2 - Math and Dynamics Review and The Two Body Problem

Dmitry Savransky

Cornell University

MAE 4060

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Vector Space Properties

A vector space is a collection of vectors V over a field of scalars $\mathcal{F}$ with the following properties:

1. Commutativity of vector addition: $\forall \mathbf{a}, \mathbf{b} \in V : \mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$
2. Associativity of vector addition: $\forall \mathbf{a}, \mathbf{b}, \mathbf{c} \in V : (\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c})$
3. Identity element of vector addition: $\exists \mathbf{0} \in V$ s.t. $\mathbf{a} + \mathbf{0} = \mathbf{a}\ \forall \mathbf{a} \in V$
4. Inverse elements of vector addition: $\forall \mathbf{a} \in V\ \exists {-\mathbf{a}} \in V$ s.t. $\mathbf{a} + (-\mathbf{a}) = \mathbf{0}$
5. Compatibility of scalar multiplication: $\forall x, y \in \mathcal{F},\ \mathbf{a} \in V : x(y\mathbf{a}) = (xy)\mathbf{a}$
6. Distributivity of scalar multiplication over vector addition: $\forall x \in \mathcal{F},\ \mathbf{a}, \mathbf{b} \in V : x(\mathbf{a} + \mathbf{b}) = x\mathbf{a} + x\mathbf{b}$
7. Distributivity of scalar multiplication over scalar addition: $\forall x, y \in \mathcal{F},\ \mathbf{a} \in V : (x + y)\mathbf{a} = x\mathbf{a} + y\mathbf{a}$
8. Identity element of scalar multiplication: $\exists 1 \in \mathcal{F}$ s.t. $1\mathbf{a} = \mathbf{a}\ \forall \mathbf{a} \in V$

Vector Products

(Scalar) Dot Product

- $\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\|\|\mathbf{b}\| \cos\theta$
- $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$
- $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$
- $x\mathbf{a} \cdot y\mathbf{b} = xy(\mathbf{a} \cdot \mathbf{b})$

(Vector) Cross Product

- $\mathbf{a} \times \mathbf{b} = \|\mathbf{a}\|\|\mathbf{b}\| \sin\theta \hat{\mathbf{c}}$ where $\mathbf{c} \perp \mathbf{a}, \mathbf{b}$
- $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$
- $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$
- $y\mathbf{a} \times \mathbf{b} = y\left(\mathbf{a} \times \mathbf{b}\right) = \mathbf{a} \times y\mathbf{b}$

All Vector Operations Can Be Written as Matrix Multiplications

$$\mathcal{I} = (O, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) \qquad \mathbf{a} = \textstyle\sum_i a_i \mathbf{e}_i \Rightarrow a_i = \mathbf{a} \cdot \mathbf{e}_i \qquad \mathbf{b} = \textstyle\sum_i b_i \mathbf{e}_i \Rightarrow b_i = \mathbf{b} \cdot \mathbf{e}_i$$

$$[\mathbf{a}]_{\mathcal{I}} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}_{\mathcal{I}} \qquad [\mathbf{b}]_{\mathcal{I}} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}_{\mathcal{I}}$$

$$\begin{aligned} [\mathbf{a} \cdot \mathbf{b}]_{\mathcal{I}} &= [\mathbf{a}]_{\mathcal{I}}^T [\mathbf{b}]_{\mathcal{I}} \\ [\mathbf{a} \times \mathbf{b}]_{\mathcal{I}} &= [\mathbf{a}\times]_{\mathcal{I}} [\mathbf{b}]_{\mathcal{I}} \\ [\mathbf{b} \times \mathbf{a}]_{\mathcal{I}} &= [\mathbf{b}\times]_{\mathcal{I}} [\mathbf{a}]_{\mathcal{I}} = -[\mathbf{a}\times]_{\mathcal{I}} [\mathbf{b}]_{\mathcal{I}} \end{aligned}$$

$$[\mathbf{a}\times]_{\mathcal{I}} = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}_{\mathcal{I}}$$

Vector Derivatives

- A vector $\mathbf{r}_{P/O} = a_1\mathbf{a}_1 + a_2\mathbf{a}_2 + a_3\mathbf{a}_3$ is differentiable in time at a time t_1 with respect to frame $\mathcal{A} = (O, \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3)$ if $a_1(t), a_2(t), a_3(t)$ are differentiable at $t = t_1$. Then:

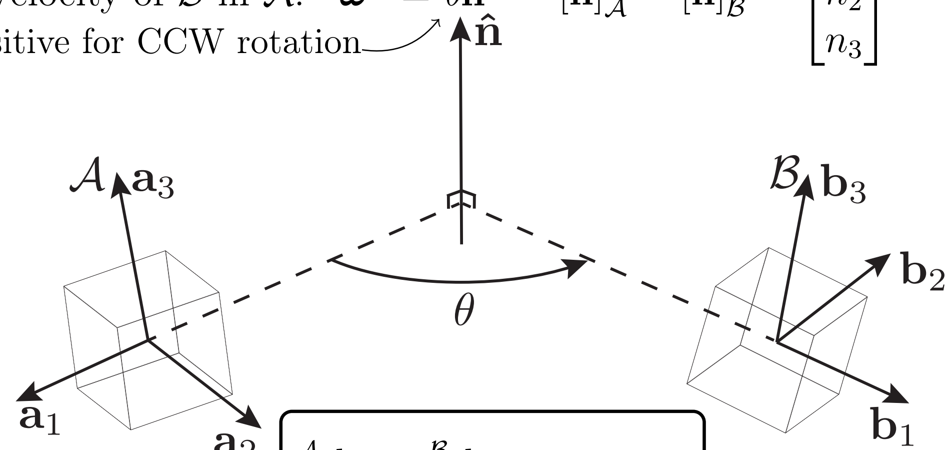
$$\left.\frac{{}^{\mathcal{A}}\mathrm{d}}{\mathrm{d}t}\mathbf{r}_{P/O}\right|_{t=t_1} = \left.\frac{\mathrm{d}a_1}{\mathrm{d}t}\right|_{t=t_1}\mathbf{a}_1 + \left.\frac{\mathrm{d}a_2}{\mathrm{d}t}\right|_{t=t_1}\mathbf{a}_2 + \left.\frac{\mathrm{d}a_3}{\mathrm{d}t}\right|_{t=t_1}\mathbf{a}_3$$

- The unit vectors defining a frame **always** have zero time derivatives with respect to that frame (but not necessarily to other frames)

Vector Differentiation Across Reference Frames

Angular Velocity of $\mathcal{B}$ in $\mathcal{A}$: ${}^{\mathcal{A}}\boldsymbol{\omega}^{\mathcal{B}} \triangleq \dot{\theta}\hat{\mathbf{n}}$ $\quad [\hat{\mathbf{n}}]_{\mathcal{A}} = [\hat{\mathbf{n}}]_{\mathcal{B}} = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}$

Positive for CCW rotation



$$\boxed{\frac{{}^{\mathcal{A}}\mathrm{d}}{\mathrm{d}t}\mathbf{c} = \frac{{}^{\mathcal{B}}\mathrm{d}}{\mathrm{d}t}\mathbf{c} + {}^{\mathcal{A}}\boldsymbol{\omega}^{\mathcal{B}} \times \mathbf{c}}$$

$${}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{F}_\mathcal{N}} = {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{F}_1} + {}^{\mathcal{F}_1}\boldsymbol{\omega}^{\mathcal{F}_2} + {}^{\mathcal{F}_2}\boldsymbol{\omega}^{\mathcal{F}_3} + \ldots + {}^{\mathcal{F}_{\mathcal{N}-1}}\boldsymbol{\omega}^{\mathcal{F}_\mathcal{N}}$$

Newton's Laws of Motion

1. *Corpus omne perseverare in statu suo quiescendi vel movendi uniformiter in directum, nisi quatenus a viribus impressis cogitur statum illum mutare*
   Every body preserves in its state of rest, or of uniform motion in a right line, unless it is compelled to change that state by forces impressed thereon
2. *Mutationem motus proportionalem esse vi motrici impressae; et fieri secundum lineam rectam qua vis illa imprimitur*
   The alteration of motion is ever proportional to the motive force impressed; and is made in the direction of the right line in which that force is impressed
3. *Actioni contrariam semper et aequalem esse reactionem: sive corporum duorum actiones in se mutuo semper esse aequales et in partes contrarias dirigi*
   To every action there is always opposed an equal reaction; or the mutual actions of two bodies upon each other are always equal, and directed to contrary parts

