5 - Orbits in 3D, Reference Frames and Coordinate Systems

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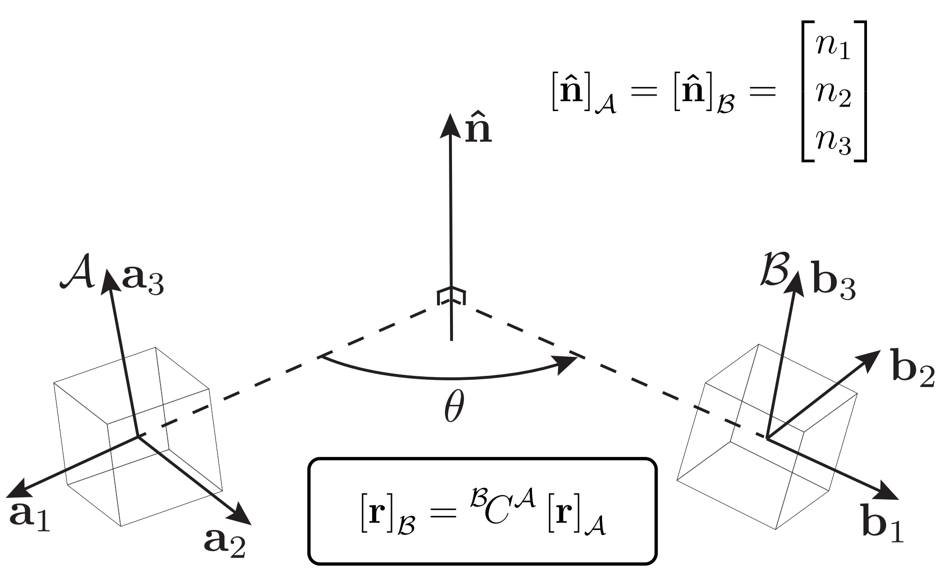
MAE 4060

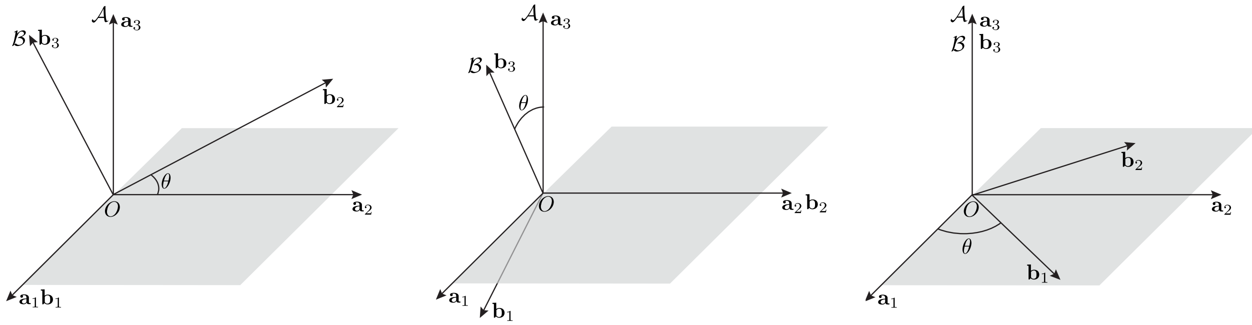
Direction Cosine Matrices



$$ {}^{\mathcal{A}}C^{\mathcal{B}} = \left({}^{\mathcal{B}}C^{\mathcal{A}}\right)^{-1} = \left({}^{\mathcal{B}}C^{\mathcal{A}}\right)^{T} $$

