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Catastrophe theory in physics

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Catastrophe theory in physics

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Abstract

We present a broadly based discussion of 'catastrophe theory', a mathematical discipline commonly associated with the names of Thom and Zeeman, placing emphasis on the developmental feedback between the mathematics and its applications, especially to the physical sciences. While the paper is in no sense a comprehensive survey of the theory and its applications, it aims to present a typical selection of current work.

Among the more prominent of the concepts that have emerged from this work are co-dimension, determinacy, unfoldings and organising centres. We show, using specific applications as motivation, how these concepts may be used, and generalised to areas not obviously within the formal purview of 'catastrophe theory' as it is often presented. Structural stability, a concept from topological dynamics that provides a philosophical background for catastrophe theory, is also discussed.

The mathematics of the subject has advanced considerably over the past decade, and in doing so has lost many of its original limitations. Some of these new directions are exhibited. On occasion, the catastrophe-theoretic methods are compared to more traditional ones.

Technical details within the mathematics are ignored as far as appeared wise: the object is to convey the *spirit* of the mathematics.

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1. Introduction

The subject loosely known as 'catastrophe theory' grew from the morphogenetic speculations of Thom (1972, 1975) and their development and exposition by Zeeman (e.g. 1977). Its ancestry can be traced back further—for example, to the work of Archimedes on floating bodies—since it sits firmly in the general area of non-linear phenomena, where there is no shortage of 'pre-history'. In particular, Thom's ideas owe much to the theory of topological dynamics developed by the Russian and American schools in the 1960s, notably by Smale.

Unlike much modern mathematics, catastrophe theory was developed with a view (albeit sometimes speculative) to applications in the sciences. For various reasons it attracted what for mathematics was an unusual level of criticism and comment; not all of this has proved productive. By now, it has found a broad range of successful applications in physics and is beginning to demonstrate its potential in biology. The more tentative suggestions for social and economic modelling remain interesting but speculative, bedevilled by the same problems that afflict all attempts to apply mathematics to these areas.

This paper is not a comprehensive survey. A number of surveys and texts exist (Poston and Stewart 1978, Zeeman 1980, Berry and Upstill 1980, Gilmore 1981, Stewart 1981) and an extensive bibliography has been prepared by Zeeman and Wetherilt (1981). The aim here is to demonstrate how catastrophe theory has responded to the needs of applications by modifying its technical standpoint while preserving its basic mathematical conceptual view. We concentrate on the physical sciences (to which, popular misconceptions notwithstanding, it has been most widely applied).

Differential topology is a technical discipline not easily penetrated by non-specialists. For ease of exposition we have ignored many mathematical fine points. *The reader who desires complete technical details must look elsewhere.* The object is to capture the *spirit* of the mathematics, and to recognise its role in applications.

For the same reason, the applications presented are chosen to illuminate the mathematical ideas, and to indicate the kind of work that has been done. There are many different ways to use a given mathematical technique; and catastrophe theory possesses many techniques. It is not *limited* to the types of application presented below; but these are fairly typical of what is being done at the moment. No doubt, if the phrase 'catastrophe theory' is taken in a sufficiently restricted sense, the subject thus denoted will be so severely restricted that its utility will be vanishingly small; but it is clear that on any reasonable interpretation the subject has matured to a point at which it can demonstrate several successes, and a growing potential.

2. Structural stability

We begin with an important concept from topological dynamics. Consider a dynamical system, i.e. a system of first-order autonomous differential equations

$$\mathrm{d}x_i/\mathrm{d}t = F_i(x_1, \ldots, x_n) \qquad (i = 1, \ldots, n) \tag{2.1}$$

where the x_i are local coordinates on an n-dimensional manifold M (*phase space*). The integral curves, or *trajectories*, define a *flow* on M, and the configuration of flowlines is the *phase portrait* (for details, see Hirsch and Smale (1974) or Abraham and Marsden (1978)). The aim of topological dynamics is to obtain information about the phase portrait by direct topological means, as opposed to the classical approach by approximate analytic solutions.

The emphasis is less on ways to analyse specific systems, and more on obtaining a general overview. An ambitious project here would be *to classify all possible dynamical systems* up to some kind of topological equivalence. But the world of dynamical systems is a jungle, and the variety and pathology apparently endless. A certain amount of order can be discerned by asking for less: *is there a class of systems which is* (*a*) *simple enough to classify, and* (*b*) *complicated enough to be typical*? This, sometimes called the *yin–yang problem* because of its conflicting yet complementary requirements, may also prove too ambitious; but it is a step in the right direction.

The first serious candidate has turned out to be of independent interest. Say that a system (2.1) is *structurally stable* if its phase portrait is not changed (up to topological equivalence) by sufficiently small perturbations of the F_i. For systems in the plane, this notion goes back to Andronov and Pontryagin (1937). Its deeper mathematical significance was recognised by Lefschetz (1957). Peixoto (1959) showed that for $n = 2$ the class of structurally stable systems solves the yin–yang problem. Smale (1967) defined structural stability in n dimensions, and made it the basis of his influential contributions to topological dynamics, but also showed in 1966 that it no longer solves the yin–yang problem. There are many variations, and the crucial issues are not all resolved, but the concept remains of central importance.

A structurally stable system preserves its basic form when its *equations* are perturbed: it is robust, not only to small changes in initial data, but to small changes in its own specification. There are philosophical reasons to prefer such systems when modelling nature, especially if the model contains parameters which are to be estimated from observations; Thom (1972) advocated this stance as a guiding principle, though he was by no means the first to do so. It is wise to interpret it with intelligence, and not as an inflexible rule; but when used this way, it can pay dividends.

For later use, we exhibit some structurally stable or unstable systems.

Example 2.1. The one-dimensional system

$$\mathrm{d}x/\mathrm{d}t = x^2 \tag{2.2}$$

is structurally unstable. Its phase portrait is shown in figure 1 with a (degenerate)

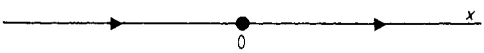

Figure 1. Phase portrait for equation (2.2) is structurally unstable.

singular point at $x = 0$. Under perturbation to

$$\mathrm{d}x/\mathrm{d}t = x^2 + \epsilon \tag{2.3}$$

this singular point either disappears ($\epsilon > 0$) to give a different phase portrait (figure 2) or splits ($\epsilon < 0$) to give figure 3 which is different again.

Figure 2. Perturbation to (2.3) with $\epsilon > 0$ destroys the singularity.

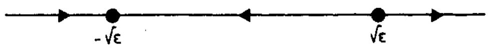

Figure 3. Perturbation to (2.3) with $\epsilon < 0$ splits the singularity into two.

If $\epsilon \neq 0$, however, then (2.3) is structurally stable: perturbations 'smaller than ϵ' preserve the phase portrait.

Example 2.2. Simple harmonic motion. Work in the plane, with

$$\mathrm{d}x/\mathrm{d}t = -y \qquad \mathrm{d}y/\mathrm{d}t = x.$$

The phase portrait consists of concentric circles, demonstrating periodic motion (figure 4). Again, this is structurally unstable. One way to see this is to transform to polar coordinates, getting

$$\mathrm{d}\theta/\mathrm{d}t = 1 \qquad \mathrm{d}r/\mathrm{d}t = 0.$$

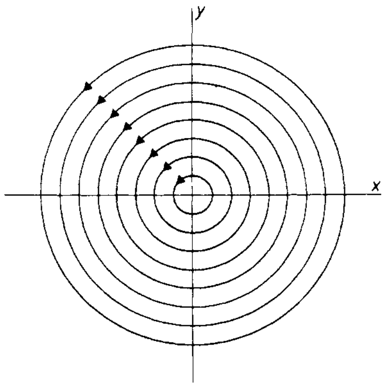

Figure 4. Simple harmonic motion is also structurally unstable.

Perturb the r equation to

$$\mathrm{d}r/\mathrm{d}t = \Phi(r)$$

where the graph of Φ is as shown in figure 5. Trajectories have constant angular velocity, composed with an outward radial movement if $r < R$, an inward one if $r > R$.

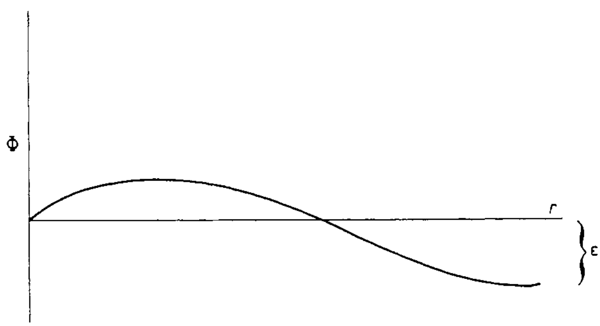

Figure 5. A possible perturbation of the radial velocity equation.

Thus the circle $r = R$ becomes an *attracting* (or *stable*) *limit cycle* (figure 6) and the topology of the phase portrait is quite different. More complicated choices of Φ with more zeros give nested limit cycles, and worse is possible. Not only is simple harmonic motion structurally unstable—it can change its topology in an infinite number of ways.

In contrast, a system like figure 6 is structurally stable, and this is intuitively clear, even if a rigorous proof is hard.

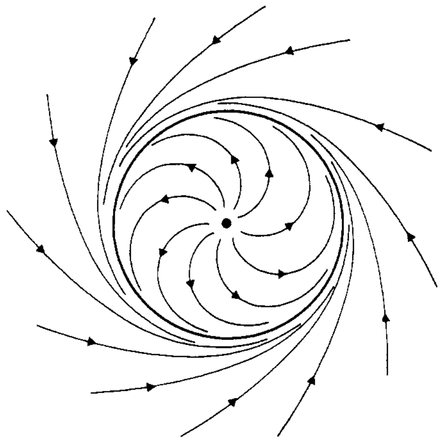

Figure 6. The resulting phase portrait has a limit cycle, and is now structurally stable.

Example 2.3. Lotka–Volterra equations. This is a simple predator–prey model (see May 1978). For positive constants a, b, α and β let

$$dx/dt = ax - \alpha xy \qquad dy/dt = -by + \beta xy. \tag{2.4}$$

The phase portrait is like figure 4 except that the loops are not circular (see below for reasons); it is therefore structurally unstable.

Whether or not a structurally unstable system is plausible as a model of nature depends on what questions it is required to answer. (And also on the precise notion of structural stability chosen: there are many variations.) For example, simple harmonic motion has a time-honoured place in engineering and physics, and for good reasons. Are we to dismiss it on questionable philosophical grounds, because of its structural instability? To resolve the problem, we ask *why* it is structurally unstable. The fundamental reason is that it is a *Hamiltonian* system. There is a function

$$H(x, y) = \tfrac{1}{2}(x^2 + y^2)$$

and (2.4) can be written as

$$dx/dt = -\partial H/\partial y \qquad dy/dt = \partial H/\partial x.$$

In consequence, H is conserved on trajectories.

The instability is now clear. Small perturbations can destroy the Hamiltonian form that the system takes, and with it the conservation law and the phase portrait (dissipative terms lead to energy loss).

We are not claiming that no Hamiltonian system can be useful as a model of nature. But what is 'useful' depends on the context. In the long run there are few exactly Hamiltonian systems, but in the medium term (up to millions of years) many systems are better modelled by a Hamiltonian one than by one with dissipative effects. Given the wealth of evidence for energy conservation, it is necessary to deal with Hamiltonian systems in their own right. The '*correct*' notion of structural stability is then: *perturb the Hamiltonian and the topology does not change.* There are severe technical problems in making this idea precise and workable, but it is pretty clear that simple harmonic motion is structurally stable in this Hamiltonian sense.

In contrast, consider the Lotka–Volterra equations as an ecological model (where x and y are population levels of two species approximated by continuous variables). The equations are not in Hamiltonian form; but if we set

$$u = \log(\beta x) \qquad v = -\log(\alpha y)$$

they become Hamiltonian, with

$$H(u, v) = bu - e^u - av - e^{-v}.$$

This gives a conserved quantity in the (u, v) variables, hence also in the (x, y) variables; it is now easy to prove that trajectories close.

