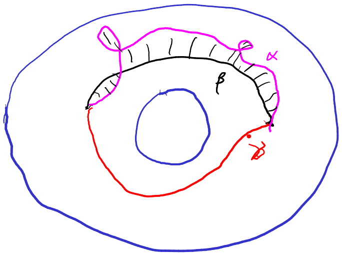
Homotopy of paths

Let $\alpha, \beta : [0,1] \to X$ be paths, i.e., continuous functions.
A homotopy fixing endpoints from α to β is a continuous function

$$H : [0,1] \times [0,1] \to X$$

such that

* $H(s,0) = \alpha(s) \forall s \in [0,1]$
* $H(s,1) = \beta(s) \forall s \in [0,1]$
* $t \mapsto H(0,t)$ is a constant function.
* $t \mapsto H(1,t)$ is a constant function.



Theorem: Homotopy fixing endpoints is an equivalence relation between paths (with given endp
Proof:
* Reflexivity:

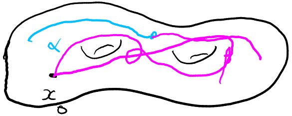
* Symmetry:

* Transitivity:

Transitivity:

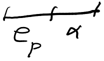
$$\alpha * \beta = \begin{cases} \alpha(2s), & 0 \le s \le \frac{1}{2} \\ \beta(2s-1), & \frac{1}{2} \le s \le 1 \end{cases}$$

$$\beta(2s-1) = \beta\left(2\left(s-\tfrac{1}{2}\right)\right)$$

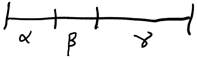


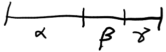
Not (yet) a group

- $*$ is partially defined
- $*$, e_p etc. do not satisfy group laws.

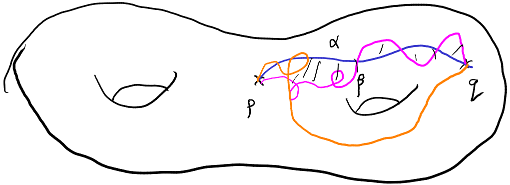


  - $e_p * \alpha \neq \alpha$ $\quad (\alpha(0) = p)$
  - $(\alpha * \beta) * \gamma \neq \alpha * (\beta * \gamma)$



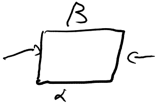


Homotopy of paths fixing endpoints



$H : \underset{s}{[0,1]} \times \underset{t}{[0,1]} \longrightarrow X$ continuous

- H is a homotopy fixing endpoints from α to β if



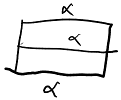
  - $H(s,0) = \alpha(s)$
  - $H(s,1) = \beta(s)$
- The functions $t \mapsto H(0,t)$ and $t \mapsto H(1,t)$ are constant
- $\alpha \sim \beta$ if $\exists H$ as above.

Theorem: ~ gives equivalence relations

$\sim$ on $\Omega(X)$

$\sim$ on $\Omega(X; p-q)$

(1) Reflexive : $\alpha \sim \alpha$; $H(s,t) = \alpha(s)$

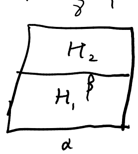


(2) Symmetric : $\alpha \sim \beta \Rightarrow \exists H : [0,1] \times [0,1] \to X$ s.t. $H(s,0) = \alpha(s)$, $H(s,1) = \beta(s)$

Define $\bar{H}(s,t) = H(s, 1-t)$

(3) Transitive : $\alpha \sim \beta$; $\beta \sim \gamma$, let H_1 , H_2 be the homotopies

Define H =

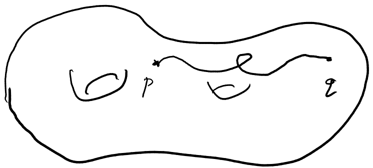


; $H(s,t) = \begin{cases} H_1(s, 2t), & 0 \le t \le 1/2 \\ H_2(s, 2t-1), & \frac{1}{2} \le t \le 1 \end{cases}$

$\pi_1(X; p, q) = \Omega(X; p, q)/\sim$

Algebraic structure up to homotopy

Theorem: $\alpha : [0,1] \to X$ path with $\alpha(0) = p$ and $\alpha(1) = q$

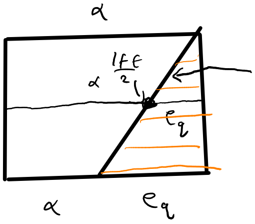


(a) $e_p * \alpha \sim \alpha$

(b) $\alpha * e_q \sim \alpha$

Proof (b): $H : [0,1] \times [0,1] \to X$

$$H(s,t) = \begin{cases} \alpha\left(\dfrac{2s}{1+t}\right) & 0 \le s \le \dfrac{1+t}{2} \\ q & 1 \ge s \ge \dfrac{1+t}{2} \end{cases}$$



$s = at + b$

$t = 0,\ s = \frac{1}{2} \Rightarrow b = \frac{1}{2}$

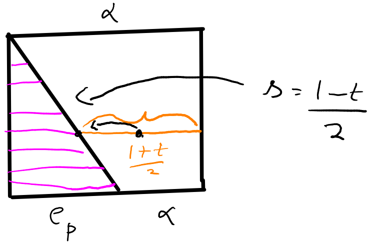
$t = 1,\ s = 1 \Rightarrow a \cdot 1 + \frac{1}{2} = 1 \Rightarrow a = \frac{1}{2}$

$s = \dfrac{(1+t)}{2}$

$\alpha(cs)$; $\alpha\left(c \cdot \left(\frac{1+t}{2}\right)\right) = \alpha(1) \Leftarrow c = \dfrac{2}{1+t}$

(a) $e_p * \alpha \sim \alpha$

$$H(s,t) = \begin{cases} e_p, & 0 \le s \le \frac{1-t}{2} \\ \alpha\left(\frac{2s-1+t}{1+t}\right), & \frac{1-t}{2} \le s \le 1 \end{cases}$$



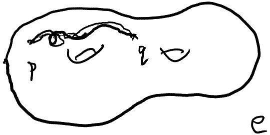
gives a homotopy between $e_p * \alpha$ & α

$$\alpha\left(\frac{2}{1+t}\left(s - \left(\frac{1-t}{2}\right)\right)\right)$$

$$\shortparallel$$

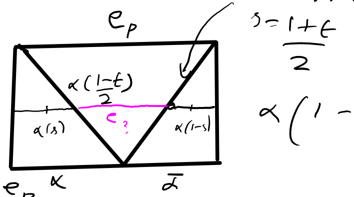
$$\alpha\left(\frac{2s-1+t}{1+t}\right)$$

Theorem: $\alpha * \bar{\alpha} \sim e_p$



Pf:
$$H(s,t) = \begin{cases} \alpha(s), & 0 \le s \le \frac{1-t}{2} \\ \alpha\left(\frac{1-t}{2}\right), & \frac{1-t}{2} \le s \le \frac{1+t}{2} \\ \alpha(1-s), & \frac{1+t}{2} \le s \le 1 \end{cases}$$

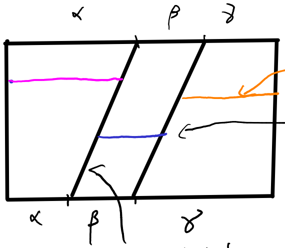
$\overset{\shortparallel}{\bar{\alpha}(s)}$



$$\alpha\left(1 - \frac{1+t}{2}\right) = \alpha\left(\frac{1-t}{2}\right)$$

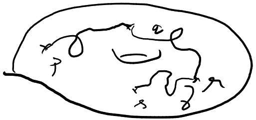
Gives a homotopy between $\alpha * \bar{\alpha}$ and e_p

Theorem: $(\alpha * \beta) * \gamma \sim \alpha * (\beta * \gamma)$



$s = \dfrac{2+t}{4}$

$s = \dfrac{1+t}{4}$



$1 - \dfrac{2+t}{4} = \dfrac{4-(2+t)}{4} = \dfrac{2-t}{4}$

$\gamma\left(\dfrac{4}{2-t}\left(s - \dfrac{t+2}{4}\right)\right)$

$\gamma\left(\dfrac{4s-t-2}{2-t}\right)$

$\beta\left(4\left(s-\left(\dfrac{1+t}{4}\right)\right)\right)$

$= \beta(4s-1-t)$

Pf: A homotopy is given by

$$H(s,t) = \begin{cases} \alpha\left(\dfrac{4s}{1+t}\right), & 0 \le s \le \dfrac{1+t}{4} \\[2mm] \beta(4s-1-t), & \dfrac{1+t}{4} \le s \le \dfrac{2+t}{4} \\[2mm] \gamma\left(\dfrac{4s-t-2}{2-t}\right), & \dfrac{2+t}{4} \le s \le 1 \end{cases}$$

Groupoids:

- X space

  $\mathrm{Homeo}(X) = \{ f : X \to X \mid f \text{ invertible} \}$

- V vector space

  $GL(V) = \{ L : V \to V \mid \text{invertible, linear} \}$

Groups

E.g. X space, $\{ f : U \to V \mid U, V \subset X \text{ open}, f \text{ invertible} \}$

$\{ L : W_1 \to W_1 \mid W_1, W_2 \subset V \text{ subspaces}, L \text{ invertible} \}$

Rk: $\mathrm{Homeo}(X)$ are morphisms of $\mathcal{C}$

- $\mathrm{Ob}(\mathcal{C}) = \{X\}$
- $\mathrm{Mor}(X, X) = \{ f : X \to X \mid f \text{ invertible} \}$

Categories from Groups: Fix G a group

$\mathcal{C}_G$ - category ; $Ob(\mathcal{C}) = \{*\}$, $Mor(*, *) = G$, composition: group multiplication

$\mathcal{C}_G$ special

(1) Small category : Objects, Morphisms Sets.

