PERIOD THREE IMPLIES CHAOS

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1. Introduction. The way phenomena or processes evolve or change in time is often described by differential equations or difference equations. One of the simplest mathematical situations occurs when the phenomenon can be described by a single number as, for example, when the number of children susceptible to some disease at the beginning of a school year can be estimated purely as a function of the number for the previous year. That is, when the number x_{n+1} at the beginning of the $n+1$st year (or time period) can be written

$$x_{n+1} = F(x_n), \tag{1.1}$$

where F maps an interval J into itself. Of course such a model for the year by year progress of the disease would be very simplistic and would contain only a shadow of the more complicated phenomena. For other phenomena this model might be more accurate. This equation has been used successfully to model the distribution of points of impact on a spinning bit for oil well drilling, as mentioned in [**8, 11**], knowing this distribution is helpful in predicting uneven wear of the bit. For another example, if a population of insects has discrete generations, the size of the $n+1$st generation will be a function of the nth. A reasonable model would then be a generalized logistic equation

$$x_{n+1} = rx_n[1 - x_n/K]. \tag{1.2}$$

A related model for insect populations was discussed by Utida in [**10**]. See also Oster *et al* [**14, 15**].

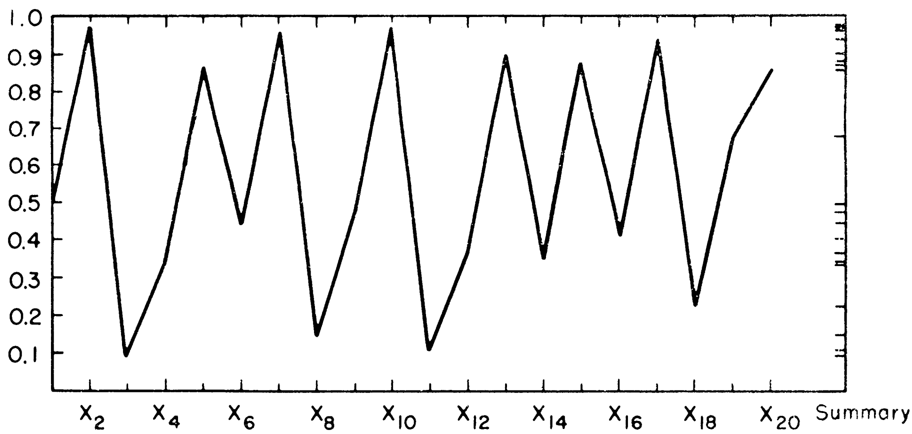

FIG. 1. For $K = 1$, $r = 3.9$, with $x_1 = .5$, the above graph is obtained by iterating Eq. (1.2) 19 times. At right the 20 values are repeated in summary. No value occurs twice. While $x_2 = .975$ and $x_{10} = .973$ are close together, the behavior is not periodic with period 8 since $x_{18} = .222$.

These models are highly simplified, yet even this apparently simple equation (1.2) may have surprisingly complicated dynamic behavior. See Figure 1. We approach these equations with the viewpoint that irregularities and chaotic oscillations of complicated phenomena may sometimes be understood in terms of the simple model, even if that model is not sufficiently sophisticated to allow accurate numerical predictions. Lorenz [**1–4**] took this point of view in studying turbulent behavior in a fascinating series of papers. He showed that a certain complicated fluid flow could be modelled by such a sequence $x, F(x), F^2(x), \cdots$, which retained some of the chaotic aspects of the original flow. See Figure 2. In this paper we analyze a situation in which the sequence $\{F^n(x)\}$ is non-periodic and might be called "chaotic." Theorem 1 shows that chaotic behavior for (1.1) will result in any situation in which a "population" of size x can grow for two or more successive generations and then having reached an unsustainable height, a population bust follows to the level x or below.