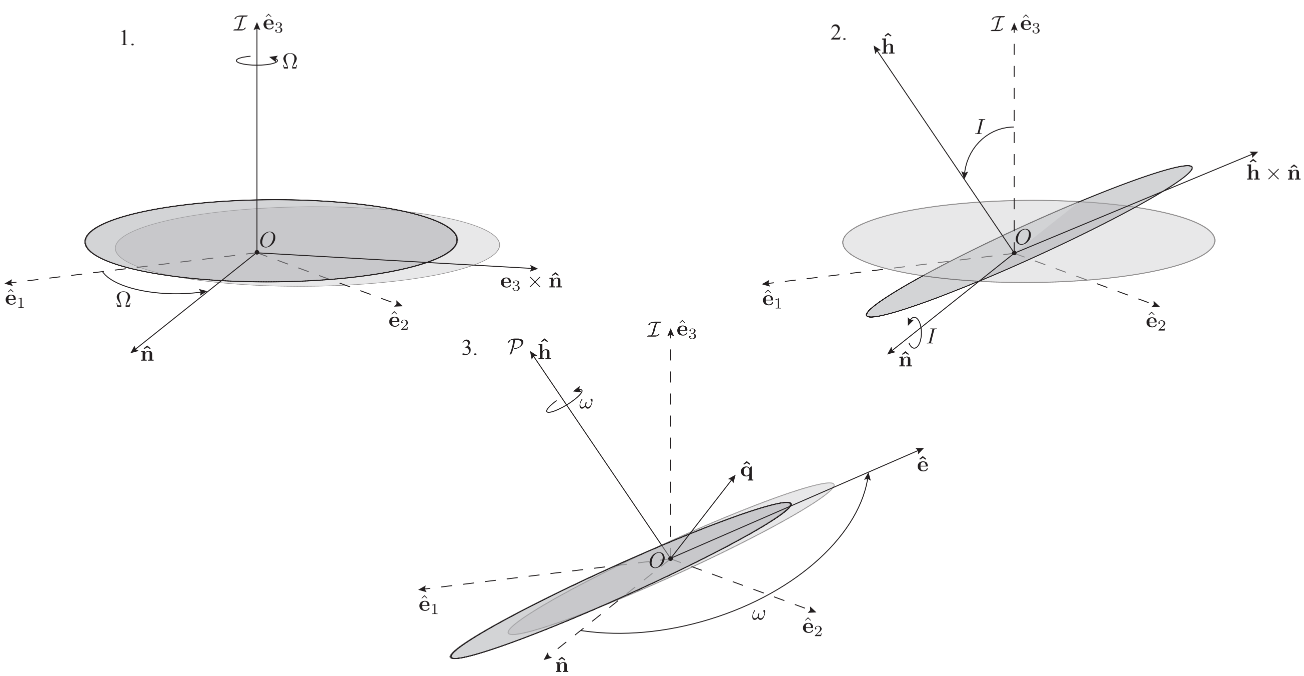
$$ {}^{\mathcal{I}}C^{\mathcal{F}_1}\,{}^{\mathcal{F}_1}C^{\mathcal{F}_2}\,{}^{\mathcal{F}_2}C^{\mathcal{F}_3}\ldots{}^{\mathcal{F}_{\mathcal{N}-1}}C^{\mathcal{F}_{\mathcal{N}}} = {}^{\mathcal{I}}C^{\mathcal{F}_{\mathcal{N}}} $$

