16 - The Rocket Equation, Chemical Propulsion, and Staging

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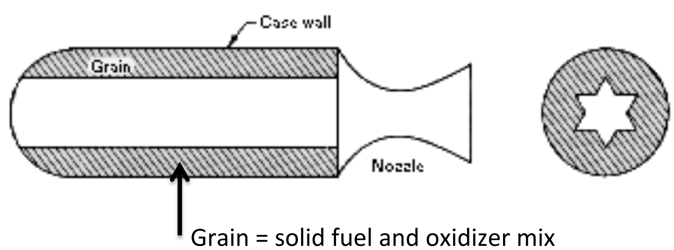
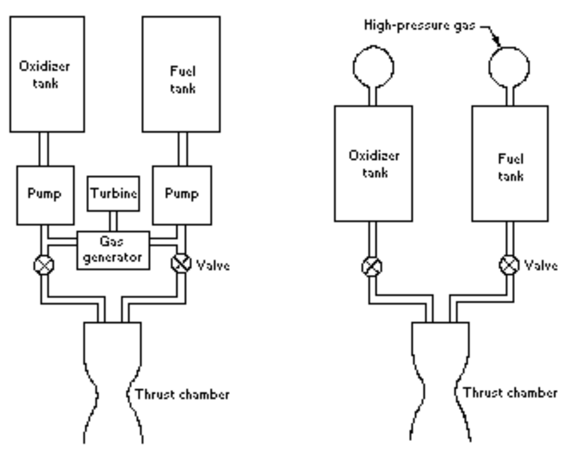
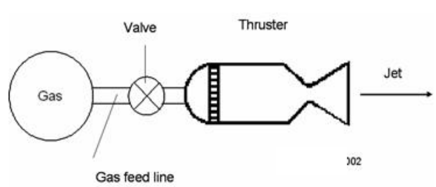
MAE 4060

