Catastrophe Theory*

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Ladies and gentlemen, it is privilege and pleasure to speak in honour of René Thom. It is also a very European occasion that you invite an Englishman to deliver the laudatio for a Frenchman receiving an honorary degree at a German university.

As Professor Güttinger has observed, the unifying factor behind Thom's work, behind his mathematics, his science and his philosophy, is that he is a geometer. Now a geometer needs an overall picture, a "Gestalt" point of view; he needs to classify his forms so that he can turn them over in his hand, and develop an affection for them until they are old friends. This is exactly what Thom does both in his early great work on manifolds prior to his Fields Medal [6] *in 1958, and also in his later work on singularities and applications, which has become known as catastrophe theory, and which (in my opinion) will prove to be even more far-reaching.*

But before I get on to catastrophe theory, let me first describe briefly René Thom's early work on the classification of manifolds up to cobordism. Two n-manifolds X and Y are said to cobound if together they bound an (n+1)-manifold.

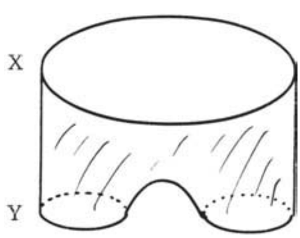

Figure 1: Cobounding manifolds

For example in the picture n=1, X is the circle at the top, Y consists of the two circles at the bottom, and together they bound the pair of trousers. This is an equivalence relation; let Ω^n denote the set of equivalence classes, and $\Omega=\Sigma\Omega^n$. Then Ω can be given a ring structure, defining addition by disjoint union, subtraction by reversal of orientation, and multiplication by the cartesian product. Of course, what made this definition significant was the fact that Thom was able to compute the ring in terms of the homotopy groups of an auxilary space manufactured out of the orthogonal group.

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* Verbatim laudatio delivered by E.C. Zeeman in honour of R. Thom on the occasion of the latter being awarded an honorary doctorate by the Faculty of Physics, University of Tübingen, in 1978.

W. Güttinger et al. (eds.), *Structural Stability in Physics*

To appreciate the significance of this achievement, you should be aware of the geometric despair of topology around the 1950's, because at that time the subject had been all but gobbled up by algebra. Surfaces (= 2-manifolds) had been classified in the nineteenth century, and when Poincaré began his attempt to classify 3-manifolds in 1899 he ran up against the famous Poincaré Conjecture: is a simply connected 3-manifold a sphere? The failure to solve this conjecture - indeed it is still unsolved today - blocked all further attempts at classification. Meanwhile topologists began to bypass the problem: they invented the tools of algebraic topology, the homology and homotopy groups, which proved so rich and splendid that topologists were seduced away from the real subject matter of topology and in effect all became algebraists. I remember as a student in 1951 the first time I heard Henry Whitehead lecture he confessed rather sadly that topologists had given up studying the homeomorphism problem because it was too difficult. Moreover this tendency was exaggerated in France, where algebra reigns supreme. So you must imagine Thom's predicament as a geometer, starting out as a lone figure swimming against the stream and in struggling to express what homology really meant in geometric terms, he had the temerity to cut right through this despair, by classifying the cobordism classes of manifolds. This was the first step towards the general classification of manifolds, which is still very much at the centre of research in topology today. It was not that he avoided the algebra; on the contrary he used it with mastery, but as a servant to the geometry, which is its proper place in topology.

After his Fields Medal, Thom retired into his shell for a while, for nearly a decade, to think about the next step. What was the next step? Whenever one has a category in mathematics, the first step is to study the objects, the next step is to study the maps, then the automaps, and so on. Therefore in the category of smooth manifolds the first step was the classification problem, the next step was to study smooth maps between manifolds, then the diffeomorphism problem, and so on. The study of smooth maps resolved itself into a local problem, singularity theory, and a global problem of how these singularities fit together. During the last two decades singularity theory has flourished, starting with Whitney, followed by Thom, who was responsible for several of the fundamental concepts concerning stability, transversality, unfolding, etc., and who in turn stimulated Arnold, Malgrange, Mather and many others. Meanwhile the global problem has hardly been touched upon as yet; as Thom puts it, we need to develop a language, in which the phonemes are singularities, the words are configurations of singularities determined by sections of higher dimensional singularities, and in which the grammar is the constraint imposed by the global structure, including the algebraic topology (illustrated for instance by the Thom polynomials).