Either H is an important quantity for predator–prey interactions, or it is an artefact. While there is evidence for periodicity in animal populations, there is none to support a phase portrait like figure 4 where the system returns to any assigned initial state. A limit cycle like figure 6, with a single well-defined amplitude, more accurately represents most experimental data. We conclude that the insight into periodicity provided by the Lotka–Volterra equations may be misleading. Ecologists are well aware of this, but there is still a tendency to over-use the Lotka–Volterra equations, apparently *because* of their misleadingly simple form.

3. Bifurcation

The phenomenon of bifurcation has been known at least since Poincaré, and possesses a vast literature. For simplicity we shall consider it only for dynamical systems; this is not to deny its importance in, say, partial differential equations. (At a suitably deep level, the distinction between partial and ordinary differential equations becomes blurred: see § 9.)

Example 3.1. The pitchfork. This is a commonly encountered bifurcation. In static form, the number of zeros of the equation

$$x^3 - \lambda x = 0 \tag{3.1}$$

is either 1 or 3, depending on the sign of the parameter λ. If the solutions x are plotted against λ we obtain the familiar *bifurcation diagram* (figure 7). To obtain this

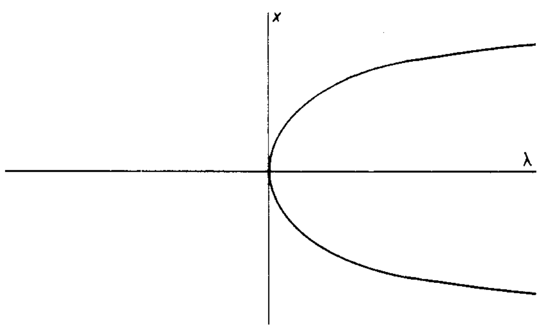

Figure 7. 'Pitchfork' bifurcation diagram for equation (3.1).

in a 'dynamic' form, we might consider stationary solutions to the differential equation $dx/dt = x^3 - \lambda x$.

Example 3.2. Limit point. We embed equations (2.2) and (2.3) above in the parametrised system

$$dx/dt = x^2 + \lambda. \tag{3.2}$$

Now there are two stationary solutions $dx/dt = 0$ for $\lambda < 0$, none for $\lambda > 0$. The point $\lambda = 0$ is a *limit point.*

Example 3.3. Hopf bifurcation. There is a general criterion for this, due to Hopf (1942), but for the moment we use a simple example taken from Zeeman (1980), written in polar coordinates as

$$d\theta/dt = 1 \qquad dr/dt = -(r^3 - \lambda r). \tag{3.3}$$

When $\lambda < 0$, the radial motion is inward, and the origin is a sink. When $\lambda > 0$, the radial component is outward for $r < \sqrt{\lambda}$, inward for $r > \sqrt{\lambda}$; the origin becomes a source and there is a stable limit cycle thrown off. The bifurcation diagram is represented in figure 8. The point $\lambda = 0$ is a *Hopf bifurcation point.*

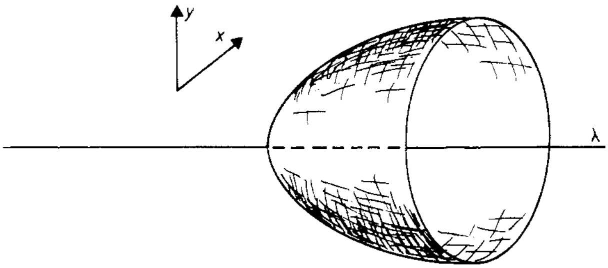

Figure 8. Hopf bifurcation. Here λ is the bifurcation parameter, and the stationary points and periodic cycles are plotted in the (x, y) plane. The amplitude grows parabolically. While the phase portrait at the bifurcation point $\lambda = 0$ is, of itself, structurally unstable, the entire process has its own kind of structural stability.

Observe that equations (3.1) and (3.3) have a lot in common: they reveal the Hopf bifurcation as a 'spun pitchfork'. This observation, which must have been made a thousand times, is actually evidence for a beautiful and powerful connection between catastrophe theory and degenerate Hopf bifurcation, recently discovered by Golubitsky and Langford (1980) and treated below in § 11.

These are the examples; but what actually *is* a bifurcation? In the classical literature, it is often defined like this. There is a 'zero solution' (here $x = 0$ or $(x, y) = 0$) which *loses stability* at some value of the bifurcation parameter λ (here $\lambda = 0$) and this leads to a *branching of solutions.* The 'zero solution' just makes one's computational life easier—example 3.2 hasn't got one, but nobody has ever denied it is a bifurcation—and the 'branching' idea does not cover everything; nevertheless, there is a persistent tendency to talk of, and think of, new solutions splitting off the original 'branch'—especially in more complicated cases where multiple branching occurs.

Catastrophe theory takes a different viewpoint, following a key idea of Smale (1970). It defines a bifurcation point to be a value of the parameter λ at which the topology of the phase portrait changes. This includes branching and limit points, but it also includes much more, and is well adapted to the startling new phenomena now being recognised as typical of dynamical systems, such as 'chaotic dynamics' (see Zeeman (1980) for a description in the spirit of this article: the idea is again due to Smale in its modern form). This implies that a *non*-bifurcation point is one at which the topology does *not* change, i.e. a point at which the system is structurally stable. Hence λ is a bifurcation point if and only if the system is structurally *un*stable for that value of λ. The difference is crucial: bifurcation is seen as a loss of (structural) stability of the *system*, rather than the loss of stability of a particular *solution*.

In Thom's terminology, a 'topological bifurcation' of the kind we have just described is called a *catastrophe*—and this is why Zeeman (1971) dubbed Thom's ideas 'catastrophe theory'. Because Thom places emphasis on discontinuous change, Zeeman (1980) has recently suggested that the terms *bifurcation* and *catastrophe* be applied to two distinct situations, depending on whether a measure associated with the system changes continuously or discontinuously. The two approaches are not so neatly divided, in practice, and it is increasingly recognised that they represent two sides of the same coin. Catastrophe theory and classical bifurcation theory complement each other; they are not adversaries.

The emphasis on structural stability leads to a capital discovery: many bifurcations (despite being caused by structural instability) *happen in a structurally stable way.* That is, *a structurally unstable event may occur within a structurally stable process.*

This is best explained by examples. If a system of equations undergoing Hopf bifurcation is slightly perturbed, it can be shown that the resulting system will still undergo Hopf bifurcation. The overall sequence of phase portraits stays the same, even though the one on the threshold of bifurcation is structurally unstable. Thus the *process* of Hopf bifurcation has its own kind of structural stability.

The same goes for the limit point bifurcation.

The pitchfork, however, lacks even this kind of stability. If the defining equation is perturbed to

$$x^2 - \lambda x + \epsilon = 0 \tag{3.4}$$

then the bifurcation diagram looks like figure 9 which, while in various ways resembling

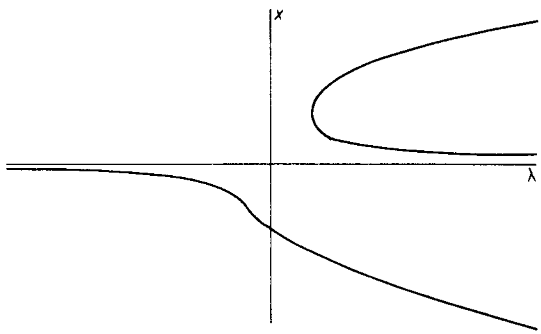

Figure 9. A perturbation of the pitchfork (figure 7) demonstrating its structural instability, even as a process.

the pitchfork, is topologically distinct from it (there is no true branching, for example, and the diagram is disconnected).

In exactly the same way that the jungle of possible dynamical systems can be tamed by considering the structurally stable ones, so can the jungle of bifurcations be tamed by using a suitable type of structural stability. This is the essence of the catastrophe theory programme: to obtain a general understanding of what kinds of bifurcations can typically occur, what kinds are special accidents requiring further explanation, and what distinguishes the two. There is more, including the first steps in carrying the programme out; but it is fair to say that on the mathematical level this is the essence. Ideally, systems more general than dynamical systems are included in the programme; in practice little is yet known (but see below).

4. From general to elementary catastrophes

While the catastrophe theory programme is broad, the earliest results were applied to a more restricted area, now known as elementary catastrophe theory. (Popular expositions tend to drop the adjective for simplicity, leaving ample scope for confusion.) Instead of studying the phase portraits of parametrised dynamical systems:

$$\mathrm{d}x_i/\mathrm{d}t = F_i(x_1, \ldots, x_n; a_1, \ldots, a_r) \qquad (i = 1, \ldots, n) \tag{4.1}$$

where the a_i are adjustable parameters, attention is first restricted to stationary solutions $\mathrm{d}x_i/\mathrm{d}t = 0$, i.e. to (no longer differential) equations

$$0 = F_i(x_1, \ldots, x_n; a_1, \ldots, a_r). \tag{4.2}$$

There are technical difficulties in handling n different functions F_i (or equivalently the vector-valued function $F = (F_1, \ldots, F_n)$), which disappear either when $n = 1$ or when F is a gradient, i.e. there is a function G such that $F_i = \partial G/\partial x_i$. In this case the problem reduces to finding the critical points of G, given by

$$0 = \partial G/\partial x_i \tag{4.3}$$

and seeing how they vary with $a_1, \ldots, a_r$. This is still a difficult problem for general analysis, and we restrict further, asking only for results valid *locally*, i.e. close to some chosen point. The simplest way to get an equation like (4.3) from one like (4.1) is if (4.1) starts out as a gradient equation:

$$\mathrm{d}x_i/\mathrm{d}t = \partial G/\partial x_i. \tag{4.4}$$

This sequence of specialisations is often taken as implying that elementary catastrophe theory applies only to equations in gradient form (4.4). There are no logical grounds for such an inference: (4.4) may be the most *obvious* form leading to a gradient equation (4.3), but it is not the *only* form possible. A simple example (of no especial physical interest) would be the equation

$$\mathrm{d}x/\mathrm{d}t = x^3 + xy^2 + x \qquad \mathrm{d}y/\mathrm{d}t = y.$$

This is not in gradient form (if $x^3 + xy^2 + x = \partial G/\partial x$, $y = \partial G/\partial y$, then $\partial(x^3 + xy^2 + x)/\partial y = \partial y/\partial x$, so $2xy = 0$, which is not so), but the stationary solutions are given by

$$0 = x^3 + xy^2 + x = x(x^2 + y^2 + 1) \qquad 0 = y$$

and, since $x^2 + y^2 + 1$ is never zero, these reduce to

$$0 = x \qquad 0 = y$$

which is in the form (4.3) with $G = \frac{1}{2}(x^2 + y^2)$. Artificial as this example may be, it should serve as a warning: the utility of elementary catastrophe theory cannot be decided by a superficial look at the equations that motivate it. A great many problems (including many in partial differential equations) can be reduced, by various transformations, to problems like (4.3).

The 'gradient' restriction on F is nevertheless a strong one; it would be desirable to relax it and tackle problems like (4.4) head-on. *The gradient restriction is not essential.* The main results of elementary catastrophe theory generalise (with appropriate modification) to the non-gradient case: the result is usually known as *singularity theory*. While this generalisation is not trivial, it is identical in spirit with elementary catastrophe theory and uses the same basic techniques. As the sequence of equations (4.1)–(4.3) shows, the relation between these three areas is

$$\mathrm{ECT} \subseteq \mathrm{ST} \subseteq \mathrm{GCT}. \tag{4.5}$$

5. Elementary catastrophes: classification

With the above as motivation, we turn to the study of critical points of smooth functions, placing emphasis on structural stability. We consider only the local structure near to a given critical point (which we may assume is 0): the global problem is less tractable, though some results do exist (Hayden 1980a, b). The local analysis is in any case a necessary preliminary to a global study.

