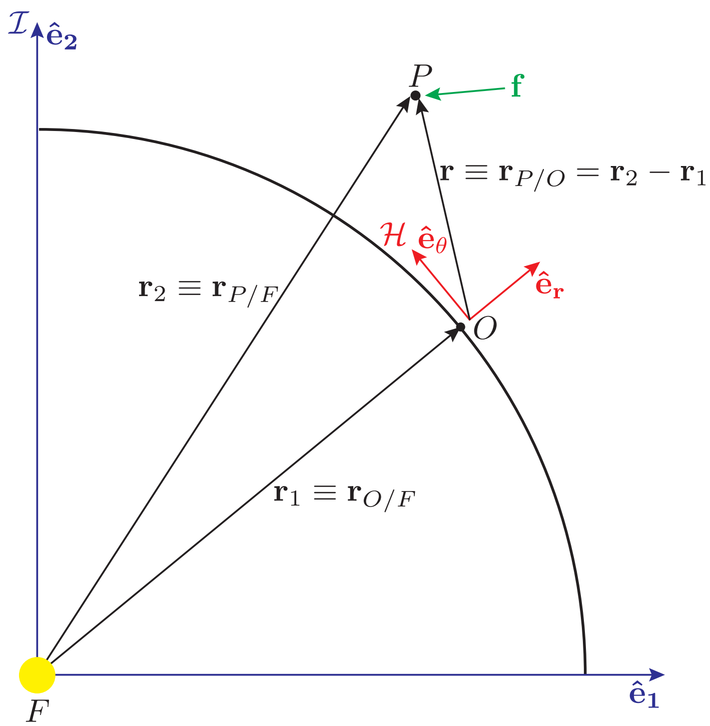
13 - Relative Motion and The Circular Restricted Three Body Problem

Dmitry Savransky

Cornell University

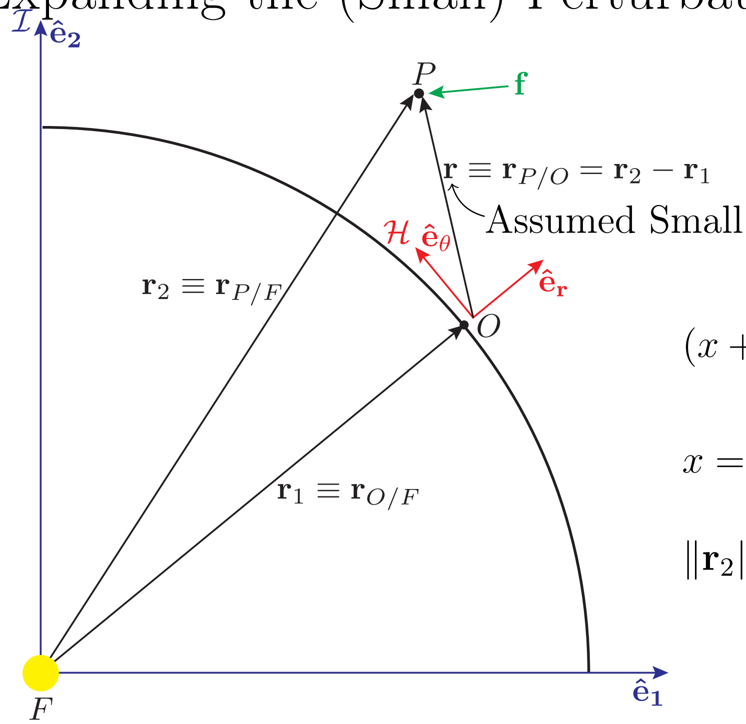
MAE 4060

The Euler-Hill Frame



$$
{}^{\mathcal{I}}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathbf{r} = \frac{\mu}{\|\mathbf{r}_1\|^3}\left(\mathbf{r} - \left(\frac{\|\mathbf{r}_1\|}{\|\mathbf{r}_2\|}\right)^3 \mathbf{r}_2\right) + \mathbf{f}
$$

