7 - Orbital Perturbations, Osculating Orbital Elements, Gauss's Equations, Lagrange's Planetary Equations, Cowell's Method and Encke's Method

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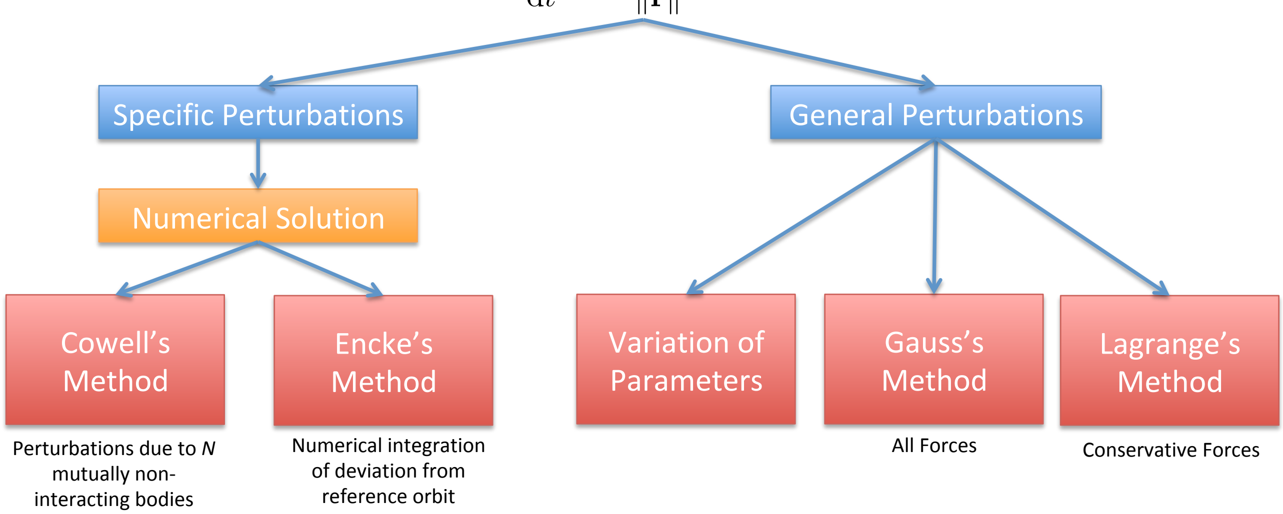
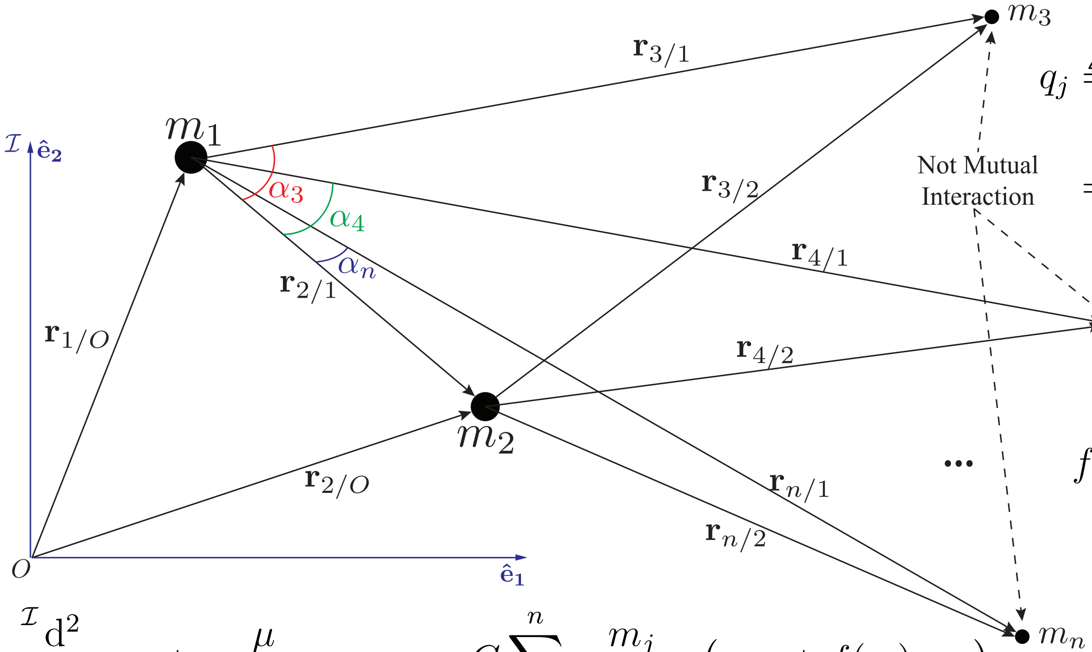
Perturbations Roadmap

