Topological problems:

Example: Is there a continuous function $f : \mathbb{C} \to \mathbb{C}$ such that $(f(z))^2 = z \ \forall z \in \mathbb{C}$?

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Topological problems:

Example: Is there a continuous function $f : \mathbb{C} \to \mathbb{C}$ such that $(f(z))^2 = z \ \forall z \in \mathbb{C}$?

* Step 1: Reformulate in terms of commutative diagrams

$$\begin{tikzcd} & \mathbb{C} \arrow[d, "z \mapsto z^2"] \\ \mathbb{C} \arrow[ur, dashed, "f"] \arrow[r, "\mathbb{1}"'] & \mathbb{C} \end{tikzcd}$$

$f(z)$

z $\quad z \quad z$

$(f(z))^2 = z$

* Step 2: Translate to an algebraic problem using a Functor.

Space $\longrightarrow$ Group, Module, Set

Continuous functions $\longrightarrow$ Homomorphism, ..., function.

In this case, we actually need to modify the original problem

$$\begin{tikzcd} & \mathbb{C}\setminus\{0\} \arrow[d, "z \mapsto z^2"] \\ \mathbb{C}\setminus\{0\} \arrow[ur, dashed, "f"] \arrow[r, "\mathbb{1}"'] & \mathbb{C}\setminus\{0\} \end{tikzcd}$$

If the original problem has a positive solution, then so does this.

* Corresponding Algebraic Problem

$$\begin{tikzcd} & \mathbb{Z} \ni k \arrow[d, "n \mapsto 2n"] \\ 1 \in \mathbb{Z} \arrow[ur, dashed, "f_*"] \arrow[r, "\mathbb{1}"'] & \mathbb{Z} \ni 1 = 2k \end{tikzcd}$$

$\longleftarrow$ has no solution.

We see: if the topological problem has a solution, so does the algebraic problem.

Lifting problems

$$\begin{tikzcd}
 & Z = \mathbb{C} \setminus \{0\} \arrow[d, "p = z \mapsto z^2"] \\
X \arrow[ur, dashed, "f"] \arrow[r, "\varphi = \mathbb{1}"'] & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \mathbb{Z} \arrow[d, "\times 2"] \\
\mathbb{Z} \arrow[ur, dashed, "f_*"] \arrow[r, "\mathbb{1}"'] & \mathbb{Z}
\end{tikzcd}$$

$X = \mathbb{C} \setminus \{0\}$, $Y = \mathbb{C} \setminus \{0\}$

		Topological spaces	Groups	Sets	Vector Spaces
X, Y, Z :	Objects	Topological spaces	Groups	Sets	Vector Spaces
$\varphi, f, \ldots$:	Morphisms	Continuous functions	Homomorphisms	functions	Linear Transformation

Morphisms — maps

To state : · Composition of morphisms
· Identities

A Category C consists of
* a class of objects ob(C) topological spaces, groups
* for objects X, Y a class of morphisms Mor(X, Y) continuous functions, homomorphisms
* for objects X, an identity morphism in Mor(X) $\mathbb{1}_X \in Mor(X)$
* for objects X, Y, Z, a composition of morphisms.
* associativity and identity properties.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[rr, bend right, "g \circ f"'] & Y \arrow[r, "g"] & Z \end{tikzcd}$$

$$f \circ (g \circ h) = (f \circ g) \circ h$$

$$\mathbb{1}_Y \circ f = f = f \circ \mathbb{1}_X \quad \text{if } f: X \to Y$$

Lifting Problem:

$$\begin{tikzcd} & Z = \tilde{Y} \arrow[d, "p"] \\ X \arrow[r, "\varphi"'] \arrow[ur, dashed, "\tilde{\varphi} = f"] & Y \end{tikzcd}$$

Given C category

$X, Y, Z \in ob(C)$

p, φ - Morphisms

Want: $f \in Mor(X, Y)$ s.t. $p \circ f = \varphi$

A Functor is a map between categories C and D, which associates
* to each object X in C an object F(X) in D
* for a morphism f from X to Y a morphism F(f) from F(X) to F(Y)
* this respects composition and identity morphisms.

$$\begin{tikzcd} & Z \arrow[d, "p"] \\ X \arrow[ur, dashed, "f"] \arrow[r, "\varphi"'] & Y \end{tikzcd}$$

$$F(f) \circ F(g) = F(f \circ g)$$

$$g : X \to Y \ , \ f : Y \to Z$$

$$F(g) : F(X) \to F(Y) \ , \ F(f) : F(Y) \to F(Z)$$

$$F(\mathbb{1}_X) = \mathbb{1}_{F(X)}$$

Suppose $F : \mathrm{Top} \to \mathrm{Group}$ is a functor, given

$$\begin{tikzcd} & Z \arrow[d, "p"] \\ X \arrow[ur, dashed, "f"] \arrow[r, "\varphi"'] & Y \end{tikzcd}$$

we get

$$\begin{tikzcd} & F(Z) \arrow[d, "F(p)"] \\ F(X) \arrow[ur, dashed, "F(f)"] \arrow[r, "F(\varphi)"'] & F(Y) \end{tikzcd}$$

$$p \circ f = \varphi \implies F(p \circ f) = F(p) \circ F(f) = F(\varphi)$$

Given

- Category $\mathcal{C}$ (e.g. Top = category of topological spaces)

- Lifting problem on $\mathcal{C}$ (many other problems)

$$(*)\qquad \begin{tikzcd} & Z \arrow[d, "p"] \\ X \arrow[r, "\varphi"'] \arrow[ur, dashed, "f"] & Y \end{tikzcd}$$

; Question: $\exists ?\ f$ s.t. $p \circ f = \varphi$

Auxiliary: Given a functor $F : \mathrm{Top} \to \mathcal{D} = \mathrm{Group}$

We get a lifting problem

$$(**)\qquad \begin{tikzcd} & Z \arrow[d, "F(p)"] \\ F(X) \arrow[r, "F(\varphi)"'] \arrow[ur, dashed, "\hat{f}"] & F(Y) \end{tikzcd}$$

Propn: A solution f to $(*)$ gives a solution $F(f) = \hat{f}$ to $(**)$.

Not Conversely: $\hat{f}$ may exist but

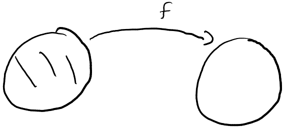
- $\hat{f}$ may not be $F(f)$
- Even if $F(f) = \hat{f}$, $p \circ f = \varphi$ may be false.

More topological problems

Extension Problem: $A \subset X$, $f: A \to Y$

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & Y \\ X \arrow[ur, dotted, "F"'] & \end{tikzcd}$$

<u>E.g.</u> $A = Y = S^{n-1}$, $X = B^n$



Functor $F: \mathrm{Top} \to \mathcal{C}$

gives extension problem in $\mathcal{C}$

Homeomorphism problem: Are X and Y homeomorphic?

$$\begin{tikzcd} X \arrow[loop left, "\mathbb{1}_X"] \arrow[r, bend left, dashed, "f"] & Y \arrow[l, bend left, dashed, "g"] \arrow[loop right, "\mathbb{1}_Y"] \end{tikzcd}$$

Given $F : \mathrm{Top} \to \mathrm{Group}$

- If X and Y are homeomorphic, we get

$$\begin{tikzcd} F(X) \arrow[loop left, "\mathbb{1}_{F(X)}"] \arrow[r, bend left, dashed, "\varphi = F(f)"] & F(Y) \arrow[l, bend left, dashed, "\psi = F(g)"] \arrow[loop right, "\mathbb{1}_{F(Y)}"] \end{tikzcd}$$

, i.e. $F(X) \cong F(Y)$

Fixed point: Given $f : X \to X$ is there a point such that $f(x) = x$?

Sorry, missing notes again

Images on the next pages

Fixed point: Given $f : X \to X$ is there a point such that $f(x) = x$?

Question: Is there a function on X without fixed points?

Graph(f) : $\Gamma(f)$

$$\begin{tikzcd} & & X \times X \arrow[dl, "p_1"'] \arrow[dd] \\ & X & \\ X \arrow[uurr, bend left, "\Gamma(f)"] \arrow[ur, "\mathbb{1}"] \arrow[rr, "f"'] & & X \end{tikzcd}$$

$$\begin{array}{c} (x, f(x)) \\ \downarrow \\ f(x) \end{array}$$

$\Delta = \{(x,x) : x \in X^2$

Graphs in terms of commutative diagrams: $f : X \to Y$; $\Gamma(f)(x) = (x, f(x))$

$$\begin{tikzcd} & & X \times Y \arrow[dl, "p_1"'] \arrow[dd, "p_2"] \\ & X & \\ X \arrow[uurr, bend left, "\Gamma(f)"] \arrow[ur, "\mathbb{1}"] \arrow[rr, "f"'] & & Y \end{tikzcd}$$

Commutes

Fixed-point free functions: Look for functions and graphs

set complements

$$\begin{tikzcd} X & (X\times X)\setminus\Delta \arrow[l, "p_1"'] \arrow[d, "p_2"] \\ X \arrow[u, "\mathbb{1}"] \arrow[ur, dashed, "\Phi"] \arrow[r, dashed, "f"'] & X \end{tikzcd}$$

$\downarrow$ apply F

$$\begin{tikzcd} F((X\times X)\setminus\Delta) \arrow[d] \\ F(X) \end{tikzcd}$$

Using a simple Functor

$\underline{\pi_0(X)}$: the set of path-connected components of a topological space

This is a functor from Top to Set, $\pi_0 : Top \longrightarrow Set$

An Extension Problem:
$$\begin{tikzcd}
S^0 \arrow[r] \arrow[d, hook] & S^0 = \{-1, 1\} \\
D^1 = [-1, 1] \arrow[ur, dashed, "f"'] &
\end{tikzcd}$$

$\pi_0(D^1) = \{*\}$; $\pi_0(S^1) = \{\text{☺}, \text{☹}\}$

$\underline{\text{Apply } \pi_0}$
$$\begin{tikzcd}
\{\text{☺}, \text{☹}\} \arrow[r, "\mathbb{1}"] \arrow[d, "i"'] & \{\text{☺}, \text{☹}\} \\
\{*\} \arrow[ur, dashed, "f"'] &
\end{tikzcd}$$

Suppose $f(*) = \text{☺}$

Then $i \circ f(\text{☹}) = \text{☺} \neq \mathbb{1}(\text{☹})$

$\underline{\text{No solution}}$

Extension of lifting problem

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[d, "i"'] & Z \arrow[d, "p"] \\ X \arrow[r, "f"'] \arrow[ur, dashed, "F"] & Y \end{tikzcd}$$

Includes lifting:

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & Z \arrow[d, "p"] \\ X \arrow[r] \arrow[ur, dashed, "F"] & Y \end{tikzcd}$$

Includes Extension:

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[d, "i"'] & Z \arrow[d] \\ X \arrow[r] \arrow[ur, dashed, "F"] & \{x\} \end{tikzcd}$$

Duality

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[d, "i"'] & Y \\ X \arrow[ur, dashed, "f"'] & \end{tikzcd} \qquad \begin{tikzcd} & X \arrow[d, "p"] \\ Y \arrow[ur, dashed, "f"] \arrow[r, "\varphi"'] & A \end{tikzcd}$$

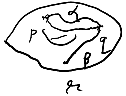
Co-functor $F : \mathcal{C} \to \mathcal{D}$; $f : X \to Y$ then $F(f) : F(Y) \to F(X)$

$$F(f \circ g) = F(g) \circ F(f)$$

Paths

Defn: For a space X, we define $\pi_0(X)$ to be the set of path components.

$\pi_0(X) = X/\sim$; we define $\sim$



$$\Omega(X;p,q) = \{\alpha : [0,1] \to X : \alpha(0) = p, \alpha(1) = q, \alpha \text{ continuous}\}$$

$$p \sim q \iff \Omega(X;p,q) \neq \phi$$

Theorem: ~ is an equivalence relation.
Proof:

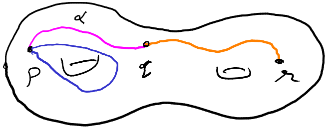
* Reflexive: $\Omega(X;p,p) \neq \emptyset$ as $e_p \in \Omega(X;p,p)$, $e_p(s) = p \ \forall s \in [0,1]$

* Symmetric: $p \sim q \Rightarrow \Omega(X;p,q) \neq \emptyset$; $\Omega(X;p,q) \to \Omega(X;q,p)$
  $\alpha \mapsto \bar{\alpha}$; $\bar{\alpha}(s) = \alpha(1-s)$

* Transitive: $p \sim q$, $q \sim r$
  $\Omega(X;p,q) \neq \emptyset \neq \Omega(X;q,r)$ | We use $\underset{\alpha}{\Omega(X;p,q)} \times \underset{\beta}{\Omega(X;q,r)} \to \Omega(X;p,r)$
  $\longmapsto \alpha * \beta$

$$\Omega(X; p, q) \times \Omega(X; q, r) \longrightarrow \Omega(X; p, r)$$

$$\alpha \qquad\qquad \beta \qquad\qquad \alpha * \beta$$



$$\alpha * \beta : [0,1] \longrightarrow X$$

$$\alpha * \beta = \begin{cases} \alpha(2s), & 0 \le s \le 1/2 \\ \beta(2s-1), & 1/2 \le s \le 1 \end{cases}$$

<u>But</u>: $\quad \underset{\substack{\uparrow \\ p = \alpha(0)}}{e_p} * \alpha \neq \alpha \quad ; \quad \underbrace{(\overset{\times 4}{\alpha} * \overset{\times 4}{\beta})}_{\times 2} * \underbrace{\gamma}_{\times 2} \neq \alpha * (\beta * \gamma)$

Switch to the "Whiteboard 2"

