AUTOMATING MATHEMATICS?

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http://math.iisc.ac.in/~gadgil/

https://github.com/siddhartha-gadgil/ProvingGround

1997

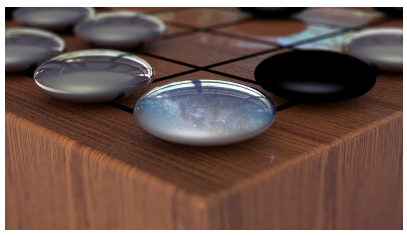
1997

2017

1997

2017

1997



2017

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Whether?
When?
How?

WHAT IS MATHEMATICS?

PRIME NUMBERS

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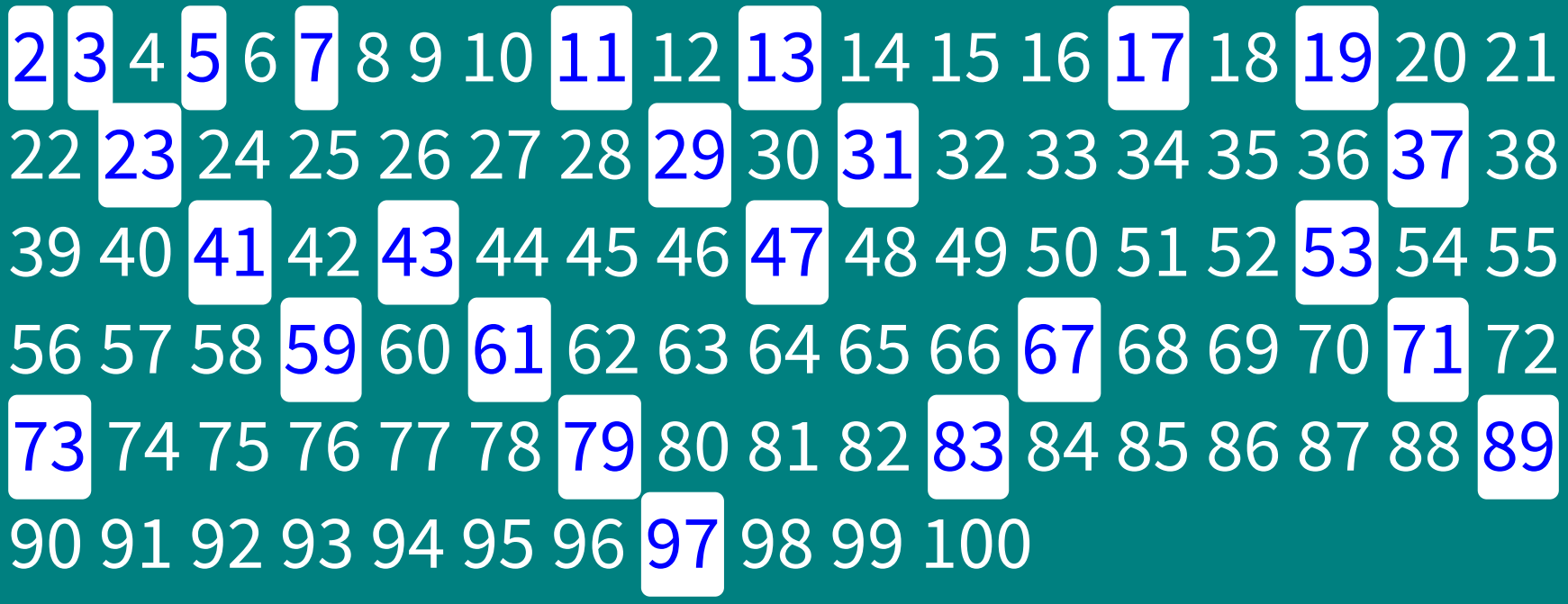
- Primes are a basic object of study in Mathematics.
- We will discuss what we study first, and then why.

HOW MANY PRIMES?



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HOW ABOUT TWIN PRIMES?

2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21
22 23 24 25 26 27 28 29 30 31 32 33 34 35 36 37 38
39 40 41 42 43 44 45 46 47 48 49 50 51 52 53 54 55
56 57 58 59 60 61 62 63 64 65 66 67 68 69 70 71 72
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90 91 92 93 94 95 96 97 98 99 100

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ARITHMETIC PROGRESSIONS IN PRIMES

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- We consider arithmetic progressions all of whose terms are primes, such as the third example.
- For experiments, we take the first million pairs of primes and extend these while we get primes.

- Some prime arithmetic progressions of length 3:

3 5 7	3 7 11	3 11 19	3 13 23	3 17 31	3 23 43	3 31 59	3 37 71	3 41 79	3 43 83
3 53 103	5 29 53	7 13 19	7 43 79	11 29 47	11 47 83	11 59 107	13 37 61		
17 23 29	17 53 89	17 59 101	19 31 43	19 43 67	19 61 103	23 41 59	23 47 71		
29 41 53	29 59 89	31 37 43	47 53 59						

- There are 110386 of these among the first million.