Chemical Rockets

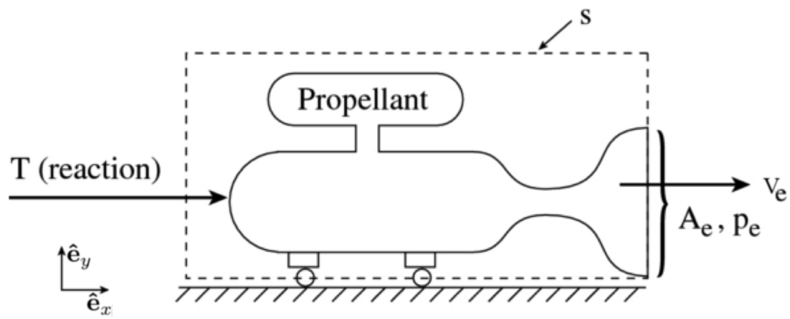






From: http://history.nasa.gov/conghand/propulsn.htm

Rocket Propulsion



From: http://web.mit.edu/16.unified/www/FALL/thermodynamics/notes/node77.html

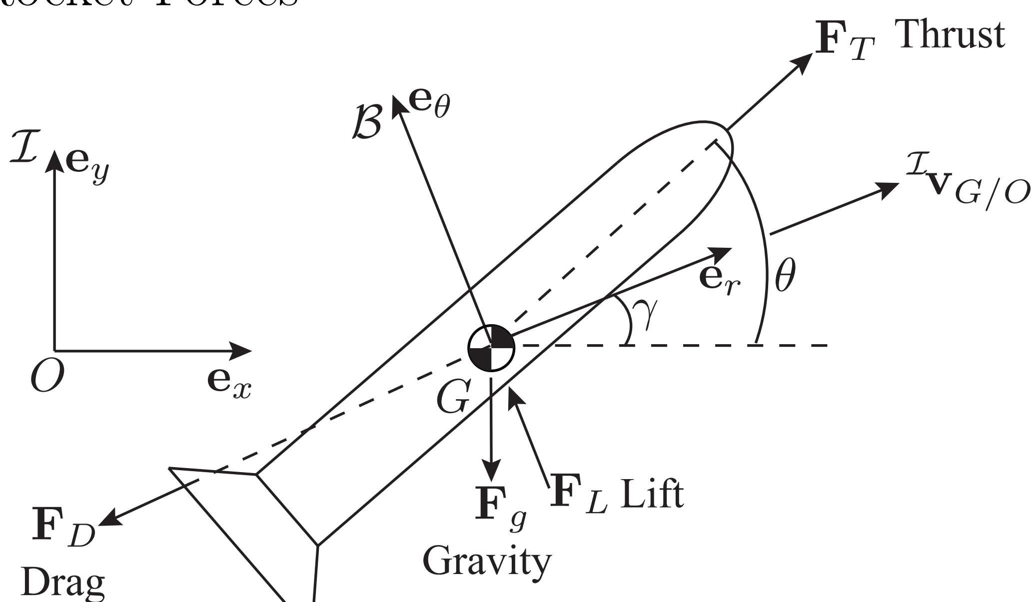
$$\underbrace{\Delta\mathbf{p}}_{\text{Change in Momentum}} = \underbrace{\dot{m}}_{\text{Mass Flow Rate}}\ \overbrace{V_e}^{\text{Nozzle Exit Velocity}}$$

$$\sum \mathbf{F}\cdot\hat{\mathbf{e}}_{\mathbf{x}} = \underbrace{T}_{\text{Thrust Force Magnitude}} - \underbrace{A_e}_{\text{Nozzle Exit Area}}(\underbrace{p_e}_{\text{Nozzle Exit Pressure}} - \underbrace{p_a}_{\text{Ambient Pressure}})$$

$$F_T = \dot{m}V_e + A_e(p_e - p_a)$$

$$v_{\text{eff}} \triangleq \frac{F_T}{\dot{m}} = V_e + \frac{A_e}{\dot{m}}(p_e - p_a)$$

Rocket Forces



NB: Rocket mass (m) is **not** constant

$$\begin{aligned} m\,{}^{\mathcal{I}}\mathbf{a}_{G/O} &= \sum F = \mathbf{F}_g + \mathbf{F}_L + \mathbf{F}_D + \mathbf{F}_T \\ &= -mg\mathbf{e}_y + F_L\mathbf{e}_\theta - F_D\mathbf{e}_r + F_T\left(\cos(\theta-\gamma)\mathbf{e}_r + \sin(\theta-\gamma)\mathbf{e}_\theta\right) \end{aligned}$$

The Tsiolkovsky (Ideal) Rocket Equation

$$
{}^{\mathcal{I}}\mathbf{a}_{G/O} = \frac{{}^{\mathcal{B}}\mathrm{d}}{\mathrm{d}t} \underbrace{{}^{\mathcal{I}}\mathbf{V}_{G/O}}_{\equiv v\mathbf{e}_r} + \underbrace{{}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}}}_{\equiv \dot{\gamma}\mathbf{e}_3} \times \underbrace{{}^{\mathcal{I}}\mathbf{V}_{G/O}}_{\equiv v\mathbf{e}_r} = \frac{\mathrm{d}v}{\mathrm{d}t}\mathbf{e}_r + v\frac{\mathrm{d}\gamma}{\mathrm{d}t}\mathbf{e}_\theta
$$

$$
\begin{aligned}
\mathbf{e}_r : \quad & \frac{\mathrm{d}v}{\mathrm{d}t} = -g\sin\gamma - \frac{F_D}{m} + \frac{F_T}{m}\cos(\theta - \gamma) \\
\mathbf{e}_\theta : \quad & v\frac{\mathrm{d}\gamma}{\mathrm{d}t} = -g\cos\gamma + \frac{F_L}{m} + \frac{F_T}{m}\sin(\theta - \gamma) \\
\Delta v \triangleq v(t_f) - v(t_0) = & \int_{t_0}^{t_f} \left[-g\sin\gamma - \frac{F_D}{m} + \frac{F_T}{m}\cos(\theta - \gamma) \right] \mathrm{d}t
\end{aligned}
$$