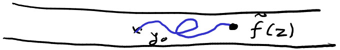
Coverings, Subgroups, Symmetries

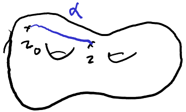
All spaces will be assumed to be $\underbrace{\text{locally path-connected}}_{\text{l. p. c.}}$ and connected, hence path-connected

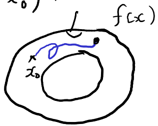
Uniqueness in Map Lifting:

Assume $\tilde{f}$ exists

$$\begin{tikzcd} & (Y, y_0) \arrow[d, "p"] \\ (Z, z_0) \arrow[ur, dashed, "\tilde{f}"] \arrow[r, "f"] & (X, x_0) \end{tikzcd}$$







By uniqueness of lifts,

$$\widetilde{f\circ\alpha} = \underbrace{\tilde{f}\circ\alpha}_{\text{is a lift}}$$

Hence, $\tilde{f}(z) = (\tilde{f}\circ\alpha)(1) = \underbrace{\widetilde{f\circ\alpha}(1)}_{\text{determined by } f \,\&\, \alpha}$ for <u>any</u> lift $\tilde{f}$ of f

Isomorphisms of (based) covers A homeomorphism $F : Y \to Y'$ (fixing basepoints) s.t. the diagram commutes

$$\begin{tikzcd}
(Y, y_0) \arrow[rr, dashed, "F"] \arrow[dr, "p"'] & & (Y', y_0') \arrow[dl, "p'"] \\
& (X, x_0) &
\end{tikzcd}$$

E.g. $\mathbb{R} \to S^1$

$t \mapsto e^{2\pi i t}$,

$\tau(t) = t+1$ is an isomorphism, but not of based covers

$$\begin{tikzcd}
(\mathbb{R}, 0) \arrow[rr, "\tau"] \arrow[dr] & & (\mathbb{R}, 0) \arrow[dl] \\
& (S^1, 1) &
\end{tikzcd}$$

Theorem: Two based covers $p : (Y, y_0) \to (X, x_0),\ p' : (Y', y_0') \to (X, x_0)$ are isomorphic if and only if

$$\underbrace{p_*(\pi_1(Y, y_0))}_{\text{injects}} = p'_*(\pi_1(Y', y_0')) \subset \pi_1(X, x_0)$$

Proof: Use map lifting

$$\begin{tikzcd}
(Y, y_0) \arrow[r, dashed, "F"] \arrow[rr, dashed, bend left=40, "F' \circ F"', "\mathbb{1}_{(Y,y_0)}"] \arrow[dr, "p"'] & (Y', y_0) \arrow[r, dashed, "F'"] \arrow[d, "p'"] & (Y, y_0) \arrow[dl, "p"] \\
& (X, x_0) &
\end{tikzcd}$$

By uniqueness

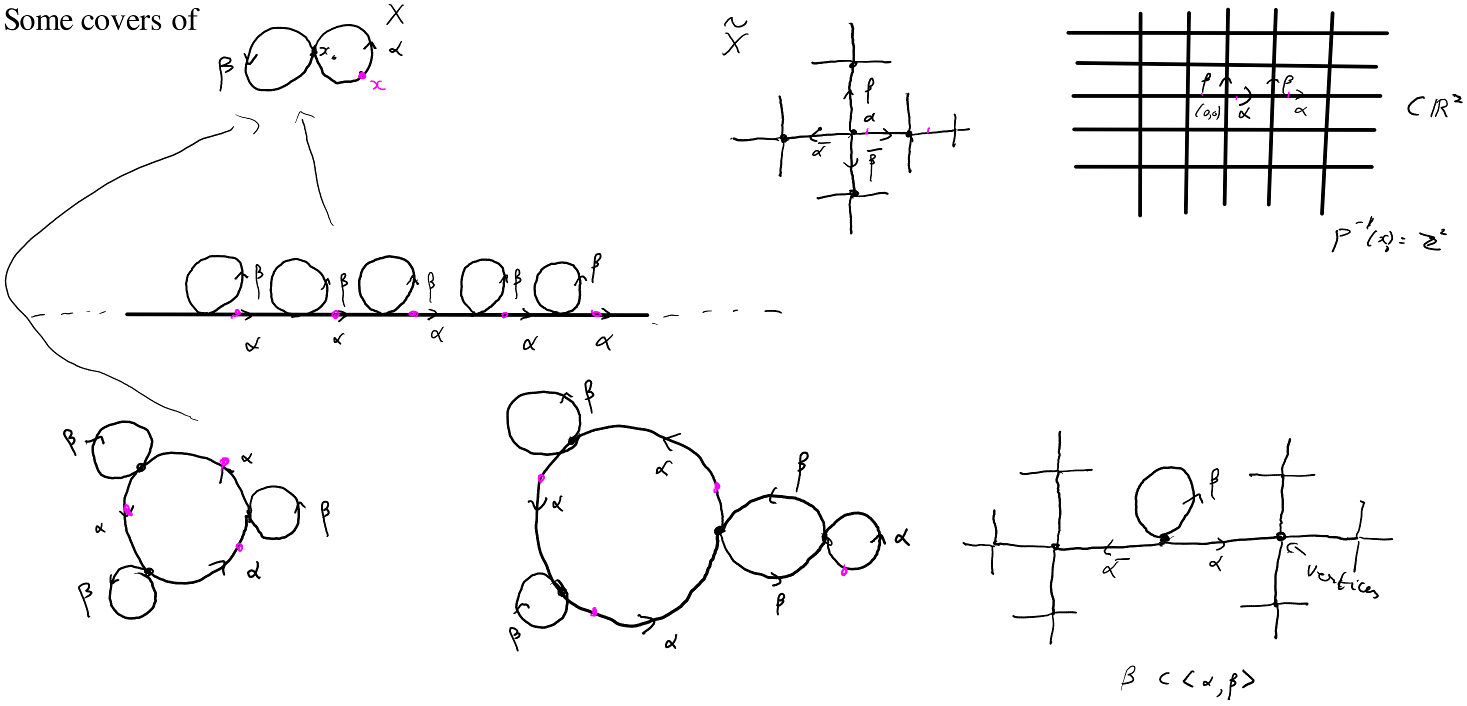
$F' \circ F = \mathbb{1}_{(Y, y_0)}$

Similarly

$F \circ F' = \mathbb{1}_{(Y', y_0')}$

F exists as $p_*(\pi_1(Y, y_0)) \subset p'_*(\pi_1(Y', y_0'))$

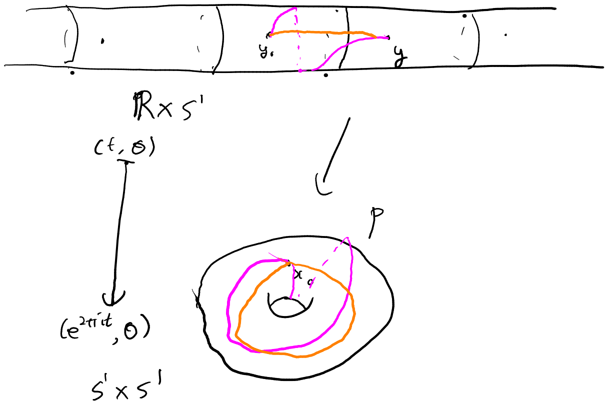
Some covers of



Fiber of a covering map

$$p:(Y,y_0) \longrightarrow (X,x_0)$$

$$\underbrace{p_*(\pi_1(Y,y_0))}_{=:\,H} \subset \underbrace{\pi_1(X,x_0)}_{=:\,G}$$



- Given $y \in p^{-1}(x_0)$
- Pick α from y_0 to y;

$$[p\circ\alpha] \in \pi_1(X,x_0) = G$$

- If β is another path, then $\beta = \underbrace{(\beta*\bar{\alpha})}_{\text{loop at } y_0} * \alpha$

$$\therefore \quad p\circ\beta \sim (p\circ(\beta*\bar{\alpha})) * (p\circ\alpha)$$

$$g' := [p\circ\beta] = \underset{\uparrow H}{p_*([\beta*\bar{\alpha}])} * [p\circ\alpha] =: g$$

Thus, $g' = Hg$, i.e. we get a well-defined <u>right coset</u>

$\varphi : p^{-1}(x_0) \longrightarrow H\backslash G$

· We construct an inverse.

Given $g \in H$,

$g = [\alpha]$,

Define $\psi(Hg) = \tilde{\alpha}(1)$

· if $g' \in Hg$, i.e. $g' = \underbrace{h}_{[\lambda]} \cdot \underbrace{g}_{\alpha}$

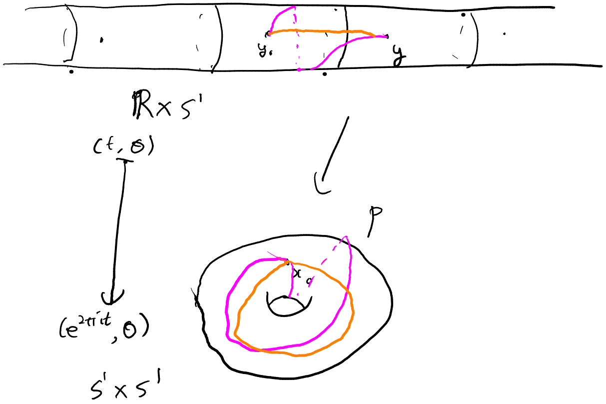
Then $g' = [\lambda * \alpha]$, $[\lambda] \in p_*(\pi_1(Y, y_0)) \Rightarrow \lambda \sim p \circ \eta$, where $\eta \in \Omega(Y, y_0)$

· <u>Claim</u>: $\tilde{\lambda}(1) = y_0$

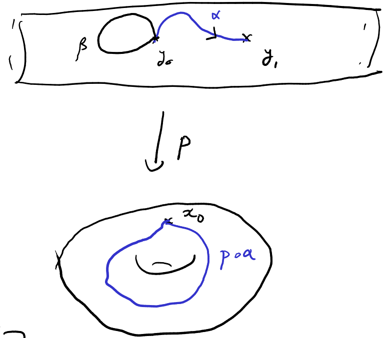
$\therefore \tilde{\lambda}(1) = \eta(1) = y_0$ as $\eta = \widetilde{p \circ \eta}$

$\therefore \widetilde{\lambda * \alpha} = \tilde{\lambda} * \tilde{\alpha} \Rightarrow \widetilde{\lambda * \alpha}(1) = \tilde{\lambda} * \tilde{\alpha}(1)$
$= \tilde{\alpha}(1)$

i.e. $\psi(Hg') = \psi(Hg)$



Change of basepoint in Covering Spaces



$$p : (Y, y_0) \longrightarrow (X, x_0)$$

$$y \in p^{-1}(y_0) = \{Hg : g \in G\} = {}_H\backslash G$$

$$G = \pi_1(X, x_0), \quad H = p_*(\pi_1(Y, y_0))$$

Question: What is $p_*(\pi_1(Y, y_1)) \subset \pi_1(X, x_0)$?

Soln: $\pi_1(Y, y_1) = \{\bar\alpha \beta \alpha : \beta \in \pi_1(Y, y_0)\}$

$\therefore \quad p_*(\pi_1(Y, y_1)) = \{p(\bar\alpha \beta \alpha) : \beta \in \pi_1(Y, y_0)\} = \{g^{-1} h g \,;\, h \in H\} = g^{-1} H g$

Let $[p \circ \alpha] = g$

Rk: If $g' = h \cdot g$, $\quad g^{-1} H g = \underbrace{(g')^{-1}}_{g^{-1}\cdot h^{-1}} \cdot H \cdot \underbrace{g'}_{hg}$

Rk: If H' is conjugate to H, $H' = g^{-1} H g$ for some $g = [\alpha]$; hence

$$H' = p_*(\pi_1(Y, \overset{\tilde\alpha(1)}{y_1''}))$$

Question: When are two coverings isomorphic?

$$\begin{tikzcd}
Y, y_0 \arrow[rr, dotted, "F"] \arrow[dr, "p"'] & & Y' \arrow[dl, "p'"] \\
& X, x_0 &
\end{tikzcd}$$

, F homeomorphic

Let $H = p_*(\pi_1(Y, y_0))$

$H' = p'_*(\pi_1(Y', y_0'))$

Then $\exists y_1'$ with the below diagram $\iff$ H' is conjugate to H.

Let $y_0' \in (p')^{-1}(x_0)$

Want for some $y_1' \in (p')^{-1}(x_0)$,

$$\begin{tikzcd}
(Y, y_0) \arrow[rr, "F", "\text{homeomorphism}"'] \arrow[dr, "p"'] & & (Y', y_1') \arrow[dl, "p'"] \\
& (X, x_0) &
\end{tikzcd}$$

$y_1' := F(y_0)$

not fixed

Deck transformations

$$\begin{tikzcd} Y \arrow[rr, "F"] \arrow[dr, "p"'] & & Y \arrow[dl, "p"] \\ & X & \end{tikzcd}$$

F homeomorphism

- Pick $x_0 \in X$, $y_0 \in p^{-1}(x_0)$
- $F(y) \in p^{-1}(x_0)$, and F determined by $F(y)$,
- $$\begin{tikzcd} (Y, y_0) \arrow[rr, "F"] \arrow[dr, "p"'] & & (Y, F(y_0)) \arrow[dl, "p"] \\ & (X, x_0) & \end{tikzcd}$$ $F(y_0) \in p^{-1}(x_0)$

$g^{-1} H g = H \iff$

$g \in N_G(H)$

$= \{g^{-1}Hg = H : g \in G\}$

Deck transformations are

$N_G(H)/H$

- For $y_1 \in p^{-1}(x_0)$, $\exists F$ deck transformation s.t. $F(y_0) = y_1 \iff$

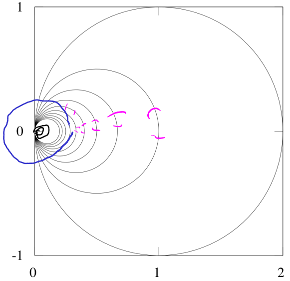
$$H = p_*(\pi_1(Y, y_0)) = p_*(\pi_1(Y, y_1)) = g^{-1}Hg, \quad \text{s.t. } y_1 \leftrightarrow Hg$$

Defn: A covering map is said to be Galois if deck transformations act transitively on the fibre. $\iff H \lhd G$

Hawaiian Earrings and Universal Covers

Definition: The universal covering of a space X is a covering map $p : \tilde{X} \to X$ such that $\tilde{X}$ is simply-connected.