- Some prime arithmetic progressions of length 4:

5 23 41 59	5 59 113 167	5 89 173 257	7 19 31 43	7 79 151 223	11 17 23 29
13 43 73 103	13 61 109 157	17 29 41 53	19 73 127 181	19 79 139 199	23 53 83 113
29 83 137 191	31 67 103 139	41 47 53 59	41 71 101 131	43 61 79 97	47 59 71 83
47 89 131 173	53 71 89 107	59 83 107 131	61 67 73 79		

- There are 27836 of these among the first million.

- Some prime arithmetic progressions of length 5:

5 11 17 23 29	5 17 29 41 53	5 47 89 131 173	5 53 101 149 197	5 101 197 293 389
5 131 257 383 509	11 41 71 101 131	13 163 313 463 613	17 167 317 467 617	
29 149 269 389 509	37 67 97 127 157	43 103 163 223 283	61 151 241 331 421	
71 131 191 251 311	83 173 263 353 443	89 179 269 359 449	113 173 233 293 353	
137 167 197 227 257				

- There are 3665 of these among the first million.

- Some prime arithmetic progressions of length 6:

7 37 67 97 127 157	11 71 131 191 251 311	13 103 193 283 373 463
13 223 433 643 853 1063	23 263 503 743 983 1223	41 461 881 1301 1721 2141
53 113 173 233 293 353	73 223 373 523 673 823	83 383 683 983 1283 1583
107 137 167 197 227 257	127 457 787 1117 1447 1777	157 307 457 607 757 907
239 359 479 599 719 839	281 401 521 641 761 881	359 389 419 449 479 509

- There are 980 of these among the first million.

- Some prime arithmetic progressions of length 7:

7 157 307 457 607 757 907

47 257 467 677 887 1097 1307

53 1103 2153 3203 4253 5303 6353

71 2381 4691 7001 9311 11621 13931

179 389 599 809 1019 1229 1439

193 613 1033 1453 1873 2293 2713

359 1619 2879 4139 5399 6659 7919

829 1039 1249 1459 1669 1879 2089

1061 1901 2741 3581 4421 5261 6101

1091 1301 1511 1721 1931 2141 2351

1453 1663 1873 2083 2293 2503 2713

- There are 127 of these among the first million.

- Some prime arithmetic progressions of length 8:

619 829 1039 1249 1459 1669 1879 2089

881 1091 1301 1511 1721 1931 2141 2351

1019 3329 5639 7949 10259 12569 14879 17189

1091 3821 6551 9281 12011 14741 17471 20201

1289 2969 4649 6329 8009 9689 11369 13049

1637 2267 2897 3527 4157 4787 5417 6047

1847 3947 6047 8147 10247 12347 14447 16547

2239 2659 3079 3499 3919 4339 4759 5179

2693 4583 6473 8363 10253 12143 14033 15923

3323 4583 5843 7103 8363 9623 10883 12143

- There are 46 of these among the first million.

17 6947 13877 20807 27737 34667 41597 48527 55457

137 8117 16097 24077 32057 40037 48017 55997 63977

409 619 829 1039 1249 1459 1669 1879 2089

433 3583 6733 9883 13033 16183 19333 22483 25633

1699 5689 9679 13669 17659 21649 25639 29629 33619

2063 3323 4583 5843 7103 8363 9623 10883 12143

3499 3709 3919 4129 4339 4549 4759 4969 5179

3823 6133 8443 10753 13063 15373 17683 19993 22303

4721 7451 10181 12911 15641 18371 21101 23831 26561

6043 6883 7723 8563 9403 10243 11083 11923 12763

- There is just 1 arithmetic progressions of length 10 among the first million.

```
199 409 619 829 1039 1249 1459 1669 1879 2089
```

- The largest explicitly known arithmetic progression is of length 27.

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- **Status:** We still do not know.

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- This was further developed, and generalized to higher dimensions by Riemann.
- Einstein's **general relativity** was based on intrinsic curvature.

MATHEMATICAL PROOFS

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  - But p is the smallest factor of n, so $q = p$.
  - So the only factors of p are 1 and p, i.e., p is a prime.

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  - p divides $1 = (n! + 1) - n!$, which is absurd.

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- A formal proof can be checked mechanically.

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reserve n,p for Nat;
theorem Euclid: ex p st p is prime & p > n
proof
set k = n! + 1;
n! > 0 by NEWTON:23;
then n! >= 0 + 1 by NAT_1:38;
then k >= 1 + 1 by REAL_1:55;
then consider p such that
A1: p is prime & p divides k by INT_2:48;
A2: p <> 0 & p > 1 by A1,INT_2:def 5;
take p;
thus p is prime by A1;
assume p <= n;
then p divides n! by A2,NAT_LAT:16;
then p divides 1 by A1,NAT_1:57;
hence contradiction by A2,NAT_1:54;
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Full proof of infinitude of primes is 44 lines long.