Let $f: R^n \to R$ be smooth, with a critical point at the origin. For f to be structurally stable, we require the *form* of the critical point to persist under small perturbations of f. The immediate task is to make this idea precise, and to this end we consider two simple examples.

Example 5.1. $f: R \to R$, $f(x) = x^2$. The critical point at 0 is a 'parabolic' minimum. Perturb f, say to $f + \epsilon g$ where ϵ is small; suppose for simplicity that $g'(0) = 0$. Then

$$(f(x) + \epsilon g(x))' = 2x + \epsilon g'(x)$$

which vanishes at 0, so 0 remains a critical point; and

$$(f(x) + \epsilon g(x))'' = 2 + \epsilon g''(x)$$

which, for small enough ϵ, is positive at $x = 0$. Hence this critical point is also a local minimum, and the shape is approximately a parabola. Thus we expect $f(x) = x^2$ to be structurally stable.

If $g'(0) \neq 0$ the analysis is not quite so simple; but for small ϵ the critical point still exists: it just moves a little away from 0.

Example 5.2. $f: R \to R$, $f(x) = x^3$. There is a unique critical point at $x = 0$ of 'inflexion' type. The small perturbation $x^3 + \epsilon x^2$, however, has *two* critical points: a local minimum or maximum at $x = 0$ and a local maximum or minimum at $x = -2\epsilon/3$. This change occurs however small ϵ is, and we conclude that $f(x) = x^3$ is structurally *un*stable.

Now to the general case. 'Smallness' of a perturbation is dealt with by defining a topology, the Whitney C^∞ topology (Golubitsky and Guillemin 1973). This is a little technical to describe; but in practice a function is 'small' if the values of *all* of its partial derivatives of arbitrarily high order are small.

It is more important to acquire a good feeling for what it means for two critical points to have the 'same form'. Let f, $g: R^n \to R$ be smooth functions defined near

0. We say they are (*right*) *equivalent* if

$$f(x)=g(\varphi(x))+\gamma \tag{5.1}$$

where $\varphi: R^n \to R^n$ is a (*local*) *diffeomorphism* and γ is constant. A diffeomorphism is a smooth function whose inverse exists and is smooth; it corresponds to a smooth change of coordinates.

For example, $f(x)=4x^2-x^4+17$ is equivalent to $g(x)=x^2$. Set $\varphi(x)=x\sqrt{(4-x^2)}$, $\varphi=17$. Both have a 'parabolic' minimum at $x=0$.

On the other hand, $f(x)=x^3$ and $g(x)=x^5$ are *not* equivalent, because the equation $x^3=(\varphi(x))^5+\gamma$ implies $\gamma=0$ and $\varphi(x)=x^{3/5}$. But this φ is not differentiable at 0, hence not smooth there.

The effect of φ is to distort the graph of f 'sideways'—parallel to the x-coordinate hyperplane. The constant γ merely shifts the whole graph up or down (having no essential effect on critical points). The features preserved are those that depend on how the graph wobbles up and down—exactly what we need to know about the shape near a critical point, where the tangent is horizontal. Thus equivalent functions have the 'same local form' for their graphs. Note that this must be interpreted in a differentiable sense: both x^3 and x^5 have points of inflexion, but the x^5 inflexion is 'flatter', in that more derivatives vanish there.

We can now define a (critical point of *a*) function to be structurally stable if all nearby critical points of functions (in the Whitney C^∞ topology) are equivalent to it. We can then prove:

Theorem 5.3. Morse Lemma. Let $f: R^n \to R$ be smooth, with a critical point at 0. The following are equivalent.

(a) f is structurally stable.

(b) f is non-degenerate, i.e. the Hessian matrix $[\partial^2 f/\partial x_i\, \partial x_j]$ has non-zero determinant at $x=0$.

(c) f is equivalent to a *Morse function* $\pm x_1^2 \pm x_2^2 \ldots \pm x_n^2$.

For a simple proof that (b)→(c) see Milnor (1963) or Poston and Stewart (1978). (a) follows easily. Note that (b) implies that 'almost all' critical points are structurally stable, i.e. a 'random' critical point has 'probability 0' of being structurally unstable—because the determinant of the Hessian is almost always non-zero. We can express this as follows: structural stability of critical points is a *generic* property. (Often this is phrased as follows: *a generic critical point is structurally stable.* Despite Sussmann and Zahler (1978), this way of talking is common in topological dynamics.)

Thus in two dimensions, the only structurally stable types of critical point are x^2+y^2 (local minimum), x^2-y^2 (saddle), and $-x^2-y^2$ (local maximum). Does this mean that in practice only these types will arise?

The answer is 'no', and the reason is that in many physical applications one deals, not with a single critical point, but with the critical points of *families* of functions, having adjustable parameters. It is possible for such a family *generically* to include structurally unstable critical points. Roughly speaking, they are 'trapped' between structurally stable ones of different types.

A bad example of this is the family $f_a(x)=ax^2$ where a runs from -1 to 1. If $a<0$ then f_a has a critical point of type $-x^2$, a local maximum; if $a>0$ then the type is x^2, a local minimum. But *at* $a=0$ the function is $f_0=0$, which is structurally unstable, and has a whole *line* of critical points. Now, it is impossible to connect a critical point of type $-x^2$ to one of type x^2 by a family consisting entirely of structurally stable critical points—because the structural stability itself implies that the type cannot change within the family. So, having pegged the ends of the family to two different types, we force something structurally unstable to occur between.

This is a bad example, however, in that we do *not* force the occurrence of the particular type $f_0 = 0$. Other ways of connecting the ends give different kinds of structural instability.

In contrast, consider the family $g_a(x) = x^3 + ax$. This has two Morse critical points for $a < 0$, none for $a > 0$, and a structurally unstable x^3 point at $a = 0$. It can be shown that any small perturbation *of the whole family* has exactly the same kind of structure *as a family*. To get from $x^3 - x$ to $x^3 + x$, say, using a family near to $x^3 + ax$, it is necessary to pass through some function near to $g_0(x) = x^3$ which has a structurally unstable critical point equivalent to x^3.

In other words, while individual members of the family may lack structural stability, the family as a whole does not.

This is exactly the situation discussed in § 3, and the above family exemplifies the structural stability of limit-point bifurcation. So our problem has now become: describe structurally stable *families*. To do this, the notion of equivalence is modified to apply to families. If $f_a(x)$ and $g_a(x)$ are two families, they will be equivalent if

$$g_a(x) = f_{\beta(a)}(\varphi_a(x)) + \gamma(a)$$

for suitable functions β, φ, γ. The exact prescription is spelled out in the standard literature, and is given in 'classical' symbolism in Poston and Stewart (1978), but the idea is that we make a change of coordinates in parameter space (β), a *parameter-dependent* change in x space (φ_a) and a parameter-dependent addition of a constant (γ). This is the natural way to extend right equivalence while preserving parameter dependence.

Having done this, we seek 'generic' families, defined by a mathematical condition of transversality which implies structural stability. It turns out that with up to five parameters, this condition does indeed characterise 'almost all' families. The result may be stated as:

Theorem 5.4. Thom classification theorem. Let $f: R^n \times R^r \to R$ be a parametrised family of smooth functions $f_a : R^n \to R$, where $a \in R^r$. Suppose that $r \leq 5$. For simplicity write $(a_1, \ldots, a_r) = (a, b, c, \ldots)$ and $(x_1, \ldots, x_n) = (x, y, \ldots)$. Then almost all such f are equivalent to one of the following list:

x	(non-critical)	[ ]
$\pm x_1^2 \pm \ldots \pm x_n^2$	(Morse)	[A_1]
$x^3 + ax + (M)$	(fold)	[A_2]
$\pm(x^4 + ax^2 + bx) + (M)$	(cusp)	[A_3]
$x^5 + ax^3 + bx^2 + cx + (M)$	(swallowtail)	[A_4]
$\pm(x^6 + ax^4 + bx^3 + cx^2 + dx) + (M)$	(butterfly)	[A_5]
$x^7 + ax^5 + bx^4 + cx^3 + dx^2 + ex + (M)$	(wigwam)	[A_6]
$x^2y - y^3 + ax^2 + by + cx + (N)$	(elliptic umbilic)	[D_4^-]
$x^2y + y^3 + ax^2 + by + cx + (N)$	(hyperbolic umbilic)	[D_4^+]
$\pm(x^2y + y^4 + ax^2 + by^2 + cx + dy) + (N)$	(parabolic umbilic)	[D_5]
$x^2y - y^5 + ay^3 + by^2 + cx^2 + dx + ey + (N)$	(second elliptic umbilic)	[D_6^-]
$x^2y + y^5 + ay^3 + by^2 + cx^2 + dx + ey + (N)$	(second hyperbolic umbilic)	[D_6^+]
$\pm(x^3 + y^4 + axy^2 + by^2 + cxy + dx + ey) + (N)$	(symbolic umbilic)	[E_6].

Here

$$(M) = \pm x_2^2 \pm \ldots \pm x_n^2$$
$$(N) = \pm x_3^2 \pm \ldots \pm x_n^2$$

are Morse functions in variables not explicitly included as x, y. The names in round brackets are standard 'pet' names; the symbol in square brackets is part of a systematic notation due to Arnol'd (1968). These families of functions, and their generalisations to larger numbers of parameters, are the *elementary catastrophes*. The reason for insisting that $r \leq 5$ is that, for larger r, the classification becomes *infinite*. However, this is not an insuperable obstacle: Arnol'd (1976) has extended the list (for complex functions) to 14 parameters.

That the above theorem is a *classification* has led some commentators to assume that 'catastrophe theory' is basically *taxonomic*. In fact, as we show below, even taxonomy has its uses; but for application it is probably the techniques used to prove theorem 4.4, and certain of its consequences, that are more important. The classification itself can be used to organise and unify otherwise scattered results, and it is evidence for a strong mathematical grip on the objects of study. Classifications in the applied literature tend to be *ad hoc* and incomplete.

Among the concepts and techniques stimulated by theorem 4.4 we mention in particular these: *co-dimension*, *determinacy*, *unfoldings* and *organising centres*. But before discussing these, we shall illustrate that even taxonomy has quantitative consequences.

Application 5.5. The statistics of twinkling starlight (*Berry 1977*). Optical caustics may be described in terms of elementary catastrophes (Poston and Stewart 1978, Berry and Upstill 1980), either in a ray-optical formulation or in a semiclassical quantum formulation. From the extensive work on this topic, which has repercussions outside optics, we select some on the diffraction of radiation by a Gaussian random medium. It turns out that the statistics of the wavefunction ψ, and its intensity I, are highly *non*-Gaussian. To obtain statistical information, Berry (1977) proceeds as follows.

Consider an ensemble S of random phases (the limit as $n \to \infty$ of an n-dimensional torus, corresponding to random phases $\theta_1, \theta_2, \ldots$). Define the nth moment of intensity to be $I_n = \langle |\psi|^{2n} \rangle$, averaged over S. Let λ be the wavelength, and define critical exponents ν_n so that $I_n \sim \lambda^{\nu_n}$ as $\lambda \to 0$.

Catastrophes occurring in the caustic structure contribute to the moments. The more complicated ones contribute rare fluctuations of high intensity; it turns out that for a given intensity moment, only one or two catastrophes dominate the asymptotics. By running through the classification we can find these dominant catastrophes, and compute ν_n. The results (modulo certain conjectures on the full classification for the case of light propagating in a three-dimensional medium) are given in table 1.