$X = (S^1 \times \mathbb{N})/\sim$, where $(1,n) \sim (1,m)$ $\forall n, m$

$X =$ 

$x_0 = [(1,n)]$, $n \in \mathbb{N}$

Basic open sets: $U \subset X$, $U \cap (S^1 \times \{n\})$ open $\forall n \in \mathbb{N}$

and either (1) $x_0 \notin U$

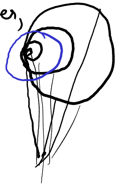
or (2) $(x_0 \in U$ and$)$ $\exists N$ s.t. $(S^1 \times \{n\}) \subset U$ for all $n > N$.

Propn: Let $\gamma_n : [0,1] \to X$, $\gamma_n(s) = (e^{2\pi i s}, n)$. Then $[\gamma_n] \neq \{e\}$ in $\pi_1(X, x_0)$

Pf: Define $\varphi_n : X \to S^1$ and $\psi_n : S^1 \to X$ by

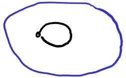
$\varphi_n((\theta, m)) = \begin{cases} 1 & \text{if } m \neq n \\ \theta & \text{if } m = n \end{cases}$ $\psi_n(\theta) = [(\theta, n)]$

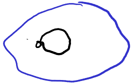
Then $\varphi_n \circ \psi_n : S^1 \to S^1$ is the identity, $\varphi_{n*}(\underset{\neq 0}{[\gamma_n]}) = \underset{\neq 0}{1} \in \pi_1(S^1, 1)$

CX (cone of X) has a universal cover,  namely itself.

Theorem: The Hawaiian earrings do not have a universal cover.
Proof: Suppose the universal cover exists.

$$\begin{array}{c} \tilde{X},\ \tilde{x}_0 \\ \big\downarrow p \\ X \ni x_0 \end{array}$$





Let U be an evenly covered neighborhood of x_0. Let $p^{-1}(U) = \coprod_{\alpha \in A} V_\alpha$, $V_0 \ni \tilde{x}_0$

· Let $\gamma \in \Omega(U, x_0)$, $(p|_{V_0})^{-1}(\gamma) \in \Omega(\tilde{X}_0, x_0)$

$\simeq e$

i.e. $[(p|_{V_0})^{-1}(\gamma)] = e$ in $\pi_1(\tilde{x}_0, x_0)$

$\therefore$ $[\gamma] = p_*([(p|_{V_0})^{-1}(\gamma)]) = e$ in $\pi_1(X, x_0)$

i.e., if $[\gamma] \in \pi_1(U, x_0)$, $i_*([\gamma]) = e$ in $\pi_1(X, x_0)$ | False for X

Definition: A space X is semi-locally simply connected if

- given $x \in X$
- V open in X, $x \in V$

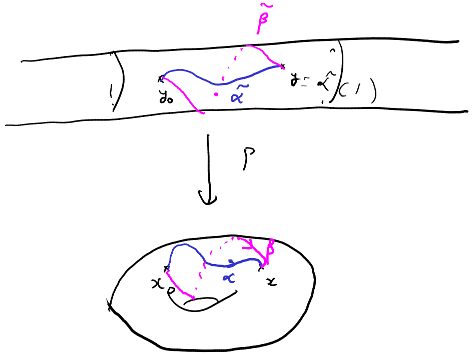
there exists $U \subset X$ open, $x \in U$ s.t.

- $i_* : \pi_1(U, x) \longrightarrow \pi_1(X, x)$ is the trivial homomorphism, where $i : U \to X$ is the inclusion.

Constructing covers

$G = \pi_1(X, x_0)$

$H = p_*(\pi_1(X, x_0))$



- Fiber over x:

$$\{\text{Paths } \alpha \text{ from } x_0 \text{ to } x\}/\sim$$

$$\Omega(X; x_0, x)/\sim$$

where $\alpha \sim \beta \iff [\alpha * \bar\beta] \in H$

<u>Description of Y:</u> <u>Given</u> H, G and X

$$\hat\Omega(X; x_0) = \{\alpha : [0,1] \to X : \alpha(0) = x_0\}$$

$$Y = \hat\Omega(X; x_0)/\sim \quad \text{where}$$

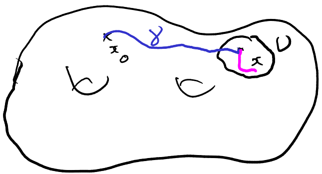
$$\alpha \sim \beta \iff \alpha(1) = \beta(1) \;\&\; [\alpha * \bar\beta] \in H$$

$p : Y \to X$, $p([\alpha]) = \alpha(1)$; p is surjective $\Leftarrow$ X is path-connected

$p^{-1}(x) =$ 'correct fibre'.

Topology: $Y := \hat{\Omega}(X, x_0)/\sim$

where $\alpha \sim \beta \iff \alpha(1) = \beta(1)$ & $[\alpha * \bar{\beta}] \in H \subset G = \pi_1(X, x_0)$

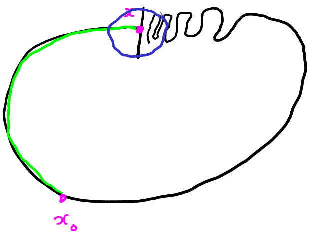


Basic open sets:

Let $U \subset X$ be open,

$\gamma : [0,1] \to X$, $\gamma(0) = x_0$, $\gamma(1) = x$

Define $U_\gamma = \{ [\gamma * \eta] : \eta : [0,1] \to U, \ \eta(0) = x \}$



Rk: $[\gamma] = [\gamma * e_x] \in U_\gamma$

· U_γ forms a basis for our topology

Assume X is connected, locally path-connected, semi-locally simply connected

$G = \pi_1(X, x_0)$, $\quad H \subset G$

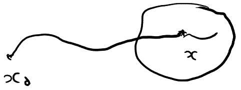
$Y = \hat{\Omega}(X; x_0)/\sim$, $\quad \alpha \sim \beta \iff \alpha(1) = \beta(1)$ & $[\alpha * \bar{\beta}] \in H$

$p : Y \to X$, $\quad p(\alpha) = \alpha(1)$

On Y, basic open sets $U_\gamma = \{[\gamma * \eta] : \eta : [0,1] \to U,\ \eta(0) = \gamma(1)\}$

Theorem: $p : Y \to X$ is a covering map.

Pf: Let $x \in X$, let $U \subset X$ be open set.



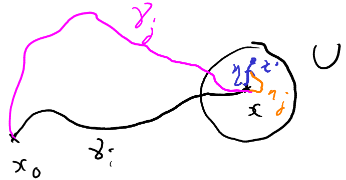
- U is path-connected
- $\pi_1(U, x) \longrightarrow \pi_1(X, x)$ is trivial
- $x \in U$

$\Big\}\ \exists ?$

Let γ be a fixed path from x_0 to x.

Claim: U is evenly covered.

- Fiber over $x = \Omega(X; x_0, x)/\sim$
- Pick representatives $\{\gamma_i\}_{i\in I}$, i.e. fiber $= \{[\gamma_i]_{i\in I}\}$
- Let $V_i = U_{\gamma_i}$; $i \in I$
- By construction, $p: V_i \longrightarrow U$



Lemma: $p|V_i : V_i \longrightarrow U$ is onto

Pf: Given $x' \in U$, let η be a path in U from x to x';

then $[\gamma_i * \eta] \in U_{\gamma_i}$ and $p([\gamma_i * \eta]) = x'$

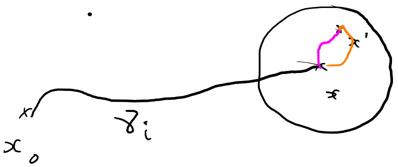
Lemma: If $i \neq j \in I$, then $V_i \cap V_j = \emptyset$

Pf: Suppose $[\gamma_i * \eta_i] = [\gamma_j * \eta_j]$; hence $[\gamma_i * \underbrace{(\eta_i * \bar{\eta}_j)}_{\sim e} * \bar{\gamma}_j] \in H \Rightarrow [\gamma_i * \bar{\gamma}_j] \in H$, i.e. $[\gamma_i] \sim [\gamma_j] \Rightarrow i = j$. ∎

Local inverse:

$p: U \longrightarrow V_i$

$p(x') = [\gamma_i * \eta]$, independent of η



Which cover?

$$\begin{array}{c} \tilde{\Omega}(X;x_0) \\ \rotatebox{90}{=} \end{array}$$

$$p : (Y, y_0) \longrightarrow (X, x_0)$$

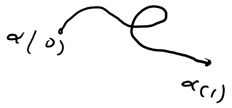
$$y_0 = [e], \quad e = e_{x_0}$$

In general

$$p_*(\pi_1(Y, y_0)) = \{ [\alpha] : \alpha \in \Omega(X; x_0),\ \tilde{\alpha}(1) = y_0 \}$$

- In our case, $\tilde{\alpha}(s) = \big[(\theta : [0,1]) \mapsto \alpha(\theta s) \big]$

  $\tilde{\alpha}(s)(1) = \alpha(s)$



- $\tilde{\alpha}(1) = [\alpha]$

- Hence $\tilde{\alpha}(1) = y_0 \iff [\alpha] \sim [e] \iff [\alpha * \bar{e}] \in H$

  $[\alpha * \bar{e}] = [\alpha]$

Thus $p_*(\pi_1(Y, y_0)) = \{ [\alpha] : [\alpha] \in H \} = H$

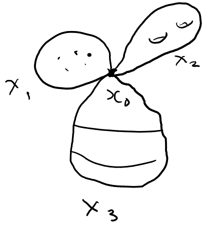
$G = \pi_1(X, x_0)$, X connected, l.p.c., s.l.s.c.

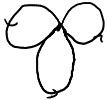
Then

- Based covers $p : (Y, y_0) \to (X, x_0) \leftrightarrow$ Subgroups $H \subset G$;
  fibre $\leftrightarrow H \backslash G$
- Covers $p : Y \to X \leftrightarrow$ Conjugacy classes of subgroups of G
- Deck transformations $\leftrightarrow N_G(H)/H$; these are transitive iff $H \lhd G$.

Free products $\{G_\alpha\}_{\alpha \in A}$ groups

$X =$





The free product is:

- A group $*_\alpha G_\alpha$ (e.g. $G_1 * G_2$)
- $\forall \alpha \in A$, $i_\alpha : G_\alpha \to *_{\alpha \in A} G_\alpha$

Universal property: Given H,

- $\varphi_\alpha : G_\alpha \to H \quad \forall \alpha \in H$

$\exists!\ \Phi : *_{\alpha \in A} G_\alpha \longrightarrow H$ s.t. $\forall \alpha \in A$

$$\begin{tikzcd} G_\alpha \arrow[r, "\varphi_\alpha"] \arrow[d, "i_\alpha"'] & H \\ *_\alpha G_\alpha \arrow[ur, dashed, "\Phi"'] & \end{tikzcd}$$

commutes

Free Groups and Free Products: Let S be a set

$\langle S \rangle$ - the free group generated by S

<u>Propn:</u> $\langle S \rangle = *_{s\in S} \underset{\mathbb{Z}_s}{\underset{\shortparallel}{\mathbb{Z}}}$

<u>Pf:</u> $*_{s\in S}\mathbb{Z}_s$ is a free group, with $i : S \to *_{s\in S}\mathbb{Z}_s$

$i(s) = i_s(\underset{\mathbb{Z}=\mathbb{Z}_s}{1}) \quad ; \quad i_s : \mathbb{Z}_s \to *_{s\in S}\mathbb{Z}_s$

Given H, $f : S \to H$, define

$\varphi_s : \underset{\mathbb{Z}_s}{\underset{\shortparallel}{\mathbb{Z}}} \to H ; \quad \varphi_s(n) = (f(s))^n$

Observe $\Phi : *_{s\in S}\mathbb{Z} \to H$ satisfies

$$\begin{tikzcd} S \arrow[r, "f"] \arrow[d, "i_s"'] & H \\ *_{s\in S}\mathbb{Z} \arrow[ur, "\Phi"'] & \end{tikzcd}$$

Commutes

Other way, uniqueness are similar $\square$

<u>Rk:</u> Given $\varphi_s : \mathbb{Z}_s \to H$, $f : S \to H$ be

$f(s) = \varphi_s(1)$

Constructing Free Products $\{G_\alpha\}_{\alpha \in A}$

- $W = \{(g_1, \dots, g_n) : \exists \alpha_1, \dots, \alpha_n \text{ s.t. } g_i \in G_{\alpha_i},\ n \geq 0\}$
- $(g_1, \dots, g_n) * (g_1', \dots, g_m') = (g_1, \dots, g_n, g_1', \dots, g_m')$
- $\sim$ on W generated by:
  - if $\alpha_i = \alpha_{i+1}$, $(g_1, g_2, \dots, g_i, g_{i+1}, \dots, g_n) \sim (g_1, \dots, g_{i-1}, g_i g_{i+1}, g_{i+2}, \dots, g_n)$
  - if $g_i = e \in G_{\alpha_i}$, $(g_1, g_2, \dots, \underset{\overset{\shortparallel}{e}}{g_i}, \dots, g_n) \sim (g_1, \dots, g_{i-1}, g_{i+1}, \dots, g_n)$, i.e.

