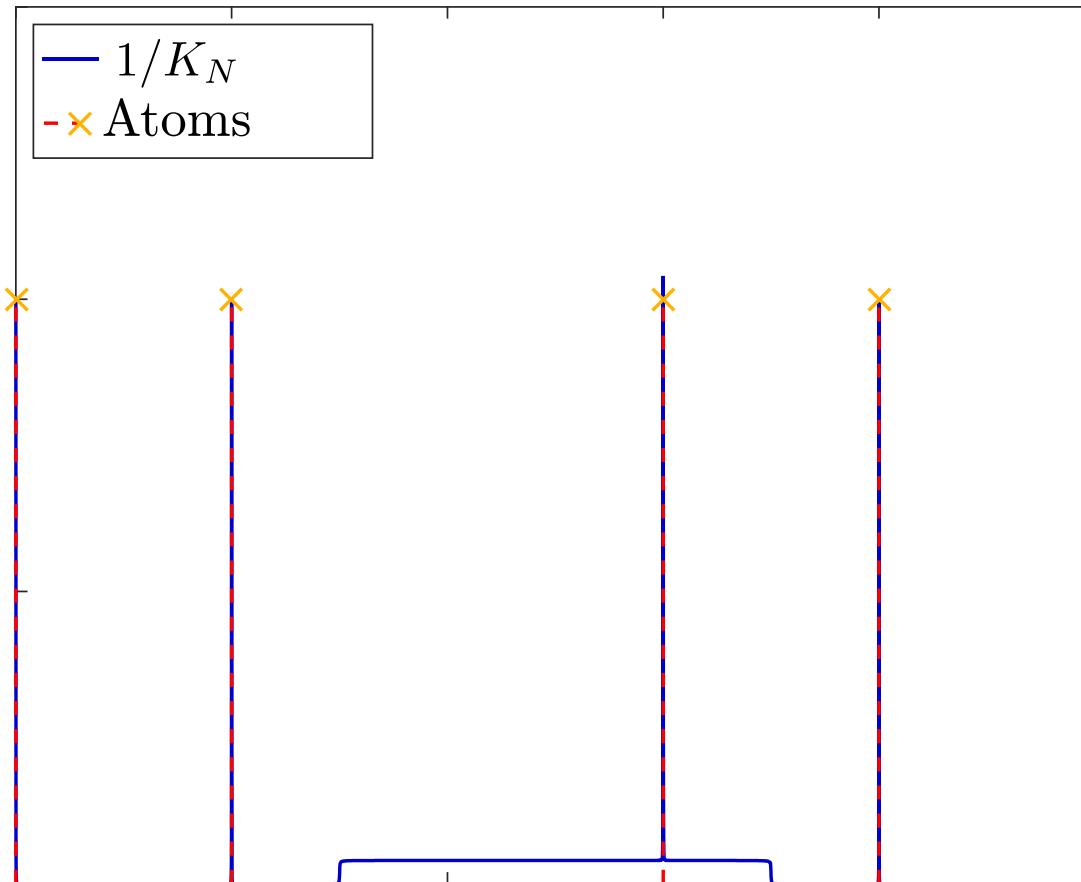
Point spectrum,  $N = 100$



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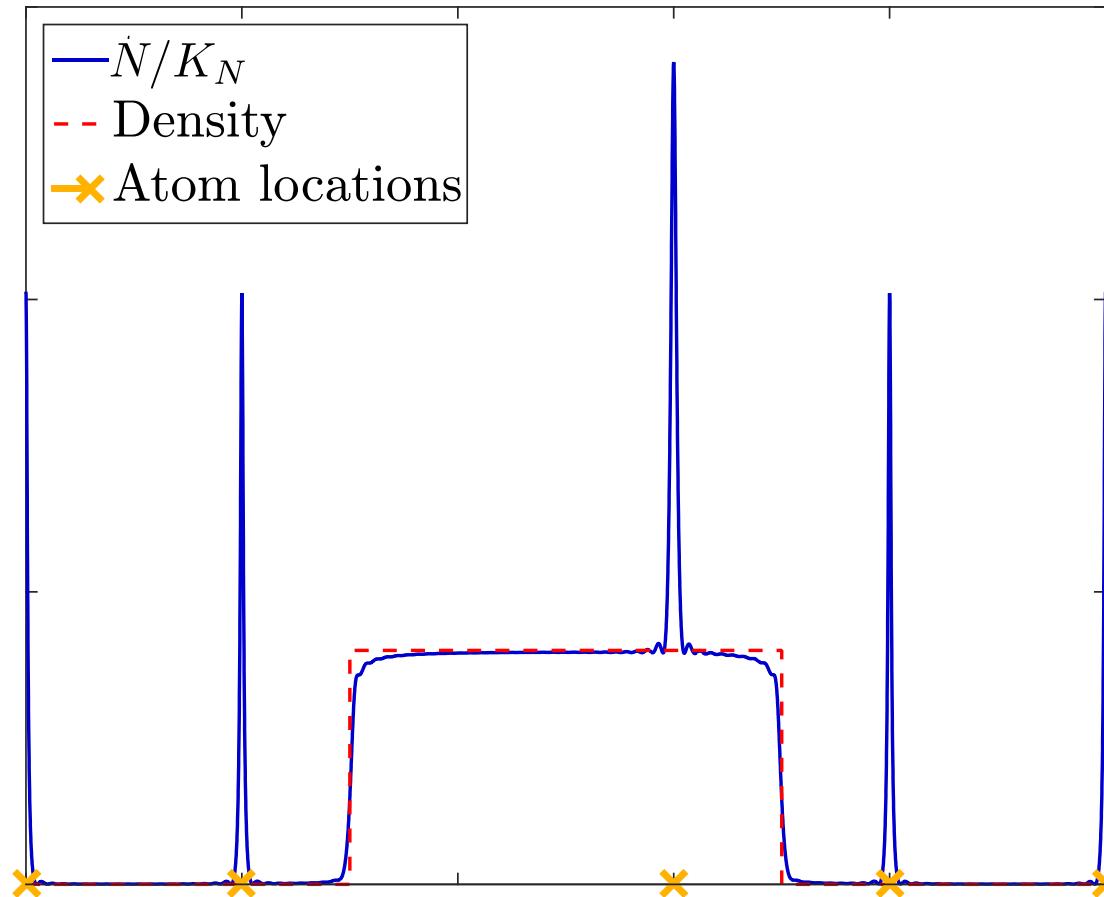
Point spectrum, N = 1000



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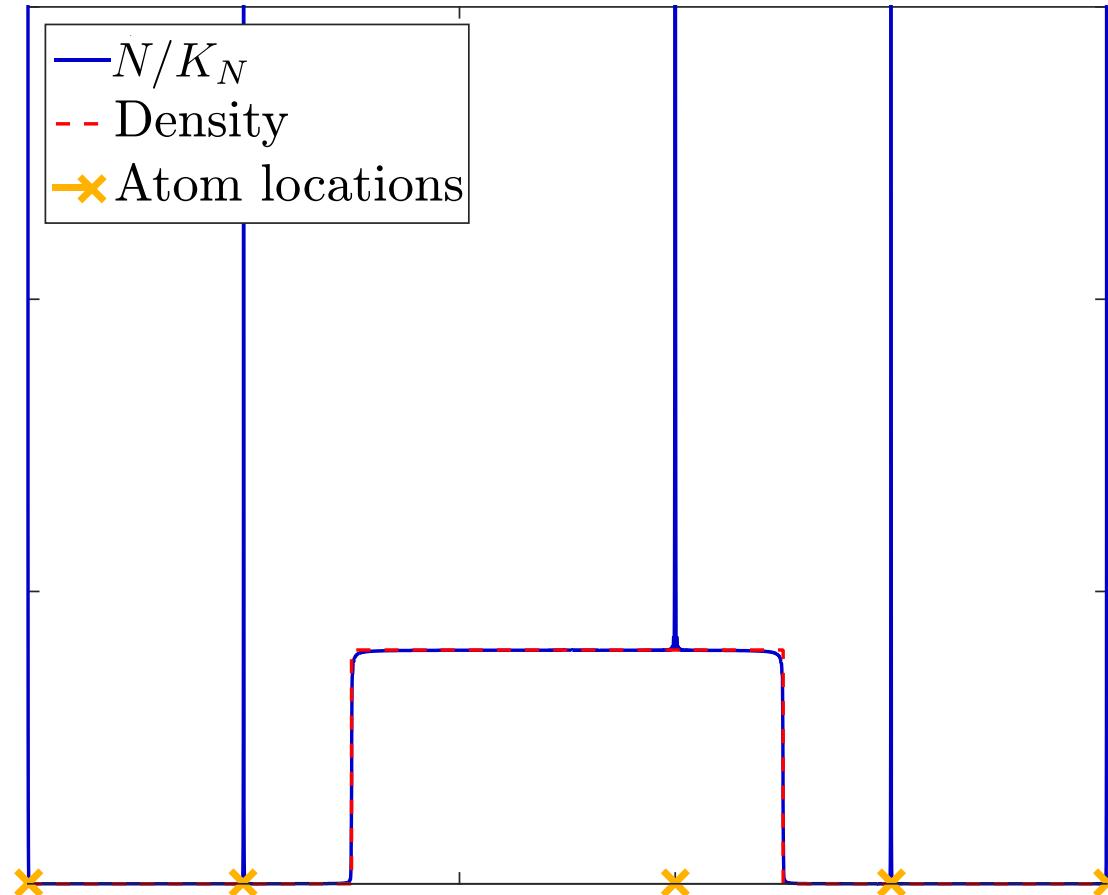
AC part,  $N = 100$



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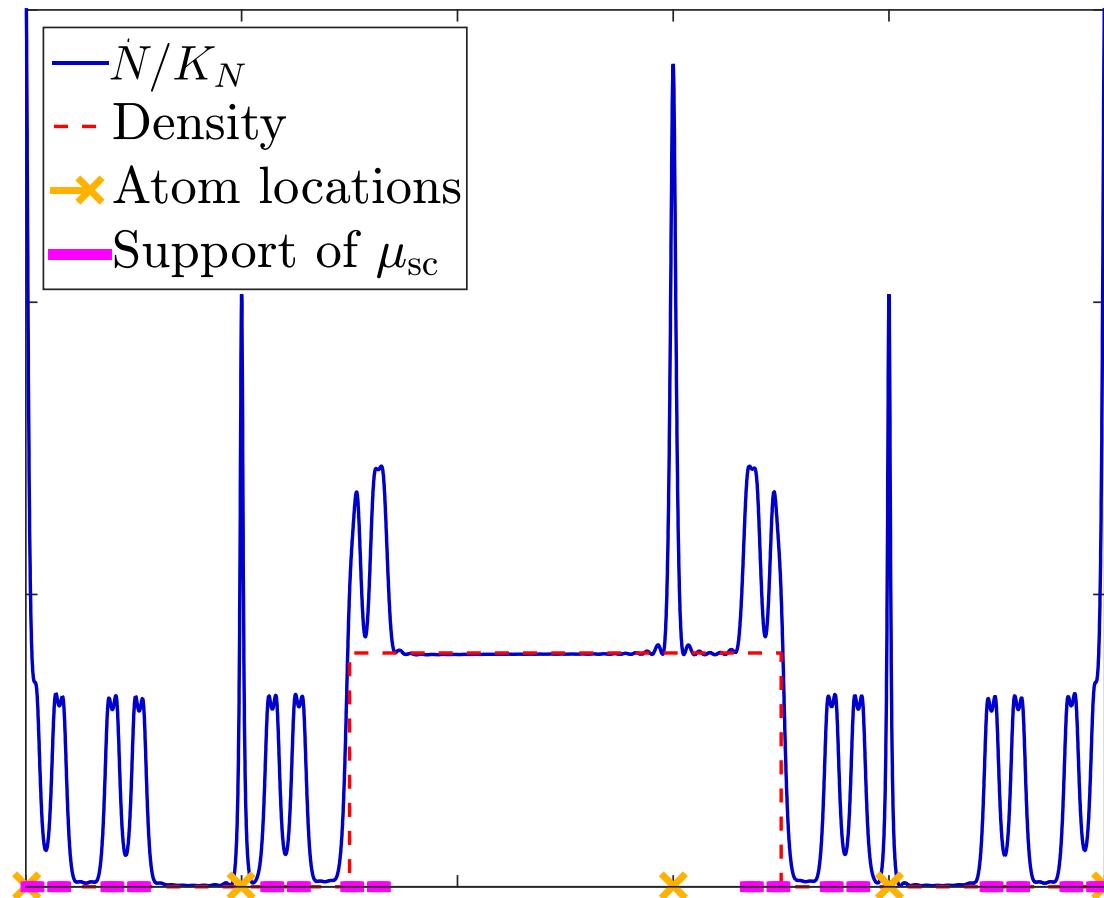
AC part, N = 1000



# Reconstructing $\mu_f$ from moments - Example

$$\mu = 4I_{[0.3,0.7]} d\theta + 0.1(\delta_0 + \delta_{0.2} + \delta_{0.6} + \delta_{0.8}) + \text{Cantor}$$

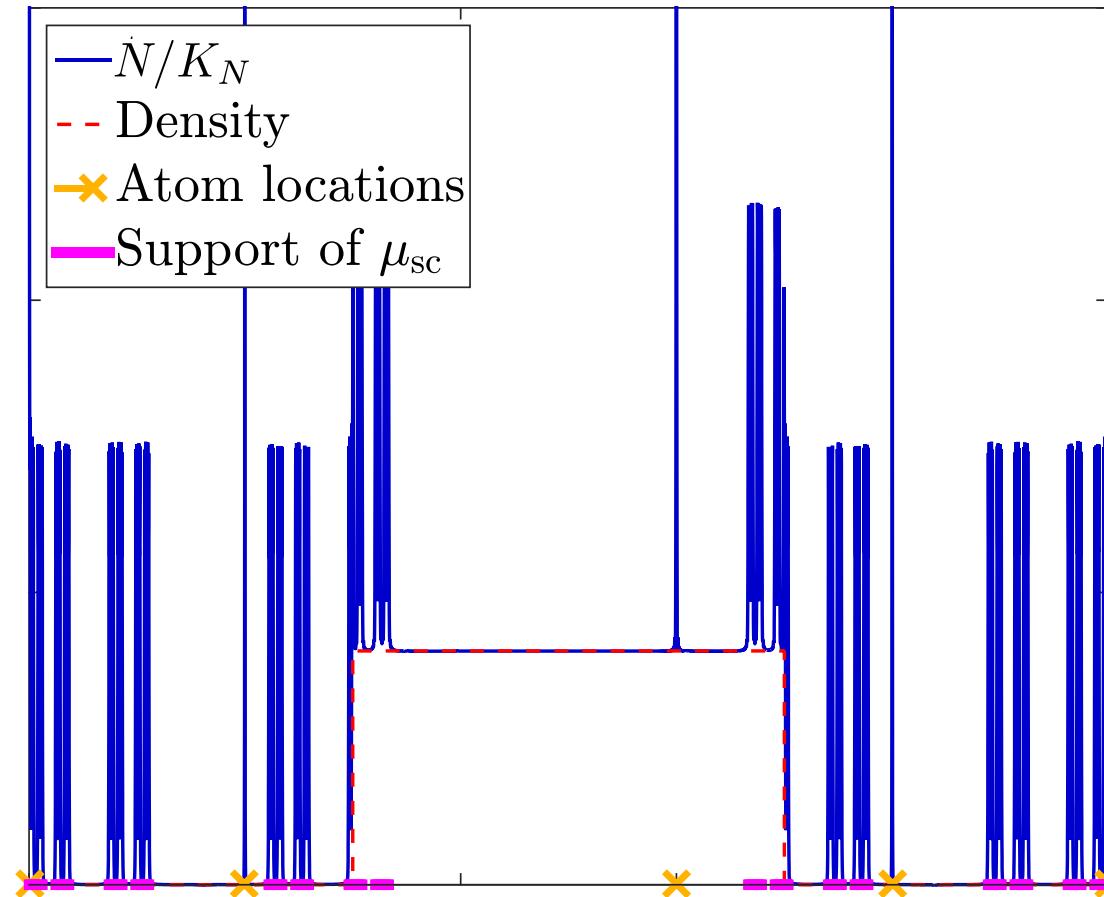
Adding SC spectrum, N = 100



# Reconstructing $\mu_f$ from moments - Example

$$\mu = 4I_{[0.3,0.7]} d\theta + 0.1(\delta_0 + \delta_{0.2} + \delta_{0.6} + \delta_{0.8}) + \text{Cantor}$$