$$ {}^{\mathcal{I}}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathbf{r} + \frac{\mu}{\|\mathbf{r}\|^3}\mathbf{r} = \mathbf{f} $$



Cowell's Method

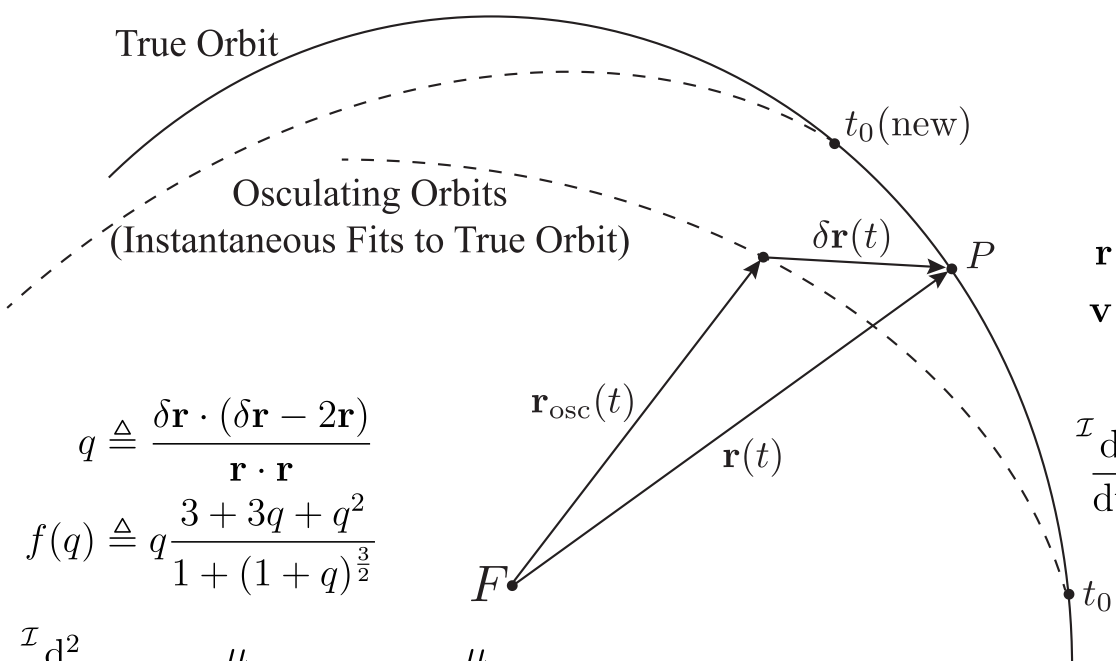


$$q_j \triangleq \frac{\mathbf{r}_{2/1} \cdot \left(\mathbf{r}_{2/1} - 2\mathbf{r}_{j/1}\right)}{\mathbf{r}_{j/1} \cdot \mathbf{r}_{j/1}} = \frac{r_{2/1}}{r_{j/1}} \left(\frac{r_{2/1}}{r_{j/1}} - 2\cos\alpha_j \right)$$

$$f(q) \triangleq q \left(\frac{3 + 3q + q^2}{1 + (1+q)^{\frac{3}{2}}} \right)$$

$$\frac{{}^{\mathcal{I}}\mathrm{d}^2}{\mathrm{d}t^2}\mathbf{r}_{2/1} + \frac{\mu}{\|\mathbf{r}_{2/1}\|^3}\mathbf{r}_{2/1} = -G\sum_{j=3}^{n} \frac{m_j}{\|\mathbf{r}_{2/j}\|^3} \left(\mathbf{r}_{2/1} + f(q_j)\mathbf{r}_{j/1}\right)$$

