Topological properties:

E.g. · Cannot distinguish $\overset{X}{[0,1]}$ from $\overset{Z}{[0,2]}$ in terms of continuous functions.

· Can distinguish $[0,1] = X$ from $Y = [0,1] \cup [2,3]$. Namely

Property ☺: X has ☺ if $\forall f: X \to \{0,1\}$ continuous map, f is constant.

Property ☆: X has ☆ if $\forall x, y \in X$, $\exists f: [0,1] \to X$ continuous map s.t. $f(0) = x$ & $f(1) = y$.

Dangers with sets:

$$X = \{ S \text{ a set} : S \notin S \} \quad \left\{ \text{Not permitted} \right\} \quad \%$$

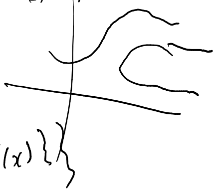
Question: Does $X \in X$?

- If $X \notin X$, by defn $X \in X$.
- If $X \in X$, by defn $X \notin X$.

Solution:
- Carefully define 'well-formed' expressions (which give sets)
- Axioms that give existence for sets. (in terms of other sets)

Carefully define arbitrary collections.

- Define $\{X_\alpha\}_{\alpha \in A}$, A 'index set' (e.g. $\mathbb{N}$). $\begin{cases} (a,b) \leftrightarrow \{\{a\},\{a,b\}\} \\ (c,d) \end{cases}$



- Try: function on A – to what?
- What is a function anyway? $\{\{x\},\{x,f(x)\}\}$
- A function $f: X \to Y$ is identified with its graph; what is this?

$$\Gamma(f) = \{(x, f(x)) : x \in X\} \subset X \times Y$$

- But not all subsets are graphs; have properties that characterize graphs
  - $\forall x \in X$, $\exists\, y \in Y$ s.t. $(x,y) \in \Gamma$
  - $\forall x \in X$, $\forall y_1, y_2 \in Y$ if $(x,y_1) \in \Gamma$ & $(x,y_2) \in \Gamma$ then $y_1 = y_2$

For collections, no <u>codomain</u>, instead have 'graph-like sets'

$\{X_\alpha\}_{\alpha\in A}$: 'Graph-like set' corresponding to $\Gamma = \{(\alpha, X_\alpha) : \alpha\in A\}$

Formally, Γ. $\Gamma(X_\alpha)$ is a set

all collection of α sets, not a set

we do not have $\Gamma\subset X\times Y$, instead $\Gamma\subset X\times Set$

- If $p\in\Gamma$, then $\exists$ S set s.t. $p = (\alpha, S)$, $\alpha\in A$.
- If $\alpha\in A$, $\exists$ S set s.t. $(\alpha, S)\in\Gamma$
- If $\alpha\in A$, S_1, S_2 sets s.t. $(\alpha, S_1)\in\Gamma$ & $(\alpha, S_2)\in\Gamma$ then $S_1 = S_2$

Then X_α is the unique set s.t. $(\alpha, X_\alpha)\in\Gamma$.

Furstenberg's topology : On $\mathbb{Z}$,

Basis: Arithmetic progressions $S(a,b) = \{a + nb : n \in \mathbb{Z}\} \subset \mathbb{Z}$

- This is a basis as finite intersections of arithmetic progressions are empty or arithmetic progressions (Ex)

$\times \cdot \cdot \cdot \cdot \times \cdot \cdot \cdot \cdot \times$

- Notice: · Basic sets are closed.

  · Finite non-empty sets are not open, i.e. complements of finite sets are not closed.

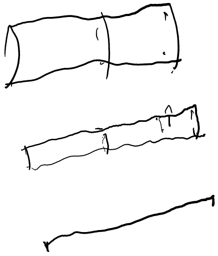
Suppose there were only finitely many primes then

$$\underbrace{\mathbb{Z} \setminus \{-1, 1\}}_{\substack{\text{contradiction}\\ \to \text{closed}}} = \underbrace{\bigcup_{p \text{ prime}}}_{\text{finite union}} \underbrace{S(0,p)}_{\text{closed}}$$

<u>Why Ψ-metrics:</u> Let $d_n : X \times X \to X$ be metrics s.t.