$$(g_1, g_2, \dots, g_{i-1}, e, g_{i+1}, \dots, g_n) \sim (g_1, \dots, g_{i-1}, g_{i+1}, \dots, g_n)$$

- $*$ induces an operation on $W/\sim$
- Let $*_{\alpha \in A} G_\alpha := W/\sim$, $i_\alpha : G_\alpha \longrightarrow *_{\alpha \in A} G_\alpha$

$$i_\alpha(g) = [(g)]$$

Rk: $i_\alpha(g) * i_\alpha(g')$

$$= [(g) * (g')]$$

$$= [(g, g')] = [(gg')] = i_\alpha(\underset{G_\alpha}{\underset{\uparrow}{g * g'}})$$

Group and Universal property

- $e = ()$ is the identity
- $[(g_1, \dots, g_n)]$ has inverse $[(g_n^{-1}, \dots, g_1^{-1})]$ as

$$[(g_1, \dots, g_n)] * [(g_n^{-1}, \dots, g_1^{-1})] = [(g_1, \dots, \overset{\in G_{\alpha_n}}{g_n}, \overset{\in G_{\alpha_n}}{g_n^{-1}}, \dots, g_1^{-1})]$$

$$= [(g_1, \dots, \overset{= e}{g_n g_n^{-1}}, \dots, g_1^{-1})]$$

$$= [(g_1, \dots, g_{n-1}, g_{n-1}^{-1}, \dots, g_1^{-1})]$$ - rest by induction

<u>Universal property</u>: Given H, $\varphi_\alpha : G_\alpha \to H$,

define $\tilde{\Phi} : W \to H$ by $\tilde{\Phi}((g_1, \dots, g_n)) = \varphi_{\alpha_1}(g_1) * \dots * \varphi_{\alpha_n}(g_n)$

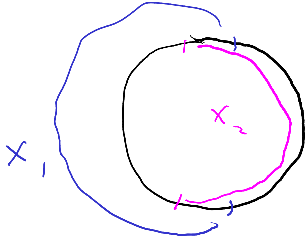
. This induces $\Phi : *_{\alpha \in A} G_\alpha \to H$

and this uniquely satisfies $\forall \alpha \in A$

$$\begin{tikzcd} G_\alpha \arrow[r, "\varphi_\alpha"] \arrow[d, "i_\alpha"'] & H \\ *_{\alpha \in A} G_\alpha \arrow[ur, "\Phi"'] & \end{tikzcd}$$

commutes.

Seifert-Van Kampen Theorem: Surjection part



- $\{X_j\}_{j \in J}$ topological spaces,
- X_j path-connected $\forall j$
- $X_j \cap X_{j'}$ path-connected $\forall j, j'$
- $x_0 \in \bigcap_{j \in J} X_j$; $X_j \subset X$ open, $X = \bigcup_{j \in J} X_j$

Then, $i_{j*} : \pi_1(X_j, x_0) \longrightarrow \pi_1(X, x_0)$

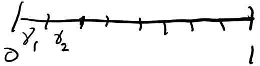
Hence we have $\Phi : *_{j \in J} \pi_1(X_j, x_0) \longrightarrow \pi_1(X, x_0)$

__Theorem:__ Φ is surjective.

Theorem: Φ is surjective.

Proof: Let $[\gamma] \in \pi_1(X, x_0)$

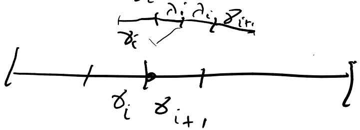
- $\{\gamma^{-1}(X_j) : j \in J\}$ is an open cover of $[0,1]$, hence has a Lesbesgue number.
- Hence $\gamma \sim \gamma_1 * \gamma_2 * \cdots * \gamma_n$ s.t. $\gamma_i : [0,1] \to X_{j_i}$ for some j_i



Claim: $\gamma \sim \gamma_1' * \gamma_2' \cdots * \gamma_n'$ with $\gamma_i' \in \Omega(X_{j_i}, x_0)$ for some i

Pf: We modify by inserting paths;

Suppose $j_i \neq j_{i+1}$ (or $j_i = j_{i+1}$), pick a path λ_i from $\gamma_i(1) = \gamma_{i+1}(0)$ to x_0 in $X_{j_i} \cap X_{j_{i+1}}$



Thus, $\gamma \sim \underbrace{(\gamma_1 * \lambda_1)}_{\gamma_1'} * \underbrace{\bar\lambda_1 * \gamma_2 * \lambda_2}_{\gamma_2'} \cdots * \underbrace{\bar\lambda_{n-1} * \gamma_n}_{\gamma_n'}$, $\gamma_i' := \bar\lambda_{i-1} * \gamma_i * \lambda_i$

Claim $\Rightarrow$ $[\gamma] = \underset{\pi_1(X_{j_1}, x_0)}{[\gamma_1']} * [\gamma_2'] * \underset{i_{j_i *}(\pi_1(X_{j_i}, x_0))}{[\gamma_i']} * [\gamma_n']$

$\Phi(*_{j \in J} \pi_1(X_j, x_0))$

Categorical Products and Coproducts (Direct Sum) $\{X_\alpha\}_{\alpha \in A}$ objects in $\mathcal{C}$

<u>Product:</u>

- Object $\prod_{\alpha \in A} X_\alpha$
- Morphisms $p : \prod_{\alpha \in A} X_\alpha$

s.t. Given Y object, morphisms $\varphi_\alpha : Y \longrightarrow X_\alpha$

$\exists!\ \Phi : Y \longrightarrow \prod_{\alpha \in A} X_\alpha$ s.t.

$$\begin{tikzcd} Y \arrow[r, "\Phi"] \arrow[dr, "\varphi_\alpha"'] & \prod_{\alpha \in A} X_\alpha \arrow[d, "p_\alpha"] \\ & X_\alpha \end{tikzcd}$$

Co-product: $\{X_\alpha\}_{\alpha \in A}$

- Object $\bigoplus_{\alpha \in A} X_\alpha$ or $\coprod_{\alpha \in A} X_\alpha$
- $i_\alpha : X_\alpha \longrightarrow \coprod_{\alpha \in A} X_\alpha$

s.t. Given · object Y

· morphisms $\varphi_\alpha : X_\alpha \longrightarrow Y$

$\exists !$ morphism $\Phi : \coprod_{\alpha \in A} X$ s.t. $\forall \alpha \in A$

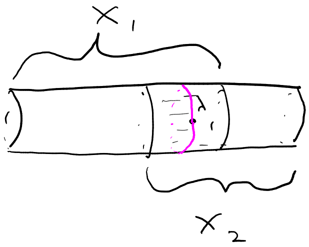
$$\begin{tikzcd}
X_\alpha \arrow[rr, "\varphi_\alpha"] \arrow[dr, "i_\alpha"'] & & Y \arrow[ll, bend left, "q_\alpha"] \arrow[dl, bend left, "\Psi"] \\
& \coprod_{\alpha \in A} X \arrow[ur, "\Phi"'] \arrow[ul, bend left, "p_\alpha"] &
\end{tikzcd}$$

$\prod_{\alpha \in A}$

Rk: Products & co-products are unique if they exist.

Exercise: Coproduct of based topological spaces.

Siefert-Van Kampen Theorem: Relations



$\{X_\alpha\}_{\alpha\in A}$ - subspaces of X, open

$x_0 \in \bigcap_{\alpha\in A} X_\alpha$, $\bigcup_{\alpha\in A} X_\alpha = X$

- $\forall \alpha, \beta$, $X_\alpha \cap X_\beta$ path-connected
- $\forall \alpha$ X_α path-connected

$\Phi : *_{\alpha\in A} \pi_1(X_\alpha, x_0) \longrightarrow \pi_1(X, x_0)$ is onto

$\hat{i}_{\alpha *} : \pi_1(X_\alpha, x_0) \longrightarrow *_{\beta\in A} \pi_1(X_\beta, x_0)$

$j_{\alpha\beta} : X_\alpha \cap X_\beta \longrightarrow X_\alpha$ inclusion.

$N = \langle\langle R \rangle\rangle$ the normal subgroup generated by R.

$$R = \left\{ \left(\hat{i}_\alpha\left(j_{\alpha\beta *}([\lambda])\right)\right)\left(\hat{i}_\beta\left(j_{\beta\alpha *}([\lambda])\right)\right)^{-1} : \lambda \in \underset{\alpha,\beta\in A}{\Omega}(X_\alpha \cap X_\beta, x_0)\right\}$$

$\in *_{\alpha\in A} \pi_1(X_\alpha, x_0)$

$[\lambda][\bar{\lambda}] = e \in \pi_1(X, x_0)$

Theorem: $\mathrm{Ker}(\bar{\Phi}) = N$, i.e. $\bar{\Phi}$ induces

$$\underbrace{\overbrace{\ast_{\alpha\in A}\pi_1(X_\alpha, x_0)}^{=\mathbb{F}}}_{\mathbb{F}} \Big/ N \xrightarrow{\ \cong\ } \pi_1(X, x_0).$$

provided $\forall \alpha, \beta, \gamma \in A$, $X_\alpha \cap X_\beta \cap X_\gamma$ is path-connected

Propn: $\mathrm{Ker}(\bar{\Phi}) \supset N$

Pf: Enough to show $\mathrm{ker}(\bar{\Phi}) \supset R$

Factorization: Elements in $\mathbb{F}$ are represented by factorizations

$$g = [\gamma_1] * [\gamma_2] \dots * [\gamma_n], \quad \exists \alpha_1, \dots, \alpha_n \quad \text{s.t. } \gamma_i \in \pi_1(X_{\alpha_i}, x_0)$$

Equivalence relation: (generated by)

- If $\alpha_i = \alpha_{i+1}$, then
$$[\gamma_1] * [\gamma_2] * \dots * [\gamma_i] * [\gamma_{i+1}] * \dots * [\gamma_n] \sim [\gamma_1] * \dots * [\gamma_i \gamma_{i+1}] * \dots * [\gamma_n]$$
- If $\gamma_i \in \Omega(X_{\alpha_i} \cap X_\beta, x_0)$, then regarding γ_i as a loop in $\pi_1(X_\beta, x_0)$ gives an equivalent factorization.

Propn: If factorizations are equivalent, then they give the same element in $\mathbb{F}/N$.

Proof of Seifert Van-Kampen theorem

$\{X_\alpha\}_{\alpha \in A}$, $x_0 \in \bigcap_{\alpha \in A} X_\alpha$

$$\Phi : \underbrace{*_{\alpha \in A} \pi_1(X_\alpha, x_0)}_{= \mathbb{F}} \longrightarrow \pi_1(X, x_0) \quad \text{- surjective}$$

$N \subset \Phi$ normal subgroup generated by $g_\lambda = \underline{\hat{i}_\alpha}(j_{\alpha\beta *}([\lambda])) \cdot (\hat{i}_\beta(j_{\beta\alpha *}([\lambda])))^{-1}$,

$\lambda \in \Omega(X_\alpha \cap X_\beta, x_0)$

$$\Phi(g) = [\lambda] \cdot [\lambda]^{-1} = e$$

Hence $N \subset \ker(\Phi)$; so we have

$$\bar{\Phi} : *_{\alpha \in A} \pi_1(X_\alpha, x_0) / N \longrightarrow \pi_1(X, x_0)$$

<u>Factorisations.</u>

$g = [\gamma_1] * [\gamma_2] * \dots * [\gamma_n]$, $\gamma_j \in \Omega(X_{\alpha_j}, x_0)$

gives element in $\mathbb{F}$, hence $\mathbb{F}/N$, and $\varphi(g) \in \pi_1(X, x_0)$

$g' = [\gamma_1'] * \dots * [\gamma_n']$, $\varphi(g') = \varphi(g)$, we <u>show</u> $g' \sim g$

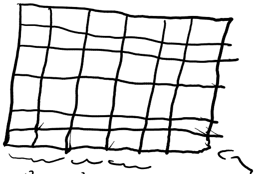
hence have the same image in $\mathbb{F}/N$

Homotopic factorizations are equivalent

$\gamma = [\gamma_1] * \dots * [\gamma_n]$

$\gamma' = [\gamma_1'] * \dots * [\gamma_n']$

Suppose $\gamma \sim \gamma'$, then $\exists\, H : [0,1] \to [0,1] \longrightarrow X$ homotopy; use Lebesgue etc.

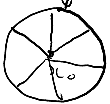


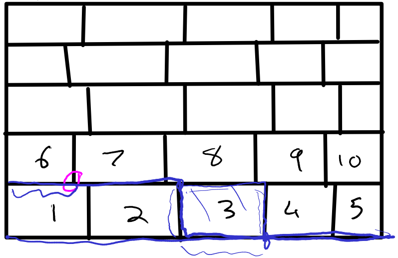
$\leftarrow$ equivalent to $[\gamma_1'] * \dots * [\gamma_n']$

$\leftarrow$ equivalent to $[\gamma_1] * \dots * [\gamma_n]$

$\gamma_1 \quad \gamma_2 \quad \gamma_3$

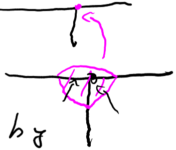
constant in $X_\alpha \cap X_\beta \cap X_\gamma$



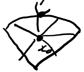


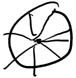
Can ensure vertices map to x_0.

- First make H constant near the vertex
- Radially join the boundary to x_0 in the intersection of the regions and replace the interiors



$y \in X_\alpha \cap X_\beta \cap X_\gamma$

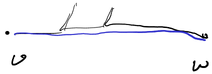




Proof: Transform factorizations by shifting across bricks.