Adding SC spectrum, N = 1000



# Detecting SC spectrum

$$F_N^{CS}(t) = \mu_N^{CS}([0, t])$$

$$F_N^Q(t) = \mu_N^Q([0, t])$$

Converge to  $F = \mu([0, t])$   
(at points of continuity)

$$F_N^K(t) = \int_0^t \frac{N}{K(e^{i2\pi\theta}, e^{i2\pi\theta})} d\theta$$

Converges to  $F = \mu([0, t])$  **only if**  $\mu$  is AC  
Otherwise **underestimates**  $F$

# Detecting SC spectrum

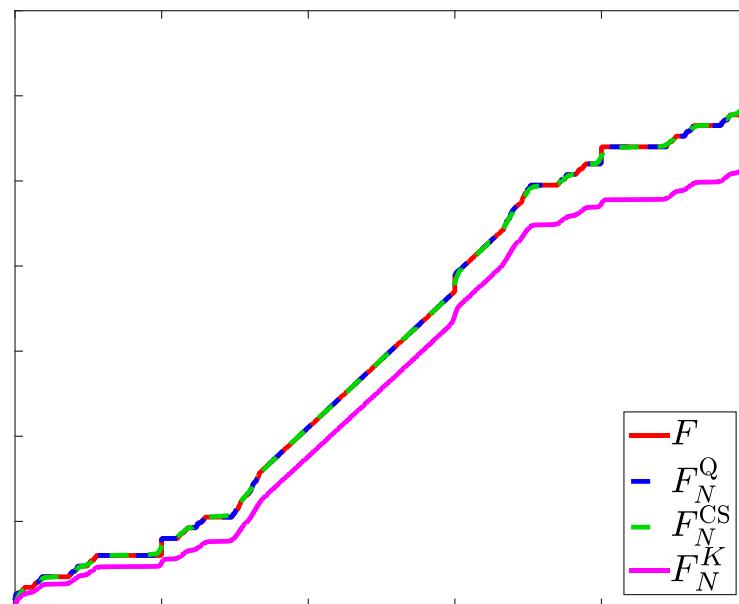
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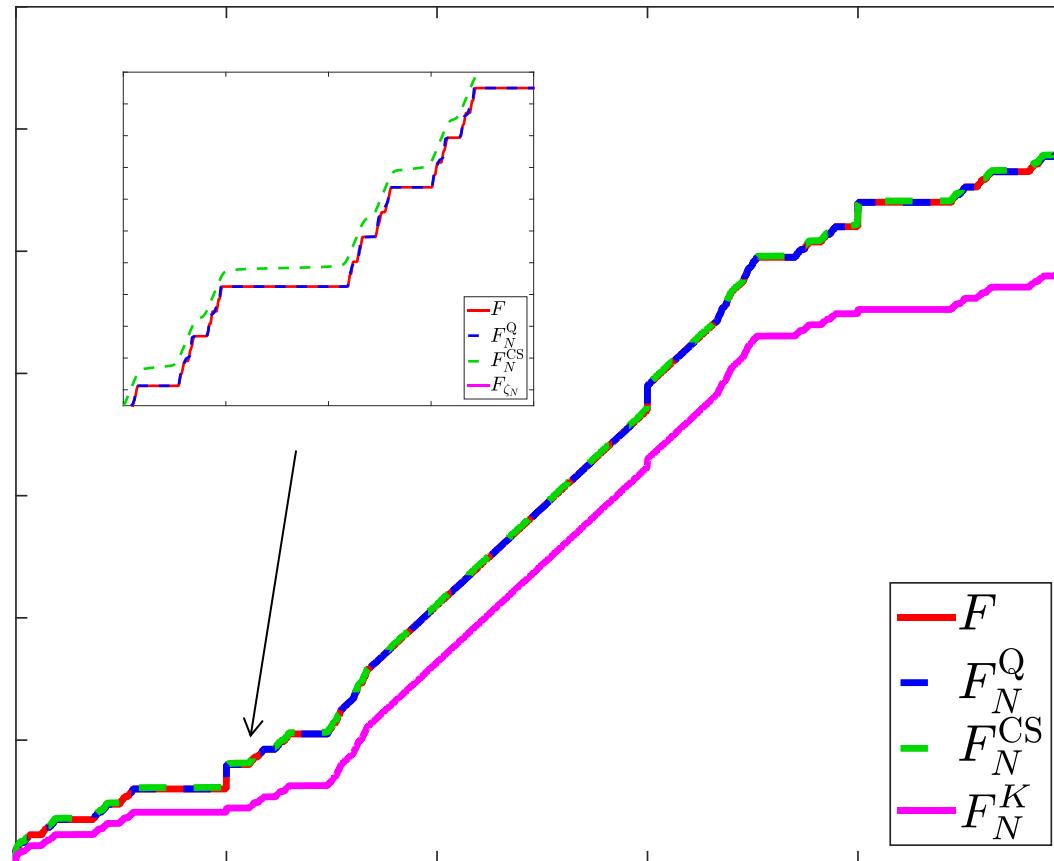
Converges to  $F = \mu([0, t])$  **only if**  $\mu$  is AC  
Otherwise **underestimates**  $F$



# Detecting SC spectrum - CDFs

$$\mu = 4I_{[0.3,0.7]} d\theta + 0.1(\delta_0 + \delta_{0.2} + \delta_{0.6} + \delta_{0.8}) + \text{Cantor}$$

$N = 1000$



# Detecting SC spectrum

$$F_N^{CS}(t) = \mu_N^{CS}([0, t])$$

$$F_N^Q(t) = \mu_N^Q([0, t])$$

Converge to  $F = \mu([0, t])$   
(at points of continuity)

$$F_N^K(t) = \int_0^t \frac{N}{K(e^{i2\pi\theta}, e^{i2\pi\theta})} d\theta$$

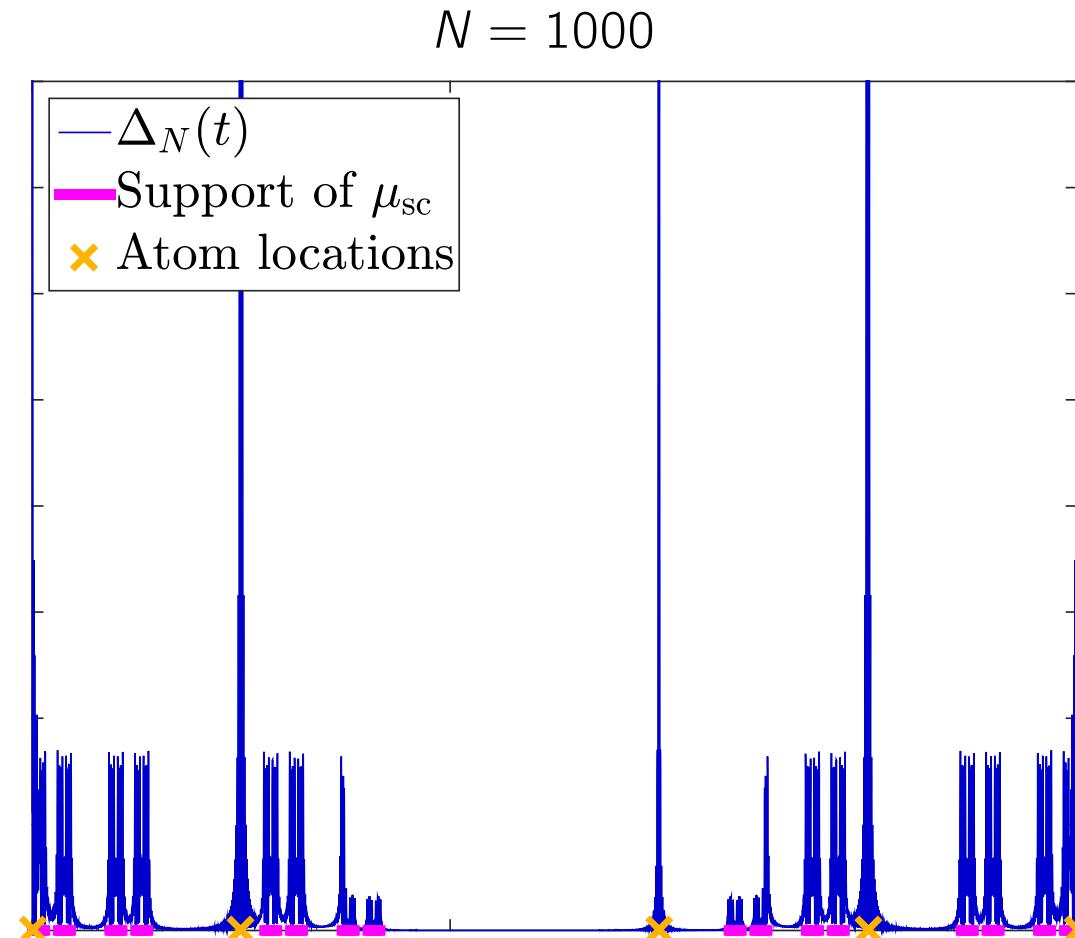
Converges to  $F = \mu([0, t])$  **only if**  $\mu$  is AC  
Otherwise **underestimates**  $F$

## Singularity indicator

$$\Delta_N(t) = \frac{F_N^{CS}(t + \delta) - F_N^{CS}(t - \delta)}{F_N^K(t + \delta) - F_N^K(t - \delta)} - 1$$

# Detecting SC spectrum – Singularity indicator

$$\mu = 4I_{[0.3,0.7]} d\theta + 0.1(\delta_0 + \delta_{0.2} + \delta_{0.6} + \delta_{0.8}) + \text{Cantor}$$



# Cat map

$$\begin{aligned}x_1^+ &= 2x_1 + x_2 \mod 1 \\x_2^+ &= x_1 + x_2 \mod 1\end{aligned}$$

$$N = 100, M = 10^5$$

Spectrum known analytically [Govindarajan et al., 2017]

$$\begin{aligned}f_1 &= e^{i2\pi(2x_1+x_2)} + \frac{1}{2}e^{i2\pi(5x_1+3x_2)} \\ \rho_{f_1} &= \frac{5}{4} + \cos(2\pi\theta)\end{aligned}$$

$$\begin{aligned}f_2 &= e^{i2\pi(2x_1+x_2)} + \frac{1}{2}e^{i2\pi(5x_1+3x_2)} + \frac{1}{4}e^{i2\pi(13x_1+8x_2)} \\ \rho_{f_2} &= \frac{21}{16} + (5/4)\cos(2\pi\theta) + \frac{1}{2}\cos(4\pi\theta)\end{aligned}$$

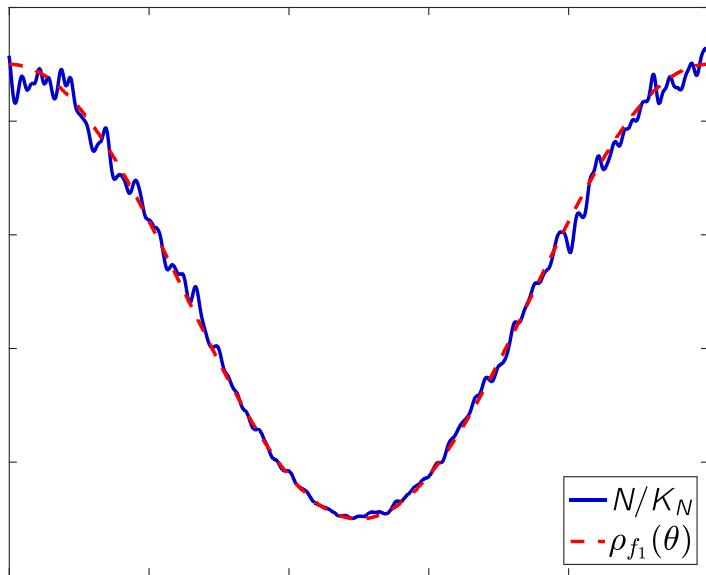
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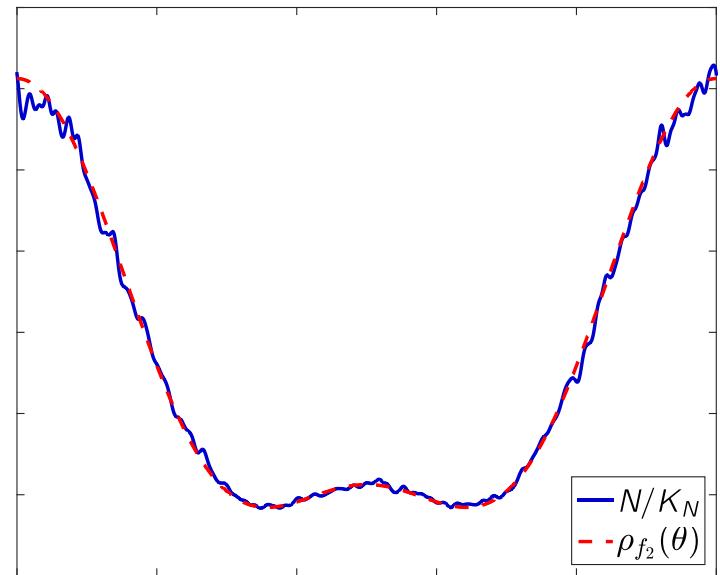
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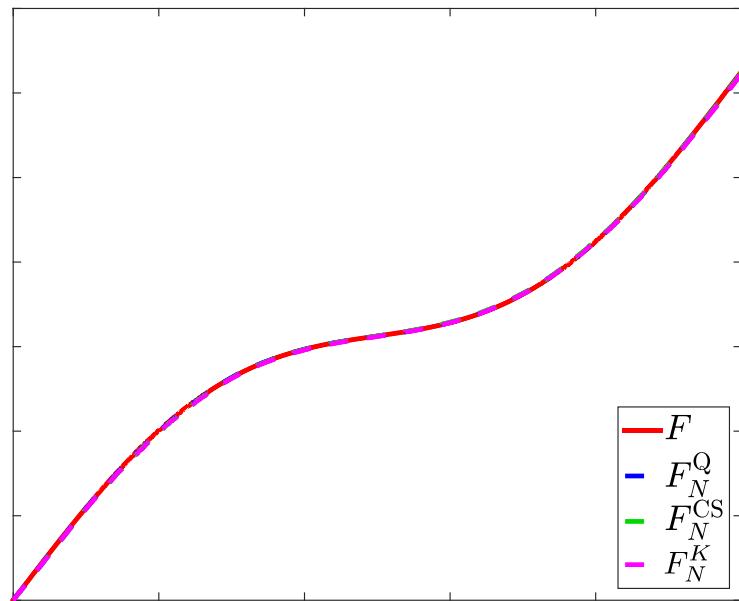