Assuming gravity and drag are negligible:
$g = 0, F_D = 0, \theta = \gamma$, and v_{eff} is constant:

$$
\Delta v = \int_{t_0}^{t_f} \frac{F_T}{m}\,\mathrm{d}t = v_{\text{eff}} \int_{t_0}^{t_f} \frac{\dot{m}}{m}\,\mathrm{d}t = -v_{\text{eff}} \int_{m_0}^{m_f} \frac{\mathrm{d}m}{m}
$$

$$
\boxed{\Delta v = v_{\text{eff}} \ln\left(\frac{m_0}{m_f}\right)}
$$

Specific Impulse

Specific Impulse

Total Impulse

Assuming Constant Thrust and Mass Flow Rate

$$
I_{sp} = \frac{1}{w_p} \overbrace{\int_{t_0}^{t_f} F_t\,\mathrm{d}t}^{\text{Total Impulse}} = \frac{F_T}{\dot{m}g_0} = \frac{v_{\text{eff}}}{g_0}
$$

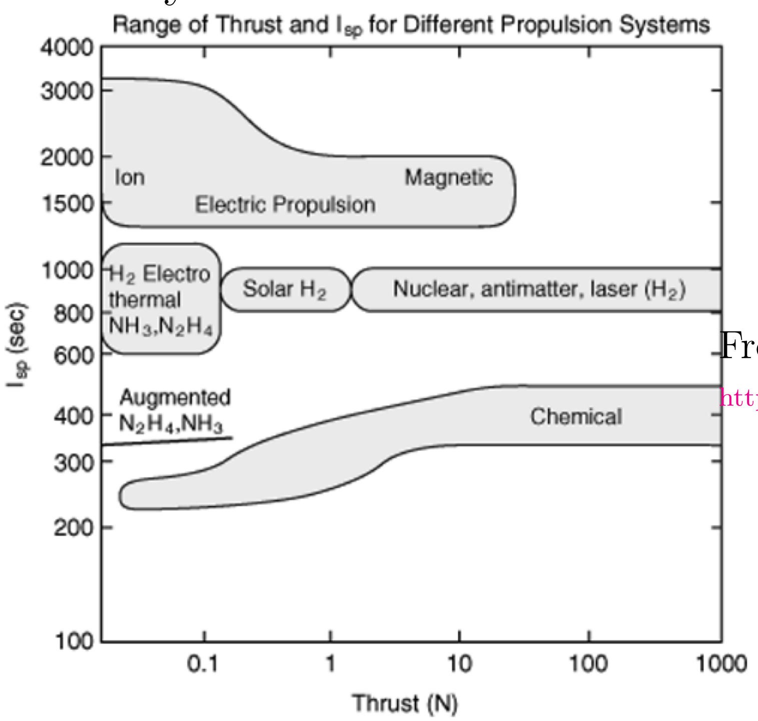
Propellant Weight

"Standard Gravity": Gravity at Earth's Surface. $g_0 = 9.80665\ \mathrm{m/s^2}$

$$
\Delta v = I_{sp} g_0 \ln\left(\frac{m_0}{m_f}\right)
$$

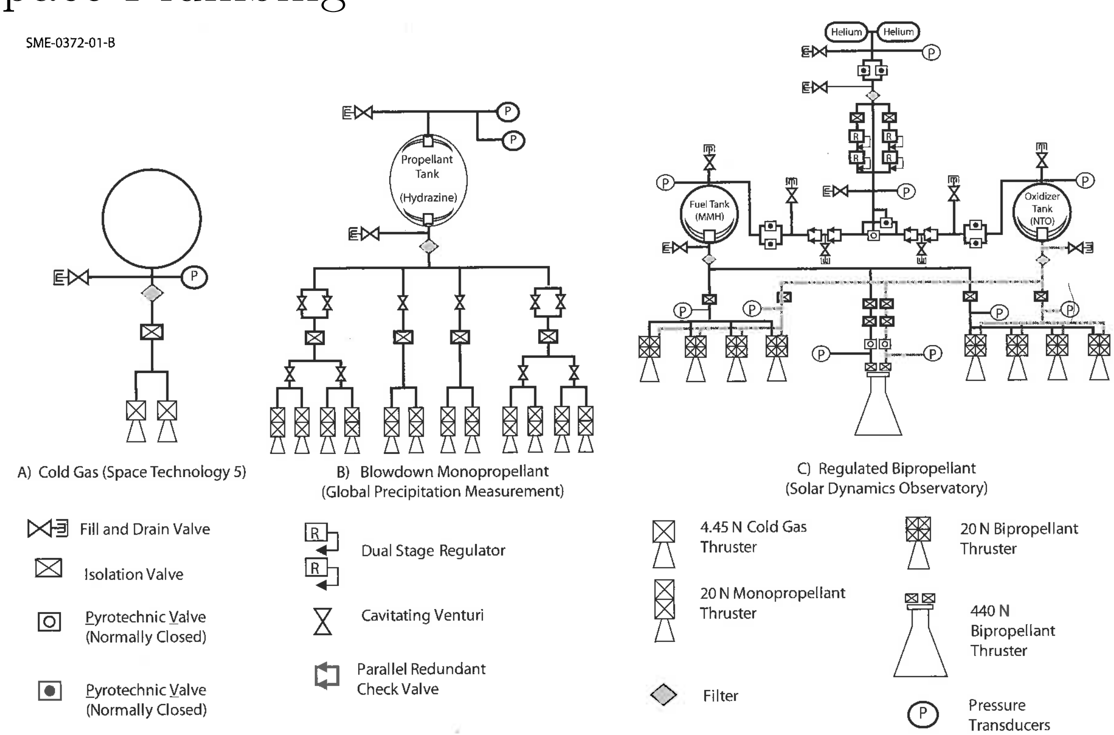
$$
m_0 - m_f = m_0 \left(1 - e^{-\frac{\Delta v}{I_{sp} g_0}}\right)
$$

Efficiency vs. Thrust



From:
http://dawn.jpl.nasa.gov/mission/ion_prop.asp

Space Plumbing



SMAD-SME Fig. 8-18

An Illustrative Example

- You wish to launch an H_2-O_2 rocket ($v_{\text{eff}} = 4000$ m/s) to a 600 km circular orbit:

$$\Delta v = v_{\text{circ}} = \sqrt{\frac{\mu}{r}} = \sqrt{\frac{3.986 \times 10^{14}\,\text{m}^3\text{s}^{-2}}{600\text{km} + 6371\text{km}}} \approx 7.5\text{km/s}$$

- Typically require an additional 1.5 km/s for atmospheric drag and gravity compensation for a total of 9 km/s

$$\Delta v = v_{\text{eff}} \ln\left(\frac{m_0}{m_f}\right) \Rightarrow \frac{m_0}{m_f} = e^{\Delta v / v_{\text{eff}}} \approx 9.5$$

- Your rocket must be 89.5% fuel by mass (and we still haven't even included all other losses)

Staging

$$\boxed{\Delta v = \sum_{i=1}^{n} \Delta v_i = \sum_{i=1}^{n} v_{\text{eff}_i} \ln\left(\frac{m_{0_i}}{m_{f_i}}\right)}$$

- m_{0_i}: Total mass before stage i ignition
- m_{f_i}: Total mass after stage i fuel expended but **before** stage i separation