PUZZLES, GAMES,
REASONING

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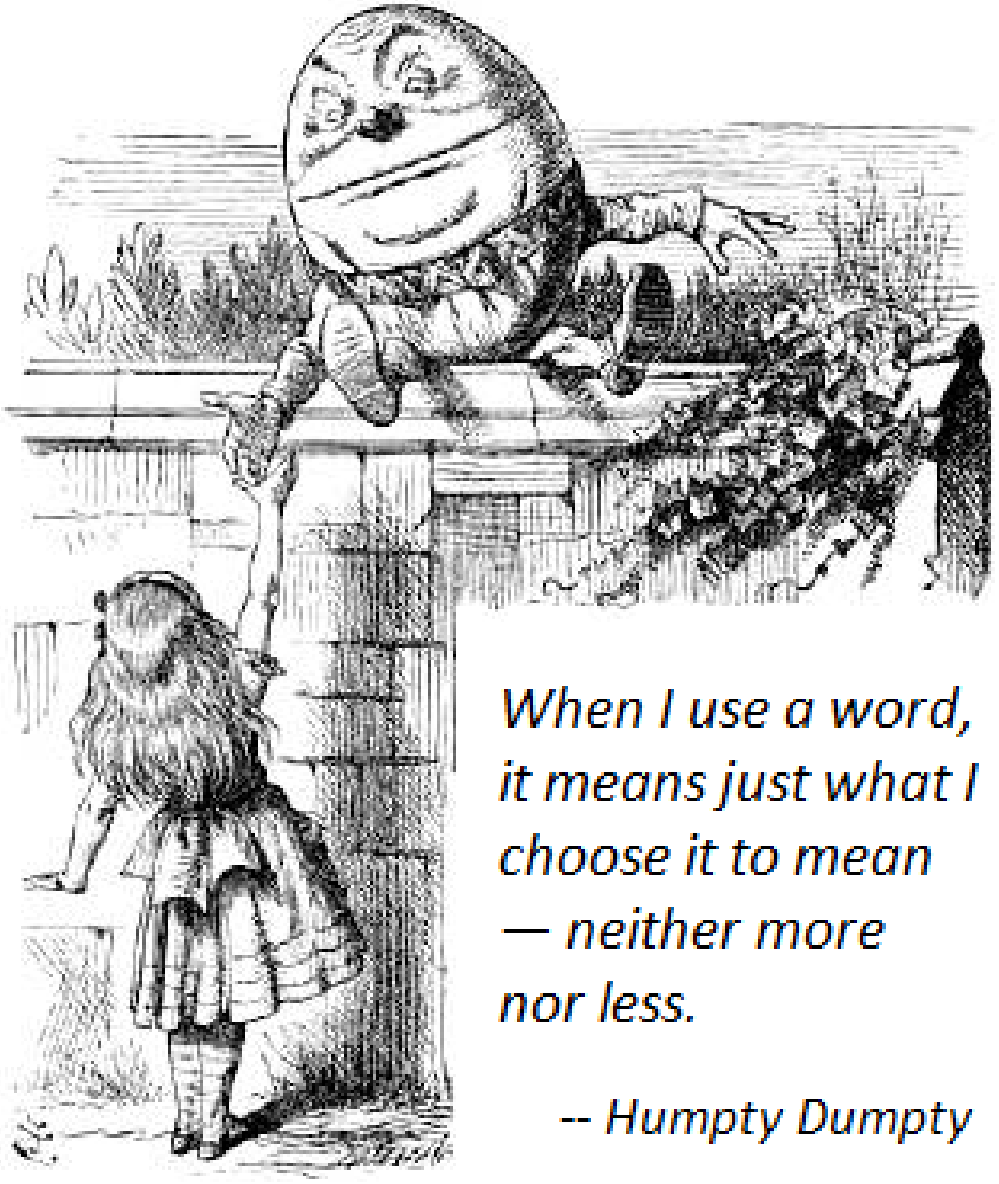
- We solve puzzles by a mixture of *deduction* (algorithms) and *intuition*.



When I use a word, it means just what I choose it to mean — neither more nor less.

-- Humpty Dumpty

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- **Examples:** riding a bicycle, speaking a language.
- Experts have a lot of tacit knowledge, typically learned through experience.
- Intuition is based on tacit knowledge.
- While intuition is sometimes wrong, to be useful it should be correct *often enough*.
    - A *hunch* is sometimes correct.
    - A *judgement* is often correct.

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  - deciding how promising the present approach/situation is - the *value*, which may use intuitive *judgements*.
- A computer following a purely algorithmic approach can compensate by following up on far more options, and looking more moves ahead before deciding the value.

COMPUTERS AND GAMES

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- *Standard openings* also give a **policy** during the early stages of the game, as do *endgame tables*.
- Deep Blue, and chess theory, extend these to elaborate, rule based values and policies.
- The value and policy functions of Kasparov were far better, but compensated for by Deep Blue being able to consider far more move sequences.

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- AlphaGo considered fewer sequences of moves than Deep Blue.
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- AlphaZero "had a dynamic, open style", and "prioritizes piece activity over material, preferring positions that looked risky and aggressive."


ARTIFICIAL
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ELSEWHERE.

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- The concise representation factorized by colour, shape and size (among other things).

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- For example the discriminative network tries to distinguish between real images and synthetic ones generated by the generative network.

