EE364m Exercise Set 1

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1.1

a). V is a uniformly convex Banach Space. We have to show that

$$\pi_C(x) := \operatorname*{argmin}_{y \in C} ||y - x||$$

exists and is unique.

WLOG, Assume $x = 0$ and $0 \notin C$. Therefore,

$$\pi_C(x) := \operatorname*{argmin}_{y \in C} ||y||$$

From the property of norm, since $0 \notin C$, $||y|| > 0$. Therefore, $\pi_C(x)$ exists.

Suppose $\pi_C(x)$ was not unique. i.e $\exists y_1$ and y_2 both of which minimise $||y||$.

WLOG, let

$$||y_1|| = ||y_2|| = 1$$

. (V is closed under scalar multiplication).

$$y_1 \neq y_2 \implies ||y_1 - y_2|| \neq 0$$

$$\text{i.e } \exists \text{ an } \epsilon \in \mathbb{R} \text{ s.t } ||y_1 - y_2|| > \epsilon > 0$$

Since $||y_1|| = ||y_2|| = 1$, $||y_1 - y_2|| \leq 2$. Since V is uniformly convex, the above conditions imply that $\exists \delta = \delta(\epsilon) > 0$ s.t

$$\left\| \frac{y_1 + y_2}{2} \right\| \leq 1 - \delta$$

. Contradiction, as $||y_1|| = ||y_2|| = 1$ were minimisers of $|| \cdot ||$.

b). Let $\|\cdot\| = \|\cdot\|_1$ be the ℓ_1-norm. Consider $x = 0$. Consider the Convex set C to be the line segment

$$||\mathbf{x}||_1 = 1, |x_1| \leq 0, \ \ |x_i| \geq 0 \ \ \forall i > 2$$

All points on C satisfy $||y||_1 = 1$,

Therefore, $\pi_C(x) = \operatorname*{argmin}_{y \in C} ||y - x|| = \ ||y||$ is not unique.

Hence, $\mathbb{R}^n$ is not uniformly convex with ℓ_1-norm

c). Let $\|\cdot\| = \|\cdot\|_\infty$ be the ℓ_∞-norm. Consider $x = 0$. Consider the Convex set C to be the line segment

$$||\mathbf{x}||_\infty = 1, x_1 = 1$$

All points on C satisfy $||y||_\infty = 1$
Therefore, $\pi_C(x) = \underset{y \in C}{\operatorname{argmin}} ||y - x|| = \ ||y||$ is not unique.

Hence, $\mathbb{R}^n$ is not uniformly convex with ℓ_∞-norm

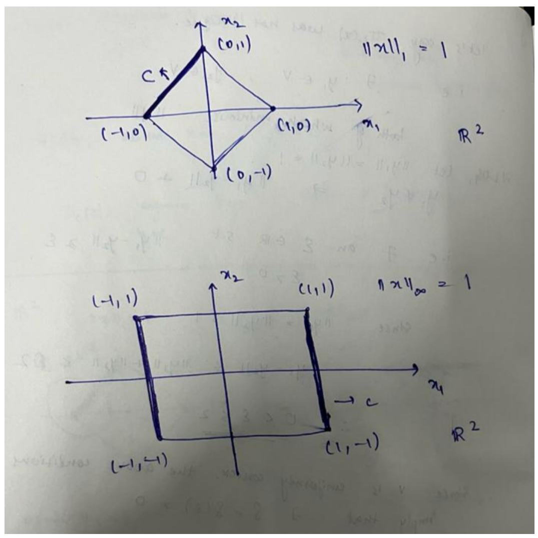

Figure 1: Illustration for n=2.