Expanding the (Small) Perturbation



$$\frac{{}^{\mathcal{I}}\mathrm{d}^2}{\mathrm{d}t^2}\mathbf{r} = \underbrace{\frac{\mu}{\|\mathbf{r}_1\|^3}}_{n^2}\left(\mathbf{r} - \left(\frac{\|\mathbf{r}_1\|}{\underbrace{\|\mathbf{r}_2\|}}\right)^3 \mathbf{r}_2\right) + \mathbf{f}$$

$$\|\mathbf{r}_2\|^{-3} = \left[(\mathbf{r}_1 + \mathbf{r}) \cdot (\mathbf{r}_1 + \mathbf{r})\right]^{-\frac{3}{2}}$$

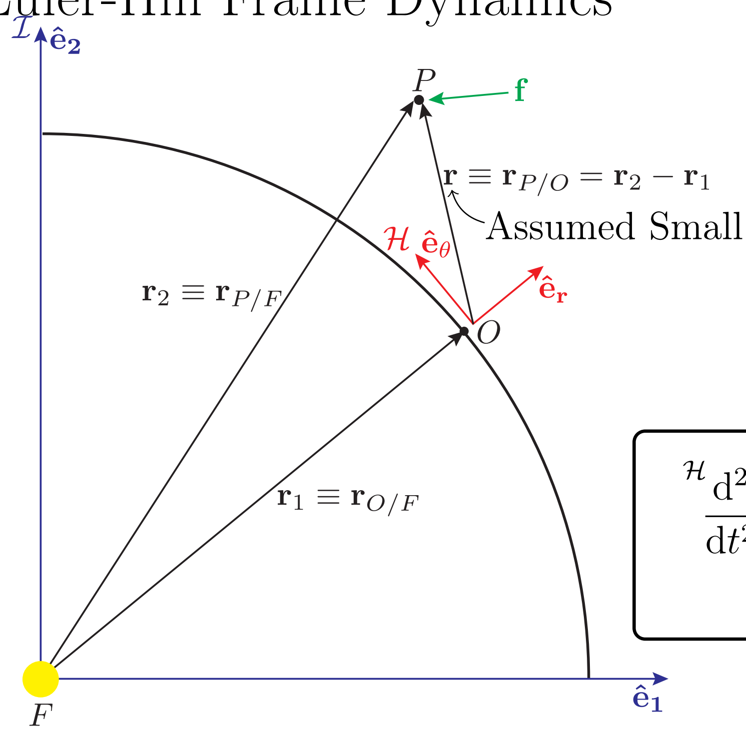
$$(x+y)^r = \sum_{k=0}^{\infty} \binom{r}{k} x^{r-k} y^k = x^r + r x^{r-1} y + \ldots$$

$$x = \|\mathbf{r}_1\|^2 \qquad y = 2\mathbf{r} \cdot \mathbf{r}_1 + \|\mathbf{r}\|^2 \qquad r = -\frac{3}{2}$$

$$\|\mathbf{r}_2\|^{-3} = \|\mathbf{r}_1\|^{-3}\left(1 - \frac{3}{2}\left(\frac{2\mathbf{r} \cdot \mathbf{r}_1}{\|\mathbf{r}_1\|^2}\right) + \mathcal{O}\left(\mathbf{r}^2\right)\right)$$

$$\frac{{}^{\mathcal{I}}\mathrm{d}^2}{\mathrm{d}t^2}\mathbf{r} \approx n^2\left(-\mathbf{r} + 3\frac{\mathbf{r}_1 \cdot \mathbf{r}}{\|\mathbf{r}_1\|^2}\mathbf{r}_1\right) + \mathbf{f}$$

