Differential Equation & It's solution

An equation which involves differential coefficien is called differential eqn.

Eg:- $\frac{dy}{dx} = \frac{1-x^2}{1-y^2}$ —— (1)

$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 8y = 0.$ —— (2)

$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$ —— (3)

$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial z}{\partial y}$ —— (2)

Types of Differential Eqn -

1. Ordinary D.E - A D.E involving derivate with respect to a single independent variable is called O.D.E.

2. Partial DE An Eqn involving partial derivative w.r.t more than one independent variable is called P.D.E.

Order & Degree of the Diff. Eqn :-

- The order of a diff. eqn is the order of the highest diff. coeff. present in the eqn.
- The Degree of a diff. eqn is the power of highest derivative after removing the

radicle signs & fractions.

Eg –

$$\left[1+\left(\frac{dy}{dx}\right)^2\right]^3 = \left(\frac{d^2y}{dx^2}\right)^2$$

Order – 2

degree – 2

Solution of a diff. eqn. –

An eqn containing dependent variable y & independent variable x and free from derivatives, which satisfies the diff. eqn. is called sol. of the diff. eqn.

Differential Eqn. of 2nd Order.

The general form of linear diff. eqn of 2nd Order with constant coeffis is

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R.$$

where P & Q are constant & R is a function of x and maybe a constant.

Sol. of 2nd Order diff. eqn. linear

Solution = complementary function + Particular Integral

Methods to find out complementary functions (C.F)

To find C.F, Ist we right the Auxiliary eqn by replacing

$\frac{d^3y}{dx^3}$ by m^3, $\frac{d^2y}{dx^2}$ by m^2, $\frac{dy}{dx}$ by m and so on.

Then we get eqⁿ of the for.

$m^2 + Pm + Qy = 0$

Then we factorize the auxiliary eqⁿ.

(I) If the roots of auxiliary eqⁿ are ~~real~~ distinct then say m_1, m_2 -- etc. then

C.F is $y = C_1 e^{m_1 x} + C_2 e^{m_2 x}$.

(II) If the roots of auxiliary eqⁿ are real but repeated say $m_1 = m_2 = m$

then C.F is $y = (C_1 + C_2 x) e^{mx}$

(III) If $m_1 = m_2 = m_3 = m$.

C.F is $y = (C_1 + C_2 x + C_3 x^2) e^{mx}$

(IV) When the roots are imaginary say $\alpha \pm i\beta$

then C.F is $y = e^{\alpha x}(C_1 \cos \beta x + C_2 \sin \beta x)$

(V) ~~Wh~~ When the roots of auxiliary eqⁿ are imginary but repeated ~~say~~

$y = e^{\alpha x}\left((C_1 + C_2 x)\cos \beta x + (C_3 + C_4 x)\sin \beta x\right)$

Ques - Solve $\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 15y = 0$.

$\boxed{D = \frac{d}{dx}}$

$(D^2 - 8D + 15)y = 0$

Replace D by m.

A.E is $(m^2 - 8m + 15) = 0$

$m^2 - 3m - 5m + 15 = 0$.

$m(m-3) - 5(m-3) = 0$

$m = 3, 5$.

C.F is $\boxed{y = C_1 e^{3x} + C_2 e^{5x}}$

Ques - Solve $\frac{d^2y}{dx} + 4\frac{dy}{dx} + 5y = 0$.

$(D^2 + 4D + 5)y = 0$.

$m^2 + 4m + 5 = 0$,

$m = -2 \pm i$

$$\boxed{y = e^{-2x}(c_1 \cos x + c_2 \sin x)}$$

Ques → (1) $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0.$

(2) $\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 16y = 0.$

(3) $\frac{d^5y}{dx^5} - \frac{d^3y}{dx^3} = 0.$

1. $(D^2 - 3D + 2)y = 0.$

Replace D by m.

$m^2 - 3m + 2 = 0.$

$m^2 - m - 2m + 2 = 0$

$m(m-1) - 2(m-1) = 0$

$(m-2)(m-1) = 0.$

$m = 2, 1$

$$\boxed{y = C_1 e^{2x} + C_2 e^x}$$

2. $(D^2 - 8D + 16)y = 0$

Replace D by m

$m^2 - 8m + 16 = 0$

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

$$m = \frac{-8 \pm \sqrt{64 - (4)(1)(16)}}{2(1)}$$

$$m = \frac{-8 \pm \sqrt{64 - 64}}{2}$$

$m = -4.$

$y = (C_1 + C_2 x)e^{-4x}.$

Solution of Linear differential equation with contant coefficients

Solution = C.F + P.I.

Methods to find out P.I. (Perticular Integral)

Type I. If $X = e^{ax}$

Write $\frac{1}{f(D)} e^{ax}$.

put $D = a$.

$e^{ax} \frac{1}{f(a)}$ if $f(a) \neq 0$.

Ques - Solve $(D^2+6D+5)y = 16e^{3x}$.

For C.F., A.E is $m^2+6m+5=0$

$m = -1, -5$

$C_1e^{-x} + C_2e^{-5x}$.

P.I. $\frac{1}{D^2+6D+5} 16e^{3x}$

$= 16 \frac{e^{3x}}{9+6\times3+5}$

$= \frac{16}{32} e^{3x}$

$= \frac{e^{3x}}{2}$

$y = C_1e^{-x} + C_2e^{-5x} + \frac{e^{3x}}{2}$.

~~Type II~~ If $X = e^{ax}$.

$\frac{1}{f(D)} e^{ax}$.

but if $f(a) = 0$.

find $f'(D)$

Replace D by a in $f'(D)$

And the P.I will be

$\frac{x e^{ax}}{f'(a)}$

Ques -

~~For the above question~~

~~if $f(a) = 0$.~~

Ques - $(D^2+6D+5)y = 7e^{-x} + 8e^{0x}$

C.F., $C_1 e^{-x} + C_2 e^{-5x}$.

P.I$_1$. $\dfrac{1}{D^2+6D+5} 7e^{-x}$

$f(D) = D^2+6D+5$.

$f(a) = 1-6+5$

$f(a) = 0$.

$f'(D) = 2D+6$, $f'(a) = 4$.

$(P.I)_1 = \dfrac{7xe^{-x}}{4}$.

$(P.I)_2 \quad \dfrac{1}{D^2+6D+5} 8e^{0x}$.

$(P.I)_2 = \dfrac{8e^{0x}}{5}$

$(P.I) = \dfrac{8}{5}$.

~~Sol~~ $y = C_1 e^{-x} + C_2 e^{-5x} + \dfrac{7xe^{-x}}{4} + \dfrac{8}{5}$

If $f'(a) = 0$.
Calculate $f''(D)$ find $f''(a)$

P.I $= \dfrac{x^2 e^{ax}}{f''(a)}$

Type II $X = \cos ax$. or $\sin ax$.

