ECHELON FORM OF A MATRIX

A matrix is said to be in echelon form if following property holds–

1. Every row of a matrix 'A', which has all entries zero occur below every row which has a non zero entries.
2. The first non zero entry in each non zero row is equal to 1. (It could be 1 or couldn't be 1)
3. Number of zero's are increasing when we go down in the matrix.

Example – $A = \begin{bmatrix} 1 & 3 & 2 & 6 \\ 0 & 1 & 4 & 2 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ → non zero row (third row), → zero row (fourth row)

Rank of a matrix – If A is a matrix, then rank of matrix A is denoted by $\rho(A)$

$\rho(A)$ = No. of non zero rows in Echelon matrix.

Rank (another method) – No. of non zero row in upper triangular matrix.

\# A no. 'r' is said to be the rank of the matrix if there is alleast one non zero minor of order 'r' whose determinant is not equal to zero & all the determinant of order (r+1) are zero.

Some Important Properties about the rank

1. Rank of A & A^T are always same.
2. Rank of a null matrix is zero.
3. Rank of a non singular matrix of order n is n.

4. Rank of a identity matrix of order n is n.
5. The Rank of a product of two matrices can't exceed the rank of either matrix.
6. for a n×n matrix A, if rank of A, $\rho(A)$, equal to n, then determinant A is not equal to zero, It mean the matrix is non-singular.
7. For any square matrix A, $\rho(A) < n$, then $|A| = 0$ & it mean matrix A is singular.
8. For a rectangular matrix A of order m×n rank of A, $\rho(A) \le \min(m, n)$

Ques Find the rank of the matrix A.

$$A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & 3 & 2 & 2 \\ 2 & 4 & 3 & 4 \\ 3 & 7 & 4 & 6 \end{bmatrix}$$

$R_2 \to R_2 - R_1 \quad R_3 \to R_3 - 2R_1 \quad R_4 \to R_4 - 3R_1$

$$A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

$R_4 \to R_4 - R_2.$

$$A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_2 \to R_2 - R_3$

$$A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_3 \leftrightarrow C_4$

$$\begin{bmatrix} 1 & 2 & 2 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_2 \leftrightarrow C_3$

$$\begin{bmatrix} 1 & 2 & 2 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\rho(A) =$ number of non zero rows.

$= 3.$

Ques Find the rank of the matrix A.

$$A = \begin{bmatrix} 1 & 3 & 4 & 3 \\ 3 & 9 & 12 & 9 \\ 1 & 3 & 4 & 1 \end{bmatrix}$$

$R_2 \rightarrow R_2 - 3R_1 \qquad R_3 \rightarrow R_3 - R_1$

$$A = \begin{bmatrix} 1 & 3 & 4 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}$$

$R_2 \leftrightarrow R_3$

$$A = \begin{bmatrix} 1 & 3 & 4 & 3 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\rho(A) = 2.$

Ques Find the rank of the following matrices.

i) $A = \begin{bmatrix} 1 & -1 & 3 & 6 \\ 1 & 3 & -3 & -4 \\ 5 & 3 & 3 & 11 \end{bmatrix}$

ii) $B = \begin{bmatrix} 2 & 3 & 7 \\ 3 & -2 & 4 \\ 1 & -3 & -1 \end{bmatrix}$

iii) $C = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 7 & 13 \\ 4 & -3 & -1 \end{bmatrix}$

iv) $D = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$

v) $E = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & 4 & 6 & 2 \\ 1 & 2 & 3 & 2 \end{bmatrix}$

Canonical Form Or Normal Form –

$[I_r]$ $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ $[I_r \ 0]$ $\begin{bmatrix} I_r \\ 0 \end{bmatrix}$

Here I_r is the identity matrix of order r.

Ques. i

ii

Home Work

Ques. i) $A = \begin{bmatrix} 1 & -1 & 3 & 6 \\ 1 & 3 & -3 & -4 \\ 5 & 3 & 3 & 11 \end{bmatrix}$

$R_2 \to R_2 - R_1$ $\quad$ $R_3 \to R_3 - 5R_1$

$A = \begin{bmatrix} 1 & -1 & 3 & 6 \\ 0 & 4 & -6 & -10 \\ 0 & 8 & -12 & -19 \end{bmatrix}$

$R_3 \to R_3 - 2R_2$.

$A = \begin{bmatrix} 1 & -1 & 3 & 6 \\ 0 & 4 & -6 & -10 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$\rho(A) = 3$.

ii) $B = \begin{bmatrix} 2 & 3 & 7 \\ 3 & -2 & 4 \\ 1 & -3 & -1 \end{bmatrix}$

$R_1 \leftrightarrow R_2$ $\quad$ $R_3 \leftrightarrow R_1$

$B = \begin{bmatrix} 1 & -3 & -1 \\ 2 & 3 & 7 \\ 3 & -2 & 4 \end{bmatrix}$

$R_2 \to R_2 - 2R_1$ $\quad$ $R_3 \to R_3 - 3R_1$

$B = \begin{bmatrix} 1 & -3 & -1 \\ 0 & 9 & 9 \\ 0 & 7 & 7 \end{bmatrix}$

$R_3 \to R_3 - 7/9\, R_2$.

$B = \begin{bmatrix} 1 & -3 & -1 \\ 0 & 9 & 9 \\ 0 & 0 & 0 \end{bmatrix}$

$\rho(A) = 2$.

iii) $C = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 7 & 13 \\ 4 & -3 & -1 \end{bmatrix}$

$C_1 \leftrightarrow C_2$.

$C = \begin{bmatrix} 1 & 2 & 3 \\ 7 & 4 & 13 \\ -3 & 4 & -1 \end{bmatrix}$

$R_2 \to R_2 - 7R_1$ $\quad$ $R_3 \to R_3 + 3R_1$

$C = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -10 & -8 \\ 0 & 10 & 8 \end{bmatrix}$

$R_3 \to R_3 + R_2$.

$C = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -10 & -8 \\ 0 & 0 & 0 \end{bmatrix}$

$\rho(A) = 2$

iv) $D = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$

$R_2 \leftrightarrow R_1$

$D = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$

$R_3 \to R_3 - 3R_1$ $\quad$ $R_4 \to R_4 - R_1$

$D = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \end{bmatrix}$

$R_3 \to R_3 + R_2$ $\quad$ $R_4 \to R_4 + R_2$

$$D = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\rho(A) = 2.$

v) $$E = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & 4 & 6 & 2 \\ 1 & 2 & 3 & 2 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$ $\quad$ $R_3 \to R_3 - R_1$

$$E = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_2 \leftrightarrow R_3$

$$E = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\rho(A) = 2.$

Rank by Normal form

Ques Reduce the matrix in normal form and find the rank.

$$A = \begin{bmatrix} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \\ 9 & 10 & 11 & 12 \end{bmatrix}$$

$R_2 \to R_2 - \frac{3}{2} R_1 \quad R_3 \to R_3 - 2R_1 \quad R_4 \to R_4 - \frac{9}{2} R_1$

$$\begin{bmatrix} 2 & 3 & 4 & 5 \\ 0 & -\frac{1}{2} & -1 & -3/2 \\ 0 & -1 & -2 & -3 \\ 0 & -\frac{7}{2} & -7 & -21/2 \end{bmatrix}$$

$C_2 \to C_2 - 3/2\, C_1 \quad C_3 \to C_3 - 2C_1$
$C_4 \to C_4 - \frac{5}{2} C_1$

$$\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & -1/2 & -1 & -3/2 \\ 0 & -1 & -2 & -3 \\ 0 & -7/2 & -7 & -21/2 \end{bmatrix}$$

$C_4 \to -\frac{\cancel{14}}{7} C_4 \quad C_1 \to \frac{1}{2} C_1 \quad C_2 \to -2C_2 \quad C_3 \to -C_3$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 3 \\ 0 & 2 & 2 & 6 \\ 0 & 7 & 7 & 21 \end{bmatrix}$$

$C_2 \to C_2 - C_3$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 7 & 21 \end{bmatrix}$$

$C_4 \to C_4 - 3C_3$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 7 & 0 \end{bmatrix}$$

$R_3 \to \frac{R_3}{2}$ $\qquad$ $R_4 \to \frac{R_4}{7}$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$R_3 \to R_3 - R_2$ $\qquad$ $R_4 \to R_4 - R_3$.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_2 \leftrightarrow C_3$

$$\begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$$

Rank of matrix, $\rho(A) = 2$

Ques. Reduce the matrix in normal form and find the rank.

$$A = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 4 & 1 & 2 & 1 \\ 3 & -1 & 1 & 2 \\ 1 & 2 & 0 & 1 \end{bmatrix}$$

$R_2 \to R_2 - 4R_1$, $R_3 \to R_3 - 3R_1$, $R_4 \to R_4 - R_1$

$$\begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -7 & 6 & -11 \\ 0 & -7 & 4 & -7 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -7 & 6 & -11 \\ 0 & -7 & 4 & -7 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

$R_3 \rightarrow R_3 + R_2$

$$\begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -7 & 6 & -11 \\ 0 & 0 & -2 & 4 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

$C_2 \rightarrow C_2 - 2C_1 \quad C_3 \rightarrow C_3 + C_1 \quad C_4 \rightarrow C_4 - 3C_1$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -7 & 6 & -11 \\ 0 & 0 & -2 & 4 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

$R_3 \rightarrow R_3 + 2R_4$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -7 & 6 & -11 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

$R_4 \leftrightarrow R_3$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -7 & 6 & -11 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_2 \rightarrow -\frac{1}{7} C_2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 6 & -11 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_4 \rightarrow -C_4$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 6 & 11 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_4 \rightarrow C_4 - 2C_3$.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 6 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_4 \rightarrow -C_4$.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 6 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_4 \rightarrow C_4 - C_2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 6 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_3 \rightarrow C_3 - 6C_2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix}$$

Rank of matrix, $\rho(A) = 3$.

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Ques 1. Reduce the matrix in normal form and find the rank.