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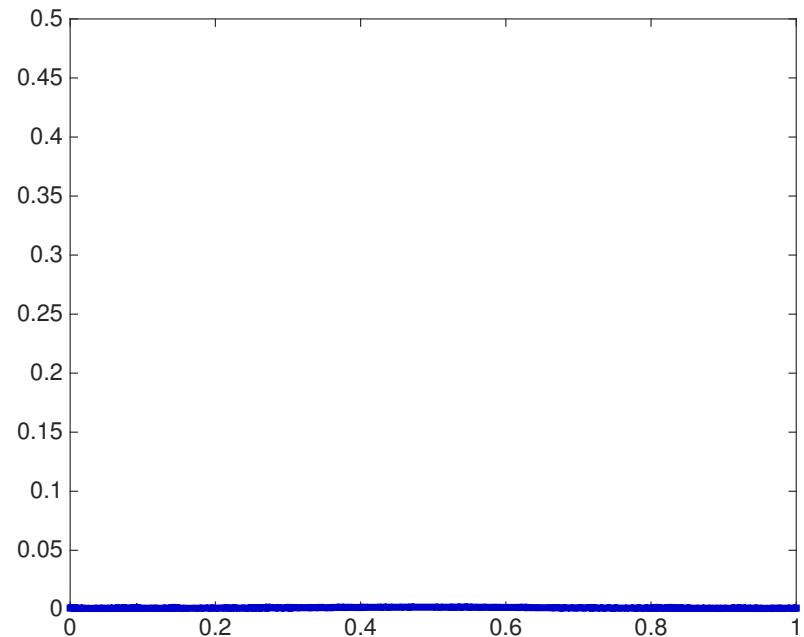
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$N = 100, M = 10^5$

Distribution functions

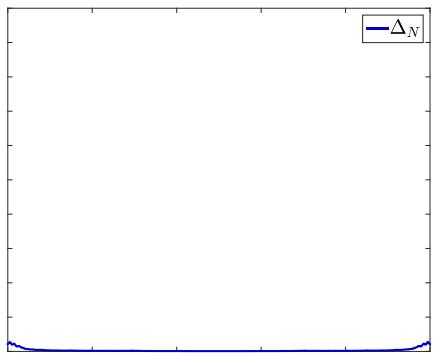
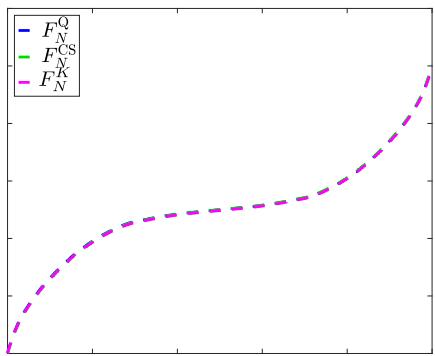
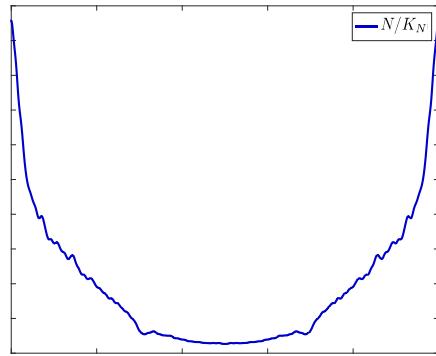


Singularity indicator

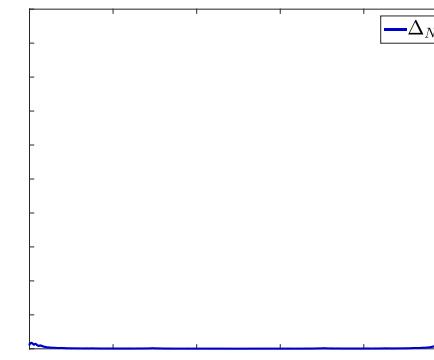
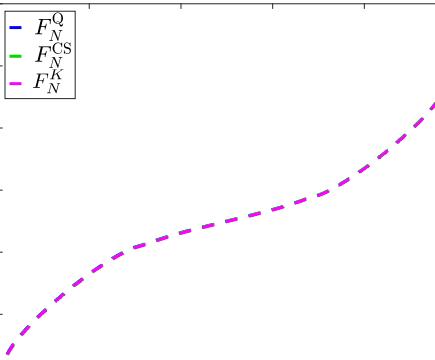
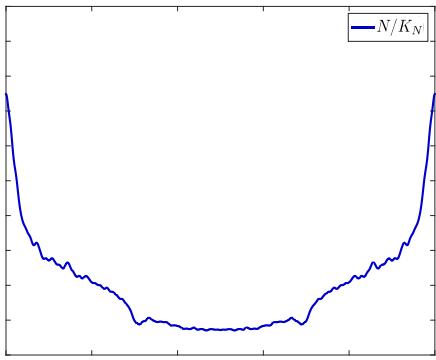


# Lorenz system

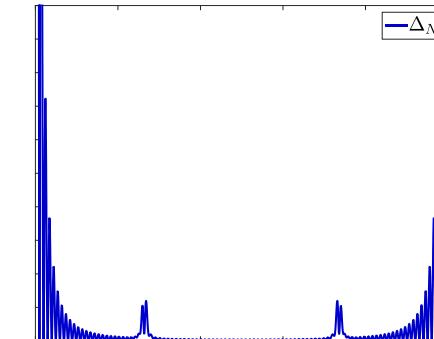
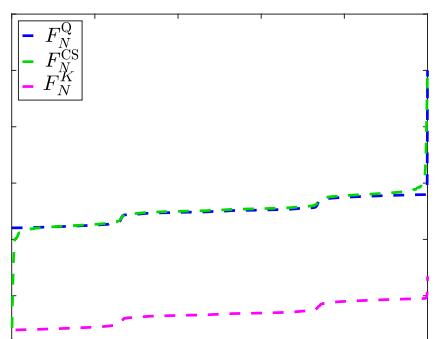
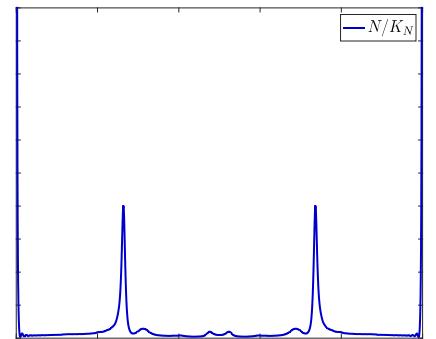
Observable  $x_1$



Observable  $x_2$

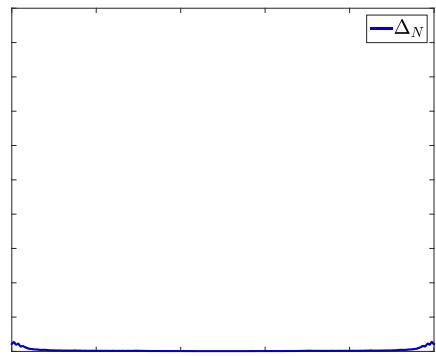
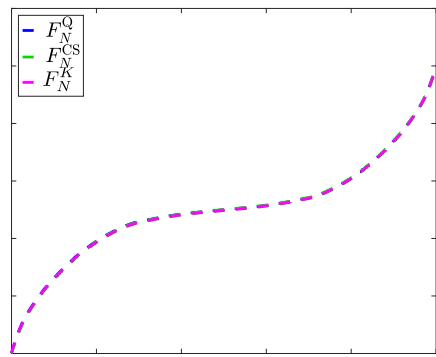
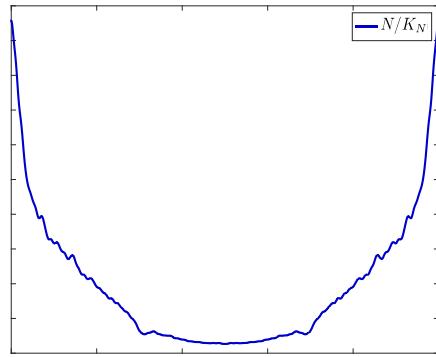


Observable  $x_3$

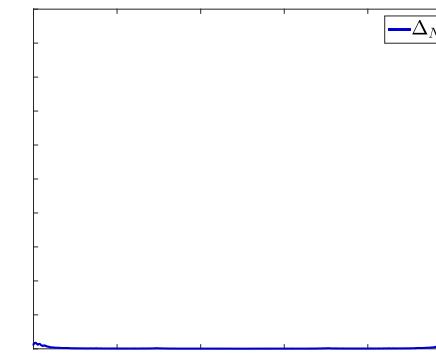
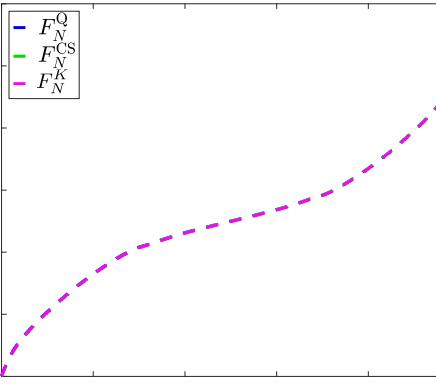
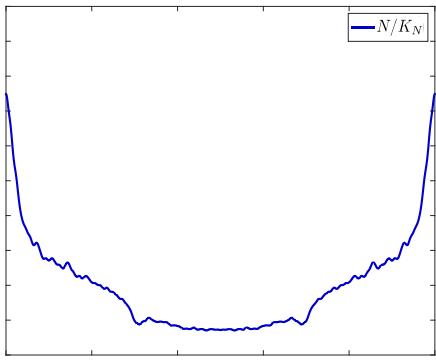


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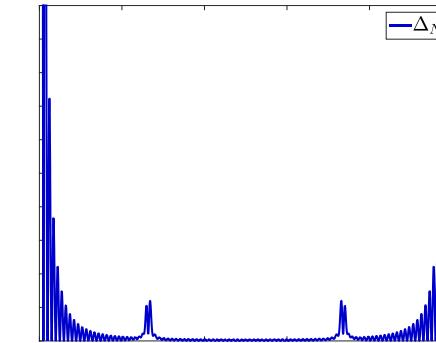
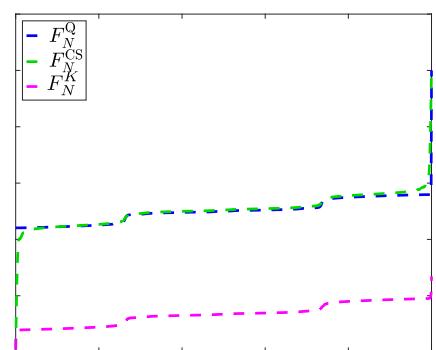
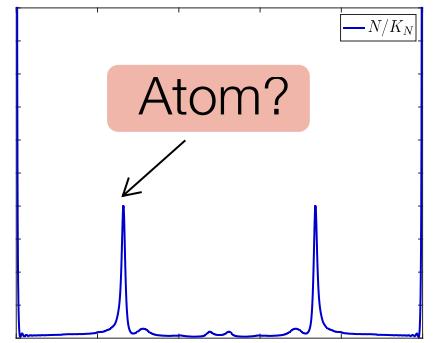
Observable  $x_1$



Observable  $x_2$

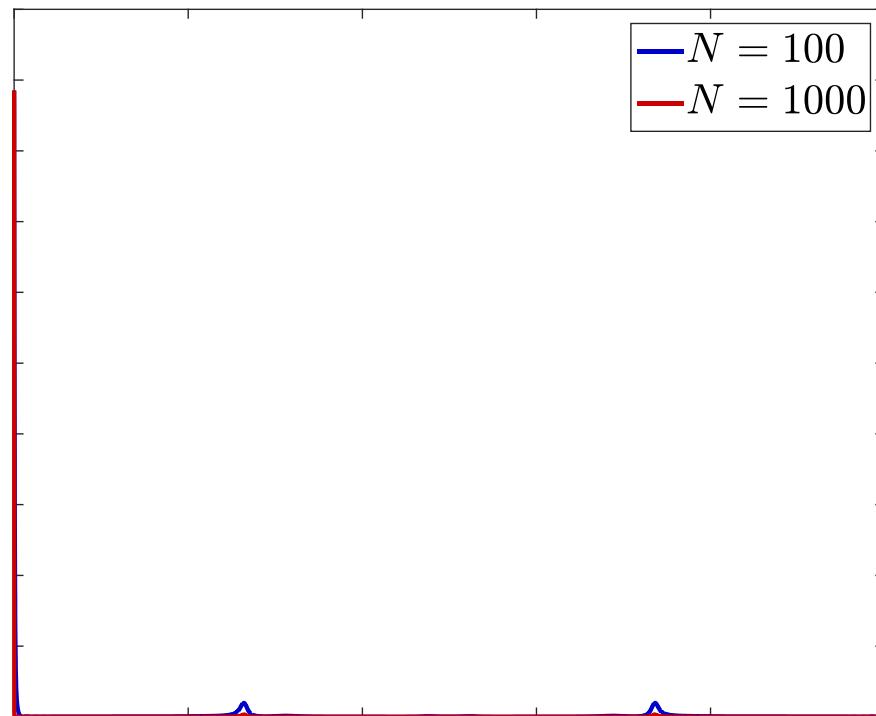


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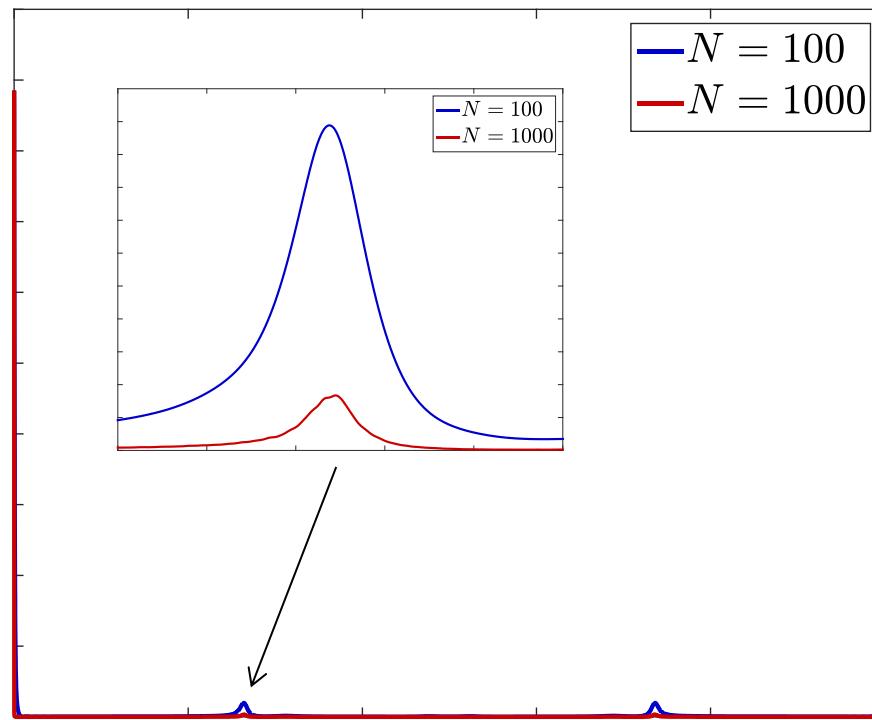
# Lorenz system