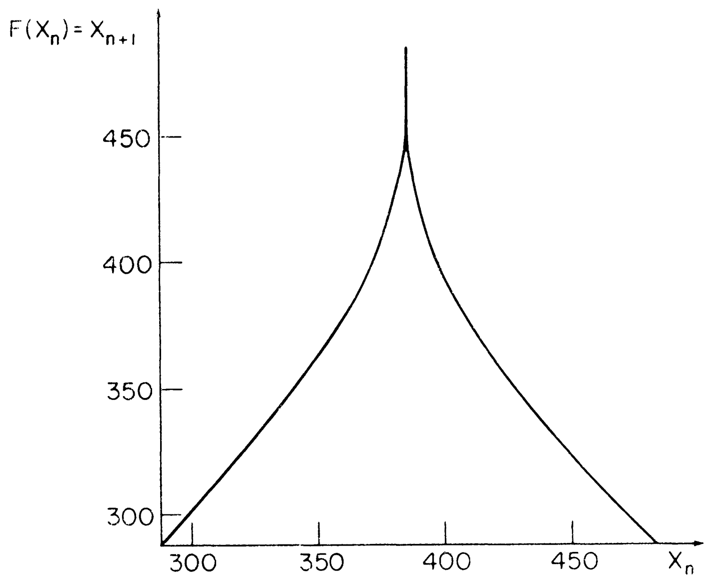

FIG. 2. Lorenz [1] studied the equations for a rotating water-filled vessel which is circularly symmetric about its vertical axis. The vessel is heated near the rim and cooled near its center. When the vessel is annular in shape and the rotation rate high, waves develop and alter their shape irregularly. From a simplified set of equations solved numerically, Lorenz let X_n be in essence the maximum kinetic energy of successive waves. Plotting X_{n+1} against X_n, and connecting the points, the above graph is obtained.

In section 3 we give a well-known simple condition which guarantees that a periodic point is stable and then in section 4 we quote a result applicable when F is like the one in Figure 2. It implies that there is an interval $J_\infty \subset J$ such that for almost every $x \in J$, the set of limit points of the sequence $\{F^n(x)\}$ is J_∞.

A number of questions remain unanswered. For example, is the closure of the periodic points an interval or at least a finite union of intervals? Other questions are mentioned later.

Added in proof. May has recently discovered other strong properties of these maps in his independent study of how the behavior changes as a parameter is varied [17].

2. The main theorem. Let $F: J \to J$. For $x \in J$, $F^0(x)$ denotes x and $F^{n+1}(x)$ denotes $F(F^n(x))$ for $n = 0, 1, \cdots$. We will say p is a **periodic point with period** n if $p \in J$ and $p = F^n(p)$ and $p \neq F^k(p)$ for $1 \leqq k < n$. We say p is **periodic** or is a **periodic point** if p is periodic for some $n \geqq 1$. We say q is **eventually periodic** if for some positive integer m, $p = F^m(q)$ is periodic. Since F need not be one-to-one, there may be points which are eventually periodic but are not periodic. Our objective is to understand the situations in which iterates of a point are very irregular. A special case of our main result says that if there is a periodic point with period 3, then for each integer $n = 1, 2, 3, \cdots$, there is a periodic point with period n. Furthermore, there is an uncountable subset of points x in J which are not even "asymptotically periodic."

THEOREM 1. *Let J be an interval and let $F: J \to J$ be continuous. Assume there is a point $a \in J$ for which the points $b = F(a)$, $c = F^2(a)$ and $d = F^3(a)$, satisfy*

$$d \leqq a < b < c \text{ (or } d \geqq a > b > c).$$

Then

T1: *for every $k = 1, 2, \cdots$ there is a periodic point in J having period k.*

Furthermore,

T2: *there is an uncountable set $S \subset J$ (containing no periodic points), which satisfies the following conditions:*

(A) *For every $p, q \in S$ with $p \neq q$,*

$$\limsup_{n \to \infty} |F^n(p) - F^n(q)| > 0 \tag{2.1}$$

and

$$\liminf_{n \to \infty} |F^n(p) - F^n(q)| = 0. \tag{2.2}$$

(B) *For every $p \in S$ and periodic point $q \in J$,*

$$\limsup_{n \to \infty} |F^n(p) - F^n(q)| > 0.$$

REMARKS. Notice that if there is a periodic point with period 3, then the hypothesis of the theorem will be satisfied.