Replace D^2 by $(-a^2)$ in $f(D^2)$

if $f(-a^2) \neq 0$.

P.I, $\dfrac{\sin ax}{f(-a^2)}$ or $\dfrac{\cos ax}{f(-a^2)}$

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Find PI of $(D^3+1)y = \sin(2x+3)$

$$P.I = \frac{1}{D^3+1}\sin(2x+3)$$

$$\frac{1}{(D^2 \cdot D+1)}\sin(2x+3)$$

$$\frac{1}{-4D+1}\sin(2x+3)$$

$(a+b)(a-b)$
$= a^2-b^2$

$\frac{d}{dx}$

$$\frac{1+4D}{(1-4D)(1+4D)}\sin(2x+3)$$

$$\frac{1+4D}{1-16D^2}\sin(2x+3)$$

$D^2 = -(a^2)$

$$\frac{(1+4D)(\sin(2x+3))}{65}$$

$D = \frac{d}{dx}$

$$\frac{1}{65}\left[\sin(2x+3) + 2\times4\cos(2x+3)\right]$$

$$\frac{1}{65}\left[\sin(2x+3) + 8\cos(2x+3)\right]$$

Ques -
Ans -

$$\boxed{\begin{array}{l}\text{When } f(-a^2) = 0. \\ \text{Calculate } f'(D^2) \text{ \& replace } D^2 \text{ by } -a^2 \text{ in } f'(D^2) \\ P.I. \quad x \cdot \dfrac{1}{f'(-a^2)} \sin ax \text{ or } \cos ax\end{array}}$$

Find P.I. of $(D^2+4)y = \cos 2x$.

$$P.I = \frac{1}{D^2+4}\cos 2x$$

$f(-a^2) = 0$.
$f'(D) = 2D$

$D^2 = -a^2$

$$\frac{D}{2D^2}\cos 2x.$$

$$= \frac{D}{2D^2}\cos 2x = x\frac{D}{-8}\cos 2x = +\frac{x}{4}\sin 2x.$$

$a^2 - b^2 = (a+b)(a-b)$

Ques- Solve $(D^3 + D^2 - D - 1)y = \cos 2x + 7$

Ans- For C.F., A.E is $m^3 + m^2 - m - 1$

Put $m = 1$

$$m-1 \overline{\big)\, m^3 + m^2 - m - 1 \,\big(}\; m^2 + 2m + 1$$

$$\begin{array}{l} m^3 - m^2 \\ \hline 2m^2 - m - 1 \\ 2m^2 - 2m - 1 \\ \hline m - 1 \\ m - 1 \\ \hline 0 \end{array}$$

$m^3 + m^2 - m - 1$

$m^2(m+1) - 1(m+1)$

$(m+1)(m^2-1)$

$m = -1$, $m = \pm 1$

$m^2 + 2m + 1$

$m^2 + m + m + 1$

$m(m+1) + 1(m+1)$

$\boxed{m = -1, 1, -1}$

$$\boxed{C.F. = C_1 e^{-x} + C_2 e^{x} + C_3 e^{-x}}$$

$$P.I. = \frac{1}{D^3 + D^2 - D - 1} \cos 2x + 7$$

$$= \frac{1}{-4D + (-4) - D - 1} \cos 2x + 7.$$

$$= \frac{1}{-5D - 5} \cos 2x + 7$$

$$= \frac{1}{-5} \frac{1}{(D+1)} \cos 2x + 7$$

$$= \frac{D-1}{-5(D+1)(D-1)} \cos 2x + 7.$$

$$= \frac{D-1}{(-5)D^2 - 1} \cos 2x + 7.$$

$$= \frac{D-1}{(-5)(-5)} \cos 2x + 7 = \frac{(D-1)\cos 2x + 7}{25}$$

$$= \frac{1}{25}\left[\frac{d}{dx} \cos 2x + 7 - \cos 2x + 7\right]$$

$P.I = \frac{1}{25}\left(-2\sin(2x+7) - \cos(2x+7)\right)$

$$\boxed{y = C_1 e^{-x} + C_2 e^{x} + C_3 e^{-x} - \frac{1}{25}\left(2\sin(2x+7) + \cos(2x+7)\right)}$$

When $X = x^n$

1. write P.I. as $\frac{1}{f(D)} x^n$
2. make $f(D)$ in form of $(1+D)^n$ or $(1-D)^n$
   Apply the binomial expansion in $(1+D)^n$ & $(1-D)^n$
4. Operate the resulting expression on x^n

→ Solve $(D^3 - D^2 - 6D)y = 1 + x^2$

→ for C.F.

A.E is $m^3 - m^2 - 6m = 0$

$m(m^2 - m - 6) = 0$

$m(m^2 - 3m + 2m - 6) = 0$

$m = 0, -2, 3$

CF is $C_1 e^{0x} + C_2 e^{-2x} + C_3 e^{3x}$

$P.I = \frac{1}{D^3 - D^2 - 6D}(1 + x^2)$

$1 - x$

$= \frac{1}{-6D\left[1 - \left(\frac{D^3 - D^2}{6}\right)\right]}(1 + x^2)$

$= -\frac{1}{6D\left[1 - \left(\frac{D^2}{6} - \frac{D}{6}\right)\right]}(1 + x^2)$

$Dy = 2e^{ax}$

$= -\frac{1}{6D}\left[1 + \left(\frac{D}{6} - \frac{D^2}{6}\right)\right]^{-1}(1 + x^2)$

$\frac{1}{D}$

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note – $(1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 \ldots$
$(1-x)^{-1} = 1 + x + x^2 + x^3 + x^4 \ldots$
$(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + 5x^4 \ldots$
$(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + 5x^4 \ldots$
$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots$

$$= \frac{-1}{6D}\left[1 - \left(\frac{D}{6} - \frac{D^2}{6}\right) + \left(\frac{D}{6} - \frac{D^2}{6}\right)^2 + \ldots\right](1+x^2)$$

$$= \frac{-1}{6D}\left[1 - \frac{D}{6} + \frac{D^2}{6} + \frac{D^2}{36}\right](1+x^2)$$

$$= \frac{-1}{6D}\left[1 - \frac{D}{6} + \frac{7}{36}D^2 + \ldots\right](1+x^2)$$

$$= \frac{-1}{6D}\left[(1+x^2) - \frac{1}{6}(2x) + \frac{7}{36}\times 2\right]$$

I $$= \frac{-1}{6D}\left[1 + x^2 - \frac{1}{3}x + \frac{7}{18}\right]$$

$\boxed{\frac{1}{D} = \text{Integration}}$

$$= -\frac{1}{6}x - \frac{1}{6}\frac{x^3}{3} + \frac{1}{18}\frac{x^2}{2} - \frac{7}{18\times 6}x$$

$$= \frac{-7-18x}{18\times 6} + \frac{x^2}{36} - \frac{1}{18}x^3$$

$$\boxed{= \frac{-25x}{108} + \frac{x^2}{36} - \frac{1}{18}x^3.}$$

$$y = C_1 + C_2e^{-2x} + C_3e^{3x} - \frac{25}{108}x + \frac{x^2}{36} - \frac{1}{18}x^3$$

when $X = \boxed{e^{ax}}\boxed{V} \rightarrow x^n \,/\, \sin ax$

1. $e^{ax}\dfrac{1}{f(D+a)}V$

2. Now apply the same rules for V as it is the case of x^n, $\sin ax$, $\cos ax$.

Solve $(D^2-4D+4)y = x^3 e^{2x}$.