$d_n \to d_\infty$, i.e. $\forall\, x, y \in X$, $d_n(x,y) \to d_\infty(x,y)$

$[0,1] \times S^1$



<u>Then:</u>

- $d_\infty(x,y) \geq 0 \quad \forall\, x, y \in X$
- $d_\infty(x,y) = d_\infty(y,x) \quad \forall\, x, y \in X$
- $d_\infty(x,z) \leq d_\infty(x,y) + d_\infty(y,z)$ (Exercise)

<u>But:</u> $d_\infty(x,y) = 0$ can hold for $x \neq y$.

<u>E.g.</u> $X = \mathbb{R}^2$, $d_n((x_1,y_1),(x_2,y_2)) = |x_1 - x_2| + \frac{1}{n}|y_1 - y_2|$

Then $d_n \to d_\infty$, $d_\infty((x_1,y_1),(x_2,y_2)) = |x_1 - x_2|$

Characterizing interior : $S \subset X$, X topological space

- 'largest' w.r.t. partial order $A \subset B$
- may not in general exist

E.g. $a, b \in \mathbb{N}$, $\gcd(a,b)$ is the largest common divisor w.r.t the partial order $m|n$, i.e.

Defn: $d = \gcd(a,b) \Leftrightarrow d|a \wedge d|b$ & $m|a$ & $m|b \Rightarrow m|d$

i.e. the set $\{d \in \mathbb{N} : d|a$ & $d|b\}$ has a maximum w.r.t. the order $m|n$.

Theorem: If d, d' are both gcd's of a & b, then $d = d'$

e.g. For polynomials with integer coefficients $\mathbb{Z}[x]$, we do not have a g.c.d. in general, e.g. $2, x$

$\mathrm{int}(S)$: (1) $\mathrm{int}(S) \subset S$

(2) $\mathrm{int}(S)$ is open

(3) If $V \subset S$ is open, $V \subset \mathrm{int}(S)$ (maximality)

X space

E.g. We say $F \subset X$ is the 'finite-core' of X if

(a) F is finite

(b) $F \subset X$

(c) If A is finite and $A \subset X$ then $A \subset F$

Propn: If F_1 & F_2 are both finite cores of X, then $F_1 = F_2$

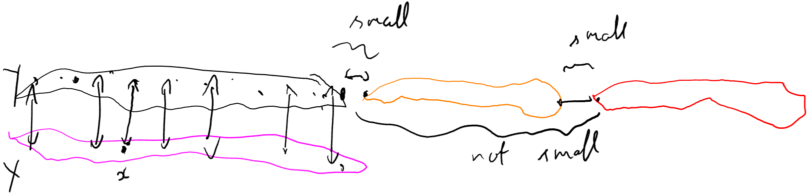
Pf: As F_1 is a finite core & F_2 satisfies (a) & (b), take $A = F_2$ in (c) to show $F_2 \subset F_1$

IIIly $F_1 \subset F_2$,

Thus $F_1 = F_2$

□

Distances between sets:



Sets $X, Y \subset Z$ bounded

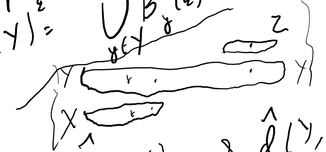
Candidates: $\cdot \ \inf\{d_Z(x,y) : x \in X, y \in Y\}$

for $d_Z(X,Y)$ $\cdot \ \sup\{d_Z(x,y) : x \in X, y \in Y\}$: $d_Z(X,X) \neq 0$ in general

What work: minimax:

$$d^+(X,Y) = \sup_{x \in X} \Big(\inf_{y \in Y} d_Z(x,y) \Big) = \inf\{\varepsilon > 0 : N_\varepsilon(Y) \supset X\}$$ where $N_\varepsilon(Y) = \bigcup_{y \in Y} B_y(\varepsilon)$