If all stages have the same effective exhaust velocity:

$$e^{\Delta v / v_{\text{eff}}} = \prod_{i=1}^{n} \frac{m_{0_i}}{m_{f_i}}$$

Stages can be optimized for thrust, or efficiency, or maximized payload mass

Another Illustrative Example

Consider a 2-stage rocket with a 6000 m/s Δv requirement and 4500 kg total launch mass. Both stages have the same thrusters: 300 s I_{sp}, 3000 m/s v_{eff}, and 0.88 fuel mass fraction (ξ). m_1, m_2 are the stage wet masses (with fuel).

$$m_{0_1} = m_1 + m_2 + m_{\text{payload}} \qquad m_{0_2} = m_2 + m_{\text{payload}}$$
$$m_{f_1} = m_1(1-\xi) + m_2 + m_{\text{payload}} \qquad m_{f_2} = m_2(1-\xi) + m_{\text{payload}}$$

Equal Mass Stages

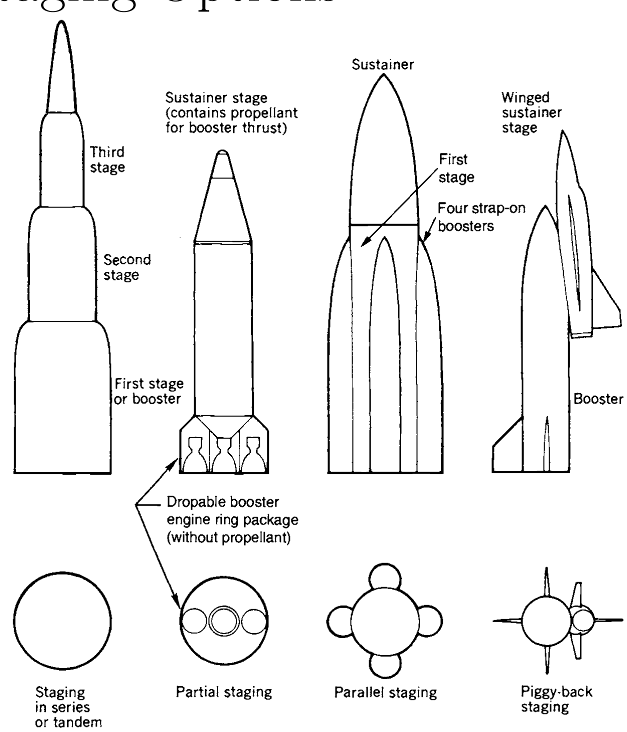
$$m_1 = m_2 \triangleq m \qquad m_{\text{tot}} = 2m + m_{\text{payload}}$$
$$c \triangleq e^{\Delta v/v_{\text{eff}}}$$
$$= \frac{m_{\text{tot}}}{m(1-\xi) + m + m_{\text{payload}}} \left(\frac{m + m_{\text{payload}}}{m(1-\xi) + m_{\text{payload}}} \right)$$
$$m_{\text{payload}} = \frac{m_{\text{tot}}}{c\xi(\xi+1)} \left(c\left(\xi^2 - \xi - 1\right) + \sqrt{c^2 + 4c\xi^2 - 2c + 1} + 1 \right)$$
$$\approx 300\,\text{kg}$$

Equal Mass-Ratio Stages

$$\frac{m_{0_1}}{m_{f_1}} = \frac{m_{0_2}}{m_{f_2}} \qquad m_{\text{tot}} = m_1 + m_2 + m_{\text{payload}}$$
$$\frac{m_{\text{tot}}}{m_{\text{tot}} - \xi m_1} = \frac{m_2 + m_{\text{payload}}}{m_2(1-\xi) + m_{\text{payload}}}$$
$$c \triangleq e^{\Delta v/v_{\text{eff}}} = \left(\frac{m_{\text{tot}}}{m_{\text{tot}} - m_1\xi} \right)^2$$
$$m_{\text{payload}} = \frac{m_{\text{tot}}}{c^2\xi^2} \left(c^2\left(\xi^2 - 2\xi + 1\right) + c + 2\sqrt{c^3}\,(\xi - 1) \right)$$
$$\approx 357\,\text{kg}$$