At the same time as studying singularities, Thom's established position in mathematics gave him the leisure, authority and opportunity to think about the creation and evolution of forms in nature, which had always fascinated him and been close to his heart. Gradually he put these two ideas together, on the one hand the mathematics of stability, singularities and dynamical systems, and on the other hand the applications to morphogenesis in several branches of science. Each idea fed upon and stimulated the other, and the result was his remarkable book "Structural Stability and Morphogenesis", [11]. This is primarily a work in the philosophy of science, but, unlike most works in that field, is oriented towards the future rather than the past: it not only offers novel approaches, but reassesses the objectives of applied mathematics, and proposes broad programmes for future research in both mathematics and the physical and biological sciences. It is this mixture of philosophy of science, applications, programmes of research, theorems and conjectures drawn from several related branches of mathematics, that has become loosely known as catastrophe theory in the popular press. Now, the main subject of my talk is to explain a little of catastrophe theory to those of you who have not heard it before, and so, at the risk of boring those who already know it, I will start with an elementary introduction.

The name "catastrophe" refers to the unexpectedness of discontinuous effects when they are produced by continuous causes, thus violating our intuition, which would

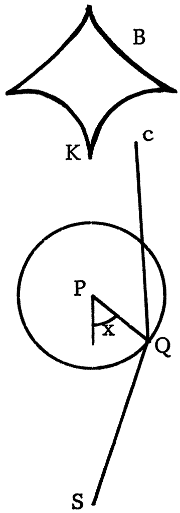


Figure 2: Catastrophe Machine

normally lead us to expect continuous effects. The little machine shown in Figure 2 is designed to illustrate this phenomenon. It consists of a disk, freely pivoted at P, with two elastic bands attached at Q, one of which has its other end fixed at S. The other end of the other elastic band is held at a point c, which we call the control point. The disk seeks a stable equilibrium position, which is a local minimum of the energy V_c in the elastic bands; we measure this position by the angle x = QPS. Here x lies in a circle, which we denote by X. Let C denote the plane. If we move c smoothly around in C, then most of the time the disk responds by moving smoothly, but sometimes it jumps. The jumps occur when c crosses a diamond-shaped curve B, which has four cusps; moreover the jumps only occur upon exit after entering from the other side. The reason for the jumps is that when c lies inside B then V_c has two local minima, but when c lies outside B then V_c has only one minimum. As c grosses B going outwards, the one minimum disappears (as in Figure 3) and so a stable equilibrium breaks down; if the disk happened to be sitting in that equilibrium, then it will jump into the other surviving stable equilibrium. B is called the bifurcation set, because that is the set above which the equivalence class of V_c changes, and hence the qualitative nature of the statics and dynamics change.

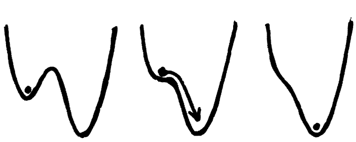


Figure 3: Energy Graphs

If we plot the stationary values of V in XxC, for all c$\in$C, we obtain the graph M of cause and effect, where c = cause and x = effect. Restricting attention to a neighbourhood of the cusp point K in C, and a neighbourhood of 0 in X, the resulting subset of the graph is shown in Figure 4. The equilibria form a smooth surface M, single-sheeted over the outside of B, triple-sheeted over the inside of B, with the upper and lower sheets corresponding to stable equilibria (local minima of V) and the middle shaded sheet corresponding to unstable equilibria (local maxima of V); the projection M $\rightarrow$ C has fold singularities over B-K and a cusp singularity

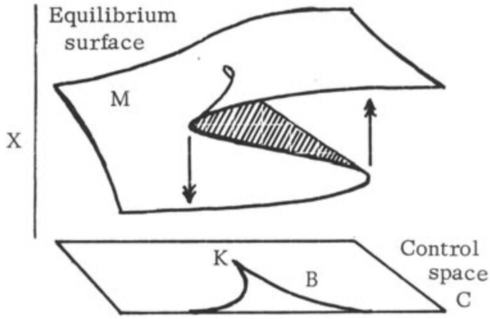


Figure 4: Cusp Catastrophe

over K. The two vertical arrows represent the catastrophic jumps exhibited by the disk when the control point is moved to the left and right across the front of the picture. The significance of the surface in Figure 4 is contained in the following theorem.