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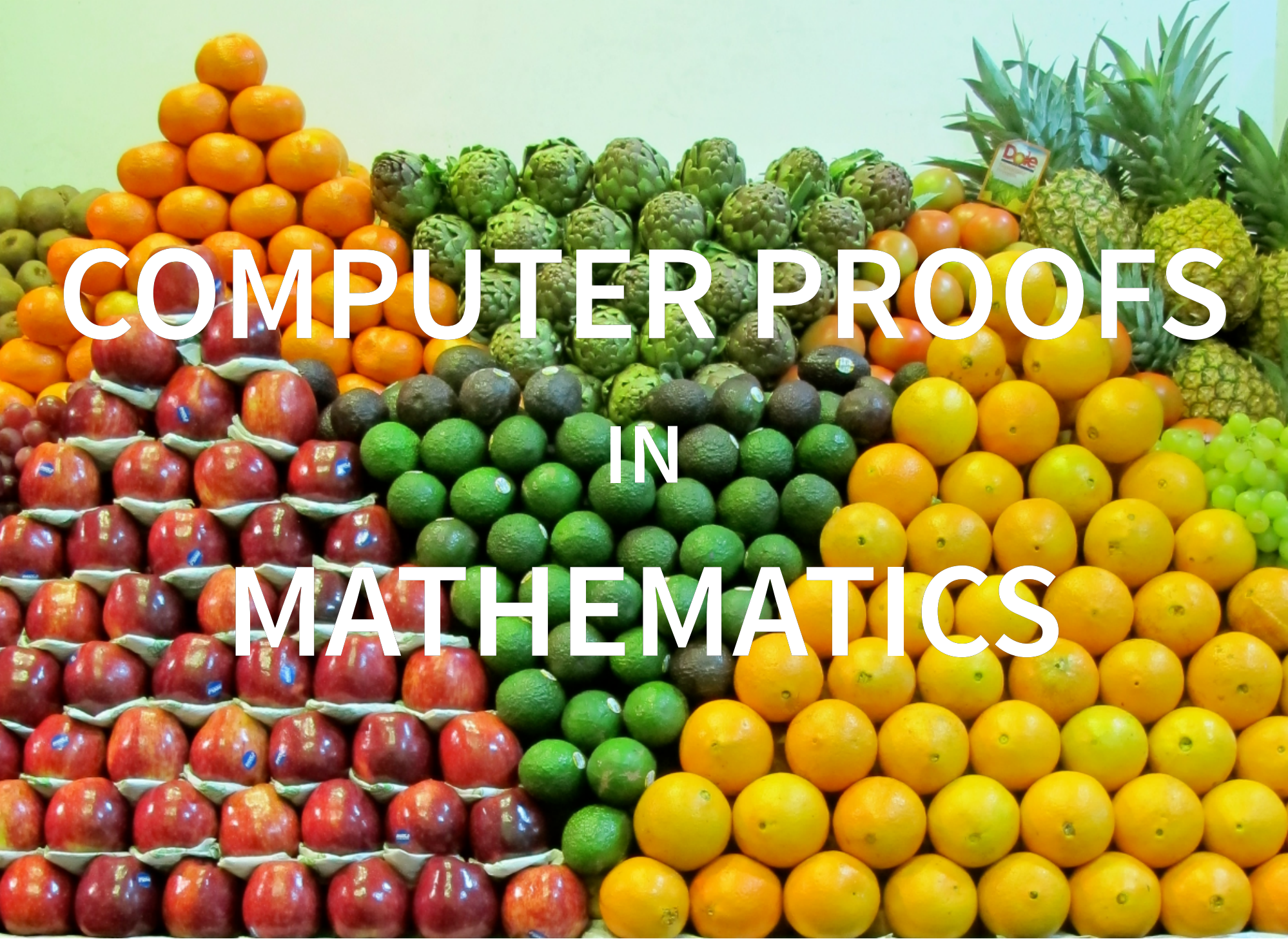
- Recently, similar distributions of *dopamine* cells was found in the brains of mice.


COMPUTER PROOFS
IN
MATHEMATICS

UNIVERSAL DEDUCER?

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- By results of Church, Gödel, Turing, such a program is impossible.
- Practically, we can conclude that there is no best deducer, as any given proof can be found by some deducer but no deducer can find all proofs.

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- All these proofs are long (maybe unavoidable).

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- This was conjectured in the 1930s, and finally proved in 1996 using the automated theorem prover **EQP**, which is a Resolution Theorem Prover with Paramodulation.
- So far, this seems to be the only major success of deductive theorem provers.