For example, in a two-dimensional medium, the only catastrophes that are relevant are the *cuspoids*

$$A_j = \pm(x^{j+1} + a_{j-1}x^{j-1} + \ldots + a_1 x).$$

It can be shown that A_j contributes to the nth moment I_n according to the (asymptotic) exponent $\lambda^{\nu_{nj}}$, where

$$\nu_{nj} = (j-1)(2n-j-2)/2(j+1).$$

Table 1. Critical exponents for light diffracted by a Gaussian random medium (after Berry 1977).

	Propagation in two dimensions	Propagation in three dimensions
n	ν_n	ν_n
2	0	0
3	1/3	1/3
4	3/4	1
5	5/4	5/3
6	9/5	5/2
7	12/5	7/2
8	3	9/2
9	11/3	11/2
10	13/3	13/2
11	5	38/5
12	40/7	87/10
13	45/7	157/16

The maximum value of this occurs when

$$j=[2\sqrt{(n-\tfrac{1}{2})}-1]$$

and the component ν_n is equal to ν_{nj} for *this* value of j.

6. Elementary catastrophes: determinacy and co-dimension

Suppose $f: R^n \to R$ is smooth, and defined near 0. Choose a coordinate system $(x_1, \ldots, x_n)$ on R^n (which need not be, but usually is, the standard one). Make the following definitions.

The *jet* jf is the Taylor series of f at 0.

The *k-jet* $j^k f$ is the Taylor series up to and including terms of order k. (In the rigorous mathematics these concepts are defined in a coordinate-free way, but our simplified approach has them coordinate-dependent. The jet is defined as a *formal* power series and convergence problems do not arise; as a result, elementary catastrophe can deal with smooth functions rather than just analytic ones.)

For example

$$j^6(\cos(x)) = 1-\tfrac{1}{2}x^2+\tfrac{1}{24}x^4-\tfrac{1}{720}x^6.$$

We say that f is *k-determined* if, whenever $j^k g = j^k f$, it follows that g is equivalent to f. (The k-jet $j^k f$ is then *k-sufficient.* If f is k-determined then f is equivalent to $j^k f$, thought of as a polynomial *function*; but the converse need not be true.)

Let E_n be the set of all functions $R^n \to R$; let F_n be the set of all formal power series in n variables. Then taking the jet gives a map $j: E_n \to F_n$. Inside E_n define a subset m_n consisting of all f such that $f(0)=0$; then m_n^k (the set of all linear combinations of products of k elements of m_n) consists of all f such that all partial derivatives of order less than k vanish at 0. These powers of m_n form a descending chain

$$E_n \supseteq m_n \supseteq m_n^2 \supseteq m_n^3 \supseteq \ldots.$$

Similarly in F_n we can define M_n to be the set of power series with zero constant term, and then M_n^k is the set of power series whose lowest terms have degree k or more. The M_n^k also form a descending chain. Note that $j(m_n^k) = M_n^k$.

The *Jacobian ideal* $\Delta = \Delta(f)$ is the set of all functions of the form

$$g_1 \frac{\partial f}{\partial x_1} + \ldots + g_n \frac{\partial f}{\partial x_n}$$

for arbitrary functions g_i. Its image $j\Delta(f)$ in F_n may be defined analogously.

We can now define an invariant that in some sense measures the 'complexity' of a singularity. We say that f is a *singularity* if it has a critical point at 0, i.e. $(\partial f/\partial x_i)(0) = 0$ for all i. Then $\Delta \subseteq m_n$; and since both are real vector spaces we may define the *co-dimension*

$$\operatorname{cod}(f) = \dim_R m_n/\Delta(f).$$

The analogous concept in F_n is

$$\operatorname{cod}(jf) = \dim_R M_n/j\Delta(f).$$

Why define both? The reason is that cod(f) is the natural object of study, but cod (jf) is more easily computed; and we can prove:

Theorem 6.1. If either of cod (f) and cod (jf) is finite, then so is the other, and they are then equal.

Morse functions (theorem 4.3) are precisely those of co-dimension 0. A small perturbation of a function of co-dimension c can have at most $c+1$ critical points.

We give a (somewhat sketchy) example of the computation of co-dimension in the case of F_2. It is based on representing F_2 in terms of its monomials, arrayed as in figure 10. In future we use only a corresponding array of dots. Then the descending chain M_2^k can be pictured as in figure 11, where, for example, we have shaded in M_2^3.

1	x	x^2	x^3	x^4	$\ldots$
y	xy	x^2y	x^3y	$x^4y \ldots$	
y^2	xy^2	x^2y^2	x^3y^2	$x^4y^2 \ldots$	
y^3	xy^3	x^2y^3	x^3y^3	$x^4y^3 \ldots$	
y^4	xy^4	x^2y^4	x^3y^4	$x^4y^4 \ldots$	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ldots$

Figure 10. Monomials in F_2, the formal power series in two variables x and y, in the standard array.

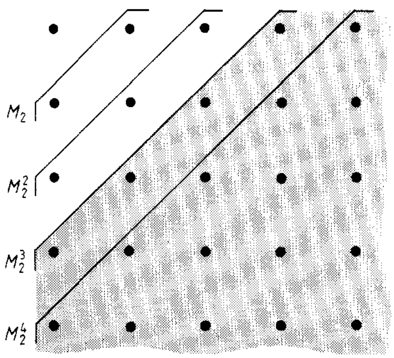

Figure 11. The chain of subspaces M_2^k of F_2. For example, power series in M_2^3 are those having non-zero terms in the shaded region.

(A shaded region thus represents all power series which have *zero* coefficients for terms outside the region.)

In simple cases, the Jacobian ideal may also have such a representation; and when it does, the co-dimension is very easy to compute.

Example 6.2. $f(x, y) = x^3 + y^3$. Here the Jacobian ideal j consists of all power series of the form $g_1 3x^2 + g_2 3y^2$, or equivalently $x^2 g_1 + y^2 g_2$ if we absorb the threes into the g. If we let Δ_1 be the set of power series $x^2 g_1$, and Δ_2 the set of $y^2 g_2$, we can represent these as in figure 12. Exactly three monomials, x, y and xy, are in M_2 but missing from the region formed by both Δ_1 and Δ_2 (which represents $j\Delta$). It follows that cod $(jf) = 3$. Hence also cod $(f) = 3$.

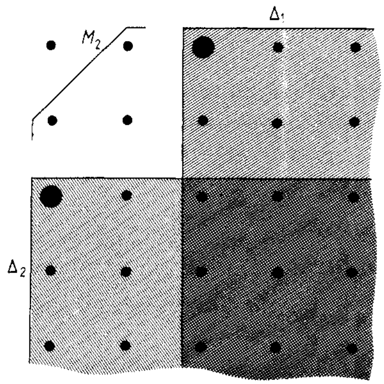

Figure 12. The Jacobian ideal of $f(x, y) = x^3 + y^3$. The three missing monomials x, y, and xy show that the co-dimension is 3. They also define a universal unfolding (see example 7.2).

When the first partials of f are not monomials, this method cannot be used directly, but various modifications can (see Arnol'd (1974) and Poston and Stewart (1978) for example).

Similar computations can be used to test for determinacy. Here there are several criteria, used for different purposes.

Theorem 6.3. With the above notation;

(a) If $M_n^k \subseteq M_n j\Delta$ then f is k-determined.

(b) If f is k-determined then $M_n^{k+1} \subseteq M_n j\Delta$.

(c) f is k-determined if and only if $M_n^{k+1} \subseteq M_n j\Delta(f+g)$ for all $g \in M_n^{k+1}$.

Example 6.4. $f(x, y) = x^3 + y^3$ is three-determined. In figure 13 we draw the regions representing $M_2\Delta_1$ and $M_2\Delta_2$. It is clear that between them they contain that for M_2^3. Hence $M_2^3 \subseteq M_2 j\Delta$, and theorem 6.3(a) gives the result.

In applications, this technique is used to justify 'approximating' a function by some k-jet. The point to note is that it is not just an approximation, but an *equivalence*: the truncated k-jet form is *exact* in *some* coordinate system. The advantage of knowing this is that it is unclear what properties approximations preserve, whereas it is generally easy to see what properties an equivalence preserves. Taylor series truncation is common in the applied literature, but it is all too often applied in a very cavalier way. Now that a technique exists to justify it, it becomes possible to *build* upon sound foundations. (A good example is the work of Schaeffer and Golubitsky (1979) on mode-jumping (see § 10). A large part of this could have been done much earlier by

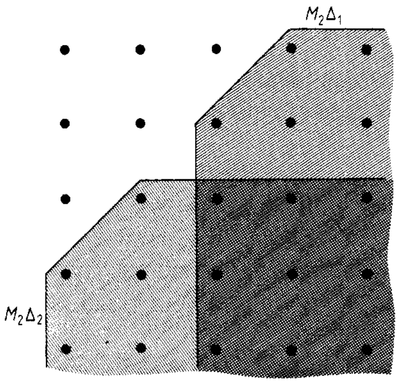

Figure 13. Proof that x^3+y^3 is three-determined.

Taylor series truncation, but the computations are extremely complicated and were not, in fact, carried out until there was good reason to have confidence in the truncation. There are many examples where 'correct' truncations are counterintuitive, especially in the presence of symmetries.)

Application 6.5. Umbilic catastrophes in buckling models (*Thompson and Gaspar 1977*). Consider a rigid link resting on a universal pivot and supported by three inclined springs spaced at angles 0, α, 2α (figure 14). Here there is a total potential

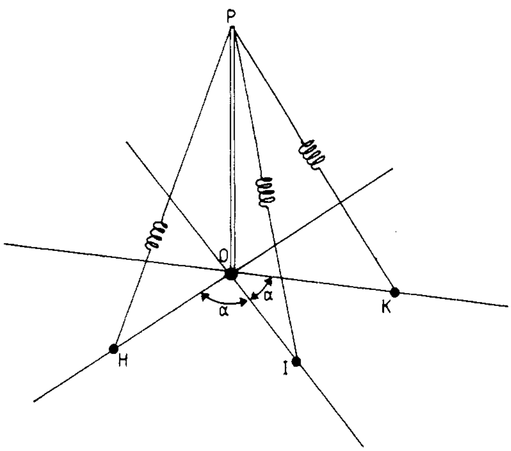

Figure 14. A rigid link supported by inclined springs. O is a universal pivot, and distances OH, OI, OK and OP are all equal to L. Angles HOI and IOK are both equal to α.

energy function

$$V(u_1, u_2, P) = U - PE$$

where

$$E = -L(1-u_1^2-u_2^2)^{1/2}$$

$$U = \tfrac{1}{2}c_1e^2(0, u_1, u_2) + \tfrac{1}{2}c_2[e^2(\alpha, u_1, u_2) + e^2(-\alpha, u_1, u_2)]$$

and

$$e(\varphi, u_1, u_2) = L([(\sin\varphi - u_1)^2 + (\cos\varphi - u_2)^2 + 1 - u_1^2 - u_2^2]^{1/2} - [(\sin\varphi - u_1^0)^2 + (\cos\varphi - u_2^0)^2 + 1 - (u_1^0)^2 - (u_2^0)^2]^{1/2}).$$

The variables are defined precisely in the original; for our purposes everything except u_1 and u_2 is a parameter.

At critical loads, where buckling occurs, the two-jet is zero. We can write the three-jet as

$$\tfrac{1}{6}(V_{111}^c u_1^3 + V_{112}^c u_1^2 u_2 + V_{122}^c u_1 u_2^2 + V_{222}^c u_2^3)$$

where

$$V_{111}^c = 0 \qquad V_{122}^c = 0$$

$$V_{112}^c = \tfrac{3}{8}KL^2\cos\alpha \qquad V_{222}^c = \tfrac{3}{8}KL^2\frac{\sin^2\alpha + \cos\alpha}{1+\cos\alpha}.$$

The condition for a cubic polynomial

$$Au_1^2u_2 + Bu_2^3$$

to be three-determined is that $A \neq 0$ and $B \neq 0$.