Euler-Hill Frame Dynamics



$$\frac{{}^{\mathcal{I}}\mathrm{d}^2}{\mathrm{d}t^2}\mathbf{r} = \underbrace{\frac{\mu}{\|\mathbf{r}_1\|^3}}_{n^2}\left(\mathbf{r} - \left(\frac{\|\mathbf{r}_1\|}{\|\mathbf{r}_2\|}\right)^3 \mathbf{r}_2\right) + \mathbf{f}$$

$$\approx n^2\left(-\mathbf{r} + 3\frac{\mathbf{r}_1 \cdot \mathbf{r}}{\|\mathbf{r}_1\|^2}\mathbf{r}_1\right) + \mathbf{f}$$

$$\boxed{\begin{aligned}\frac{{}^{\mathcal{H}}\mathrm{d}^2}{\mathrm{d}t^2}\mathbf{r} = &- 2n\hat{\mathbf{e}}_3 \times \frac{{}^{\mathcal{H}}\mathrm{d}}{\mathrm{d}t}\mathbf{r} - n^2\left(\hat{\mathbf{e}}_3 \times \left(\hat{\mathbf{e}}_3 \times \mathbf{r}\right)\right) \\ &- n^2\left(\mathbf{r} - 3\left(\hat{\mathbf{e}}_\mathbf{r} \cdot \mathbf{r}\right)\hat{\mathbf{e}}_\mathbf{r}\right) + \mathbf{f}\end{aligned}}$$

Euler-Hill/Clohessy-Wiltshire Equations

$$\begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{bmatrix}_{\mathcal{H}} = \underbrace{\begin{bmatrix} 2n\dot{y} \\ -2n\dot{x} \\ 0 \end{bmatrix}_{\mathcal{H}} + \begin{bmatrix} n^2 x \\ n^2 y \\ 0 \end{bmatrix}_{\mathcal{H}}}_{\text{Rotating Frame}} - \underbrace{\begin{bmatrix} n^2 x \\ n^2 y \\ n^2 z \end{bmatrix}_{\mathcal{H}} + \begin{bmatrix} 3n^2 x \\ 0 \\ 0 \end{bmatrix}_{\mathcal{H}}}_{\text{Gravity Perturbations}} + \underbrace{[\mathbf{f}]_{\mathcal{H}}}_{\text{Other Perturbations}}$$

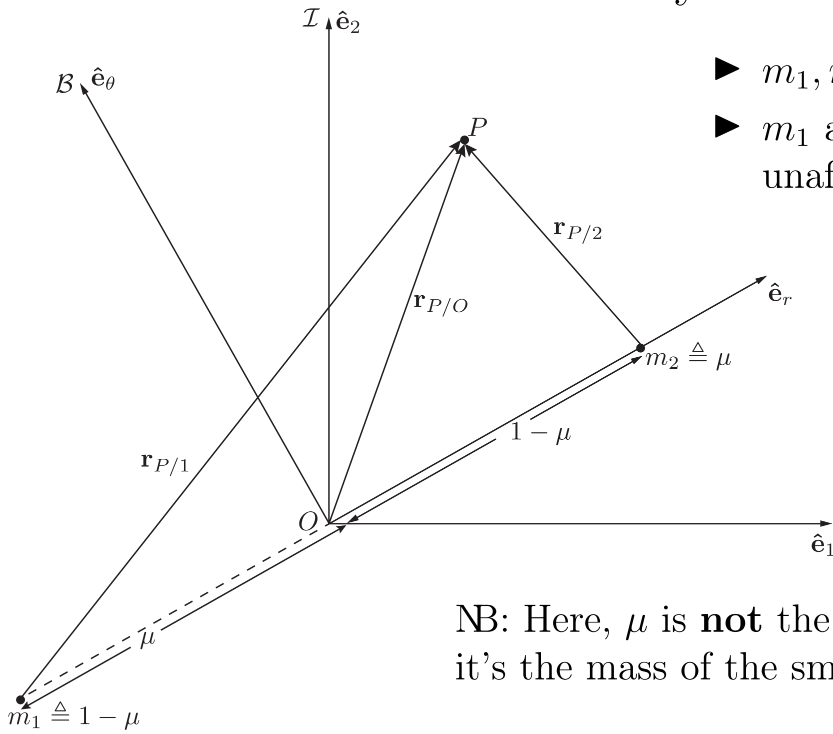
$$\begin{aligned} \ddot{x} - 2n\dot{y} - 3n^2 x &= \mathbf{f} \cdot \hat{\mathbf{e}}_{\mathbf{r}} \triangleq f_x \\ \ddot{y} + 2n\dot{x} &= \mathbf{f} \cdot \hat{\mathbf{e}}_{\theta} \triangleq f_y \\ \ddot{z} + n^2 z &= \mathbf{f} \cdot \hat{\mathbf{e}}_{\mathbf{3}} \triangleq f_z \end{aligned}$$