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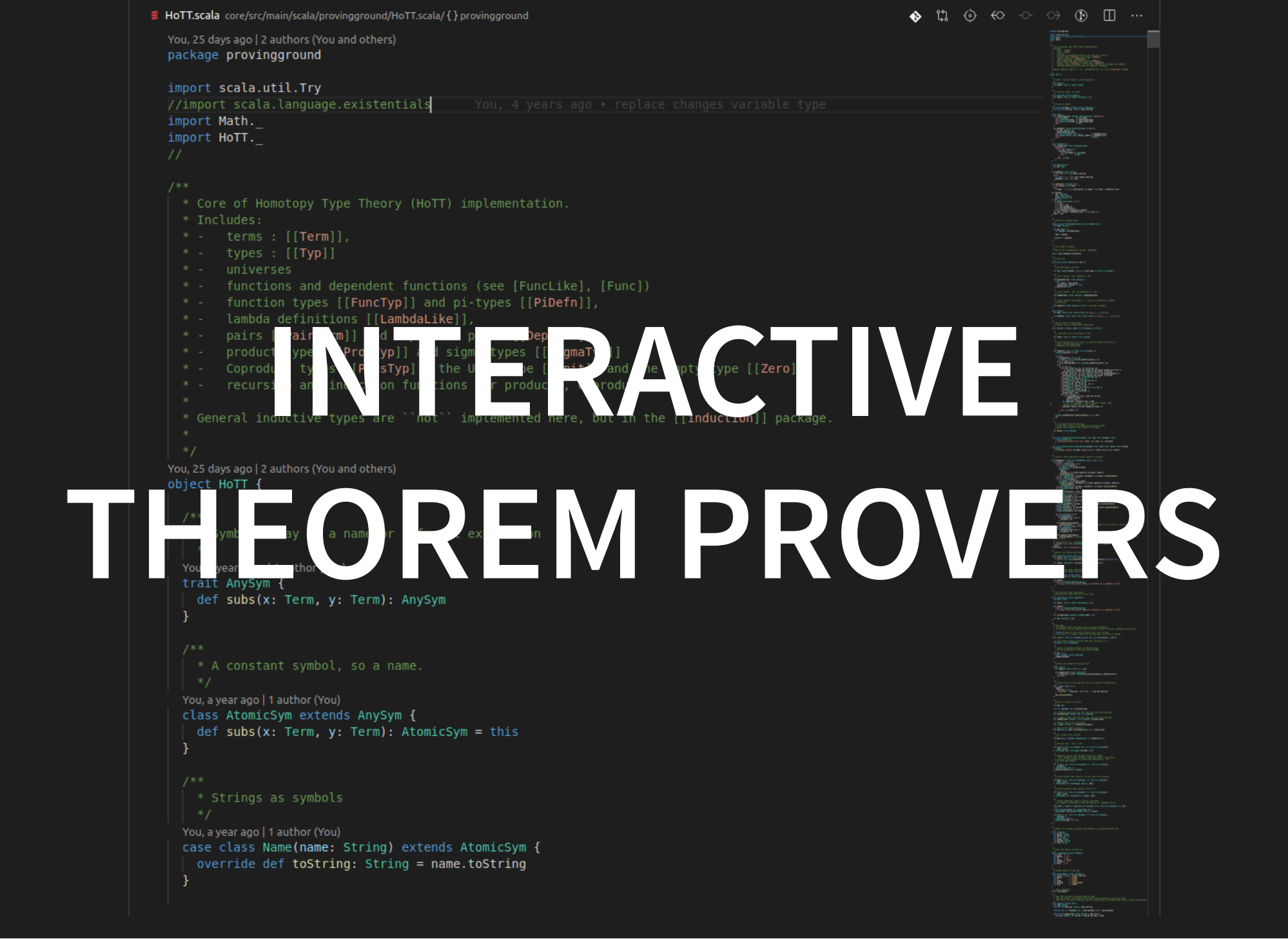
Gross proof time

Here is a computer generated proof of a bound on the length of the commutator $ab\bar{a}\bar{b}$ for a *linear* norm on the free group with the lengths of the generators bounded above by 1.

1. $|\bar{a}| \leq 1.0$
2. $|\bar{b}\bar{a}b| \leq 1.0$ using $|\bar{a}| \leq 1.0$
3. $|\bar{b}| \leq 1.0$
4. $|a\bar{b}\bar{a}| \leq 1.0$ using $|\bar{b}| \leq 1.0$
5. $|\bar{a}\bar{b}ab\bar{a}\bar{b}| \leq 2.0$ using $|\bar{a}\bar{b}a| \leq 1.0$ and $|b\bar{a}\bar{b}| \leq 1.0$
6. $|a| \leq 1.0$
7. $|ba\bar{b}| \leq 1.0$ using $|a| \leq 1.0$
8. $|b| \leq 1.0$
9. $|\bar{a}ba| \leq 1.0$ using $|b| \leq 1.0$
10. $|ab\bar{a}\bar{b}ab| \leq 2.0$ using $|ab\bar{a}| \leq 1.0$ and $|\bar{b}ab| \leq 1.0$
11. $|\bar{a}ab\bar{a}\bar{b}aba| \leq 2.0$ using $|ab\bar{a}\bar{b}ab| \leq 2.0$
12. $|b\bar{a}\bar{b}aab\bar{a}\bar{b}ab\bar{a}| \leq 3.0$ using $|b\bar{a}\bar{b}| \leq 1.0$ and $|aab\bar{a}\bar{b}ab\bar{a}| \leq 2.0$
13. $|\bar{a}\bar{b}ab\bar{a}\bar{b}aab\bar{a}\bar{b}ab\bar{a}| \leq 4.0$ using $|\bar{a}\bar{b}a| \leq 1.0$ and $|b\bar{a}\bar{b}aab\bar{a}\bar{b}ab\bar{a}| \leq 3.0$
14. $|\bar{b}\bar{a}\bar{b}aab\bar{a}\bar{b}ab\bar{a}\bar{b}| \leq 4.0$ using $|\bar{b}\bar{a}\bar{b}aab\bar{a}\bar{b}ab\bar{a}| \leq 3.0$