Elementary Catastrophe Theorem (Thom, Mather).

Given an n-dimensional behaviour space X, a 2-dimensional control space C, and a smooth locally-stable function V on X parametrised by C, then the stationary points of V form a smooth 2-dimensional surface $M \subset X \times C$, and the only singularities of the projection $M \to C$ are folds and cusps. Moreover, most functions are locally stable (more precisely, there is an open dense set of them of measure one in the space of all smooth functions).

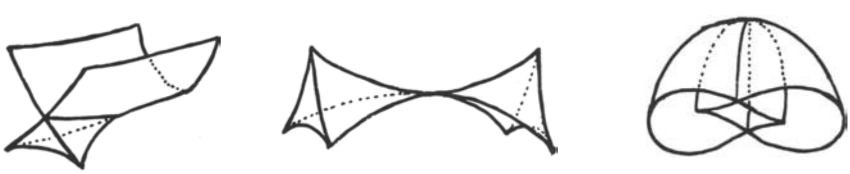
In other words Figure 4 illustrates the most complicated possible local structure of M; of course globally there may be many cusps - for instance there are four in Figure 2. Since the hypothesis is so general, one would expect the surface illustrated in Figure 4 to crop up frequently, as indeed it does, in many branches of science. The theorem also extends to higher dimensional control spaces.

Elementary Catastrophe Theorem (continued)

If C is k-dimensional then M is a smooth k-manifold, and the number of types of singularities is given by:

k	1	2	3	4	5	6	...
number of new singularities	1	1	3	2	4	4	...

When k = 1 only the fold appears; when k = 2 the cusp appears. When k = 3 three new singularities appear, whose bifurcation sets are shown in Figure 5.



Swallowtail Elliptic umbilic Hyperbolic umbilic

Figure 5: Bifurcation sets, k=3

The first surprising thing about the elementary catastrophe theorem is that classification should be possible at all; there are relatively few branches of mathematics that are sufficiently clear for the stable objects to be dense. But it is exactly this stability that is so desirable in applied mathematics for scientific modelling.

The next surprise is the fewness of singularities on Thom's list. With a little imagination one can visualise all sorts of the ways for M to map onto C, so why don't they occur? For example why can't we have a winding point, such as occurs in the map $z \to z^2$ of the complex plane? The answer is that this singularity is unstable, as Whitney [13] showed, because the perturbation $z \to z^2 + \varepsilon\bar{z}$ converts the singular point at the origin into a bifurcation set with three cusps, in fact a triangular hypocycloid.

Whitney's work in the 1950's on the classification of stable singularities was the inspiration for Thom's theory.

During the 1960's Thom introduced the essential concepts and laid down the main structure for the proof of the elementary catastrophe theorem. He persuaded others to provide the necessary subsidiary results, and fill in details of proofs: Malgrange proved the preparation theorem, and Mather proved the uniqueness of unfoldings. A complete exposition of the proof for $k \leq 5$ can be found in [14]. Thom emphasised the first seven singularities corresponding to $k \leq 4$ because he was particularly interested in applications to embryology, in which C represented 4-dimensional space-time. In dimension $k \geq 5$ the classification has been considerably extended by Arnold [1] and the Russian school.

At first sight it seemed as though Thom's classification was the same as Whitney's, because Whitney had already proved that the only stable singularities of maps between surfaces are folds and cups; it turns out that the two theories are subtly different, because the surface M that appears in the elementary catastrophe theorem above is not an arbitrary surface, but is constrained to be the critical set (= set of stationary points) of a function V. In other words Thom classifies singularities that are stable under this constraint. Although the constraint makes no difference when $k = 2$, it causes the two theories to diverge when $k \geq 3$. Most of Whitney's singularities do not appear on Thom's list since they cannot satisfy the constraint, and conversely most of Thom's singularities do not appear on Whitney's list because if the contraint is removed they become unstable. For example if $k = 3$, the three singularities on Thom's list are shown in Figure 5, but when the constraint is removed then the two umbilics become unstable, leaving only the swallowtail on Whitney's list. In order to distinguish between the two lists, and since Whitney had already pre-empted the term <u>stable singularities,</u> Thom called his list the <u>elementary catastrophes.</u> Arnold <u>prefers to call them Lagrangian singularities</u> <u>because the same singularities</u> also appear in the more <u>specialised context of</u> Lagrangian manifolds in symplectic geometry.