$m^2-4m+4=0$

$(m-2)^2=0$

$m=2,2$.

CF is $(C_1+C_2x)e^{2x}$

P.I. $\dfrac{1}{D^2-4D+4} x^3 e^{2x}$

$e^{2x} \dfrac{1}{(D+2)^2-4(D+2)+4} \cdot x^3$

$e^{2x} \dfrac{1}{D^2+4+4D-4D-8+4} \cdot x^3$

$e^{2x} \dfrac{1}{D^2} x^3$

$\frac{1}{D} \cdot \frac{1}{D}$ $\quad e^{2x} \dfrac{1}{D^2} \cdot x^3$

D $\quad e^{2x} \dfrac{1}{D} \cdot \dfrac{x^4}{4}$.

$\downarrow$

$D+a$ $\quad e^{2x} \dfrac{x^5}{20}$

x

PI is $\dfrac{x^5}{20} e^{2x}$.

$\frac{1}{D} \cdot \frac{1}{D}$ $\quad y = (C_1+C_2x)e^{2x} + \dfrac{x^5}{20} e^{2x}$.

$\dfrac{d^3y}{dx^3} - 7\dfrac{d^2y}{dx^2} + 10\dfrac{dy}{dx} = e^{2x} \sin x$.

$(D^3-7D^2+10D)y = e^{2x}\sin x$.

A.E is $m^3-7m^2+10m=0$

$m(m^2-7m+10)=0$.

$m(m^2-5m-2m+10)=0$

$m=0,5,2$.

CF is $C_1 e^{0x} + C_2 e^{2x} + C_3 e^{5x}$

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P.I. $\dfrac{1}{D^3 - 7D^2 + 10D} e^{2x} \sin x.$

$e^{2x} \dfrac{1}{(D+2)^3 - 7(D+2)^2 + 10(D+2)} \sin x.$

$e^{2x} \dfrac{1}{D^3 + 6D^2 + 12D + 8 - 7D^2 - 28 - 28D + 10D + 20} \sin x.$

$e^{2x} \dfrac{1}{D^3 - D^2 - 6D - 0} \sin x.$

$e^{2x} \dfrac{1}{D^3 - D^2 - 6D} \sin x.$

$e^{2x} \dfrac{1}{-D + 1 - 6D} \sin x.$

$e^{2x} \dfrac{1}{1 - 7D} \sin x.$

$e^{2x} \dfrac{1 + 7D}{1 - 49D^2} \sin x.$

$e^{2x} \dfrac{1 + 7D}{1 + 49} \sin x.$

$e^{2x} \dfrac{\sin x + 7\cos x}{50}.$

$$y = C_1 + C_2 e^{2x} + C_3 e^{5x} + e^{2x} \frac{\sin x + 7\cos x}{50}.$$

1. Solve $(D^3 + 4D^2 + 4D) y = x^2 e^{-2x}$
2. Solve $(D^2 + 2D + 3) y = 5x^2$

$x + y = 5$
$2x + 3y = 10$

Simultaneous differential Equation

$\frac{dx}{dt} = y + 1 \qquad \frac{dy}{dt} = x + 1$

$\frac{d}{dt} = D$ Re-writing the eqn.

$Dx - y = 1$ —— ①

$-x + Dy = 1$. —— ②

$e^{ot} = 1$

Multiply ① by D and add in ②

$D^2 x - Dy = D$

$-x + Dy = 1$

$D^2 x - x = D+1$

$\Rightarrow (D^2-1)x = D+1$.

$(D^2-1)x = 1 + D \cdot 1$

$(D^2-1)x = 1$.

$\boxed{(D^2-1)x = e^{0t}}$

for C.F

A.E $m^2 - 1 = 0$

$m^2 = 1$

$m = 1, -1$.

C.F $C_1 e^{t} + C_2 e^{-t}$.

for P.I.

$\frac{1}{D^2-1} e^{0t}$.

$\frac{1}{-1} e^{0t}$.

$-e^{0t}$

-1.

$x = C_1 e^{t} + C_2 e^{-t} - 1$.

for y,

$y = \frac{d}{dt}(C_1 e^{t} + C_2 e^{-t} - 1) - 1$

CF

P.S

$y = c_1 e^{t} - c_2 e^{-t} - 1.$

Solve $\frac{dx}{dt} + 5x + y = e^{t}, \quad \frac{dy}{dt} + x + 5y = e^{5t}.$

$Dx + 5x + y = e^{t}, \quad Dy + x + 5y = e^{5t}$

$(D+5)x + y = e^{t}, \quad (D+5)y + x = e^{5t}$

Multiply ① by $D+5$.

$$(D+5)^2 x + \cancel{(D+5)y} = (D+5)e^{t}$$

$$x + \cancel{(D+5)y} = e^{5t}$$

$$((D+5)^2 - 1)x = (D+5)e^{t} - e^{5t}$$

$$((D+5)^2 - 1)x = e^{t} + 5e^{t} - e^{5t}$$

$$(D^2 + 25 + 10D - 1)x = 6e^{t} - e^{5t}$$

For C.F.

