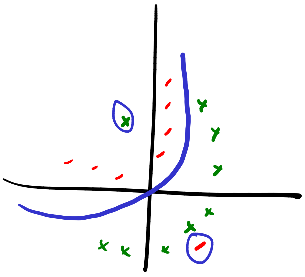


$$\min_{\substack{w \in \mathbb{R}^d,\\ \xi}} \quad \frac{1}{2}\|w\|^2 + C\sum_{i=1}^{n} \xi_i$$

st $\rightarrow (w^T x_i) y_i + \xi_i \geq 1 \quad \forall i \longrightarrow 1 - w^T x_i y_i - \xi_i \leq 0$

$\rightarrow \xi_i \geq 0 \quad \forall i \longrightarrow -\xi_i \leq 0$

$$L(w, \xi, \alpha, \beta) = \frac{1}{2}\|w\|^2 + C\sum_{i=1}^{n}\xi_i + \sum_{i=1}^{n}\alpha_i\left(1 - w^T x_i y_i - \xi_i\right) + \sum_{i=1}^{n}\beta_i(-\xi_i)$$

DUAL PROBLEM

$$\min_{w,\xi}\left[\max_{\substack{\alpha \geq 0\\ \beta \geq 0}} \frac{1}{2}\|w\|^2 + C\sum_{i=1}^{n}\xi_i + \sum_{i=1}^{n}\alpha_i\left(1 - w^T x_i y_i - \xi_i\right) + \sum_{i=1}^{n}\beta_i(-\xi_i)\right]$$

$\parallel\!\!\mid$ Duality

$$\max_{\substack{\alpha \geq 0\\ \beta \geq 0}}\left[\min_{w,\xi}\underbrace{\left[\frac{1}{2}\|w\|^2 + C\sum_{i=1}^{n}\xi_i + \sum_{i=1}^{n}\alpha_i\left(1 - w^T x_i y_i - \xi_i\right) + \sum_{i=1}^{n}\beta_i(-\xi_i)\right]}_{L}\right]$$

Fix α, β

$$\frac{\partial L}{\partial w} = 0 \Rightarrow \boxed{w^*_{\alpha,\beta} = \sum_{i=1}^{n}\alpha_i x_i y_i} \quad -① \qquad w^*_{\alpha,\beta} = XY\alpha$$

$$\frac{\partial L}{\partial \xi_i} = 0 \Rightarrow C + \alpha_i(-1) + \beta_i(-1) = 0$$

$$\Rightarrow \boxed{\alpha_i + \beta_i = C} \quad -②$$

Back Substituting $w^*_{\alpha,\beta}$ into the Lagrangian

Dual problem

$$\max_{\substack{\alpha \geq 0 \\ \beta \geq 0 \\ \alpha + \beta = C\mathbf{1}}} \quad \alpha^T \mathbf{1} - \frac{1}{2} \alpha^T y^T x^T x \, y \, \alpha$$

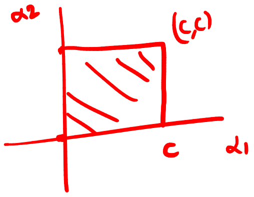
↑ no β term

$\alpha + \beta = C\mathbf{1} \;\Rightarrow\; \alpha_i + \beta_i = C \;\; \forall i$

|||

$$\max_{0 \leq \alpha \leq C} \quad \alpha^T \mathbf{1} - \frac{1}{2} \alpha^T y^T x^T x \, y \, \alpha \;\longrightarrow\; \text{Kernelizable}$$

Box Constraints.



If $C = 0 \Rightarrow \alpha^* = 0 \in \mathbb{R}^n \Rightarrow w^* = \sum \alpha_i^* x_i y_i$

$\Rightarrow \boxed{w^* = 0}$

$C = \infty \Rightarrow$ Hard-margin

What do the COMPLEMENTARY SLACKNESS (C.S) say about the Soft-margin SVM?

Let (w^*, ξ^*) be the primal optimal solutions.

Let (α^*, β^*) be the dual optimal solutions.

C.S

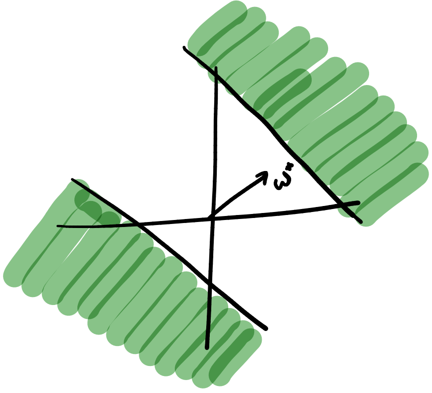
1. $\forall i \quad \alpha_i^* \left(1 - w^{*T} x_i y_i - \xi_i^*\right) = 0$ ←
2. $\forall i \quad \beta_i^* \left(\xi_i^*\right) = 0$

Various cases possible

1. $\alpha_i^* = 0 \Rightarrow \beta_i^* = C \quad [\alpha_i^* + \beta_i^* = C]$

$\Downarrow$ C.S 2

$\xi_i^* = 0$



we know $w^{*T} x_i y_i + \xi_i^* \geq 1$

$\Rightarrow \quad w^{*T} x_i y_i \geq 1 \quad \Rightarrow \quad w^*$ classifies (x_i, y_i) correctly !