In fact there are good reasons for both points of view, which we now explain. In many applications there is, in addition to the function V, an associated dynamic that locally minimises V (in other words a dynamic for which V is a Lyapunov function), making the critical set M into a set of equilibria. For example in the machine described above, V is the energy in the elastic bands, given by Hooke's law, and the associated dynamic is given by Newton's law and friction, in other words by the damped Hamiltonian system (see below). The presence of a dynamic has several consequences: it implies that some points of M will be attractors, and other points repellors, like the shaded middle sheet in Figure 4. It implies that when the control point crosses a fold, going in the right direction, then there will be a breakdown of equilibrium, and a catastrophic jump to another equilibrium. It implies delay and hysteresis. To emphasise all these additional qualities Thom coined the term "elementary catastrophe".

The word elementary refers to the fact that the associated dynamic is gradient-like. Non-elementary catastrophes can occur in non-gradient dynamics; the classical example is the Hopf bifurcation, where an attractor point bifurcates into a repellor point inside a small attracting cycle, as occurs for instance at the onset of aircraft flutter, or a bicycle speed wobble. Recent work of Takens [10] on bifurcations of differential equations in the plane shows that the Hopf bifurcation is so inextricably intermingled with the elementary catastrophes that they must be considered as belonging to the same theory. Thus in this context (non-elementary) catastrophe theory is the same as multiparameter bifurcation theory, although it would perhaps be more accurate to call them both a programme of research rather than a theory, because in higher dimensions knowledge is still at the level of exploring examples, and as yet far from any general classification.

Returning to elementary catastrophe theory, in some applications there is a global minimising principle of the function V (rather than a local minimising principle like a dynamic) causing the system to seek the absolute minimum of V rather than a local minimum - for example in the maximisation of entropy in phase transition, or in the minimisation of risk in Bayesian decision making [9]. In these applications the equilibrium surface is again decomposed into attractors and repellors, and also meta-attractors, corresponding to meta-stable states where the minimum is local but not global. Again there are catastrophic jumps (except that these now occur at Maxwell points rather than at bifurcation points). Therefore it is again appropriate to use Thom's term "elementary catastrophes" in these contexts.

However, there are other applications in which there is no minimising principle. Therefore there are no catastrophic jumps. Nor is there any meaningful distinction between the maxima, minima and saddles of V, and therefore no meaningful decomposition of M into attractors and repellors (the middle sheet in Figure 4 should no longer be shaded). In fact M should no longer be called an equilibrium set, but more appropriately a critical set. If, furthermore, M has arisen from Hamiltonian-Jacobi equations, then it is more appropriate to follow Arnold's terminology and call the singularities Lagrangian. The classical example is light caustics; Figure 5 shows the three possible types of singular point that a generic 3-dimensional light caustic can have. Incidentally it was while experimenting with light caustics in the early 1960's that Thom first discovered catastrophe theory. According to Whitney's list he expected to find only the swallowtail, and in addition he found the two umbilics appearing stably, to his surprise, until he realised the nature of the mathematical constraint that was in effect being imposed by Fermat's principle (or equivalently the method of stationary phase). Thus light caustics gave birth to catastrophe theory; and today in return, catastrophe theory is breathing new life into optics. On the theoretical side the knowledge of the Lagrangian singularities has improved the understanding of oscillatory integrals [4]. Berry and Nye have extended the classical interference patterns of Airy (for the fold) and Pearcey (for the cusp) to higher singularities. Nye [8] has done beautiful experiments revealing the geometry of important 6- and 8-dimensional singularities through their sections, by shining laser beams through water droplets. Berry [3] had used caustics to explain the twinkling of star-light, and to predict how the n-th moment of the intensity depends upon some exponent of the wave number. This is perhaps the most sophisticated application of catastrophe theory to date, since to calculate the exponents he needed to use Arnold's complete list.