A.E. $m^2 + 25 + 10m - 1 = 0$

$m^2 + 10m + 24 = 0.$

$m = -4, -6.$

C.F is $c_1 e^{-4t} + c_2 e^{-6t}.$

P.I, $\frac{1}{D^2 + 10D + 24} \, 6e^{t} - e^{5t}.$

$\frac{6e^{t}}{35} - \frac{e^{5t}}{99}.$

$$x = c_1 e^{-4t} + c_2 e^{-6t} + \frac{6}{35}e^{t} - \frac{1}{99}e^{5t}$$

$$\therefore y = e^{t} - \frac{dx}{dt} - 5x.$$

$$= e^{t} - \frac{d}{dt}\left(c_1 e^{-4t} + c_2 e^{-6t} + \frac{6}{35}e^{t} - \frac{1}{99}e^{5t}\right) - 5(x)$$

$$= e^{t} - \left(-4c_1 e^{-4t} - 6c_2 e^{-6t} + \frac{6}{35}e^{t} - \frac{5}{99}e^{5t}\right) - 5\left(c_1 e^{-4t} + c_2 e^{-6t} + \frac{6}{35}e^{t} - \frac{1}{99}e^{5t}\right)$$

$$= e^{t} + 4c_1 e^{-4t} + 6c_2 e^{-6t} - \frac{6}{35}e^{t} + \frac{5}{99}e^{5t} - 5c_1 e^{-4t} - 5c_2 e^{-6t} - \frac{30e^{t}}{35} + \frac{5}{99}e^{5t}$$

$$y = e^{t} - C_1 e^{-4t} + C_2 e^{-6t} - \frac{36}{35} e^{t} + \frac{10}{99} e^{5t}$$

$$y = \frac{-e^{t}}{35} - C_1 e^{-4t} + C_2 e^{-6t} + \frac{10}{99} e^{5t}$$

* $(D-3)x + 2(D+2)y = e^{2t}$

$2(D+1)x + (D-1)y = 0$

* $\frac{dx}{dt} - y = t$

$\frac{dy}{dt} + x = t^2$

* $\frac{dx}{dt} + y = \sin t$

$\frac{dy}{dt} + x = \cos t$

* $(2D-3)x + Dy = e^{t}$

$Dx + (D+2)y = \cos 2t$

→ $(D-3)x + 2(D+2)y = e^{2t}$ — ①

$2(D+1)x + (D-1)y = 0.$ — ②

Multiply ① by $2(D+1)$ & ② by $(D-3)$. Then substract ① from ②.

$$2(D+1)(D-3)x + 2(D+1)2(D+2)y = 2(D+1)e^{2t}$$
$$2(D+1)(D-3)x + (D-3)(D-1)y = 0$$

$$[4(D+1)(D+2) - (D-3)(D-1)]y = 2(D+1)e^{2t}$$

$\therefore$ $[4(D^2+2D+D+2) - D^2-D-3D+3]y = 2(D+1)e^{2t}$

$[4(D^2+3D+2) - (D^2-4D+3)]y = 2(D+1)e^{2t}.$

$[3D^2+16D+5]y = (2D+2)e^{2t}$

$$\cancel{\left[\frac{3D^2+16D+5}{2D+2}\right] y = e^{2t}}$$

~~for CF, A.E $\rightarrow \frac{3m^2+16m+5}{2m+2} = 0$.~~

$[3D^2+16D+5]\, y = 4e^{2t} + 2e^{2t}$

$(3D^2+16D+5)\, y = 6e^{2t}$

for C.F, A.E is $3m^2+16m+5=0$.

$$m = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$= \frac{-16 \pm \sqrt{256-60}}{6}$$

$$= \frac{-16 \pm \sqrt{196}}{6}.$$

$$= \frac{-16 \pm 14}{6} = \frac{-2}{6} \text{ or } \frac{-30}{6} = \frac{-1}{3} \text{ or } -5.$$

CF is $C_1 e^{-t/3} + C_2 e^{-5t}$

Or.

Multiply ① by $(D-1)$ & ② by $2(D+2)$.

$(D-3)(D-1)x + \cancel{2(D+2)(D-1)y} = e^{2t}(D-1)$

$2(D+2)(2)(D+1)x + \cancel{2(D+2)(D-1)y} = 0.$

$[(D^2-4D+3) - 4(D^2+3D+2)]\, x = e^{2t}(D-1)$

$(-3D^2-16D-5)\, x = e^{2t}(D-1)$

$(3D^2+16D+5)\, x = e^{2t} - 2e^{2t}$

$(3D^2+16D+5)\, x = -e^{2t}.$

for C.F, A.E is $3m^2+16m+5=0$.

$$m = \frac{-16 \pm \sqrt{256-60}}{6}$$

$m = \frac{-1}{3}$ or -5.

CF is $C_1 e^{-t/3} + C_2 e^{-5t}$

P.I. $\frac{1}{3D^2+16D+5} \; -e^{2t}.$

$$\frac{1}{+59} - e^{2t}$$

PI is $\frac{-e^{2t}}{59}$

$$x = C_1 e^{-t/3} + C_2 e^{-5t} - e^{2t}/59.$$

$$y = \frac{-2(D+1)}{D-1} x$$

$$y = \frac{2D+2}{1-D} x.$$

$$y = \frac{2D+2}{1-D} x = \frac{2(D+1)^2}{1^2 - D^2} x$$

$$= -(2D+2)(1-D)^{-1} x$$

$$D'y = \frac{dy}{dz}$$

$$\frac{32}{59}$$

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EULER'S HOMOGENEOUS LINEAR DIFFERENTIAL EQUATIONS

$$a_n x^n \frac{d^n x}{dy^n} + a_{n-1} x^{n-1} \frac{d^{n-1} x}{dy^{n-1}} + \dots + a_1 x \frac{dy}{dx} + a y = f(x)$$

$$x = e^z$$

$$z = \log x$$

$$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx}.$$

$$\frac{dy}{dx} = \frac{1}{x} \frac{dy}{dz}$$

$$x \frac{dy}{dx} = \frac{dy}{dz}$$

$$x D y = D' y. \quad \text{— (1)}$$

$$x^2 D^2 y = D'(D'-1) y. \quad \text{— (2)}$$

$$x^3 D^3 y = D'(D'-1)(D'-2) y. \quad \text{— (3)}$$

To solve Euler's Equation first we reduce the eqn using (1), (2) & (2)

Then the eqn will be transformed into linear differential eqn with constt coeff. and we will follow the rules to ~~follo~~ solve it.

After finding the soln substitute $z = \log x$ we will get the final soln.

Solve $\boxed{x^3} y'''(x) + 2x^2 y''(x) - x y'(x) + y(x) = \log x$.

Take $x = e^z$

& $\boxed{z = \log x}$.

Replace $x D y = D' y$

$$x^2 D^2 y = D'(D'-1) y$$

$$x^3 D^3 y = D'(D'-1)(D'-2) y.$$

$y'''(x)$

$\frac{d^3 y}{dx^3}$

$$[D'(D'-1)(D'-2) + 2D'(D'-1) - D' + 1] y = z.$$

$$[D'(D'^2 - 3D' + 2) + 2D'^2 - 2D' - D' + 1] y = z$$

$$[D'^3 - 3D'^2 + 2D' + 2D'^2 - 2D' - D' + 1] y = z.$$

$\frac{dy}{dx}$

$$[D'^3 - D'^2 - D' + 1] y = z.$$

For C.F, A.E is $m^3 - m^2 - m + 1 = 0$

$(m^2-1)(m-1) = 0$

$m = 1, 1, -1$.

