10 - Impulsive Orbital Maneuvers and Δv, Hohmann Transfers, Bi-elliptic Transfers, Inclination Changes, and Intercept & Rendezvous

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Review of Linear Momentum and Impulse

$$\int_{t_1}^{t_2} \mathbf{F}_P \,\mathrm{d}t = \int_{t_1}^{t_2} \frac{^\mathcal{I}\mathrm{d}}{\mathrm{d}t}({}^\mathcal{I}\mathbf{p}_{P/O})\,\mathrm{d}t = {}^\mathcal{I}\mathbf{p}_{P/O}(t_2) - {}^\mathcal{I}\mathbf{p}_{P/O}(t_1)$$

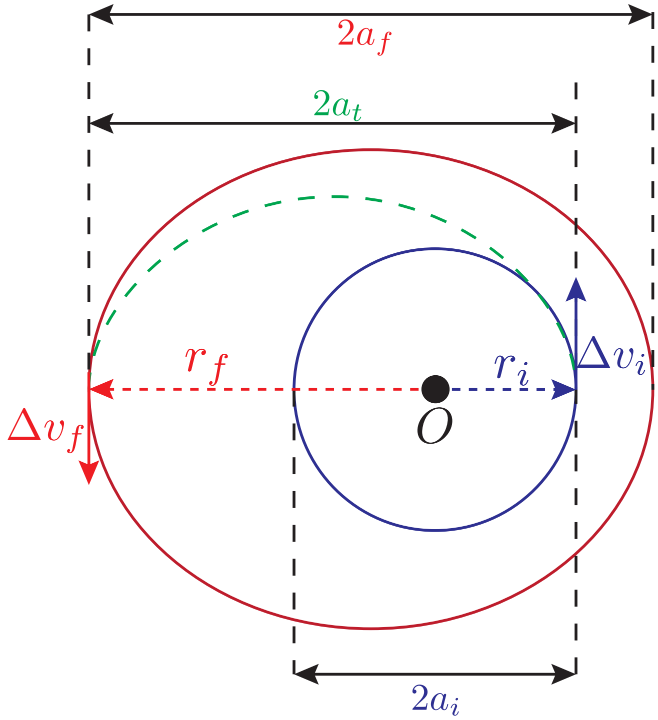
$$m_P {}^\mathcal{I}\mathbf{v}_{P/O}(t_2) = m_P {}^\mathcal{I}\mathbf{v}_{P/O}(t_1) + \int_{t_1}^{t_2} \mathbf{F}_P \,\mathrm{d}t$$

- The change in linear momentum (proportional to the change in velocity) from t_1 to t_2 is equal to the integral of the total force applied
- Define **linear impulse**:

$$\bar{\mathbf{F}}_P(t_1, t_2) \triangleq \int_{t_1}^{t_2} \mathbf{F}_P \,\mathrm{d}t$$

