Homework 1

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November 2023

2.1 Let $\mathcal{C} \subseteq \mathbf{R}^n$ be a convex set, with $x_1, ..., x_k \in \mathcal{C}$, and let $\theta_1, ..., \theta_k \in \mathbf{R}$ satisfy $\theta_i \geq 0$, $\theta_1 + ... + \theta_k = 1$. Show that $\theta_1 x_1 + ... + \theta_k x_k \in \mathcal{C}$. (The definition of convexity is that this holds for $k = 2$; you must show it for arbitrary k.) Hint. Use induction on k.

Solution : Base case: $k = 2$.
From definition of convexity, $x_1, x_2 \in \mathcal{C}$ (a convex set),
any convex combination of x_1, x_2 also lies in $\mathcal{C}$
i.e $\theta_1 x_1 + \theta_2 x_2 \in \mathcal{C}$ with $\theta_1 + \theta_2 = 1$ Therefore, Base case is true.

Induction Hypothesis:
With $x_1, ..., x_k \in \mathcal{C}$,
Let $\theta_1 x_1 + ... + \theta_k x_k \in \mathcal{C}$ with $\theta_1 + ... + \theta_k = 1$ be true.

Induction Step: For k+1, For $x_1, x_2 ...., x_k, x_{k+1} \in \mathcal{C}$,
Consider $\theta_1 x_1 + \theta_2 x_2 + ... + \theta_k x_k + \theta_{k+1} x_{k+1}$ with $\theta_1 + ... + \theta_{k+1} = 1$.
$\theta_1 + ... + \theta_k = 1 - \theta_{k+1}$
$\frac{\theta_1}{1-\theta_{k+1}} + ... + \frac{\theta_k}{1-\theta_{k+1}} = 1$
Therefore, from Induction Hypothesis, $\frac{\theta_1}{1-\theta_{k+1}} x_1 + ... + \frac{\theta_k}{1-\theta_{k+1}} x_k \in \mathcal{C}$
Therefore,

$$\theta_1 x_1 + \theta_2 x_2 + ... + \theta_k x_k + \theta_{k+1} x_{k+1} = (1 - \theta_{k+1})(\frac{\theta_1}{1 - \theta_{k+1}} x_1 + ... + \frac{\theta_k}{1 - \theta_{k+1}} x_k) + \theta_{k+1} x_{k+1}$$

which $\in \mathcal{C}$ from the definition of convexity. ■

2.9 *Voronoi sets and polyhedral decomposition.* Let $x_0, ..., x_K \in \mathbf{R^n}$ be distinct. Consider the set of points that are closer (in Euclidean norm) to x_0 than the other x_i, i.e.,

$$V = \{x \in \mathbf{R}^n \mid ||x - x_0||_2 \leq ||x - x_i||_2, i = 1, .., K\}.$$

V is called the *Voronoi region* around x_0 with respect to $x_1, ..., x_K$

(a) Show that V is a polyhedron. Express V in the form $V = \{x \mid Ax \preceq b\}$.

Solution The point x will be closer to x_0 than x_i in the Euclidean sense if it is inside the hyperplane separating x_0 and x_i closer to x_0.

The normal vector for the hyperplane is $(x_i - x_0)$ and the hyperplane passes through $(x_0 + x_i)/2$, their midpoint.

Therefore, the equation of the hyperplane is :

$$(x_i - x_0)^T x = (x_i - x_0)^T (x_i + x_0)/2$$

Therefore the *Voronoi region around* x_0 *w.r.t* $x_1, ..., x_K$ can also be described as:

$$\{x \mid Ax \preceq b\}$$

where $A = \begin{bmatrix} --(x_1 - x_0)^T -- \\ . \\ . \\ . \\ --(x_K - x_0)^T -- \end{bmatrix}$ and $b = \begin{bmatrix} (x_1^T x_1 - x_0^T x_0)/2 \\ . \\ . \\ . \\ (x_K^T x_K - x_0^T x_0)/2 \end{bmatrix}$

Hence V is a polyhedron. ■

(b) Conversely, given a polyhedron P with nonempty interior, show how to find $x_0, ..., x_K$ so that the polyhedron is the Voronoi region of x_0 with respect to $x_1, ..., x_K$.

Solution In the previous problem, we proved that a given Voronoi region is a polyhedron and described it as a set of finitely many linear inequalities.

Now, given a polyhedron P with non-empty interior, we can express that polyhedron as a solution set of finitely many linear inequalities as well. We now compare the coefficients of the 2 forms and equate them.