Natural Motion

$$\begin{aligned} \ddot{x} - 2n\dot{y} - 3n^2 x &= 0 \\ \ddot{y} + 2n\dot{x} &= 0 \\ \ddot{z} + n^2 z &= 0 \end{aligned}$$

$$\begin{aligned} x(t) &= 4x_0 - 3x_0 \cos(nt) + \frac{\dot{x}_0}{n} \sin(nt) + 2\frac{\dot{y}_0}{n} - 2\frac{\dot{y}_0 \cos(nt)}{n} \\ y(t) &= -6x_0 nt + 6x_0 \sin(nt) + 2\cos(nt)\frac{\dot{x}_0}{n} - 2\frac{\dot{x}_0}{n} + \frac{\dot{y}_0}{n}\left(4\sin(nt) - 3nt\right) + y_0 \\ z(t) &= z_0 \cos(nt) + \frac{\dot{z}_0}{n} \sin(nt) \end{aligned}$$

The Circular Restricted 3-Body Problem (CR3BP)



- $m_1, m_2 \gg m_p$
- m_1 and m_2 in mutual circular orbits, unaffected by m_p

$$\mathbf{r}_{1/O} = -\mu\hat{\mathbf{e_r}}$$
$$\mathbf{r}_{2/O} = (1-\mu)\hat{\mathbf{e_r}}$$

NB: Here, μ is **not** the gravitational parameter—it's the mass of the smaller major body

CR3BP Dynamics

$$\mathbf{F}_P = -\frac{Gm_1m_P}{\|\mathbf{r}_{P/1}\|^3}\mathbf{r}_{P/1} - \frac{Gm_2m_P}{\|\mathbf{r}_{P/2}\|^3}\mathbf{r}_{P/2}$$

$$^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}} = n\hat{\mathbf{e}}_3 = \hat{\mathbf{e}}_3$$

$$^{\mathcal{B}}\mathbf{a}_{P/O} + 2\hat{\mathbf{e}}_3 \times {}^{\mathcal{B}}\mathbf{v}_{P/O} + \hat{\mathbf{e}}_3 \times \left(\hat{\mathbf{e}}_3 \times \mathbf{r}_{P/O}\right) = -G\left(\frac{m_1}{\|\mathbf{r}_{P/1}\|^3}\mathbf{r}_{P/1} + \frac{m_2}{\|\mathbf{r}_{P/2}\|^3}\mathbf{r}_{P/2}\right)$$

$$\mathbf{F}_P = -\nabla V$$
$$V = -\left(\frac{1-\mu}{r_1} + \frac{\mu}{r_2}\right)$$
$$r_1 \triangleq \|\mathbf{r}_{P/1}\| = \sqrt{(x+\mu)^2 + y^2 + z^2}$$
$$r_2 \triangleq \|\mathbf{r}_{P/2}\| = \sqrt{(x-(1-\mu))^2 + y^2 + z^2}$$

$$\boxed{\begin{aligned} \ddot{x} - 2\dot{y} - x &= -\frac{\partial V}{\partial x} \\ \ddot{y} + 2\dot{x} - y &= -\frac{\partial V}{\partial y} \\ \ddot{z} &= -\frac{\partial V}{\partial z} \end{aligned}}$$

