11 - Trajectories Between Two Points, Lambert's Time of Flight Theorem, and Interplanetary Trajectories

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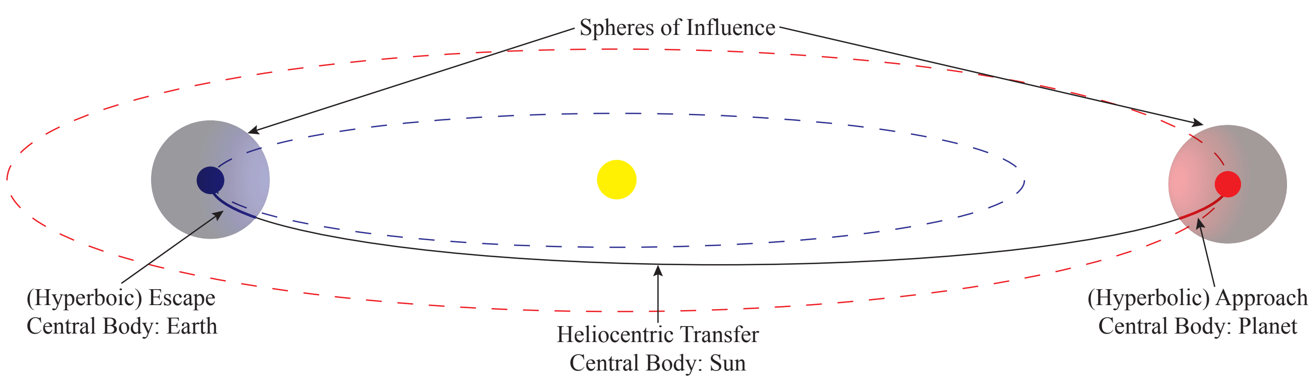
MAE 4060

