4 - Kepler's Time Equation and Orbit Propagation

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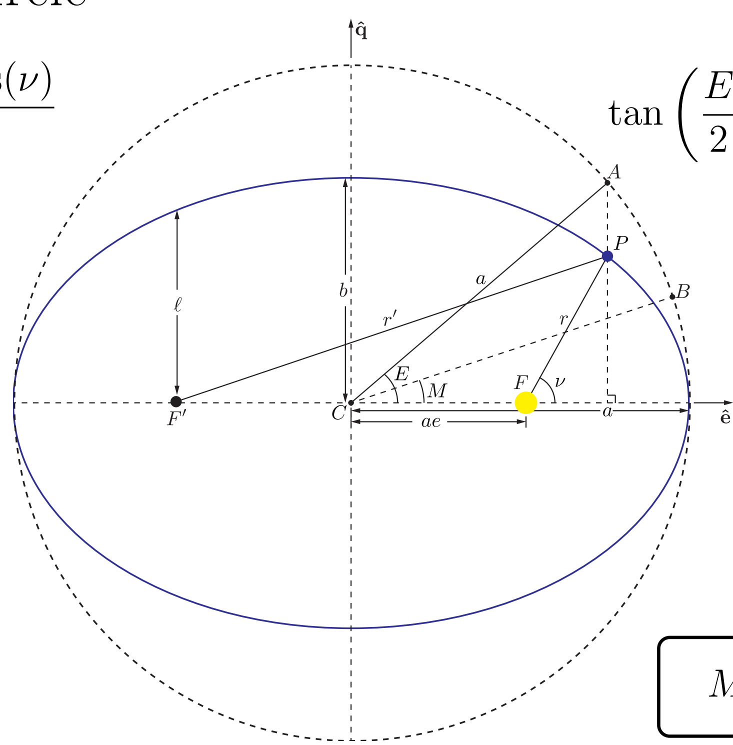
The Auxiliary Circle

$$\cos(E) = \frac{ae + r\cos(\nu)}{a}$$

$$\tan\left(\frac{E}{2}\right) = \sqrt{\frac{1-e}{1+e}}\tan\left(\frac{\nu}{2}\right)$$



$$\boxed{M = E - e\sin(E)}$$

Newton-Raphson Iteration

- Given: $x : f(x) = 0, x \in \mathbb{R}; \quad f'(x) = \frac{\mathrm{d}f}{\mathrm{d}x}$
- Iterate: $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$
- Until converged (answer stops changing to your desired precision)

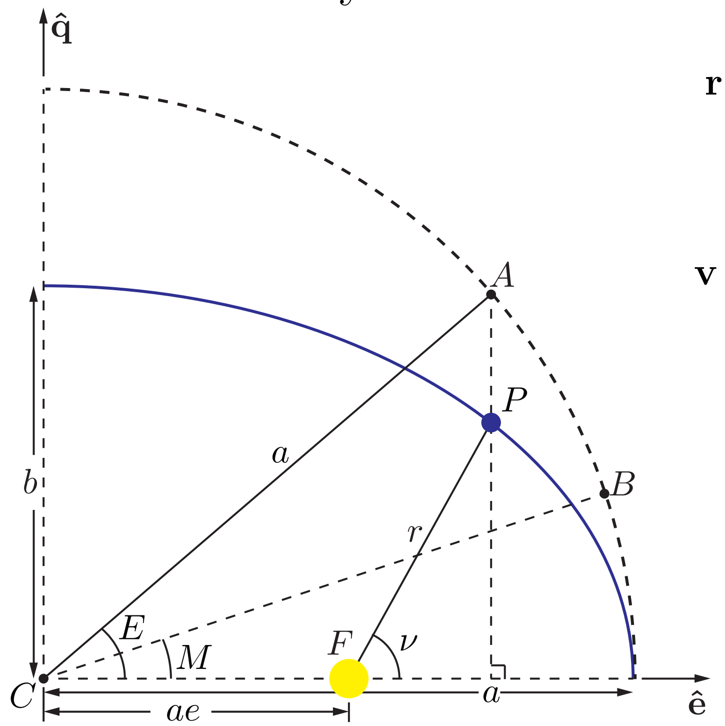
Newton-Raphson Iteration for Kepler's Time Equation

$$M - (E - e\sin(E)) = 0$$

$$E_{n+1} = E_n - \frac{M - E_n + e\sin(E_n)}{e\cos(E_n) - 1}$$

$$E_0 = \begin{cases} \frac{M}{1-e} & \frac{M}{1-e} < \sqrt{\frac{6(1-e)}{e}} \\ \left(\frac{6M}{e}\right)^{\frac{1}{3}} & \text{else} \end{cases}$$