(1) $\begin{bmatrix} 1 & 3 & 4 & 2 \\ 2 & -1 & 3 & 2 \\ 3 & -5 & 2 & 2 \\ 6 & -3 & 8 & 6 \end{bmatrix}$ (2) $\begin{bmatrix} 1 & 2 & 0 & -1 \\ 3 & 4 & 1 & 2 \\ -2 & 3 & 2 & 5 \end{bmatrix}$

(3) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \\ 4 & 3 & 1 & 2 \end{bmatrix}$ (4) $\begin{bmatrix} 1 & 2 & -1 & 4 \\ 2 & 4 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ -1 & -2 & 6 & -7 \end{bmatrix}$

Ques 2 Find the rank of matrices by determinant method.

1. $A = \begin{bmatrix} 1 & 3 & 5 & 7 \\ -3 & 2 & 4 & 5 \end{bmatrix}$

2. $A = \begin{bmatrix} 1 & 2 & 3 & -4 \\ -2 & 3 & 7 & -1 \\ 1 & 9 & 16 & -13 \end{bmatrix}$

Ques 3 For which value of b the rank of matrix is 2

$A = \begin{bmatrix} 1 & 5 & 4 \\ 0 & 3 & 2 \\ b & 13 & 10 \end{bmatrix}$

Ques 4 Find the rank of matrix after reducing it Echelon form.

(1.) $\begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 1 \\ 1 & 3 & 4 & 5 \end{bmatrix}$ (2) $\begin{bmatrix} -1 & 2 & 3 & -2 \\ 2 & -5 & 1 & 2 \\ 3 & -8 & 5 & 2 \\ 5 & -12 & -1 & 6 \end{bmatrix}$

(3) $$\begin{bmatrix} 0 & 1 & 2 & -2 \\ 4 & 0 & 2 & 6 \\ 2 & 1 & 3 & 1 \end{bmatrix}$$

Home Work

Ques 1. (1) $$\begin{bmatrix} 1 & 3 & 4 & 2 \\ 2 & -1 & 3 & 2 \\ 3 & -5 & 2 & 2 \\ 6 & -3 & 8 & 6 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 3R_1$, $R_4 \to R_4 - 6R_1$

$$\begin{bmatrix} 1 & 3 & 4 & 2 \\ 0 & -7 & -5 & -2 \\ 0 & -14 & -10 & -4 \\ 0 & -21 & -16 & -6 \end{bmatrix}$$

$C_2 \to C_2 - 3C_1$, $C_3 \to C_3 - 4C_1$, $C_4 \to C_4 - 2C_1$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -7 & -5 & -2 \\ 0 & -14 & -10 & -4 \\ 0 & -21 & -16 & -6 \end{bmatrix}$$

$R_3 \to R_3 - 2R_2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -7 & -5 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & -21 & -16 & -6 \end{bmatrix}$$

$R_4 \to R_4 - 3R_2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -7 & -5 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

$C_2 \to -C_2 \qquad C_3 \to -C_3 \qquad C_4 \to -C_4.$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 7 & 5 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$R_2 \to 2R_2.$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 14 & 10 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$C_4 \to \frac{1}{2} C_4$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 14 & 10 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$R_2 \to R_2 - 10R_4.$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 14 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$R_2 \to \frac{1}{2} R_2.$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 7 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$C_2 \to C_2 - 7C_4.$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$R_4 \leftrightarrow R_3$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_2 \leftrightarrow C_4$.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix}$$

$\rho(A) = 3$.

2.
$$\begin{bmatrix} 1 & 2 & 0 & -1 \\ 3 & 4 & 1 & 2 \\ -2 & 3 & 2 & 5 \end{bmatrix}$$

$R_2 \to R_2 - 3R_1$, $R_3 \to R_2 + 2R_1$.

$$\begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & -2 & 1 & -1 \\ 0 & 7 & 2 & 3 \end{bmatrix}$$

$C_2 \to C_2 - 2C_1$, $C_4 \to C_4 + C_1$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -2 & 1 & -1 \\ 0 & 7 & 2 & 3 \end{bmatrix}$$

$C_2 \to C_2 + 2C_3$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 11 & 2 & 3 \end{bmatrix}$$

$C_4 \to C_4 + C_3$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 11 & 2 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 11 & 2 & 5 \end{bmatrix}$$

$C_2 \rightarrow 5C_2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 55 & 2 & 5 \end{bmatrix}$$

$\cancel{C_2 \rightarrow C_2 - 11C_3}$. $R_3 \rightarrow R_3 - 2R_2$.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 55 & 0 & 5 \end{bmatrix}$$

$C_4 \rightarrow 11C_4$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 55 & 0 & 55 \end{bmatrix}$$

$C_2 \rightarrow C_2 - C_4$.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 55 \end{bmatrix}$$

$C_4 \rightarrow \frac{1}{55} C_4$.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$C_2 \leftrightarrow C_3$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$C_3 \leftrightarrow C_4$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

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$$\begin{bmatrix} I_3 & 0 \end{bmatrix}$$

$$\rho(A) = 3.$$

3. $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \\ 4 & 3 & 1 & 2 \end{bmatrix}$

$R_2 \to R_2 - 3R_1$, $R_3 \to R_3 - 4R_1$

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -2 & -8 & -10 \\ 0 & -5 & -11 & -14 \end{bmatrix}$$

$C_2 \to C_2 - 2C_1$, $C_3 \to C_3 - 3C_1$, $C_4 \to C_4 - 4C_1$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -2 & -8 & -10 \\ 0 & -5 & -11 & -14 \end{bmatrix}$$

$C_2 \to -C_2$, $C_3 \to -C_3$, $C_4 \to -C_4$.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 8 & 10 \\ 0 & 5 & 11 & 14 \end{bmatrix}$$

$C_3 \to C_3 - 4C_2$, $C_4 \to C_4 - 5C_2$.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 5 & -9 & -11 \end{bmatrix}$$

$C_3 \to -\frac{1}{9}C_3$, $C_4 \to -\frac{1}{11}C_4$.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 5 & 1 & 1 \end{bmatrix}$$

$C_3 \to C_3 - C_4$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 5 & 0 & 1 \end{bmatrix}$$

~~$C_4 \to 10C_4$~~

$\cancel{C_4 \to 10C_4}$

$$\cancel{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 5 & 0 & 10 \end{bmatrix}}$$

$C_2 \to C_2 - 5C_4$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$C_2 \to \frac{1}{2} C_2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$C_3 \leftrightarrow C_4$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Ques 2 (1) $A = \begin{bmatrix} 1 & 3 & 5 & 7 \\ -3 & 2 & 4 & 5 \end{bmatrix}$

Minor's Matrix –

$\begin{vmatrix} 1 & 3 \\ -3 & 2 \end{vmatrix} = 2 - (-9) = 2 + 9 = 11 \neq 0$

$\begin{vmatrix} 3 & 5 \\ 2 & 4 \end{vmatrix} = 12 - 10 = 2 \neq 0$

$\begin{vmatrix} 5 & 7 \\ 4 & 5 \end{vmatrix} = 25 - 28 = -3 \neq 0$

$\begin{vmatrix} 1 & 7 \\ -3 & 5 \end{vmatrix} = 5 + 21 = 26 \neq 0$

$\rho(A) = 2$

(2.) $A = \begin{bmatrix} 1 & 2 & 3 & -4 \\ -2 & 3 & 7 & -1 \\ 1 & 9 & 16 & -13 \end{bmatrix}$

$$\begin{vmatrix} 1 & 2 & 3 \\ -2 & 3 & 7 \\ 1 & 9 & 16 \end{vmatrix} = 1\begin{vmatrix} 3 & 7 \\ 9 & 16 \end{vmatrix} - 2\begin{vmatrix} -2 & 7 \\ 1 & 16 \end{vmatrix} + 3\begin{vmatrix} -2 & 3 \\ 1 & 9 \end{vmatrix}$$

$= (48-63) - 2(-32-7) + 3(-18-3)$

$= -15 - 2(-39) + 3(-21)$

$= -15 + 78 - 63$

$= 78 - 78 = 0$

$$\begin{vmatrix} 2 & 3 & -4 \\ 3 & 7 & -1 \\ 9 & 16 & -13 \end{vmatrix} = 2\begin{vmatrix} 7 & -1 \\ 16 & -13 \end{vmatrix} - 3\begin{vmatrix} 3 & -1 \\ 9 & -13 \end{vmatrix} + (-4)\begin{vmatrix} 3 & 7 \\ 9 & 16 \end{vmatrix}$$

$= 2(-91+16) - 3(-39+9) - 4(48-63)$

$= 2(-75) - 3(-30) - 4(-15)$

$= 60 - 150 + 90$

$= 150 - 150 = 0$

$\therefore \rho(A) \neq 3.$

$\begin{vmatrix} 1 & 2 \\ -2 & 3 \end{vmatrix} = 8 \neq 0$

$\therefore \rho(A) = 2.$

Linear System Of Equations

Homogeneous | Non Homogeneous

Homogeneous linear System Of Equations

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = 0.$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = 0$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = 0.$$

In matrix form it is $\boxed{AX = 0}$.

where $A = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{1n} \\ a_{22} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m\times n} \rightarrow$ Coefficient Matrix

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1} \quad \& \quad O = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Application

1. Traffic flow
2. Electric field
3. Consumption and uses of in bussiness

Non - Homogeneous linear system of Equations

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2.$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_n$$

$AX = B$ (In Matrix form)

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \quad X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Augmented Matrix

$$[A : B] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_n \end{array}\right]$$

Charactization Of Equations by Solution

Types Of EQUATION

1. Consistent.
2. Non-consistent or inconsistent

Consistent

A system of eqn is said to be consistent if it has unique soln or more than one soln.

Inconsistent –

If a system of eqn has no soln than it is known as inconsistent.

Example's – **Consistent**

$$x + 2y = 4$$
$$3x + 2y = 2$$
$$\underline{- \quad - \quad\quad -}$$
$$-2x = 2.$$
$$x = -1.$$

$$2y = 4 + 1 = 5$$
$$y = \frac{5}{2}.$$

Inconsistent

$x + 2y = 4$
$3x + 2y = 6.$
Not soluble.