CF is $(C_1 + C_2 z)e^{z} + C_3 e^{-z}$

P.I $\frac{1}{(D'^3 - D'^2 - D' + 1)} z$

$\frac{1}{1 + (D'^3 - D'^2 - D')} z$.

$[1 + (D'^3 - D'^2 - D')]^{-1} z$

$[1 - (D'^3 - D'^2 - D') + \cdots] z$.

$[1 + D'] z$.

$z + 1$

$y = (C_1 + C_2 z)e^{z} + C_3 e^{-z} + z + 1$.

$D' = \frac{d}{dz}$

$$\boxed{y = (C_1 + C_2 \log x)x + C_3 \frac{1}{x} + \log x + 1.}$$

$m^3 - m^2 - m + 1$

$m^2(m-1) - 1(m-1)$

$(m^2-1)(m-1)$

→ Solve $[x^2D^2 - 3xD + 4]y = x^2(\cos(\log x))$

Take $x = e^{z}$ & $z = \log x$.

Replace $xDy = D'y$.

$x^2D^2y = D'(D'-1)y$.

$[D'(D'-1) - 3D' + 4]y = e^{2z}\cos z$

A.E $m^2 - 4m + 4 = 0$. $\Rightarrow (m-2)^2 = 0$.

$\Rightarrow m = 2, 2$

CF is $(C_1 + C_2 z)e^{2z}$

P.I $\frac{1}{D'^2 - 4D' + 4} e^{2z}\cos z$.

$e^{2z} \frac{1}{(D'+2)^2 - 4(D'+2) + 4} \cos z$.

$e^{2z} \frac{1}{D'^2 + 4D' + 4 - 4D' - 8 + 4} \cos z$

$e^{2z} \frac{1}{D'^2} \cos z$.

$e^{2z} \dfrac{1}{-1^2} \cos z.$

$- e^{2z} \cos z.$

$y = (C_1 + C_2 z) e^{2z} - e^{2z} \cos z.$

$= (C_1 + C_2 \log x) x^2 - x^2 \cos(\log x.$

Solve $x^2 y'' - x y' + y = x^2$.

Take $x = e^z$ & $z = \log x$.

Replace $x^2 D^2 y = D' y \cdot (D' - 1)$

$x D y = D' y.$

$[D'(D'-1) - D' + 1] y = e^{2z}.$

$[D'^2 - D' - D' + 1] y = e^{2z}$

$[D'^2 - 2D' + 1] y = e^{2z}$

A.E $m^2 - 2m + 1$

$m^2 - m - m + 1$

$m(m-1) - 1(m-1)$

$m = 1, 1$

CF. is $(C_1 + C_2 z) e^z$

P.I $\dfrac{1}{D'^2 - 2D' + 1} e^{2z}$

$e^{2z} \dfrac{1}{(D'-2)^2 - 2(D'-2) + 1}$

$e^{2z} \dfrac{1}{D'^2 - 4D' + 4 - 2D' + 4 + 1}$

$e^{2z} \dfrac{1}{D'^2 - 2D' + 9}$

$e^{2z} \dfrac{1}{1 - 2 + 9}$

$\dfrac{e^{2z}}{8}.$

$\dfrac{e^{2z}}{4 - 4 + 1}$

$\dfrac{e^{2z}}{1}$

P.I is e^{2z}.

$y = (C_1 + C_2 z)e^z + \frac{e^{2z}}{8}$

$= (C_1 + C_2 \log x)x + \frac{x^2}{8}$

LEGENDERE'S LINEAR DIFFERENTIAL EQUATIONS

Eqn's reducable to homogeneous form.

$(ax+b)^n \frac{d^n y}{dx^n} + (ax+b)^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \dots + (ax+b)\frac{dy}{dx} + y = f.$

$(ax+b) = e^z$, $z = \log(ax+b)$

$\boxed{(ax+b)} Dy = aD'y.$

$(ax+b)^2 D^2 y = a^2 D'(D'-1)y.$

$(ax+b)^3 D^3 y = a^3 D'(D'-1)(D'-2)y.$

Then Solve the eqn by usual method.

Solve $(2x+3)^2 y'' - (2x+3)y' - 12y = 6x.$

$2x+3 = e^z$ & $z = \log(2x+3)$

$[4D'(D'-1) - 2D' - 12]y = 6\left(\frac{e^z - 3}{2}\right)$

$[4(D'^2 - D') - 2D' - 12]y = 3e^z - 9$

$[4D'^2 - 4D' - 2D' - 12]y = 3e^z - 9.$

$[4D'^2 - 6D' - 12]y = 3e^z - 9.$

A.E $= 4m^2 - 6m - 12.$

$m =$

(1) $x^2 y'' - x y' + y = \log x + \pi$

(2) $x^2 \frac{d^2y}{dx^2} - 7x \frac{dy}{dx} + 12y = x^2$

(3) $[x^3D^3 + 3x^2D^2 + xD - 8]y = 7x - 6$

(4) $x^3 \frac{d^3y}{dx^3} + 2x^2 \frac{d^2y}{dx^2} + 2y = 10\left(x + \frac{1}{x}\right)$

(5) $[(2x-1)^2D^2 - 4(2x-1)D - 8]y = 8x$

Partial Differentiation

$y = f(x)$

$\frac{dy}{dx} = f'(x)$

$y = \cos x.$

$\frac{dy}{dx} = -\sin x.$

$z = f(x, y)$

$\frac{\partial z}{\partial x}$

$\boxed{z = ax^2 + 2hxy + by^2}$

$\frac{\partial z}{\partial x} = \boxed{2ax + 2hy}$

$\frac{\partial z}{\partial y} = 2hx + 2by$

$\frac{\partial^2 z}{\partial y \, \partial x} = 2h$

$\frac{\partial^2 z}{\partial x \, \partial y} = 2h$

When we have to diff. the function w.r.t x we treat y as a constt. & diff. the function by usual diff. rules.
Ily for y we treat x as a constt.