Going back to the elementary catastrophe theorem above, one of the surprising features is that the result is independent of n. This enables it to be used in contexts where n is large and k is small. For example there might be implicitly too many behaviour variables (n large) to measure or compute, but nevertheless the behaviour might be modellable by an explicit low dimensional manifold M^k (k small) using only a few of those variables that happened to be convenient to measure. For example in an economic model n might be 10^3 to represent the important variables in an economy; in an embryological model n might be 10^4 to represent protein concentrations in a cell; in a brain model n might be 10^{10} to represent neuronal activity; in a physical model n might be infinite to represent possible eigenvalues.

Another point in the elementary catastrophe theorem that is worth drawing attention to is the little word "are" in the phrase "the only singularities are folds and cusps". What does it mean to say that a singularity "is" a cusp? If we look up the formula for the cusp catastrophe we find some nice little canonical polynomial such as $V = ax + bx^2 + x^4$, where $c = (a,b)$ is the control point and x the behaviour variable. On the other hand if we actually calculate V in some specific application, for example, if we work out the energy in the elastic bands in the machine described above, we obtain a Taylor series $V = \Sigma\lambda_i x^i$, where the coefficients λ_i depend upon c. I well remember as a student struggling, and failing, to acquire the proper applied mathematician's approach towards approximating a Taylor series by its first few terms; but even on those rare occasions when I could prove convergence, and satisfactorily estimate that boundedness of the error, I confess I could never cheerfully forget that small truncated tail of the Taylor series. Perhaps this intolerance of imperfection springs from too great an affection for the perfection of geometric forms, a penalty that any geometrically minded mathematician may have to pay. One of the attractions of theorems like the elementary catastrophe theorem is that it resolves this difficulty. In effect it says that if at a certain control point, the first non-vanishing Taylor coefficient is λ_4, then there is a smooth change of coordinates with respect to which V will have the canonical form; there is no need to approximate because the Taylor series is a polynomial of degree 4, all higher terms being zero. In other words if the problem satisfies a computable algebraic test, then there are coordinates with respect to which it has a perfect geometric form. This may explain why certain physical laws can take polynomial form, even in the absence of symmetry, when at first sight polynomials would appear to be much too special. No doubt, the aesthetic appeal of this approach will attract more geometrically minded mathematicians towards applied mathematics, and I would venture to suggest that this is an ingredient of Thom's own mathematical taste.

Of course, to make quantitative estimates and quantitative predictions in terms of the original coordinates one may have to return to the original Taylor series, but to obtain qualitative understanding and make qualitative predictions one can utilise the known geometry of the canonical models. Also different problems may be "solved" at different levels of detail. For instance, it may be possible to solve

I. the full dynamics
II. only the statics (and not the dynamics)
III. only the bifurcation set (and not even the statics)
IV. only statistical properties.

We illustrate the four levels by examples.

I. Consider a parametrised damped Hamiltonian system; for each value of the parameter (or control) the damping causes the system to seek a local minimum of the Hamiltonian. A simple example is given by the catastrophe machine described above. Before we do the mathematics, let me tell a story about a physicist friend of mine, who made a machine for himself; only instead of using a cardboard disk and drawing pins like a mathematician would, he got his departmental workshop to make it, and they used heavy steel disk pivoted on high quality ball bearings. Of course, it didn't work, because when the control point crossed the bifurcation set, instead of jumping into the other equilibrium position, the disk went into large steady oscillations. When I laughingly pointed out that it didn't work, he replied that it was my mathematics that didn't work, because it didn't describe his physical model. Of course, the correct mathematics does describe both his machine and my machine; they have the same statics, and qualitatively similar dynamics, but differ quantitatively in the size of the friction term, as follows.

The configuration space of the disk is a circle, Q say. The state space of the disk, $X = T^*Q$, is the cotangent bundle of Q, which is topologically a cylinder. The state of state of the disk is given by a point $x \varepsilon X$. We can write x in coordinates, $x = (p,q)$, where $q \varepsilon Q$ denotes the position of the disk and $p \varepsilon T^*_q Q$ denotes the angular momentum. The Hamiltonian $H{:}X \times C \to R$ is given by

$$H(x,c) = V_c(q) + \frac{p^2}{2I}$$

where $V_c(q)$ is the potential energy in the elastic when the control point is at c and the disk at q, and $p^2/2I$ is the kinetic energy, I being the moment of inertia of the disk. The dynamic is given by the damped Hamiltonian equations

$$\dot{q} = \frac{\partial H}{\partial p} = \frac{p}{I}$$

$$\dot{p} = -\frac{\partial H}{\partial q} - \varepsilon p = \frac{\partial V}{\partial q} - \varepsilon p \quad ,$$

where $\varepsilon > 0$ denotes friction. Therefore the dynamic locally minimises H, because $\dot{H} = -\varepsilon p^2/I < 0$, if $p \neq 0$.