$1/K_N$  – atomic part



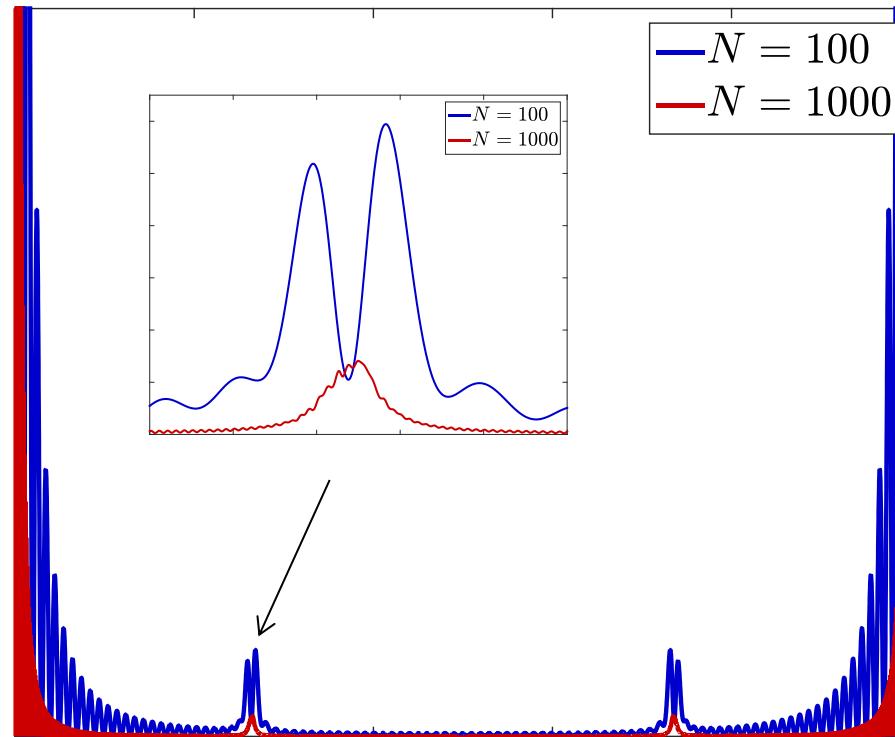
# Lorenz system

$1/K_N$  – atomic part



# Lorenz system

Singularity indicator

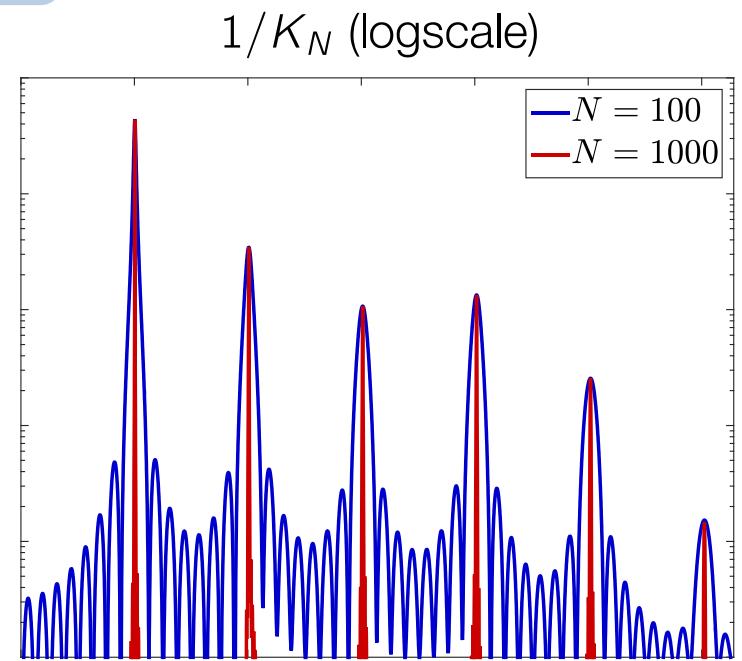
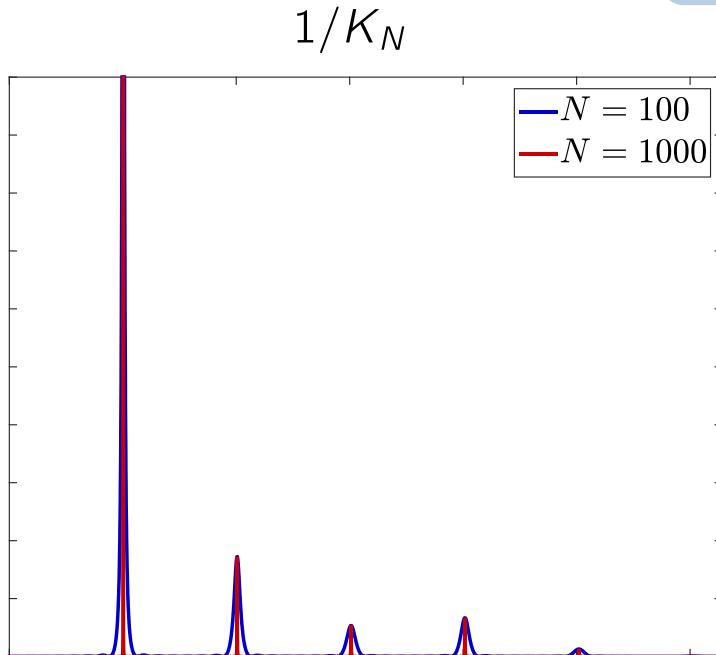


# Cavity flow

2-D lid-driven cavity flow

Observable: point measurement of stream function

Re = 13k



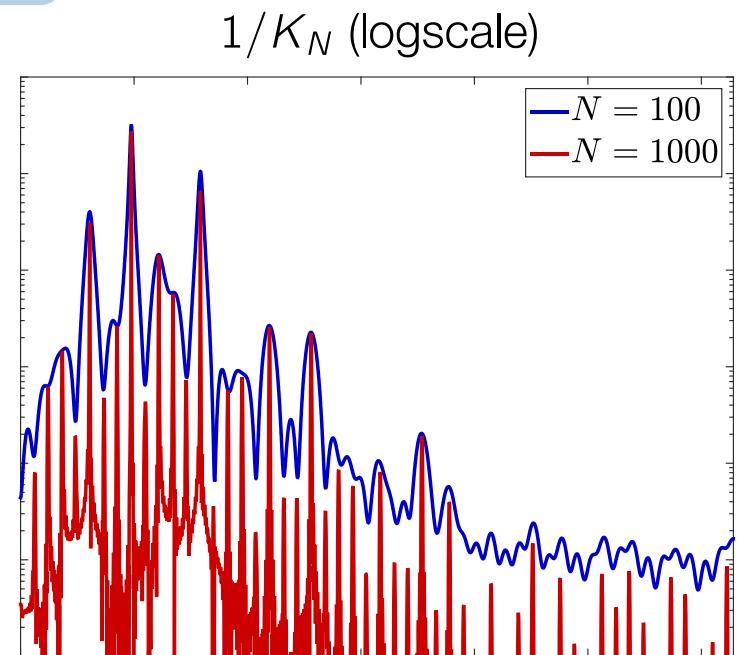
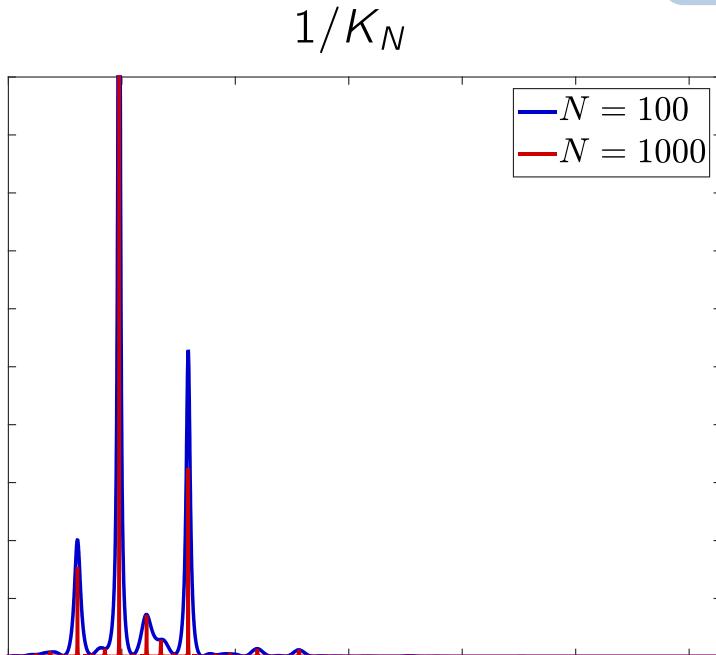
[Data courtesy of H. Arbabi]

# Cavity flow

2-D lid-driven cavity flow

Observable: point measurement of stream function

Re = 16k



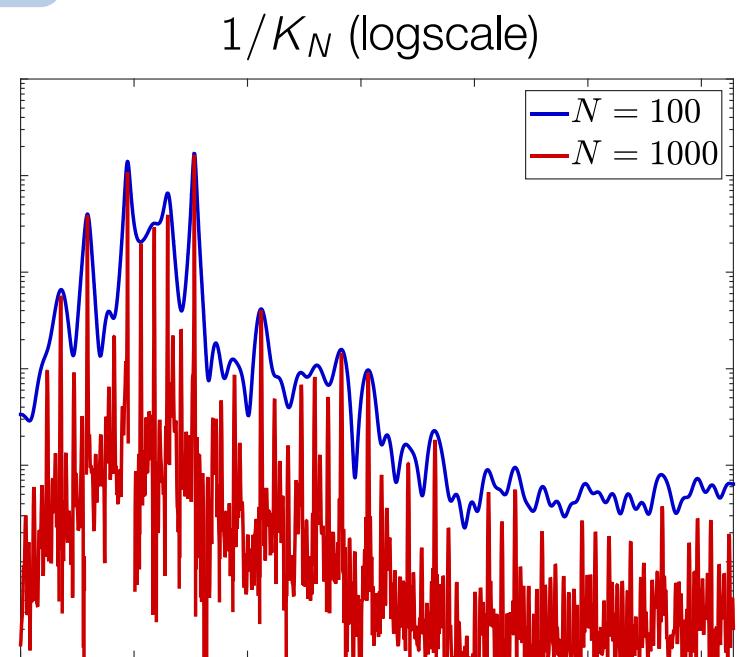
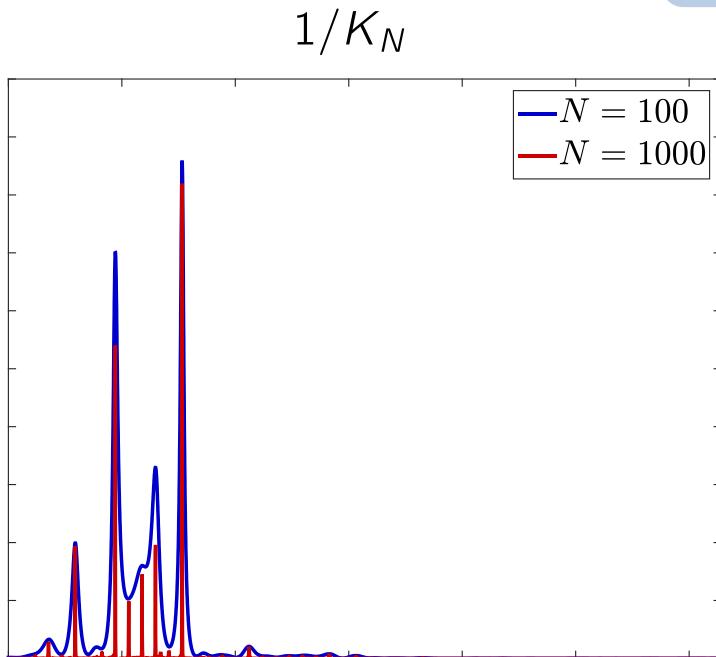
[Data courtesy of H. Arbabi]

# Cavity flow

2-D lid-driven cavity flow

Observable: point measurement of stream function

Re = 19k



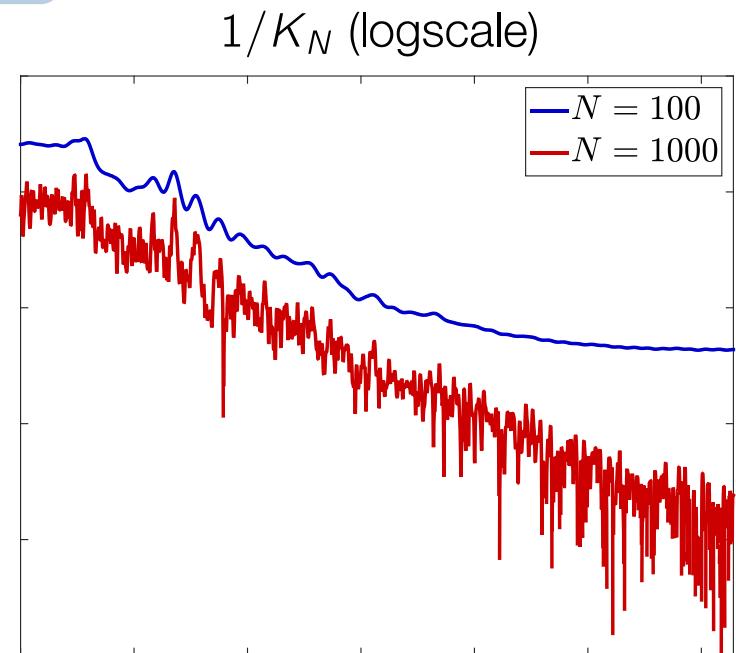
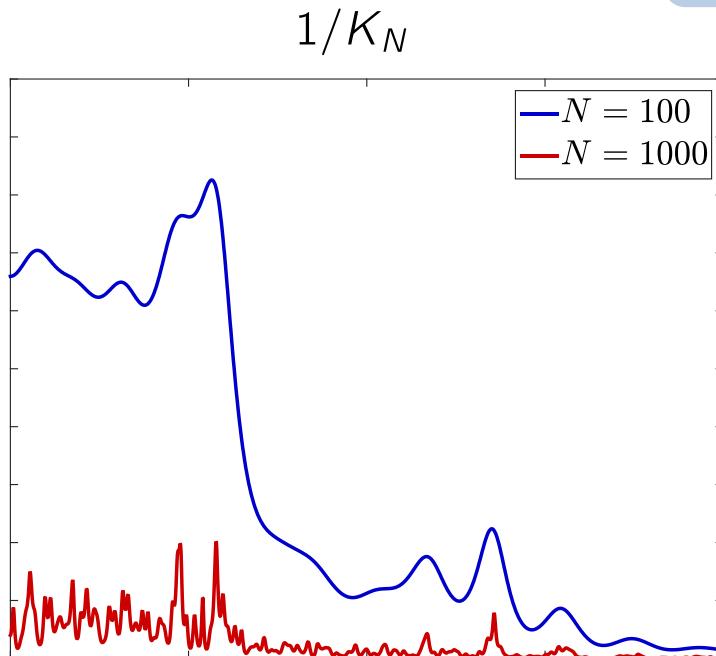
[Data courtesy of H. Arbabi]