Simple Direction Cosine Matrices

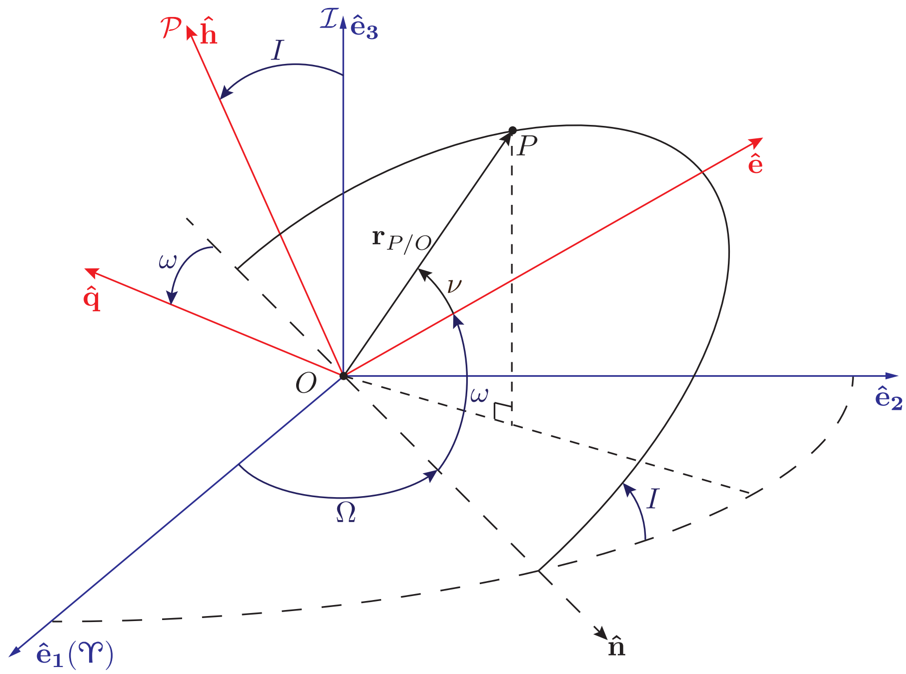


$$
{}^{\mathcal{B}}C^{\mathcal{A}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \qquad \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \qquad \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}
$$

Inertial $\rightarrow$ Perifocal: 3-1-3 (Ω, I, ω) Body-2 Rotation



Orbits in 3D



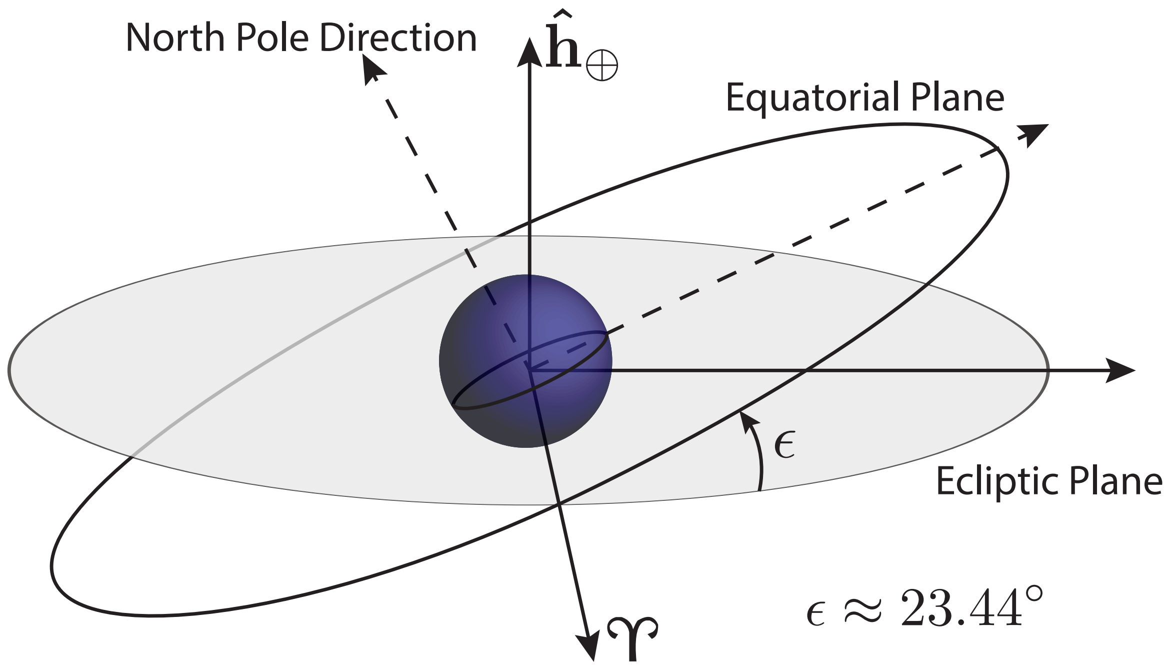
Orbits in 3D (Math Version)

$$
{}^{\mathcal{P}}C^{\mathcal{I}} = \begin{bmatrix} \cos(\omega) & \sin(\omega) & 0 \\ -\sin(\omega) & \cos(\omega) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(I) & \sin(I) \\ 0 & -\sin(I) & \cos(I) \end{bmatrix} \begin{bmatrix} \cos(\Omega) & \sin(\Omega) & 0 \\ -\sin(\Omega) & \cos(\Omega) & 0 \\ 0 & 0 & 1 \end{bmatrix} =
$$