HoTT.scala core/src/main/scala/provinggroundHoTT.scala/{} provinggroundYou, 25 days ago | 2 authors (You and others)
package provinggroundimport scala.util.Try//import scala.language.existentialsYou, 4 years ago • replace changes variable typeimport Math._import HoTT._//
/**
* Core of Homotopy Type Theory (HoTT) implementation.
* Includes:
* - terms : [[Term]],
* - types : [[Typ]]
* - universes
* - functions and dependent functions (see [FuncLike], [Func])
* - function types [[FuncTyp]] and pi-types [[PiDefn]],
* - lambda definitions [[LambdaLike]],
* - pairs
* - product
* - Coprodu
* - recursi
*
* General inductive types are ``not`` implemented here, but in the [[Induction]] package.
*
*/
You, 25 days ago | 2 authors (You and others)
object HoTT {
/*
trait AnySym {
def subs(x: Term, y: Term): AnySym
}
/**
* A constant symbol, so a name.
*/
You, a year ago | 1 author (You)
class AtomicSym extends AnySym {
def subs(x: Term, y: Term): AtomicSym = this
}
/**
* Strings as symbols
*/
You, a year ago | 1 author (You)
case class Name(name: String) extends AtomicSym {
override def toString: String = name.toString
}
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- Some very large mathematical proofs have been verified by such systems.
- Formal methods are also based on interactive provers.

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- For example, the *lean theorem prover* has three (small) proof checkers written in three languages.

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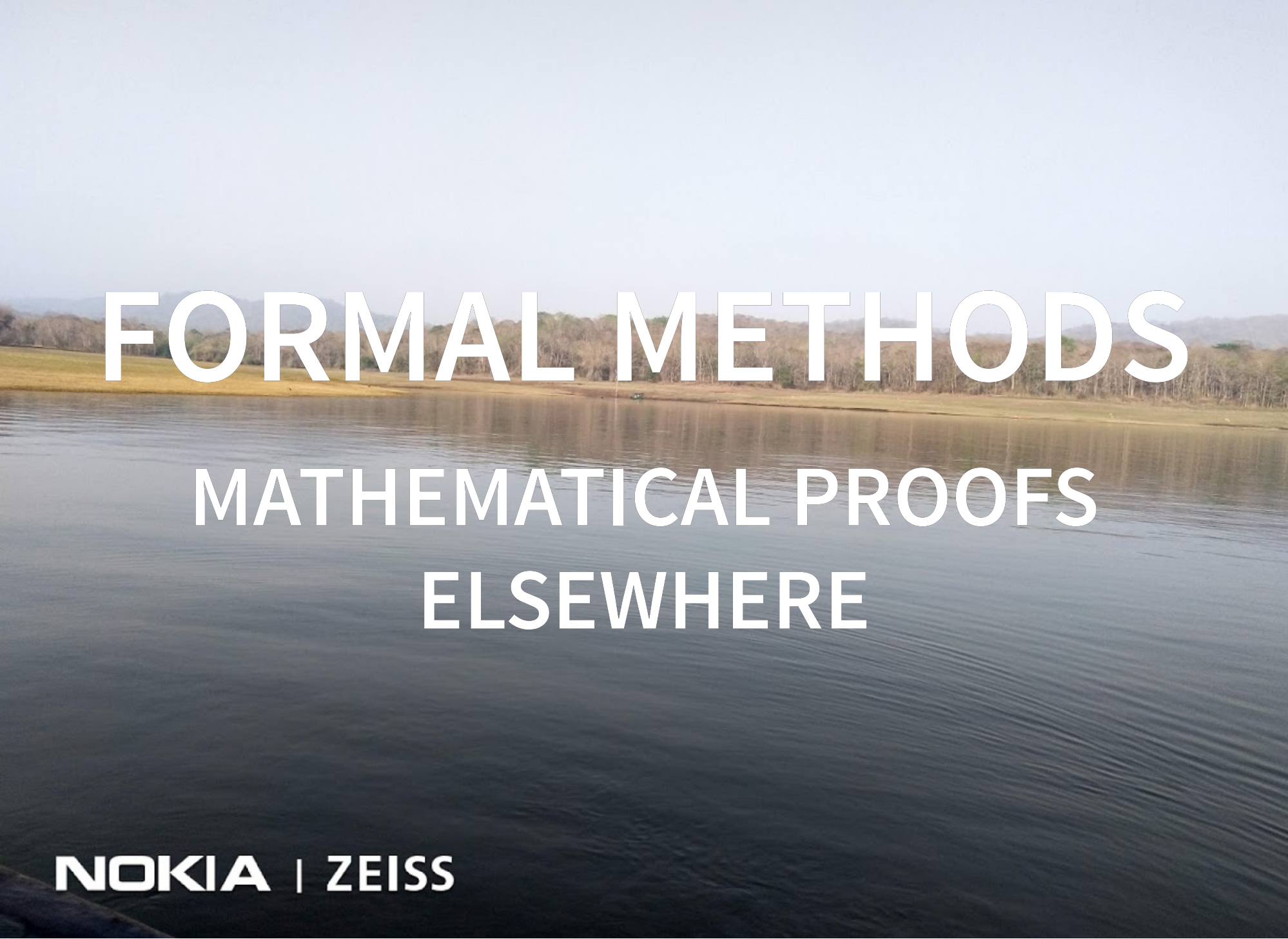
- However, formal proofs in these foundation become long and opaque, and don't resemble real life mathematics.
- Instead, interactive proof systems use richer foundations that have evolved over time.
- A dramatic advance in the new foundations came about with the recent discovery of connections with *topology*, a branch of mathematics.