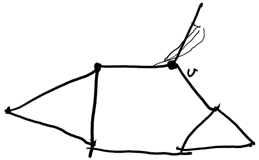
Graphs and Paths: $\Gamma = \Gamma(E, V)$

- An edge path is a sequence $e_1, \dots, e_n$; $e_j \in E$ s.t. $\forall i$, $\tau(e_i) = i(e_{i+1})$
  'from': $i(e_1)$
  'to': $\tau(e_n)$

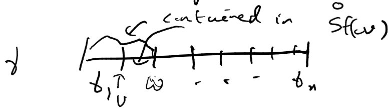


- Edge path is reduced if $e_i \neq \bar{e}_{i+1}$ $\forall i$
- These have geometric realizations $e_1 * e_2 * \dots * e_n$

Propn: Any path $\gamma : [0,1] \to |\Gamma|$ s.t. $\gamma(0) = v \in V$, $\gamma(1) = w \in V$ is homotopic to a reduced edge path



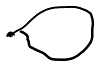
Idea of pf: Use Lebesgue number w.r.t. $\mathring{St}(v)$



- Modify s.t. segments connect vertices
- Observe that if stars intersect, then there is an edge between these vertices.

Cor: $|\Gamma|$ is connected iff $\forall v, w \in V(\Gamma)$, $v \neq w$, $\exists$ reduced edge path from v to w.

- An edge loop is an edge path from v to v, $v \in V(\Gamma)$



- **Defn:** A Tree T is a graph with $V \neq \emptyset$ s.t. $\forall v, w \in V(T)$, $v \neq w$, $\exists!$ reduced edge path from v to w

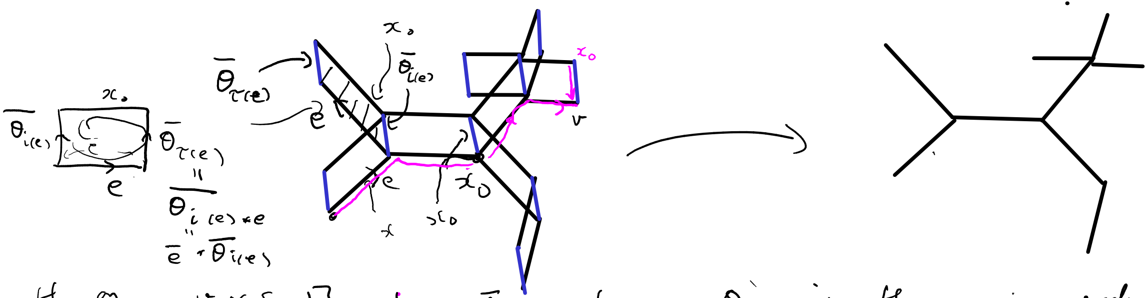
Propn: T is a tree $\iff$ T is connected and has no reduced edge loops.

Trees are contractible

T tree, $x_0 \in V(T)$

$$H : \tilde{T} \times \widetilde{[0,1]} \longrightarrow T$$

$$H(x,0) = x, \quad H(x,1) = x_0, \quad H(x_0,t) = x_0 \quad \forall x \ \forall t$$



- Define H on $v \times [0,1]$ by $\bar{\theta}_v$ where θ_v is the unique reduced edge path to v from x_0
- Extends over $e \times [0,1]$ $\forall e$:

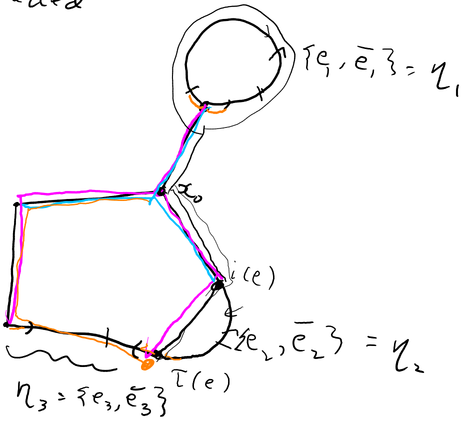
- Consider reduced edge path $e_1, \dots, e_n$ to $i(e)$

<u>Case 1</u>: $e_n = \bar{e}$; $\theta_{i(e)} = \theta_{\tau(e)} * \bar{e}$

<u>Case 2</u>: $e_n \neq \bar{e}$; $\theta_{\tau(e)} = \theta_{i(e)} * e$

Fundamental groups of Graphs

$\Gamma = \Gamma(E, V)$, $x_0 \in V$, Γ connected



Propn: (a) Γ has a maximal tree T

(b) $V(T) = V(\Gamma)$

$\square$

- Unoriented edge η is a pair $\{e, \bar{e}\}$
- Let S be the unoriented edges not in T

Theorem: $\pi_1(|\Gamma|, x_0) = \langle S \rangle$ = free group generated by S

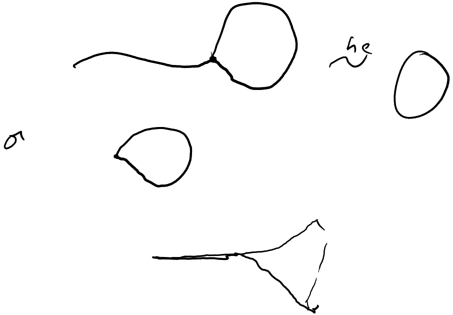
Pf: By Van Kampen theorem with $X_0 = N(T) = {}^{\backprime}T{}^{\prime}$

- For $\eta = \{e, \bar{e}\} \in S$, $X_e = N(\theta_{i(e)} \cup \theta_{\tau(e)} \cup e)$
- $X = X_0 \cup \bigcup_{\{e,\bar{e}\} \in S} X_e$

$\pi_1(X_0, x_0) = \{e\}$

$\pi_1(X_e, x_0) = \mathbb{Z}$

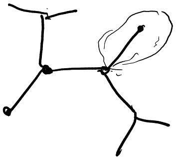
$X_e =$



Hence

$$\pi_1(|\Gamma|, x_0) = \{e\} * \mathop{\ast}_{s \in S} \mathbb{Z} = \langle S \rangle$$

<u>Rk:</u> $\{e\} * G = G$ $\square$



<u>Addendum:</u> For a finite tree $|V| - |E| = 1$, $\underbrace{|\hat{E}| - |V| + 1 = 0}$

Hence if $|V|, |E| < \infty$,

(rank of the free group) $= |S| = (\underbrace{|\hat{E}|}_{\text{unoriented edges}} - |V|) + 1$

$F_n \cong F_m \Rightarrow \mathbb{Z}^n \cong \mathbb{Z}^m$ $\otimes \mathbb{Q}$

$\mathbb{Q}^n \cong \mathbb{Q}^m$

Covers of graphs and Fenchel-Nielsen Theorem

<u>Theorem:</u> A cover $p : \hat{X} \to X = |\Gamma|$ of a graph is induced by a graph covering $\varphi : \hat{\Gamma} \to \Gamma$, $\hat{X} = |\hat{\Gamma}|$, $p = |\varphi|$

<u>Pf:</u> Define $\hat{V} = p^{-1}(V)$ (discrete)

$\hat{E}$ = lifts of e, $e \in E$

$= \{ \hat{e} : e$ lift of $e \in E \}$

$\hat{X}$

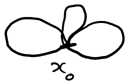
$i(\hat{e}) = \hat{e}(0) \in \hat{V}$

<u>Cor:</u> The fundamental group of $\hat{X}$ is free

$|\Gamma|$

<u>Theorem (Fenchel - Nielsen)</u>: Any subgroup of a free group is free.

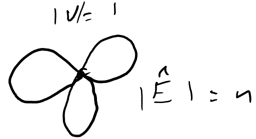
<u>Pf:</u> If $F = \langle S \rangle$, $F = \pi_1(\bigvee_{s \in S} S^1)$



· Given $G \subset F$, $\exists$ cover with $\pi_1(\hat{X}) = G$

Γ, $V = \{x_0\}$, $E = S \cup \bar{S}$

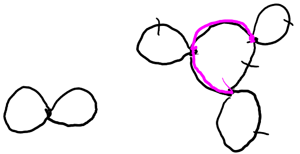
Addendum: If $|S| = n$, $\langle S \rangle = F_n$,



$G \subset F$ has index k

Then $G = F_m$, where $m = nk - k + 1 = n(k-1)+1$

Rk: We have $F_2 \hookrightarrow F_3$ and $F_3 \hookrightarrow F_2$



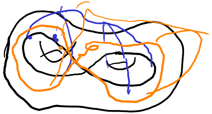
A 'converse' holds

Theorem (Stallings): Any group G that is torsion-free and contains a free subgroup F of finite index is free

F_n, k index

Attaching cells

$k = 0, 1, 2, \ldots$

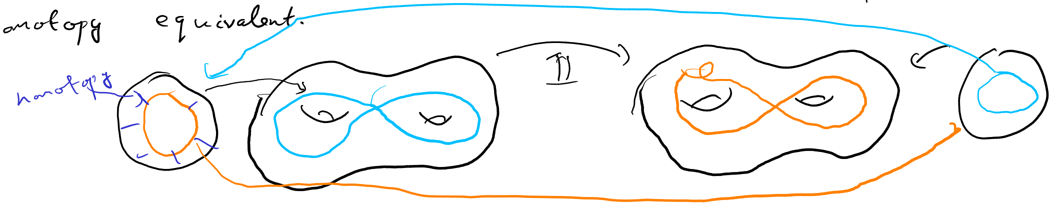


- X space
- $\varphi : S^{k-1} \longrightarrow X$

The result of attaching a k-cell along φ is

$$X_\varphi := \left(X \amalg D^k \right)/_\sim \quad , \quad \theta \in \partial D^k = S^{k-1} \sim \varphi(\theta)$$

<u>Theorem:</u> $\varphi, \psi : S^{k-1} \to X$ and $\varphi \sim \psi$, then X_φ and X_ψ are homotopy equivalent.



<u>pf:</u> Construct $f : X_\varphi \to X_\psi$ and $g : X_\psi \to X_\varphi$ & homotopies

- Given $\{\varphi_\alpha : S^{k-1} \to X\}_{\alpha \in A}$, we can attach corresponding k-cells.

- If $k=0$, $S^{k-1} = \emptyset$

- We can attach 0-cells corresponding to a set A; we get A with discrete topology

- If we attach 1-cells to a discrete spaces, we get a graph.

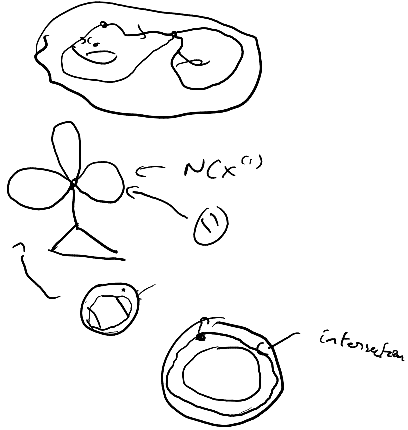
Two complexes: The result of attaching 2-cells to a graph X result of attaching 2-cells

Given by , Γ, $\pi_1(X^{(1)}, x_0)$ $\in |\Gamma|$ is a free group.

· $\varphi_\alpha : S^1 \longrightarrow \pi_1(X^{(1)}, x_0)$, $[\varphi_\alpha] \in \pi_1(X^{(1)}, \varphi_\alpha(1))$

well-defined conjugacy-class in $\pi_1(X^{(1)}, x_0)$

· Let r_α be representatives of these



Theorem: $\pi_1(X, x_0) = \pi_1(X, x_0) / \langle\langle \{r_\alpha\}_{\alpha \in A} \rangle\rangle$

Pf: By Van Kampen, using $N(X^{(1)})$, interior of 2-cells

$$\pi_1(X^{(1)}, x_0) * \left(*_{\alpha \in A} \{e\} \right) / \langle\langle r_\alpha \cdot e^{-1} \rangle\rangle$$

$\pi_1(X^{(1)}, x_0) = \langle S \rangle$; $R \subset \langle S \rangle$, words in $S \cup \bar{S}$

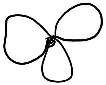
<u>Defn:</u> $\langle S \rangle / \langle\langle R \rangle\rangle$ is the group with <u>presentation</u> $\langle S \mid R \rangle$ " $\langle S ; R \rangle$

<u>E.g.</u> $\mathbb{Z}^2 = \langle \alpha, \beta ; \alpha\beta\bar{\alpha}\bar{\beta} \rangle$

<u>Propn:</u> Every group G has a presentation.

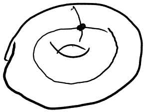
<u>Theorem:</u> If G is a group, there exists (X, x_0) 2-complex s.t.

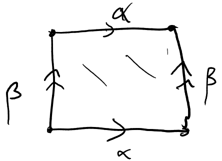
$$G = \pi_1(X, x_0)$$

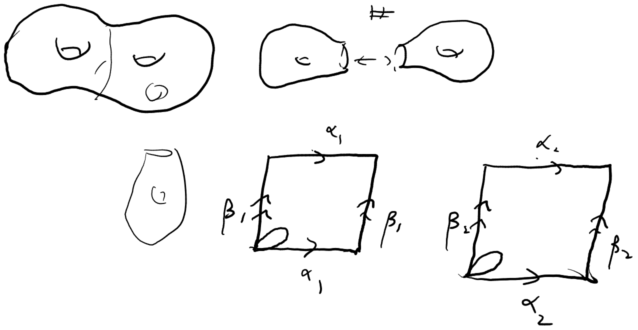
<u>Pf:</u> Let $G = \langle S ; R \rangle$, take  $\begin{cases} \text{- one 0-cell} \\ \text{- one edge (1-cell) for each } s \in S \end{cases}$

- For each $r \in R$, attach a 2-cell.