$\boxed{u = ax^3 + bx^2 y + cxy^2 + dy^3}$

$\frac{\partial u}{\partial x} = 3ax^2 + 2bxy + cy^2$

$\frac{\partial u}{\partial y} = bx^2 + 2cxy + 3dy^2$

$\frac{\partial^2 u}{\partial x^2} = 6ax + 2by$

$$\frac{\partial^3 u}{\partial x^3} = 6a$$

$$\frac{\partial^2 u}{\partial y^2} = 2cx + 6dy$$

$$\frac{\partial^3 u}{\partial y^3} = 6d$$

$$\frac{\partial^2 u}{\partial y \partial x} = 2bx + 2cy$$

$$\frac{\partial^2 u}{\partial x \partial y} = 2bx + 2cy$$

→ $u = \log(x^3 + y^3 + z^3 - 3xyz)$

$$\frac{\partial u}{\partial x} = \frac{1}{x^3 + y^3 + z^3 - 3xyz}(3x^2 - 3yz)$$

$$\frac{\partial u}{\partial y} = \frac{1}{x^3 + y^3 + z^3 - 3xyz}(3y^2 - 3xz)$$

→ $z = (1 - 2xy + y^2)^{-1/2}$

calculate $x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}$

$$x\frac{\partial z}{\partial x} = -\frac{1}{2}(1 - 2xy + y^2)^{-3/2}(-2yx)$$

$$y\frac{\partial z}{\partial y} = -\frac{1}{2}(1 - 2xy + y^2)^{-3/2}(-2xy + 2y^2)$$

$$x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y} = -\frac{1}{2}(1 - 2xy + y^2)^{-3/2}(-2y^2)$$

$$= (1 - 2xy + y^2)^{-3/2} y^2 = y^2 z^3 = y^2 z^3$$

→ $u = x^2 \tan^{-1}\frac{y}{x} - y^2 \tan^{-1}\frac{x}{y}$

Calculate $\frac{\partial^2 u}{\partial x \partial y}$ & $\frac{\partial^2 u}{\partial y \partial x}$.

$$\frac{\partial u}{\partial y} = x^2 \frac{1}{1+(y/x)^2}\left(\frac{1}{x}\right) - \left[2y \tan^{-1}\frac{x}{y} + y^2\left(\frac{1}{1+(x/y)^2}\right)\left(-\frac{x}{y^2}\right)\right]$$

$$= x \frac{1}{1+(y/x)^2} - \left[2y \tan^{-1}\frac{x}{y} - \frac{x}{1+(x/y)^2}\right]$$

$$= \left[\frac{1}{1+(y/x)^2} + \frac{1}{1+(x/y)^2}\right]x - 2y \tan^{-1}\frac{x}{y}$$

→ $z = e^{ax+by} f(ax-by)$

Prove that $b\frac{\partial z}{\partial x} + a\frac{\partial z}{\partial y} = 2abz$.

$$\frac{\partial z}{\partial x} = a e^{ax+by} f(ax-by) + a e^{ax+by} f'(ax-by)$$

$x^2 + x$
$\partial y = by$

$$\frac{\partial z}{\partial y} = b e^{ax+by} f(ax-by) - b e^{ax+by} f'(ax-by)$$

$$b\frac{\partial z}{\partial x} + a\frac{\partial z}{\partial y} = ba e^{ax+by} f(ax-by) + ba e^{ax+by} f'(ax-by) + ab e^{ax+by} f(ax-by) - ab e^{ax+by} f'(ax-by)$$

$$= 2ab e^{ax+by} f(ax-by)$$

$$= 2abz.$$

Ques— Prove that

If $y = f(x+at) + g(x-at)$

$$\frac{\partial^2 y}{\partial t^2} = a^2\left(\frac{\partial^2 y}{\partial x^2}\right)$$

Homogeneous Function.

A function $f(x,y)$ is said to be homogeneous function in which the power of each term is same.

A function $f(x,y)$ is a homogeneous function of order n is the degree of each term in x & y is equal to n.

$a_0x^n + a_1x^{n-1}y + a_2x^{n-2}y^2 + \dots + a_{n-1}xy^{n-1} + a_ny^n$

This polynomial can also be written as

$x^n\left[a_0 + a_1\left(\frac{y}{x}\right) + a_2\left(\frac{y}{x}\right)^2 + \dots + a_{n-1}\left(\frac{y}{x}\right)^{n-1} + a_n\left(\frac{y}{x}\right)^n\right]$

$= x^n\phi(y/x)$

Testing of homogeneous function

To test the function $f(x,y,z)$ to be homogeneous put $x = tx$, $y = ty$ & $z = tz$.

Then find $f(tx, ty, tz)$

If we can write it as $t^n f(x,y,z)$ then function will be homogeneous of degree n.

where n need not be integer.

Example $f(x,y) = \dfrac{\sqrt{x}+\sqrt{y}}{x^2+y^2}$

$$f(tx,ty) = \frac{\sqrt{tx}+\sqrt{ty}}{t^2x^2+t^2y^2}$$

$$= \frac{t^{1/2}(\sqrt{x}+\sqrt{y})}{t^2(x^2+y^2)}$$

$$= t^{-3/2} f(x,y)$$

It is homogeneous function of degree $-\frac{3}{2}$.

→ $f(x,y,z) = x^3 \sin\left(\frac{y}{x}\right) + y^3 \tan^{-1}\left(\frac{y}{x}\right)$

$f(tx,ty,tz) = t^3x^3 \sin\left(\frac{ty}{tx}\right) + t^3y^3 \tan^{-1}\left(\frac{ty}{tx}\right)$

$= t^3 f(x,y)$

It is homogeneous function of degree 3.

Euler's Theorem ~~of~~ on Homogeneous Function

If z is a homogeneous Function of x, y of order n then $\boxed{x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz}$.

Proof - Since z is homogeneous function of x, y of order n. Then z can be written as

$$z = x^n f\left(\frac{y}{x}\right) \quad \text{—— (1)}$$

Diff. (1) w.r.t x.

$$\frac{\partial z}{\partial x} = nx^{n-1} f\left(\frac{y}{x}\right) + x^n f'\left(\frac{y}{x}\right)\left(-\frac{y}{x^2}\right) \quad \text{— (2)}$$

Diff. (1) w.r.t y.

$$\frac{\partial z}{\partial y} = \frac{x^n}{x} f'\left(\frac{y}{x}\right)$$

$$= x^{n-1} f'\left(\frac{y}{x}\right) \quad \text{—— (3)}$$

Multiply (2) by x & ~~(3)~~ add with eq^n (3) $\times y$.