Encke's Method



$$\mathbf{r}(t_0) \equiv \mathbf{r}_{\rm osc}(t_0)$$
$$\mathbf{v}(t_0) \equiv \mathbf{v}_{\rm osc}(t_0)$$

$$\mathbf{r}(t) = \mathbf{r}_{\rm osc}(t) + \delta\mathbf{r}(t)$$
$$\mathbf{v}(t) = \mathbf{v}_{\rm osc}(t) + \delta\mathbf{v}(t)$$

$$q \triangleq \frac{\delta\mathbf{r} \cdot (\delta\mathbf{r} - 2\mathbf{r})}{\mathbf{r} \cdot \mathbf{r}}$$

$$f(q) \triangleq q\frac{3 + 3q + q^2}{1 + (1+q)^{\frac{3}{2}}}$$

$$\frac{{}^{\mathcal{I}}\mathrm{d}^2}{\mathrm{d}t^2}\mathbf{r}_{\rm osc} + \frac{\mu}{\|\mathbf{r}_{\rm osc}\|^3}\mathbf{r} = 0$$

$$\frac{{}^{\mathcal{I}}\mathrm{d}^2}{\mathrm{d}t^2}\mathbf{r} + \frac{\mu}{\|\mathbf{r}\|^3}\mathbf{r} = \mathbf{f}$$

$$\frac{{}^{\mathcal{I}}\mathrm{d}^2}{\mathrm{d}t^2}\delta\mathbf{r} + \frac{\mu}{\|\mathbf{r}_{\rm osc}\|^3}\delta\mathbf{r} = -\frac{\mu}{\|\mathbf{r}_{\rm osc}\|^3}f(q)\mathbf{r} + \mathbf{f}$$

Variation of Parameters (Orbital Elements)

$$\mathbf{h} = \mathbf{r} \times \mathbf{v}$$

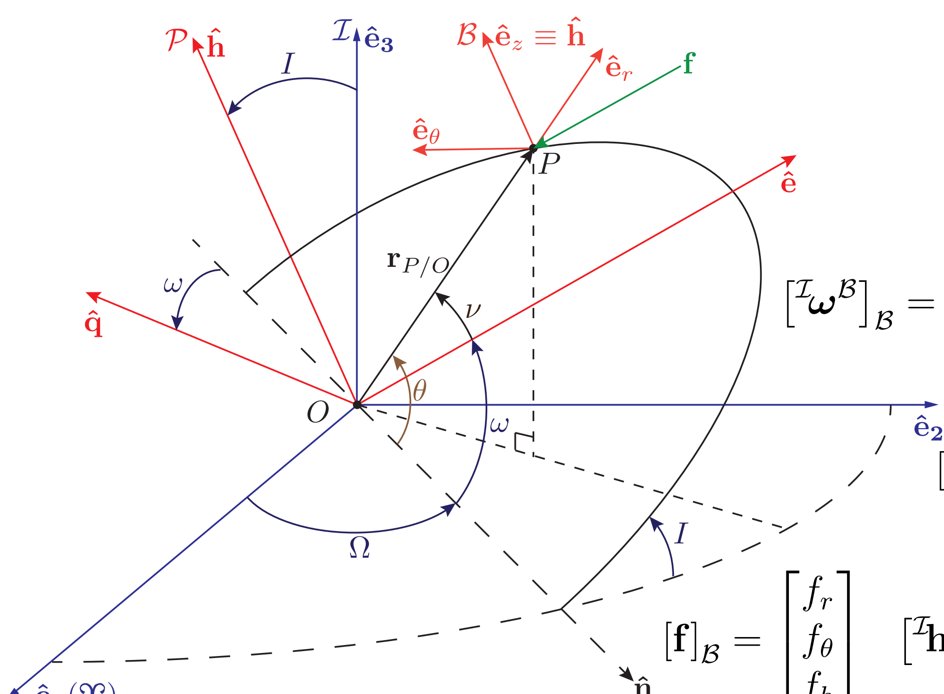
$$\mathbf{e} = \frac{\mathbf{v} \times \mathbf{h}}{\mu} - \frac{\mathbf{r}}{\|\mathbf{r}\|}$$