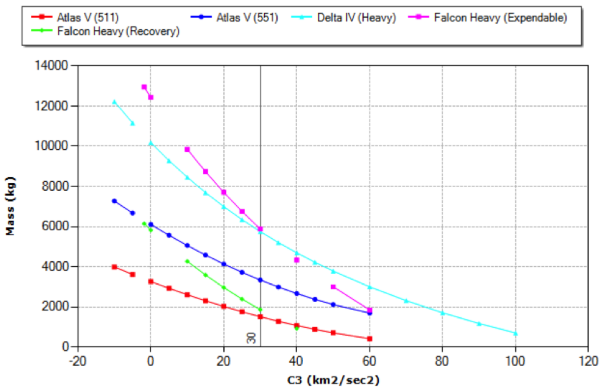
The Patched Conic Approximation

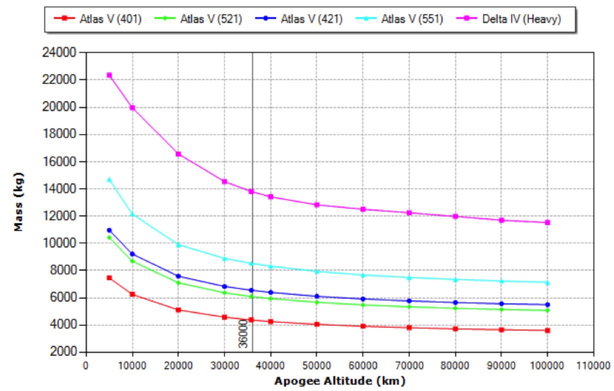


$$r_{\text{SOI}} = a_p \left(\frac{m_p}{m_\odot} \right)^{\frac{2}{5}}$$

C_3 and Launch Vehicle Performance

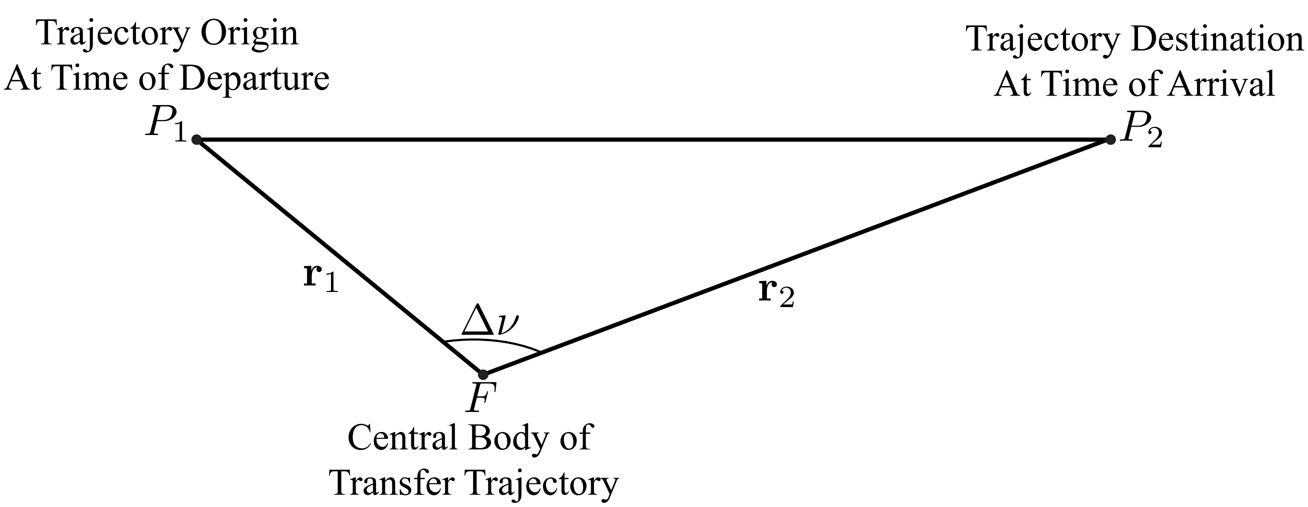
$$C_3 \triangleq 2E = v_\infty^2 = v^2 - v_{\text{esc}}^2$$





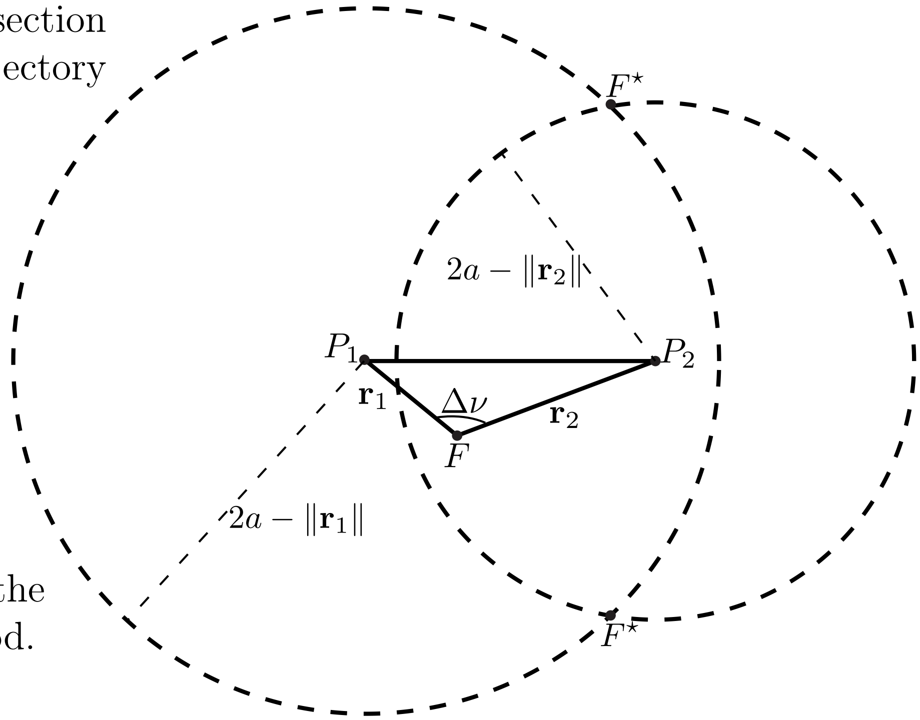
From: https://elvperf.ksc.nasa.gov/Pages/Query.aspx

Lambert's Problem



Lambert's Problem: Location of the Vacant Focus

Vacant focus must be at intersection of two circles centered at trajectory endpoints