(2) Every morphism is invertible

(3) $Ob(\mathcal{C})$ is a singleton. (say $\{*\}$)

. Given such a category $\mathcal{C}$, $\mathcal{C} = \mathcal{C}_G$, $G = Mor(*, *)$

Definition: A groupoid is a small category where every morphism is invertible

E.g. (1) $\{h : W_1 \to W_1 : W_1, W_2 \subset V,\ h \text{ invertible}\}$ is a 'groupoid'

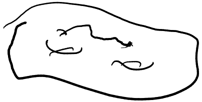
$$\mathrm{Ob}(\mathcal{C}) = \{W \subset V \text{ subspace}\}$$

$$\mathrm{Mor}(W_1, W_2) = \{h : W_1 \to W_2 \mid h \text{ invertible}\}$$

(2) X topological space

Fundamental groupoid

$$\pi_1(X; p, q) = \Omega(X; p, q)/\sim$$



Define $\mathcal{C}_X$ by $\mathrm{Ob}(\mathcal{C}_X) = X$ (as a set)

$$\mathrm{Mor}(\underset{\in X}{p}, \underset{\in X}{q}) = \pi_1(X; p, q) = \{[\alpha] : \alpha \in \Omega(X; p, q)\}$$

- 'Composition' is $*$
- $e_p \in \mathrm{Mor}(p, p)$ - identity

Category from 'morphisms', $*$ (appropriate)

- In a group, e is characterized by $e^2 = e$
- Given a groupoid, with elements G

$$Ob(\mathcal{C}) = \{ e \in G : e * e \text{ is defined and } e * e = e \}$$

$$Mor(e_1, e_2) = \{ g \in G : e_1 * g \text{ and } g * e_2 \text{ are defined and } e_1 * g = e_1 , g * e_2 = g \}$$

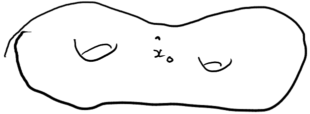
Groupoid $\to$ Group

given $X \in ob(\mathcal{C})$

$$G = Mor(X, X)$$

Fundamental Group:

- X topological space
- $x_0 \in X$ basepoint



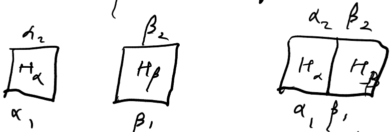
- $\Omega(X, x_0) := \Omega(X; x_0, x_0) = \{\alpha : [0,1] \to X \,;\, \alpha(0) = \alpha(1) = x_0\}$
- $*$, $\alpha \mapsto \bar{\alpha}$ are defined on $\Omega(X, x_0)$
- $\pi_1(X, x_0) = \Omega(X, x_0)/\sim$

Theorem: $\alpha_1 \sim \alpha_2$, $\beta_1 \sim \beta_2$; $\alpha_1 * \beta_1$ is defined; $\alpha_1(1) = \beta_1(0)$

(a) $\alpha_2 * \beta_2 \sim \alpha_1 * \beta_1$

(b) $\bar{\alpha}_1 \sim \bar{\alpha}_2$

Pf (a)



(b) 'flip' horizontally, $\hat{H}_\alpha(s,t) = H_\alpha(1-s, t)$

Thus, we have induced operations

- $$\begin{array}{ccccc} \pi_1(X,x_0) & \times & \pi_1(X,x_0) & \longrightarrow & \pi_1(X,x_0) \\ [\alpha] & & [\beta] & \longmapsto & [\alpha * \beta] \\ \shortparallel & & \shortparallel & & \shortparallel \\ [\alpha_1] & & [\beta_1] & & [\alpha_1 * \beta_1] \end{array}$$

- $$\begin{array}{ccc} \pi_1(X,x_0) & \longrightarrow & \pi_1(X,x_0) \\ [\alpha] & \longmapsto & [\bar{\alpha}] \end{array}$$

<u>Theorem:</u> $\pi_1(X,x_0)$ with these operations is a group

<u>Pf:</u> (a) $*$ is associative: $([\alpha]*[\beta])*[\gamma] = [(\alpha*\beta)*\gamma] = [\alpha*(\beta*\gamma)]$ as $(\alpha*\beta)*\gamma \sim \alpha*(\beta*\gamma)$

(b) $e = [e_{x_0}]$, $e*[\alpha] = [\alpha]*e = [\alpha]$

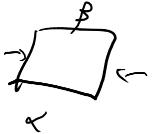
$e*[\alpha] = [e*\alpha] = [\alpha]$

(c) $[\alpha]*[\bar{\alpha}] = e = [\bar{\alpha}]*[\alpha]$

$\pi_1(X, x_0)$ is called the <u>fundamental group</u> of (X, x_0).

<u>E.g.</u> $X \subseteq \mathbb{R}^n$ is convex.

<u>Propn:</u> $\alpha, \beta : [0,1] \to X$, then $\alpha \sim \beta \iff \alpha(0) = \beta(0)$ and $\alpha(1) = \beta(1)$

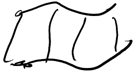


<u>Pf:</u> · If $\alpha \sim \beta$, H is a homotopy, as

$t \mapsto H(0,t)$ is constant,

$\alpha(0) = H(0,0) = H(0,1) = \beta(0)$; $///^{ly}$ $\alpha(1) = \beta(1)$

· Conversely, if $\alpha(0) = \beta(0)$, $\alpha(1) = \beta(1)$, define



$$H(s,t) = (1-t)\,\alpha(s) + t \cdot \beta(s)$$

$t \mapsto H(0,t)$ is $t \mapsto (1-t)\alpha(0) + t\,\overset{\alpha(0)}{\overset{\shortparallel}{\beta(0)}} = \alpha(0)$ <u>constant</u>

<u>Cor:</u> $\pi_1(X, x_0) = \{e\}$ 

Theorem: $\pi_1(S^1, 1) = \mathbb{Z}$

- We define $\tilde{\varphi} : \Omega(S^1, 1) \to \mathbb{Z}$ (path lifting)
- We show $\alpha \sim \beta$, $\tilde{\varphi}(\alpha) = \tilde{\varphi}(\beta)$, (homotopy lifting)

  so have $\varphi : \pi_1(S^1, 1) \to \mathbb{Z}$
- In fact, φ is an isomorphism.

We have

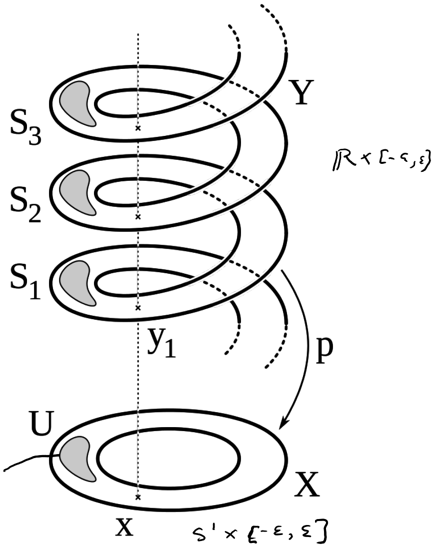
$$p : \mathbb{R} \to S^1, \quad p(t) = e^{2\pi i t}$$

Defn: Given $p : \hat{X} \to X$ continuous, surjective. Then $U \subset X$ open is evenly covered if

$$p^{-1}(U) = \coprod_{\alpha \in A} V_\alpha, \quad V_\alpha \text{ open in } \hat{X},$$

$p|_{V_\alpha} : V_\alpha \to U$ is a homeomorphism

Evenly covered open set



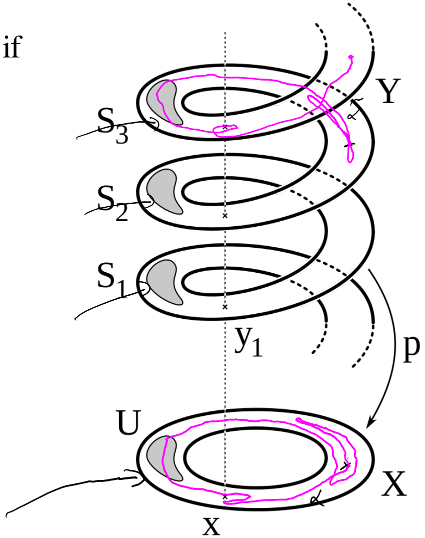
* We have $p : \mathbb{R} \to S^1,\ p(t) = e^{2\pi i t}$

* Given $p : Y \to X$ continuous, we say that an open set U is evenly covered if

$$p^{-1}(U) = \coprod_{\alpha} V_\alpha,\quad V\alpha \text{ open in Y}$$

$$p|_{V_\alpha} : V_\alpha \to U \text{ is a homeomorphism}$$

* A covering map is a surjective continuous function such that every point has an evenly covered neighbourhood.



Goal: Associate to $\alpha : [0,1] \to S^1$, $\alpha(0) = \alpha(1) = 1$ an element in $\mathbb{Z}$
$\qquad S^1 = X$

· Given $\alpha : [0,1] \to S^1$, $\tilde{\alpha}$ satisfies

$$\begin{tikzcd} & \mathbb{R} = Y \arrow[d, "p"] \\ {[0,1]} \arrow[r, "\alpha"] \arrow[ur, dashed, "\tilde{\alpha}"] & S^1 = X \end{tikzcd}$$

lift

and $\tilde{\alpha}(0)$ is known

Define $\varphi(\alpha) = \tilde{\alpha}(1) - \tilde{\alpha}(0)$
$\alpha \in \Omega(S^1, 1)$

We proceed inductively, by solving instances of
Relative Lifting problem $W \subset Z$ closed

$$\begin{tikzcd} W \arrow[r, "\tilde{f}|_W"] \arrow[d, hook, bend right] & Y \arrow[d, "p"] \\ Z \arrow[r, "f"] \arrow[ur, dashed, "\tilde{f}"] & X \end{tikzcd}$$

; p covering map

Problem: find $\tilde{f}$

Trivial case: $p = \text{identity}$, $Y = X$
Here $\tilde{f} = p^{-1} \circ f$

Local lifting lemma: $U \subset X$ evenly covered, $f(Z) \subset U$

$p^{-1}(U) = \coprod_\alpha V_\alpha$

(a) Suppose W is connected, then $\tilde{f}$ exists
(b) Suppose $W \neq \emptyset$ and Z connected, $\tilde{f}$ unique.

Pf: (a) As W is connected, $\exists \alpha_0$ s.t. $\tilde{f}(W) \subset V_{\alpha_0}$

Define $\tilde{f} = (p|_{V_{\alpha_0}})^{-1} \circ f$; $f(Z) \subset V_{\alpha_0} \Rightarrow \tilde{f}|_Z = (p|_{V_{\alpha_0}})^{-1} \circ f$

(b) As Z is connected, $W \neq \emptyset$ $\tilde{f}(Z) \subset V_{\alpha_0}$, so $\tilde{f} = (p|V_{\alpha_0})^{-1} \circ f$

Path lifting: $p : Y \to X$ covering map

· $\alpha : [0,1] \to X$ continuous function

· $x_0 = \alpha(0)$, $\tilde{x}_0 \in p^{-1}(x_0)$

Theorem: $\exists!\, \tilde{\alpha} : [0,1] \to Y$ continuous s.t.

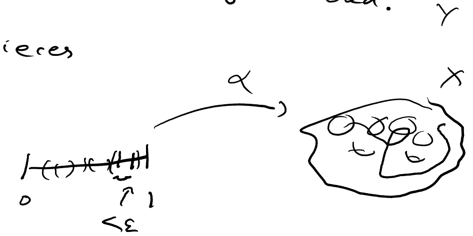
$$\begin{tikzcd} & Y \arrow[d] \\ {[0,1]} \arrow[ur, dashed, "\tilde{\alpha}"] \arrow[r, "\alpha"] & X \end{tikzcd}$$

commutes and $\tilde{\alpha}(0) = \tilde{x}_0$.