An example of a function satisfying the hypotheses of the theorem is $F(x) = rx[1 - x/K]$ as in (1.2) for $r \in (3.84, 4]$ with $J = [0, K]$ and for $r > 4$, $F(x) = \max\{0, rx[1 - x/K]\}$ with $J = [0, K]$. See [**2**] for a detailed description of iterates of this function for $r \in [0, 4)$. The case $r = 4$ is discussed in [**6, 7, 12**].

While the existence of a point of period 3 implies the existence of one of period 5, the converse is false. (See Appendix 1).

We say $x \in J$ is **asymptotically periodic** if there is a periodic point p for which

$$F^n(x) - F^n(p) \to 0 \quad \text{as} \quad n \to \infty. \tag{2.3}$$

It follows from (B) that the set S contains no asymptotically periodic points. We remark that it is unknown what the infimum of r is for which the equation (1.2) has points which are not asymptotically periodic.

Proof of Theorem 1. The proof of T1 introduces the main ideas for both T1 and T2. We now give the proof of T1 with necessary lemmas and relegate the tedious proof of T2 to Appendix 2.

LEMMA 0. *Let $G: I \to R$ be continuous, where I is an interval. For any compact interval $I_1 \subset G(I)$ there is a compact interval $Q \subset I$ such that $G(Q) = I_1$.*

Proof. Let $I_1 = [G(p), G(q)]$, where $p, q \in I$. If $p < q$, let r be the last point of $[p, q]$ where $G(r) = G(p)$ and let s be the first point after r where $G(s) = G(q)$. Then $G([r, s]) = I_1$. Similar reasoning applies when $p > q$.

LEMMA 1. *Let $F: J \to J$ be continuous and let $\{I_n\}_{n=0}^{\infty}$ be a sequence of compact intervals with $I_n \subset J$ and $I_{n+1} \subset F(I_n)$ for all n. Then there is a sequence of compact intervals Q_n such that $Q_{n+1} \subset Q_n \subset I_0$ and $F^n(Q_n) = I_n$ for $n \geqq 0$. For any $x \in Q = \cap Q_n$ we have $F^n(x) \in I_n$ for all n.*

Proof. Define $Q_0 = I_0$. Then $F^0(Q_0) = I_0$. If Q_{n-1} has been defined so that $F^{n-1}(Q_{n-1}) = I_{n-1}$, then $I_n \subset F(I_{n-1}) = F^n(Q_{n-1})$. By Lemma 0 applied to $G = F^n$ on Q_{n-1} there is a compact interval $Q_n \subset Q_{n-1}$ such that $F^n(Q_n) = I_n$. This completes the induction.

The technique of studying how certain sequences of sets are mapped into or onto each other is often used in studying dynamical systems. For instance, Smale uses this method in his famous "horseshoe example" in which he shows how a homeomorphism on the plane can have infinitely many periodic points **[13]**.

LEMMA 2. *Let $G: J \to R$ be continuous. Let $I \subset J$ be a compact interval. Assume $I \subset G(I)$. Then there is a point $p \in I$ such that $G(p) = p$.*

Proof. Let $I = [\beta_0, \beta_1]$. Choose $\alpha_i (i = 0, 1)$ in I such that $G(\alpha_i) = \beta_i$. It follows $\alpha_0 - G(\alpha_0) \geqq 0$ and $\alpha_1 - G(\alpha_1) \leqq 0$ and so continuity implies $G(\beta) - \beta$ must be 0 for some β in I.

Assume $d \leqq a < b < c$ as in the theorem. The proof for the case $d \geqq a > b > c$ is similar and so is omitted. Write $K = [a, b]$ and $L = [b, c]$.