A system of non-homogeneous Eqn
$AX = B$
| Find $\rho(A)$ & $\rho(C)$

Solution exist
system is consistent
If $\rho(A) = \rho(C)$

No Solution
system is inconsistent
$\rho(A) \neq \rho(C)$

Unique Solution
$\rho(A) = \rho(C) = n$
n → no. of unknown

Infinite No. of solution
$\rho(A) = \rho(C) < n$ (no. of unknown)

Ques - Solve $2x + 6y = -11$
$6x + 20y - 6z = -3$
$6y - 18z = -1.$

Ans - $C = (A \vdots B) = \left[\begin{array}{ccc:c} 2 & 6 & 0 & -11 \\ 6 & 20 & -6 & -3 \\ 0 & 6 & -18 & -1 \end{array}\right]$

$R_2 \rightarrow R_2 - 3R_1$

$\left[\begin{array}{ccc:c} 2 & 6 & 0 & -11 \\ 0 & 2 & -6 & 30 \\ 0 & 6 & -18 & -1 \end{array}\right]$

$R_3 \rightarrow R_3 - 3R_2$

$\left[\begin{array}{ccc:c} 2 & 6 & 0 & -1 \\ 0 & 2 & -6 & 30 \\ 0 & 0 & 0 & -91 \end{array}\right]$

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$\rho(A) = 2$

$\rho(C) = 3$

$\rho(A) \neq \rho(C)$

$\Rightarrow$ system has no solution.

Ques. Solve $2x + 3y + 4z = 11$

$x + 5y + 7z = 15$

$3x + 11y + 13z = 25$

Ans - Augmented matrix

$$C = (A : B) = \left[\begin{array}{ccc|c} 2 & 3 & 4 & 11 \\ 1 & 5 & 7 & 15 \\ 3 & 11 & 13 & 25 \end{array}\right]$$

$R_2 \leftrightarrow R_1$

$$\left[\begin{array}{ccc|c} 1 & 5 & 7 & 15 \\ 2 & 3 & 4 & 11 \\ 3 & 11 & 13 & 25 \end{array}\right]$$

$R_2 \to R_2 - 2R_1, \; R_3 \to R_3 - 3R_1$

$$\left[\begin{array}{ccc|c} 1 & 5 & 7 & 15 \\ 0 & -7 & -10 & -19 \\ 0 & -4 & -8 & -20 \end{array}\right]$$

$R_2 \to \frac{-R_2}{7}, \quad R_3 \to \frac{-R_3}{4}$

$$\left[\begin{array}{ccc|c} 1 & 5 & 7 & 15 \\ 0 & 1 & 10/7 & 19/7 \\ 0 & 1 & 2 & 5 \end{array}\right]$$

$R_3 \to R_3 - R_2$

$$\left[\begin{array}{ccc|c} 1 & 5 & 7 & 15 \\ 0 & 1 & 10/7 & 19/7 \\ 0 & 0 & 4/7 & 16/7 \end{array}\right]$$

$\rho(C) = 3 = \rho(A)$

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Matrix Method

$$\begin{bmatrix} 1 & 5 & 7 \\ 0 & 1 & 10/7 \\ 0 & 0 & 4/7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 15 \\ 19/7 \\ 16/7 \end{bmatrix}$$

$$\frac{4}{7} z = \frac{16}{7}$$

$$4z = 16$$

$$z = 4.$$

$$y + \frac{10}{7} z = \frac{19}{7}$$

$$y = \frac{19}{7} - \frac{40}{7}$$

$$= \frac{-21}{7} = -3$$

$$x + 5y + 7z = 15$$

$$x + 5(-3) + 7(4) = 15$$

$$x + 28 - 15 = 15$$

$$x = 15 + 15 - 28$$

$$x = 2$$

$\therefore$ $x = 2$, $y = -3$, $z = 4$.

Ques - Solve $3x + 3y + 2z = 1.$

$$x + 2y = 4$$

$$10y + 3z = -2$$

$$2x - 3y - z = 5.$$

Augmented Matrix

$$C = (A \vdots B) = \left[\begin{array}{ccc|c} 3 & 3 & 2 & 1 \\ 1 & 2 & 0 & 4 \\ 0 & 10 & 3 & -2 \\ 2 & -3 & -1 & 5 \end{array}\right]$$

$R_2 \leftrightarrow R_1$

$$\left[\begin{array}{ccc:c} 1 & 2 & 0 & 4 \\ 3 & 3 & 2 & 1 \\ 0 & 10 & 3 & -2 \\ 2 & -3 & -1 & 5 \end{array}\right]$$

$R_2 \rightarrow R_2 - 3R_1 \quad , \quad R_4 \rightarrow R_4 - 2R_1$

$$\left[\begin{array}{ccc:c} 1 & 2 & 0 & 4 \\ 0 & -3 & 2 & -11 \\ 0 & 10 & 3 & -2 \\ 0 & -7 & -1 & -3 \end{array}\right]$$

$R_4 \rightarrow R_4 + \frac{1}{2} R_2$

$$\left[\begin{array}{ccc:c} 1 & 2 & 0 & 4 \\ 0 & -3 & 2 & -11 \\ 0 & 10 & 3 & -2 \\ 0 & -17/2 & 0 & -17/2 \end{array}\right]$$

$R_4 \rightarrow \frac{-2}{17} R_4$

$$\left[\begin{array}{ccc:c} 1 & 2 & 0 & 4 \\ 0 & -3 & 2 & -11 \\ 0 & 10 & 3 & -2 \\ 0 & 1 & 0 & 1\cancel{17/2} \end{array}\right]$$

$R_1 \rightarrow R_1 - 2R_4$

$$\left[\begin{array}{ccc:c} 1 & 0 & 0 & 2 \\ 0 & -3 & 2 & -11 \\ 0 & 10 & 3 & -2 \\ 0 & 1 & 0 & 1 \end{array}\right]$$

$R_2 \rightarrow R_2 + 3R_4$

$$\left[\begin{array}{ccc:c} 1 & 0 & 0 & 2 \\ 0 & 0 & 2 & -8 \\ 0 & 10 & 3 & -2 \\ 0 & 1 & 0 & 1 \end{array}\right]$$

AX = B

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$R_3 \to R_3 - 10R_4.$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 0 & 2 & -8 \\ 0 & 0 & 3 & -12 \\ 0 & 1 & 0 & 1 \end{array}\right]$$

$R_2 \to 3R_2$ — $\qquad R_3 \to 2R_3.$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 0 & 6 & -24 \\ 0 & 0 & 6 & -24 \\ 0 & 1 & 0 & 1 \end{array}\right]$$

$R_3 \to R_3 - R_2.$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 0 & 6 & -24 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{array}\right]$$

$R_3 \leftrightarrow R_4$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 0 & 6 & -24 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$R_3 \leftrightarrow R_2.$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 6 & -24 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\rho(A) = 3 = \rho(C)$$

Matrix Method —

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 6 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -24 \\ 0 \end{bmatrix}$$

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$x = 2$
$y = 1$
$z = -4$

Homogeneous Equations

$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = 0$
$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = 0$
$\vdots$
$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = 0$

$$\boxed{AX = 0}$$

This system of Eqⁿ has the zero solution if the rank of the coefficent matrix $(\rho(A))$ is equal to the no. of unknowns varibles.

$\rho(A) = $ no. of unknown.

This system of eqⁿ has infinitly many sol. if rank of A, $\rho(A) <$ no. of unknown.

Ques - $x + 3y - 2z = 0$
$2x - y + 4z = 0$
$x - 11y + 14z = 0.$

$$A = \begin{bmatrix} 1 & 3 & -2 \\ 2 & -1 & 4 \\ 1 & -11 & 14 \end{bmatrix}$$

$$|A| = 1\begin{vmatrix} -1 & 4 \\ -11 & 14 \end{vmatrix} - 2\begin{vmatrix} 3 & -2 \\ -11 & 14 \end{vmatrix} + 1\begin{vmatrix} 3 & -2 \\ -1 & 4 \end{vmatrix}$$

$|A| = 0$
$\Rightarrow \rho(A) \neq 3$
$\Rightarrow \rho(A) < 3$

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$\rho(A) <$ no. of unknown's.

$\Rightarrow$ Infinitly many solutions.

$$\begin{bmatrix} 1 & 3 & -2 \\ 2 & -1 & 4 \\ 1 & -11 & -14 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$

$R_3 \to R_3 - R_1$

$$\begin{bmatrix} 1 & 3 & -2 \\ 0 & -7 & 8 \\ 0 & -14 & 16 \end{bmatrix}$$

$R_3 \to R_3 - 2R_2$

$$\begin{bmatrix} 1 & 3 & -2 \\ 0 & -7 & 8 \\ 0 & 0 & 0 \end{bmatrix}$$

Now, $\begin{bmatrix} 1 & 3 & -2 \\ 0 & -7 & 8 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$-7y + 8z = 0$$

$$x + 3y - 2z = 0.$$

assume $y = K$.

$$z = +\frac{7y}{8}$$

$$z = \frac{7K}{8}$$

$$x = -3y + 2z$$

$$= -3K + 2\left(\frac{7K}{8}\right) = \frac{-10K}{8}.$$

$x = \frac{-10K}{8}$, $y = K$, $z = \frac{7K}{8}$.

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Investigate the value of λ & μ in the following system of linear eqn.

$x + y + z = 6$.

$x + 2y + 5z = 10$

$2x + 3y + \lambda z = \mu$.

has i) a unique sol.

ii) no sol.

iii) infinite sol.