where $t_2 - t_1$ is typically very small

Hohmann Transfers



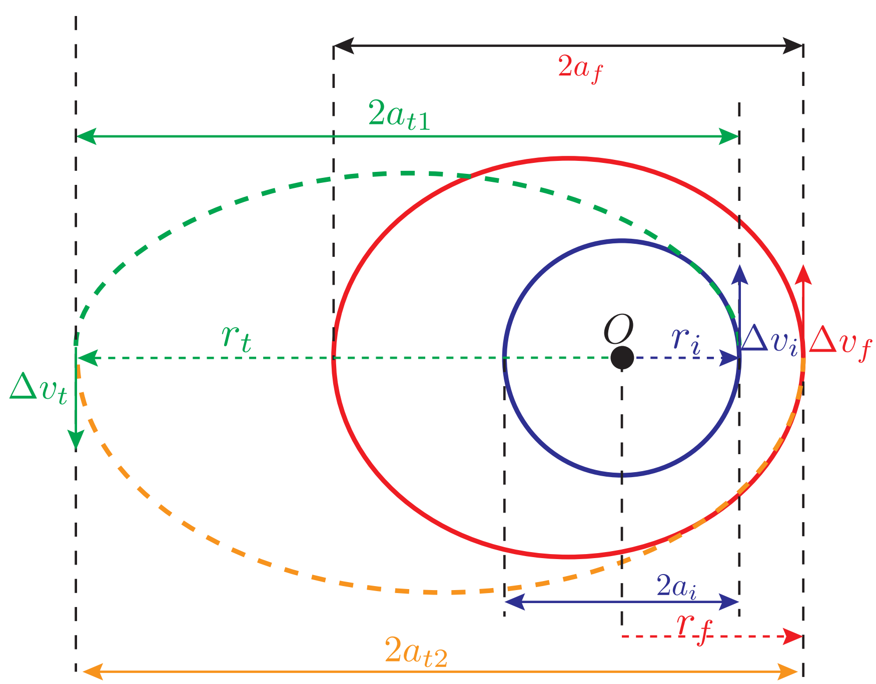
$$a_t = \frac{r_i + r_f}{2}$$

$$t_{\text{transfer}} = \frac{1}{2} T_P^{\text{transfer}} = \pi \sqrt{\frac{a_t^3}{\mu}}$$

$$v = \sqrt{\frac{2\mu}{r} - \frac{\mu}{a}}$$

$$\Delta v = |\Delta v_i| + |\Delta v_f|$$

Bi-Elliptic Transfers



Hohmann vs. Bi-Elliptic

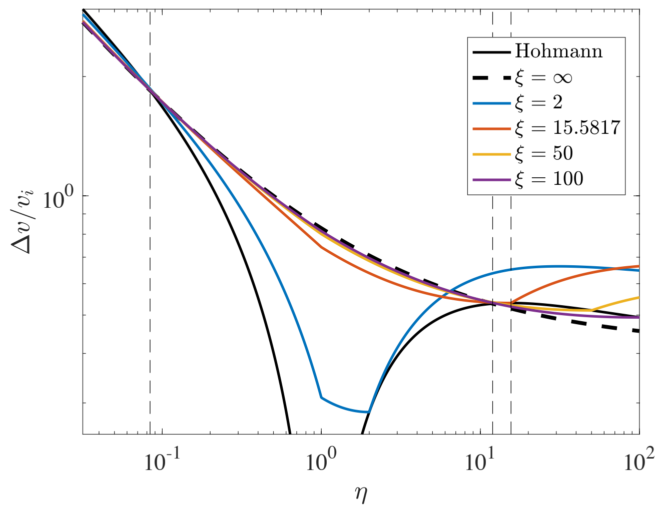
$$\text{Hohmann}: \quad \frac{\|\Delta v_i\| + \|\Delta v_f\|}{v_i} = \left| \sqrt{\frac{2\eta}{1+\eta}} + \sqrt{\frac{1}{\eta}} - \left(1 + \sqrt{\frac{2}{\eta(1+\eta)}}\right) \right|$$

$$\text{Bi}-\text{Elliptic}: \frac{\|\Delta v_i\| + \|\Delta v_t\| + \|\Delta v_f\|}{v_i} = \left| \sqrt{\frac{2\xi}{1+\xi}} - 1 \right| + \left| \sqrt{\frac{2\eta}{\xi(\eta+\xi)}} - \sqrt{\frac{2}{\xi(1+\xi)}} \right| + \left| \sqrt{\frac{1}{\eta}} - \sqrt{\frac{2\xi}{\eta(\eta+\xi)}} \right|$$

$$\Rightarrow \lim_{r_t \to \infty} \left(\frac{\|\Delta v_i\| + \|\Delta v_t\| + \|\Delta v_f\|}{v_i} \right) = \sqrt{2} - 1 + \left| \sqrt{\frac{1}{\eta}} - \sqrt{\frac{2}{\eta}} \right|$$