Let the matrix A obtained from the polyhedron P be:

$A = \begin{bmatrix} --(a_1)^T -- \\ . \\ . \\ . \\ --(a_K)^T -- \end{bmatrix}$ and $b = \begin{bmatrix} b_1 \\ . \\ . \\ . \\ b_K \end{bmatrix}$

by equating coefficients, we get:

$$x_i = a_i + x_0, b_i = (x_i^T x_i - x_0^T x_0)/2$$

Substituting x_i, we get (K equations but we have n variables in x_0):

$$2b_i - a_i^T a_i = 2a_i^T x_0$$

If $K \leq n$, we will have a few free variables and can choose coordinate of x_0 freely by ensuring that the conditions are satisfied.

If $K \geq n$, we will have to find x_0 by solving all the equations and check for consistency as well. Once we have x_0, we can obtain all remaining $x_i's$ by substituting x_0 in $x_i = a_i + x_0$. ■

(c) We can also consider the sets

$$V_k = \{x \in \mathbf{R}^n \mid ||x - x_k||_2 \leq ||x - x_i||_2, i \neq k\}$$

The set V_k consists of points in $\mathbf{R}^n$ for which the closest point in the set $x_0, ..., x_K$ is x_k.

The sets $V_0, ..., V_K$ give a polyhedral decomposition of $\mathbf{R}^n$. More precisely, the sets V_k are polyhedra with nonempty interior, $\bigcup_{k=0}^{K} V_k = \mathbf{R}^n$, and $\mathbf{int}V_i \bigcap \mathbf{int}V_j = \phi$ for $i \neq j$, i.e., V_i and V_j intersect at most along a boundary.

Suppose that $P_1, \ldots, P_m$ are polyhedra with nonempty interior such that $\bigcup_{i=1}^{m} P_i = \mathbf{R}^n$, $\mathbf{int}P_i \bigcap \mathbf{int}P_j = \phi$ for $i \neq j$. Can this polyhedral decomposition of $\mathbf{R}^n$ be described as the Voronoi regions generated by an appropriate set of points?

Solution With just one polyhedron, we were able to calculate coordinates of all the points for the corresponding Voronoi regions (if the system of equations so obtained were consistent). Now we repeat the same procedure for all the polyhedrons and obtain m set of coordinates. If all are the same, we can break it down into a set of Voronoi regions, else, we cannot.

2.12 Which of the following sets are convex?

(a) A *slab* ,*i.e.*,a set of the form $A = \{x \in \mathbf{R}^n \mid \alpha \leq a^T x \leq \beta\}$.

Solution Consider 2 points $x_1, x_2 \in A$.
Consider their convex combination

$$\lambda x_1 + (1 - \lambda)x_2 \geq \lambda\alpha + (1 - \lambda)\alpha$$

$$\lambda x_1 + (1 - \lambda)x_2 \leq \lambda\beta + (1 - \lambda)\beta$$

Therefore $\lambda x_1 + (1 - \lambda)x_2 \in A$. Thus, a slab is a convex set.

(b) A *rectangle, i.e.*, a set of the form $\{x \in \mathbf{R}^n \mid \alpha_i \leq x_i \leq \beta_i, i = 1, ..., n\}$. A rectangle is sometimes called a hyperrectangle when $n > 2$.

Solution Similar to (a), we can prove that a convex combination of points also lies in the set (bound the value of each coordinate in the convex combination) and conclude that a rectangle is a convex set.

(c) A *wedge,i.e.*, $\{x \in \mathbf{R}^n \mid a_1^T x \leq b_1, a_2^T x \leq b_2\}$.

Solution Same procedure as (a)

(d) The set of points closer to a given point than a given set, *i.e.*,

$$\{x \mid ||x - x_0||_2 \leq ||x - y||_2 \forall y \in \mathcal{S}\}$$

where $\mathcal{S} \subseteq \mathbf{R}^n$.

Solution We can express this set as

$$\bigcap_{y \in \mathcal{S}} \{x \mid ||x - x_0||_2 \leq ||x - y||_2\}$$

which is an intersection of halfspaces. (Voronoi description of a halfspace). Hence it is convex.