② $\quad \alpha_i^* \in (0, C) \qquad 0 < \alpha_i^* < C$

$\Downarrow$ [c.s ①]

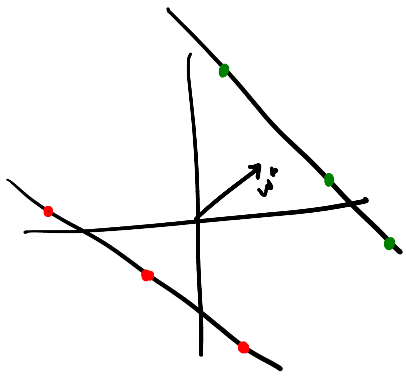
$1 - w^{*T} x_i y_i - \xi_i^* = 0$

$\Downarrow$

$1 - w^{*T} x_i y_i = 0$

$\Rightarrow \quad \beta_i^* \in (0, C) \qquad 0 < \beta_i^* < C$

$\Rightarrow \quad \xi_i^* = 0 \quad$ [c.s ②]



$\Rightarrow \quad \boxed{w^{*T} x_i y_i = 1}$

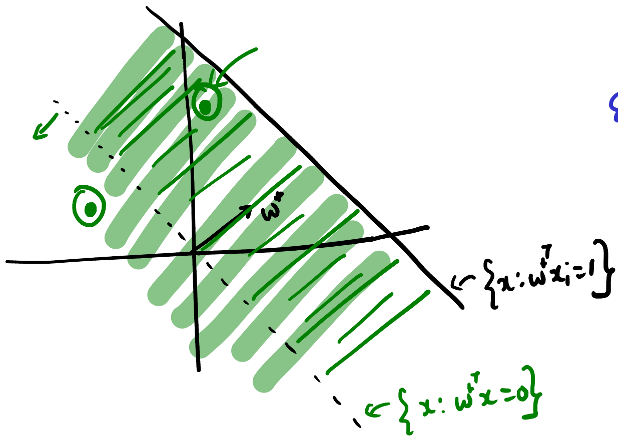
③ $\quad \alpha_i^* = C \quad \xrightarrow{c.s ①} \quad 1 - w^{*T} x_i y_i - \xi_i^* = 0$

$\Downarrow$

$\beta_i^* = 0 \qquad \xi_i^* = 1 - w^{*T} x_i y_i \geq 0$

$\Downarrow$

$\xi_i^* \geq 0 \qquad \Rightarrow \quad \boxed{w^{*T} x_i y_i \leq 1}$



Points where either x_i is incorrectly classified by w^* or correctly classified but with margin ≤ 1

Let's see this from the primal point of view.

Case ①

$w^{*T} x_i y_i < 1$

$w^{*T} x_i y_i + \xi_i^* \geq 1 \quad \Rightarrow \quad \xi_i^* \geq 1 - w^{*T} x_i y_i$

$\Rightarrow \quad \xi_i^* > 0 \quad \xrightarrow{c.s ②} \quad \beta_i^* = 0 \Rightarrow \boxed{\alpha_i^* = C}$

Case ②

$$w^{*T} x_i y_i = 1$$

$$\xi_i^* \geq 1 - w^{*T} x_i y_i$$

$$\xi_i^* \geq 0 \Rightarrow \alpha_i^* \in [0, C]$$

Case ③

$$\boxed{w^{*T} x_i y_i > 1}$$

$$\Rightarrow \quad \underbrace{1 - w^{*T} x_i y_i} \; - \underbrace{\xi_i^*} \; < 0$$

$$\overset{cs\,①}{\Rightarrow} \quad \boxed{\alpha_i^* = 0}$$

SUMMARY

$$\alpha_i^* = 0 \Rightarrow w^{*T} x_i y_i \geq 1$$

$$0 < \alpha_i^* < C \Rightarrow w^{*T} x_i y_i = 1$$

$$\alpha_i^* = C \Rightarrow w^{*T} x_i y_i \leq 1$$

$$w^{*T} x_i y_i < 1 \Rightarrow \alpha_i^* = C$$

$$w^{*T} x_i y_i = 1 \Rightarrow 0 \leq \alpha_i^* \leq C$$

$$w^{*T} x_i y_i > 1 \Rightarrow \alpha_i^* = 0$$