Pf: By Lebesgue number theorem, $\exists\, \varepsilon > 0$ s.t. if $J \subset [0,1]$, $\mathrm{diam}(J) < \varepsilon$, then $\alpha(J)$ is evenly covered.

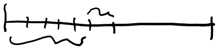
· Pick $n > 1/\varepsilon$, subdivide $[0,1]$ into n pieces

· Define inductively $\tilde{\alpha}$ on $[0, \frac{k}{n}]$ using local lifting lemma.

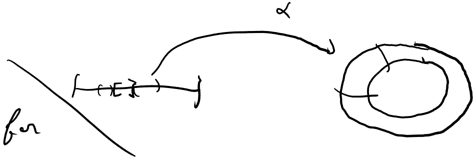


Path lifting proof again

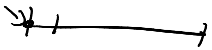
* We decompose the interval $[0,1] = J_1 \cup J_2 \cup \cdots \cup J_n$ into closed sets. $\alpha(J_i)$ is evenly covered
* $\tilde{\alpha}(0)$ is given
The point 0 is in the first set. Take $J_0 = \{0\}$.
* We have connectedness of $J_k \cap \left(\bigcup_{i=1}^{k-1} J_i\right)$



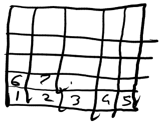
* We inductively define the lift on $\bigcup_{i=1}^{k} J_i$;



• given $\tilde{\alpha}|_{\bigcup_{i=1}^{k-1} J_i}$, we solve local lifting for J_n relative to $J_k \cap \left(\bigcup_{i=0}^{k-1} J_i\right)$ to get $\tilde{\alpha}|_{J_n}$



• By gluing lemma, $\tilde{\alpha}$ is defined on $\bigcup_{i=0}^{k} J_i$.



Homotopy Lifting:

- $p : Y \to X$
- $H : [0,1] \times [0,1] \to X$
- $H(0,0) = x_0$; $\tilde{x}_0 \in p^{-1}(x_0)$

$$
\begin{tikzcd}
 & Y \ni \tilde{x}_0 \arrow[d] \\
(0,0) \in [0,1]\times[0,1] \arrow[r] \arrow[ur, dotted] & X \ni x_0
\end{tikzcd}
$$

Theorem: There exists a unique $\tilde{H} : [0,1] \times [0,1] \to Y$ s.t.

- $p \circ \tilde{H} = H$
- $\tilde{H}(0,0) = \tilde{x}_0$

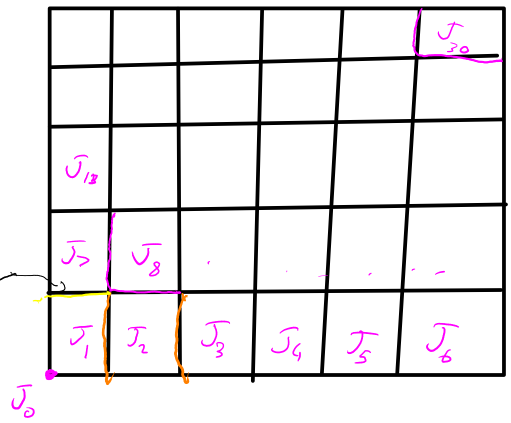
Further, if H fixes endpoints, so does $\tilde{H}$.

Proof:
- X has an open cover $\{U_\beta\}_\beta$ by evenly covered neighbourhoods
- $\{H^{-1}(U_\beta)\}_\beta$ is an open cover of $[0,1] \times [0,1]$
- Hence, $\exists\, \varepsilon > 0$ s.t. if $J \subset [0,1] \times [0,1]$ has $\operatorname{diam}(J) < \varepsilon$, then $\exists\, \beta_0$ s.t. $J \subset H^{-1}(U_{\beta_0})$, i.e. $H(J) \subset U_{\beta_0}$.

- We have sets $J_1, J_2, \ldots, J_n$
  - J_i is connected, closed
  - $J_k \cap \left(\bigcup_{i=0}^{k-1} J_i \right)$ connected
  - $\operatorname{diam}(J_i) < \varepsilon \quad \forall i$
- By local lifting lemma, we define inductively, $\tilde{H}|_{\bigcup_{i=1}^{k} J_i}$,,

$$\left(\bigcup_{i=0}^{k-1} J_i \right) \cup J_k$$

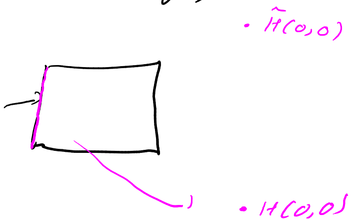
by extending $\left(\tilde{H}|_{\bigcup_{i=0}^{k-1} J_i} \right) \Big|_{J_k \cap \left(\bigcup_{i=0}^{k-1} J_i \right)}$ to J_k



Addendum: Suppose H fixes endpoints

- then $t \mapsto H(0,t)$ is a constant function, (path)
- a <u>lift</u> of this, hence the lift at $\tilde{H}(0,0)$, is constant
- i.e., $t \mapsto \tilde{H}(0,t)$ is constant (as it is the lift)

$$\begin{tikzcd} & Y \arrow[d] \\ {[0,1]} \arrow[ur, "t \mapsto \tilde{H}(0,t)"] \arrow[r, "t \mapsto H(0,t)"'] & X \end{tikzcd}$$



- Thus, $\tilde{H}$ fixes endpoints

Propn: If $\alpha \sim \beta$, then $\alpha(0) = \beta(0)$ and $\alpha(1) = \beta(1)$

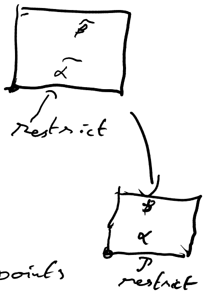
Theorem: $\pi_1(S^1, 1) = \mathbb{Z}$

Proof:
$$\begin{array}{c}\Phi : \Omega(S^1, 1) \longrightarrow \mathbb{Z} \\ \rotatebox{90}{$\in$} \\ \alpha\end{array} \qquad\qquad \begin{array}{c}\mathbb{R} \\ \Big\downarrow p \\ S^1\end{array}$$

- Let $\tilde{\alpha} : [0,1] \longrightarrow \mathbb{R}$ be <u>the</u> lift of α with $\tilde{\alpha}(0) = 0$

- Define $\Phi(\alpha) = \tilde{\alpha}(1) \in p^{-1}(1) = \mathbb{Z}$

<u>Lemma:</u> $\alpha \sim \beta \Rightarrow \Phi(\alpha) = \Phi(\beta)$



<u>Pf:</u> · Let H be a homotopy from α to β fixing endpoints

· Let $\tilde{H}$ be its lift with $\tilde{H}(0,0) = 0 \in \mathbb{R}$

<u>Claim:</u> $\tilde{H}$ is a homotopy from $\tilde{\alpha}$ to $\tilde{\beta}$ fixing endpoints

<u>Pf:</u> $s \mapsto \tilde{H}(s,0)$ is a lift of $s \mapsto H(s,0) = \alpha(s)$ with $\tilde{H}(0,0) = 0$, hence by <u>uniqueness</u> $s \mapsto \tilde{H}(s,0) = \tilde{\alpha}$, i.e. $\tilde{H}(s,0) = \tilde{\alpha}(s)\ \forall s \in [0,1]$

· $|||^{ly}$ $\tilde{H}(s,1) = \tilde{\beta}(s)\ \forall s \in [0,1]$

· Hence $\tilde{\alpha} \sim \tilde{\beta} \Rightarrow \Phi(\alpha) = \tilde{\alpha}(1) = \tilde{\beta}(1) = \Phi(\beta)$ $\square$

Thus, Φ induces

$$\varphi : \pi_1(S^1, 1) \longrightarrow \mathbb{Z}$$

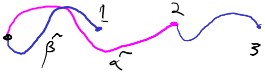
$$\pi_1(S^1,1) = \Omega(S^1, 1)/\sim$$

To show:

· φ is a homomorphism
· φ is injective
· φ is surjective

Homomorphism:

$\alpha, \beta \in \Omega(S^1, 1)$; $[\alpha], [\beta] \in \pi_1(S^1, 1)$



- $\tilde{\alpha}, \tilde{\beta}$ as before
- We construct the lift of $\alpha * \beta$ starting at 0

Let $$\tilde{\eta} = \begin{cases} \tilde{\alpha}(2s), & 0 \leq s \leq \frac{1}{2} \\ \tilde{\beta}(2s-1) + \tilde{\alpha}(1), & \frac{1}{2} \leq s \leq 1 \end{cases}$$

- Observe $\tilde{\eta}$ is continuous and $p \circ \tilde{\eta} = \alpha * \beta$; $\underset{\tilde{\alpha}(0)}{\underset{\shortparallel}{\tilde{\eta}(0)}} = 0$.
- Thus $\tilde{\eta} = \widetilde{\alpha * \beta}$

$$\therefore \quad \varphi(\underset{[\alpha]*[\beta]}{\underset{\shortparallel}{[\alpha * \beta]}}) = \tilde{\eta}(1) = \tilde{\alpha}(1) + \tilde{\beta}(1) = \varphi([\alpha]) + \varphi([\beta])$$

Injectivity: Assume $\varphi([\alpha]) = \varphi([\beta])$

<u>Claim:</u> $\alpha \sim \beta$ ($\Rightarrow [\alpha] = [\beta]$)

<u>Pf:</u> $\tilde{\alpha}(1) = \varphi([\alpha]) = \varphi([\beta]) = \tilde{\beta}(1)$ and $\tilde{\alpha}(0) = 0 = \tilde{\beta}(0)$

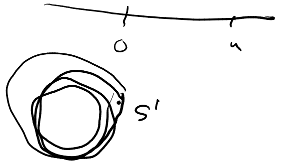
Thus $\tilde{\alpha}, \tilde{\beta} : [0,1] \longrightarrow \mathbb{R}$ have $\tilde{\alpha}(0) = \tilde{\beta}(0)$ & $\tilde{\alpha}(1) = \tilde{\beta}(1)$

$\exists \tilde{H}$ homotopy fixing endpoints from $\tilde{\alpha}$ to $\tilde{\beta}$

- Let $H = p \circ \tilde{H}$. Then H is a homotopy fixing endpoints from α to β.

Surjectivity: Let $n \in \mathbb{Z}$, define

$\alpha_n(s) = e^{2\pi i n s}$, $s \in [0,1]$, $[\alpha_n] \in \pi_1(S^1, 1)$



- We see $\tilde{\alpha}_n(s) = ns$, $\tilde{\alpha}_n : [0,1] \longrightarrow \mathbb{R}$
- Thus $\varphi([\alpha_n]) = \tilde{\alpha}_n(1) = n$.