Also find the solution for $\lambda = 2$ & $\mu = 8$.

$$\underset{A}{\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 5 \\ 2 & 3 & \lambda \end{bmatrix}} \underset{X}{\begin{bmatrix} x \\ y \\ z \end{bmatrix}} = \begin{bmatrix} 6 \\ 10 \\ \mu \end{bmatrix}$$

$$\left[\begin{array}{ccc:c} 1 & 1 & 1 & 6 \\ 1 & 2 & 5 & 10 \\ 2 & 3 & \lambda & \mu \end{array}\right]$$

$R_2 \to R_2 - R_1$, $R_3 \to R_3 - 2R_1$.

$$\sim \left[\begin{array}{ccc:c} 1 & 1 & 1 & 6 \\ 0 & 1 & 4 & 4 \\ 0 & 1 & \lambda - 2 & \mu - 12 \end{array}\right]$$

$R_3 \to R_3 - R_2$.

$$\sim \left[\begin{array}{ccc:c} 1 & 1 & 1 & 6 \\ 0 & 1 & 4 & 4 \\ 0 & 0 & \lambda - 6 & \mu - 16 \end{array}\right]$$

i) unique solution exists.

when $\rho(A) = \rho(A:B) = 3$.

for $R(A:B) = 3 = \text{Rank}(A)$

~~$\mu - 16 \neq 0$~~ $\lambda - 6 \neq 0$.

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ii) no - soln exist
when $R(A) \neq R(A:B)$
$\lambda - 6 = 0$, $\mu - 16 \neq 0$
$\lambda = 6$, $\mu \neq 16$.

iii) infinite soln exist
when $R(A) = R(A:B) < n$.
$\lambda - 6 = 0$, $\mu - 16 = 0$
$\lambda = 6$, $\mu = 16$

Now put $\lambda = 2$ & $\mu = 8$.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 4 & 4 \\ 0 & 0 & -4 & -8 \end{array}\right]$$

Now,

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 4 \\ 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ -8 \end{bmatrix}$$

$x + y + z = 6$
$y + 4z = 4$
$-4z = -8$
$\boxed{z = 2}$

$y = 4 - 8$
$\boxed{y = -4}$

$x = 6 + 4 - 2$
$\boxed{x = 8}$

$x = 8$, $y = -4$, $z = 2$

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Ques – Determine the value of λ & μ has
i) No solⁿ, ii) a unique solⁿ, iii) infinite solⁿ.

$x + y + z = 6$

$x + 2y + 3z = 10$

$x + 2y + \lambda z = \mu$.

Ans –

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \lambda & \mu \end{array}\right]$$

$R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & \lambda-1 & \mu-6 \end{array}\right]$$

$R_3 \to R_3 - R_2$.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & \lambda-3 & \mu-10 \end{array}\right]$$

i) No solⁿ

$R(A) \neq R(A:B)$

$\lambda - 3 = 0$, $\mu - 10 \neq 0$

$\lambda = 3$, $\mu \neq 10$

ii) a unique solⁿ.

$R(A) = R(A:B) = 3$.

$\lambda - 3 \neq 0$.

$\lambda \neq 3$ μ (can have any value)

iii) Infinite solⁿ.

$R(A) = R(A:B) \neq 3$

$\lambda - 3 = 0$, $\mu - 10 = 0$

$\lambda = 3$, $\mu = 10$.

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EIGENVALUES AND EIGENVECTORS

If A is a non zero square matrices and if there exist a scalar λ & a non zero coloum vector (matrix) X such that $AX = \lambda X$ then the scalar λ is know as Eigenvalues or characteristic value and the X is known as eigenvector or characteristic vectors.

$$AX = \lambda X$$

$$AX - \lambda X = 0$$

$$(A - \lambda I)X = 0$$

$$|A - \lambda I| = 0 \longrightarrow \text{characteristic eq}^n.$$

If ~~eq^n~~ matrix A is off order $n \times n$ then the degree of characteristic eqn is n.

Ques - Find the Eigen values of the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 5 & 4 \end{bmatrix}$$

Ans - $|A - \lambda I| = 0$.

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 2 \\ 5 & 4-\lambda \end{vmatrix} = 0.$$

$$(1-\lambda)(4-\lambda) - 10 = 0$$

$$4 - 4\lambda - \lambda + \lambda^2 - 10 = 0$$

$$\lambda^2 - 5\lambda - 6 = 0 \quad \text{— characteristic eq}^n.$$

$$\lambda^2 - 6\lambda + \lambda - 6 = 0$$

$$\lambda(\lambda - 6) + 1(\lambda - 6) = 0$$

$$(\lambda + 1)(\lambda - 6) = 0$$

$$\lambda = -1, \quad \lambda = 6.$$

Eigenvalues are -1 & 6.

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Find the characteristic roots of the matrix

$$A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$$

$$|A - \lambda I| = 0$$

$$|A - \lambda I| = \begin{vmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{vmatrix} = 0.$$

$$(2-\lambda)\begin{vmatrix} 3-\lambda & 1 \\ 2 & 2-\lambda \end{vmatrix} - 2\begin{vmatrix} 1 & 1 \\ 1 & 2-\lambda \end{vmatrix} + 1\begin{vmatrix} 1 & 3-\lambda \\ 1 & 2 \end{vmatrix}$$

$$(2-\lambda)[(3-\lambda)(2-\lambda)-2] - 2(2-\lambda-1) + [2-3+\lambda]$$

$$(2-\lambda)[6-2\lambda-3\lambda+\lambda^2-2] - 2(1-\lambda) + [-1+\lambda]$$

$$(2-\lambda)(4-5\lambda+\lambda^2) - 2 + 2\lambda + \lambda - 1$$

$$8 - 10\lambda + 2\lambda^2 - 4\lambda + 5\lambda^2 - \lambda^3 - 2 + 3\lambda - 1$$

$$-\lambda^3 + 7\lambda^2 - 11\lambda + 5 = 0. \quad —(1)$$

Put $\lambda = 1$

$$-1 + 7 - 11 + 5 = 0$$

$$0 = 0.$$

$\Rightarrow (\lambda - 1)$ is a factor of eqn (1)

$$\lambda - 1 \;\big)\; -\lambda^3 + 7\lambda^2 - 11\lambda + 5 = 0 \;\big(\; \lambda^2 - 6\lambda + 5.$$

$$-\lambda^3 + 7\lambda^2 - 11\lambda + 5$$

$$0$$

$$\lambda^2 - 6\lambda + 5$$

$$\lambda^2 - \lambda - 5\lambda + 5$$

$$\lambda(\lambda - 1) - 5(\lambda - 1)$$

$$(\lambda - 5)(\lambda - 1)$$

$$\lambda = 1, 5, 1$$

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Properties Of Eigenvalues

- → Any square matrix A & it ~~tr~~ transpose $\bar{A}$ has same eigenvalue.
- → The sum of eigenvalue of a matrix is equal to the trace of the matrix.
  Trace = Sum of diagonal elements.
- → The product of eigen values of a matrix A is equal to the determinant of A.
- → If $\lambda_1, \lambda_2 --- \lambda_n$ are the eigenvalue of A then the ~~eq~~ eigenvalue of
  - i) KA are $K\lambda_1, K\lambda_2 ---- K\lambda_n$
  - ii) A^m are $\lambda_1^m, \lambda_2^m ---- \lambda_n^m$
  - iii) A^{-1} are $\frac{1}{\lambda_1}, \frac{1}{\lambda_2} ---- \frac{1}{\lambda_n}$.

Ques - Find the Eigenvalues of $3A^3+5A^2-6A+2I$, where

$$A = \begin{bmatrix} 1 & 2 & -3 \\ 0 & 3 & 2 \\ 0 & 0 & -2 \end{bmatrix}$$

$$|A-\lambda I| = \begin{vmatrix} 1-\lambda & 2 & -3 \\ 0 & 3-\lambda & 2 \\ 0 & 0 & -2-\lambda \end{vmatrix}$$

$$= (1-\lambda)(3-\lambda)(-2-\lambda) = 0$$

$$\lambda = 1, \ \lambda = 3, \ \lambda = -2.$$

Eigenvalue of A are 1, 3, −2.

" " A^3 are 1, 27, −8

" " A^2 are 1, 9, 4

" " 6A are 6, 18, −12

" " 2I are 2, 2, 2

First Eigenvalue of $3A^3+5A^2-6A+2I$ is $3\times1+5\times1-6+2=4$

" " " " is $3\times27+5\times9-6\times3+2=11$

Third Eigenvalue of $3A^3+5A^2-6A+2I$ is $3\times-8+5\times4+12+2=10$

Ques. Find the Eigenvalue of the matrix $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

b) $A = \begin{bmatrix} 2 & -3 & 1 \\ 3 & 1 & 3 \\ -5 & 2 & -4 \end{bmatrix}$

Ques - Find the Eigenvalues of $3A^3+5A^2+6A+I$, where
$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & 6 \\ 0 & 0 & -3 \end{bmatrix}$

The characteristic roots of a triangular matrix are its diagonal elements.

Ques - Find the Eigenvalue of $A = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$ and Eigenvalue of A^{25}, $A+2I$.

$|A-\lambda I| = 0$

$\begin{vmatrix} 3-\lambda & 0 \\ 8 & -1-\lambda \end{vmatrix} = (3-\lambda)(-1-\lambda) - 0$
$= -3-3\lambda+\lambda+\lambda^2 = \lambda^2-2\lambda-3$

$\lambda^2-2\lambda-3$

$\lambda^2+\lambda-3\lambda-3$

$\lambda(\lambda+1)-3(\lambda+1)$

$(\lambda-3)(\lambda+1)$ $\qquad \lambda = 3, -1$

Eigenvalue of A^{25} are $3^{25}, -1^{25}$

Eigenvalue of $A+2I$ are $5, 1$

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CAYLEY HAMILTON THEOREM –

Statement - Every $n \times n$ matrix satisfies its own characteristic eqn.