Surfaces





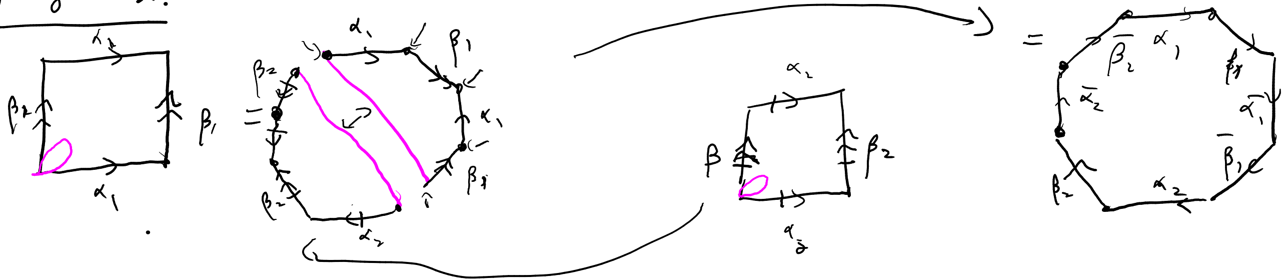


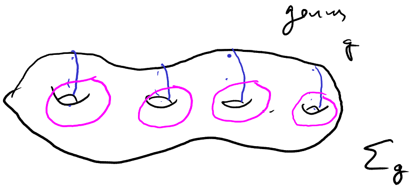
2-complex:
- 1 0-cell
- 2 1-cells
- 1 2-cell

$$\pi_1(T^2, \{x_0\}) = \langle \alpha, \beta ; \alpha\beta\bar{\alpha}\bar{\beta} \rangle = \langle \alpha, \beta ; \alpha\beta\bar{\alpha}\bar{\beta} = 1 \rangle$$

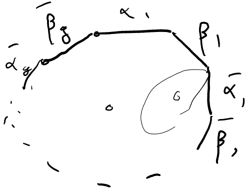
$$= \langle \alpha, \beta ; \alpha\beta = \beta\alpha \rangle \cong \mathbb{Z}^2$$

Surface of genus 2:





2-complex:



$$[x,y] = x\,y\,\bar{x}\,\bar{y} = x\,y\,x^{-1}y^{-1}$$

Thus $\quad \pi_1(\Sigma_g, x_0) = \Big\langle \alpha_1, \beta_1, \dots, \alpha_g, \beta_g \;;\; \prod_{i=1}^{g} \underbrace{[\alpha_g, \beta_g]}_{=e \text{ in } G/[G,G]} \Big\rangle$

Theorem: $\Sigma_g = \Sigma_{g'} \Rightarrow g = g'$

Pf: G group, $[G,G] = \langle\langle ghg^{-1}h^{-1} : g,h \in G \rangle\rangle$, $G/\langle\langle G,G\rangle\rangle$ abelian

$$\pi_1(\Sigma_g, x_0) \Big/ [\pi_1(\Sigma_g, x_0), \pi_1(\Sigma_g, x_0)] = F_{2g}\Big/[F_{2g,2g}] = \mathbb{Z}^{2g}$$

CW complexes Cell - Weak

* (-1)-dimensional: $\emptyset$
* inductively, n-dimensional: ; $\pi_0(X) = \emptyset \Longleftrightarrow X = \emptyset$

$X^{(n-1)}$ – $(n-1)$-dimensional CW-complex

- An n-dimensional CW-complex $X^{(n)}$ is a space obtained by attaching a collection of n-cells to $X^{(n-1)}$, for some $X^{(n-1)}$

- Attaching: determined by $\{\varphi_\alpha^n\}_{\alpha \in \underline{\underline{A_n}}}$; $\varphi_\alpha : S^{n-1} \longrightarrow X^{(n-1)}$

- Let $X^{(0)} \hookrightarrow X^{(1)} \hookrightarrow X^{(2)} \hookrightarrow \cdots X^{(n)} \hookrightarrow \cdots$ be CW-complexes, $X^{(n)}$ is obtained from $X^{(n-1)}$ as above.

Then $X = \varinjlim X^{(n)}$ is a CW-complex

- $X^{(k)}$ is the k-skeleton.

Set: $\varphi^n_\alpha : S^{n-1} \longrightarrow X, \quad \alpha \in A_n$

Characteristic maps:

Recall

$$X^{(n)} = \left(X^{(n-1)} \coprod \left(\coprod_{\alpha \in A_n} \underset{\underset{D^n}{\parallel}}{D^n_\alpha} \right) \right) \Big/ \sim$$

- We have

$$\Phi_\alpha : \underset{\underset{D^n_\alpha}{\parallel}}{\mathbb{D}^n} \longrightarrow X \quad \text{characteristic maps}$$

I. $\Phi_\alpha |_{\mathring{D}}$ is a homeomorphism

II. $X = \coprod_{\substack{n \geq 0 \\ \alpha \in A_n}} \Phi_\alpha(\mathring{D}^n)$ as a set

Topology: Cell weak

- $V \subset X$ is open $\iff$ $\forall n \geq 0, \alpha \in A_n$, $\Phi_\alpha^{-1}(V)$ is open

III. $f : X \to Y$ is continuous $\iff$

$$f \circ \Phi_\alpha : \mathbb{D}^n \longrightarrow Y$$

is continuous, $\forall n \geq 0$, $\alpha \in A_n$.

Alternative defn: Space X, together with $n \geq 0$, A_n, $\Phi_\alpha : D^n \to X$ s.t. I, II, III &

$$\alpha \in A_n, \quad \Phi_\alpha(\partial D^n) \subset \bigcup_{\substack{k < n \\ \beta \in A_k}} \Phi_\beta(\mathring{D}^k)$$

Simplifying maps

X CW complex

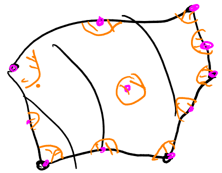
- K compact, $f : K \to X$

<u>Theorem</u>: $\exists\, g : K \to X$, $f \sim g$ s.t. $\#\{\alpha \in A_n : n \geq 0,\ g(K) \cap \Phi_\alpha(\mathring{D}^n)\} < \infty$

<u>Pf</u>: . $f(K)$ is compact

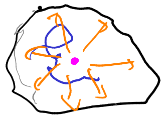
- Let $\mu_\alpha = \Phi_\alpha(\underset{\substack{\cap \\ D^n}}{0})$, $n \geq 0$, $\alpha \in A_n$ - midpoints

- $M = \{\mu_\alpha\}_{\alpha \in \cup A_n}$ is a <u>discrete set</u>

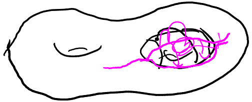


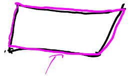
- Hence $f(K) \cap M$ is finite

- If $\mu_\alpha \notin f(K)$, we can homotop f to be disjoint from $\Phi_\alpha(\mathring{D}^n)$



Pushing maps off discs.



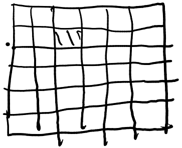


- Y is obtained from X by attaching n-discs along $\{\varphi_\alpha\}_{\alpha \in A}$, $\varphi_\alpha : S^{n-1} \to X$
- Let $k < n$, $f : D^k \to Y$ s.t. $f(\partial D^k) \subset X$

Theorem: $f \simeq g$ rel ∂D^k with $g(D^k) \subset X$

Pf: Idea:
- Homotop f to a map that is linear 'near $0 \in D^n$'.
- Then 'push off'

- $D^k = [0,1]^k$

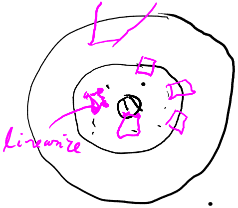


Subdivide, ensure by Lesbesgue number theorem that for any cubelet C, $f(C)$ is
- disjoint from the mantle
- is fully in D^n $(= \Phi_\alpha(D^n))$

and has diameter $< \varepsilon$ (to be specified)

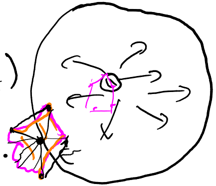
Modify f: No change on cubelets c with $f(c)$ disjoint from mantle

- If $f(c) \subset$ 'mantle', replace $f|_c$ by a linear map

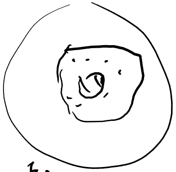


- If $f(c) \cap$ 'mantle' $\neq \emptyset$ but $f(c) \not\subset$ 'mantle', extend by 'convexity': these are disjoint from core.

- New map $g \sim f$. (convex homotopy)



Consequence: If $g(c) \cap \text{core} \neq \emptyset$, $g|_c$ is linear.



- Hence $g([0,1]^k) \cap \underset{\subset B^n}{\text{core}}$ is contained in the image of k-dimensional spaces,

So $B^n \setminus g([0,1]^k) \neq \emptyset$, w.l.g. $\mu \notin g([0,1]^k)$

- <u>Radial homotopy</u>

(Higher) homotopy groups $\pi_n(X, x_0)$

On $\{ f : [0,1]^n \longrightarrow X : f(\partial[0,1]^n) = \{x_0\}\}$, we have $*$ ($\sim +$)

$$f * g(x_1, x_2, \dots, x_n) = \begin{cases} f(2x_1, x_2, \dots, x_n), & 0 \le x_1 \le 1/2 \\ g(2x_1 - 1, \dots, x_n), & 1/2 \le x_1 \le 1 \end{cases}$$

- Homotopy rel ∂, $f \sim g$ if $\exists\, H : [0,1]^{n+1} \longrightarrow X$ s.t.

$$H(x_1, \dots, x_n, 0) = f(x_1, \dots, x_n) \quad \forall \dots$$
$$H(x_1, \dots, x_n, 1) = g$$
$$H(\partial[0,1]^{n+1}) = \{x_0\}$$

Theorem: $\pi_n(X, x_0)$ is a group.

Defn: $\pi_n(X, x_0)$ is maps $f : ([0,1]^n, \partial[0,1]^n) \longrightarrow (X, \{x_0\})$ up to relative homotopy with $*$ as an operation.

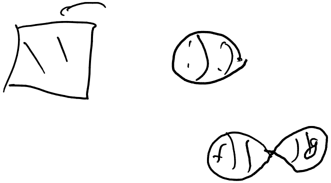
Theorem: (a) $\pi_n(X, x_0)$ with $*$ is a group.

(b) $f : (X, x_0) \longrightarrow (Y, y_0)$ induces $f_* : \pi_n(X, x_0) \longrightarrow \pi_n(Y, y_0)$

given by $\left(\Theta : ([0,1]^n, \partial[0,1]^n) \longrightarrow (X, x_0) \right) \longmapsto f \circ \Theta : ([0,1]^n, \partial[0,1]^n) \longrightarrow (Y, y_0)$

(c) $f \sim g$ rel basepoint, $f_* = g_*$

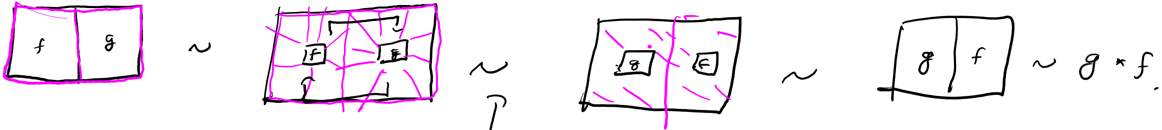
Cor: If $(X, x_0) \overset{h.e.}{\sim} (Y, y_0)$, then $\pi_n(X, x_0) \cong \pi_n(Y, y_0)$ $\forall n \geq 1$



Higher homotopy groups are Abelian

$n \geq 2$

Theorem: If $n \geq 2$, $f * g \sim g * f$, $f, g : ([0,1]^n, \partial [0,1]^n) \to (X, x_0)$

Pf: $f * g =$  $\sim g * f$.

pre-compose with a family of homeomorphisms rel ∂ of $[0,1]^n$

Higher homotopy groups and covering spaces

Let $p:(Y, y_0) \longrightarrow (X, x_0)$ be a covering map; $n \geq 2$

Theorem: $p_*: \pi_n(Y, y_0) \longrightarrow \pi_n(X, x_0)$ is an isomorphism.

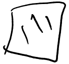
Pf: We construct an inverse $\Phi: \pi_n(X, x_0) \to \pi_n(Y, y_0)$ as follows:

Given

$$\theta: ([0,1]^n, \partial[0,1]^n) \longrightarrow (X, x_0)$$

By map lifting, we have $\tilde{\theta}: [0,1]^n \longrightarrow Y$, $\tilde{\theta}((0,\dots,0)) = y_0$

$\tilde{\theta}(\underbrace{\partial[0,1]^n}_{\text{connected as } n \geq 2}) \subset p^{-1}(x_0)$ which is discrete.

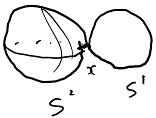


$\therefore$ $\tilde{\theta}|_{\partial[0,1]^n}$ is constant, $\therefore$ $\tilde{\theta}: ([0,1]^n, \partial[0,1]^n) \longrightarrow (Y, y_0)$

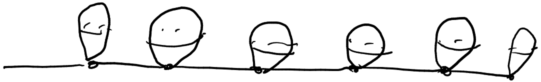
Define $\Phi([\theta]) := [\tilde{\theta}] \in \pi_n(Y, y_0)$

$n \geq 2$

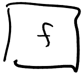
. $\pi_n(X, x_0)$ is a $\mathbb{Z}[\pi_1(X, x_0)]$-module.

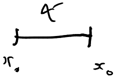
E.g. $X =$ 

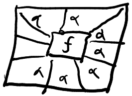
$= S^2 \vee S^1$

$\pi_2(X, x_0) \cong \pi_2(\tilde{X}, \tilde{x}_0)$ $\longrightarrow$ 

$\longrightarrow$ deck transformation.