Functoriality:

* ~~A based~~ space is a pair (X, x_0) consisting of a topological space and a (base) point in it.
* ~~Morphisms~~ are functions $f:(X,x_0)\to(Y,y_0)$, i.e. $f: X\to Y,\ f(x_0)=y_0$, f continuous

Theorem: $(X,x_0)\mapsto \pi_1(X,x_0)$ is a functor.

$$\begin{tikzcd}
\{0,1\} \arrow[r] \arrow[d, hook] & \{x_0\} \arrow[r] & \{y_0\} \\
{[0,1]} \arrow[r, "\alpha"] \arrow[rr, bend right] & X \arrow[r, "f"] & Y
\end{tikzcd}$$

Map on morphism: $f:(X,x_0)\longrightarrow(Y,y_0)$

Define $f_*([\alpha]) = [f\circ\alpha]$

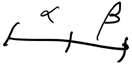
Propn: (a) f_* is well-defined on $\Omega(X,x_0)/\sim\ = \pi_1(X,x_0)$

(b) f_* is a homomorphism.

Pf: (a) Suppose $[\alpha]=[\beta]$, let H be a homotopy from α to β fixing endpoints.

Then $f\circ H : [0,1]\times[0,1]\longrightarrow Y$ is a homotopy from $f(\alpha)$ to $f(\beta)$ fixing endpoints,

$\therefore\ [f\circ\alpha]=[f\circ\beta]$



(b) $f_*([\alpha*\beta]) = f\circ(\alpha*\beta) = (f\circ\alpha)*(f\circ\beta) = f_*([\alpha])*f_*([\beta])$

Homotopy of maps :

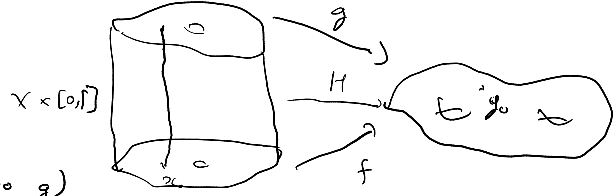
$f, g : X \longrightarrow Y$ continuous function

<u>Defn:</u> A homotopy from f to g is a map

$$H : X \times [0,1] \longrightarrow Y$$

s.t. $\forall x \in X$, $H(x,0) = f(x)$

& $H(x,1) = g(x)$



- We say $f \sim g$ (f is homotopic to g)
- This is an equivalence relation

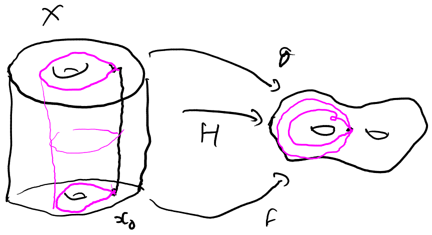
<u>Based spaces:</u> $f, g : (X, x_0) \longrightarrow (Y, y_0)$ then **also** require

$$H(x_0, t) = y_0 \quad \forall\, t \in [0,1]$$

Theorem: $f, g : (X, x_0) \longrightarrow (Y, y_0)$

$f \sim g$ (as maps of <u>based</u> spaces)

Then $f_* = g_* : \pi_1(X, x_0) \longrightarrow \pi_1(Y, y_0)$



<u>Proof:</u> Pick $[\alpha] \in \pi_1(X, x_0)$ arbitrary

$f_*([\alpha]) = [f \circ \alpha]$; $g_*([\alpha]) = [g \circ \alpha]$

Let H be the homotopy from f to g.

<u>Claim:</u> $f \circ \alpha \sim g \circ \alpha$

<u>Pf:</u> A homotopy is given by

$$\hat{H} : [0,1] \times [0,1] \longrightarrow Y$$

$$\hat{H}(s,t) = H(\alpha(s), t)$$

when $t = 0$, $\hat{H}(s,t) = H(\alpha(s), 0)$
$= f(\alpha(s)) = f\alpha(s)$

when $s = 0$, $\hat{H}(0,t) = H(\alpha(0), t)$
$= H(x_0, t) = y_0$

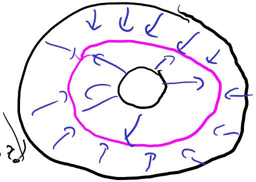
<u>(Strong) deformation retracts</u>: $A \subset X$ subspace is said to be a (strong) deformation retract if

$$\exists\, H : X \times [0,1] \to X$$

s.t. $\cdot\ H(x,0) = x \quad \forall x \in X \quad$ (i.e. $x \mapsto H(x,0) = \mathbb{1}_{(X,?)}$)

$\cdot\ H(x,1) \in A \quad \forall x \in A$

and $\cdot\ H(a,t) = a \quad \forall a \in A.$



Let $x_0 \in A$, then $i : (A, x_0) \to (X, x_0)$ is a map of based spaces

$p : (X, x_0) \to (A, a_0)$; $\quad p(x) = H(x,1)$

<u>Propn</u>: (a) $p \circ i : (A, a_0) \to (A, a_0) = \mathbb{1}_{(A,a_0)}$; (b) $i \circ p : (X, x_0) \to (X, x_0) \sim \mathbb{1}_{(X,x_0)}$

<u>Pf</u>: (b) H gives a homotopy $\mathbb{1}_{(X,x_0)} \sim i \circ p$

Homotopy equivalence of (based) spaces

Defn: $f: X \to Y$ is a homotopy equivalence if $\exists g: Y \to X$ s.t.

$(f: (X, x_0) \to (Y, y_0))$ $\exists g: (Y, y_0)$ & homotopy of based spaces

$f \circ g \sim \mathbb{1}_Y$ and $g \circ f \sim \mathbb{1}_X$

Theorem: If $f: (X, x_0) \to (Y, y_0)$ is a homotopy equivalence of based spaces, then $f_*: \pi_1(X, x_0) \xrightarrow{\cong} \pi_1(Y, y_0)$

Pf: Let g be as above, then

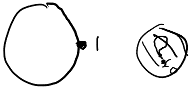
$\mathbb{1}_Y = (\mathbb{1}_Y)_* = (f \circ g)_* = f_* \circ g_*$ and $g_* \circ f_* = \mathbb{1}_X$

. Hence g_* is an inverse of f.

Cor: If $A \subset X$ is a deformation retract, then $i_*: (A, x_0) \to (X, x_0)$ is an isomorphism [Pf: ρ is the 'inverse' up to homotopy]

Dependence on base-point

Suppose $\hat{X} \subset X$ is the path-component containing $x_0 \in X$,

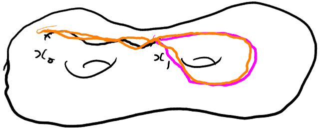
$X =$ 

Then $\Omega(X, x_0) = \Omega(\hat{X}, x_0)$

$\therefore$ Propn: $\pi_1(X, x_0) \cong \pi_1(\hat{X}, x_0)$

- Suppose $x_0, x_1 \in X$, $\gamma : [0,1] \to X$ is a path from x_0 to x_1
- Define $\tau_\gamma : \pi_1(X, x_1) \longrightarrow \pi_1(X, x_0)$

$$\tau_\gamma([\alpha]) := [\gamma * \alpha * \bar{\gamma}]$$



- well-defined: $\alpha \sim \alpha'$, then $\gamma * \alpha * \bar{\gamma} \sim \gamma * \alpha' * \bar{\gamma}$
- homomorphism: $\tau_\gamma([\alpha_1]) * \tau_\gamma([\alpha_2]) = [\gamma * \alpha_1 * \bar{\gamma}] * [\gamma * \alpha_2 * \bar{\gamma}] = [\gamma * \alpha_1 * \bar{\gamma} * \gamma * \alpha_2 * \bar{\gamma}]$
$= [\gamma * (\alpha_1 * \alpha_2) * \bar{\gamma}] = \tau_\gamma(\alpha_1 * \alpha_2)$

- $\tau_{\bar{\gamma}} \circ \tau_\gamma = \mathbb{1}_{\pi_1(X, x_1)}$ as $\tau_{\bar{\gamma}}(\tau_\gamma([\alpha])) = [\overbrace{\bar{\gamma} * \gamma}^{\sim e} * \alpha * \overbrace{\bar{\gamma} * \underbrace{\gamma}_{\bar{\bar{\gamma}}}}^{\sim e}] = [\alpha]$

& $\tau_\gamma \circ \tau_{\bar{\gamma}} = \mathbb{1}_{\pi_1(X, x_0)}$

> Thus,
> $$\pi_1(X, x_1) \cong \pi_1(X, x_0)$$
> with isomorphism τ_γ if γ is a path from x_0 to x_1

No retraction theorem: There is no map $r : \mathbb{D}^2 \to S^1$ s.t.

$$\begin{tikzcd} S^1 \arrow[r, hook, "i"] \arrow[dr] & \mathbb{D}^2 \arrow[d, dashed, "r"] \\ & S^1 \end{tikzcd}$$

commutes, i.e., $r \circ i = \mathbb{1}_{S^1}$

Proof: If r exists, we get

$$\begin{tikzcd} (S^1,1) \arrow[r, "i"] \arrow[dr, "\mathbb{1}_{(S^1,1)}"'] & (\mathbb{D}^2,1) \arrow[d, "r"] \\ & (S^1,1) \end{tikzcd}$$

, hence

we get

$$\begin{tikzcd} \mathbb{Z} = \pi_1(S^1,1) \arrow[r, "i_*", "0"'] \arrow[dr, "\mathbb{1}_* = \mathbb{1}_{\mathbb{Z}}"'] & \pi_1(\mathbb{D}^2,1) = 0 \arrow[d, "r_*", "0"'] \\ & \pi_1(S^1,1) = \mathbb{Z} \end{tikzcd}$$

which is impossible.

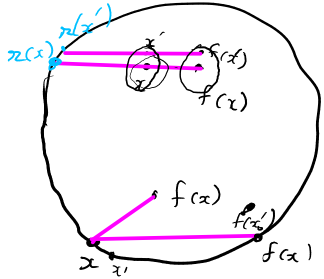
Brouwer fixed point theorem: Any map $f : D^2 \longrightarrow D^2$ has a fixed point, i.e. $\exists\, x \in D^2$ s.t. $f(x) = x$

<u>Proof:</u> Suppose $f : D^2 \longrightarrow D^2$ has no fixed points.

- We construct a retraction $r : D^2 \longrightarrow S^1$,
- Let $\rho_x : [0, \infty) \longrightarrow \mathbb{R}^2$ be

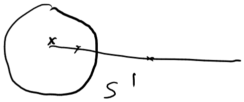
$$\rho_x(t) = (1-t)\, f(x) + t\, x, \quad t \geq 0$$

- Let $T_x = \inf\{ t \geq 1 : \|\rho_x(t)\| \geq 1 \}$
- Let $r(x) = \rho_x(T_x)$



- <u>Observe:</u> $x \in S^1$, $T_x = 1$ & $r(x) = x$
- r is continuous; show if x' is close to x, $T_{x'}$ close to T_x and the $r(x')$ is close to $r(x)$

Theorem: $\mathbb{R}^2$ is not homeomorphic to $\mathbb{R}^3$.