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nx^n f\left(\frac{y}{x}\right) - x^{n-1} y f'\left(\frac{y}{x}\right) + x^{n-1} y f'\left(\frac{y}{x}\right)$$

$$= nx^n f\left(\frac{y}{x}\right)$$

$$= nz \quad (\text{using (1)})$$

Note : $\tan^{-1}\left(\frac{y}{x}\right)$ or $\cos\left(\frac{x}{y}\right)$ or $\sin\left(\frac{y}{x}\right)$

These all are homogeneous function of degree 0

$(x^2t^2 + xt \times yt + yt)$

$t^{-2}(x^2 + xy + y^2)$

Ques - Verify Euler's Theorem for,

$$z = (x^2 + xy + y^2)^{-1}$$

$$z = (x^2 + xy + y^2)^{-1}$$

$$= \left[x^2\left(1 + \frac{y}{x} + \frac{y^2}{x^2}\right)\right]^{-1}$$

$$= x^{-2}\phi(y/x)$$

$x = tx$ & ty

$(x^2t^2 + xtyt + y^2t^2)^{-1}$

x^{-2}

$u = \cos^{-1}\left(\frac{x}{y}\right)$

$\cos u = \frac{x}{y}$

$\sin u \frac{du}{dx}$

$\sin u \frac{du}{dy}$

$\boxed{\text{Degree of function is } -2}$

$$\frac{\partial z}{\partial x} = -(x^2 + xy + y^2)^{-2}(2x + y)$$

$$x\frac{\partial z}{\partial x} = -x(x^2 + xy + y^2)^{-2}(2x + y) \quad \text{— (1)}$$

$$\frac{\partial z}{\partial y} = -(x^2 + xy + y^2)^{-2}(2y + x)$$

$$y\frac{\partial z}{\partial y} = -y(x^2 + xy + y^2)^{-2}(2y + x) \quad \text{— (2)}$$

Add (1) & (2),

$$= -(x^2 + xy + y^2)^{-2}\left[(2y^2 + xy) + 2x^2 + yx\right]$$

$$= -(x^2 + xy + y^2)^{-2}\cancel{(3y + 3x)}(2y^2 + 2x^2 + 2xy)$$

$$= -2(x^2 + xy + y^2)^{-2}(x^2 + xy + y^2)$$

$$= -2z.$$

Ques - If $u = \sin^{-1}\left(\frac{x^2 + y^2}{x + y}\right)$ Then prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \tan u.$

$$f(x, y) = \sin u = \frac{x^2 + y^2}{x + y}.$$

Degree of function is 1.

$f(V) = f(x,y) = \sin u$ is of degree 1.

$$x\frac{\partial v}{\partial y} + y\frac{\partial v}{\partial y} = 1 \cdot v$$

$$x\cos u\frac{\partial u}{\partial x} + y\cos u\frac{\partial u}{\partial y} = 1 \cdot \sin u$$

$$\cos u\left[x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right] = 1 \cdot \sin u.$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \tan u$$

→ $u = \tan^{-1}\left(\frac{x^3+y^3}{x+y}\right)$ Prove $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u$

$$\tan u = \frac{x^3+y^3}{x+y}$$

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = 2V$$

$$x\sec^2 u\frac{\partial u}{\partial x} + y\sec^2 u\frac{\partial u}{\partial x} = 2\tan u$$

$$\sec^2 u\left[x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right] = 2\tan u$$

$$\left[x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right] = \frac{2\sin u}{\cancel{\cos u}}\ \cancel{\cos u}\ \cos u$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2\sin u\cos u$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u$$

Taylor's Series

If $f(x,y)$ have continues partial derivative of n^{th} order in the neighbourhood of a point x,y and if $(x+h, y+k)$ is any point of its neighbourhood $f(x+h, y+k)$

$$= f(x,y) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)f + \frac{1}{2!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f + \frac{1}{3!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^3 f + \cdots$$

If we want to expand a function $f(x,y)$ on the neighbourhood point then we use the Taylor's Series

Case I, $f(a+h, b+k) = f(a,b) + [h f_x(a,b) + k f_y(a,b)] + \left[\frac{h^2}{2!} f_{xx}(a,b) + hk f_{xy}(a,b) + \frac{k^2}{2!} f_{yy}(a,b)\right] + \cdots$

Case II, When $a=0, b=0, h=x$ & $k=y$ [Expansion of function in powers of x and y]

$$f(x,y) = f(0,0) + [x f_x(0,0) + y f_y(0,0)] + \left[\frac{x^2}{2!} f_{xx}(0,0) + xy f_{xy}(0,0) + \frac{y^2}{2!} f_{yy}(a,b)\right] + \cdots$$

It is also known as Maclaurin's Series

Case III Expansion in power's of $(x-a)$ & $(y-b)$

$$f(x,y) = f(a,b) + (x-a) f_x(a,b) + (y-b) f_y(a,b) + \left[\frac{(x-a)^2}{2!} f_{xx}(a,b) + (x-a)(y-b) f_{xy}(a,b) + \frac{(y-b)^2}{2!} f_{yy}(a,b)\right] + \cdots$$

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II

Ques – Expand $e^x \cos y$ in power of x, y upto 2nd degree term.

$f(x,y) = e^x \cos y$.

By the Taylor Series –

$$f(x,y) = f(0,0) + [x f_x(0,0) + y f_y(0,0)] + \left[\frac{x^2}{2!} f_{xx}(0,0) + xy f_{xy}(0,0) + \frac{y^2}{2!} f_{yy}(0,0)\right]$$

$f(0,0) = e^0 \cos 0$

$= 1$.

$f_x = \frac{\partial z}{\partial x} = e^x \cos y$

$f_x(0,0) = e^0 \cos 0 = 1$.

$f_y(0,0) = \frac{\partial z}{\partial y} = -e^x \sin y = -e^0 \sin 0 = 0$.

$f_{xx}(0,0) = e^x \cos y = 1$.

$f_{yy}(0,0) = \frac{\partial}{\partial y}(-e^x \sin y) = -e^x \cos y = -1$.

$f_{xy}(0,0) = -\frac{\partial}{\partial x} e^x \sin y = -e^x \sin y = 0$

$$\therefore f(x,y) = 1 + [x(1) + y(0)] + \left[\frac{x^2}{2!}(1) + xy(0) + \frac{y^2}{2!}(-1)\right]$$

$$= 1 + x + \frac{x^2}{2!} - \frac{y^2}{2!}$$

Ques – Expand $e^x \sin y$ in power of x, y upto 3rd degree term.

Ques –

Ques- Expand $e^x \cos y$ in the neighbourhood of $(1, \pi/4)$

$f(x,y) = e^x \cos y$.

$f(1, \frac{\pi}{4}) = e^1 \cos\frac{\pi}{4} = \frac{e}{\sqrt{2}}$

$f_x = e^x \cos y \Rightarrow f_x(1, \pi/4) = \frac{e}{\sqrt{2}}$

$f_{xx} = e^x \cos y \Rightarrow f_{xx}(1, \pi/4) = \frac{e}{\sqrt{2}}$.

$f_y = -e^x \sin y \Rightarrow f_y(1, \pi/4) = \frac{-e}{\sqrt{2}}$.

$f_{yy} = -e^x \cos y \Rightarrow f_{yy}(1, \pi/4) = \frac{-e}{\sqrt{2}}$.

$f_{xy} = -e^x \sin y \Rightarrow f_{xy}(1, \pi/4) = \frac{-e}{\sqrt{2}}$.