If the damping, ε, is large then the behaviour is dominated by the statics: the system homes rapidly towards the equilibrium surface M, which is given by

$$\frac{\partial H}{\partial p} = \frac{\partial H}{\partial q} = 0, \quad \text{in other words by } p = 0, \ \frac{\partial V}{\partial q} = 0.$$

Now M is contained in the 4-dimensional space T*QxC, but since p = 0 it is actually sitting inside the 3-dimensional subspace QxD, where Q is identified with the zero section of the bundle T*Q. Therefore if we are only interested in the statics, in other words in the equilibria and the catastrophic jumps (rather than the full dynamics), then it suffices to consider only the embedding $M \subset QxC$, given by $\partial V/\partial q=0$, as illustrated in Figure 4. The advantage of Figure 4 is that it gives good geometric intuition of the statics, but the disadvantage is that it does not contain the dynamics. For instance, a catastrophic jump really looks like Figure 6, the damped oscillations being the image, under the projection $T^*Q \to Q$, of an orbit spiralling in towards the attractor, as in my physicist friend's machine.

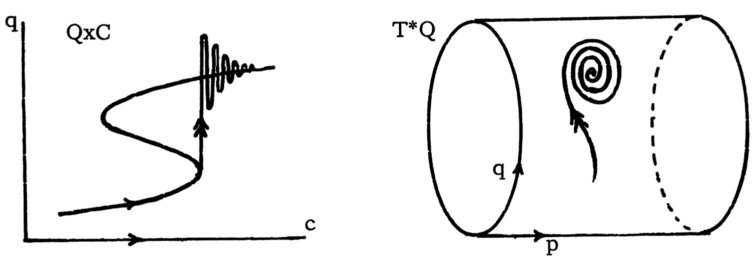


Figure 6: Damped Hamiltonian Catastrophe

Exactly the same equations and geometry occur in the rolling of a ship in still water [14]. Here the parameter c represents the position of the centre of gravity of the ship, and the diamond-shaped bifurcation set represents the evolute of the buoyancy locus determined by the shape of the hull; the catastrophic jump corresponds to capsizing. Modern ships have a more rectangular hull, which produces a butterfly-catastrophe modification of the bifurcation set, implying greater stability. More complicated singularities occur in the 3-dimensional problem, for instance in the study of the stability of icebergs. To represent the effect of waves a forcing term must be added to the equations, and in this case the dynamics is as yet unsolved; moreover, this is a serious problem because even today the number of ships that capsize in heavy weather is still high. Whether or not our improved understanding of the singularities of the statics in still water can lead to a better qualitative understanding of the dynamics in the seaway remains to be seen.

II. In some problems one is more interested in the statics than the dynamics, and here the classification of higher dimensional singularities is proving to be useful. For example, when looking at the von Karman equations for a buckling plate, Magnus and Poston [7] identified a double-eigenvalue as a double-cusp catastrophe, and knowing that the latter was 8-dimensional, were able to select 8 parameters that gave a full unfolding of the equations. By imposing symmetries on the problem Golubitsky and Schaeffer [5] have given a 1-dimensional unfolding.

III. In some problems both the statics and dynamics are insoluble, but nevertheless the bifurcation set B in the parameter space C may be not only identifiable, but measurable and usable for quantitative prediction. For example Benjamin [2] observed the formation of Taylor cells in viscous fluid flow inside a cylinder with D-shaped cross section, the curved part of whose boundary was being rotated with constant velocity. He used two parameters

a = rotation of boundary ($\sim$ Reynolds number)
b = length of cylinder.

The state space X is the infinite dimensional space of all possible flows, and the dynamic on X is the evolution determined by the Navier-Stokes equations, which of course depend upon the parameters. The equilibrium states were observed to be stable steady flows, and the set of equilibria formed a cusp catastrophe surface M over the parameter space, as shown in Figure 7.