Hohmann maximum (for $\eta > 1$) occurs at $\eta = 15.5817$

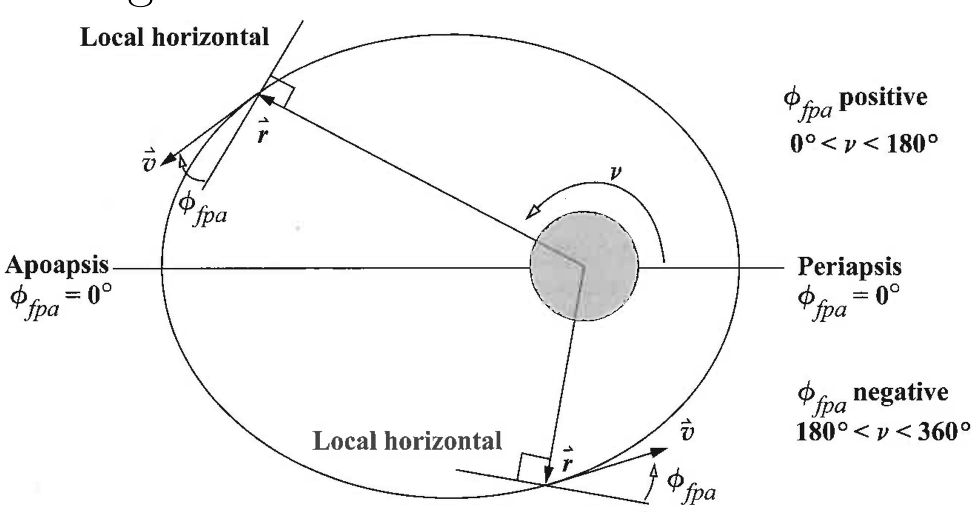
Hohmann and $\eta = \infty$ intersect at $\eta = 11.93876^{\pm 1}$