Newton's Second Law

Inertial Frame Derivative

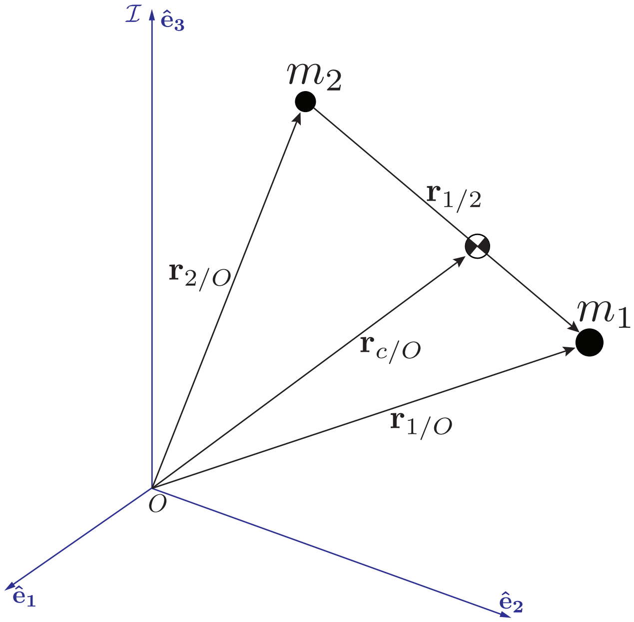
$$\mathbf{F}_P = \frac{{}^{\mathcal{I}}\mathrm{d}}{\mathrm{d}t}\left({}^{\mathcal{I}}\mathbf{p}_{P/O}\right) = \frac{{}^{\mathcal{I}}\mathrm{d}}{\mathrm{d}t}\left(m_P {}^{\mathcal{I}}\mathbf{v}_{P/O}\right) = m_P \, {}^{\mathcal{I}}\mathbf{a}_{P/O}$$

Inertially Fixed Point

Mass Assumed Constant

$$\mathbf{M}_{P/O} = \frac{{}^{\mathcal{I}}\mathrm{d}}{\mathrm{d}t}\left({}^{\mathcal{I}}\mathbf{h}_{P/O}\right) = \frac{{}^{\mathcal{I}}\mathrm{d}}{\mathrm{d}t}\left(\mathbf{r}_{P/O} \times {}^{\mathcal{I}}\mathbf{p}_{P/O}\right) = \mathbf{r}_{P/O} \times \mathbf{F}_P$$

Newton's Law of Gravity and the Two Body Problem



Gravitational Constant

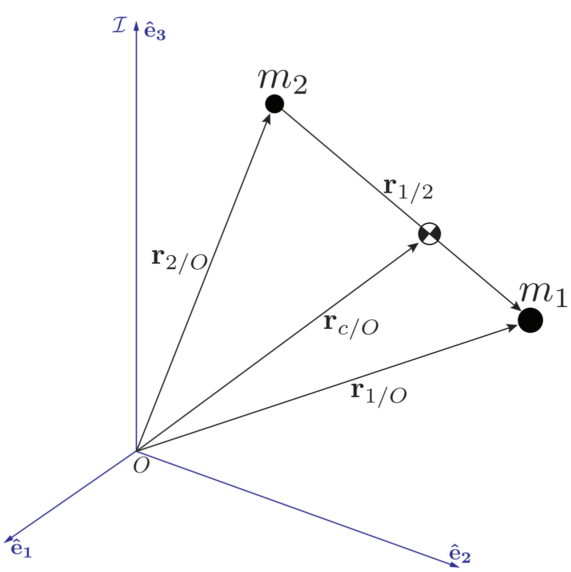
$$\mathbf{F}_1 = -\mathbf{F}_2 = -\frac{G\, m_1 m_2}{\|\mathbf{r}_{1/2}\|^3}\mathbf{r}_{1/2}$$

Orbital Radius: $\mathbf{r} \equiv \mathbf{r}_{1/2}$

Gravitational Parameter: $\mu \triangleq G(m_1 + m_2)$

$$\boxed{\frac{{}^{\mathcal{I}}\mathrm{d}^2}{\mathrm{d}t^2}\mathbf{r} + \frac{\mu}{\|\mathbf{r}\|^3}\mathbf{r} = 0}$$

The Two Body Problem



Specific Angular Momentum: $\mathbf{h} \triangleq \mathbf{r} \times \frac{{}^{\mathcal{I}}\mathrm{d}}{\mathrm{d}t}\mathbf{r}$

$$\frac{{}^{\mathcal{I}}\mathrm{d}^2}{\mathrm{d}t^2}\mathbf{r} + \frac{\mu}{\|\mathbf{r}\|^3}\mathbf{r} = 0 \Rightarrow \frac{{}^{\mathcal{I}}\mathrm{d}^2}{\mathrm{d}t^2}\mathbf{r} \times \mathbf{h} = \frac{{}^{\mathcal{I}}\mathrm{d}}{\mathrm{d}t}\left(\frac{\mu}{\|\mathbf{r}\|}\mathbf{r}\right)$$

$$\Rightarrow \quad \frac{{}^{\mathcal{I}}\mathrm{d}}{\mathrm{d}t}\mathbf{r} \times \mathbf{h} = \mu\left(\frac{\mathbf{r}}{\|\mathbf{r}\|} + \mathbf{e}\right)$$

Constant of Integration

$$\boxed{r \triangleq \|\mathbf{r}\| = \frac{h^2/\mu}{1 + e\cos(\nu)}}$$

$$h \triangleq \|\mathbf{h}\| \qquad e \triangleq \|\mathbf{e}\| \qquad \mathbf{r}\cdot\mathbf{e} = re\cos\nu$$