$f(x,y) = f(a,b) + (x-a) f_x(a,b) + (y-b) f_y(a,b) + \frac{(x-a)^2}{2!}$

$f_{xx}(a,b) + (x-a)(y-b) f_{xy}(a,b) + \frac{(y-b)^2}{2!} f_{yy}(a,b)$

$= \frac{e}{\sqrt{2}} + (x-1)\frac{e}{\sqrt{2}} - (y-\frac{\pi}{4})\frac{e}{\sqrt{2}} + \frac{(x-1)^2}{2!}\frac{e}{\sqrt{2}} - (x-1)(y-\frac{\pi}{4})\frac{e}{\sqrt{2}} - \frac{(y-\pi/4)^2}{2!}\frac{e}{\sqrt{2}}$

$= \frac{e}{\sqrt{2}}\left[1 + (x-1) - (y-\frac{\pi}{4}) + \frac{(x-1)^2}{2!} - (x-1)(y-\frac{\pi}{4}) - \frac{(y-\frac{\pi}{4})^2}{2!}\right]$

$= \frac{e}{\sqrt{2}}\left[x - y - \frac{\pi}{4} + \frac{(x-1)^2}{2!} - (x-1)(y-\frac{\pi}{4}) - \frac{(y-\pi/4)^2}{2!}\right]$

Ques – Expand $x^2y + 3y - 2$ in powers of $(x-1)$ & $(y+2)$ upto 3rd degree.

Ques – Expand $e^x \log(1+y)$ in powers of x & y upto 3rd

Ques – $\frac{(x+h)(y+k)}{x+h+y+k}$ in powers of h & k upto 2nd

Maxima & Minima.

Calculate $f_x, f_y, f_{xx}, f_{yx}, f_{yy}$.

Consider $f_{xx} \to r$, $f_{xy} \to s$ (or f_{yx}), $f_{yy} \to t$

Calculate $f_x(a,b) = 0$ & $f_y(a,b) = 0$

i) $f(a,b)$ is max. if $rt - s^2 > 0$ (and $r < 0$ or $t < 0$)

ii) $f(a,b)$ is min. if $rt - s^2 > 0$ (and $r > 0$ or $t > 0$)

iii) $f(a,b)$ is not exterimum (or neither max. nor min.) if $rt - s^2 < 0$ then (a,b) is saddle point)

If $rt - s^2 = 0$ then doubtful case.

Ques. Find the maxima & minima of $f(x,y) = x^2 + y^2 + 6x + 12$.

$f_x = 2x + 6$.

$f_y = 2y$

$f_{xx} = 2$.

$f_{yy} = 2$.

$f_{xy} = 0$.

put $f_x = 0$ & $f_y = 0$.

$2x + 6 = 0$ & $2y = 0$.

$x = -3$ $y = 0$.

$(a,b) = (-3,0)$

$rt - s^2$ at $(-3,0)$

$(2)(2) - (0)^2$

$4 > 0$

$r = 2 > 0$

$t = 2 > 0$

$f(x,y)$ is minima at $(-3,0)$ & minimum value is $f(-3,0) = 9 - 18 + 12 = 3$.

Ques Find the maxima & minima of $f(x,y) = x^3 + y^3 - 3axy$

Ques Find the maxima & minima of $f(x,y) = x^4 + y^4 + 4xy - 2x^2 - 2y^2$.

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Implicit Function

If we can't able to write y as a function of x or x as a function of y then function is known as implicit function

$$y \neq f(x)$$
$$x \neq f(y)$$
$$f(x,y) = 0.$$

Total derivative

$$\frac{dy}{dx} = -\frac{f_x}{f_y}$$

$$\frac{d^2y}{dx^2} = -\left[\frac{f_{xx}(f_y)^2 - 2f_{xy}f_x f_y + f_{yy}(f_x)^2}{(f_y)^3}\right]$$

$(\cos x)^y = (\sin y)^x$ Find $\frac{dy}{dx}$?

Taking log both side

$y \log \cos x = x \log \sin y$

$z = f(x,y) = y \log \cos x - x \log \sin y.$

$f_x = -y \tan x - \log \sin y.$

$f_y = \log \cos x - x \cot y.$

$$\frac{dy}{dx} = \frac{y \tan x + \log \sin y}{\log \cos x - x \cot y}.$$

If $y^3 - 3ax^2 + x^3 = 0$ prove that $\frac{d^2y}{dx^2} + \frac{2a^2x^2}{y^5} = 0$.

$f(x,y) = y^3 - 3ax^2 + x^3$

$f_x = -6ax + 3x^2$

$f_{xx} = -6a + 6x$.

$f_y = 3y^2$

$f_{yy} = 6y$

$f_{xy} = 0$

$$\frac{d^2y}{dx^2} = -\frac{(6x-6a)(9y^4) + 6y(3x^2-6ax)^2}{27y^6}$$

$$= -\left[\frac{+2(x-a)(y^3) + 2(x^2-2ax)^2}{y^5}\right]$$

$$= -2\left[\frac{(x-a)(3ax^2-x^3) + (x^2-2ax)^2}{y^5}\right]$$

$$= -2\left[\frac{3ax^3 - x^4 - 3a^2x^2 + x^3a + x^4 + 4a^2x^2 - 4x^3a}{y^5}\right]$$

$$= -2\left[\frac{a^2x^2}{y^5}\right]$$

$$\frac{d^2y}{dx^2} + \frac{2a^2x^2}{y^5} = 0.$$

$x^y = y^x$ find $\frac{dy}{dx}$.

Taking log both side

$y \log x = x \log y$.

$f(x,y) = y \log x - x \log y = 0$.

$f_x = \frac{y}{x} - \log y$

$f_y = \log x - \frac{x}{y}$

$\frac{dy}{dx} = \frac{y/x - \log y}{\log x - \frac{x}{y}}.$

Q — $x^y + y^x = a^k$ Find $\frac{dy}{dx}$.

Q — If $u = \phi(x,y)$, $f(x,y) = 0$
Prove that $\frac{du}{dx} = \frac{\phi_x f_y - \phi_y f_x}{f_y}$

Q — $ax^2 + by^2 + 2gx + 2fy + c = 0$. Find $\frac{dy}{dx} = ?$

Q — $u = x\log(xy)$ & $x^3 + y^3 + 3axy = 0$
Find $\frac{du}{dx}$.

$\cancel{\frac{du}{dy}} = \cancel{\frac{dy}{dy/dx}}$

$\cancel{\frac{du}{dy}} = \cancel{\frac{dx}{dy}}$

$\cancel{\frac{dy}{dy}} = \cancel{\frac{du}{dx}}$

$\cancel{\frac{dx}{dy}}$