$$\eta \triangleq \frac{a_f}{a_i}$$

$$\xi \triangleq \frac{r_t}{a_i}$$

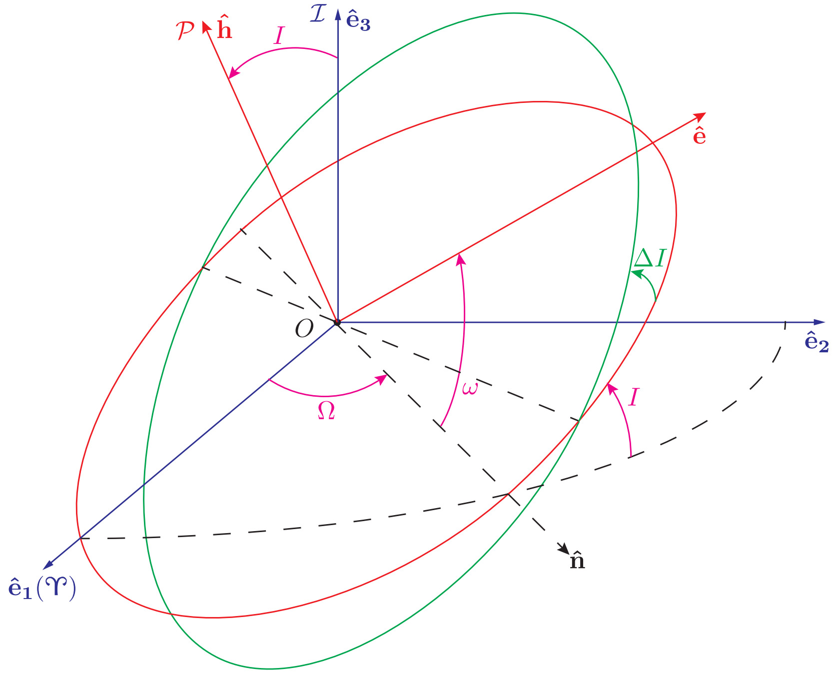
Flight Path Angle



Vallado (2013) Fig. 1-10

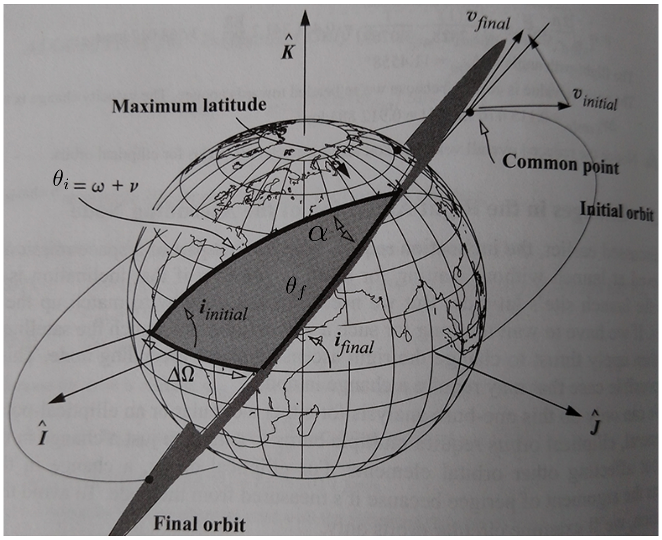
$$\cos\phi_{fpa} = \frac{r\dot{\nu}}{v} = \frac{1 + e\cos\nu}{\sqrt{1 + 2e\cos\nu + e^2}} \qquad \sin\phi_{fpa} = \frac{\dot{r}}{v} = \frac{e\sin\nu}{\sqrt{1 + 2e\cos\nu + e^2}}$$

Inclination Changes (Super Costly!)



$$\Delta v = 2v_i \cos(\phi_{fpa}) \sin\left(\frac{\Delta I}{2}\right)$$

Ascending Node Change



Vallado (2013) Fig. 6-11

$$I_i = I_f \triangleq I$$

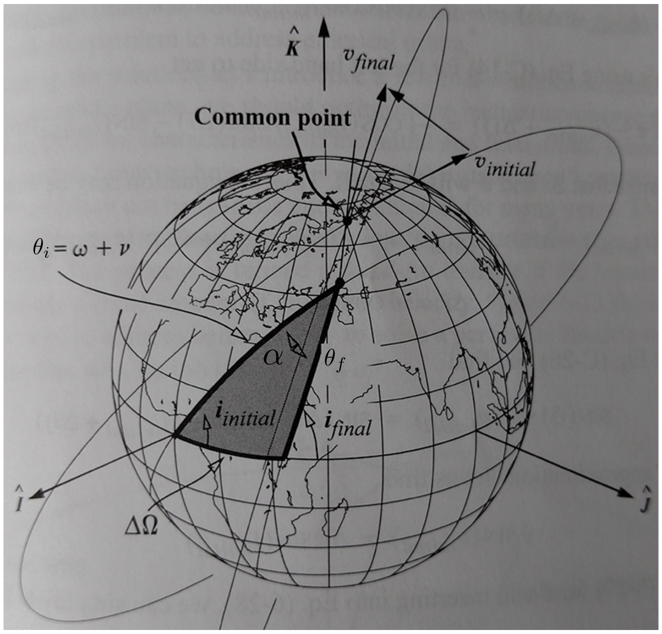
$$\cos(\theta_i) = \tan I \left(\frac{\cos(\Delta\Omega) - \cos\alpha}{\sin\alpha}\right)$$

$$\cos(\theta_f) = \cos I \sin I \left(\frac{1 - \cos(\Delta\Omega)}{\sin\alpha}\right)$$