Proof: Let $f: \mathbb{R}^2 \to \mathbb{R}^3$ be a homeomorphism.

W.l.g. $f(0,0) = (0,0,0)$, so

$$\mathbb{R}^2 \setminus \{(0,0)\} \xrightarrow[\text{homeo}]{f} \mathbb{R}^3 \setminus \{(0,0,0)\}$$

$$\Rightarrow \quad \pi_1(\mathbb{R}^2 \setminus \{(0,0)\}, (1,0)) \cong \pi_1(\mathbb{R}^3 \setminus \{(0,0,0)\}, f(1,0))$$

$$\cong \pi_1(\mathbb{R}^3 \setminus \{(0,0,0)\}, (1,0,0))$$

Lemma: $\mathbb{R}^n \setminus \{0\}$ deformation retracts to S^{n-1}.

Pf: Define $H: \mathbb{R}^n \setminus \{0\} \times [0,1] \longrightarrow \mathbb{R}^{n-1}$

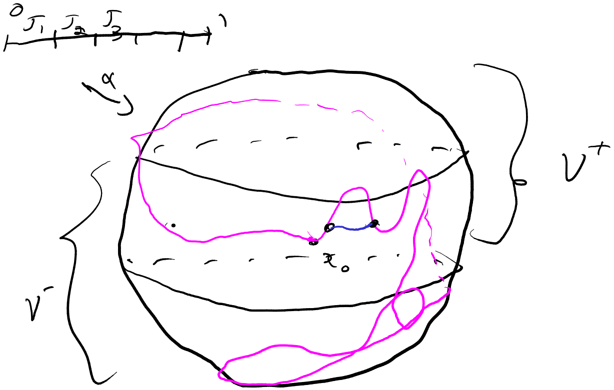
$$H(x,t) = \frac{x}{(1-t) + t\|x\|}; \quad H(x,0) = x, \quad H(x,1) = \frac{x}{\|x\|} \in S^{n-1}$$

$$x \in S^{n-1} \Rightarrow \|x\| = 1 \Rightarrow H(x,t) = \frac{x}{(1-t)+t} = x$$

$$\pi_1(\mathbb{R}^2\setminus\{0\}, -) \cong \pi_1(\mathbb{R}^3\setminus\{0\}, -)$$

$$\begin{array}{ccc} \pi_1(\mathbb{R}^2\setminus\{0\}, -) & & \pi_1(\mathbb{R}^3\setminus\{0\}, -) \\ \wr\! \| & & \wr\! \| \\ \pi_1(S^1, 1) & & \pi_1(S^2, (1,0,0)) \\ \wr\! \| & & \wr\! \| \\ \mathbb{Z} & & \{e\} \end{array}$$

Contradiction.



<u>Theorem:</u> $\pi_1(S^2, (1,0,0)) \cong \{e\}$

<u>Pf:</u> Let $\alpha : [0,1] \longrightarrow S^2$.

By Lesbesgue number theorem, we can subdivide $[0,1]$ s.t. $\forall i$ $\alpha(J_i) \subset V^+$ or $\alpha(J_i) \subset V^-$

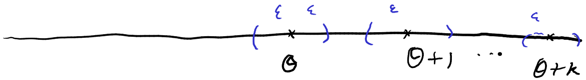
- If J_i & $J_{i+1} \subset V^+$, merge.
- Hence each intermediate point is in $V^+ \cap V^-$
- Hence $\alpha|_{J_i}$ is homotopic to $\beta_i : J_i \to V^+ \cap V^-$
  $\cap$ $V^+ \cong D^2$

By all these, $\alpha \simeq \beta$, $\beta : [0,1] \to V^+ \cap V^- \subset V^+ \cong D^2$

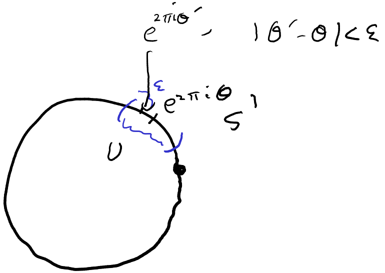
Hence $\beta \simeq e$.

More Covering maps : $p : \mathbb{R} \to S^1$; $t \mapsto e^{2\pi i t}$

$p^{-1}(U) = \coprod_{k \in \mathbb{Z}} V_k$, $V_k = (\theta + k - \varepsilon, \theta + k + \varepsilon) \subset \mathbb{R}$, $\varepsilon < 1$



· V_k are open



· $p|_{V_k} : V_k \to U$ is a homeomorphism,
$t \mapsto e^{2\pi i t}$

as it has inverse $U \to V_k$
$e^{2\pi i \theta'} \mapsto \theta' + k$

where θ' is s.t. $|\theta' - \theta| < \varepsilon$

· This inverse is continuous

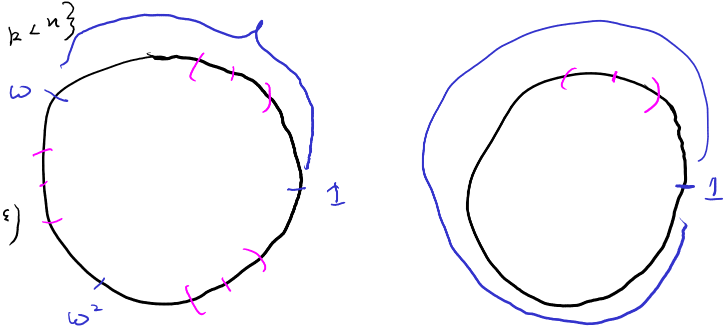
<u>Other covers</u>: $p_n : S^1 \longrightarrow S^1$ by $p_n(z) = z$

If $z = e^{2\pi i \theta}$

$$p^{-1}(z) = \left\{ e^{2\pi i \left(\frac{\theta + k}{n}\right)},\ 0 \le k < n \right\}$$

$$U = \{ e^{2\pi i \theta'} : |\theta' - \theta| < \varepsilon \}$$

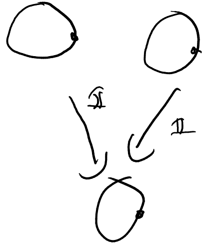
$$p^{-1}(U) = \coprod_{k=0}^{n} \underbrace{e^{\left(\frac{\theta + k - \varepsilon}{n},\ \frac{\theta + k + \varepsilon}{n}\right)}}_{= V_k}$$



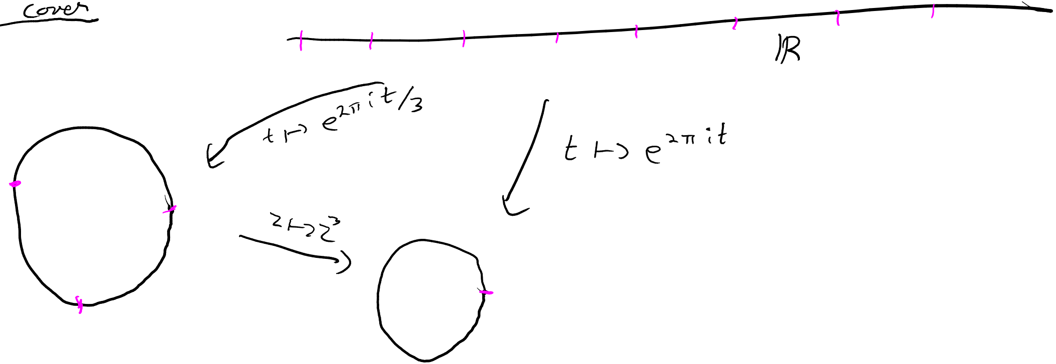
Then $p|_{V_k} : V_k \longrightarrow U$ is a homeomorphism

- Disconnected covers:

$$\begin{tikzcd} S^1 \sqcup S^1 \arrow[r] \arrow[r, bend left, "z \mapsto z^m"] \arrow[r, bend right, "z \mapsto z^n"'] & S^1 \end{tikzcd}$$



- Universal cover



Theorem: $p : (Y, y_0) \longrightarrow (X, x_0)$ is a covering map, then

$$p_* : \pi_1(Y, y_0) \longrightarrow \pi_1(X, x_0) \quad \text{is } \underline{\text{injective}}$$

$\hat{\alpha}$ is ~~a~~ the lift of α starting at y_0

Proof: · Let $[\hat{\alpha}] \in \pi_1(Y, y_0)$, $\boxed{\alpha = p \circ \hat{\alpha}}$, so $[\alpha] = p_*([\hat{\alpha}])$

· Suppose $[\alpha] = e$, then $\alpha \sim e_{x_0}$ by a homotopy H (fixing endpoints)

· By homotopy lifting, we have a homotopy fixing endpoints

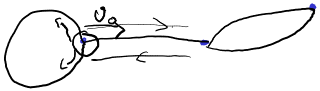
$$\tilde{H} : [0,1] \times [0,1] \longrightarrow (Y, y_0), \quad \tilde{H}(0,0) = y_0$$

between the lifts of α and e_{x_0} starting at y_0

i.e., $\hat{\alpha}$ and e_{y_0}; thus $\hat{\alpha} \sim e_{y_0}$ so $[\hat{\alpha}] = e$

Thus: $p_*(\pi_1(Y, y_0)) \hookrightarrow \pi_1(X, x_0)$ is a subgroup

Graphs: $\Gamma = \Gamma(V, E)$ is given by



- V set
- E set with a fixed-point free involution

$$e \mapsto \bar{e} \; ; \quad \bar{\bar{e}} = e, \quad e \neq \bar{e}$$

- $i : E \longrightarrow V$ (initial point)

Define: $\tau : E \longrightarrow V$ by $\tau(e) = i(\bar{e})$

$$lk(v; \Gamma) = \{ e \in E : i(e) = e \}$$

Geometric realisation : $|\Gamma| = |\Gamma(E,V)|$ is

$$|\Gamma| = \Big(\underbrace{V}_{\text{discrete}} \ \coprod \ \big(\underbrace{E}_{\text{discrete}} \times [0,1] \big) \Big) \Big/ \sim$$

with $\sim$ generated by

- $\forall e \in E$, $(e, s) \sim (\bar{e}, 1-s)$, with $(e,s) \in E \times [0,1]$, $s \in [0,1]$ — Quotient topology
- $\forall e \in E$, $(e, 0) \sim i(e)$, with $(e,0) \in E \times [0,1]$, $i(e) \in V$

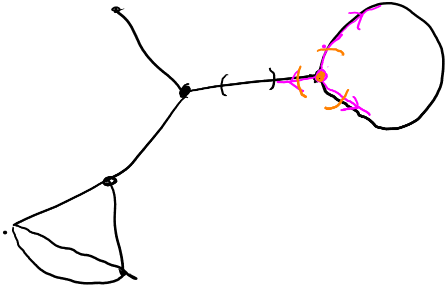
follows $(e,1) \sim \tau(e)$ $\forall e$

Basic neighbourhoods:

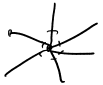
- For $e \in E$, $(a,b) \subset (0,1)$, $\overbrace{\{e\} \times (a,b)}^{\text{image of}}$
- For $v \in V$, specify $\varepsilon : \mathrm{lk}(v) \to (0, \frac{1}{2})$

  define $U_{\varepsilon, v}$ be the image of

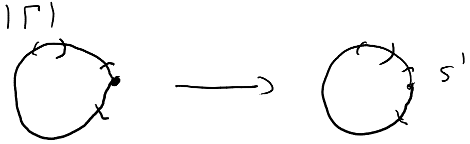
$$\{v\} \cup \bigcup_{e \in \mathrm{lk}(v)} \{(e,s) : 0 \le s < \varepsilon(e)\}.$$



- Usually assume $|\mathrm{lk}(v)| < \infty \quad \forall v \in V$



$\underline{\underline{S^1}}$: $V = \{1\}$, $E = \{e, \bar{e}\}$, $i(e) = 1$



Define: $\varphi : \underset{\overset{\parallel}{\cdots/\sim}}{|\Gamma|} \longrightarrow S^1$ as the quotient of

$$\varphi((e,s)) = e^{2\pi i s}$$

$$\varphi((\bar{e},s)) = e^{2\pi i (1-s)}$$

$$\varphi(\underset{\overset{\parallel}{1}}{v}) = 1$$

Morphisms of Graphs : $\Gamma = \Gamma(E, V)$; $\Gamma' = \Gamma'(E', V')$

Morphism $\varphi : \Gamma \longrightarrow \Gamma'$

is $\varphi : E \longrightarrow E'$

$\varphi : V \longrightarrow V'$

s.t. $\varphi(\bar{e}) = \overline{\varphi(e)}$; $\varphi(i(e)) = i(\varphi(e))$

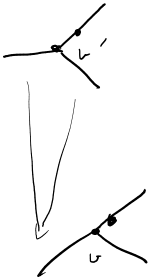
- Induces $|\varphi| : |\Gamma| \longrightarrow |\Gamma'|$

$$\begin{array}{ccc} |\Gamma| & \longrightarrow & |\Gamma'| \\ (e, s) & \longmapsto & (\varphi(e), s) \end{array}$$

- $|\mathbb{1}_\Gamma| = \mathbb{1}_{|\Gamma|}$
- $|\varphi \circ \psi| = |\varphi| \circ |\psi|$

Criterion for covering: if $v \in V$,

$$\underbrace{\varphi}_{\text{on edges}}\big|_{lk(v)} = \varphi_v : lk(v) \longrightarrow lk(v')$$



Propn: φ is a covering iff φ surjects on vertices (and edges) and $\forall v \in V$, $\varphi_v : lk(v) \longrightarrow lk(\varphi(v))$ is a bijection.

Pf:

- Let $B_{1/3}(v) = \{v\} \cup \{(e,s) : s \in [0, \tfrac{1}{3})\}$
- Any interval in $E' \times (0,1)$ is evenly covered
- Pick $v' \in V'$; then $\underset{|\varphi|}{\overset{\shortparallel}{p^{-1}}}(B_{1/3}(v')) = \coprod_{v \in \varphi^{-1}(v')} B_{1/3}(v)$

$|\varphi|\big|_{B_{1/3}(v)} : B_{1/3}(v) \to B_{1/3}(v')$ has inverse

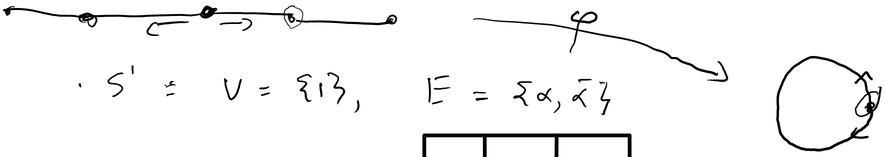
$f(v') = v$

$f((e,s)) = (\varphi_v^{-1}(e), s)$

Examples: $\mathbb{R}$ graph, $V = \mathbb{Z}$, $E = E^+ \cup E^-$

$E^+ = \{(n, n+1) : n \in \mathbb{Z}\}$

$E^- = \{(n+1, n) : n \in \mathbb{Z}\}$

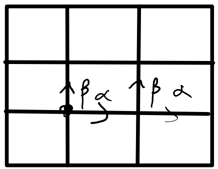


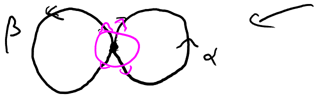
S^1: $V = \{1\}$, $E = \{\alpha, \bar{\alpha}\}$

$e \in E^+ \Rightarrow \varphi(e) = \alpha$

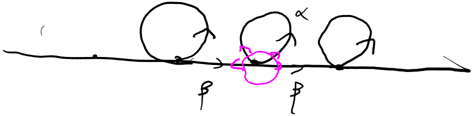
$e \in E^- \Rightarrow \varphi(e) = \bar{\alpha}$

$\varphi(n) = 1$

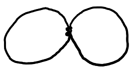




X



Free Groups: Let S be a set

$\pi_1($  $)$

$F(S)$ = free group generated by S, is

'group with basis S'

Let V be a vector space and S a subset of V, $i : S \hookrightarrow V$

S is a basis if : · for W vector space

· $f : S \to W$

$\exists!\ L : V \to W$ linear s.t.

$L(s) = f(s) \ \forall s \in S$

$$\begin{tikzcd} S \arrow[r, "f"] \arrow[d, hook, "i"'] & W \\ V \arrow[ur, dashed, "L"'] & \end{tikzcd}$$

A Free Group F(S) generated by S is a group together with an inclusion map $i : S \to F(S)$ such that

given · H a group

· $f : S \longrightarrow H$

$\exists!$ $\varphi : G \longrightarrow H$ homomorphism s.t.

$\forall s \in S$, $\varphi(s) = f(s)$, i.e.

$$\begin{tikzcd} S \arrow[r, "f"] \arrow[d, bend right, "i"'] & H \\ F(S) \arrow[ur, dashed, "\varphi"'] & \end{tikzcd} \quad , \boxed{f = \varphi \circ i}$$

<u>E.g.</u> $\mathbb{Z}$ is a free group generated by 1.

$f : \{1\} \longrightarrow H$

we get $\varphi : \mathbb{Z} \longrightarrow H$ given by $\varphi(n) = \left(f(\{1\})\right)^n$

Uniqueness : Suppose $F'(S)$ and $F(S)$ are free groups generated by S,

$i' : S \longrightarrow F'(S)$, $\quad i : S \longrightarrow F(S)$

<u>Theorem:</u> $\exists\, \varphi : F(S) \xrightarrow{\ \simeq\ } F'(S)$ s.t.

$$\begin{tikzcd} & S \arrow[dl, "i"'] \arrow[dr, "i'"] & \\ F(S) \arrow[rr, "\varphi", "\simeq"'] & & F'(S) \end{tikzcd}$$

commutes.

<u>Proof:</u>

- As $F'(S)$ is a group and $F(S)$ is a free group, given
  $i' : S \longrightarrow F'(S)$
  $\exists\, \varphi : F(S) \longrightarrow F'(S)$ homomorphism with $i' = \varphi \circ i$
- Similarly, as $F'(S)$ is a free group, given $i : S \to F(S)$
  $\exists\, \psi : F'(S) \to F(S)$ s.t. $i = \psi \circ i'$ homomorphism
-

$$\begin{tikzcd} & S \arrow[dl, "i"'] \arrow[d, "i'"] \arrow[dr, "i"] & \\ F(S) \arrow[r, "\varphi"'] \arrow[rr, bend right, "\varphi\circ\psi"'] & F'(S) \arrow[r, "\psi"] & F(S) \end{tikzcd}$$

$\exists!\, \eta : F(S) \to F(S)$, s.t. $\eta \circ i = i$

- $\eta = \varphi \circ \psi$ is as above

as is $\mathbb{1}_{F(S)}$. Hence $\varphi \circ \psi = \mathbb{1}$

$\square$

Constructing the Free group:
Let S be a set.

$\bar{S} = \{\bar{s} : s \in S\}$, notation $\bar{\bar{s}} = s$

$$\begin{aligned} S \cup \bar{S} &\longrightarrow S \cup \bar{S} \\ s &\longmapsto \bar{s} \\ \bar{s} &\longmapsto s = \bar{\bar{s}} \end{aligned}$$

- $W = \{(l_1, \ldots, l_n) : n \geq 0,\ l_i \in S \cup \bar{S}\ \forall i\}$, $e = () \in W$
- $w_1 = (l_1, \ldots, l_n)$, $w_2 = (l_1', \ldots, l_m')$, then $w_1 * w_2 := (l_1, \ldots, l_n, l_1', \ldots, l_m')$
- Let $\sim$ be the equivalence relation on W generated by

$$(l_1, \ldots, l_k, \underset{\substack{\in \\ S \cup \bar{S}}}{\lambda}, \bar{\lambda}, l_{k+1}, \ldots, l_n) \sim (l_1, \ldots, l_k, l_{k+1}, \ldots, l_n)$$

- Equivalently: related by finitely many moves: cancel, uncancel

$F = W/\sim$

<u>Theorem:</u> • $*$ induces $F \times F \to F$

• F with $*$ is a group.

<u>Pf:</u> • $*$ induces $F \times F \to F$ as this holds for a single move, hence

$w_1 \sim w_1'$ & $w_2 \sim w_2'$ then $w_1 * w_2 \sim w_1' * w_2 \sim w_1' * w_2'$

• e is the identity as $w * e = e * w$

• $(l_1, l_2, \ldots, l_n)$ has inverse $(\bar{l}_n, \bar{l}_{n-1}, \ldots, \bar{l}_1)$ as

$(l_1, \ldots, l_n, \bar{l}_n, \ldots, \bar{l}_1)$
$\sim$
$(l_1, \ldots, l_{n-1}, \bar{l}_{n-1}, \ldots, \bar{l}_1)$
$\sim$
e

F is the Free Group generated by S

- $i : S \to F$, $i(s) = (s) = s$; extend f to $\bar{S}$ by $f(\bar{s}) = (f(s))^{-1}$

Given H, $f : S \to H$,

define $\Phi : W \to H$,

$$\Phi((l_1, \dots, l_n)) = f(l_1) \cdot f(l_2) \dots f(l_n)$$

- $$\Phi((l_1, \dots, l_k, \lambda, \bar{\lambda}, l_{k+1}, \dots, l_n)) = f(l_1) \dots f(l_k) \cdot f(\lambda) \cdot \underbrace{f(\bar{\lambda})}_{= (f(\lambda))^{-1}} \cdot f(l_{k+1}) \dots f(l_n)$$

$$= f(l_1) \dots f(l_k) f(l_{k+1}) \dots f(l_n) = \Phi((l_1, \dots, l_n))$$

- Hence Φ induces $\varphi : F \to H$
- φ is a homomorphism by construction
- φ is unique as $\varphi((l_1, \dots, l_n)) = \varphi((l_1) * (l_2) \dots * (l_n)) = f(l_1) * f(l_2) \dots * f(l_n)$

Free Objects and Concrete Categories:

A Concrete category is a

* ~~Category~~ $\mathcal{C}$
*

Functor $\sigma : \mathcal{C} \longrightarrow Set$

- Faithful: $A, B \in \mathcal{C}$; $Mor_{\mathcal{C}}(A, B) \hookrightarrow Mor_{Set}(\sigma(A), \sigma(B))$ is injective

$$f \longmapsto \sigma(f)$$

- S a set, the free object on S in $\mathcal{C}$ is
  - $F \in \mathcal{C}$
  - $i : S \hookrightarrow \sigma(F)$

s.t. $\forall\ A \in \mathcal{C}$, $f : S \longrightarrow \sigma(A)$, $\exists!\ \varphi \in Mor_{\mathcal{C}}(F, A)$ s.t.