A New Potential

$$\text{Define}: U \triangleq -\frac{1}{2}(x^2+y^2) - \left(\frac{1-\mu}{r_1} + \frac{\mu}{r_2}\right)$$

$$\left.\begin{aligned} \ddot{x} - 2\dot{y} - x &= -\tfrac{\partial V}{\partial x} \\ \ddot{y} + 2\dot{x} - y &= -\tfrac{\partial V}{\partial y} \\ \ddot{z} &= -\tfrac{\partial V}{\partial z} \end{aligned}\right\} \Rightarrow \qquad \boxed{\begin{aligned} \ddot{x} &= -\frac{\partial U}{\partial x} + 2\dot{y} \\ \ddot{y} &= -\frac{\partial U}{\partial y} - 2\dot{x} \\ \ddot{z} &= -\frac{\partial U}{\partial z} \end{aligned}}$$

$$\text{NB}: \ \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\dot{x}^2+\dot{y}^2+\dot{z}^2\right) = -\frac{\mathrm{d}U}{\mathrm{d}t}$$

The Jacobi Constant

$$\frac{1}{2}\left({}^{\mathcal{B}}\mathbf{v}_{P/O}\cdot{}^{\mathcal{B}}\mathbf{v}_{P/O}\right) + U(x,y,z) = C \triangleq \text{Jacobi Constant}$$

$${}^{\mathcal{B}}\mathbf{v}_{P/O} = {}^{\mathcal{I}}\mathbf{v}_{P/O} - \hat{\mathbf{e}}_3 \times \mathbf{r}_{P/O}$$

$$\underbrace{\frac{1}{2}\left({}^{\mathcal{I}}\mathbf{v}_{P/O}\cdot{}^{\mathcal{I}}\mathbf{v}_{P/O}\right) - \left(\frac{1-\mu}{r_1}+\frac{\mu}{r_2}\right)}_{\text{KE+PE}} - \underbrace{{}^{\mathcal{I}}\mathbf{v}_{P/O}\cdot\left(\hat{\mathbf{e}}_3\times\mathbf{r}_{P/O}\right)}_{\hat{\mathbf{e}}_3\cdot(\mathbf{r}_{P/O}\times{}^{\mathcal{I}}\mathbf{v}_{P/O}) = \hat{\mathbf{e}}_3\cdot{}^{\mathcal{I}}\mathbf{h}_{P/O}} = C$$