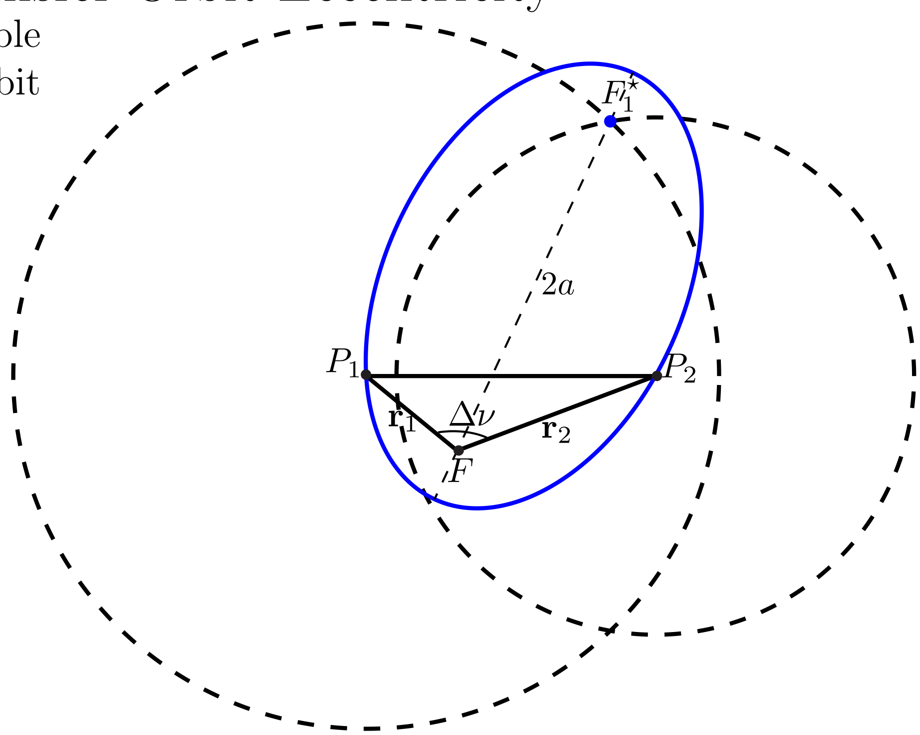


Selecting transfer orbit a sets the transfer orbit energy and period.

Lambert's Problem: Transfer Orbit Eccentricity

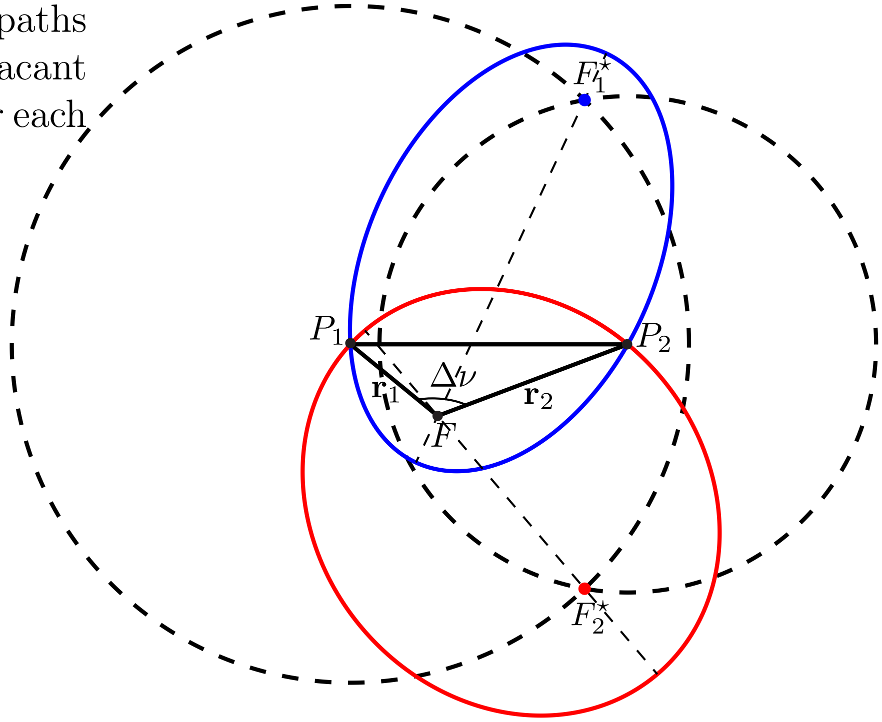
Selecting one of the two possible vacant foci sets the transfer orbit eccentricity:

$$\overline{FF^\star} = 2ae$$



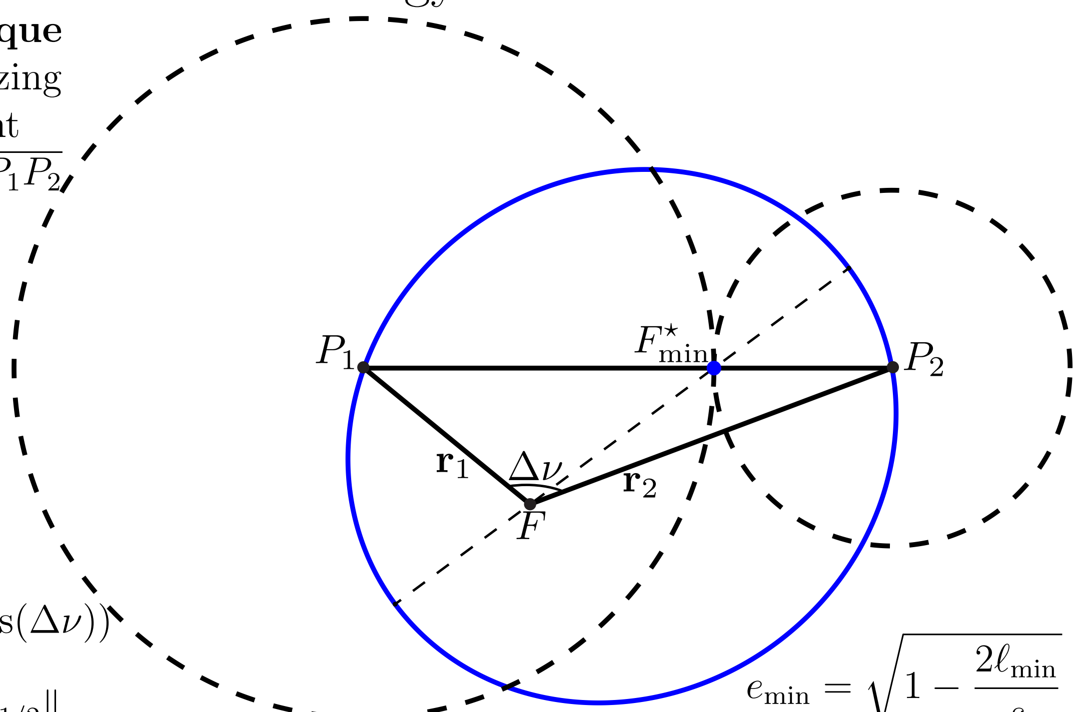
Lambert's Problem: Closed Transfer Orbits

There are four possible transfer paths for each semi-major axis: 2 vacant foci, and 2 directions of travel for each



Lambert's Problem: Minimum Energy Transfer

There is always a **unique** elliptical orbit minimizing energy, where the vacant focus lies on the chord $\overline{P_1P_2}$



$$\ell_{\text{min}} = \frac{\|\mathbf{r}_1\|\|\mathbf{r}_2\|}{\|\mathbf{r}_{1/2}\|}\left(1-\cos(\Delta\nu)\right)$$

$$2s = \|\mathbf{r}_1\| + \|\mathbf{r}_2\| + \|\mathbf{r}_{1/2}\|$$

$$e_{\text{min}} = \sqrt{1-\frac{2\ell_{\text{min}}}{s}}$$

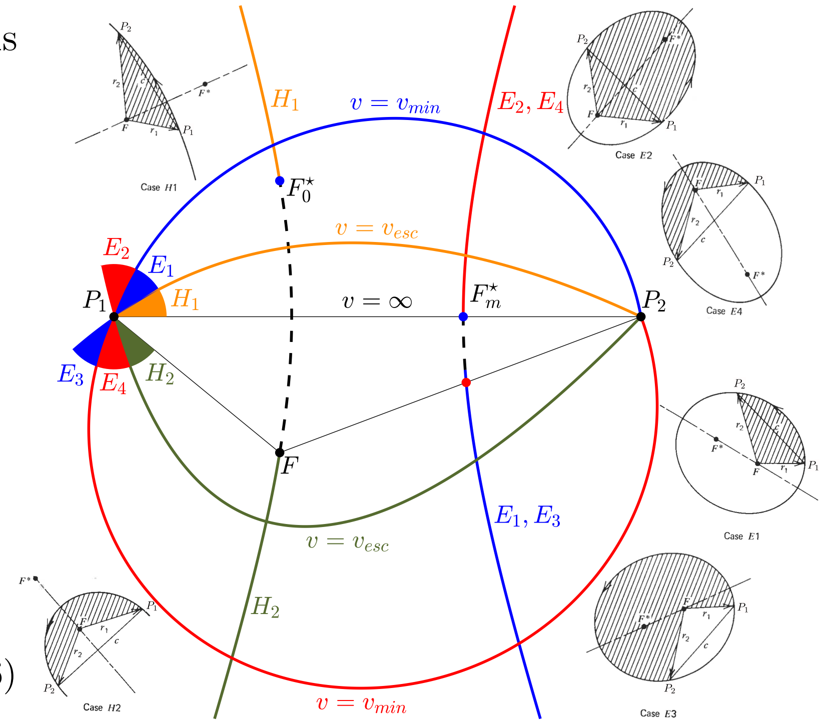
Lambert's Problem: All Possible Transfers