Action: Given 

, 

, $\alpha * f =$ 

Homotopy Groups and Extending maps

$f : S^n \longrightarrow X$, does this extend to $F : D^{n+1} \to X$

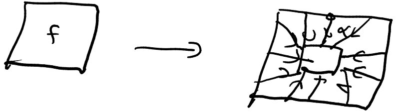
- $n \geq 2$, X path-connected, $x_0 \in X$

Theorem: Every $f : S^n \to X$ extends to $F : D^n \to X$ iff $\pi_n(X, x_0)$ is trivial.

Pf: Propn: If X is path-connected, $x_1, x_0 \in X$, then $\pi_n(X, x_0) \cong \pi_n(X, x_1)$

Pf: Given a path α from x_0 to x_1, define

$$\pi_n(X, x_1) \longrightarrow \pi_n(X, x_0)$$
$$f \mapsto \alpha * f$$



- Observe this is a group action;

$$\bar{\alpha} * \alpha * f \sim e * f \sim f$$

Thus this is an isomorphism.

(1) Suppose every $f: S^n \to X$ extends,

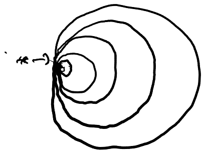
let $\theta : ([0,1]^n, \partial([0,1]^n)) \to (X, x_0)$

get $\bar{\theta} : (S^n, \{x\}) \to (X, x_0)$ by $S^n = [0,1]^n / (\partial[0,1])^n$

Extend this to $H : D^{n+1} \to X$

· We define a homotopy from θ to e.

· family of maps $S^n \to X$, in fact $(S^n, \{x\}) \to (X, \{x\})$

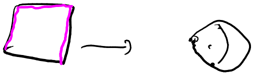


· compose with quotient $I^n / \partial I^n \to S^n$

Suppose $\pi_n(X, x_0) = 0 \Rightarrow \forall x_1 \in X, \ \pi_n(X, x_0) = 0$

Given $f: S^n \to X$, say

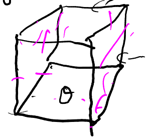
$$\begin{tikzcd} f: (S^n, \{*\}) \arrow[r] & (X, x_0) \\ {[0,1]^n / \partial([0,1]^n)} \arrow[u, "\cong"', "q"] & \end{tikzcd}$$



Define $\theta : ([0,1]^n, \partial[0,1]^n) \to (X, x_0)$

By hypothesis, we have $H : ([0,1]^{n+1}, \cancel{\partial[0,1]^{n+1}}) \to (X, x_0)$

Quotient by all faces except the base to get



$$F : D^{n+1} \to X$$

extending $f : S^n \to X$

Product: X, Y CW complexes, cells indexed by $\{A_n\}_{n\geq 0}$ & $\{B_m\}_{m\geq 0}$

Theorem: $X\times Y$ is a CW-complex with n-cells corresponding to $\displaystyle\bigcup_{\substack{k+l=n\\ k,l\geq 0}} A_k\times B_m$

Pf: . $D^k\times D^l \underset{\text{homeo}}{\cong} D^{k+l}$

. We define a cell-structure using the attaching maps:

$\alpha\in A_k$, $\beta\in B_l$, $k+l=n$

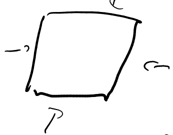
$$\varphi_{(\alpha,\beta)}: \partial(D^k\times D^l) \longrightarrow (X\times Y)^{(n-1)}$$

$$\parallel$$

$$(\partial D^k\times D^l)\cup(D^k\times\partial D^l)$$

$$\varphi_{(\alpha,\beta)}(\underset{\partial D^k}{x}, \underset{D^l}{y}) = (\underset{X^{(k-1)}}{\varphi_\alpha(x)}, \underset{Y^{(l)}}{y}) \in (X\times Y)^{(n-1)}$$

$$\varphi_{(\alpha,\beta)}(\underset{D^k}{x}, \underset{\partial D^l}{y}) = (x, \varphi_\beta(y)) \in (X\times Y)^{(n-1)}$$



Verify: If $(x,y)\in\partial D^k\times\partial D^l$

$$\varphi_{(\alpha,\beta)}(x,y) = (\varphi_\alpha(x), \underset{\in\partial D^l}{y})$$

$$\underset{\text{in quotient}}{=} (\varphi_\alpha(x), \varphi_\beta(y))$$

$$\parallel$$

$$(x, \varphi_\beta(y))$$

<u>Subcomplex</u>: $A \subset X$, · consists of cells in X

· if a cell corresponding to $\alpha \in A_n$ is in A, then every cell intersecting $\varphi_\alpha(\partial D^n)$ is in A.

<u>Extension problem</u>: $f: A \to Y$, want to extend to $F: X \to Y$

· Enough to define on cells not in A, inductively on dimension,

i.e. we extend to $\underset{A \cup X^{(-1)}}{A}$, $A \cup X^{(0)}$, $A \cup X^{(1)}$, $A \cup X^{(2)}$, ...

· $A \cup X^{(n)}$ is obtained from $A \cup X^{(n-1)}$ by attaching n-cells along $\{\varphi_\alpha\}_{\alpha \in B_n}$

<u>Theorem</u>: If $\pi_n(Y) = \pi_n(\{*\})$ for $n \geq 0$, then any map $f: A \to Y$ extends
(n=0 path-connected)
to $F: X \to Y$

<u>Pf</u>: $A \cup X^{(n)} = \left(\underbrace{(A \cup X^{(n-1)})}_{\text{assume defined}} \coprod \{D_\alpha^n\}_{\alpha \in B_n} \right) / \sim$; $\begin{cases} \text{· on } \partial D_\alpha^n,\ F_{|\partial D_\alpha^n}(x) = F_{|A \cup X^{(n-1)}}(\varphi_\alpha(x)) \\ \text{define } F_{|D_\alpha^n} \text{ by extending map from } S^{n-1} \text{ to } D^n. \end{cases}$

Theorem: X CW complex, $\pi_n(X) = \pi_n(\{*\})$ $\forall n \geq 0$, then X is contractible. (and conversely). $\left\{ \text{Rk: } \pi_n(\{*\}) = 0, \ n \geq 2. \right.$

Pf: Define $H : X \times [0,1] \to X$, extending

- pick $x_0 \in X^{(0)}$,
- $A = X \times \{0,1\} \cup x_0 \times [0,1]$.
- $H_{|A}(x,0) = x$

  $H_{|A}(x,1) = x_0$

  $H_{|A}(x_0,t) = x_0$
- This extends to give contraction H.

Cor: X connected CW-complex, $\pi_n(X, x_0) = 0$ $\forall n \geq 2$, then the universal cover $\tilde{X}$ is contractible (and conversely).

Eilenberg-MacLane Space: X CW-complex, path-connected, $\pi_n(X,x_0)=0 \quad \forall n\geq 2$

Equivalently: $\tilde{X}$, universal cover of X, is contractible.

Eg: S^1, $T^2 = S^1\times S^1$

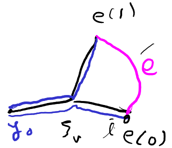
Theorem: (Y,y_0) path-connected CW complex, X Eilenberg-MacLane space $K(G,1)$, $G = \pi_1(X,x_0)$

Given $\varphi : \pi_1(Y,y_0) \to \pi_1(X,x_0)$, $\exists\, f:(Y,y_0)\to(X,x_0)$ s.t. $\varphi = f_*$

Pf: We inductively define on $Y^{(1)}$ (graph), $Y^{(2)}$, ...

- Let T be a maximal tree in $Y^{(1)}$,

  define $f_{|T}: T\to X$ by $f_{|T}(y) = x_0$



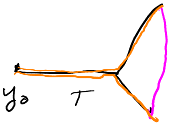
- Any edge (1-cell) e not in T gives an element $\theta_e \in \pi_1(Y,y_0)$

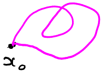
  $\theta_e = [\xi_{e(0)} * e * \overline{\xi_{e(1)}}]$

- Then $\varphi(e) \in \pi_1(X,x_0) = [\lambda]$ for a loop $\lambda:[0,1]\to(X,x_0)$
- Define f on $D^1=[0,1]$ by $f(s) = \lambda(s)$; do this for each $e\notin T$.

Extending to 2-cells: $\mu_e = \xi_{e(0)} * e * \overline{\xi_{e(1)}}$

$$f_*([\mu_e]) = [f\circ\mu_e] = [f\circ e] = \varphi([\mu_e])$$





As μ_e generate, $f_* = \varphi$.

· In particular, $\lambda \in \Omega(Y, y_0)$ with $\lambda \sim e \Rightarrow f\circ\lambda \sim e \Rightarrow f\circ\lambda$ extends to a disc.

$\Updownarrow$

λ extends to a disc

$\hookrightarrow \lambda : (S^1, 1) \to (Y, y_0)$

By change of basepoint, if $\lambda : S^1 \to Y$ extends to $\Lambda : D^2 \to Y$, so does $f\circ\lambda$.

Define on $Y^{(2)} = \left(\underbrace{Y^{(1)}}_{f \text{ is defined}} \amalg \left(\amalg D^2_\alpha\right)\right)/\sim$ given by $\varphi_\alpha : S^1 \to Y^{(1)}$

· Extension problem: $\forall \alpha \in A_2$, $f\circ\varphi_\alpha : S^1 \to X$ extends to $D^2 \to X$

But $\varphi_\alpha : S^1 \to Y$ extends to a disc,

hence $f\circ\varphi_\alpha$ does. $\square$

- Extend inductively to $Y^{(n)}$, $n \geq 3$
- Take direct limit
- At each stage,

$$Y^{(n)} = \left(Y^{(n-1)} \amalg \left(\coprod_{\alpha \in A_n} D^n_\alpha \right) \right) / \sim$$

with f defined on $Y^{(n-1)}$, hence on ∂D^n_α by $f|_{X^{(n-1)}} \circ \varphi_\alpha$

- We can extend $f|_{\partial D^n_\alpha = S^{n-1}}$ to D^n_α as $\pi_n(X, x) = \emptyset$

Theorem: (X, x_0) be a $K(G,1)$, Y path-connected

$f, g : (Y, y_0) \to (X, x_0)$ s.t. $\underline{f_* = g_*} : \pi_1(Y, y_0) \to \pi_1(X, x_0)$.

Then $f \sim g$ (fixing basepoints)

Pf: We define $H : Y \times [0,1] \to X$ extending

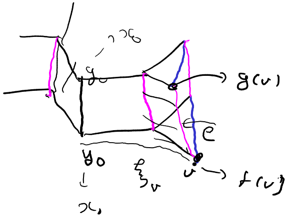
$$H(y,0) = f(y) \quad \forall y \in Y$$
$$H(y,1) = g(y) \quad \forall y \in Y$$
$$H(y_0, s) = x_0 \quad \forall s \in [0,1]$$

· We do this cell-by-cell, inductively on dimension.

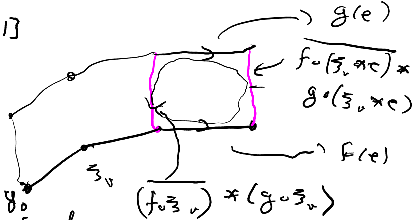
Extend to $(Y \times [0,1])^{(1)}$: T maximal tree, ξ_v reduced path from y_0 to v

· Extend to $\overbrace{v \times [0,1]}^{e}$ by
edge of $[0,1]$

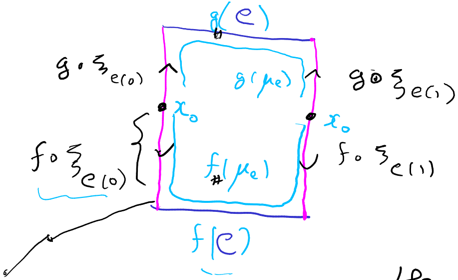
$\overline{(f \circ \xi_v)} * (g \circ \xi_v)$



Given $e \subset T$, extend to $e \times [0,1]$



$\partial \sim e$; extend



$\mu_e = \xi_{e(0)} * e * \overline{\xi_{e(1)}}$ – generator of $\pi_1(Y^{(1)}, y_0)$

We know $f_* = g_*$,

hence $f \circ \mu_e \sim g \circ \mu_e$

Hence the boundary is homotopic to e,

hence the map on the boundary extends to D^2.

- We can extend to $Y^{(3)}, Y^{(4)}, \ldots,$ using $\pi_n(X, x_0) = 0 \ \forall n \geq 2$. $\square$

<u>Theorem</u>: Let X, Y be Eilenberg-MacLane spaces. Then $X \overset{he}{\sim} Y \iff \pi_1(X, x_0) = \pi_1(Y, y_0)$

Constructing Eilenberg-MacLane spaces : G, a group

Theorem: X a CW complex, $x_0 \in X^{(0)}$. Then $\pi_n(X^{(n)}, x_0) \twoheadrightarrow \pi_n(X, x_0)$, $n \geq 2$.

Pf: Let $\theta : ([0,1]^n, \partial([0,1]^n)) \to (X, x_0)$. Then up to homotopy (fixing ∂)

- θ intersects interiors of only finitely many cells
- θ has image disjoint from the interior of m-cells for $m > n$, i.e. θ has image in $X^{(n)}$

p

Construction:

- Construct $X^{(2)}$ s.t. $\pi_1(X^{(2)}, x_0) = G$, $x_0 \in X^{(0)}$.
- Take generators of $\pi_2(X^{(2)}, x_0)$, θ_α : $\alpha \in A_2$. These give

$\theta_\alpha : ([0,1]^2, \partial[0,1]^2) \to (X^{(2)}, x_0)$

$\varphi_\alpha : S^2 \to X^{(2)}$. Attach 3-cells along these.