$\varphi : F \to A$ gives

$\sigma(\varphi) : \sigma(F) \to \sigma(A)$;

$$\begin{tikzcd}
 & S \arrow[dl, "i"'] \arrow[dr, "f"] & \\
\sigma(F) \arrow[rr, "\sigma(\varphi)"] & & \sigma(A)
\end{tikzcd}$$

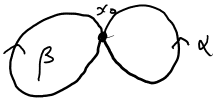
commutes

<u>Examples:</u>

- Free groups
- Free Top : discrete space
- Free Abelian groups: $\mathbb{Z}^n$, $S=[n]$

$$\begin{tikzcd} & S \arrow[dl] \arrow[dr, dashed] & \\ \mathbb{Z}^n \arrow[rr, "\varphi"'] & & A \end{tikzcd} \quad , A \text{ abelian}$$

- No free <u>finite groups</u>
- No free fields

$X =$ 

; $X = |\Gamma|$, $\Gamma = \Gamma(V, \overset{E(\Gamma)}{\overset{\shortparallel}{E}})$, $V = \{x_0\}$, $E = \{\alpha, \beta, \bar{\alpha}, \bar{\beta}\}$

$i(e) = x_0$

<u>Theorem</u>: $\pi_1(X, x_0) = \langle \alpha, \beta \rangle$

<u>Rk</u>: $e \in E(\Gamma)$, we have $\underset{\substack{\shortparallel \\ e}}{\chi_e} : [0,1] \longrightarrow |\Gamma|$, $\chi_e(s) := [(e,s)]$

· Define $\varphi : \langle \alpha, \beta \rangle \longrightarrow \pi_1(X, x_0)$; $\alpha \longmapsto \underset{\substack{\uparrow \\ \pi_1(X, x_0)}}{[\alpha]}$, $\beta \longmapsto [\beta]$

$$\varphi((l_1, \dots, l_n)) = \varphi(l_1 \dots l_n) = \varphi(l_1) * \varphi(l_2) * \dots * \varphi(l_n)$$
$$= [l_1 * l_2 * \dots * l_n]$$

Reduced words: A word $l_1 l_2 \dots l_n$ is said to be reduced if $l_i \neq \overline{l_{i+1}} \ \forall i$.

E.g. $\alpha \beta \bar{\beta} \bar{\alpha} \alpha \beta$ not reduced.

$\alpha \beta \bar{\beta} \beta \qquad \alpha \bar{\alpha} \alpha \beta$

$\rightsquigarrow \alpha \beta = \alpha \beta$

Propn: Every word $\omega = l_1 \dots l_n$ is equivalent to a reduced word.

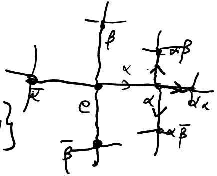
Pf: Keep cancelling. $\square$

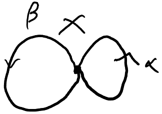
We construct: Graph $\tilde{\Gamma} = \tilde{\Gamma}(\tilde{E}, \tilde{V})$, with $|\tilde{\Gamma}| = \tilde{X}$

- $\tilde{V} = \{ l_1 \dots l_n : l_i$ reduced word in $\alpha, \beta, \bar{\alpha}$ & $\bar{\beta} \}$
- $\tilde{E} = \tilde{E}^+ \sqcup \tilde{E}^-$

$\tilde{E}^+ = \{ (\omega, \omega\lambda) : \lambda \in \{\alpha, \beta, \bar{\alpha}, \bar{\beta}\},\ \omega, \omega\lambda$ both reduced$\}$

$\tilde{E}^- = \{ (\omega\lambda, \omega) : \lambda \in \dots \}$





$\overline{(\omega, \omega\lambda)} = (\omega\lambda, \omega)$

- $i((\omega, \omega\lambda)) = \omega$; $i((\omega\lambda, \omega)) = \omega\lambda$

Conclusion: $|\varphi|$ is a covering space.

$\tilde{X} = |\tilde{\Gamma}|$, $X = |\Gamma|$

- We construct a graph morphism, φ

$$v \in \tilde{V} \longmapsto x_0$$

- $(\tilde{\omega}, \omega\lambda) \longmapsto \lambda \in \{\alpha, \beta, \bar{\alpha}, \bar{\beta}\}$

$(\omega\lambda, \omega) \longmapsto \bar{\lambda}$

$\overset{\shortparallel}{\overline{(\omega, \omega\lambda)}}$

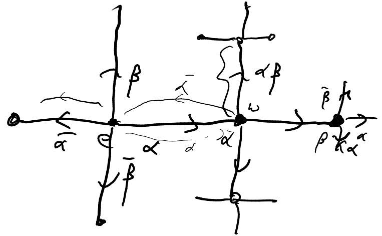
- This is a graph morphism

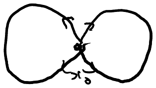
- $lk_\Gamma(x_0) = \{\alpha, \beta, \bar{\alpha}, \bar{\beta}\}$

- $lk_{\tilde{\Gamma}}(e) = \{(e, e\lambda) : \lambda \in \{\alpha, \beta, \bar{\alpha}, \bar{\beta}\}\}$; $\varphi_e : lk_{\tilde{\Gamma}}(e) \to lk_\Gamma(x_0)$ is a bijection

- if $\omega = l_1 \dots l_n$, $l_n = \alpha$ (w.l.g.)

$lk_{\tilde{\Gamma}}(\omega) = \{\underbrace{(\omega, \omega\alpha), (\omega, \omega\beta), (\omega, \omega\bar{\beta})}_{\tilde{E}^+}, (\omega, \underbrace{l_1 l_2 \dots l_{n-1}}_{\tilde{E}^-})\}$; $\varphi_\omega : lk_{\tilde{\Gamma}}(\omega) \to lk_\Gamma(x_0)$ is a bijection.



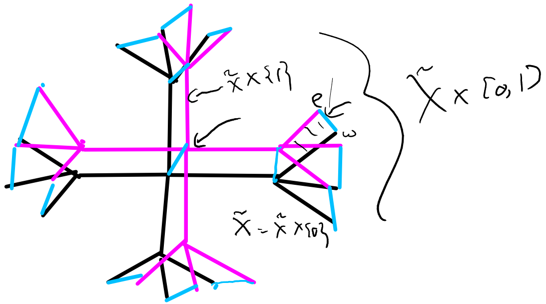


Contractibility of $\tilde{X}$

Defn: A space X is contractible if it deformation retracts to a point.

Idea: Construct $H: \tilde{X} \times [0,1] \to \tilde{X}$ inductively

Given H on $\underset{\text{||}}{X \times \{0\}}$, $\underset{\{e\}}{X \times \{1\}}$, $\underset{\{e\}}{\{e\} \times [0,1]}$



$H(x,0) = x \quad \forall x \in \tilde{X}$

$H(x,1) = e \quad \forall x \in \tilde{X}$

$H(e,t) = e \quad \forall t \in [0,1]$

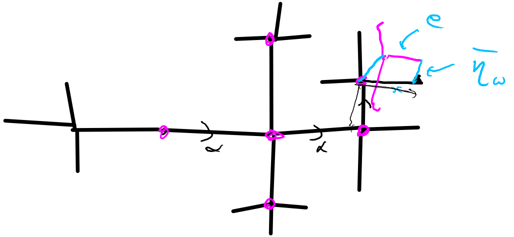
- Define H on $\tilde{V} \times [0,1]$
- Extend to all the squares

Recall $\tilde{V} = \{ l_1 \ldots l_n : l_1 \ldots l_n$ reduced word$\}$

- There is a path to $l_1 \ldots l_n$, namely $(e, l_1), (l_1, l_2), (l_2, l_3), \ldots$
- sequence of edges
- edges $\longleftrightarrow$ paths

- For $w \in \tilde{V}$, let $\eta_w = (e,l_1) * (l_1,l_2) * \dots * (l_{n-1},l_n)$
  $w = (l_1 \dots l_n)$

- Define $H(w,s) = \bar{\eta}_w(s)$



- Finally, extend H to <u>squares</u>

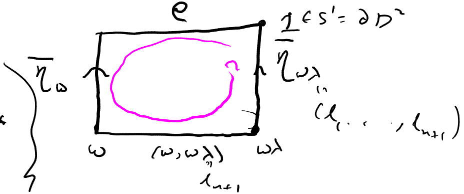
  given $(w, w\lambda)$, w & $w\lambda$ reduced

  $D^2 = [w, w\lambda] \times [0,1]$

- H is defined on its boundary

- On ∂D^2, we get the loop

$$\theta = e * (e,l_1) * (l_1,l_2) * \dots (l_{n-1},l_n) * (l_n,l_{n+1}) * \overline{(l_n,l_{n+1})} * (l_{n+1},l_n) \dots * \overline{(e,l)}$$



We see $\Theta = e * (e, l_1) * \dots * (e, l_n) * (l_n, l_{n+1}) * \overline{(l_{n+1}, l_n)} * \dots * \overline{(e, l_1)}$

$\sim e$

<u>Propn:</u> $f : S^1 \longrightarrow X$ extends to $F : D^2 \to X$ iff

$f_*(\underset{\substack{\cap \\ \pi_1(S^1, 1)}}{1}) \in \pi_1(X, f(1))$ is trivial.

Thus, H extends on all squares.

By gluing lemma, H is continuous and as required

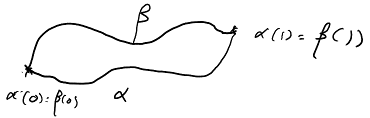
<u>Cor:</u> $\pi_1(\tilde{X}, e) = \{e\}$

Definition: A space X is simply-connected if X is path-connected and $\pi_1(X, x_0) = \{e\}$ for $x_0 \in X$.

Proposition: Two paths in a simply-connected space are homotopic fixing endpoints if and only if their endpoints coincide.