# Cavity flow

2-D lid-driven cavity flow

Observable: point measurement of stream function

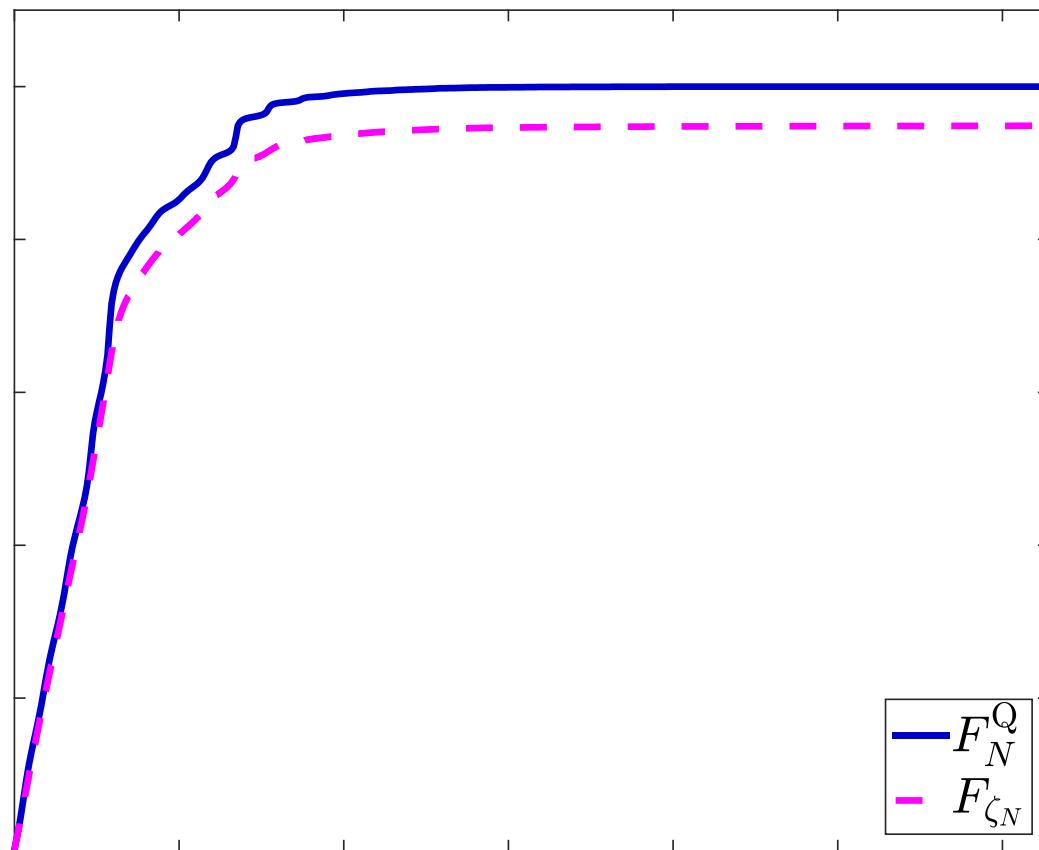
Re = 30k



[Data courtesy of H. Arbabi]

# Cavity flow – distribution functions

Re = 30k

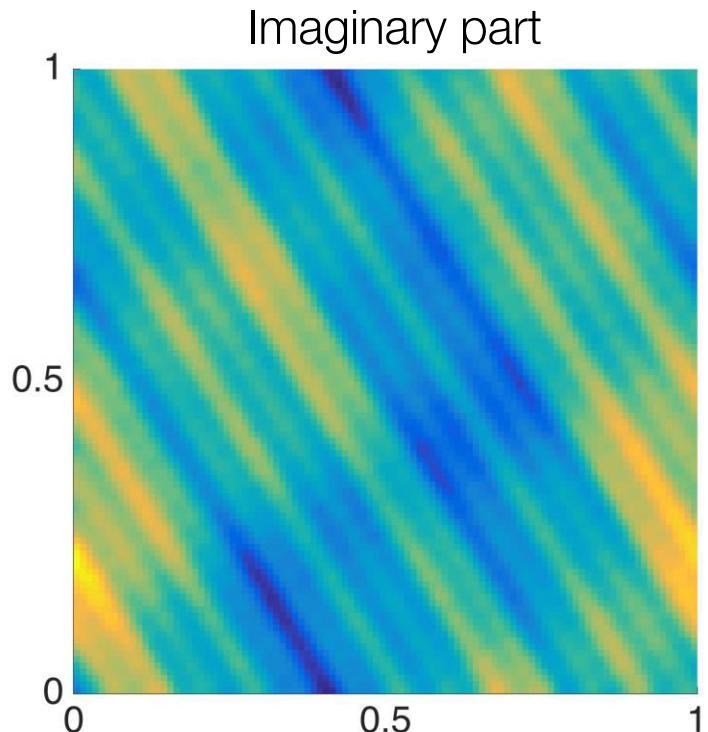
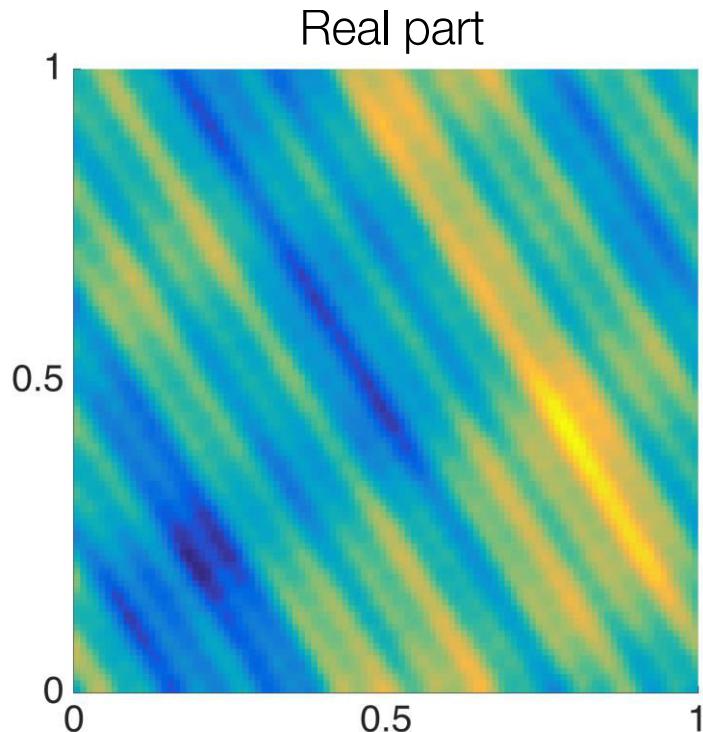


# Example – Cat map

$$\begin{aligned}x_1^+ &= 2x_1 + x_2 \mod 1 \\x_2^+ &= x_1 + x_2 \mod 1\end{aligned}$$

Data: Single trajectory of length  $10^5$

Approximation of  $P_A$  with  $A = [1/8, 3/8]$  and  $N = 100$

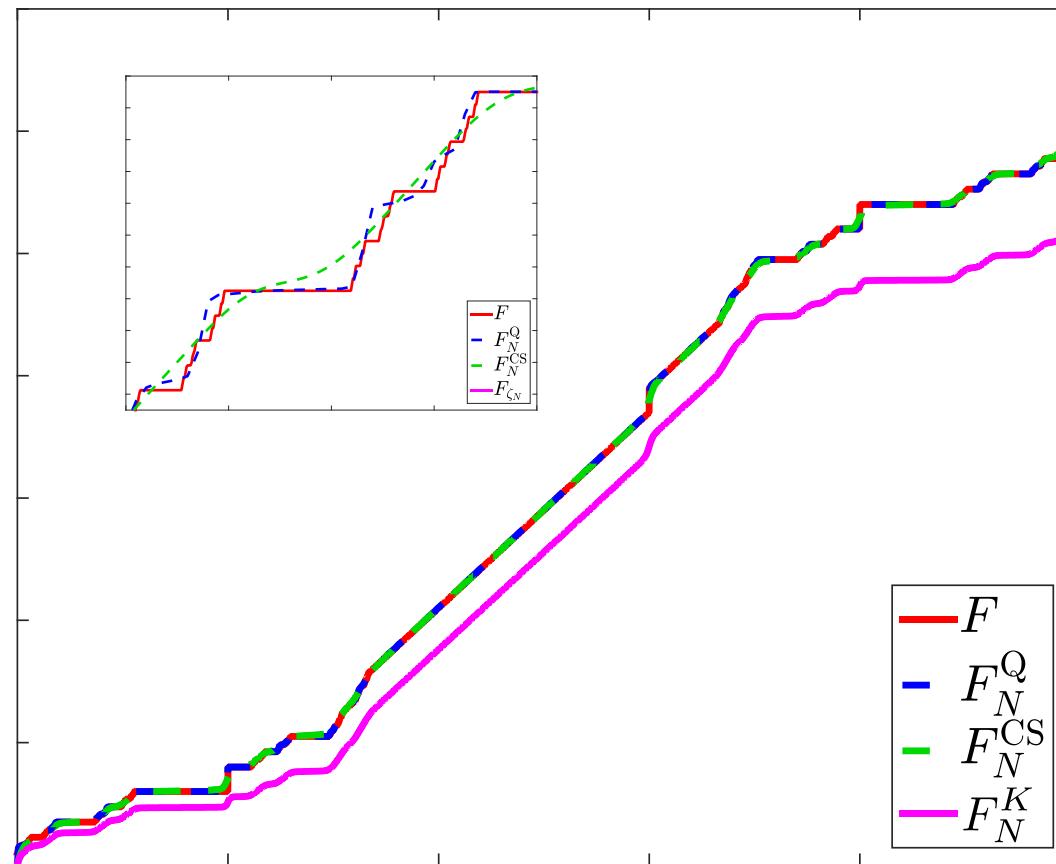


cf. [Govindarajan et al., 2017]

# Detecting SC spectrum - CDFs

$$\mu = 4I_{[0.3,0.7]} d\theta + 0.1(\delta_0 + \delta_{0.2} + \delta_{0.6} + \delta_{0.8}) + \text{Cantor}$$

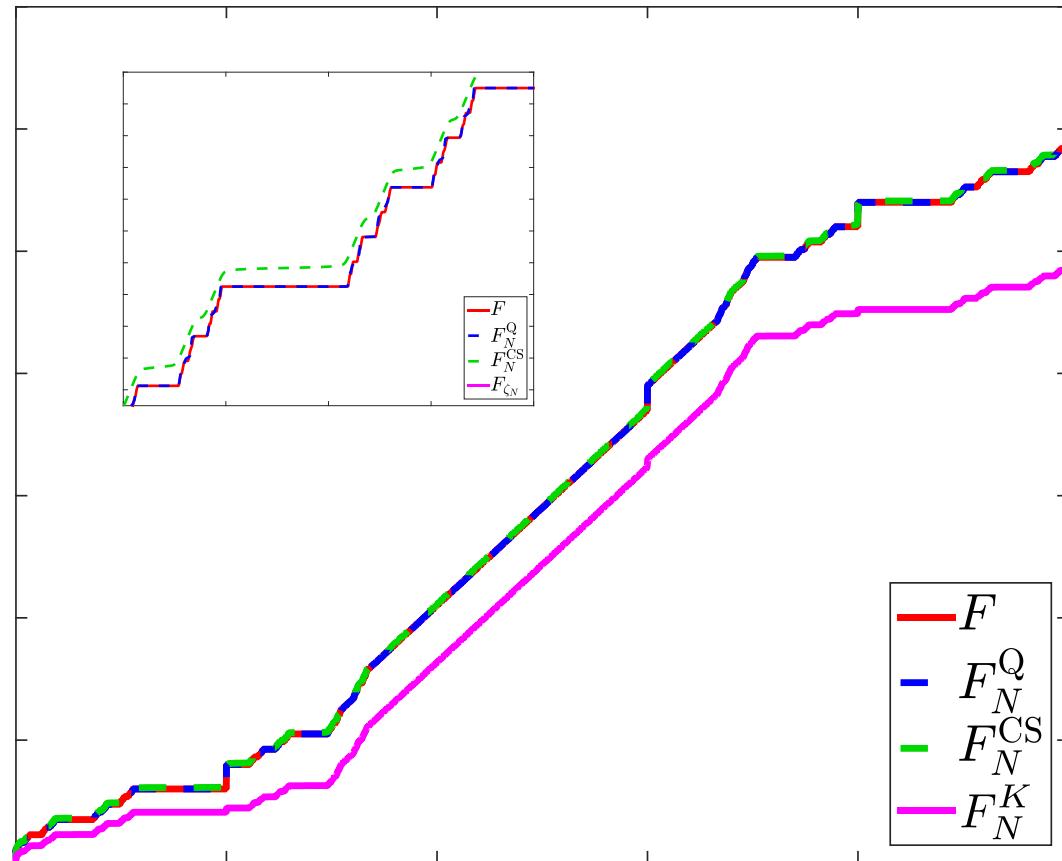
$N = 100$



# Detecting SC spectrum - CDFs

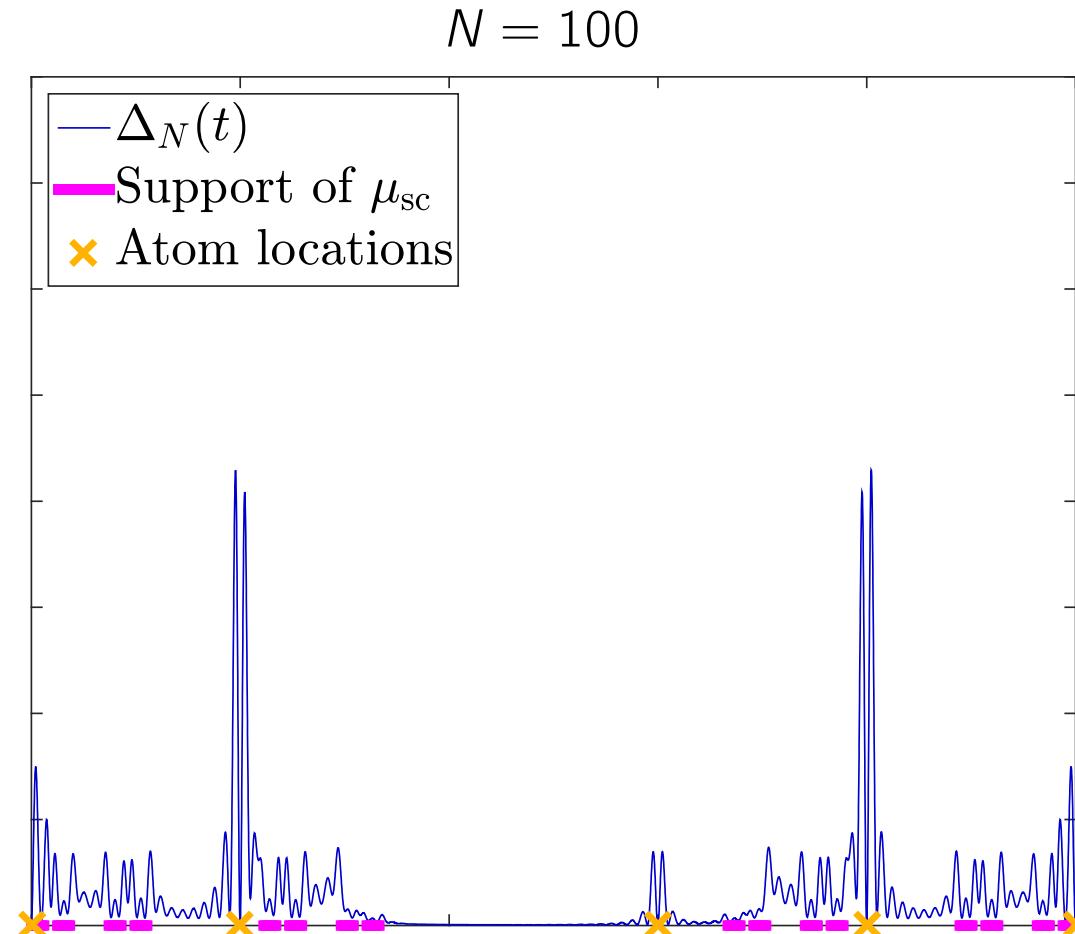
$$\mu = 4I_{[0.3,0.7]} d\theta + 0.1(\delta_0 + \delta_{0.2} + \delta_{0.6} + \delta_{0.8}) + \text{Cantor}$$