Eccentric Anomaly Revisited



$$
\begin{aligned}
\mathbf{r} &= r\cos(\nu)\hat{\mathbf{e}} + r\sin(\nu)\hat{\mathbf{q}} \\
&= a\left(\cos(E) - e\right)\hat{\mathbf{e}} + b\sin(E)\hat{\mathbf{q}}
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{v} &= -a\dot{E}\sin(E)\hat{\mathbf{e}} + b\dot{E}\cos(E)\hat{\mathbf{q}} \\
&= \frac{an}{r}\left(-a\sin(E)\hat{\mathbf{e}} + b\cos(E)\hat{\mathbf{q}}\right)
\end{aligned}
$$

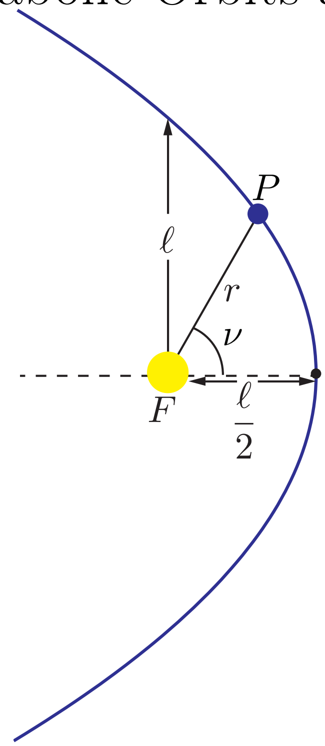
$$\boxed{\dot{E} = \frac{n}{1 - e\cos(E)}}$$

f and g Functions

$$r_0 \triangleq r(t) \qquad r_1 \triangleq r(t + \Delta t)$$

$$
\begin{aligned}
f &= 1 - \frac{r_1}{\ell}\left(1 - \cos(\Delta\nu)\right) &&= \frac{a}{r_1}\left(\cos(\Delta E) - 1\right) + 1 \\
g &= \frac{r_1 r_0}{\sqrt{\mu\ell}}\sin(\Delta\nu) &&= \frac{1}{n}\left(\sin(\Delta E) - \Delta E\right) + \Delta t \\
\dot{f} &= \sqrt{\frac{\mu}{\ell}}\tan\left(\frac{\Delta\nu}{2}\right)\left(\frac{1 - \cos(\Delta\nu)}{\ell} - \frac{1}{r_0} - \frac{1}{r_1}\right) &&= -\frac{a^2 n}{r_1 r_0}\sin(\Delta E) \\
\dot{g} &= 1 - \frac{r_0}{\ell}\left(1 - \cos(\Delta\nu)\right) &&= \frac{a}{r_1}\left(\cos(\Delta E) - 1\right) + 1
\end{aligned}
$$

Parabolic Orbits and Barker's Equation



$$B \triangleq \tan\left(\frac{\nu}{2}\right)$$

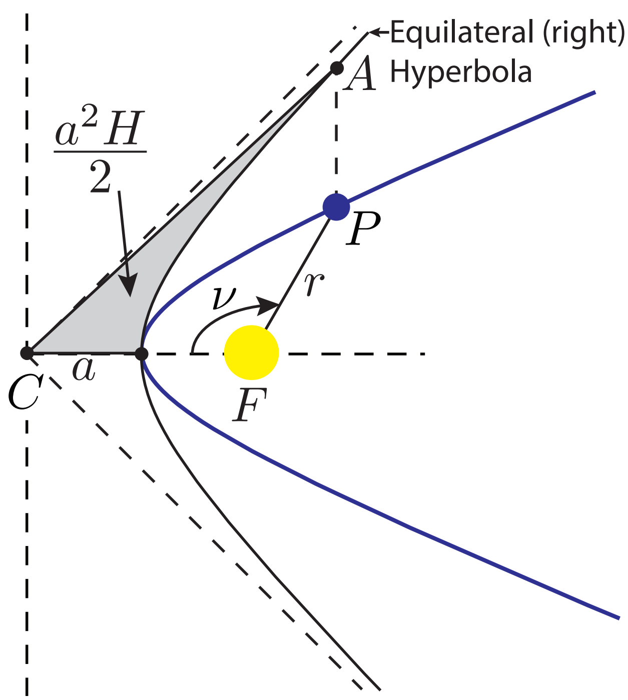
$$r = \frac{\ell}{2}\left(1 + B^2\right)$$

$$\nu = \sin^{-1}\left(\frac{\ell B}{r}\right)$$

$$n_p(t - t_p) = B + \frac{B^3}{3}$$

$$n_p \triangleq 2\sqrt{\frac{\mu}{\ell^3}}$$

Hyperbolic Orbits



$$\sinh(H) = -\frac{r\sin(\nu)}{a\sqrt{1 - e^2}}$$

$$\cosh(H) = \frac{ae + r\cos(\nu)}{a}$$

$$r = a\left(1 - e\cosh(H)\right)$$

$$\tan\left(\frac{\nu}{2}\right) = \sqrt{\frac{e+1}{e-1}}\tanh\left(\frac{H}{2}\right)$$

$$n_h\left(t - t_p\right) = e\sinh(H) - H$$

$$n_h \triangleq \sqrt{-\frac{\mu}{a^3}}$$