$$\boxed{E - h\cos(I) = C}$$

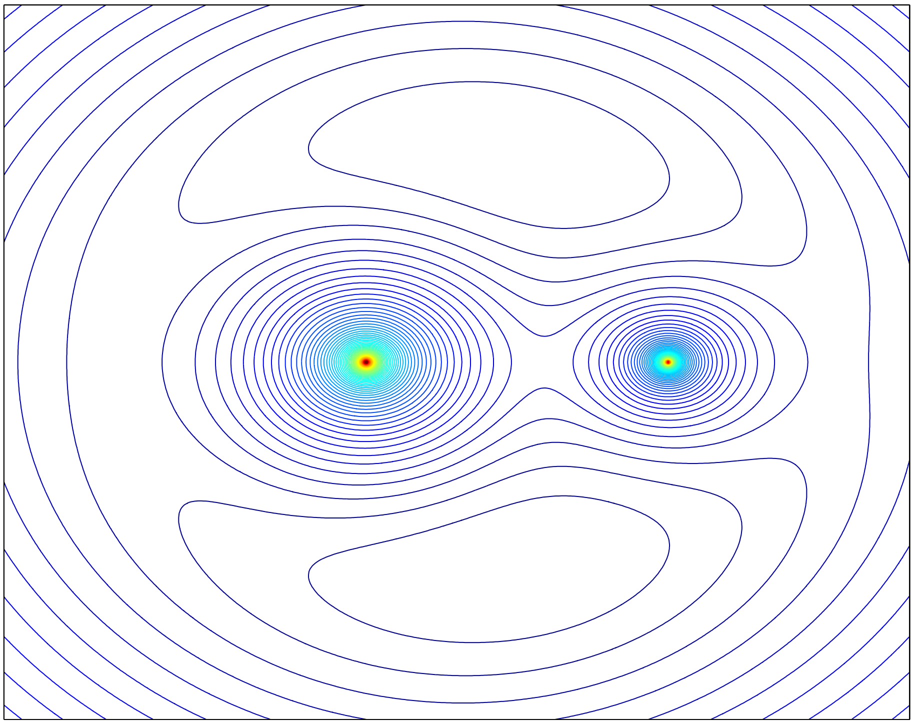
Set by initial conditions

Total Energy of Mass P

Angular Momentum of Mass P

Angle between orbit of mass P and $\hat{\mathbf{e}}_1 - \hat{\mathbf{e}}_2$ plane

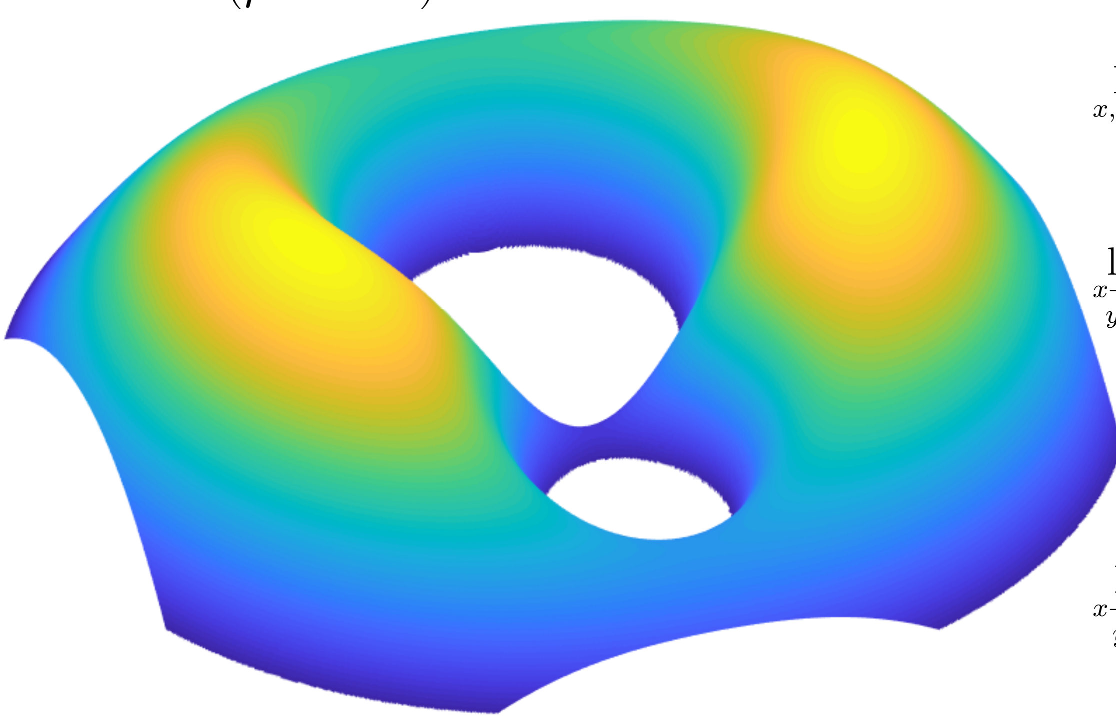
Hill Curves ($\mu = 0.3$)



$$U(x,y) = U(x,-y)$$

$$U(x,y) \neq U(-x,y)$$

Hill Curves ($\mu = 0.3$)



$$\lim_{x,y\to\infty} U = -\frac{x^2+y^2}{2}$$

$$\lim_{\substack{x\to-\mu \\ y\to 0}} U = -\frac{1-\mu}{r_1}$$

$$\lim_{\substack{x\to 1-\mu \\ y\to 0}} U = -\frac{\mu}{r_2}$$