Proof of T1: Let k be a positive integer. For $k > 1$ let $\{I_n\}$ be the sequence of intervals $I_n = L$ for $n = 0, \cdots, k-2$ and $I_{k-1} = K$, and define I_n to be periodic inductively, $I_{n+k} = I_n$ for $n = 0, 1, 2, \cdots$. If $k = 1$, let $I_n = L$ for all n.

Let Q_n be the sets in the proof of Lemma 1. Then notice that $Q_k \subset Q_0$ and $F^k(Q_k) = Q_0$ and so by Lemma 2, $G = F^k$ has a fixed point p_k in Q_k. It is clear that p_k cannot have period less than k for F; otherwise we would need to have $F^{k-1}(p_k) = b$, contrary to $F^{k+1}(p_k) \in L$. The point p_k is a periodic point of period k for F.

3. Behavior near a periodic point. For some functions F, the asymptotic behavior of iterates of a point can be understood simply by studying the periodic points. For

$$F(x) = ax(1-x) \tag{3.1}$$

a detailed discussion of the points of period 1 and 2 may be found in **[1]** for $a \in [0, 4]$ and we now summarize some of those results. For $a \in [0, 4]$, F: $[0, 1] \to [0, 1]$.

For $a \in [0, 1]$, $x = 0$ is the only point of period 1; in fact, for $x \in [0, 1]$, the sequence $F^n(x) \to 0$ as $n \to \infty$.

For $a \in (1, 3]$, there are two points of period 1, namely 0 and $1 - a^{-1}$, and for $x \in (0, 1)$, $F^n(x) \to 1 - a^{-1}$ as $n \to \infty$.

For $a > 3$ there are also two points of period 2 which we may call p and q and of course $F(p) = q$ and $F(q) = p$. For $a \in (3, 1 + \sqrt{6} \approx 3.449)$ and $x \in (0, 1)$, $F^{2n}(x)$ converges to either p or q while $F^{2n+1}(x)$ converges to the other, except for those x for which there is an n for which $F^n(x)$ equals the point $1 - a^{-1}$ of period 1. There are only a countable number of such points so that the behavior of $\{F^n(x)\}$ can be understood by studying the periodic points.

For $a > 1 + \sqrt{6}$, there are 4 points of period 4 and for a slightly greater than $1 + \sqrt{6}$, $F^{4n}(x)$ tends to one of these 4 unless for some n, $F^n(x)$ equals one of the points of period 1 or 2. Therefore we may summarize this situation by saying that each point in [0, 1] is asymptotically periodic.

For those values of a for which each point is asymptotically periodic, it is sufficient to study only the periodic points and their "stability properties." For any function F a point $y \in J$ with period k is said to be **asymptotically stable** if for some interval $I = (y - \delta, y + \delta)$ we have

$$|F^k(x) - y| < |x - y| \qquad \text{for all} \qquad x \in I.$$

If F is differentiable at the points $y, F(y), \cdots, F^{k-1}(y)$, there is a simple condition that will guarantee this behavior, namely

$$\left| \frac{d}{dx} F^k(x) \right| < 1.$$

By the chain rule

$$
\begin{aligned}
\frac{d}{dx} F^k(y) &= \frac{d}{dx} F(F^{k-1}(y)) \cdot \frac{d}{dx} F^{k-1}(y) \\
&= \frac{d}{dx} F(F^{k-1}(y)) \times \frac{d}{dx} F(F^{k-2}(y)) \times \cdots \times \frac{d}{dx} F(y) \qquad (3.2) \\
&= \prod_{n=0}^{k-1} \frac{d}{dx} F(y_n),
\end{aligned}
$$

where y_n is the nth iterate, $F^n(y)$. Therefore y is asymptotically stable if

$$\left| \prod_{i=0}^{k-1} \frac{d}{dx} F(y_i) \right| < 1, \qquad \text{where} \qquad y_i = F^i(y).$$

This condition of course guarantees nothing about the limiting behavior of points which do not start "near" the periodic point or one of its iterates. The function in Figure 2 which was studied by Lorenz has the opposite behavior, namely, where the derivative exists we have

$$\left| \frac{d}{dx} F(x) \right| > 1.$$

For such a function every periodic point is "unstable" since for x near a periodic point y of period k, the kth iterate $F^k(x)$ is further from y than x is. To see this, approximate $F^k(x)$ by

$$F^k(y) + \frac{d}{dx} F^k(y)[y-x] = y + \frac{d}{dx} F^k(y)[y-x].$$

Thus for x near y, $|F^k(x) - y|$ is approximately $|x-y|\,|(d/dx)F^k(y)|$. From (3.2) $|(d/dx)F^k(y)|$ is greater than 1. Therefore $F^n(x)$ is further from y than x is.