Combined with the condition for the occurrence of compound buckling ($\pi/4 \leqslant \alpha \leqslant 3\pi/4$) these imply that the buckling is given by an elliptic umbilic if $\pi/2 < \alpha < \cos^{-1}(1-\sqrt{5})/2$; and by a hyperbolic umbilic if $\pi/4 \leqslant \alpha < \pi/2$ or $\cos^{-1}(1-\sqrt{5})/2 < \alpha \leqslant 3\pi/4$. In the transitional cases, analysis of higher-order terms and further use of determinacy shows that we have a symbolic umbilic for $\alpha = \pi/2$ and a parabolic umbilic for $\alpha = \cos^{-1}(1-\sqrt{5})/2$.

Particularly in these transitional cases, incorrect truncation would produce incorrect descriptions of the buckling behaviour.

In one dimension, the determinacy rule amounts to a triviality: the correct place to truncate a Taylor series is at the first non-zero term. In more than one dimension, this is no longer the case. Some simple examples (whose properties follow from theorem 6.3) may help to illustrate some of the pitfalls: more complicated problems can also arise.

The function $x^2y + y^k$ ($k \geqslant 4$) cannot be truncated at its first non-zero term x^2y. One might imagine that the degeneracy of this truncation, x^2y, with a repeated factor x, might be responsible. However ....

The function $x^5 + y^5 + x^3y^3$ cannot be truncated at degree 5, even though that truncation is non-degenerate.

Finally: the function $\tfrac{3}{2}x^2 + x^3 - 3xy^2$ is not three-dimensional but it *is* four-determined, even though it has no term of degree 4. That is, while we cannot guarantee that a function with the same three-jet is equivalent to it, we can do so if we know in addition that the order-four terms vanish.

When symmetries are present, intuition must largely be abandoned. Terms of high degree may be crucial, while those of lower degree may not matter at all; terms of given degree may be dominated by terms of arbitrarily high degree. The representation theory of the symmetry group interacts violently with the singularity theory. But there exists a determinacy rule, and it is a natural generalisation of 6.3.

7. Elementary catastrophes: unfoldings

Unfolding theory is a much more powerful tool than determinacy. It can be viewed as a sophisticated kind of singular perturbation theory. It gives a way to capture all possible perturbations of a given singularity, and even all parametrised families of perturbations, in a single 'universal' family.

Let f be a function in E_n. An r-parameter *unfolding* of f is a function F in E_{n+r}, with r new variables, such that $F(x, 0) = f(x)$. That is, setting the new variables to zero gives the original function. For example, $F(x, y, a, b, c) = x^5 - 7xy^2 + 44a^2xy - (\sin b)y^{17} + 3abc^2x^6y\cos(x+y)$ is an unfolding of $f(x, y) = x^5 - 7xy^2$. This example shows that in principle unfoldings can be very complicated. However, up to equivalence, most of the complexity is an illusion.

Say that another unfolding G, with s new parameters, is *induced* from F if

$$G(x, v) = F(\xi_v(x), \psi(v)) + \gamma(v)$$

where $v = (v_1, \ldots, v_s)$ are the new G variables, $\xi_v : R^n \to R^n$, $\psi : R^s \to R^r$, and $\gamma : R^r \to R$ are smooth. (The exact details of the relationship are not crucial here: the point is that we can build G out of F by using some arbitrarily chosen auxiliary functions. So in principle F already captures G.)

Say that two unfoldings are *equivalent* if each can be induced from the other. An r-parameter unfolding F is *versal* if all other unfoldings can be induced from it; *universal* if, in addition, r is minimal.

Theorem 7.1. A function f has a universal unfolding if and only if it has finite co-dimension. In this case a universal unfolding is given by

$$F(x, u) = f(x) + u_1c_1(x) + \ldots + u_rc_r(x)$$

where $c_1, \ldots, c_r$ form a basis for M_n modulo $j\Delta(f)$.

Example 7.2. $f(x, y) = x^3 + y^3$. A basis for M_2 modulo $j\Delta(f)$ is given, from example 6.2, by x, y and xy (representing c_1, c_2, c_3 in the above notation). So a universal unfolding is

$$F(x, y, u_1, u_2, u_3) = x^3 + y^3 + u_1x + u_2y + u_3xy.$$

This is the hyperbolic umbilic; up to certain linear coordinate changes compared to the canonical form given in theorem 4.4.

Application 7.3. Polymer flow (Berry and Mackley 1977, Poston and Stewart 1978). Steady two-dimensional flow of a fluid in a 'six-roll mill', consisting of six cylindrical rollers in a hexagonal pattern, may (plausibly, by a genericity argument) be described in the ideal symmetric case by a stream function equivalent to $x^3 - 3xy^2$. Symmetry-breaking perturbations (that keep the flow steady and two-dimensional) should be described by stream functions in the universal unfolding

$$F(x, y; a, b, c) = x^3 - 3xy^2 + a(x^2 + y^2) + bx + cy.$$

The new terms may be identified as rotational and linear flow, superimposed on the ideal symmetric one. Analysis, by catastrophe theory, of the unfolding leads to a complete classification of perturbed flow patterns in good agreement with experiment, and with implications for studies of polymer solutions.

8. Elementary catastrophes: organising centres

The idea of an organising centre is based on the fact that the *global* structure of the elementary catastrophes is just a 'blown up' version of the *local* structure. To describe this more carefully we must take a brief look at catastrophes from the point of view of their associated bifurcations.

Suppose that $f(x, a)$ is a family of smooth functions, with 'state' variables $x = (x_1, \ldots, x_n)$ and 'control' parameters $a = (a_1, \ldots, a_r)$. Define the *catastrophe manifold* to be

$$M = \left\{(x, a) \,\middle|\, \frac{\partial f}{\partial x_i}(x, a) = 0 \text{ for } i = 1, \ldots, n\right\}$$

the *singularity set* to be

$$\Sigma = \left\{(x, a) \in M \,\middle|\, \det\left[\frac{\partial^2 f}{\partial x_i\, \partial x_j}(x, a)\right] = 0\right\}$$

and the *bifurcation set* to be

$$B = \{a \,|\, \text{there exists } x \text{ such that } (x, a) \in \Sigma\}.$$

Example 8.1. The cusp catastrophe. This is notorious, though the main reason for its ubiquity is that it is general enough to be interesting and simple enough to draw in three dimensions, and *not* that nobody knows about anything else. We assume f is given in the form

$$f(x, a, b) = \frac{x^4}{4} + a\frac{x^2}{2} + bx.$$

Then

$$\begin{aligned} M &= \{(x, a, b) \,|\, x^3 + ax + b = 0\} \\ &= \{(x, a, b) \,|\, x^3 + ax + b = 0, 3x^2 + a = 0\} \\ B &= \{(a, b) \,|\, \text{there exists } x \text{ such that } x^3 + ax + b = 0, 3x^2 + a = 0\} \\ &= \{(a, b) \,|\, 4a^3 + 27b^2 = 0\}. \end{aligned}$$

Every elementary catastrophe has its own characteristic bifurcation geometry: for a detailed analysis see Bröcker and Lander (1975) or Poston and Stewart (1978).

Suppose we introduce a scale chage in f, as follows: $x \to \kappa x$ for a constant κ. Then we can also scale a by $a \to \kappa^2 a$, and b by $b \to \kappa^3 b$, in such a way that f is merely multiplied by a constant. That is, if we set

$$X = \kappa x \qquad A = \kappa^2 a \qquad B = \kappa^3 b$$

then

$$f(X, A, B) = \kappa^4 f(x, a, b).$$

This change of scale preserves exactly the geometry of M, Σ and B. Moreover, the transformation is a diffeomorphism except at the origin. By using this transformation we can make the global structure 'flow towards the origin'. Then the origin captures the global system 'in microcosm', and the entire system can be understood as a blown-up version of a small family of perturbations of the singular system found at

$a = b = 0$. For example, a zig-zag curve (figure 15) is most naturally viewed as a section through the cusp catastrophe surface, passing transversely to the cusp axis and missing the origin; and this has the same topological structure as a section close to the origin. In this way the global arrangement of two folds in a zig-zag can be turned into a local object. This observation may seem trite, but in fact it is the basis of what appears to be a very powerful technique. *To understand a global arrangement of singularities, first find an organising centre by pushing them all together, and then unfold the resulting highly singular system. The original arrangement can be recovered as some section of this unfolding.*

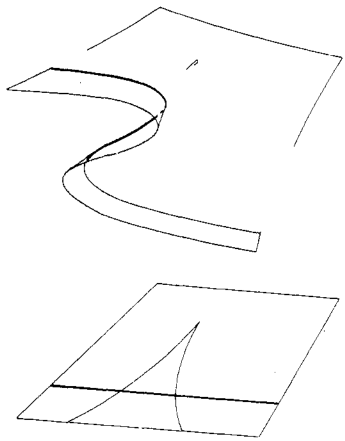

Figure 15. The cusp catastrophe acting as an organising centre for a zig-zag hysteresis loop. Generalising this idea gives a powerful method for localising global configurations of singularities.

The best examples of applications of this idea occur in the imperfect bifurcation theory of Golubitsky and Schaeffer (1979a, b) and will be discussed further below. Note that the transformation need not be a simple scaling; all that is required is that the elementary catastrophe be topologically conical. For a rigorous description of organising centres for dynamical systems, generalising this idea, see Zeeman (1980). This is likely to be one of the ways in which elementary catastrophe theory will influence general catastrophe theory and dynamical systems theory. The power of the technique resides in its ability to turn a global problem into a local one, where techniques are more readily available and computations are easier.

The use of organising centres is radically different from the most common techniques of 'classical' bifurcation theory, which seek to understand a global arrangement of singularities by making separate local analyses near each, and examining the way they are glued together. To oversimplify the position we can say that the classical techniques *avoid* the degenerate singularity at the organising centre, whereas the catastrophe-theoretic ones *deliberately seek it out.* This difference in philosophy can be traced to a difference in weaponry: most classical analyses are based on the implicit function theorem, which breaks down at degeneracies, whereas catastrophe theory is based on the more powerful Malgrange preparation theorem, which does not. The difference shows up in hidden ways: analyses using the implicit function theorem tend to be non-uniform in the parameters. In consequence, even where the classical methods give the same *pictures* as catastrophe theory, those pictures are proved valid in a smaller region, and one that *does not fill out a full neighbourhood of parameter* *space*. If catastrophe theory is to be criticised because it is 'local', it is perhaps worth noting that a lot of classical bifurcation theory is not even that.

9. Imperfect bifurcation

So far we have discussed 'classical' catastrophe theory—the subject as it was a decade ago. More recent developments, inspired directly by the needs of specific applications, build on those foundations; but various extensions and modifications are required. The key ideas, such as structural stability, unfoldings and organising centres, run as a recurrent theme.

For motivation we return to the pitchfork bifurcation of 3.1: the equation $x^3-\lambda x=0$. To fit into the above discussion we shall talk as if this derived from a critical point problem

$$\frac{d}{dx}\left(\frac{x^4}{4}-\lambda\frac{x^2}{2}\right)=0$$

although it would be possible to deal directly with the bifurcation equation as such.

The potential function $f(x,\lambda)=x^4/4-\lambda x^2/2$ is structurally unstable. In fact it represents a one-parameter unfolding of the co-dimension two function $x^4/4$. To obtain a universal unfolding we need an additional parameter μ, obtaining

$$F(x,\lambda,\mu)=\frac{x^4}{4}-\lambda\frac{x^2}{2}+\mu x$$

and a bifurcation equation

$$0=x^3-\lambda x+\mu.$$

Thus we have a cusp catastrophe. The original bifurcation problem is replaced by a two-parameter problem; the original bifurcation at $\lambda=0$ replaced by the full cusp catastrophe bifurcation *set*. Details of this approach, for the case of a buckling beam, are to be found in Zeeman (1977). The buckling behaviour, for arbitrary (small) sequences of changes in the forces acting, may be read off from the cusp catastrophe picture in a synoptic fashion. Structural stability ensures a better chance of agreement with actual experiments.