$N = 1000$

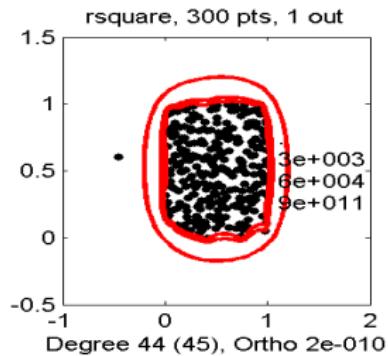
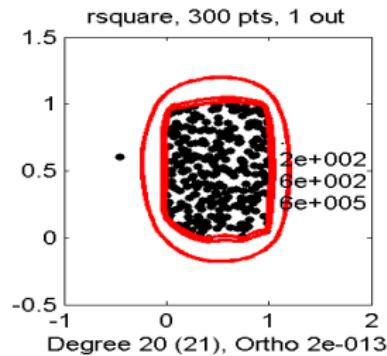
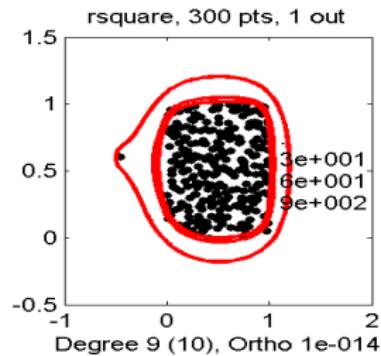
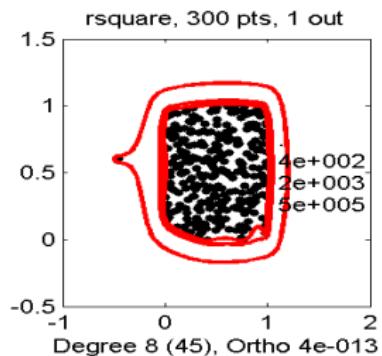
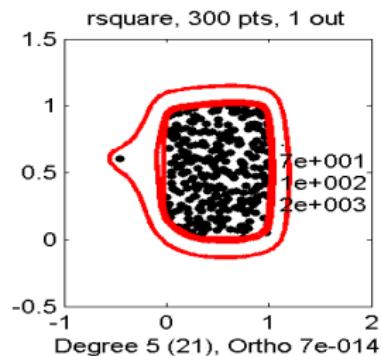
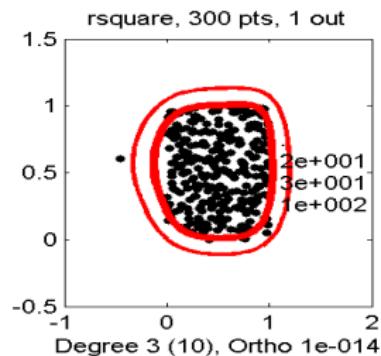


# Detecting SC spectrum – Singularity indicator

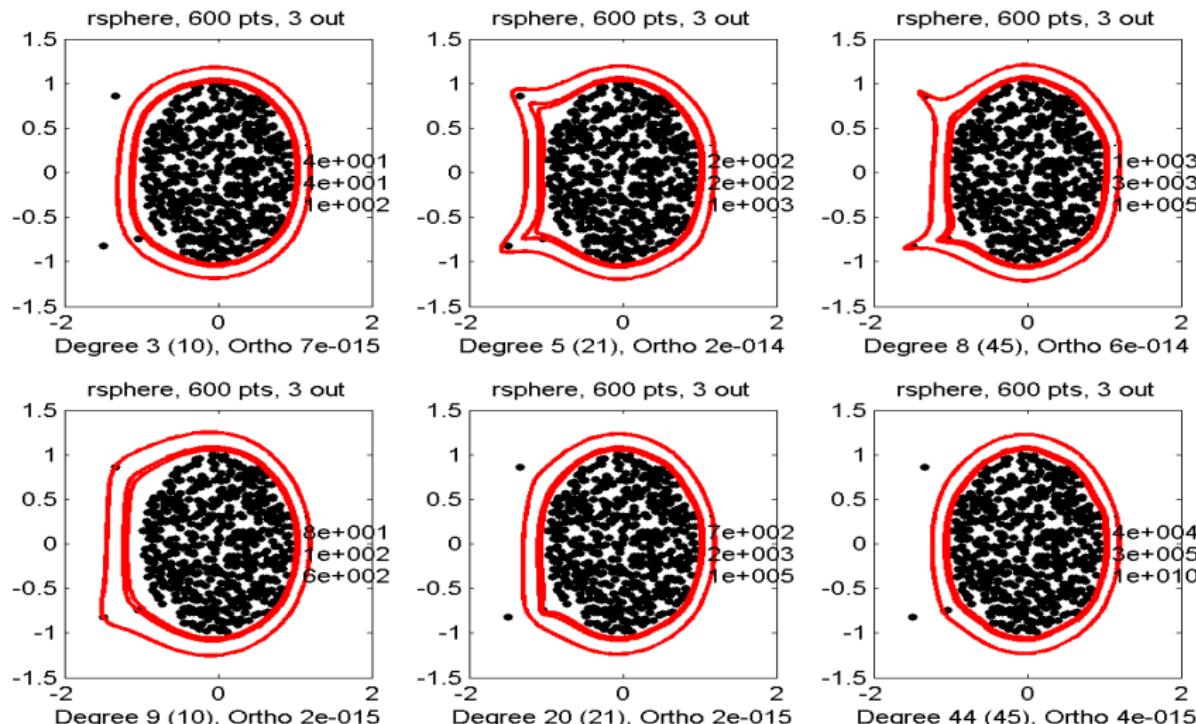
$$\mu = 4I_{[0.3,0.7]} d\theta + 0.1(\delta_0 + \delta_{0.2} + \delta_{0.6} + \delta_{0.8}) + \text{Cantor}$$



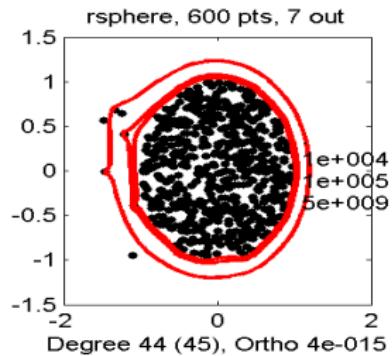
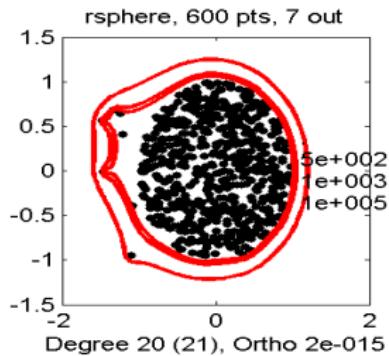
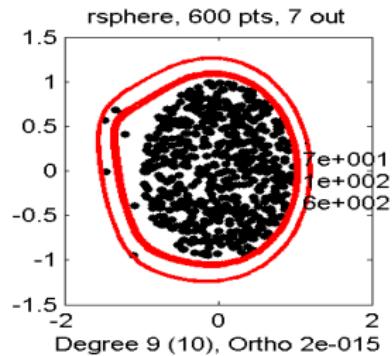
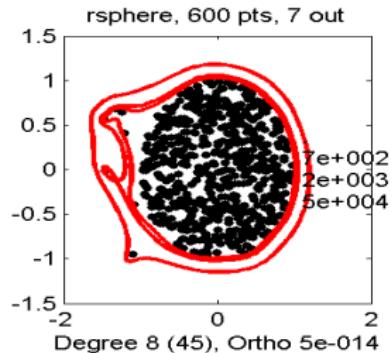
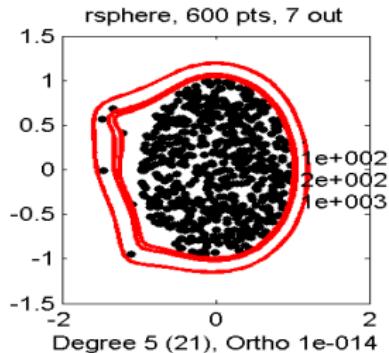
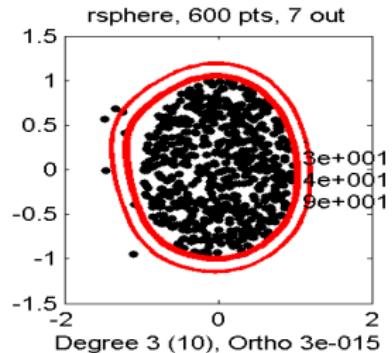
### 3. Detection of outliers in statistical data



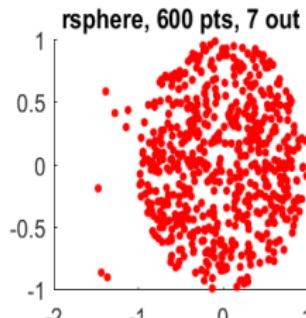
1 outlier,  $N = 300$  shape of square



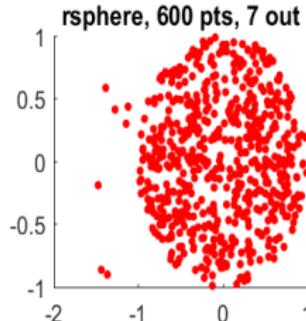
3 outliers,  $N = 300$  disk cloud, "real" level lines do not separate well,  
"complex" do



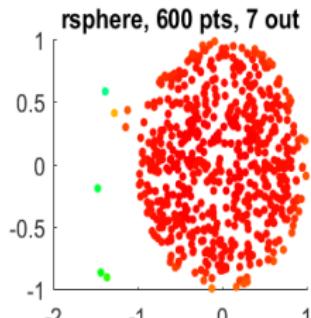
7 outliers,  $N = 300$  disk cloud, "real" level lines do not separate well,  
"complex" do



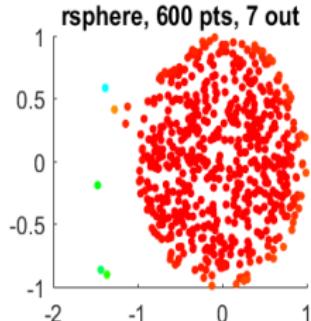
Degree 1 (3), Ortho 2e-16



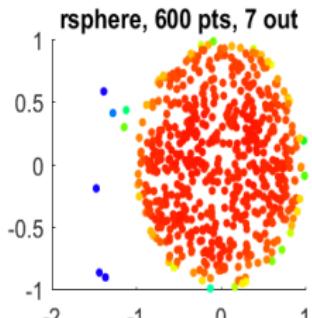
Degree 1 (2), Ortho 2e-16



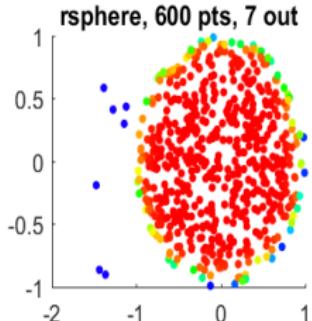
Degree 4 (15), Ortho 3e-15



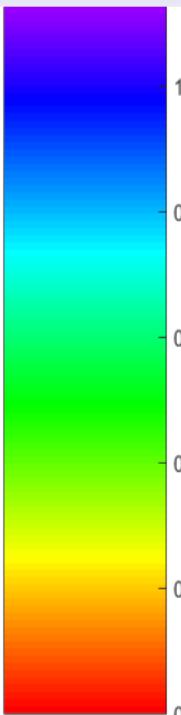
Degree 9 (10), Ortho 6e-16



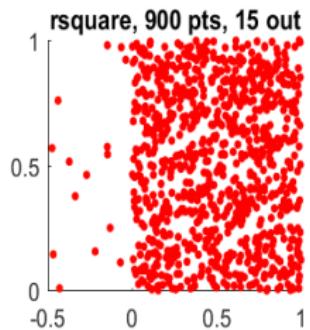
Degree 8 (45), Ortho 3e-14



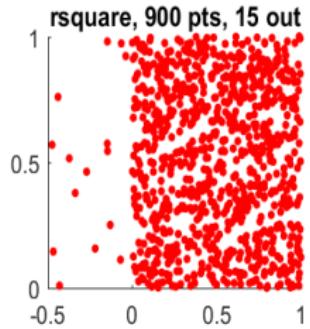
Degree 44 (45), Ortho 9e-16



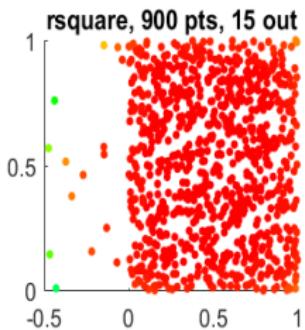
7 outliers,  $N = 600$  disk cloud.