We do not know when values of a begin to occur for which F in (3.1) has points which are not asymptotically periodic. For $a = 3.627$, F has a periodic point (which is asymptotically stable) of period 6 (approx. $x = .498$). This x is therefore a point of period 3 for F^2 and so Theorem 1 may be applied to F^2. Since F^2 has points which are not asymptotically periodic, the same is true of F.

In order to contrast the situations in this section with other possible situations discussed in the next section, we define the limit set of a point x. The point y is a **limit point** of a sequence $\{x_n\} \subset J$ if there is a subsequence x_{n_i} converging to y. The **limit set** $L(x)$ is defined to be the set of limit points of $\{F^n(x)\}$. If x is asymptotically periodic, then $L(x)$ is the set $\{y, F(y), \cdots, F^{k-1}(y)\}$ for some periodic point y of period k.

4. Statistical properties of $\{F^n(x)\}$. Theorem 1 establishes the irregularity of the behavior of iterates of points. What is also needed is a description of the regular behavior of the sequence $\{F^n(x)\}$ when F is piecewise continuously differentiable (as is Lorenz's function in Figure 2) and

$$\inf_{x \in J_1} \left| \frac{dF}{dx} \right| > 1 \qquad \text{where} \qquad J_1 = \left\{ x : \frac{dF}{dx} \text{ exists} \right\}. \qquad (4.1)$$

One approach to describing the asymptotic behavior for such functions is to describe $L(x)$, if possible. A second approach, which turns out to be related, is to examine the average behavior of $\{F^n(x)\}$. The fraction of the iterates $\{x, \cdots, F^{N-1}(x)\}$ of x that are in $[a_1, a_2]$ will be denoted by $\phi(x, N, [a_1, a_2])$. The limiting fraction will be denoted

$$\phi(x, [a_1, a_2]) = \lim_{N \to \infty} \phi(x, N, [a_1, a_2])$$

when the limit exists. The subject of ergodic theory, which studies transformation on general spaces, motivates the following definition. We say g is the **density of** x (for F) if the limiting fraction satisfies

$$\phi(x,[a_1,a_2]) = \int_{a_1}^{a_2} g(x)dx \quad \text{for all} \quad a_1, a_2 \in J; \quad a_1 < a_2.$$

The techniques for the study of densities use non-elementary techniques of measure theory and functional analysis, so that we shall only summarize the results. But their value lies in the fact that for certain F almost all $x \in J$ have the same density. Until recently the existence of such densities had not been proved, except for the simplest of functions F. The following result has recently been proved:

THEOREM 2. **[5]**. *Let F: $J \to J$ satisfy the following conditions:*

1) *F is continuous.*
2) *Except at one point $t \in J$, F is twice continuously differentiable.*
3) *F satisfies* (4.1).

Then there exists a function g: $J \to [0,\infty)$, such that for almost all $x \in J$, g is the density of x. Also for almost all $x \in J$, $L(x) = \{y: g(y) > 0\}$ which is an interval. Moreover, the set $J_\infty = \{y: g(y) > 0\}$ is an interval, and $L(x) = J_\infty$ for almost all x.

The proof makes use of results in **[8]**. The problem of computationally finding the density is solved in **[9]**.

A detailed discussion of (3.1) is given in **[16]**, describing how $L(x)$ varies as the parameter a in (3.1) varies between 3.0 and 4.0.

A major question left unsolved is whether (for some nice class of functions F) the existence of a stable periodic point implies that almost every point is asymptotically periodic.