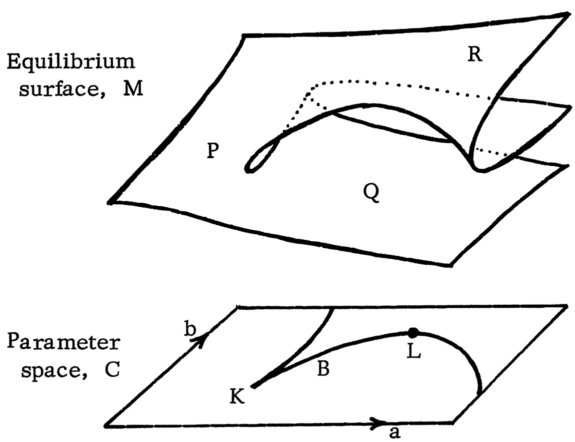


Figure 7: Fluid flow and embryological determination

The regions P, Q, R on M correspond to 0, 2, 4 Taylor cells respectively. When the rotation of the boundary was altered then the flow adapted smoothly, unless the bifurcation set was crossed in the appropriate direction, in which case a catastrophic jump occured, consisting of the cell pattern breaking up and the flow resetling relatively quickly into the other stable pattern. In particular, the behaviour exhibited hysteresis near the cusp point, since, as to be expected from generic considerations, the cusp axis was not parallel to the a-axis. Thus although neither the dynamics (the evolution) nor the statics (the steady state flow in a bounded cylinder) were soluble - the Navier-Stokes equations being as yet too difficult - nevertheless the bifurcation set was observable, measurable and predictable.

Exactly the same picture occurs in a model for differentiation in embryology [14]. Here the parameters a, b represent time, space. X represents the possible states of a biological cell, and the dynamics on X represents the homeostasis of the cell, which of course depends upon the parameters of its position, a, in the embryo and the stage, b, of development of the embryo. Neither the dynamics nor the statics is measurable, but the three qualitatively different types of cell P, Q, R were observable. For example, if the model is applied to gastrulation, P represents blastoderm before differentiation while Q and R represent mesoderm and ectoderm after differentiation. This application is due to Thom [11], and my modification was to turn the cusp axis so that it was generically not parallel to the time-axis (the a-axis in Figure 7). Wasserman [12] confirmed that the modified model was locally stable in the group of diffeomorphisms of state-space-time, XxSxT, preserving the projections XxSxT → SxT → S, in other words the group leaving invariant the basic concept of the developmental path of a cell. Furthermore, under a suitable simplicity hypothesis, it can be shown that Figure 7 is unique up to global diffeomorphism in the group.

The interesting consequence of this result is that the arc KL of the bifurcation set B represents catastrophic switches of developing cells from R to Q, in other words a wave of Q-determination sweeping through the embryo. In the case of gastrulation this can be identified with the wave of mesoderm determination, which can be measured and predicted in space-time. Moreover it appeared to be related to subsequent morphogenetic movements. When combined with a clock, it gave a simple model for somite formation, that explained regulation, and led to predictions that were subsequently confirmed by experiment [14]. Again we emphasise, here is an application of catastrophe theory in which neither the statics nor the dynamics is measurable, but the bifurcation set is.

IV. The most striking result concerning the statistics of singularities is Berry's application to twinkling of stars [3], mentioned above. The intensity of starlight is affected by many caustics arising from many waves in the turbulent atmosphere, and so it is necessary to average or integrate their effects. By analysing the oscillatory integral associated with a singularity Berry isolated two indices, α derived from the germ of the singularity, measuring the intensity of light at the caustic, and β derived from the unfolding of the singularity measuring the width of the caustic, or in other words the duration of its effect. Thus the contribution of that type of singularity to the average intensity is proportional to $\alpha\beta$, and the contribution to the n-th moment is proportional to $\alpha^n\beta$. Since the indices are proportional to the wave number raised to some exponent, each moment will be dominated by the random appearances of that singularity with the highest exponent. Thus statistical considerations led, surprisingly, to a particular singularity for each n, and the precise computation of rational exponents.

I have tried to illustrate the breadth of the impact of Thom's work by giving a brief selection from amongst the hundreds of different applications of catastrophe theory. Naturally, in each field of application the specialist has gone further than Thom himself, but I am sure that Thom's own breadth of vision enabled him to see from the beginning both the central position that his work would occupy in mathematics, and its potential use in all branches of science.

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