Location of the vacant focus is given by the hyperbola:

$$a_F = \frac{\|\mathbf{r}_1\| - \|\mathbf{r}_2\|}{2}$$

$$e_F = \left| \frac{\|\mathbf{r}_{1/2}\|}{\|\mathbf{r}_1\| - \|\mathbf{r}_2\|} \right|$$

NB: This is **not** a transfer orbit itself.

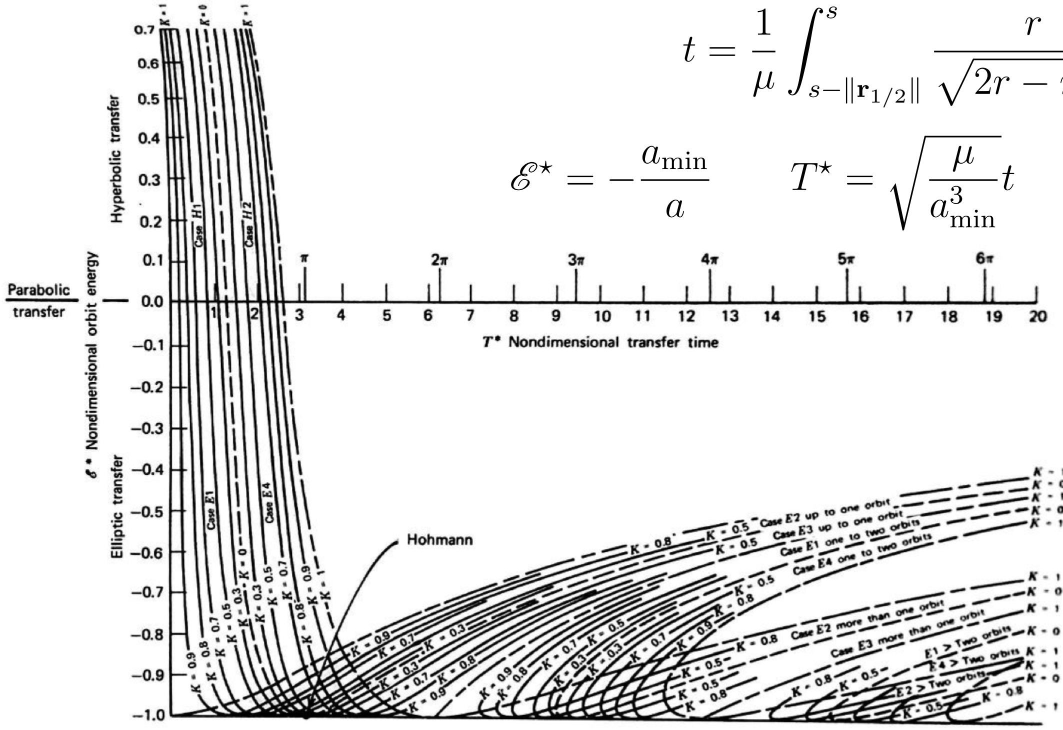


Inset figures from: Kaplan (1976)

Lambert's Problem: All Possible Transfers

$$t = \frac{1}{\mu} \int_{s - \|\mathbf{r}_{1/2}\|}^{s} \frac{r}{\sqrt{2r - r^2/a}} \, \mathrm{d}r$$

$$\mathscr{E}^\star = -\frac{a_{\min}}{a} \qquad T^\star = \sqrt{\frac{\mu}{a_{\min}^3}} t \qquad K = 1 - \frac{\|\mathbf{r}_{1/2}\|}{s}$$