Appendix 1: Period 5 does not imply period 3. In this Appendix we give an example which has a fixed point of period 5 but no fixed point of period 3.

Let F: $[1,5] \to [1,5]$, be defined such that $F(1) = 3$, $F(2) = 5$, $F(3) = 4$, $F(4) = 2$, $F(5) = 1$ and on each interval $[n, n+1]$, $1 \leqq n \leqq 4$, assume F is linear. Then

$$F^3([1,2]) = F^2([3,5]) = F([1,4]) = [2,5].$$

Hence, F^3 has no fixed points in $[1,2]$. Similarly, $F^3([2,3]) = [3,5]$ and $F^3([4,5]) = [1,4]$, so neither of these intervals contains a fixed point of F^3. On the other hand,

$$F^3([3,4]) = F^2([2,4]) = F([2,5]) = [1,5] \supset [3,4].$$

Hence, F^3 must have a fixed point in $[3,4]$. We shall now demonstrate that the fixed point of F^3 is unique and is also a fixed point of F.

Let $p \in [3,4]$ be a fixed point of F^3. Then $F(p) \in [2,4]$. If $F(p) \in [2,3]$, then $F^3(p)$ would be in $[1,2]$ which is impossible since then p could not be a fixed point. Hence $F(p) \in [3,4]$ and $F^2(p) \in [2,4]$. If $F^2(p) \in [2,3]$ we would have $F^3(p) \in [4,5]$, an impossibility. Hence p, $F(p)$, $F^2(p)$ are all in $[3,4]$. On the interval $[3,4]$, F is defined linearly and so $F(x) = 10 - 2x$. It has a fixed point 10/3 and it is easy to see that F^3 has a unique fixed point, which must be 10/3. Hence there is no point of period 3.

Appendix 2. Proof of T2 of Theorem 1. Let $\mathcal{M}$ be the set of sequences $M = \{M_n\}_{n=1}^{\infty}$ of intervals with

$$M_n = K \quad \text{or} \quad M_n \subset L, \quad \text{and} \quad F(M_n) \supset M_{n+1} \tag{A.1}$$

$$\text{if} \quad M_n = K \quad \text{then}$$

$$n \text{ is the square of an integer and } M_{n+1}, M_{n+2} \subset L, \tag{A.2}$$

where $K = [a, b]$ and $L = [b, c]$. Of course if n is the square of an integer, then $n+1$ and $n+2$ are not, so the last requirement in (A.2) is redundant. For $M \in \mathcal{M}$, let $P(M, n)$ denote the number of i's in $\{1, \cdots, n\}$ for which $M_i = K$. For each $r \in (3/4, 1)$ choose $M^r = \{M_n^r\}_{n=1}^{\infty}$ to be a sequence in $\mathcal{M}$ such that

$$\text{(A.3)} \qquad \lim_{n\to\infty} P(M^r, n^2)/n = r.$$

Let $M_0 = \{M^r : r \in (3/4, 1)\} \subset M$. Then $\mathcal{M}_0$ is uncountable since $M^{r_1} \neq M^{r_2}$ for $r_1 \neq r_2$. For each $M^r \in \mathcal{M}_0$, by Lemma 1, there exists a point x_r with $F^n(x_r) \in M_n^r$ for all n. Let $S = \{x_r : r \in (3/4, 1)\}$. Then S is also uncountable. For $x \in S$, let $P(x, n)$ denote the number of i's in $\{1, \cdots, n\}$ for which $F^i(x) \in K$. We can never have $F^k(x_r) = b$, because then x_r would eventually have period 3, contrary to (A.2). Consequently $P(x_r, n) = P(M^r, n)$ for all n, and so

$$\rho(x_r) = \lim_{n\to\infty} P(X_r, n^2) = r$$

for all r. We claim that

(A.4) for $p, q \in S$, with $p \neq q$, there exist infinitely many n's such that $F^n(p) \in K$ and $F^n(q) \in L$ or vice versa.