$$
\begin{bmatrix} -\sin(\Omega)\sin(\omega)\cos(I) + \cos(\Omega)\cos(\omega) & \sin(\Omega)\cos(\omega) + \sin(\omega)\cos(I)\cos(\Omega) & \sin(I)\sin(\omega) \\ -\sin(\Omega)\cos(I)\cos(\omega) - \sin(\omega)\cos(\Omega) & -\sin(\Omega)\sin(\omega) + \cos(I)\cos(\Omega)\cos(\omega) & \sin(I)\cos(\omega) \\ \sin(I)\sin(\Omega) & -\sin(I)\cos(\Omega) & \cos(I) \end{bmatrix}
$$

$$
\left[\mathbf{r}_{P/O}\right]_{\mathcal{I}} = {}^{\mathcal{I}}C^{\mathcal{P}} \begin{bmatrix} r\cos\nu \\ r\sin\nu \\ 0 \end{bmatrix}_{\mathcal{P}} = r \begin{bmatrix} \cos(\Omega)\cos(\nu+\omega) - \sin(\Omega)\sin(\nu+\omega)\cos(I) \\ \sin(\Omega)\cos(\nu+\omega) + \sin(\nu+\omega)\cos(I)\cos(\Omega) \\ \sin(I)\sin(\nu+\omega) \end{bmatrix}
$$

Solar System Reference Planes



Special Cases

- $I = 0$, Longitude of Periapsis:

$$\pi \equiv \varpi \triangleq \omega + \Omega$$

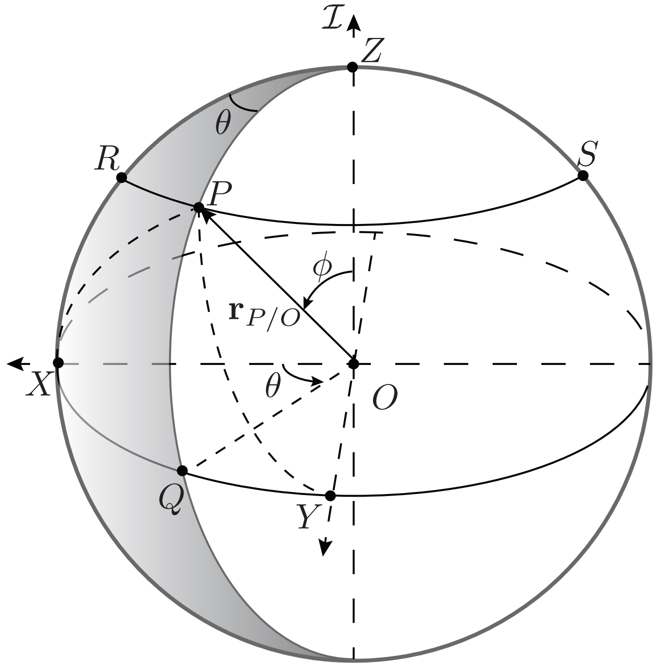
- $e = 0$, Argument of Lattitude:

$$u \equiv \theta \triangleq \nu + \omega$$

- $e = I = 0$, True Longitude:

$$l \triangleq \varpi + \nu = \Omega + \omega + \nu$$

Spherical Coordinates



Adapted from Green (1985)

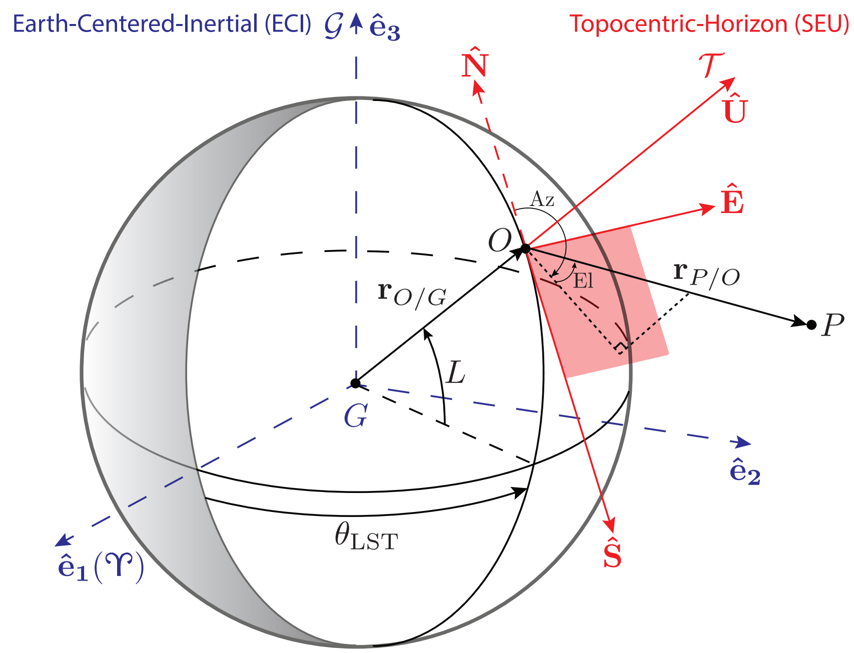
$$
[\hat{\mathbf{r}}_{P/O}]_{\mathcal{I}} = \begin{bmatrix} \cos\theta \sin\phi \\ \sin\theta\sin\phi \\ \cos\phi \end{bmatrix}_{\mathcal{I}}
$$