For specific computational problems in bifurcation theory, however, these ideas are best presented in a modified form. The possibility (and desirability) of this was first realised by Golubitsky and Schaeffer (1979a), who noted that in a bifurcation problem the variable λ which acts as 'bifurcation parameter' should be *distinguished* from the other unfolding parameters, because experimental graphs are plotted against a single parameter, assumed to be the theoretical λ. In response to the needs of applications, they developed a version of catastrophe theory that incorporated such a distinguished parameter. In spirit, and in the methods used to prove the theorems, it is a direct descendant of elementary catastrophe theory; however, making use of newer generalisations, it is not restricted to the scalar-valued case (variational problems) but works equally well—in fact better—for general functions $R^n\to R^n$. The precise formulation may be found in the original papers, and is sketched briefly in Stewart (1981); the main point to note is that the transformations allowed must be restricted to those that 'preserve λ'.

The upshot is a complete theory of (local) bifurcation problems

$$G(x, \lambda) = 0$$

where

$$G : R^n \times R \to R^n$$

is smooth. Notions of structural stability, co-dimension, unfoldings and organising centres carry over. In computing the co-dimension it is no longer appropriate to use the Jacobian ideal; this is replaced by a natural modification. There is a classification theorem (up to co-dimension three at the time of writing) in preparation.

Finite-dimensional bifurcation problems arise from, for example, partial differential equations by way of reduction techniques such as that of Lyapunov–Schmidt. As a result, catastrophe theory can be used even though the original problem is not modelled by a dynamical system.

To give the flavour of the mathematics, we shall explain how to compute co-dimensions and unfoldings for bifurcation problems $G(x, \lambda) = 0$.

Write G as

$$G(x, \lambda) = (g_1(x, \lambda), \ldots, g_n(x, \lambda)).$$

Let $E_{n+1,n}$ be the set of all functions $R^n \times R \to R^n$, and let E_{n+1} be the set of all functions $R^{n+1} \to R$. We define three subsets of $E_{n+1,n}$ as follows:

$\langle G \rangle$ is the set of all functions $u_1 g_1 + \ldots + u_n g_n$

where $u_i(x, \lambda) \in E_{n+1}$. Then $\langle G \rangle^n$ is the set of function $(\varphi_1, \ldots, \varphi_n)$ where each $\varphi_i \in \langle G \rangle$.

$E_{n+1}\{\partial G/\partial x\}$ is the set of all functions $v_1 \dfrac{\partial G}{\partial x_1} + \ldots + v_n \dfrac{\partial G}{\partial x_n}$

where $v_i \in E_{n+1}$:

$E_\lambda\{\partial G/\partial \lambda\}$ is the set of functions $\varphi(\lambda) \dfrac{\partial G}{\partial \lambda}$ where $\varphi \in E_1$.

Then set

$$TG = \langle G \rangle^n + E_{n+1}\{\partial G/\partial x\} + E_\lambda\{\partial G/\partial \lambda\}.$$

This is the analogue of the Jacobian ideal. Essentially the Jacobian ideal is derived from 'infinitesimal' coordinate changes, whereas TG is derived from 'infinitesimal' λ-preserving coordinate changes. Once TG is defined, co-dimensions and unfoldings follow the natural pattern with TG replacing the Jacobian ideal.

Rather than explain this in the general setting, we give the following example.

Example 9.1. Unfolding the pitchfork. We have $G(x, \lambda) = x^3 - \lambda x$. Then

$$\partial G/\partial x = 3x^2 - \lambda$$

$$\partial G/\partial \lambda = -x.$$

Hence

$$TG = \langle x^3 - \lambda x \rangle + \langle 3x^2 - \lambda \rangle + E_\lambda(-x).$$

We take this in pieces, and set

$$\tilde{T}G = \langle x^3 - \lambda x \rangle + \langle 3x^2 - \lambda \rangle.$$

This contains

$$x^3 = \tfrac{1}{2}x(3x^2 - \lambda) - \tfrac{1}{2}(x^3 - \lambda x)$$
$$\lambda x = 3x^3 - x(3x^2 - \lambda)$$
$$\lambda^2 = 3x\lambda x - \lambda(3x^2 - \lambda).$$

Representing this on an array of monomials, as in § 5, we get $\tilde{T}G$ in figure 16. The only terms missing are $1, x, x^2, \lambda$. Hence $\dim E_{1+1,1}/\tilde{T}G = 3$ because x^2 and λ are

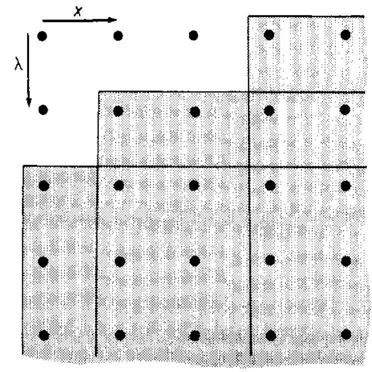

Figure 16. Unfolding the pitchfork as a bifurcation problem. Powers of x run horizontally, powers of λ vertically, as in figure 10 with λ in place of y. The shaded region represents $\tilde{T}G$, a first approximation to the analogue of the Jacobian ideal.

linearly-dependent modulo $\tilde{T}G$. To get TG we must add to $\tilde{T}G$ terms in $E_\lambda(-x)$, i.e. functions of the form $x\varphi(\lambda)$ (we absorb the minus sign into φ). The diagram now becomes figure 17, from which only $1, \lambda, x^2$ are missing. But, as before, x^2 and λ are linearly-dependent modulo TG, because $3x^2 - \lambda \in TG$. So we have only two linearly-independent terms missing, and the co-dimension is 2. Further, a universal unfolding

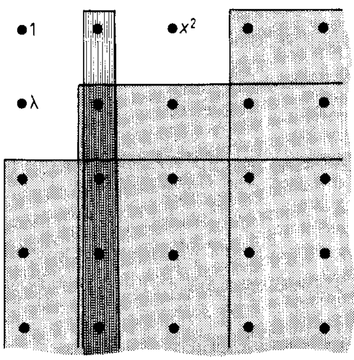

Figure 17. The full analogue of the Jacobian ideal, TG. Of the missing monomials, only two are linearly independent modulo TG, so the co-dimension of the pitchfork is 2 when it is considered as a problem with a distinguished parameter.

is of the form

$$H(x, \lambda, a, b) = x^3 - \lambda x + a\lambda + b.$$

The way that the resulting bifurcation diagrams $H(x, \lambda, a, b) = 0$ depend upon a and

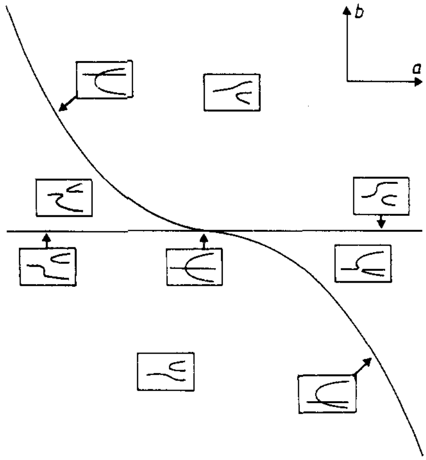

Figure 18. The full unfolding of the pitchfork bifurcation, with a and b as unfolding parameters.

b is shown in figure 18. Between them, these account for all of the possible small perturbations of the 'ideal' pitchfork bifurcation.

With these techniques (suitably reinforced by others which cannot be discussed here) it becomes possible to analyse an 'imperfect' system by *first* analysing an organising centre, the 'ideal' system that approximates it, and *then* unfolding this to find the effects of imperfections. The more traditional approach would require imperfections to be included from the start. This leads to great computational complexity, especially with methods like Lyapunov–Schmidt reduction (Marsden 1978) which can be difficult enough for the 'ideal' problem.

The value of these ideas should not be overstated. In many cases a complete imperfection analysis by way of a universal unfolding may involve large numbers of new variables and be computationally infeasible. While this is strong evidence for an inherent complexity of the corresponding problem—likely to surface in any genuine attempt to grapple with it—one should not rule out the possibility of finding better methods. However, the common approach of *ignoring* awkward terms in equations cannot be recommended for this purpose.

Application 9.2. The winged cusp. Starting from the work of Uppal *et al* (1976) and Aris (1977), Golubitsky and Keyfitz (1980) applied the above ideas to a bifurcation problem arising in a standard continuum model of the stirred tank reactor. This model leads (see the original papers or Stewart (1981)) to a steady-state bifurcation equation

$$G(y, \epsilon; B, \delta, \eta) = \eta - (1+\epsilon)y + B\epsilon\{1 + \epsilon\delta \exp[-\gamma y/(1+y)]\}^{-1} = 0. \qquad (9.1)$$

They find an *organising centre* equivalent to the bifurcation problem

$$x^3 + \lambda^2 = 0. \qquad (9.2)$$

This form is suggested by the qualitative features of the numerical results. By *unfolding* this to

$$x^3 + \lambda^2 + (a + b\lambda)x + c = 0$$

they obtain a list of seven structurally stable perturbations of the ideal problem (9.2), giving the possible qualitative behaviour in the original problem (9.1). (There are also various degenerate, unstable bifurcation diagrams that occur as perturbations, and these can be listed without difficulty.) Of these seven, five had been found numerically by Uppal *et al* (1976); the others have since been found numerically also (in independent work).

Obviously, for specific information, the numerical results are important. But an exhaustive theoretical analysis is preferable to a long computer search, if one is seeking to understand all possible types of behaviour in a given system. (Uppal *et al* also analyse Hopf bifurcation to limit cycles, a type of bifurcation not covered by the steady-state analysis; this is also important. Hopf bifurcation can also be dealt with by singularity theory methods; but in this case there is no reason to expect degenerate Hopf bifurcation, and the original Hopf theorem suffices. The Hopf bifurcation is obviously a necessary ingredient of any general 'theory of catastrophes', and since Hopf has already discovered it, it is easy enough to incorporate it into the necessary body of technique.)

10. Equivariant bifurcation

Inasmuch as the pitchfork bifurcation has co-dimension 2, a 'random' bifurcation should not resemble it. The problem is to reconcile this with the remarkably frequent occurrence of the pitchfork in the applied literature. The answer seems to be *symmetry*. The potential $x^4/4-\lambda x^2/2$ is an even function of x for each λ; the bifurcation problem $x^3-\lambda x=0$ retains this symmetry, being unchanged if we transform x into $-x$. Thus the problem is an *equivariant* bifurcation, with a symmetry group Z_2 of order 2.

If we are interested in symmetry-*breaking* imperfections, this problem has to be treated as it was above—leading to the conclusion that it has co-dimension 2 and thus requires two independent imperfections for a full unfolding.

But if we are interested only in symmetry-*preserving* imperfections, it makes more sense to work, from the beginning, in the set of functions having that symmetry, and to compute co-dimensions, unfoldings, and so on, within the symmetry class. Such an extension of the theory has been worked out by Golubitsky and Schaeffer (1979b), building on earlier work of Poénaru. It should also be noted that Sattinger (1979) has emphasised the importance of symmetry considerations in bifurcation theory, providing some of the groundwork on which the recent techniques can build. The computations are analogous to example 9.1, but involve the representation theory of the relevant symmetry group; they are sketched in Stewart (1981) and, to save space, will not be discussed further here.

Application 10.1. Mode-jumping in the rectangular plate. Experiments made by Stein (1959a, b) revealed the following phenomenon. A rectangular plate, subjected to longitudinal compression, buckles initially into a mode of wavenumber 5. On further compression it *snaps* to wavenumber 6. A number of unsuccessful attempts have been made to explain this *mode-jumping*, and it has become clear that some kind of secondary bifurcation is responsible.