FORMAL METHODS
MATHEMATICAL PROOFS
ELSEWHERE
NOKIA | ZEISS

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- Formal proofs use interactive theorem provers.

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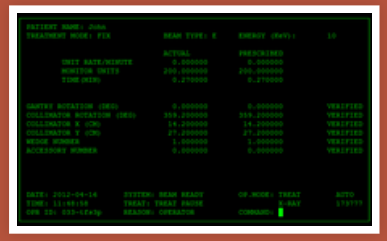

Therac 25
radiation machine

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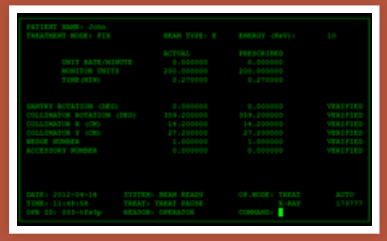	Therac 25 radiation machine	Safety critical

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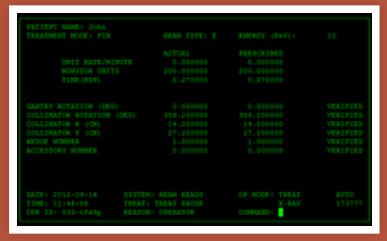

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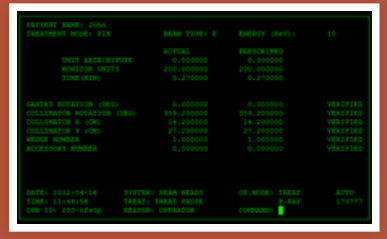	Therac 25 radiation machine	Safety critical
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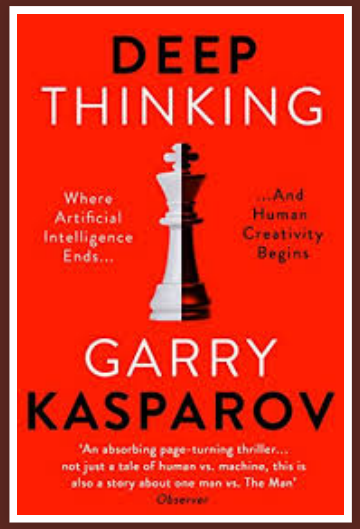
THE FUTURE OF
COMPUTER PROOFS?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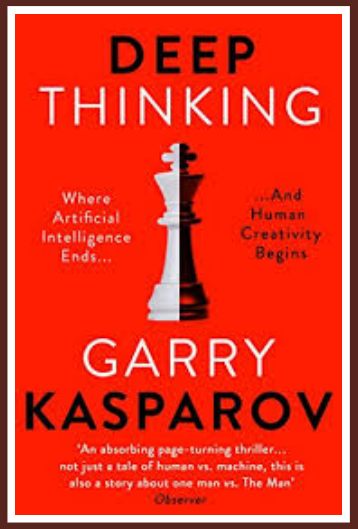
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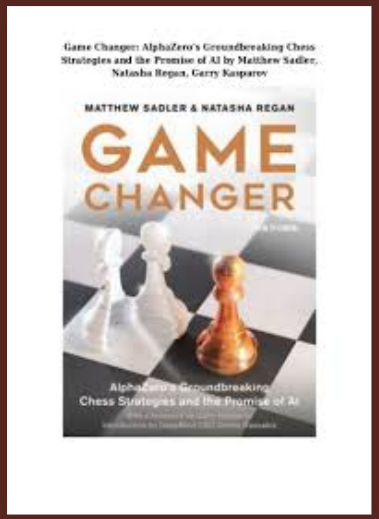
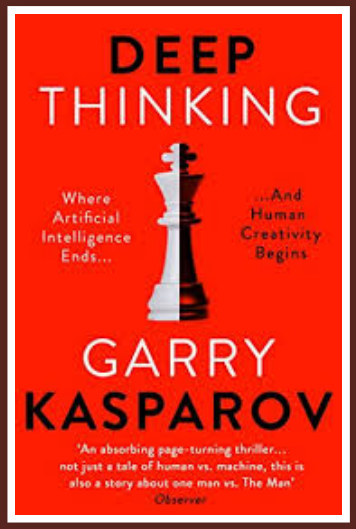