Adapted form Green (1985)

Spherical Coordinate Systems

Name	Origin	Reference Plane	Prime Direction	Azimuth Angle	Elevation Angle
Geographic	Geocentric	Equator	Prime Meridian	Longitude (λ)	Latitude (φ or L)
Horizontal (Topocentric)	Observer Location	Horizon	North	Azimuth (Az)	Altitude/Elevation (Alt/El)
Equatorial	Geocentric or Heliocentric	Celestial Equator	Vernal Equinox	Right Ascension (α)	Declination (δ)
Ecliptic	Geocentric or Heliocentric	Ecliptic	Vernal Equinox	Ecliptic Longitude (λ)	Ecliptic Latitude (β)
Galactic	Heliocentric	Galactic Plane	Galactic Center	Galactic Longitude (l)	Galactic Latitude (b)

Topocentric-Horizon Coordinate System



The Reference Geoid

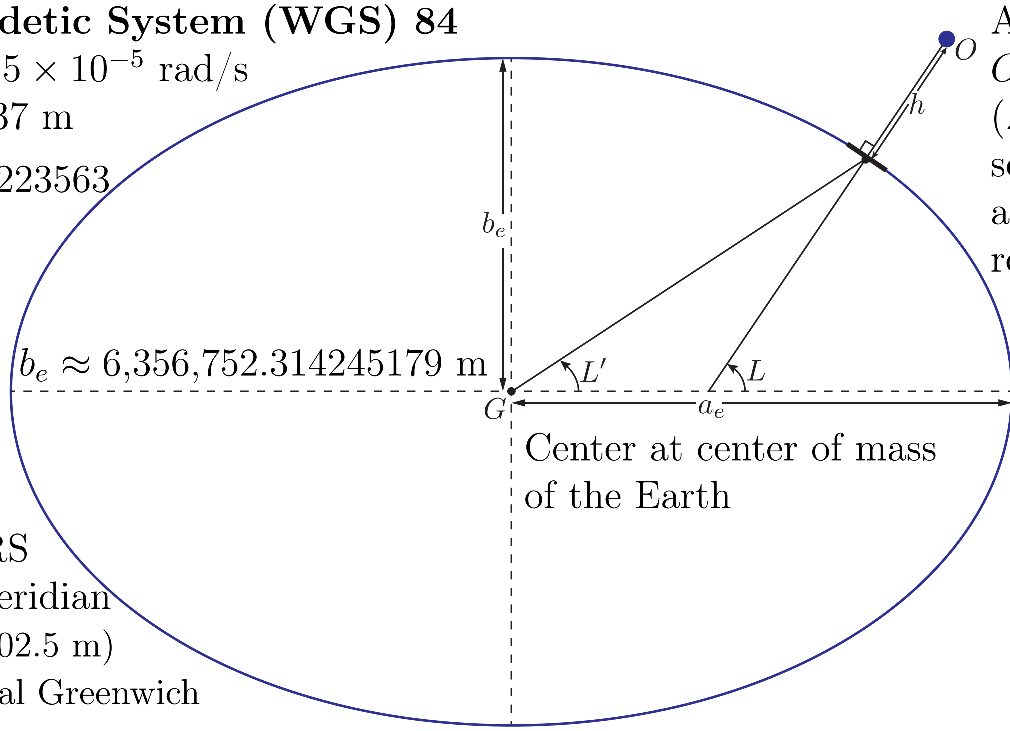
World Geodetic System (WGS) 84

$\omega_e = 7.292115 \times 10^{-5}$ rad/s

$a_e = 6{,}378{,}137$ m

$$\frac{1}{f} = 298.257223563$$

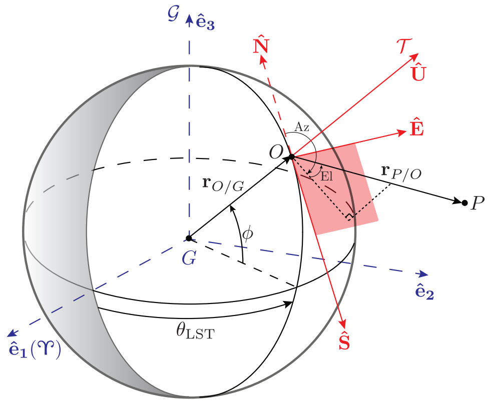
$$f \triangleq \frac{a-b}{a}$$

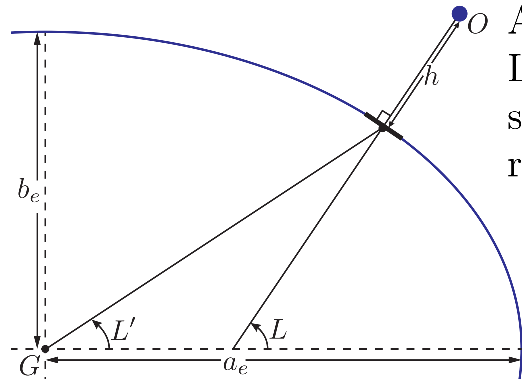


A surface point O at Lon/Lat (λ, L) is at some height h above the reference geoid

$\lambda = 0$ at IERS Reference Meridian $\sim$5.3 arcsec(102.5 m) East of original Greenwich

Finding Where You Are





A surface point O at Lon/Lat (λ, L) is at some height h above the reference geoid

Geoid described by a_e and e_e where: $e_e^2 = 2f - f^2$