Schaeffer and Golubitsky (1979) treat the system as an equivariant bifurcation problem with symmetry group $Z_2 \oplus Z_2$ (up–down symmetry in the eigenfunctions together with that part of the geometric symmetry of the rectangular plate that acts non-trivially on the eigenfunctions).

The behaviour of the plate is assumed to be modelled well by a system of partial differential equations known as the von Karman equations. Two types of boundary condition are considered: *clamped* or *simply supported.*

Linearise the von Karman equations. The resulting equations have eigenfunctions corresponding to buckled plates with wavenumber k, for various k. At a discrete set of values of the aspect ratio l of the plate, the eigenvalues are double:

$$l=[k(k+2)]^{1/2} \text{ for clamped conditions}$$

$$l=[k(k+1)]^{1/2} \text{ for simply supported conditions.}$$

The Lyapunov–Schmidt reduction method implies that near these aspect ratios the buckled state may be considered as a slight distortion of a combination of buckling modes of the corresponding wavenumbers k and $k+1$; hence we may draw the bifurcation diagram relative to coordinates x and y which give the relative strengths of the two modes. In Stein's experiments l was 5.38; when modes 5 and 6 are under consideration the values of l must be near to 5.92 (clamped) or 5.48 (simply supported). In either case we can consider the experimental value as a small perturbation from the double eigenvalue, in the negative direction.

By using equivariant singularity theory and suitably generalised determinacy criteria, Schaeffer and Golubitsky are able to find all of the possible types of bifurcation diagram that might occur in such perturbations, preserving the $Z_2 \oplus Z_2$ symmetry. They find that secondary bifurcations may occur (as is well known); but that these can only describe mode-jumping under *clamped* boundary conditions. (Most previous analysis has used simply supported conditions, which permit treatment by classical methods—although Stein himself recommended clamped conditions!) Figure 19 shows

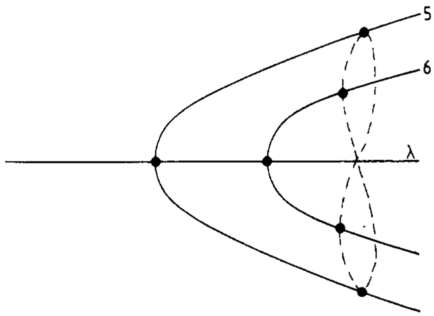

Figure 19. Mode-jumping from wavenumber 5 to 6 in the buckling rectangular plate (see application 10.1) (after Schaeffer and Golubitsky 1979).

the bifurcation diagram for this case. For clarity the direction corresponding to wavenumber 6 has been rotated into the same plane as that for wavenumber 5. The initial branching to wavenumber 5 becomes unstable where the dotted branch meets it and jump bifurcation to a stable solution with wavenumber 6 can occur.

Under simply supported boundary conditions there *is* secondary bifurcation, but the branching takes the wrong form to be a possible explanation of mode-jumping. Compare a recent paper of Matkowsky *et al* (1980), attempting to explain mode-jumping, which largely just reiterates the view that it has something to do with secondary bifurcation. The crucial point—that the choice of boundary conditions determines which kind of secondary bifurcation can occur, and that the usual choice will not work—does not appear.

Application 10.2. The spherical Bénard problem. It is well known that a fluid, heated from below, can break up into convection cells, which in 'ideal' cases have some geometric regularity. Chossat (1979) discusses heat convection within a spherical annulus, in the Boussinesq approximation. This gives an equivariant bifurcation problem, with symmetry group O(3) acting in its five-dimensional irreducible representation (spherical harmonics of order 2).

Golubitsky and Schaeffer (1980) correct and extend Chossat's results. They show that in the non-self-adjoint case (not treated by Chossat) imperfections give drastically different effects. In the self-adjoint case Chossat finds a single stable axisymmetric solution after bifurcation, and no non-axisymmetric solutions. Golubitsky and Schaeffer show that there exist non-axisymmetric solutions (which can be stable), and that there may exist two stable axisymmetric solutions in the non-self-adjoint case. Figure 20 shows a sample bifurcation diagram exhibiting a fairly unusual kind of secondary bifurcation.

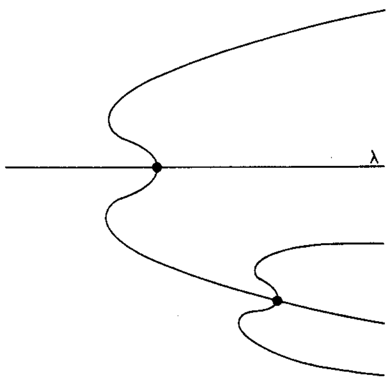

Figure 20. A possible bifurcation diagram for the spherical Bénard problem, represented schematically. Two more such branches exist, placed symmetrically at 120° angles (see application 10.2) (after Golubitsky and Schaeffer 1980).

Application 10.3. The planar Bénard problem. It is known that there are solutions with the symmetry of a hexagonal lattice. Sattinger (1978, 1980) has proposed a model of the bifurcations, which works with doubly periodic functions on the hexagonal lattice, and assumes that the bifurcation is 'as simple as possible'. By way of a Lyapunov–Schmidt reduction he obtains partial results on such bifurcations. Buzano and Golubitsky (1981) have now completed and extended his results. The problem is treated as one with symmetry group $Z \times Z \times D_6$ where D_6 is the dihedral group of order 12. The singularity theory appears indispensible here as a method for finding which higher-order terms in a Taylor expansion of the bifurcation equations are important.

The types of solution of the equations are distinguished by the patterns formed by the steady-flow equilibria. Briefly, they may be described as *hexagons* (a repeating pattern of convex closed curves roughly hexagonal in form), *rolls* (straight lines), *wavy rolls* (roughly sinusoidal lines) and *false hexagons* (like hexagons but with elongated closed curves).

The possible bifurcation diagrams are highly complicated (rather more so than the earlier truncations used by Sattinger (1978, 1980) might suggest) and include not only secondary, but tertiary bifurcations. New experimental predictions are made; for example, figure 21 illustrates a possible jump bifurcation, with hysteresis, from hexagons to rolls.

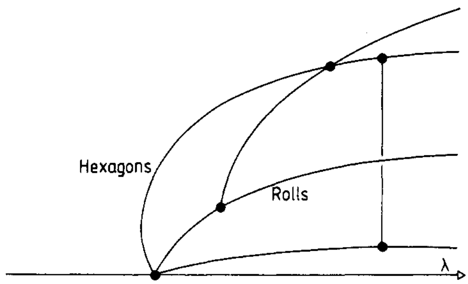

Figure 21. Bifurcations in the planar Bénard problem on a hexagonal lattice (see application 10.3) (after Buzano and Golubitsky 1981).

The calculations are extremely non-intuitive, as a glance at Buzano and Golubitsky (1981) will demonstrate. But despite this great complexity, the singularity theory methods are able to cope adequately, and the computations are in principle routine.

Application 10.4. End effects in the Taylor experiment. There can be few physicists who have not encountered the famous work of Taylor (1923) on fluid flow between concentric cylinders, modelling the formation of vortex cells. In Taylor's idealised analysis, it is assumed that end effects may be neglected, and the cylinders are taken to be infinitely long. Evidence has been steadily accumulating that end effects are not, in fact, negligible—for example, they appear to be *responsible* for the formation of vortex cells—and Benjamin (1978) proposed an approach to the observed events which differed considerably from the standard one. One of the ideas of this paper was to use the catastrophe-theoretic classification of generic types of bifurcations to select the most plausible bifurcation mechanism available.

Schaeffer (1980) extended these ideas, suggesting that the 'true' Taylor experiment, with end effects, might be interpreted as a perturbation of the idealised problem which thus acts as an organising centre for the true one. The latter may be seen as an unfolding of the degenerate singularity occurring in the idealised problem, which is fairly well understood. Symmetries inherent in the standard experimental set-up lead to a problem in equivariant singularity theory, and a detailed model of the formation of vortex cells. The theory is *not* derived rigorously from the Navier–Stokes equations (in fact, even numerical results are not available here); but plausible links between these, and the singularity theory, are exhibited.

Benjamin and Mullin (1981) have made extensive experiments, in part designed to test the qualitative theories of Benjamin (1978) and Schaeffer (1980). In particular they examine how the *number* of cells changes with the Reynolds number. We have no space to embark upon a detailed description, but the general nature of the conclusions reached may be judged by the following quotations from their paper. '. . . the qualitative results from the abstract theory are particularly serviceable at present in making sense of what is seen experimentally.' 'The theoretical framework accounts plausibly for all the phenomena observed.' (The authors are here referring to steady flows only, to which their experiments are confined.) 'Most of (the observations) cannot otherwise be satisfactorily explained.'

The observed phenomena do *not* correspond to the idealised model neglecting end effects, and include some not observed before. The effect of having a finite cylinder is seen to be quite large.

These papers provide an example of how catastrophe theory, by way of concepts such as genericity, organising centres and unfoldings, can suggest models of phenomena, even when the underlying equations cannot be solved analytically; can suggest new experiments; and can survive the results of those experiments.

11. Degenerate Hopf bifurcation

An important bifurcation theorem is the classic one due to Hopf (1942), which gives conditions under which a differential equation

$$dx/dt = f(x, \lambda) \qquad f: R^n \times R \to R^n \tag{11.1}$$

will bifurcate from a stationary solution to a limit cycle. It has been widely applied (Hassard *et al* 1981), and one mathematical reason for its recurring appearance is that, as a one-parameter process, it is structurally stable. However, in a system having several additional parameters (as is commonly the case) it is possible for the hypotheses used by Hopf to fail in a stable way. Numerous partial extensions of Hopf's theorem have been given in the past decade in attempts to deal with such failures.

Limit cycles are not possible in gradient dynamical systems. This fact has led more than one commentator to conclude that the Hopf bifurcation demonstrates an inherent inability of 'catastrophe theory' to deal with common types of bifurcation. There are two reasons to question this argument. First, it applies only to 'elementary' catastrophes, not general ones—indeed, the structural stability of the Hopf bifurcation is evidence *for* the general philosophy of catastrophe theory. The other is that a mathematical tool may be used in more than one way: while it is common to *motivate* elementary catastrophe theory through gradient differential equations, it is a logical *nonsequitur* to deduce that it is limited to such equations alone.

Nevertheless, at first sight, the Hopf bifurcation does look rather different from anything in elementary catastrophe theory. This apparent difference has led, for example, Smale (1978) to suggest that Hopf bifurcation is *deeper* than (elementary) catastrophe theory.

It turns out that this is not so, and the reason for this is instructive.

First we recall the exact hypotheses of the classical Hopf theorem. These are:

(H1) The Jacobian of f at $(0, 0)$ has simple eigenvalues $\pm i$ (perhaps after rescaling t) and (*non-resonance condition*) no other eigenvalues $\pm k i$ for integer k.

(H2) As λ varies through 0, the eigenvalues near $\pm i$ cross the imaginary axis *transversely*. That is, if $\sigma(\lambda) + i\omega(\lambda)$ is an eigenvalue for λ near 0, with $\sigma(0) = 0$ and $\omega(0) = 1$, then $\sigma'(0) \neq 0$.

It then follows that there is (locally) a unique branch of non-constant periodic solutions of (11.1) with period near 2π, bifurcating from the zero solution, and parametrised by amplitude a. Along the solution branch λ is an even function of a, given by

$$\lambda = \mu_2 a^2 + \mu_4 a^4 + \dots .$$

If further

(H3) $\mu_2 \neq 0$

then the solution branch is either super- or sub-critical (i.e. branches to the left or to the right along the λ axis) depending on the sign of μ_2.

If (H1) fails, i.e. if resonance occurs, then the situation becomes very complex (for example, bifurcations to tori may occur). Until recently, it has been possible to relax either (H2) or (H3), but not both. Now Golubitsky and Langford (1980) give an essentially complete description when both (H2) and (H3) fail (modulo a problem in singularity theory which can be solved in principle up to arbitrary co-dimension and in practice as far as is likely to be useful).