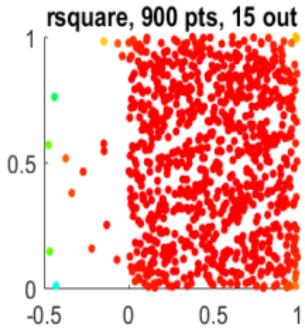
Degree 1 (3), Ortho 7e-16



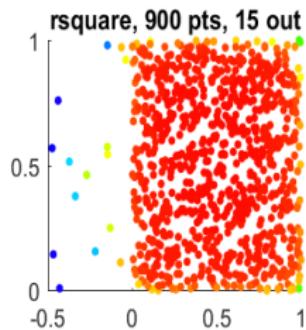
Degree 1 (2), Ortho 7e-16



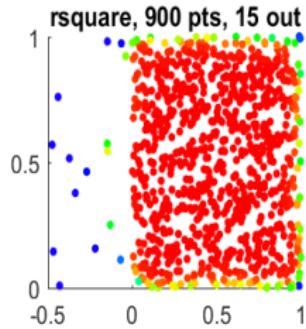
Degree 4 (15), Ortho 5e-15



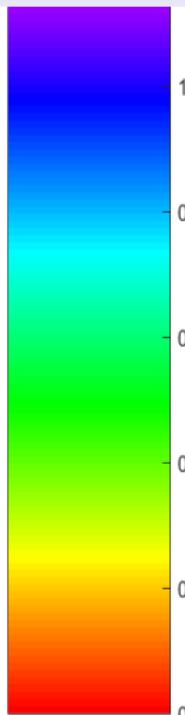
Degree 9 (10), Ortho 2e-15



Degree 8 (45), Ortho 5e-14



Degree 44 (45), Ortho 5e-14



15 outliers,  $N = 900$  square cloud.

# Perturbation of CD kernels

$$\mu_N = \tau_N + \sigma_N, \quad \tau_N = \frac{1}{N} \sum_{j=s+1}^N \delta_{z_j} \approx \tau, \quad \sigma_N = \frac{1}{N} \sum_{j=1}^s \delta_{z_j}.$$

- 1 From discrete  $\tau_N$  to continuous  $\tau$ :  
 $K_n^{\mu_N}(z, z) \approx K_n^{\tau + \sigma_N}(z, z)$ ?
- 2 Level lines of  $K_n^\tau(z, z)$  approach  $\text{supp}(\tau)$  ?
- 3 Bounds for  $K_n^{\mu_N}(z, z)$ 
  - upper bounds on  $\text{supp}(\tau + \sigma_N)$  ?
  - lower bounds outside  $\text{supp}(\tau + \sigma_N)$  ?

## Comparison of measures and kernel (1)

Set  $\mathcal{P}_n = \text{span}\{p_0^\mu, \dots, p_n^\mu\}$  (independent of  $\mu$ ...), then for all  $z \in \mathbb{C}$ ,  $n \geq 0$

$$\frac{1}{K_n^\mu(z, z)} = 1 \left/ \sum_{j=0}^n |p_j^\mu(z)|^2 \right. = \min_{p \in \mathcal{P}_n} \left( \frac{\|p\|_{2,\mu}}{|p(z)|} \right)^2.$$

**Lem 1:** if  $\mu \leq \nu$  then for all  $z$

$$K_n^\mu(z, z) \geq K_n^\nu(z, z).$$

## Comparison of measures and kernel (2)

Set  $\mathcal{P}_n = \text{span}\{p_0^\mu, \dots, p_n^\mu\}$  (independent of  $\mu$ ...), then for all  $z \in \mathbb{C}$ ,  $n \geq 0$

$$\frac{1}{K_n^\mu(z, z)} = 1 \left/ \sum_{j=0}^n |p_j^\mu(z)|^2 \right. = \min_{p \in \mathcal{P}_n} \left( \frac{\|p\|_{2,\mu}}{|p(z)|} \right)^2.$$

Consider (modified) Grammian

$$M_n(\nu, \mu) = \left( \langle p_j^\mu, p_k^\mu \rangle_{\nu, 2} \right)_{j,k=0, \dots, n}, \quad \left( \frac{\|p\|_{2,\nu}}{\|p\|_{2,\mu}} \right)^2 = \frac{\xi^* M_n(\nu, \mu) \xi}{\xi^* \xi}.$$

**Lem 2:** if  $\text{spec}(M_n(\nu, \mu)) \subset [\frac{1}{2}, \frac{3}{2}]$  then

$$\frac{1}{2} K_n^\nu(z, z) \leq K_n^\mu(z, z) \leq \frac{3}{2} K_n^\nu(z, z).$$

## From discrete $\tau_N$ to continuous $\tau$

$$\mu_N = \tau_N + \sigma_N, \quad \tau_N = \frac{1}{N} \sum_{j=s+1}^N \delta_{z_j} \approx \tau, \quad \sigma_N = \frac{1}{N} \sum_{j=1}^s \delta_{z_j}.$$

**Cor 1:** Suppose **(H)** :  $\text{spec}(M_n(\tau_N, \tau)) \subset [\frac{1}{2}, \frac{3}{2}]$ . Then for all  $z$

$$\frac{1}{2} K_n^{\tau_N}(z, z) \leq K_n^\tau(z, z) \leq \frac{3}{2} K_n^{\tau_N}(z, z),$$

$$\frac{1}{2} K_n^{\mu_N}(z, z) \leq K_n^{\tau+\sigma_N}(z, z) \leq \frac{3}{2} K_n^{\mu_N}(z, z).$$

## From discrete $\tau_N$ to continuous $\tau$

$$\mu_N = \tau_N + \sigma_N, \quad \tau_N = \frac{1}{N} \sum_{j=s+1}^N \delta_{z_j} \approx \tau, \quad \sigma_N = \frac{1}{N} \sum_{j=1}^s \delta_{z_j}.$$

**Cor 1:** Suppose **(H)** :  $\text{spec}(M_n(\tau_N, \tau)) \subset [\frac{1}{2}, \frac{3}{2}]$ . Then for all  $z$

$$\frac{1}{2} K_n^{\tau_N}(z, z) \leq K_n^\tau(z, z) \leq \frac{3}{2} K_n^{\tau_N}(z, z),$$

$$\frac{1}{2} K_n^{\mu_N}(z, z) \leq K_n^{\tau+\sigma_N}(z, z) \leq \frac{3}{2} K_n^{\mu_N}(z, z).$$

Recall, e.g., from [A. Chkifa, A. Cohen, G. Migliorati, F. Nobile, and R. Tempone'2015]: for  $s = 0$ , if  $z_1, \dots, z_N$  are samplings of i.i.d. random variables with law given by measure  $\tau$  then

$$\text{Prob}\left( (\mathbf{H}) \text{ is true} \right) \geq 1 - 2(n+1) \exp\left(-\frac{\log(\frac{e}{2})}{2} \frac{N}{\max_{z \in \text{supp}(\tau)} K_n^\tau(z, z)}\right).$$

## Level lines of $K_n^\tau(z, z)$ approach $\text{supp}(\tau)$ ?

**Thm 1:** [Lasserre & Pauwels 2017]: if  $\tau$  is **area measure** on some compact  $S \subset \mathbb{R}^d$  with  $S = \text{Clos}(\text{Int}(S))$  then there exist explicit  $\gamma_n \in \mathbb{R}$  such that the Hausdorff distance between  $S = \text{supp}(\tau)$  and

$$S_n = \{z : K_n^\tau(z, z) \leq \gamma_n\}$$

tends to zero for  $n \rightarrow \infty$ .

Two ingredients of proof (works also for  $\mathbb{C}^d$ ):

- upper bounds for  $K_n^\tau(z, z)$  for  $z$  in compact subsets of  $\text{Int}(S)$ 
  - through Lem 1 via area measures on small balls
  - classical for  $S \subset \mathbb{R}, \mathbb{C}$ , recent progress [Totik'2010] in case of  $C^2$  boundary
  - recent progress [Kroó, Lubinsky 2013] in  $\mathbb{C}^d, \mathbb{R}^d$
- lower bounds for  $K_n^\tau(z, z)$  for  $z$  in compact subsets of  $\mathbb{C} \setminus S$ 
  - through peak polynomials following [Kroó, Lubinsky 2013]
  - logarithmic potential theory (Siciak function) in  $\mathbb{C}$ , pluripotential theory in  $\mathbb{C}^d, \mathbb{R}^d$ .

## Lower bounds for $K_n^{\mu_N}(z, z)$ outside $\text{supp}(\tau + \sigma_N)$

**Lem 3:** Under hypothesis **(H)** for  $z \notin \text{supp}(\tau + \sigma_N)$  :

$$\begin{aligned} \frac{3}{2} \frac{K_n^{\mu_N}(z, z)}{K_n^\tau(z, z)} &\geq \frac{K_n^{\tau+\sigma_N}(z, z)}{K_n^\tau(z, z)} \\ &\geq \det(C_n^\tau(z_j, z_k))_{j,k=0,\dots,s} / \det(C_n^\tau(z_j, z_k))_{j,k=1,\dots,s} \end{aligned}$$

with  $z_0 = z$ , and  $C_n^\tau(z, w) = \frac{K_n^\tau(z, w)}{\sqrt{K_n^\tau(z, z)K_n^\tau(w, w)}}$ .

**Idea of proof:** Writing  $v_n(z) = (p_0^\tau(z), \dots, p_n^\tau(z))$ ,

$$K_n^{\tau+\sigma_N}(z, z) = v_n(z) M_n(\tau + \sigma_N, \tau)^{-1} v_n(z)^*$$

$$M_n(\tau + \sigma_N, \tau) = I + \frac{1}{N} V_n V_n^*, \quad V_n = (v_n(z_1), \dots, v_n(z_s)).$$

# Main Theorem for one complex variable [BB, MP, ES, NS'19]

Consider  $\tau$  area measure on compact simply connected set  $S = \text{supp}(\tau) \subset \mathbb{C}$ , and let  $n, N \rightarrow \infty$  such that **(H)** holds, and

$$\max_{z \in S} K_n^\tau(z, z) \ll N \ll \min_{j=1, \dots, s} \exp(2ng_S(z_j, \infty)).$$

Then uniformly for  $z \in S \supset \text{supp}(\tau_N)$  we have

$$\frac{1}{N} K_n^{\mu_N}(z) = o(1),$$

uniformly on compact subsets of  $\mathbb{C} \setminus \text{supp}(\tau + \sigma_N)$

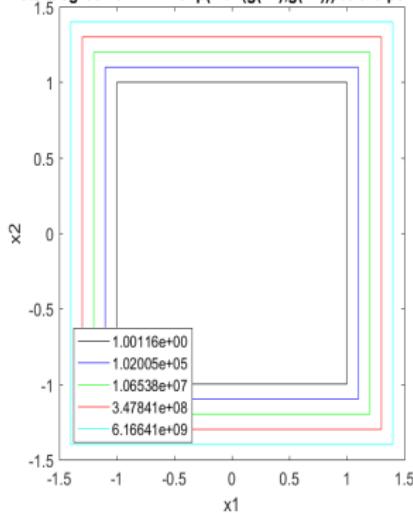
$$\frac{1}{N} K_n^{\mu_N}(z, z) \geq \frac{2}{3N} \exp(2ng_S(z, \infty)) \prod_{j=1}^s \exp(-2g_S(z_j, z))(1 + o(1)),$$

and for  $z_k \in \text{supp}(\sigma_N)$

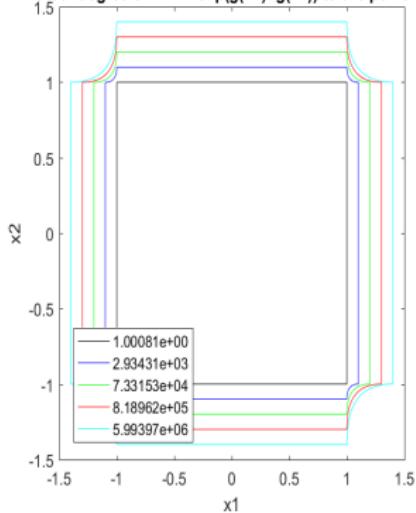
$$\begin{aligned} 1 - \frac{1}{N} K_n^{\mu_N}(z_k, z_k) &\leq \frac{N}{K^{\mu_N - \delta_{z_k}/N}(z_k, z_k)} \\ &\leq \frac{3N}{2} \exp(-2ng_S(z, \infty)) \prod_{j=1, j \neq k}^s \exp(2g_S(z_j, z_k))(1 + o(1)). \end{aligned}$$

# One complex versus 2 real variables

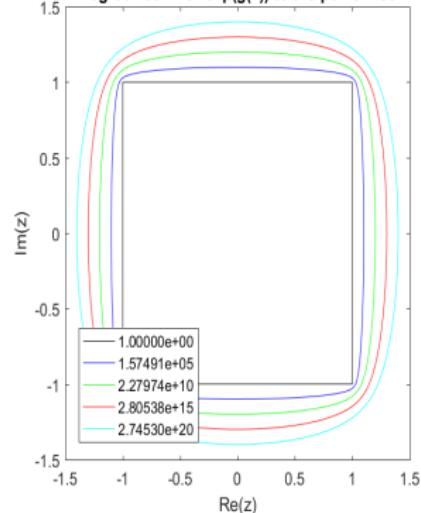
Total degree 13 in  $\mathbb{R}^2$ :  $\exp(\max(g(x_1), g(x_2)))$  to the power 26



Partial degree 9 in  $\mathbb{R}^2$ :  $\exp(g(x_1) + g(x_2))$  to the power 18

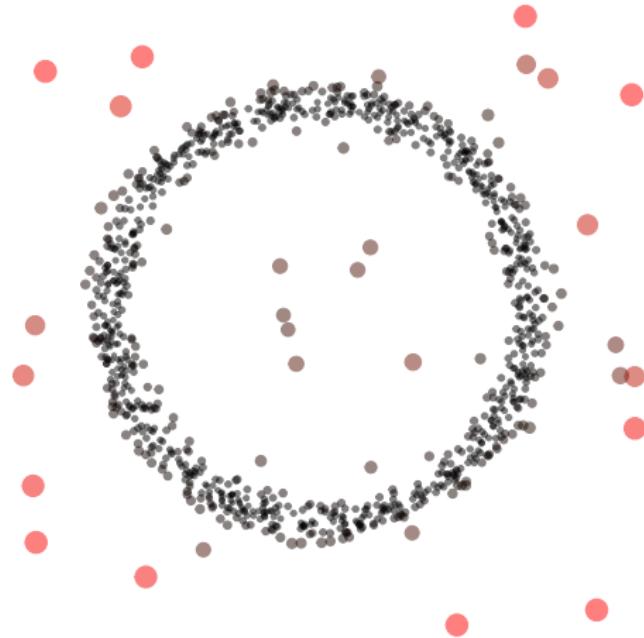


Degree 100 in  $\mathbb{C}$ :  $\exp(g(z))$  to the power 200

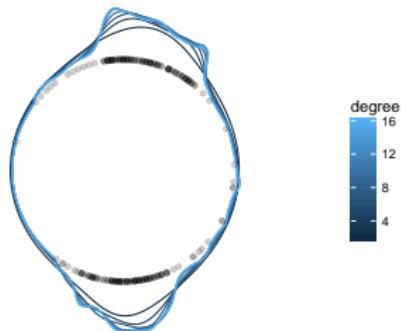


# Supports on algebraic varieties

Detecting a circle:

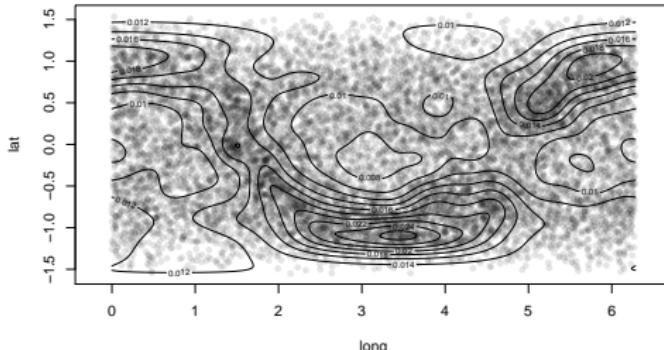


## 2D torus



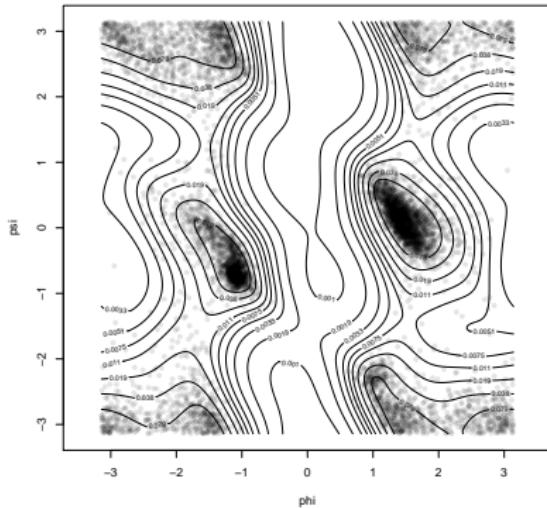
**Figure:** Dragon fly orientation with respect to the sun, on the torus. The curves represent the empirical Christoffel function for different values of the degree.