Kaplan (1976) Fig. 7.32

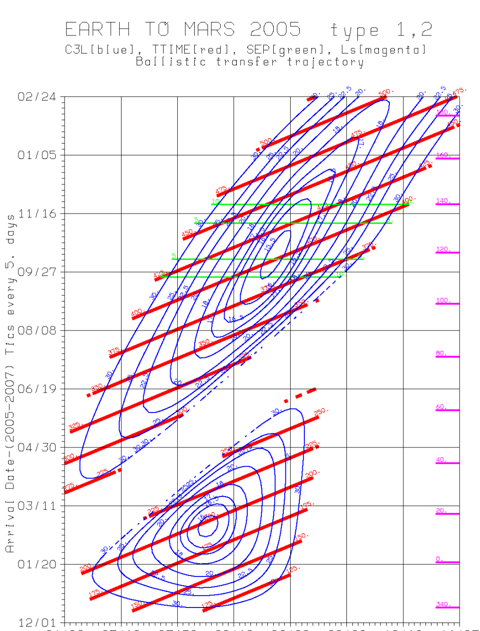
Lambert's Problem: Transfer Orbit Equations

	t	ℓ
$E1/4$	$\frac{T_P}{2\pi}\left[(\alpha - \sin\alpha) \mp (\beta - \sin\beta)\right]$	$\frac{4a(s-r_1)(s-r_2)}{c^2}\sin^2\left(\frac{\alpha \pm \beta}{2}\right)$
$E2/3$	$T_P - \frac{T_P}{2\pi}\left[(\alpha - \sin\alpha) \pm (\beta - \sin\beta)\right]$	$\frac{4a(s-r_1)(s-r_2)}{c^2}\sin^2\left(\frac{\alpha \mp \beta}{2}\right)$
Parabolas	$\frac{1}{3}\sqrt{\frac{2}{\mu}}\left[s^{\frac{3}{2}} \mp (s-c)^{\frac{3}{2}}\right]$	$\frac{4(s-r_1)(s-r_2)}{c^2}\left[\sqrt{\frac{s}{2}} \pm \sqrt{\frac{s-c}{2}}\right]^2$
$H1/2$	$\sqrt{\frac{-a^3}{\mu}}\left[(\sinh\gamma - \gamma) \mp (\sinh\delta - \delta)\right]$	$\frac{-4a(s-r_1)(s-r_2)}{c^2}\sinh^2\left(\frac{\gamma \pm \delta}{2}\right)$

$$c = \|\mathbf{r}_{1/2}\| \qquad r_1 = \|\mathbf{r}_1\| \qquad r_2 = \|\mathbf{r}_2\| \qquad 2s = \|\mathbf{r}_1\| + \|\mathbf{r}_2\| + \|\mathbf{r}_{1/2}\|$$

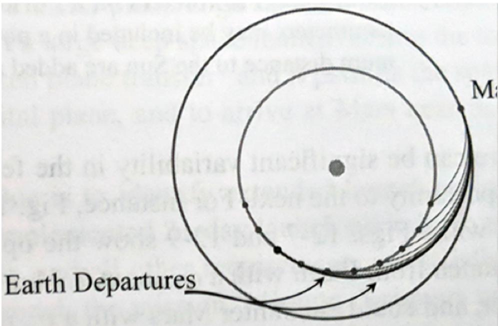
$$\sin\left(\frac{\alpha}{2}\right) = \sqrt{\frac{s}{2a}} \quad \sin\left(\frac{\beta}{2}\right) = \sqrt{\frac{s-c}{2a}} \quad \sinh\left(\frac{\gamma}{2}\right) = \sqrt{\frac{s}{-2a}} \quad \sinh\left(\frac{\delta}{2}\right) = \sqrt{\frac{s-c}{-2a}}$$

Porkchop Plots

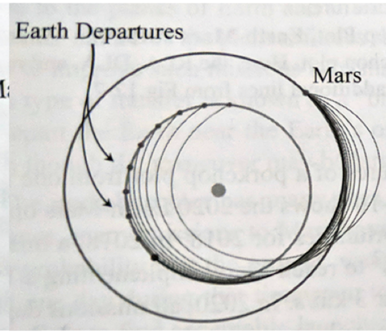


https://mars.jpl.nasa.gov/spotlight/porkchopAll.html

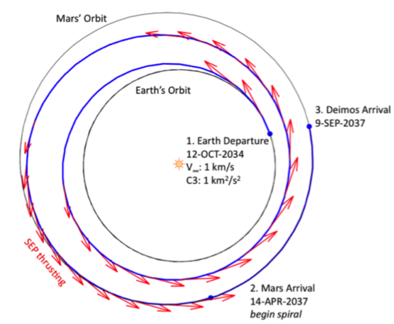
Transfer Types



Type I
$\Delta\nu < 180°$
Valldo (2013) Fig. 12-8



Type II
$180° < \Delta\nu < 360°$
Valldo (2013) Fig. 12-8



Type III
$\Delta\nu > 360°$
Strange et al. (2013)