Topological properties:

E.g. · Cannot distinguish $[0,1] =: X$ from $[0,2] =: Z$ in terms of continuous functions.

· Can distinguish $[0,1] =: X$ from $Y := [0,1] \cup [2,3]$. Namely

Property ☺ : X has ☺ if $\forall f : X \to \{0,1\}$ continuous map, f is constant.

Property ☆ : X has ☆ if $\forall x, y \in X$, $\exists f : [0,1] \to X$ continuous map s.t. $f(0) = x$ & $f(1) = y$

Dangers with sets:

$X = \{ S \text{ a set} : S \notin S \}$ {Not permitted} $0/0$

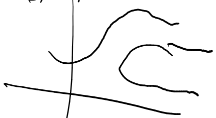
Question: Does $X \in X$?

- If $X \notin X$, by defn $X \in X$.
- If $X \in X$, by defn $X \notin X$.

Solution:
- Carefully define 'well-formed' expressions (which give sets)
- Axioms that give existence for sets. (in terms of other sets)

Carefully define arbitrary collections.

- Define $\{X_\alpha\}_{\alpha \in A}$, A 'index set' (e.g. $\mathbb{N}$). $\begin{cases} (a,b) \leftrightarrow \{\{a\},\{a,b\}\} \\ (c,d) \end{cases}$



- Try: function on A – to what?
- What is a function anyway? $\{\{x\},\{x,f(x)\}\}$
- A function $f: X \to Y$ is identified with its graph; what is this?

$$\Gamma(f) = \{(x, f(x)) : x \in X\} \subset X \times Y$$

- But not all subsets are graphs; have properties that characterize graphs
  - $\forall x \in X$, $\exists y \in Y$ s.t. $(x,y) \in \Gamma$
  - $\forall x \in X$, $\forall y_1, y_2 \in Y$ if $(x,y_1) \in \Gamma$ & $(x,y_2) \in \Gamma$ then $y_1 = y_2$

For collections, no <u>codomain</u>, instead have 'graph-like sets'

$\underline{\{X_\alpha\}_{\alpha\in A}}$: 'Graph-like set' corresponding to $\Gamma := \{(\alpha, X_\alpha) : \alpha\in A\}$

all collection of α sets, not a set

Formally, Γ. $\Gamma(X_\alpha)$ is a set

· we do not have $\Gamma \subset X \times Y$, instead $\Gamma \subset X \times Set$

· If $p \in \Gamma$, then $\exists$ S set s.t. $p = (\alpha, S)$, $\alpha\in A$.

· If $\alpha\in A$, $\exists$ S set s.t. $(\alpha, S)\in\Gamma$

· If $\alpha\in A$, S_1, S_2 sets s.t. $(\alpha, S_1)\in\Gamma$ & $(\alpha, S_2)\in\Gamma$ then $S_1 = S_2$

Then X_α is the unique set s.t. $(\alpha, X_\alpha)\in\Gamma$.

Furstenberg's topology : On $\mathbb{Z}$,

Basis: Arithmetic progressions $S(a,b) = \{a + nb : n \in \mathbb{Z}\} \subset \mathbb{Z}$

- This is a basis as finite intersections of arithmetic progressions are empty or arithmetic progressions (Ex)

- Notice: · Basic sets are closed.

  · Finite non-empty sets are not open, i.e. complements of finite sets are not closed.

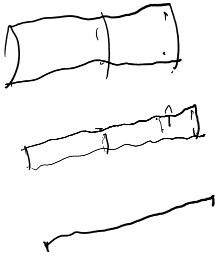
Suppose there were only finitely many primes then

$$\underbrace{\mathbb{Z} \setminus \{-1, 1\}}_{\text{contradiction} \rightarrow \text{closed}} = \underbrace{\bigcup_{p \text{ prime}}}_{\text{finite union}} \underbrace{S(0,p)}_{\text{closed}}$$

Why Ψ-metrics: Let $d_n : X \times X \to X$ be metrics s.t.

$d_n \to d_\infty$, i.e. $\forall\, x, y \in X$, $d_n(x,y) \to d_\infty(x,y)$

$[0,1] \times S^1$



Then:

- $d_\infty(x,y) \geq 0 \quad \forall\, x, y \in X$
- $d_\infty(x,y) = d_\infty(y,x) \quad \forall\, x, y \in X$
- $d_\infty(x,z) \leq d_\infty(x,y) + d_\infty(y,z)$ (Exercise)

But: $d_\infty(x,y) = 0$ can hold for $x \neq y$.

E.g. $X = \mathbb{R}^2$, $d_n((x_1,y_1),(x_2,y_2)) = |x_1 - x_2| + \frac{1}{n}|y_1 - y_2|$

Then $d_n \to d_\infty$, $d_\infty((x_1,y_1),(x_2,y_2)) = |x_1 - x_2|$