$$[\mathbf{r}_{O/G}]_{\mathcal{G}} = \begin{bmatrix} x\cos\theta_{\text{LST}} \\ x\sin\theta_{\text{LST}} \\ z \end{bmatrix}_{\mathcal{G}}$$

$$x = \left(\frac{a_e}{\sqrt{1 - e_e^2\sin^2\phi}} + h\right)\cos\phi$$

$$y = \left(\frac{a_e(1 - e_e^2)}{\sqrt{1 - e_e^2\sin^2\phi}} + h\right)\sin\phi$$

Finding When You Are

$$\theta_{\text{LST}} = \underbrace{\theta_{g0}}_{\text{Reference Value at Epoch}} + \omega_e(t - \underbrace{t_0}_{}) + \lambda_E$$

Reference Value at Epoch

OR

MEAN SIDEREAL TIME, 2019

Greenwich mean sidereal time at 0^{h} UT

	h		h		h		h
Jan. 0	6·6250	Apr. 0	12·5389	July 0	18·5185	Oct. 0	0·5638
Feb. 0	8·6620	May 0	14·5102	Aug. 0	20·5555	Nov. 0	2·6008
Mar. 0	10·5019	June 0	16·5472	Sept. 0	22·5925	Dec. 0	4·5721

Greenwich mean sidereal time (GMST) on day d of month at hour t UT

$= \text{GMST at } 0^{\text{h}} \text{ UT on day } 0 + 0^{\text{h}}\!.065\,71\, d + 1^{\text{h}}\!.002\,74\, t$

Local mean sidereal time = GMST $\begin{matrix} + \text{ east} \\ - \text{ west} \end{matrix}$ longitude

https://aa.usno.navy.mil/publications/reports/ap19_for_web.pdf

$$\theta_{g0} = 100.4606184^\circ + 36{,}000.77005361 T_{\text{UT1}} + 0.00038793 T_{\text{UT1}}^2 - 2.6 \times 10^{-8} T_{\text{UT1}}^3$$
T_{UT1} = number of Julian centuries J2000.0