## 2D sphere



**Figure:** Each point represent the observation of a double star in the sky. They live on the sphere and are associated to their longitude and latitude. The level sets are those of the empirical Christoffel functions evaluated on the sphere in  $\mathbb{R}^3$ . The degree is 8. The band which is highlighted by the level sets corresponds to the Milky Way.

## 2D torus lying on 3D sphere



**Figure:** Each point two dihedral angles for a Glycine amino acid. These angles are used to describe the global three dimensional shape of a protein. They live on the bitorus. The level sets are those of the empirical Christoffel functions evaluated on the sphere in  $\mathbb{R}^4$ . The degree is 4.

## Christoffel-Darboux Kernel for Data Analysis

J. B. Lasserre (Université Toulouse Midi-Pyrénées), E. Pauwels  
(Université Toulouse Midi-Pyrénées), M. Putinar (University of California,  
Santa Barbara and Newcastle University)

The Christoffel-Darboux kernel, a central object in approximation theory, is shown to have many potential uses in modern data analysis, including applications to machine learning. This first text book to offer a rapid introduction to the subject, combining the original effects of the kernel at a sample and bridging the gap between classical mathematics and current leading research, the authors present the topic in detail and follow a heuristic, example-based approach, assuming only a basic background in functional analysis, probability and some elementary notions of algebraic geometry. They cover new results in both pure and applied mathematics and introduce techniques that have a wide range of potential impacts on modern quantitative and qualitative science. Comprehensive notes provide historical background, discuss advanced concepts and give detailed bibliographical references. Researchers and graduate students in mathematics, statistics, engineering or economics will find new perspectives on traditional themes, along with challenging open problems.

- The first comprehensive account of the Christoffel-Darboux kernel covering a century and a half of development.
- Offers a rapid introduction to the subject as an interdisciplinary subject for non-expert readers.
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### 3. Approximation in the mean by complex polynomials

Let  $\mu$  be a positive, compactly supported measure on  $\mathbb{C}$ . We denote by  $P^2(\mu)$  the closure of complex polynomials in  $L^2(\mu)$ , and  $R^2(F, \mu)$  the closure in  $L^2(\mu)$  of rational functions with poles disjoint of  $F$ .

In general, the spectral analysis of the multiplier  $S_\mu = M_z : P^2(\mu) \longrightarrow P^2(\mu)$  reveals (constructively) the nature of the measure  $\mu$ . Beyond the spectral theorem for normal operators.

**Problem.** When is  $P^2(\mu) = L^2(\mu)$ , or  $R^2(F, \mu) = L^2(\mu)$ ?

## Thomson's Theorem, 1991

Let  $\mu$  be a positive Borel measure, compactly supported on  $\mathbb{C}$ . There exists a Borel partition  $\Delta_0, \Delta_1, \dots$  of the closed support of  $\mu$  with the following properties:

- 1)  $P^2(\mu) = L^2(\mu_0) \oplus P^2(\mu_1) \oplus P^2(\mu_2) \oplus \dots$ , where  $\mu_j = \mu|_{\Delta_j}$ ,  $j \geq 0$ ;
- 2) Every operator  $S_{\mu_j} = M_z$ ,  $j \geq 1$ , is irreducible with spectral picture:

$$\sigma(S_{\mu_j}) \setminus \sigma_{\text{ess}}(S_{\mu_j}) = G_j, \text{ simply connected,}$$

and

$$\text{supp } \mu_j \subset \overline{G_j}, \quad j \geq 1;$$

- 3) If  $\mu_0 = 0$ , then any element  $f \in P^2(\mu)$  which vanishes  $[\mu]$ -a.e. on  $G = \cup_j G_j$  is identically zero.

## Analytic bounded point evaluations

Let  $\mu$  be a positive Borel measure, compactly supported on  $\mathbb{C}$ .  
Then

$$P^2(\mu) \neq L^2(\mu)$$

if and only if there exists an open set  $U$  of *analytic bounded point evaluations*

$$|f(a)| \leq C\|f\|_{2,\mu}, \quad f \in \mathbb{C}[z], \quad a \in U,$$

where  $C$  does not depend on  $a$ .

## Brennan's Theorem, 2008

*Let  $\mu$  be a positive measure without point masses, compactly supported on  $\mathbb{C}$ . Assume the set  $F$  contains the closed support of the measure  $\mu$ , and the complement  $\mathbb{C} \setminus F$  does not have components of arbitrarily small diameter.*

*Then  $R^2(F, \mu) \neq L^2(\mu)$  if and only if  $R^2(F, \mu)$  admits analytic bounded point evaluations:*

$$|f(a)| \leq C\|f\|_{2,\mu}, \quad f \in R^2(F, \mu), \quad a \in U \text{ (open set)}.$$

Novel technical ingredient and simplification: X. Tolsa  
semi-additivity of analytic capacity.

## Classification of irreducible subnormal operators...

is a matter of complex hermitian geometry: the kernel of the linear pencil

$$z \mapsto \ker(S^* - \bar{z})$$

is, on the cloud of analytic bounded point evaluations, a hermitian line bundle. Curvature type invariants determine the unitary orbit of the generating operator  $S$ .

## Density of complex polynomials on the unit circle

**Theorem (Kolmogorov, 1941, Krein, 1945)** *For a positive measure  $\mu$  supported on  $\mathbb{T}$ , one has  $P^2(\mu) = L^2(\mu)$  if and only if*

$$\int_{-\pi}^{\pi} |\log \mu'| dt = \infty.$$

Or equivalently, the multiplier  $M_z$  is a normal operator. In the opposite case, (Szegő's limit theorem gives even more),

$$\dim \ker(M_z - w)^* = 1, \quad |w| < 1,$$

and hence the Fredholm index of  $M_z$  is  $-1$  at every point of the unit disk.

Carry the analysis on affine, algebraic curves

**Theorem (S. Biswas-M.P., 2022)** *Let  $X$  be a rational curve in  $\mathbb{C}^n$  and let  $\mu$  be a positive Borel measure without point masses, supported by a compact subset of  $X$ . Then  $P^2(\mu) \neq L^2(\mu)$  if and only if there are analytic  $P^2(\mu)$ -bounded point evaluations.*

## Idea of proof

Let  $R = (r_1, r_2, \dots, r_n)$  be an  $n$ -tuple of rational functions which properly parametrizes the rational affine curve  $X \subset \mathbb{C}^n$ . That is, denoting by  $S \subset \mathbb{C}$  the poles of  $R$ , the holomorphic map

$$R : \mathbb{C} \setminus S \longrightarrow X$$

is one to one, except finitely many points, and it covers  $X$  except finitely many points.

Let  $\rho$  denote a sufficiently large radius, so that the support of the measure  $\mu$  is contained in the ball  $B(0, \rho)$ . The pull-back

$$U = R^{-1}B(0, \rho)$$

is an open subset of  $\mathbb{C}$ , of finite connectivity, with piece-wise smooth boundary. In particular we can assume that every connected component of the complement of  $U$  has positive diameter.

The restricted analytic map

$$R : U \longrightarrow B(0, \rho)$$

has finite fibres, hence it is proper. Grauert's finiteness theorem implies that the direct image sheaf  $R_* \mathcal{O}_U$  is coherent and

$$R_* \mathcal{O}_U(B(0, \rho)) = \mathcal{O}(U).$$

The coherence of  $R_* \mathcal{O}_U$  and the injectivity of  $R$  modulo a finite set imply that

$$\dim \mathcal{O}(U)/R^* \mathcal{O}(B(0, \rho)) < \infty.$$

One can define a pull-back measure  $\nu$  on  $U$

$$\int \phi \, d\mu = \int \phi \circ R \, d\nu,$$

for every continuous function  $\phi : B(0, \rho) \longrightarrow \mathbb{C}$ .

Let  $R^2(U, \nu)$  denote the closure in  $L^2(\nu)$ , of rational functions with poles on the complement of  $U$ . Runge's approximation theorem implies that  $R^2(U, \nu)$  is also the closure of the algebra  $\mathcal{O}(U)$  in  $L^2(\nu)$ . Hence

*There exist analytic bounded point evaluations with respect to  $P^2(\mu)$  if and only if there exists analytic bounded point evaluations with respect to  $R^2(U, \nu)$ .*

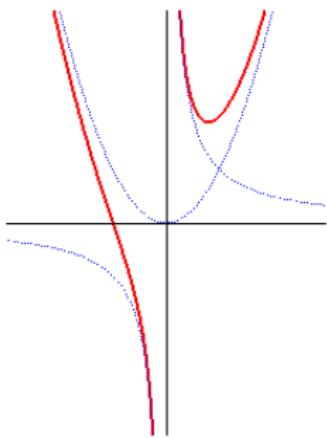
Brennan's Theorem completes the proof.

## Invariant subspaces for subnormal tuples

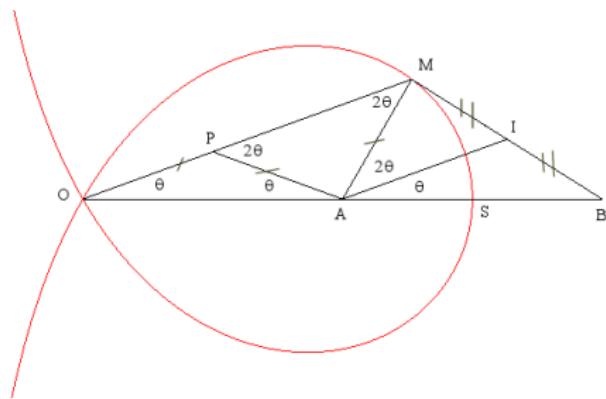
*A commuting subnormal tuple with Taylor's joint spectrum contained in a rational curve admits joint invariant subspaces.*

As a matter of fact, a continuum of invariant subspace of codimension one exists, parametrized by a relatively open subset of the joint spectrum.

# Newton's Trident: $xy = x^3 + 1$



Maclaurin trisectrix:  $x(x^2 + y^2) = a(3x^2 - y^2)$ ,  
 $x = a \frac{3-t^2}{1+t^2}$ ,  $y = tx$ .



## The general case

Resolution of singularities and our base change technique lead to the following **Open Question**:

*Let  $X$  be an open Riemann surface of finite genus, and let  $\mu$  be a positive Borel measure on  $X$  without point masses. Let  $U \subset X$  be an open, relatively compact subset of  $X$ , with finitely many components of  $X \setminus U$ , none reduced to a point. Assume the closed support of the measure  $\mu$  is contained in  $U$ . Then analytic functions  $\mathcal{O}(U)$  are dense in  $L^2(\mu)$  if and only if there are no corresponding bounded analytic point evaluations.*

# Uniform approximation on Riemann surfaces

Let  $X$  be an open Riemann surface. Assume that the measure  $\mu$  is supported by a piecewise smooth curve  $\Gamma \subset X$  with the property that the complement  $X \setminus \Gamma$  is connected. Then  $P^2(\mu) = L^2(\mu)$ .

Proof derived from **Scheinberg's Theorem** (on uniform approximation).

## Counterexample to Szegő's condition on the unit circle

Let  $Q(w)$  be a polynomial of degree at least three, with simple roots. The hyperelliptic curve

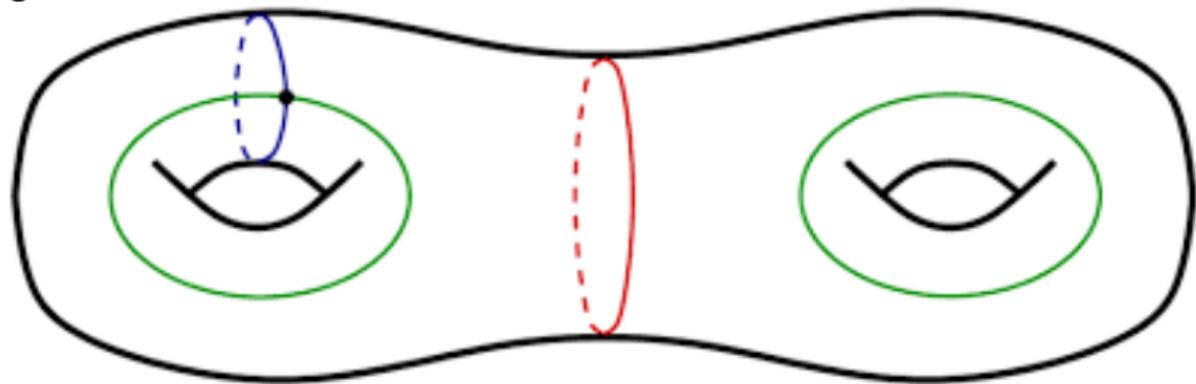
$$X = \{(z, w) \in \mathbb{C}^2; z^2 = Q(w)\}$$

has a single point at infinity, and it is not rational.

Assume  $X$  has genus 1. Let  $\Gamma \subset X$  be a smooth, simple, closed curve, non homotopically trivial on  $\hat{X}$ . Let  $\mu$  be **any** positive measure supported on  $\Gamma$ .

Then  $P^2(\mu) = L^2(\mu)$ , and indeed  $\mu$  does not admit ABPE.

genus two



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## Constructive approximation of the cloud of ABPE

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# Géza Freud, Orthogonal Polynomials and Christoffel Functions. A Case Study\*

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## DEDICATED TO THE MEMORY OF GÉZA FREUD

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# The Christoffel–Darboux Kernel

Barry Simon\*

ABSTRACT. A review of the uses of the CD kernel in the spectral theory of orthogonal polynomials, concentrating on recent results.

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