(e) The set of points closer to one set than another, *i.e.*,

$$\{x \mid \mathbf{dist}(x, \mathcal{S}) \leq \mathbf{dist}(x, \mathcal{T})\}$$

where $\mathcal{S}, \mathcal{T} \subseteq \mathbf{R}^n$, and

$$\mathbf{dist}(x, \mathcal{S}) = \inf\{||x - z||_2 \mid z \in \mathcal{S}\}$$

Solution This set is not convex.
Consider $\mathcal{S} = (-\infty, -1) \bigcup (1, \infty)$, $\mathcal{T} = (-1/2, 1/2)$

(f) The set $\{x \mid x + S_2 \subseteq S_1\}$, where $S_1, S_2 \subseteq \mathbf{R}^n$ with S_1 convex.

Solution $\{x \mid x + S_2 \subseteq S_1\} = \bigcup_{y \in S_2} \{x \mid x + y \in S_1\} = \bigcup_{y \in S_2} (S_1 - y)$ An intersection of convex sets.

(g) The set of points whose distance to a does not exceed a fixed fraction θ of the distance to b, *i.e.*, the set $\{x \mid ||x - a||_2 \leq \theta ||x - b||_2\}$. You can assume $a \neq b$ and $0 \leq \theta \leq 1$.

Solution The given set is a ball. (a halfspace for $\theta = 1$.

$$\{x \mid ||x - a||_2 \leq \theta ||x - b||_2\}$$

$$= \{x \mid ||x - a||_2^2 \leq \theta ||x - b||_2^2\}$$

$$= \{x \mid (1 - \theta^2) x^T x - 2(a - \theta^2 b)^T x + (a^T a - \theta^2 b^T b) \leq 0\}$$

$$= \{x \mid (x - x_0)^T (x - x_0) \leq R^2\}$$

where,

$$x_0 = \frac{a - \theta^2 b}{1 - \theta^2}, R = (||x_0||_2^2 + \frac{\theta^2 ||b||_2^2 - ||a||_2^2}{1 - \theta^2})^{1/2}$$

2.15 *Some sets of probability distributions.* Let x be a real-valued random variable with $\mathbf{prob}(x = a_i) = p_i, i = 1, ..., n$, where $a_1 < a_2 < ... < a_n$. Of course $p \in \mathbf{R}^n$ lies in the standard probability simplex $P = \{p | \mathbf{1}^T p = 1, p \succeq 0\}$. Which of the following conditions are convex in p? (That is, for which of the following conditions is the set of $p \in P$ that satisfy the condition convex?)

(a) $\alpha \leq \mathbf{E}(f(x)) \leq \beta$

(b) $\mathbf{P}(x > \alpha) \leq \beta$

(c) $\mathbf{E}(|x^3|) \leq \alpha \mathbf{E}(|x|)$

(d) $\mathbf{E}(x^2) \leq \alpha$

(e) $\mathbf{E}(x^2) \geq \alpha$

Solution All of the above are linear inequalities in P, hence are convex.

(f) $\mathbf{Var}(x) \leq \alpha$

Solution not convex in general. As a counterexample, we can take $n = 2, a_1 = 0, a_2 = 1$, and $\alpha = 1/5$. $p = (1, 0)$ and $p = (0, 1)$ are two points that satisfy $\mathbf{Var}(x) \leq \alpha$, but the convex combination $p = (1/2, 1/2)$ does not.

(g) $\mathbf{Var}(x) \geq \alpha$

Solution The condition is equivalent to

$$\sum_{i=1}^{n} p_i a_i^2 - (\sum_{i=1}^{n} p_i a_i)^2 \geq \alpha$$

$$= b^T p - p^T A p \geq \alpha$$

where $(b_i = a_i^2, A = aa^T \succeq 0)$

Let p_1, p_2 satisfy the above condition. Consider their convex combination $\lambda p_1 + (1-\lambda) p_2$.

$$\begin{aligned}
& b^T(\lambda p_1 + (1-\lambda)p_2) - (\lambda p_1 + (1-\lambda)p_2)^T A(\lambda p_1 + (1-\lambda)p_2) \\
&= \lambda b^T p_1 + (1-\lambda) b^T p_2 - \lambda^2 p_1^T A p_1 - (1-\lambda)^2 p_2^T A p_2 - \lambda(1-\lambda)(p_1^T A p_2 + p_2^T A p_1) \\
&\geq \lambda(1-\lambda)(b^T p_1 + b^T p_2 - p_1^T A p_2 - p_2^T A p_1) + (1 + 2\lambda^2 - 2\lambda)\alpha \\
&\geq \alpha + \lambda(1-\lambda)(b^T p_1 + b^T p_2 - p_1^T A p_2 - p_2^T A p_1 - 2\alpha) \\
&\geq \alpha + \lambda(1-\lambda)(\alpha + p_1^T A p_1 + \alpha + p_2^T A p_2 - p_1^T A p_2 - p_2^T A p_1 - 2\alpha) \\
&\geq \alpha + \lambda(1-\lambda)((p_1 - p_2)^T A(p_1 - p_2)) \\
&\geq \alpha
\end{aligned}$$