Game Changer: AlphaZero's Groundbreaking Chess Strategies and the Promise of AI by Matthew Sadler, Natasha Regan, Garry Kasparov
MATTHEW SADLER & NATASHA REGAN
GAME CHANGER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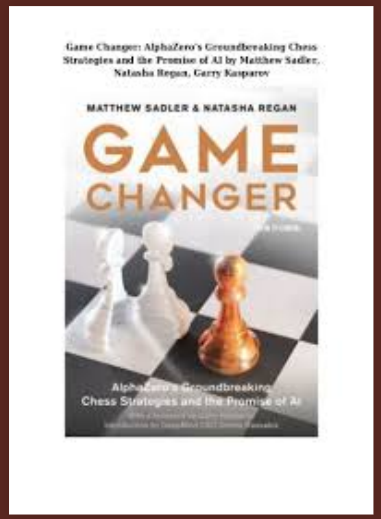
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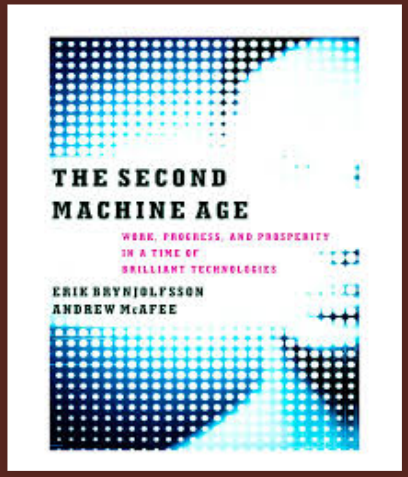
Chess Strategies and the Promise of AI


THE SECOND MACHINE AGE
WORK, PROGRESS, AND PROSPERITY IN A TIME OF BRILLIANT TECHNOLOGIES
ERIK BRYNJOLFSSON
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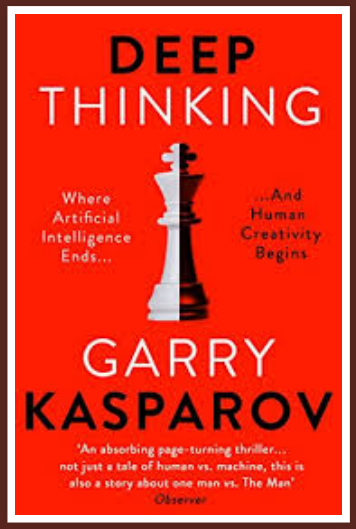
DEEP
THINKING
Where Artificial Intelligence Ends...
...And Human Creativity Begins
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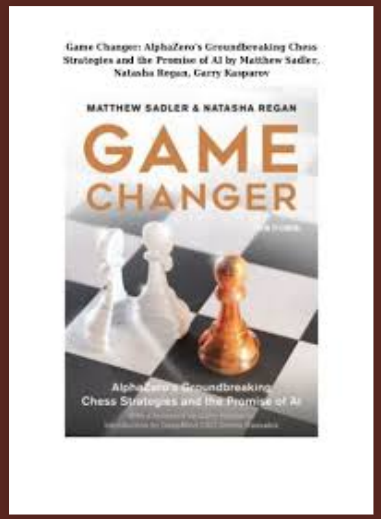
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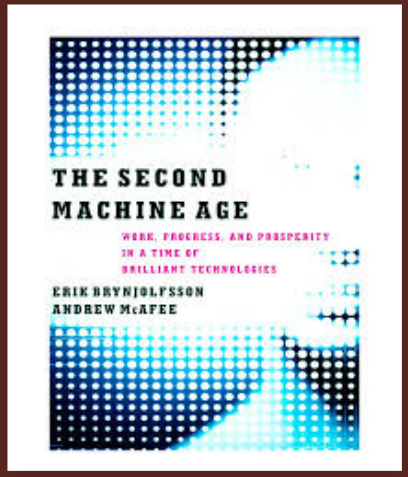
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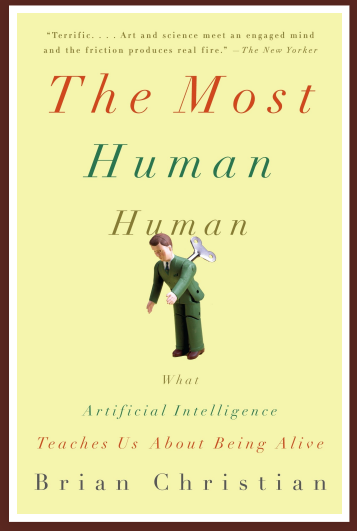
"Terrific. . . . Art and science meet an engaged mind and the friction produces real fire." —The New Yorker
The Most Human Human
What Artificial Intelligence Teaches Us About Being Alive
Brian Christian

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- **Automating mathematics?** It appears that there are clear approaches and **no evident barriers**.