Ques- Verify Cayley Hamilton Theorem & hence find A^{-1}

$$A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$$

The characteristic eqn is

$$|A - \lambda I| = 0.$$

$$\begin{vmatrix} 1-\lambda & 2 \\ 2 & -1-\lambda \end{vmatrix} = 0$$

$$-(1-\lambda)(1+\lambda) - 4 = 0.$$

$$-(1-\lambda^2) - 4 = 0$$

$$\lambda^2 - 5 = 0.$$

To prove Cayley Hamilton Theorem

We will Prove

$$A^2 - 5I = 0.$$

$$A^2 = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$$

$$A^2 - 5I = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$A^2 = 5I.$$

Multiply by A^{-1}

$$5A^{-1} = A$$

$$A^{-1} = \frac{1}{5} A$$

$$A^{-1} = \frac{1}{5}\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$$

Ques- Verify Cayley Hamilton Theorem & hence find A^{-1}.

$$A = \begin{bmatrix} 4 & 3 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 1 \end{bmatrix}$$

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 4-\lambda & 3 & 1 \\ 2 & 1-\lambda & -2 \\ 1 & 2 & 1-\lambda \end{vmatrix} = 0.$$

$$(4-\lambda)[(1-\lambda)^2 + 4] - 3[2(1-\lambda)+2] + 1[4-(1-\lambda)] = 0$$

$$(4-\lambda)[1+\lambda^2 - 2\lambda + 4] - 3[2-2\lambda+2] + [3+\lambda] = 0.$$

$$(4-\lambda)[5+\lambda^2-2\lambda] - [6-6\lambda+6] + [3+\lambda] = 0.$$

$$20 + 4\lambda^2 - 8\lambda - 5\lambda - \lambda^3 + 2\lambda^2 - 12 + 6\lambda + 3 + \lambda = 0$$

$$-\lambda^3 + 6\lambda^2 - 6\lambda + 11 = 0.$$

To prove Caylay Hamilton Theorem

We will prove.

$$A^3 - 6A^2 + 6A - 11I = 0. \quad \text{——— (1)}$$

$$A^2 = \begin{bmatrix} 4 & 3 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} 4 & 3 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 16+6+1 & 12+3+2 & 4-6+1 \\ 8+2-2 & 6+1-4 & 2-2-2 \\ 4+4+1 & 3+2+2 & 1-4+1 \end{bmatrix}$$

$$\begin{bmatrix} 23 & 17 & -1 \\ 8 & 3 & -2 \\ 9 & 7 & -2 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 23 & 17 & -1 \\ 8 & 3 & -2 \\ 9 & 7 & -2 \end{bmatrix}\begin{bmatrix} 4 & 3 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 125 \cancel{+34-1} & 69+17-2 & 23-34-1 \\ 32+6-2 & 24+3-4 & 8-6-2 \\ 36+14-2 & 27+7-4 & 9-14-2 \end{bmatrix}$$

$$\begin{bmatrix} 125 & 84 & -12 \\ 36 & 23 & 0 \\ 48 & 30 & -7 \end{bmatrix}$$

From (1)

$$\begin{bmatrix} 125 & 84 & -12 \\ 36 & 23 & 0 \\ 48 & 30 & -7 \end{bmatrix} + \begin{bmatrix} -138 & -102 & +6 \\ -48 & -18 & +12 \\ -54 & -42 & +12 \end{bmatrix} + \begin{bmatrix} 24 & 18 & 6 \\ 12 & 6 & -12 \\ 6 & 12 & 6 \end{bmatrix} + \begin{bmatrix} -11 & 0 & 0 \\ 0 & -11 & 0 \\ 0 & 0 & -11 \end{bmatrix}$$

$$\begin{bmatrix} 125-138+24-11 & 84-102+18 & -12+6+6 \\ 36-48+12+0 & 23-18+6-11 & +12-12 \\ 48-54+6+0 & 30-42+12 & -7+12+6-11 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Multiply (1) by A^{-1}

$$A^2 - 6A + 6I - 11A^{-1} = 0$$

$$11A^{-1} = A^2 - 6A + 6I.$$

$$A^{-1} = \frac{1}{11}\left[A^2 - 6A + 6I\right]$$

$$A^{-1} = \left(\frac{1}{11}\right)\begin{bmatrix} 23-24+6 & 17-18 & -1-6 \\ 8-12 & 3-6+6 & -12+12 \\ 9-6 & 7-12 & -2-6+6 \end{bmatrix}$$

$$A^{-1} = \frac{1}{11}\begin{bmatrix} 5 & -1 & -7 \\ -4 & 3 & 10 \\ 3 & -5 & -2 \end{bmatrix}$$

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Ques a) Verify Caley Hamilton Theorem & hence find A^{-1}

$$A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

Express i) $A^6 - 6A^5 + 9A^4 - 2A^3 - 12A^2 + 23A - 9I$
ii) $A^5 - 5A^4 + 3A^3 + 6A^2 - 6A + 2I$ as linear polynomial

b) $$A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$$

Prove That: $A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I = \begin{bmatrix} 8 & 5 & 5 \\ 0 & 3 & 0 \\ 5 & 5 & 8 \end{bmatrix}$

Ans a) $$A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$|A - \lambda I| = 0$

$$|A - \lambda I| = \begin{vmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{vmatrix} = 0$$

$$\begin{vmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{vmatrix} = 0$$

$(2-\lambda)[(2-\lambda)(2-\lambda)-1] + 1[\lambda - 2 + 1] + 1[1 - 2 + \lambda] = 0$

$(2-\lambda)[4 - 2\lambda - 2\lambda + \lambda^2 - 1] + [\lambda - 1] + [\lambda - 1] = 0$

$(2-\lambda)[3 - 4\lambda + \lambda^2] + 2\lambda - 2 = 0$

$6 + 8\lambda + 2\lambda^2 - \lambda^3 + 4\lambda^2 - \lambda^3 + 2\lambda - 2 = 0$

$-\lambda^3 + 6\lambda^2 - 9\lambda + 4 = 0.$

$\lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0$

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$A^3 - 6A^2 + 9A - 4 = 0$

$$A^2 = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 4+1+1 & -2-2-1 & 2+1+2 \\ -2-2+1 & 1+4+1 & -1-2+2 \\ 2+1+2 & -1-2-2 & 1+4+1 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}$$

$$A^3 = A^2 . A . = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 12+5+5 & -6-10-5 & 6+5+10 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix}$$

$$= \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix}$$

$$A^3 - 6A^2 + 9A - 4I = \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix} - \begin{bmatrix} 36 & -30 & 30 \\ -30 & 36 & -30 \\ 30 & -30 & 36 \end{bmatrix}$$

$$+ \begin{bmatrix} 18 & -9 & 9 \\ -9 & 18 & -9 \\ 9 & -9 & 18 \end{bmatrix} - \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 22-36+18-4 & -21+30-9 & 21-30+9 \\ -21+30-9 & 22-36+18-4 & -21+30-9 \\ 21-30+9 & -21+30-9 & 22-36+18-4 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

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Ques. Find the sum & product of the eigenvalues of $A = \begin{bmatrix} 1 & 6 & 1 \\ 2 & 1 & 0 \\ 0 & 5 & 3 \end{bmatrix}$

Sum of eigenvalues = trace of the matrix
trace = sum of diagonal element
$= 1 + 1 + 3 = 5$.
$|A| = -25 =$ Product of

Ques Two eigenvalues of matrix $A = \begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -5 & -2 \end{bmatrix}$ are equal and they are double of third. Find them.

Sum of eigenvalues = sum of diagonal element.
Let $\lambda_1 = 2a$, $\lambda_2 = 2a$, $\lambda_3 = \frac{a}{1}$

$2a + 2a + \frac{a}{1} = 5$

$5a = 5$

$a = 1$.

Eigenvalues are 2, 2, 1

Ques - If $A = \begin{bmatrix} 3 & 5 & 3 \\ 0 & 4 & 6 \\ 0 & 0 & 1 \end{bmatrix}$ find eigenvalues of A, A^{-1} adj A, A^5.

By Property,

Eigenvalues of A are $1, 4, 3$
" " A^{-1} are $\frac{1}{1}, \frac{1}{4}, \frac{1}{3}$
" " A^5 are $1, 4^5, 3^5$.

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Eigenvalue of adj A are $12\ (12/4)\ (12/3)$.
Eigenvalue of adj A = $|A| \times$ Eigenvalue

Property of Eigenvector

1. The eigenvector x of a matrix A is not unique
2. If $\lambda_1, \lambda_2, ---, \lambda_n$ of a $n \times n$ matrix then eigenvector $X_1, X_2, ---, X_n$ form a linearly independent set.
   $A_1 x_1 + A_2 x_2 + ---- + a_n x_n = 0$.
   $a_1 = 0, a_2 = 0, ----- a_n = 0$.
   Then $X_1, X_2, ----, X_n$ are linearly independent.
3. Eigenvectors of a symmetric matrix corresponding to different eigenvalues are orthogonal.
   $\lambda_1, \lambda_2, ---, \lambda_n$.
   $X_1, X_2, --, X_n$.
   Two vectors x_1 & x_2 are said orthogonal if
   $X_1 X_2^T = 0$
4. If two or more eigenvalues of a matrix are equal then eigenvector may be linearly dependent or linearly independent.

Method to find Eigenvector

Let $A = \begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}$

Step I - Write the characteristic eqn.
Step II - Find the Eigenvalue
Step III - Corresponding to each eigenvalue Make $[A - \lambda I]X = 0$

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Step IV – Find the solution of eqn which we get in step III.

Step V – Find x_1, x_2, x_3 put them in column $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

In above matrix eigenvalues are 3, 2, 5

$\lambda_1 = 3, \lambda_2 = 2, \lambda_3 = 5.$

for $\lambda = 2$.

$(A - \lambda I) X = 0.$

$(A - 2I) X = 0.$

$$\begin{bmatrix} 1 & 1 & 4 \\ 0 & 0 & 6 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_1 + x_2 + 4x_3 = 0.$

$6x_3 = 0$

$3x_3 = 0$

$\therefore x_3 = 0$

$x_1 + x_2 = 0.$

Let $x_1 = K$

Then $x_2 = -K$

& $x_3 = 0.$

$$X = \begin{bmatrix} K \\ -K \\ 0 \end{bmatrix}$$

$$X = K \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

For $\lambda = 3$.