We may assume $\rho(p) > \rho(q)$. Then $P(p, n) - P(q, n) \to \infty$, and so there must be infintely many n's such that $F^n(p) \in K$ and $F^n(q) \in L$.

Since $F^2(b) = d \leqq a$ and F^2 is continuous, there exists $\delta > 0$ such that $F^2(x) < (b+d)/2$ for all $x \in [b-\delta, b] \subset K$. If $p \in S$ and $F^n(p) \in K$, then (A.2) implies $F^{n+1}(p) \in L$ and $F^{n+2}(p) \in L$. Therefore $F^n(p) < b - \delta$. If $F^n(q) \in L$, then $F^n(q) \geqq b$ so

$$|F^n(p) - F^n(q)| > \delta.$$

By claim (A.4), for any $p, q \in S$, $p \neq q$, it follows

$$\limsup_{n\to\infty} |F^n(p) - F^n(q)| \geqq \delta > 0.$$

Hence (2.1) is proved. This technique may be similarly used to prove (B) is satisfied.

Proof of 2.2. Since $F(b) = c$, $F(c) = d \leqq a$, we may choose intervals $[b^n, c^n]$, $n = 0, 1, 2, \cdots$, such that

(a) $[b, c] = [b^0, c^0] \supset [b^1, c^1] \supset \cdots \supset [b^n, c^n] \supset \cdots$,
(b) $F(x) \in (b^n, c^n)$ for all $x \in (b^{n+1}, c^{n+1})$,
(c) $F(b^{n+1}) = c^n$, $F(c^{n+1}) = b^n$.

Let $A = \bigcap_{n=0}^{\infty} [b^n, c^n]$, $b^* = \inf A$ and $c^* = \sup A$, then $F(b^*) = c^*$ and $F(c^*) = b^*$, because of (c).

In order to prove (2.2) we must be more specific in our choice of the sequences M^r. In addition to our previous requirements on $M \in M$, we will assume that if $M_k = K$ for both $k = n^2$ and $(n+1)^2$ then $M_k = [b^{2n-(2j-1)}, b^*]$ for $k = n^2 + (2j-1)$, $M_k = [c^*, c^{2n-2j}]$ for $k = n^2 + 2j$ where $j = 1, \cdots, n$. For the remaining k's which are not squares of integers, we assume $M_k = L$.

It is easy to check that these requirements are consistent with (A.1) and (A.2), and that we can still choose M^r so as to satisfy (A.3). From the fact that $\rho(x)$ may be thought of as the limit of the fraction of n's for which $F^{n^2}(x) \in K$, it follows that for any $r^*, r \in (3/4, 1)$ there exist infinitely many n such that $M_k^r = M_k^{r^*} = K$ for both $k = n^2$ and $(n+1)^2$. To show (2.2), let $x_r \in S$ and $x_{r^*} \in S$. Since $b^n \to b^*$, $c^n \to c^*$ as $n \to \infty$, for any $\varepsilon > 0$ there exists N with $|b^n - b^*| < \varepsilon/2$, $|c^n - c^*| < \varepsilon/2$ for all $n > N$. Then, for any n with $n > N$ and $M_k^r = M_k^{r^*} = K$ for both $k = n^2$ and $(n+1)^2$, we have

$$F^{n^2+1}(x_r) \in M_k^r = [b^{2n-1}, b^*]$$

with $k = n^2+1$ and $F^{n^2+1}(x_r)$ and $F^{n^2+1}(x_{r^*})$ both belong to $[b^{2n-1}, b^*]$. Therefore, $|F^{n^2+1}(x_r) - F^{n^2+1}(x_{r^*})| < \varepsilon$. Since there are infinitely many n with this property, $\liminf_{n\to\infty} |F^n(x_r) - F^n(x_{r^*})| = 0$. $\square$

REMARK. The theorem can be generalized by assuming that $F\colon J \to R$ without assuming that $F(J) \subset J$ and we leave this proof to the reader. Of course $F(J) \cap J$ would be nonempty since it would contain the points a, b, and c, assuming that b, c, and d, are defined.

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