$$\frac{{}^{\mathcal{I}}\mathrm{d}^2}{\mathrm{d}t^2}\mathbf{r} = \frac{{}^{\mathcal{I}}\mathrm{d}}{\mathrm{d}t}\mathbf{v} = -\frac{\mu}{\|\mathbf{r}\|^3}\mathbf{r} + \mathbf{f}$$

$$\frac{{}^{\mathcal{I}}\mathrm{d}}{\mathrm{d}t}\mathbf{h} = \mathbf{r} \times \mathbf{f}$$

$$\frac{{}^{\mathcal{I}}\mathrm{d}}{\mathrm{d}t}\mathbf{e} = \frac{1}{\mu}\left(\mathbf{f} \times \mathbf{h} + \mathbf{v} \times \mathbf{r} \times \mathbf{f}\right)$$

Variation of Parameters Reference Frames



$${}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}} = \dot{\Omega}\hat{\mathbf{e}}_3 + \dot{I}\hat{\mathbf{n}} + \dot{\theta}\hat{\mathbf{h}}$$

$$\left[{}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}}\right]_{\mathcal{B}} = \begin{bmatrix} \dot{I}\cos(\theta) + \dot{\Omega}\sin(I)\sin(\theta) \\ -\dot{I}\sin(\theta) + \dot{\Omega}\sin(I)\cos(\theta) \\ \dot{\Omega}\cos(I) + \dot{\theta} \end{bmatrix}_{\mathcal{B}}$$

$$[\mathbf{r}]_{\mathcal{B}} = \begin{bmatrix} r \\ 0 \\ 0 \end{bmatrix}_{\mathcal{B}} \qquad \left[{}^{\mathcal{I}}\mathbf{v}\right]_{\mathcal{B}} = \begin{bmatrix} v_r \\ v_\theta \\ 0 \end{bmatrix}_{\mathcal{B}}$$

$$[\mathbf{f}]_{\mathcal{B}} = \begin{bmatrix} f_r \\ f_\theta \\ f_h \end{bmatrix}_{\mathcal{B}} \quad \left[{}^{\mathcal{I}}\mathbf{h}\right]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 0 \\ h \end{bmatrix}_{\mathcal{B}} \quad [\mathbf{e}]_{\mathcal{B}} = \begin{bmatrix} e\cos\nu \\ -e\sin\nu \\ 0 \end{bmatrix}_{\mathcal{B}}$$

Gauss's Perturbation Equations (the setup)

$$\frac{{}^{\mathcal{I}}\mathrm{d}\mathbf{h}}{\mathrm{d}t} = \frac{{}^{\mathcal{B}}\mathrm{d}\mathbf{h}}{\mathrm{d}t} + {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}} \times \mathbf{h} = \mathbf{r} \times \mathbf{f} \quad \Rightarrow$$

$$\begin{bmatrix} 0 \\ 0 \\ \dot{h} \end{bmatrix}_{\mathcal{B}} + \begin{bmatrix} h\left(-\dot{I}\sin(\theta) + \dot{\Omega}\sin(I)\cos(\theta)\right) \\ -h\left(\dot{I}\cos(\theta) + \dot{\Omega}\sin(I)\sin(\theta)\right) \\ 0 \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} 0 \\ -f_h r \\ f_\theta r \end{bmatrix}_{\mathcal{B}}$$