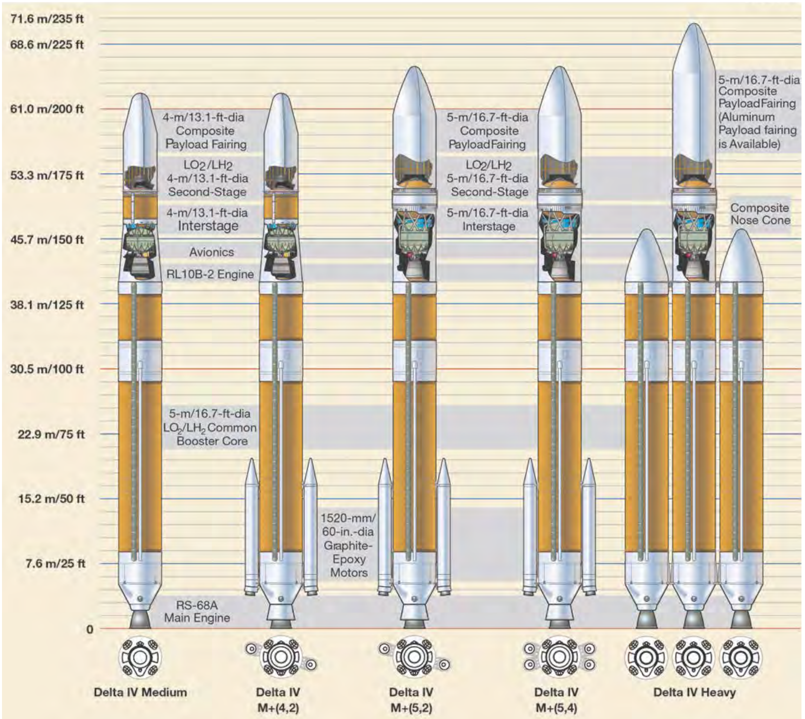
$\boxed{\sim 20\,\%\ \text{Gain!}}$

Staging Options



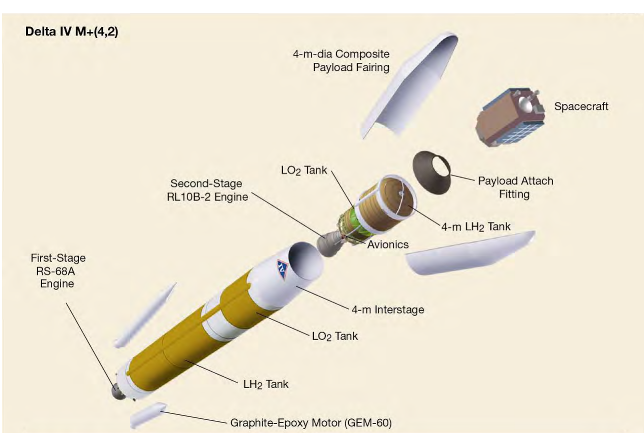
Sutton and Biblarz (2001) Fig. 4-14

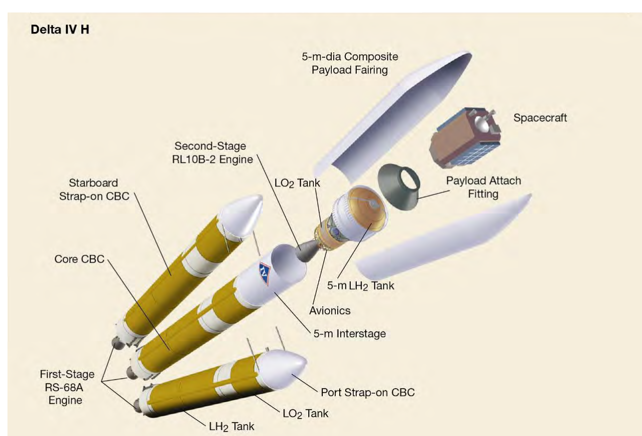
Delta IV Configurations



Credit: ULA
Delta IV User's Guide
https://www.ulalaunch.com/docs/default-source/rockets/delta-iv-user's-guide.pdf