$(A - \lambda I) X = 0.$

$(A - 3I) X = 0$

$$\begin{bmatrix} 0 & 1 & 4 \\ 0 & -1 & 6 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_2 + 4x_3 = 0$

$-x_2 + 6x_3 = 0$

$2x_3 = 0.$

$\therefore \; x_3 = 0$

$\& \; x_1 = K$

$x_2 + 4(0) = 0$

$\therefore \; x_2 = 0$

$$X = \begin{bmatrix} K \\ 0 \\ 0 \end{bmatrix} = K \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

For $\lambda = 5$.

$$\begin{bmatrix} -2 & 1 & 4 \\ 0 & -3 & 6 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-2x_1 + x_2 + 4x_3 = 0$

$-3x_2 + 6x_3 = 0.$

Let $x_3 = K$

$-3x_2 = -K$

$x_2 = \frac{6K}{3} = 2K$

$-2x_1 + \frac{12K}{3} + 4K = 0.$

$\cancel{+2x_1 = \frac{+13K}{3}}$ $\qquad -2x_1 = -6K$

$\qquad 2x_1 = 6K$

$\cancel{x_1 = \frac{13K}{6}}$ $\qquad x_1 = 3K$

$$X = \begin{bmatrix} 3k \\ 2k \\ k \end{bmatrix} = k \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

Shortcut method to find the eigenvalues & eigenvectors:

If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

$|A - \lambda I| = 0$.

$$\lambda^3 - (a_{11} + a_{22} + a_{33})\lambda^2 + \lambda(a_{11}a_{22} + a_{22}a_{33} + a_{33}a_{11} - a_{12}a_{21} - a_{13}a_{31} - a_{23}a_{32}) - |A| = 0.$$

If $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$

then, $\lambda^3 - 7\lambda^2 + 36 = 0$ CR. Eqn

Eigenvalues $-2, 3, 6$.

$(A - \lambda I)X = 0$.

$$\begin{bmatrix} 3 & 1 & 3 \\ 1 & 7 & 1 \\ 3 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Find the cofactors of I^{st} row

$A_{11} = 20$, $A_{12} = 0$, $A_{13} = -20$.

$$X_1 = \begin{bmatrix} 20 \\ 0 \\ -20 \end{bmatrix} = 20 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

If cofactor corresponding to 1^{st} row are all zero then, we calculate the cofactor ~~to~~ of 2^{nd} row.

If All cofactors are zero method fails.

Ques $A = \begin{bmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{bmatrix}$ find eigenvalue & eigenvector.

Ques. $A \begin{bmatrix} 1 & -6 & -4 \\ 0 & 4 & 2 \\ 0 & -6 & -3 \end{bmatrix}$ find eigenvalue & eigenvector

Ques. $A \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$ find eigenvalue & eigenvector.

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Diagonalization Of A Matrix

It is the process of reduction of A to a diagonal form D, we reduce the matrix A by the transformation $P^{-1}AP = D$.
If A is a symmetric matrix then transformation is $N^T A N = D$
P is known as modal matrix.

$$A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

Characteristic eqn,

$$\lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0$$

Put $\lambda = 2$.

$$8 - 12(4) + 36(2) - 32 = 0.$$

$$0 = 0.$$

Eigenvalues 2, 2, 8.

Eigenvectors $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$

$$\begin{array}{cccc} & X_1 & X_2 & X_3 \end{array}$$

$$P = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 3 & -1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 2 & -2 \\ 1 & 1 & 1 \end{bmatrix}$$

$$P^{-1} = \frac{\text{Adj } P}{|P|}$$

$$= \frac{-1}{6} \begin{bmatrix} 4 & 1 & -7 \\ -2 & -2 & 2 \\ -2 & 1 & -1 \end{bmatrix}$$

$$P^{-1}AP = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{bmatrix}$$

Ques- $A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$

$|A - \lambda I| = 0.$

$$\begin{vmatrix} 3-\lambda & -1 & 1 \\ -1 & 5-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{vmatrix} = 0.$$

$(3-\lambda)[(5-\lambda)(3-\lambda)-1] + 1[(\lambda-3)+1] + 1[1+\lambda-5]$

$(3-\lambda)[15-3\lambda+\lambda^2-5\lambda-1] + [\lambda-2] + [\lambda-4]$

$45-9\lambda+3\lambda^2-15\lambda-3-15\lambda+3\lambda^2-\lambda^3+5\lambda^2+\lambda+2\lambda-6$

$\cancel{36}\ \lambda^3-11\lambda^2+36\lambda-36=0.$

$\lambda=2.$

$8-11(4)+36(2)-36=0$

$0=0.$

Eigenvalues $2, 3, 6.$

$\lambda_1 = 2$

$$\begin{bmatrix} 1 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 1 \end{bmatrix}$$

$A_{11} = 3-(1)$
$= 2$

$\cancel{X_1 = \begin{bmatrix} 4 \\ 0 \\ -2 \end{bmatrix}}$ $X_1 = \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

$\lambda_2 = 3$

$$\begin{bmatrix} 0 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 0 \end{bmatrix}$$

$X_2 = \begin{bmatrix} +1 \\ -1 \\ -1 \end{bmatrix} = -1\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$\lambda_3 = 6$

$$\begin{bmatrix} -3 & -1 & 1 \\ -1 & -1 & -1 \\ 1 & -1 & -3 \end{bmatrix}$$

$$X_3 = \begin{bmatrix} 2 \\ -4 \\ 2 \end{bmatrix} = 2\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -2 \\ -1 & 1 & 1 \end{bmatrix}$$

$$\text{adj } P = \begin{bmatrix} 3 & 3 & 1 \\ 0 & 2 & -2 \\ -3 & +2 & 1 \end{bmatrix}^T = \begin{bmatrix} 3 & 0 & -3 \\ 2 & 2 & 2 \\ 1 & -2 & 1 \end{bmatrix}$$

$$|P| = 1(3) - (1)(-3) + 0(0)$$
$$= 3 + 3 + 0$$
$$= 6$$

$3(2-2) - 3(\cancel{0+6) + 1(-6)}$

$12 - 3(6) + 1(-6)$

$= 12 - 18 - 6$

$$P^{-1} = \frac{1}{6}\begin{bmatrix} 3 & 0 & -3 \\ 2 & 2 & 2 \\ 1 & -2 & 1 \end{bmatrix}$$

$$P^{-1}A = \frac{1}{6}\begin{bmatrix} 3 & 0 & -3 \\ 2 & 2 & 2 \\ 1 & -2 & 1 \end{bmatrix}\begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$

$$= \frac{1}{6}\begin{bmatrix} 9-3 & -3+3 & 3-9 \\ 6-2+2 & -2+10-2 & 2-2+6 \\ 3+2+1 & -1-10-1 & 1+2+3 \end{bmatrix}$$

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$$P^{-1}A = \frac{1}{6}\begin{bmatrix} 6 & 0 & -6 \\ 6 & 6 & 6 \\ 6 & -12 & 6 \end{bmatrix}$$

$$P^{-1}A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \\ 1 & -2 & 1 \end{bmatrix}$$

$$P^{-1}AP = D$$

$$P^{-1}AP = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \\ 1 & -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -2 \\ -1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+0+1 & 1+0-1 & 1+0-1 \\ 1+0-1 & 1+1+1 & 1-2+1 \\ 1+0-1 & 1-2+1 & 1+4+1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

Diagonalise the matrix $A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix}$ by orthogonal transformation

C.R. eqn $|A| = 3(9-1) - 1(3+1) + 1(-1-3)$

$|A| = 16$.

$\lambda^3 - (3+3+3)\lambda^2 + (9+9+9-1-1-1)\lambda - 16 = 0$.

$\lambda^3 - 9\lambda^2 + 24\lambda - 16 = 0$.

$\lambda = 1$

$1 - 9 + 24 - 16 = 0 \Rightarrow 0 = 0$.

$\lambda - 1 \overline{)\lambda^3 - 9\lambda^2 + 24\lambda - 16} \; \lambda^2 - 8\lambda + 16$

$\lambda^3 - 9\lambda^2 + 24\lambda - 16$

0

$\lambda = 1, 4, 4.$

$(A - I)X = 0.$

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

corresponding to $\lambda = 1.$

$A_{11} = 3, \quad A_{12} = -3, \quad A_{13} = -3.$

$$X = \begin{bmatrix} 3 \\ -3 \\ -3 \end{bmatrix} = -3 \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

Corresponding to $\lambda = 4.$

$(A - 4I)X = 0.$

$$\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-x_1 + x_2 + x_3 = 0.$

$x_1 - x_2 - x_3 = 0.$

In eqn let $x_2 = 1, x_1 = 1.$

$x_1 - x_2 - x_3 = 0.$

$\Rightarrow x_3 = 0$

$$X_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$\because$ A is symmetric matrix therefore X_1, X_2 & X_3 are orthogonal

Let $X_3 = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

$X_1^T X_3 = 0$

$$\Rightarrow \begin{bmatrix} -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0 \quad \Rightarrow -a + b + c = 0 \quad \text{—(1)}$$

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$X_2^T X_3 = 0.$

$$\begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0$$

$a + b = 0$ —— (2).

by adding eqⁿ (1) & (2).

$2a + c = 0$

$2a = -c.$

Let $c = 2$,

$\therefore 2b = -2$

$b = -1.$

In eqⁿ (2).

$a - 1 = 0$

$a = 1.$

$$X_3 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}.$$

$$P = \begin{bmatrix} -1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & 0 & 2 \end{bmatrix}$$

N is the normalized vector.

$$N = \begin{bmatrix} \frac{-1}{\sqrt{(-1)^2+1^2+1^2}} & \frac{1}{\sqrt{1^2+1^2+0^2}} & \frac{1}{\sqrt{1^2+1^2+2^2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{0}{\sqrt{2}} & \frac{2}{\sqrt{6}} \end{bmatrix}$$

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$$N^T = \begin{bmatrix} -1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{6} & -1/\sqrt{6} & 2/\sqrt{6} \end{bmatrix}$$

$$N^TA = \begin{bmatrix} -1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{6} & -1/\sqrt{6} & 2/\sqrt{6} \end{bmatrix} \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} -3/\sqrt{3} + 1/\sqrt{3} + 1/\sqrt{3} & -1/\sqrt{3} + 3/\sqrt{3} - 1/\sqrt{3} & -1/\sqrt{3} - 1/\sqrt{3} + 3/\sqrt{3} \\ 3/\sqrt{2} + 1/\sqrt{2} + 0 & 1/\sqrt{2} + 3/\sqrt{2} + 0 & 1/\sqrt{2} - 1/\sqrt{2} + 0 \\ 3/\sqrt{6} - 1/\sqrt{6} + 2/\sqrt{6} & 1/\sqrt{6} - 3/\sqrt{6} - 2/\sqrt{6} & 1/\sqrt{6} + 1/\sqrt{6} + 6/\sqrt{6} \end{bmatrix}$$

$$= \begin{bmatrix} -1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 4/\sqrt{2} & 4/\sqrt{2} & 0 \\ 4/\sqrt{6} & -4/\sqrt{6} & 8/\sqrt{6} \end{bmatrix}$$

$$N^TAN = \begin{bmatrix} -1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 4/\sqrt{2} & 4/\sqrt{2} & 0 \\ 4/\sqrt{6} & -4/\sqrt{6} & 8/\sqrt{6} \end{bmatrix} \begin{bmatrix} -1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 0 & 2/\sqrt{6} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

Ques- Diagonalise the matrix $A = \begin{bmatrix} 7 & -2 & 0 \\ -2 & 6 & -2 \\ 0 & -2 & 5 \end{bmatrix}$ by orthogonal transformation.