$$\cos(\alpha) = \cos^2 I + \sin^2 I \cos(\Delta\Omega)$$

$$\Delta v^{\mathrm{circ}} = 2v_i \sin\left(\frac{\alpha}{2}\right)$$

Ascending Node and Inclination Change

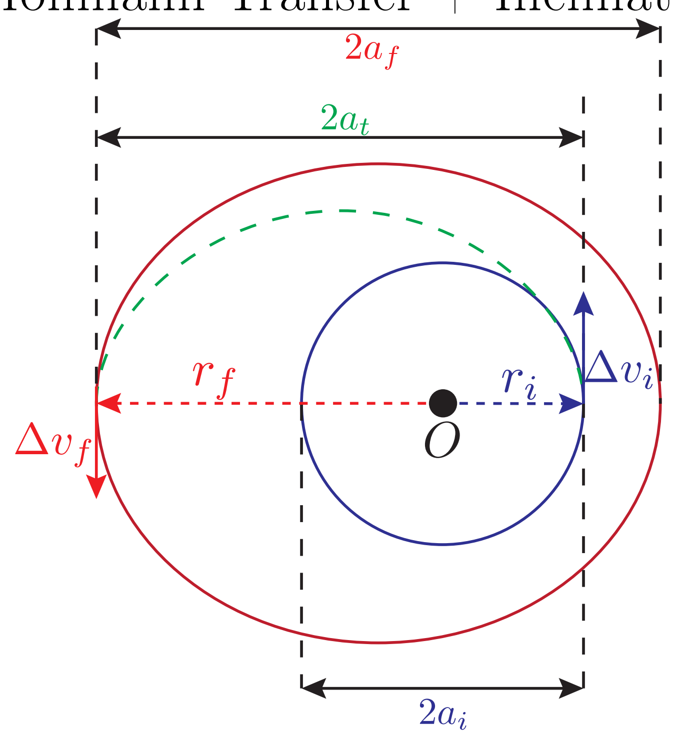


Vallado (2013) Fig. 6-12

$$\theta_i = \tan^{-1}\left(\frac{\Delta\Omega \sin(I_i)}{\Delta I}\right)$$

$$\alpha = \left((\Delta\Omega \sin(I_i))^2 + \Delta I^2\right)^{\frac{1}{2}}$$

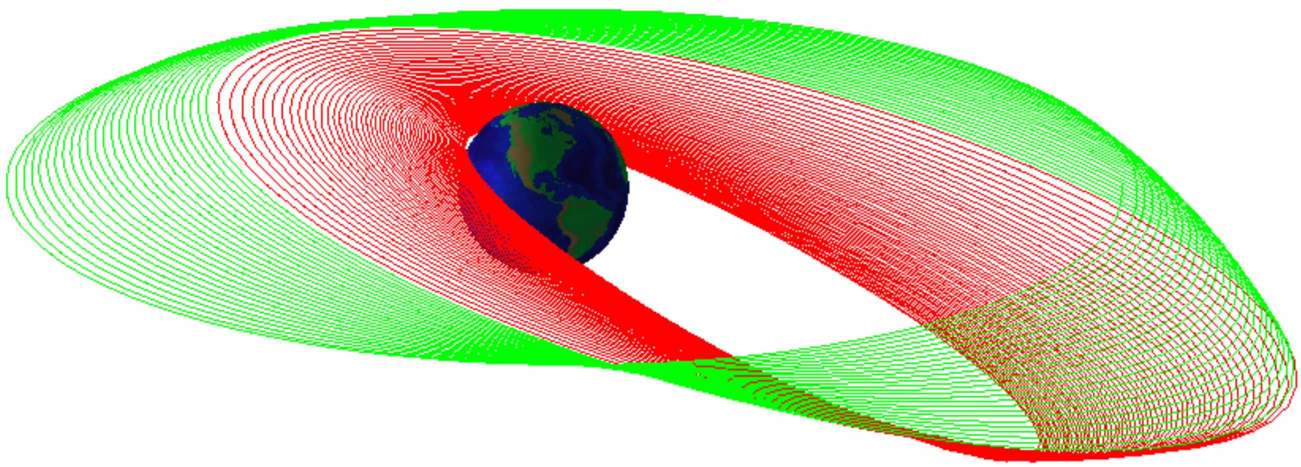
Hohmann Transfer + Inclination Change



For a total inclination change of ΔI:
Change by $x\Delta I$ on initial burn
Change by $(1-x)\Delta I$ on final burn

$$x \approx \frac{1}{\Delta I}\tan^{-1}\left(\frac{\sin(\Delta I)}{\frac{v_i v_{t_i}}{v_f v_{t_f}} + \cos(\Delta I)}\right)$$

Continuous Thrust Trajectories



Optimized GTO-GEO continuous thrust trajectory. From: Ilin et al. (2012)