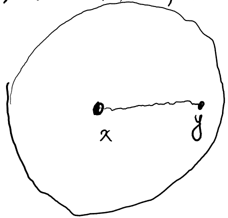
$$d(X,Y) = \max\{d^+(X,Y), d^+(Y,X)\}$$



$\hat{d}(X,Y)$ & $\hat{d}(Y,Z)$ are small, but not $\hat{d}(X,Z)$

Another candidate $\hat{d} = \min\{\ldots\}$

Rk: $d(x,y) = \inf\{\varepsilon > 0 : y \in B_\varepsilon(x)\}$



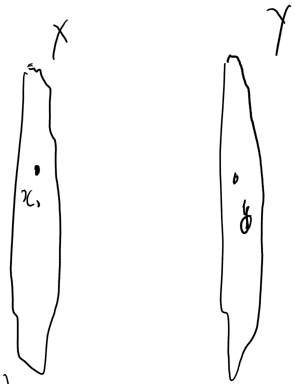
Propn: $d^+(X,Y) := \sup_{x\in X}\{\inf_{y\in Y} d_2(x,y)\} = \inf\{\varepsilon>0 : X \subset N_\varepsilon(Y)\}$

D $D-\delta \notin$, no $x \notin N_{D-\delta}(Y)$

Pf: **Lemma:** $\inf_{y\in Y} d_2(x_0,y) < \varepsilon$ for fixed $x_0 \in X \iff x_0 \in N_\varepsilon(Y)$

pf: $\inf_{y\in Y}\{d_2(x_0,y) < \varepsilon\} \Rightarrow \exists\, y_0 \in Y$ s.t. $d_2(x_0,y_0) < \varepsilon$

$\Rightarrow x_0 \in B_\varepsilon(y_0) \subset N_\varepsilon(Y) = \bigcup_{y\in Y} B_\varepsilon(y)$



Conversely, if $x_0 \in N_\varepsilon(Y)$, then $x_0 \in B_\varepsilon(y_0)$ for some $y_0 \in Y \Rightarrow d_2(x_0,y_0) < \varepsilon \Rightarrow \inf_{y\in Y}\{d_2(x_0,y)\} < \varepsilon$

Rest ~~as~~ exercise.

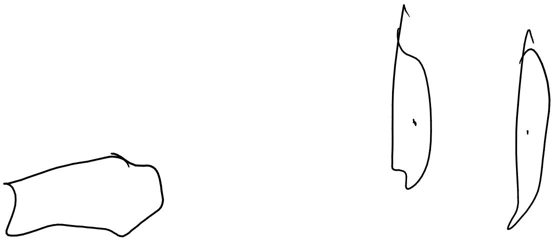
Alternative definition: $d(X,Y) = \inf\{\varepsilon>0 : X \subset N_\varepsilon(Y) \text{ and } Y \subset N_\varepsilon(X)\}$.

Then $d(\emptyset,\emptyset) = 0$ but $d(\emptyset, X) = \infty$ or D if $X \neq \emptyset$.

Modified definition if Z bounded, $\mathrm{diam}(Z) \leq D$

$$d_H^{(D)}(X,Y) = \inf\left(\{\varepsilon > 0 \,;\, X \subset N_\varepsilon(Y) \text{ and } Y \subset N_\varepsilon(X)\} \cup \{D\}\right)$$

- If $X \neq \emptyset \neq \emptyset$, $d_H(X,Y) = d_H^{(D)}(X,Y)$
- If $X \neq \emptyset$, $d_H(X,\emptyset) = D$
- $d_H(\emptyset,\emptyset) = 0$



Nowhere dense $\quad \text{inf}(\bar{A}) = \emptyset \quad ; \quad X \setminus \bar{A}$ is dense, i.e. $\overline{X \setminus \bar{A}} = X$

$$\text{inf}(\bar{A}) = \emptyset \iff \forall x \in X, \quad x \notin \text{inf}(\bar{A})$$

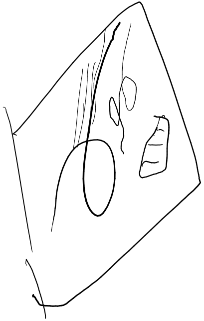
not

$$\iff \forall x \in X, \ \neg \left(\exists V \text{ open}, \ x \in V \text{ s.t. } V \subset \bar{A} \right)$$

$$\iff \forall x \in X, \ \forall V \text{ open s.t. } x \in V, \ V \not\subset \bar{A}$$

$$\iff \forall x \in X, \ \forall V \text{ open s.t. } x \in V, \ V \cap (X \setminus \bar{A}) \neq \emptyset$$

$$\iff \forall x \in X, \ x \in \overline{(X \setminus \bar{A})}$$



$$\iff X \setminus \bar{A} \text{ is dense.}$$