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Quadratic forms

A homogeneous polynomial of 2nd degree in any no. of varibles is called quadratic forms. for example. $12x^2+5xy-18y^2$

$x^2-2y^2+z^2+xy-2xz+yz$

$ax^2+by^2+2hxy+2gxz+2fyz+cz^2$

The general form is $\sum_{j=1}^{n}\sum_{i=1}^{n} b_{ij}x_i x_j$

Note: Every quadratic form can be written as in the general form $\sum_{j=1}^{n}\sum_{i=1}^{n} b_{ij}x_i x_j$ where $[b_{ij}]=B$ is always symmetric matrix.

Ques- Reduce $10x_1^2+2x_2^2+5x_3^2-4x_1x_2+6x_2x_3-10x_3x_1$ the quadratic form to a canonical form by orthogonal reduction.

$$A.\begin{bmatrix} \text{coeff of } x_1^2 & \tfrac{1}{2}\text{coeff of } x_1x_2 & \tfrac{1}{2}\text{coeff of } x_1x_3 \\ \tfrac{1}{2}\text{coeff of } x_1x_2 & \text{coeff of } x_2^2 & \tfrac{1}{2}\text{coeff of } x_2x_3 \\ \tfrac{1}{2}\text{coeff } x_1x_3 & \tfrac{1}{2}\text{coeff } x_2x_3 & \text{coeff of } x_3^2 \end{bmatrix}$$

$$A.\begin{bmatrix} 10/1 & -4/2 & -10/2 \\ -4/2 & 2/1 & 6/2 \\ -10/2 & 6/2 & 5/1 \end{bmatrix} : \begin{bmatrix} 10 & -2 & -5 \\ -2 & 2 & 3 \\ -5 & 3 & 5 \end{bmatrix}$$

steps (1) Reduce quadratic form into symmetric matrix
(2) Find CR eq?.
(3) Find Eigen values.
(4) Find Eigen vector.
(5) Find Modal matrix.

⑥ Find Normalized modal matrix. (N)

⑦ The orthogonal Transformation is given by:

$$\boxed{X = NY}$$

where $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ & $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

which is the required canonical form.

$$A = \begin{bmatrix} 10 & -2 & -5 \\ -2 & 2 & 3 \\ -5 & 3 & 5 \end{bmatrix}$$

$[A - \lambda I]\cancel{x} = 0.$

CR eqn.

$$\begin{vmatrix} 10-\lambda & -2 & -5 \\ -2 & 2-\lambda & 3 \\ -5 & 3 & 5-\lambda \end{vmatrix} = 0.$$

$$(10-\lambda)\left[(2-\lambda)(5-\lambda)-9\right] + 2\left[(5-\lambda)(-2)+15\right] + (-5)\left[(-6)-(-5)(2-\lambda)\right]$$

$$(10-\lambda)\left[10-2\lambda-5\lambda+\lambda^2-9\right] + 2\left[-10+2\lambda+15\right] - 5\left[-6+10-5\lambda\right]$$

$$(10-\lambda)\left[\lambda^2-7\lambda+1\right] + 2\left[2\lambda+5\right] - 5\left[4-5\lambda\right]$$

$$10\lambda^2 - 70\lambda + 10 - \lambda^3 + 7\lambda^2 - \lambda + 4\lambda + 10 - 20 + 25\lambda = 0$$

$$-\lambda^3 + 17\lambda^2 - 42\lambda = 0.$$

$$\lambda^3 - 17\lambda^2 + 42\lambda = 0$$

$$\lambda(\lambda^2 - 17\lambda + 42) = 0.$$

$$\lambda = 0, 3, 14.$$

Eigenvector corresponding to $\lambda = 0$.

$[A - 0I]X = 0.$

$$\begin{bmatrix} 10 & -2 & -5 \\ -2 & 2 & 3 \\ -5 & 3 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$X_1 = \begin{bmatrix} 1 \\ -5 \\ 4 \end{bmatrix}$$

Corresponding to $\lambda = 3$.

$$\begin{bmatrix} 7 & -2 & -5 \\ -2 & -1 & 3 \\ -5 & 3 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$X_2 = \begin{bmatrix} -11 \\ -11 \\ -11 \end{bmatrix}$$

$$X_2 = -11 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Corresponding to $\lambda = 14$.

$$\begin{bmatrix} -4 & -2 & -5 \\ -2 & -12 & 3 \\ -5 & 3 & -9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$X_3 = \begin{bmatrix} 99 \\ -33 \\ -66 \end{bmatrix} = -33 \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 1 & -3 \\ -5 & 1 & 1 \\ 4 & 1 & 2 \end{bmatrix}$$

$$N = \begin{bmatrix} 1/\sqrt{42} & 1/\sqrt{3} & -3/\sqrt{14} \\ -5/\sqrt{42} & 1/\sqrt{3} & 1/\sqrt{14} \\ 4/\sqrt{42} & 1/\sqrt{3} & 2/\sqrt{14} \end{bmatrix}$$

$$X = NY.$$

$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{42} & 1/\sqrt{3} & -3/\sqrt{14} \\ -5/\sqrt{42} & 1/\sqrt{3} & 1/\sqrt{14} \\ 4/\sqrt{42} & 1/\sqrt{3} & 2/\sqrt{14} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

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$$\cancel{X = \begin{bmatrix} y_1/\sqrt{42} + y_2/\sqrt{3} - 3y_3/\sqrt{14} \\ -5y_1/\sqrt{42} + y_2/\sqrt{3} + y_3/\sqrt{14} \\ 4y_1/\sqrt{42} + 0y_2/\sqrt{3} + 2y_3/\sqrt{14} \end{bmatrix}}$$

$$\cancel{= \frac{1}{\sqrt{42}} \begin{bmatrix} y_1 + \sqrt{14}\, y_2 - 3\sqrt{3}\, y_3 \\ -5y_1 + y_2\sqrt{14} - \sqrt{3}\, y_3 \\ 4y_1 + \sqrt{14}\, y_2 + 2\sqrt{3}\, y_3 \end{bmatrix}}$$

$$N^T = \begin{bmatrix} 1/\sqrt{42} & -5/\sqrt{42} & 4/\sqrt{42} \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ -3/\sqrt{14} & 1/\sqrt{14} & 2/\sqrt{14} \end{bmatrix}$$

$$N^T A = \begin{bmatrix} 1/\sqrt{42} & -5/\sqrt{42} & 4/\sqrt{42} \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ -3/\sqrt{14} & 1/\sqrt{14} & 2/\sqrt{14} \end{bmatrix} \begin{bmatrix} 10 & -2 & -5 \\ -2 & 2 & 3 \\ -5 & 3 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{10+10-20}{\sqrt{42}} & \frac{-2-10+12}{\sqrt{42}} & \frac{-5-15+20}{\sqrt{42}} \\ \frac{10-2-5}{\sqrt{3}} & \frac{-2+2+3}{\sqrt{3}} & \frac{-5+3+5}{\sqrt{3}} \\ \frac{-30-2-10}{\sqrt{14}} & \frac{6+2+6}{\sqrt{14}} & \frac{15+3+10}{\sqrt{14}} \end{bmatrix}$$

$$= \begin{bmatrix} \cancel{\frac{20}{\sqrt{42}}}\ 0 & 0 & 0 \\ 3/\sqrt{3} & 3/\sqrt{3} & 3/\sqrt{3} \\ \frac{-42}{\sqrt{14}} & 14/\sqrt{14} & 28/\sqrt{14} \end{bmatrix}$$

$$N^T A N = \begin{bmatrix} \cancel{\frac{20}{\sqrt{42}}} & 0 & 0 \\ 3/\sqrt{3} & 3/\sqrt{3} & 3/\sqrt{3} \\ -42/\sqrt{14} & 14/\sqrt{14} & 28/\sqrt{14} \end{bmatrix} \begin{bmatrix} 1/\sqrt{42} & 1/\sqrt{3} & -3/\sqrt{14} \\ -5/\sqrt{42} & 1/\sqrt{3} & 1/\sqrt{14} \\ 4/\sqrt{42} & 1/\sqrt{3} & 2/\sqrt{14} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ \frac{3-15+12}{\sqrt{3}\sqrt{42}} & \frac{3+3+3}{\sqrt{3}\sqrt{3}} & 0 \\ 0 & 0 & 14 \end{bmatrix}$$

$$D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 14 \end{bmatrix}$$

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Properties of Quadratic form :

1. Quadratic form is known as positive definate if $r = n$, $s = n$
r = rank
n = no. of variable
s = index

2. Quadratic form is known as negative definate if $r = n$ & $s = 0$.

3. Quadratic form is semi positive definate if $r < n$ $s = r$.

4. Quadratic form is negative semi definite if $r < n$, $s = 0$.

5. Otherwise Infinite.

Continued → For canonical form, $Y^T D Y$

$0y_1^2 + 3y_2^2 + 14y_3^2$

$3y_2^2 + 14y_3^2$

Index (p) – The no. of +ve terms in the canonical form is known as index.

SIGNATURE – The difference b/w +ve terms & –ve terms in the canonical form is called signature.