Proof:

Suppose $\alpha, \beta : [0,1] \to X$, $\alpha(0) = \beta(0)$, $\alpha(1) = \beta(1)$

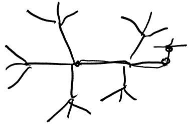


Then $\beta \sim \alpha * \bar{\alpha} * \beta \sim \alpha * \underbrace{(\bar{\alpha} * \beta)}_{\sim} \sim \alpha * e = \alpha$

$\pi_1(X, \beta(1)) = \{e\}$, $\beta(1) = \bar{\alpha}(0)$

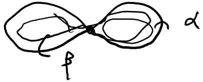
□

$\varphi : \langle \alpha, \beta \rangle \to \pi_1(X, x_0)$



ω word in $\{\alpha, \beta, \bar{\alpha}, \bar{\beta}\}$, ($\omega$ reduced)

- Defined $\eta_\omega : [0,1] \to \tilde{X}$



e.g. $\omega = \alpha\alpha\beta$

- Define $\theta_\omega = l_1 * l_2 \cdots * l_n$

if $\omega = l_1 \cdots l_n$

$\theta_\omega : [0,1] \to X$

Rk: $\eta_\omega = \tilde{\Theta}_\omega$ is the lift of ω starting at e.

$(e, l_1) * (l_1, l_1 l_2) * \dots * (l_1 \dots l_{n-1}, l_1 \dots l_n)$

$$\Big\downarrow p$$

$l_1 * l_2 * \dots * l_n = \Theta_\omega$

$\varphi(\omega) = [\Theta_\omega]$

Theorem: $\varphi : \langle \alpha, \beta \rangle \longrightarrow \pi_1(X, x_0)$ is an isomorphism

Pf: Injectivity: Suppose $g \in \langle \alpha, \beta \rangle$, then $g = [\omega]$, ω reduced

If $\varphi(g) = e$, then $\Theta_\omega \sim e \Rightarrow \tilde{\Theta}_\omega(1) = \tilde{e}(1) = e$

$\eta_\omega(1) = \omega$

$\therefore g = [\omega] = [e] = e$

Surjectivity: Let $[\alpha] \in \pi_1(X, x_0)$, let $\tilde{\alpha} : [0,1] \to \tilde{X}$ be the lift of α starting at e. Let $w = \tilde{\alpha}(1)$; w is a reduced word.

$\uparrow$
$\tilde{V}$

Thus, $\tilde{\alpha}(1) = \eta_w(1) = \tilde{\theta}_w(1)$

$\because$ As $\tilde{X}$ is simply-connected, $\tilde{\theta}_w \sim \eta_w \sim \tilde{\alpha} \Rightarrow [\alpha] = [\theta_w] = \varphi(w)$ $\square$

Theorem: If w_1 and w_2 are reduced and $w_1 \sim w_2$, then $w_1 = w_2$.

i.e. $[w_1] = [w_2]$ $\quad$ $\hat{W}$

Pf: $[w_1] = [w_2] \Rightarrow \varphi([w_1]) = \varphi([w_2])$

$\Rightarrow \theta_{w_1} \sim \theta_{w_2}$

$\Rightarrow w_1 = \tilde{\theta}_{w_1}(1) = \tilde{\theta}_{w_2}(1) = w_2$

$\square$

Map lifting: X, Y and Z path-connected topological spaces

Lifting problem:

$$\begin{tikzcd} & (Y, y_0) \arrow[d, "p"] \\ (Z, z_0) \arrow[r, "f"'] \arrow[ur, dashed, "\tilde{f}"] & (X, x_0) \end{tikzcd}$$

p covering map

Suppose $\tilde{f}$ exists

$$f_*(\pi_1(Z, z_0)) = p \circ \tilde{f}_*(\pi_1(Z, z_0))$$

$$= p_*(\tilde{f}_*(\pi_1(Z, z_0)))$$

$$\subset p_*(\pi_1(Y, y_0))$$

E.g.

$$\begin{tikzcd} & (\mathbb{R}, 0) \arrow[d] \\ (S^1, 1) \arrow[r, "\mathbb{1}"'] \arrow[ur, dashed, "\tilde{f}"] & (S^1, 1) \end{tikzcd}$$

No solution

Theorem: If X is locally path-connected (l.p.c.), then $\tilde{f}$ exists iff

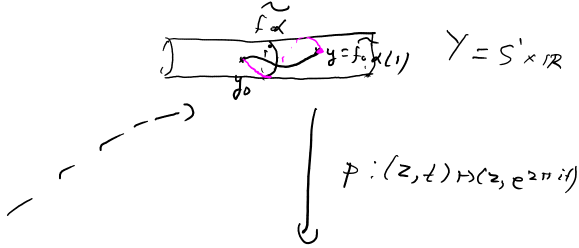
$$f_*(\pi_1(Z, z_0)) \subset p_*(\pi_1(Y, y_0))$$

Defn: X is l.p.c. if $\forall x \in X$, $U \subset X$ open $x \in U$, $\exists\, V \subset U$ open, $x \in V$ s.t. V is path-connected.

Proof: (1) We construct $\tilde{f}$

Given $z \in Z$, let $\alpha : [0,1] \to Z$ be a path from z_0 to z.

Let $\tilde{f}(z) = \widetilde{f \circ \alpha}(1)$, where $\tilde{\beta} :=$ lift of β with $\tilde{\beta}(0) = y_0$



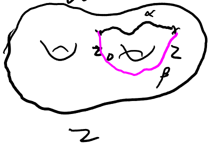
(2) Well-defined:

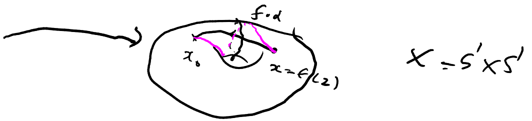
Suppose β is another path,

$[\alpha * \bar{\beta}] \in \pi_1(Z, z_0)$

$[f \circ (\alpha * \bar{\beta})] \in \pi_1(X, x_0)$ $\in p_*(\pi_1(Y, y_0))$

$= p_*([\tilde{\theta}])$





Hence $(f \circ \alpha) * (f \circ \bar{\beta}) \sim \theta = p \circ \tilde{\theta}$, $\tilde{\theta}$ loop at y_0

$f \circ \alpha \sim \theta * f \circ \beta$

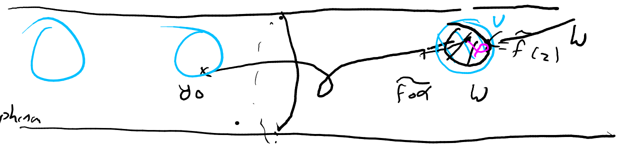
$\Rightarrow \underline{\widetilde{f \circ \alpha}(1)} = \widetilde{\theta * f \circ \beta}(1) = \tilde{\theta} * \widetilde{f \circ \beta}(1) = \underline{\widetilde{f \circ \beta}(1)}$

Continuity:

- Let $W \subset Y$ be open, $y = \tilde{f}(z) \in W$

  x has an evenly-covered neighbourhood U

  with $y \in V \subset p^{-1}(U)$, $p|_V : V \to U$ homeomorphism



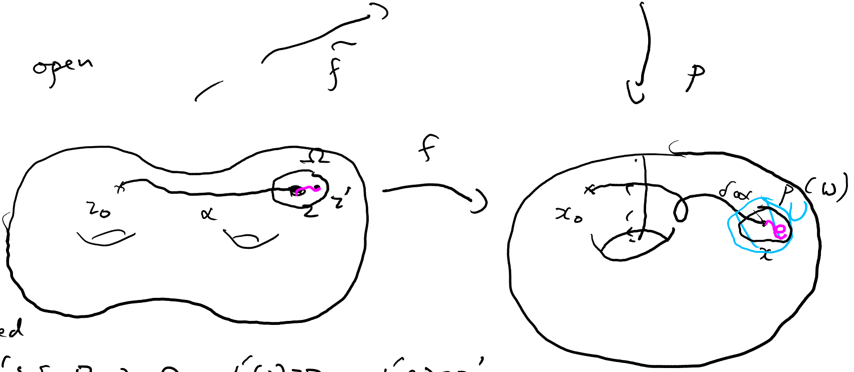
- w.l.o.g. $W \subset V$,

  so $p(W) \subset U$ is open

- As f is continuous,

  $\exists\, \Omega \subset Z$ open, $z \in \Omega$

  s.t. $f(\Omega) \subset p(W)$



- By l.p.c., can assume

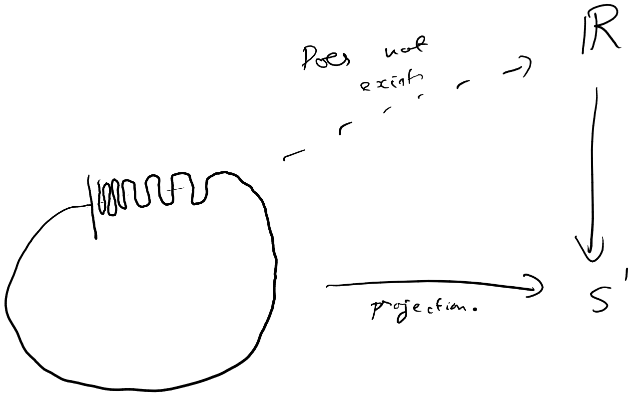
  Ω is path-connected

- Let $z' \in \Omega$, then $\exists\, \alpha' : [0,1] \to \Omega$, $\alpha'(0) = z$, $\alpha'(1) = z'$

- Then $\alpha * \alpha'$ is a path from z_0 to z', so $\tilde{f}(z') = \widetilde{f\circ(\alpha*\alpha')}(1)$

$$\widetilde{(f\circ\alpha)*(f\circ\alpha')}(1) = (\widetilde{f\circ\alpha}) * (p|_V)^{-1}\circ(f\circ\alpha')(1) = (p|_V)^{-1}(\underset{\in\, p(W)}{f\circ\alpha'(1)}) \in W$$

Counterexample:



Application: (1) $f: \mathbb{C} \to \mathbb{C} \setminus \{0\}$; $\tilde{f}$ exists, i.e. $\exists\, g: \mathbb{C} \to \mathbb{C}$ s.t. $f(z) = e^{g(z)}$

$$\begin{tikzcd} & \mathbb{C} \arrow[d, "\exp"] \\ \mathbb{C} \arrow[ur, dashed, "\tilde{f}"] \arrow[r, "f"] & \mathbb{C} \setminus \{0\} \end{tikzcd}$$

$z \mapsto e^z$.

(2) Little Picard

$$\begin{tikzcd} & \mathbb{D} \subset \mathbb{C} \arrow[d, "p"] \\ \mathbb{C} \arrow[ur, dashed, "\tilde{f}"] \arrow[r, "f"] & \mathbb{C} \setminus \{p, q\} \end{tikzcd}$$

covering map, holomorphic

Coverings, Subgroups, Symmetries

Assume all spaces are connected and locally path-connected, hence path-connected