The last inequality follows from $A \succeq 0$. Hence this condition is convex in P

2.28 Positive semidefinite cone for $n = 1, 2, 3$. Give an explicit description of the positive semidefinite cone $\mathbf{S}_+^n$, in terms of the matrix coefficients and ordinary inequalities, for $n = 1, 2, 3$. To describe a general element of $\mathbf{S}^n$, for $n = 1, 2, 3$, use the notation

$$x_1, \begin{bmatrix} x_1 & x_2 \\ x_2 & x_3 \end{bmatrix}, \begin{bmatrix} x_1 & x_2 & x_3 \\ x_2 & x_4 & x_5 \\ x_3 & x_5 & x_6 \end{bmatrix}$$

Solution **(i)** We know that $\mathbf{S}_+^n$ is a convex cone. We just need to ensure Positive semidefiniteness of each cone as they are already symmetric.

$$z^T x_1 z \geq 0 \implies x_1 \geq 0$$

(ii)

$$\begin{bmatrix}\alpha & \beta\end{bmatrix}\begin{bmatrix}x_1 & x_2 \\ x_2 & x_3\end{bmatrix}\begin{bmatrix}\alpha \\ \beta\end{bmatrix} \geq 0$$

$$\alpha^2 x_1 + 2\alpha\beta x_2 + \beta^2 x_3 \geq 0$$

$$\alpha = 0 \implies x_3 \geq 0, \beta = 0 \implies x_1 \geq 0$$

$$\alpha = 1, \beta = -1 \implies 2x_2 \leq x_1 + x_3 \implies x_2^2 \leq x_1 x_3$$

(iii)

$$\begin{bmatrix}\alpha & \beta & \gamma\end{bmatrix}\begin{bmatrix}x_1 & x_2 & x_3 \\ x_2 & x_4 & x_5 \\ x_3 & x_5 & x_6\end{bmatrix}\begin{bmatrix}\alpha \\ \beta \\ \gamma\end{bmatrix} \geq 0$$

$$\alpha^2 x_1 + \beta^2 x_4 + \gamma^2 x_6 + 2\alpha\beta x_2 + 2\beta\gamma x_5 + 2\gamma\alpha x_3 \geq 0$$

$$\beta, \gamma = 0 \implies x_1 \geq 0$$

$$\gamma, \alpha = 0 \implies x_4 \geq 0$$

$$\alpha, \beta = 0 \implies x_6 \geq 0$$

$$\alpha = 1, \beta = 0, \gamma = -1 \implies x_3^2 \leq x_1 x_6$$

$$\alpha = 1, \beta = -1, \gamma = 0 \implies x_2^2 \leq x_1 x_4$$

$$\alpha = 0, \beta = 1, \gamma = -1 \implies x_5^2 \leq x_4 x_6$$

3.15 *A family of concave utility functions.* For $0 < \alpha \leq 1$ let

$$u_\alpha(x) = \frac{x^\alpha - 1}{\alpha}$$

with **dom** $u_\alpha = \mathbf{R}_+$. We also define $u_0(x) = \log(x)$ (with **dom** $u_0 = \mathbf{R}_{++}$)

(a) Show that for $x > 0$,

$$u_0(x) = \lim_{\alpha \to 0} u_\alpha(x)$$

Solution

$$u_0(x) = \lim_{\alpha \to 0} \frac{e^{\alpha \ln x} - 1}{\alpha} \approx \lim_{\alpha \to 0} \frac{\alpha \ln x + (\alpha \ln x)^2/2}{\alpha} = \ln(x) = u_0(x)$$

(b) Show that u_α are concave, monotone increasing, and all satisfy $u_\alpha(1) = 0$.

Solution $u_\alpha^{'}(x) = x^{\alpha-1} > 0$ (monotone increasing)
$u_\alpha^{''}(x) = -(1-\alpha)x^{\alpha-2} < 0$ (concave)
$u_\alpha(1) = \frac{1^\alpha - 1}{\alpha} = 0$
These functions are often used in economics to model the benefit or utility of some quantity of goods or money. Concavity of u_α means that the marginal utility (i.e., the increase in utility obtained for a fixed increase in the goods) decreases as the amount of goods increases. In other words, concavity models the effect of *satiation*.