- Then $\pi_2(X^{(3)}, x_0) = 0$
- Similarly, given $X^{(n)}$ with $\pi_k(X^{(n)}, x_0) = 0$ for $2 \leq k < n$, we construct $X^{(n+1)}$ with $\pi_k(X^{(n)}, x_0) = 0$ $\forall k$, $2 \leq k < n+1$. Take limit

Hopf map $\pi_n(X, x_0)$ – complicated

; $\pi_n(S^n) = \mathbb{Z}$ (true)

$\pi_k(S^n) = 0$ if $k < n$

Theorem: $\pi_3(S^2) = \mathbb{Z}$

Generated by
$$\begin{tikzcd} S^3 \arrow[r] & S^2 \\ {[0,1]^3} \arrow[u] \arrow[ur] & \end{tikzcd}$$

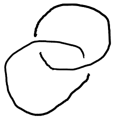
$f: S^3 \to S^2$: $S^3 \subset \mathbb{C}^2 = \{(z_1, z_2) : |z_1|^2 + |z_2|^2 = 1\}$

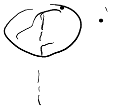
$S^2 = \mathbb{C} \cup \{\infty\} = \mathbb{C}P^1$

$f(z_1, z_2) = z_1 / z_2$

Fibers: circles, i.e., $f(z_1, z_2) = f(z_1', z_2') \iff \exists \alpha \in S^1$

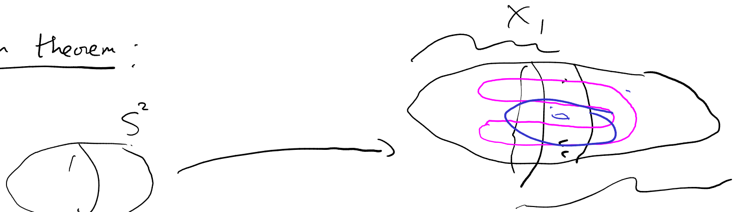
s.t. $(z_1', z_2') = \alpha (z_1, z_2)$





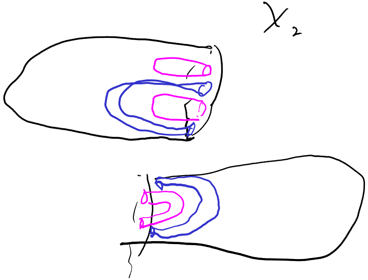
No combination theorem:

Reason:



Break into maps

from planar surface



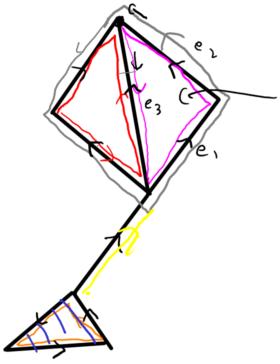
Rejoin: Maps from all

closed surface.

How many loops/holes?

↓

independent cycles



$e_1 + e_2 - e_3$

Describe loops: We consider edges (crossed) with multiplicity and sign; ignoring order.

- C_1 = free abelian group on edges $= \mathbb{Z}\langle E \rangle$
- C_0 = free abelian group on vertices
- $\partial_1 : C_1 \to C_0$, $\partial_1(e) = \tau(e) - i(e)$

Defn: 1-Cycles are

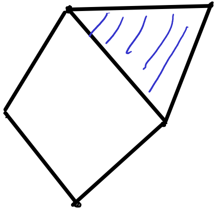
$$Z_1 = \ker(\partial_1) \cong \mathbb{Z}^3$$

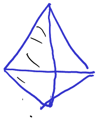
$\partial_2 : C_2 \to C_1$

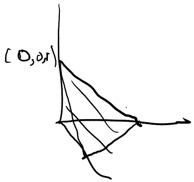
$H_1 = Z_1 / \mathrm{im}(\partial_2)$

Simplicial Complexes

$$\Delta^n = \{(a_0, \dots, a_n) \in \mathbb{R}^n : a_i \geq 0, \sum_{i=0}^{n} a_i = 1\}$$



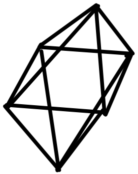


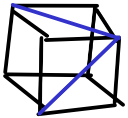


<u>Abstract simplicial complex</u>: A collection Σ of finite non-empty sets (subsets of some set S) s.t. if $\sigma \in \Sigma$, $\tau \subset \sigma$, then $\tau \in \Sigma$

<u>E.g.</u> $S = \{0, 1, \dots, n\}$

2^S is Δ^n



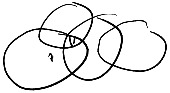


Poset complex: Given $(P, \leq)$ a partially ordered set, e.g. $P = \mathbb{N}$, $a \leq b$ iff $a | b$

$$\Sigma = \{ \sigma \subset P, \sigma \neq \emptyset : \sigma \text{ is totally ordered with } \leq \}$$

Nerve complex: X topological space, $\mathcal{U} = \{ \overset{\neq\emptyset}{U_\alpha} \}_{\alpha \in A}$ open cover.

$$N = \{ \{U_{\alpha_1}, \dots, U_{\alpha_n}\} \neq \emptyset : U_{\alpha_1} \cap \dots \cap U_{\alpha_n} \neq \emptyset \}$$



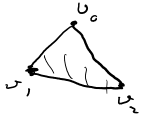
Geometric Realization Σ a simplicial complex

- Vertices: singletons $\{v_i\} \in \Sigma$; if $\sigma = \{v_0, \dots, v_n\}$, then $\{v_0\}, \dots \{v_n\} \in \Sigma$

$\sum^{n}$

- k-simplex: set in Σ of cardinality $k+1$

As a set



$$|\Sigma| = \Big\{ \underbrace{\sum_{i=0}^{n} a_i v_i}_{\text{formal sum}} : a_i \in \mathbb{R},\ a_i \geq 0,\ \sum_{i=0}^{n} a_i = 1,\ \{v_0, \dots, v_n\} \in \Sigma \Big\}$$

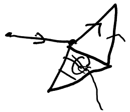
- Topology:
  - If V is finite, $|V| = N$, then $|\Sigma| \subset \mathbb{R}^N$ with subspace topology
  - If $W \subset V$ is finite, $\Sigma_W := \Sigma \cap 2^W$ is finite,

    so $|\Sigma_W|$ has a topology
  - If $W_1, W_2 \subset V$ are finite & $W_1 \subset W_2$, we have a map

    $$|\Sigma_{W_1}| \longrightarrow |\Sigma_{W_2}|$$
  - Define $|\Sigma|$ as $\varinjlim_{W} |\Sigma_W|$ as a topological space.

Simplicial homology Σ a simplicial complex

- Orientation: · For $\sigma = \{v_0, \dots, v_n\} \in \Sigma$, we fix an order (total) on $\{v_0, \dots, v_n\}$



  · If $\tau \subset \sigma$, order on σ gives order on τ.

$\partial e = v_1 - v_0$

  · $\sigma = \langle v_0, \dots, v_n \rangle$ simplex means $\{v_0, \dots, v_n\} \in \Sigma$ & $v_0 < \dots < v_n$ in σ.

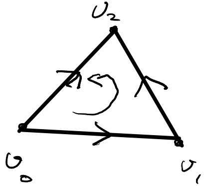
- R a ring, e.g. $R = \mathbb{Z}, \mathbb{Z}/2, \mathbb{Q}, \mathbb{R}$

- $C_k(\Sigma)$ = free R-module on k-simplices of Σ.

- $\partial_k : C_k(\Sigma) \longrightarrow C_{k-1}(\Sigma)$

$$\partial_k(\langle v_0, \dots, v_k \rangle) = \sum_{i=0}^{k} (-1)^k \langle v_0, \dots, \widehat{v_i}, \dots, v_k \rangle$$

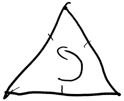
$v_0 < v_1 < v_2$



$$\partial \langle v_0, v_1, v_2 \rangle = \langle v_0, v_1 \rangle + \langle v_1, v_2 \rangle - \langle v_0, v_2 \rangle$$

Theorem: $\partial_{k-1} \circ \partial_k = 0$ for $k \geq 1$

Pf: · Let $\sigma \in C_k$, $\sigma = \langle v_0, \dots, v_k \rangle$



· $$\partial_{k-1} \circ \partial_k \sigma = \partial_{k-1} \left(\sum_{i=0}^{k} (-1)^i \langle v_0, \dots, \hat{v}_i, \dots, v_k \rangle \right)$$

$$= \sum_{j=0}^{i-1} \sum_{i=0}^{k} (-1)^i \cdot (-1)^j \langle v_0, \dots, \hat{v}_j, \dots, \hat{v}_i, \dots, v_k \rangle + \sum_{i=0}^{k} \sum_{j=i+1}^{k} (-1)^i (-1)^{j-1} \langle v_0, \hat{v}_i, \dots, \hat{v}_j, \dots, v_k \rangle$$

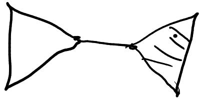
interchange i & j

$$= \sum_{0 \leq j < i \leq k} \sum (-1)^{i+j} \langle v_0, \dots, \hat{v}_j, \dots, \hat{v}_i, \dots, v_k \rangle + \sum_{0 \leq j < i \leq k} \sum (-1)^{i+j-1} \langle v_0, \dots, \hat{v}_j, \dots, \hat{v}_i, \dots, v_k \rangle$$

$$= \sum_{0 \leq j < i \leq k} \sum \underbrace{\left((-1)^{i+j} + (-1)^{i+j-1} \right)}_{0} \langle v_0, \dots, \hat{v}_j, \dots, \hat{v}_i, \dots, v_k \rangle = 0$$

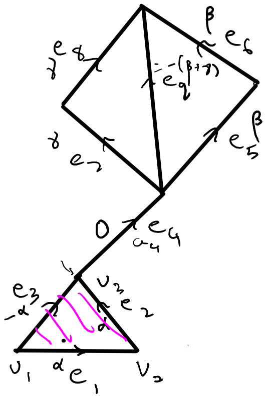
- $C_{-1} = 0$, $\partial_0 = 0$
- $\partial_k \circ \partial_{k+1} = 0 \quad \forall k \geq 0 \implies \underset{\substack{\parallel \\ B_{k+1}}}{\operatorname{im}(\partial_{k+1})} \subset \underset{\substack{\parallel \\ Z_k}}{\ker(\partial_k)}$

<u>Defn</u>: $H_k(\Sigma) = \dfrac{\ker(\partial_k)}{\operatorname{im}(\partial_{k+1})}$



Example.

$C_1 = \{ \underbrace{\sum_{i=1}^{n} a_i e_i}_{=\sigma} : a_i \in \mathbb{Z} \}$



Suppose $\partial\sigma = \sum b_j v_j = 0$, all b_j are 0.

· If $b_2 = 0$; $\partial(e_1) = v_2 - v_1$

$\partial e_2 = v_3 - v_2$

no other v_2 terms

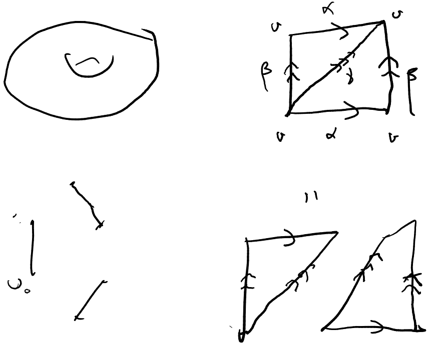
$\therefore$ $a_1 - a_2 = 0$, so $a_1 = a_2 =: \alpha$

$\therefore$ $\sigma = \alpha(\underbrace{e_1 + e_2 - e_3}_{z_\alpha}) + \beta(\underbrace{e_5 + e_6 - e_9}_{z_\beta}) + \gamma(\underbrace{e_7 + e_8 - e_9}_{z_\gamma})$

$H_1 = Z_1 = \{ \alpha z_\alpha + \beta z_\beta + \gamma \cdot z_\gamma : \alpha, \beta, \gamma \in \mathbb{Z} \}$

Δ-complexes: (finite)

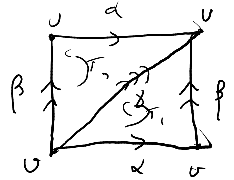
- Start with $\coprod$ simplices Δ^n for some n.
  $\simeq \left|2^{\{0,\dots,n\}}\right|$



- Face in a simplex in Δ^n
- A Δ-complex is given by
  - collection of simplices
  - collections $\{\mathcal{F}_\alpha\}_{\alpha\in A}$ with $\mathcal{F}_\alpha$ collections of faces of the simplices, with vertices ordered, and all of the same dimension d_α for fixed α.
- We have linear maps between faces in $\mathcal{F}_\alpha$

- We quotient by identifying using this.

- We can orient simplices.
- We can still define C_*, H_*



E.g.
- $C_0 = \mathbb{Z} = \mathbb{Z}\langle v \rangle$
- $C_1 \cong \mathbb{Z}^3 = \mathbb{Z}\langle \alpha, \beta, \gamma \rangle$
- $\partial_1 \equiv 0$
- $C_2 = \mathbb{Z}\langle T_1, T_2 \rangle$
- $\partial T_1 = \alpha + \beta - \gamma$; $\partial T_2 = \beta + \alpha - \gamma = \partial T_1$
- $C_3 = 0$

$$H_0 = \frac{\ker(\partial_0)}{\operatorname{im}(\partial_1)} = \mathbb{Z}/0 = \mathbb{Z}$$

$$H_1 = \frac{\ker(\partial_1)}{\operatorname{im}(\partial_2)} = \frac{\mathbb{Z}\langle \alpha, \beta, \gamma \rangle}{\mathbb{Z}\langle \alpha + \beta - \gamma \rangle}$$

$$\cong \mathbb{Z}\langle \alpha, \beta \rangle \cong \mathbb{Z}^2$$

$$H_2 = \frac{\ker(\partial_2)}{\operatorname{im}(\partial_3)} = \mathbb{Z}\langle T_1 - T_2 \rangle$$

$$\cong \mathbb{Z}.$$

$$V - E + F = 2 - 2g$$