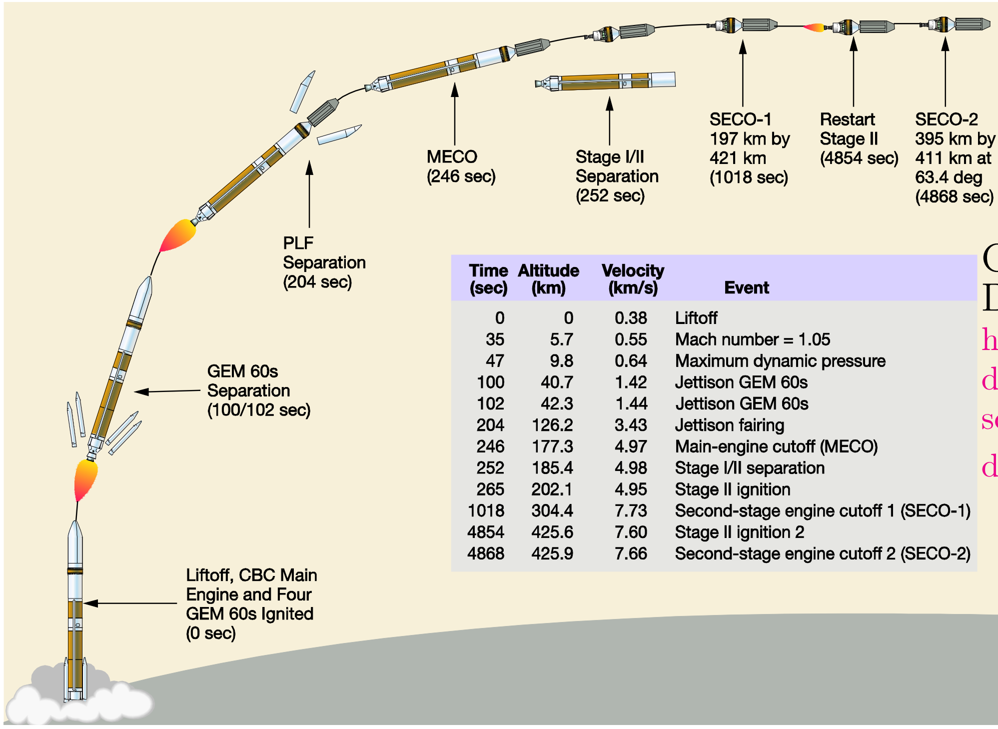
Delta IV Configurations





Credit: ULA, Delta IV User's Guide
https://www.ulalaunch.com/docs/default-source/rockets/delta-iv-user's-guide.pdf

Delta IV LEO Mission Profile



Time (sec)	Altitude (km)	Velocity (km/s)	Event
0	0	0.38	Liftoff
35	5.7	0.55	Mach number = 1.05
47	9.8	0.64	Maximum dynamic pressure
100	40.7	1.42	Jettison GEM 60s
102	42.3	1.44	Jettison GEM 60s
204	126.2	3.43	Jettison fairing
246	177.3	4.97	Main-engine cutoff (MECO)
252	185.4	4.98	Stage I/II separation
265	202.1	4.95	Stage II ignition
1018	304.4	7.73	Second-stage engine cutoff 1 (SECO-1)
4854	425.6	7.60	Stage II ignition 2
4868	425.9	7.66	Second-stage engine cutoff 2 (SECO-2)

Credit: ULA
Delta IV User's Guide
https://www.ulalaunch.com/docs/default-source/rockets/delta-iv-user's-guide.pdf