To calculated this,

$2p - r$, here r is rank.

→ If $r = p = n$, then quadratic form is +ve
n = order

$A^{m} = 0$ = +ve ; index

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definite.

→ If $r = p < n$ then quadratic form is semi definite.

→ If $p = 0$ & $r = n$, quadratic form is $-ve$ definite.

→ If $p = 0$ & $r < n$, quadratic form $-ve$ semi definite.

→ Otherwise Indefinite

Let $A = (a_{ij})_{n\times n}$ $\quad D_1 = |a_{11}|$

$$D_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

$$D_3 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$D_n = |A|$

i) A quadratic form is +ve definite if all $D_1, D_2, D_3 \cdots D_n$ ~~is all~~ are +ve.

ii) A quadratic form is $-ve$ definite if $D_1, D_3, D_5 \cdots$ are $-ve$ & $D_2, D_4, D_6 \cdots$ are +ve

iii) If some of the det. are vanish while others are +ve then Q.F. is +ve semi definite

iv) If some of the det. vanish ~~while~~ in the case (ii) then Q.F. is $-ve$ semi definite

v) Otherwise Infinite.

Find the nature of Q.F. $2x^2 + 3y^2 + 2z^2 + 2xy$.

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

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$D_1 = |2| = 2 > 0$

$D_2 = \begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} = 6-1 = 5 > 0$

$D_3 = \begin{vmatrix} 2 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 2 \end{vmatrix} = 2(6-1) = 10 > 0$

Because $D_1, D_2, D_3 > 0$ so the nature of Q.F. is +ve definite

$x_1^2 + 2x_2^2 + 3x_3^2 + 2x_2x_3 - 2x_3x_1 + 2x_1x_2$

$A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 3 \end{bmatrix}$

$D_1 = |1| = 1 > 0$

$D_2 = \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 2-1 > 0$

$D_3 = \begin{vmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 3 \end{vmatrix} = 1(6-1) - 1(3+1) + (-1)(1+2)$
$= 5-4-3$
$= -2 > 0$

Indefinite.

(1) $2x^2 + 5y^2 + 7xy$

(2) $2x_1^2 + 3x_2^2 + 6x_3^2 + 2x_2x_3 + 4x_3x_1 + 2x_1x_2$

(3) $2x_1^2 + 2x_1x_2 + 3x_2^2$

(4) $6x^2 + 17y^2 + 3z^2 - 20xy - 14yz + 8zx$.

→ Reduce the Q.F. into canonical form & discuss its nature, also rank, signature & index.

i) $-x^2 + y^2 + 4yz + 4zx$.

ii) $2xy + 2yz + 2zx$.

iii) $8x^2 + 7y^2 + 3z^2 + 4xz - 8yz - 12xy$.

1. $2x^2 + 5y^2 + 7xy.$

$$A = \begin{bmatrix} 2 & 7/2 & 0 \\ 7/2 & 5 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$D_1 = |2| = 2 > 0$

$$D_2 = \begin{vmatrix} 2 & 7/2 \\ 7/2 & 5 \end{vmatrix} = 10 - \frac{49}{4} = \frac{-9}{4} < 0$$

$$D_3 = \begin{vmatrix} 2 & 7/2 & 0 \\ 7/2 & 5 & 0 \\ 0 & 0 & 0 \end{vmatrix} = 0 = 0$$

Indefinite

2. $2x_1^2 + 3x_2^2 + 6x_3^2 + 2x_2x_3 + 4x_3x_1 + 2x_1x_2.$

$$A = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 3 & 1 \\ 2 & 1 & 6 \end{bmatrix}$$

$D_1 = |2| = 2 > 0$

$$D_2 = \begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} = 6 - 1 = 5 > 0$$

$$D_3 = \begin{vmatrix} 2 & 1 & 2 \\ 1 & 3 & 1 \\ 2 & 1 & 6 \end{vmatrix} = 2(18-1) - 1(6-2) + 2(1-6)$$

$= 34 - 4 - 10$

$= 20 > 0.$

As, $D_1, D_2, D_3 > 0$, nature of Q.F. is positive definite.

3. $2x_1^2 + 2x_1x_2 + 3x_2^2$

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$D_1 = |2| = 2 > 0$

$$D_2 = \begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} = 6 - 1 = 5 > 0$$

$D_3 = 0 = 0$, semi positive definite

4. $6x^2+17y^2+3z^2-20xy-14yz+8zx$.

$$A = \begin{bmatrix} 6 & -10 & 4 \\ -10 & 17 & -7 \\ 4 & -7 & 3 \end{bmatrix}$$

$D_1 = |6| = 6 > 0$.

$$D_2 = \begin{vmatrix} 6 & -10 \\ -10 & 17 \end{vmatrix} = 102 - 100 = 2 > 0.$$

$$D_3 = \begin{vmatrix} 6 & -10 & 4 \\ -10 & 17 & -7 \\ 4 & -7 & 3 \end{vmatrix} = 6(51-49)+10(-30+28) + 4(70-68) = 12-20+8=0$$

Semi positive definite

i) $-x^2+y^2+4yz+4zx$

$$A = \begin{bmatrix} -1 & 0 & 2 \\ 0 & 1 & 2 \\ 2 & 2 & 0 \end{bmatrix}$$

$|A-\lambda I| = 0$

$$\begin{vmatrix} -1-\lambda & 0 & 2 \\ 0 & 1-\lambda & 2 \\ 2 & 2 & -\lambda \end{vmatrix} = 0$$

$\cancel{(-1-\lambda)(-4)+2(-(2)(1-\lambda)) = 0}$

$\cancel{(4+4\lambda)+2(2\lambda-2)=0}$

$\cancel{4+4\lambda+4\lambda-4=0}$

$\cancel{8\lambda=0}$

$\cancel{\lambda=0,0,0}$

~~for $\lambda=0$.~~

$\cancel{[A-\lambda I][X]=0}$

$$\cancel{\begin{bmatrix} -1 & 0 & 2 \\ 0 & 1 & 2 \\ 2 & 2 & 0 \end{bmatrix}}$$

To be continued →

$(-1-\lambda)((1-\lambda)(1-\lambda)-4) + 2((-2)(1-\lambda)) = 0$

$+(1+\lambda)(\lambda(1-\lambda)+4) + -(4)(1-\lambda) = 0$

$(1+\lambda)(\lambda-\lambda^2+4) + 4\lambda - 4 = 0$

$\lambda + 4 - \lambda^2 + \lambda^2 - \lambda^3 + 4\lambda + 4\lambda - 4 = 0.$

$-\lambda^3 + 9\lambda = 0$

$\lambda(\lambda^2 - 9) = 0$

$\lambda = 0$

$\lambda = 3$

$\lambda = -3$

For $\lambda = 0$.

$[A - \lambda I][X] = 0.$

$$\begin{bmatrix} -1 & 0 & 2 \\ 0 & 1 & 2 \\ 2 & 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_1 = -4, \quad x_2 = 4, \quad x_3 = -2.$

$$X_1 = \begin{bmatrix} -4 \\ 4 \\ -2 \end{bmatrix} = -2 \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

For $\lambda = 3$

$[A - 3I][X] = 0$

$$\begin{bmatrix} -4 & 0 & 2 \\ 0 & -2 & 2 \\ 2 & 2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_1 = 2, \quad x_2 = 4, \quad x_3 = 4.$

$$X_2 = \begin{bmatrix} 2 \\ 4 \\ 4 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

For $\lambda = -3$.

$[A + 3I][X] = 0$

$$\begin{bmatrix} 2 & 0 & 2 \\ 0 & 4 & 2 \\ 2 & 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_1 = 8$, $x_2 = 4$, $x_3 = -8$

$$X_3 = \begin{bmatrix} 8 \\ 4 \\ -8 \end{bmatrix} = 4 \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$$

$$P = \begin{bmatrix} 2 & 1 & 2 \\ -2 & 2 & 1 \\ 1 & 2 & -2 \end{bmatrix}$$

$$N = \begin{bmatrix} 2/3 & 1/3 & 2/3 \\ -2/3 & 2/3 & 1/3 \\ 1/3 & 2/3 & -2/3 \end{bmatrix}$$

$$N^T = \begin{bmatrix} 2/3 & -2/3 & 1/3 \\ 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \end{bmatrix}$$

$$N^T A = \begin{bmatrix} 2/3 & -2/3 & 1/3 \\ 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \end{bmatrix} \begin{bmatrix} -1 & 0 & 2 \\ 0 & 1 & 2 \\ 2 & 2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{-2+2}{3} & \frac{-2+2}{3} & \frac{4-4}{3} \\ \frac{-1+4}{3} & \frac{2+4}{3} & \frac{2+4}{3} \\ \frac{-2+4}{3} & \frac{1+4}{3} & \frac{4+2}{3} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 1 & 2 & 2 \\ -2/3 & 5/3 - 1 & 2 \end{bmatrix}$$

$$N^TAN = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 2 & 2 \\ -2/3 & 5/3 - 1 & 2 \end{bmatrix} \begin{bmatrix} 2/3 & 1/3 & 2/3 \\ -2/3 & 2/3 & 1/3 \\ 1/3 & 2/3 & -2/3 \end{bmatrix}$$

$$= \begin{bmatrix} \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} \\ \cancel{\frac{2-4+2}{3}} & \cancel{\frac{1+4+4}{3}} & \cancel{2+2} & \end{bmatrix}$$

$$N^T A N = \begin{bmatrix} 0 & 0 & 0 \\ \frac{2-4+2}{3} & \frac{1+4+4}{3} & \frac{2+2-4}{3} \\ \frac{-4+2+2}{3} & \frac{-2-2+4}{3} & \frac{-4-1-4}{3} \end{bmatrix}$$

$$D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

$$Y^T D Y = \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 3y_2 & -3y_3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$Y^T D Y = 3y_2^2 - 3y_3^2$$

~~Conical~~ Canonical form,

$$3y_2^2 - 3y_3^2$$

index, $p = 1$.

$$s = 2p - r$$
$$= 2(1) - 2$$
$$= 2 - 2 = 0.$$