It has been known for some time that when (H1) holds the Hopf theorem can be proved by *Lyapunov–Schmidt reduction* (a standard technique in bifurcation theory, in essence the implicit function theorem in a Banach space, although requiring computational skill to apply in specific cases). Begin with (11.1) and rescale time by setting

$$s = (1+\tau)t \qquad u(s) = x(t)$$

so that periodic solutions have period exactly 2π in s. Now (11.1) becomes $(1+\tau)\mathrm{d}u/\mathrm{d}s = f(u(s), \lambda)$, which we write as

$$N(u, \lambda, \tau) = (1+\tau)\mathrm{d}u/\mathrm{d}s - f(u, \lambda) = 0 \tag{11.2}$$

where N is to be thought of as a non-linear operator.

Let $C_{2\pi}$, $C^1_{2\pi}$ be the Banach spaces of 2π-periodic continuous (respectively once-differentiable) functions $R \to R^n$ (with the norms defined in Golubitsky and Langford (1980)—the details are irrelevant here). Then N is a non-linear operator from $C^1_{2\pi} \times R \times R$ into $C_{2\pi}$, defined near (0, 0, 0).

The Lyapunov–Schmidt procedure replaced the infinite-dimensional problem $N(u, \lambda, \tau) = 0$ by a finite-dimensional one of the form

$$g(x, y, \lambda, \tau) = 0. \tag{11.3}$$

The classical Hopf theorem is now obtained by a careful analysis of the properties of g, to detect branching. What Golubitsky and Langford observe is that not only can all of the above be performed without assuming (H2) and (H3) (as is well known), but also that the use of equivariant singularity theory methods permits the analysis to be pushed through to a conclusion in the absence of these hypotheses. In fact, the Lyapunov–Schmidt technique fits the catastrophe theory methods like a glove.

The symmetry group appears because a phase shift on a given periodic solution automatically gives another one. This turns out to mean that $g(x, y, \lambda, \tau)$ is S^1-equivariant, where S^1 is the circle group acting by orthogonal rotation in (x, y) space.

A further reduction is now possible: restriction to the x axis gives a Z_2-equivariant bifurcation in x space. The analysis and classification of such problems is an exercise in Z_2-equivariant elementary catastrophe theory (i.e. the functions are scalar-valued) of the kind for which the results of Golubitsky and Schaeffer (1979b) were designed. In consequence it is possible to define the co-dimension of a degenerate Hopf bifurcation, and to universally unfold it.

Even if one is unwilling to accept that the methods of Golubitsky and Schaeffer (1979a, b) are ‘catastrophe theory’—a difficult stance to take since Golubitsky (1979) states that they are—they are certainly at the same *depth* as elementary catastrophe theory, and mathematically are variations on the same basic ideas (though requiring some virtuosity). Not only is the Hopf *theorem* a direct consequence of catastrophe *theory*; it is also a special low-co-dimension (zero!) case of a much more general degenerate Hopf *theory* which can be read off at once from catastrophe theory. Hopf's theorem, useful though it may be, is no more a *theory* than is the fold catastrophe; compare Smale (1978).

The results obtained by Golubitsky and Langford (1980) are by no means obvious. In particular, the classification up to co-dimension 3 runs as follows (table 2).

Table 2.

Co-dimension	Normal form
0	$x^3 \pm \lambda x$
1	$x^3 \pm \lambda^2 x$
	$x^5 \pm \lambda x$
2	$x^3 \pm \lambda^3 x$
	$x^7 \pm \lambda x$
3	$x^3 \pm \lambda^4 x$
	$x^5 + 2b\lambda x^3 + \epsilon\lambda^2 x$ ($\epsilon = \pm 1$, $b^2 \neq \epsilon$)
	$x^5 \pm 2(\lambda \pm \lambda^2)x^3 + \lambda^2 x$
	$x^5 \pm 2\lambda x^3 \pm \lambda^3 x$
	$x^7 \pm \lambda x^3 \pm \lambda^2 x$
	$x^9 \pm \lambda x$

There is a good deal more to the theory than mere classification, but the details cannot be given here. It is obvious that there is enormous potential for applications of these results. Two samples follow.

Application 11.1. Glycolytic oscillations (*Golubitsky and Langford 1980*). Experimental studies of glycolysis have suggested a model for the chemical oscillations that are observed, of the form (Tyson and Kauffman 1975)

$$dX/dt = \lambda - KX - XY^2$$

$$dY/dt = KX + XY^2 - Y.$$

Here X and Y are scaled concentration variables, λ is feed rate and K is a low activity reaction rate.

Computations show that when $K = \frac{1}{8}$ and $\lambda = \sqrt{\frac{3}{8}}$ there is a degenerate Hopf bifurcation with normal form $x^3 + \lambda^2 x = 0$. By looking at the universal unfolding $x^3 + \lambda^2 x + \alpha x = 0$, one find the bifurcation diagrams of figure 22. Note the occurrence of two back-to-back Hopf bifurcations in 22(c), giving a branch of periodic solutions that departs from the zero solution and then returns to it. Classical methods (using the Hopf theorem) can find these two branches, but give no indication whether they join up. Conceivably a sufficiently complicated classical analysis might establish this too, but only in a region depending non-uniformly on parameters. The catastrophe theory method is simple, natural and direct—in this case.

Application 11.2. Nerve impulse transmission (*Labouriau in preparation*). The full (unclamped) Hodgkin–Huxley equations modelling nerve impulse activity take the

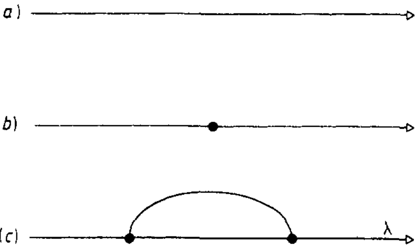

Figure 22. Degenerate Hopf bifurcation in a model of glycolytic oscillations (after Golubitsky and Langford 1980). (a) If the unfolding parameter α is positive, there is no bifurcation. (b) If $\alpha = 0$ there is a degenerate point—an incipient Hopf bifurcation that vanishes the instant it is created. (c) If $\alpha < 0$ there are two Hopf bifurcations joined back-to-back. For the singularity analysis it is diagram (b), which in classical terms does not exhibit bifurcation at all, that is crucial: the degenerate point is a point of differential-topological bifurcation at which the *multiplicity* of solutions does not actually change (see application 11.1).

form (Rinzel and Miller 1980):

$$C_m \frac{\partial v}{\partial t} = \frac{d}{4R_i} \frac{\partial^2 v}{\partial x^2} - \bar{g}_{\mathrm{Na}} m^3 h (v - \bar{v}_{\mathrm{Na}}) - \bar{g}_{\mathrm{K}} n^4 (v - \bar{v}_{\mathrm{K}}) - \bar{g}_{\mathrm{L}}(v - \bar{v}_{\mathrm{L}})$$

$$\frac{\partial m}{\partial t} = \varphi(m_\infty(v) - m)/\tau_m(v)$$

$$\frac{\partial h}{\partial t} = \varphi(h_\infty(v) - h/\tau_h(v)$$

$$\frac{\partial n}{\partial t} = \varphi(n_\infty(v) - n)/\tau_n(v).$$

Here v is the deviation of membrane potential from its resting value, $\bar{g}_{\mathrm{Na}}$ and $\bar{g}_{\mathrm{K}}$ are the sodium and potassium conductances, $\bar{g}_{\mathrm{L}}$ represents leakage due to other ions and φ models the effects of temperature T:

$$\varphi = \exp\,(\log 3)(T - 6.3\ ^\circ\mathrm{C})/10.$$

For the full specification of variables and functions see Rinzel and Miller (1980). These authors use numerical methods to find periodic solutions of the Hodgkin–Huxley equations, obtaining an amplitude bifurcation diagram as in figure 23.

The shape of this diagram, especially at 26 °C, suggests the presence 'nearby' of a degenerate Hopf bifurcation of type $x^5 + 2b\lambda x^3 + \lambda^2 x$. The defining conditions for

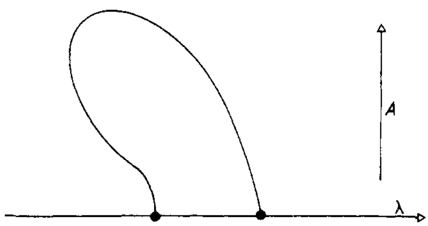

Figure 23. Qualitative representation of bifurcations of periodic solutions to the Hodgkin–Huxley nerve impulse equations at, for example, 26 °C (after Rinzel and Miller 1980). Recent work of Labouriau has exhibited this as an unfolding of a degenerate Hopf bifurcation obtained by small perturbations of the temperature and sodium conductance (see application 11.2).

such a bifurcation are (Golubitsky and Langford 1980) that $\sigma'(0) = 0 = \mu_2$—precisely the case that cannot be handled classically. The parameter b is a 'modal' parameter, affecting differential but not topological properties; the topological co-dimension is 2 rather than 3. What this means is that a two-parameter family can (structurally) stably contain a bifurcation of this type, but that the value of b may change under perturbation.

Again using numerical methods, Labouriau (in preparation) has obtained evidence for the existence of precisely this type of degenerate Hopf bifurcation in the Hodgkin–Huxley equations, acting as an organising centre for the bifurcations of figure 23. The two parameters that are varied to obtain a full unfolding are temperature T and the sodium conductance g_{Na} which must be perturbed away from its 'physical' value by about 5%. This work (which is under further development) provides a simple rationale for the occurrence of diagrams of the observed type on a qualitative level, and can in principle be extended to give more quantitative information by numerically approximating the coordinate transformations required. It also gives a very different mathematical viewpoint from that of Rinzel and Miller (1980), which may prove useful. In addition, it raises the possibility (not especially 'catastrophe-theoretic') of using Lyapunov–Schmidt reduction and curve-following algorithms to obtain numerical bifurcation diagrams for periodic solutions by reducing them to ordinary 'static' bifurcations. The status of this suggestion is as yet unclear.

12. Conclusions

Catastrophe theory has developed considerably since the late 1960s. The original elementary catastrophe theory—the local theory of scalar-valued functions—has been generalised to vector-valued functions (singularity theory); global results of various kinds are now known (e.g. Hayden 1980a, b); and the effect of symmetries or distinguished parameters can be taken into account. There is also an infinite-dimensional generalisation (e.g. Arkeryd 1979). All of these generalisations may be traced back to elementary catastrophe theory and take their place within the expanding body of general catastrophe theory.

Catastrophe theory is proving useful in applied science, in particular in bifurcation theory. It has contributed new information to at least half a dozen problems which have been unsolved for decades (such as mode-jumping in the rectangular plate, end effects in Taylor vortex cells, the planar Bénard problem, and degenerate Hopf bifurcation). Especially in the presence of symmetry, it adds conceptual and computational power of a kind that the more traditional methods, by their nature, do not possess.

However, it is in no sense a *replacement* for the traditional methods, which retain their importance. The most likely development over the next few decades is a fusion of the various schools of thought about non-linear phenomena. Catastrophe theory is a natural language for describing many basic types of non-linearity, and its influence is likely to grow as its methods and concepts become more widely understood and accepted.

At present the main demonstrable successes of the theory have been in the physical sciences. This does not rule out applications to biological or social science; neither does it necessarily imply applicability to these areas. There are some grounds for optimism in biology; the sociological applications remain speculative for reasons mostly not specific to catastrophe theory. In particular, physical scientists have no reason to shy away from these techniques, merely because of criticisms of their more tentative applications. While catastrophe theory is not a universal panacea, it *is* an important contribution to our understanding of non-linear phenomena, and its limitations are less than a superficial analysis of its formal structure might suggest. In particular it is applicable, by way of reduction techniques, to many types of partial differential equation, and is most certainly not (as is often argued or assumed) confined to ordinary differential equations of gradient type.

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