$$\frac{{}^{\mathcal{I}}\mathrm{d}\mathbf{e}}{\mathrm{d}t} = \frac{{}^{\mathcal{B}}\mathrm{d}\mathbf{e}}{\mathrm{d}t} + {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}} \times \mathbf{e} = \frac{1}{\mu}\left(\mathbf{f} \times {}^{\mathcal{I}}\mathbf{h} + {}^{\mathcal{I}}\mathbf{v} \times \mathbf{r} \times \mathbf{f}\right) \quad \Rightarrow$$

$$\begin{bmatrix} -e\left(\dot{\omega} - \dot{\theta}\right)\sin(\omega - \theta) + \dot{e}\cos(\omega - \theta) \\ e\left(\dot{\omega} - \dot{\theta}\right)\cos(\omega - \theta) + \dot{e}\sin(\omega - \theta) \\ 0 \end{bmatrix}_{\mathcal{B}} + \begin{bmatrix} -e\left(\dot{\Omega}\cos(I) + \dot{\theta}\right)\sin(\omega - \theta) \\ e\left(\dot{\Omega}\cos(I) + \dot{\theta}\right)\cos(\omega - \theta) \\ e\left(\dot{I}\sin(\omega) - \dot{\Omega}\sin(I)\cos(\omega)\right) \end{bmatrix}_{\mathcal{B}} = \frac{1}{\mu}\left(\begin{bmatrix} f_\theta h \\ -f_r h \\ 0 \end{bmatrix}_{\mathcal{B}} + \begin{bmatrix} f_\theta r v_\theta \\ -f_\theta r v_r \\ -f_h r v_r \end{bmatrix}_{\mathcal{B}}\right)$$

Gauss's Perturbation Equations (the solution)

$$\boxed{\begin{aligned} \dot{I} &= \frac{f_h r}{h}\cos(\theta) & \dot{e} &= \frac{e f_\theta}{h} r \sin^2(\nu) + \frac{f_r h}{\mu}\sin(\nu) + \frac{2 f_\theta}{\mu} h \cos(\nu) \\ \dot{\Omega} &= \frac{f_h r \sin(\theta)}{h \sin(I)} & \dot{\omega} &= -\frac{f_h r \sin(\theta)}{h \tan(I)} - \frac{f_\theta r}{2h}\sin(2\nu) - \frac{f_r h}{e\mu}\cos(\nu) + \frac{2 f_\theta h}{e\mu}\sin(\nu) \\ \dot{h} &= f_\theta r & \frac{h}{r^2} &= \dot{\Omega}\cos(I) + \dot{\theta} \end{aligned}}$$

$$\dot{a} = \frac{2a^2}{h}\left[e \sin\nu f_r + (1 + e\cos(\nu)) f_\theta\right]$$

Gauss's Perturbation Equations (other versions)

$$
\begin{aligned}
\frac{d\Omega}{dt} &= \frac{r\sin\theta}{h\sin i}a_{dh} \\
\frac{di}{dt} &= \frac{r\cos\theta}{h}a_{dh} \\
\frac{d\omega}{dt} &= \frac{1}{he}[-p\cos f\, a_{dr} + (p+r)\sin f\, a_{d\theta}] - \frac{r\sin\theta\cos i}{h\sin i}a_{dh} \\
\frac{da}{dt} &= \frac{2a^2}{h}\left(e\sin f\, a_{dr} + \frac{p}{r}a_{d\theta}\right) \\
\frac{de}{dt} &= \frac{1}{h}\{p\sin f\, a_{dr} + [(p+r)\cos f + re]a_{d\theta}\} \\
\frac{dM}{dt} &= n + \frac{b}{ahe}[(p\cos f - 2re)a_{dr} - (p+r)\sin f a_{d\theta}]
\end{aligned}
$$

Battin (1999) Eq. 10.41
NB: $f \equiv \nu$ here
$(a_{dr}, a_{d\theta}, a_{dh}) \equiv (f_r, f_\theta, f_h)$

$$
\begin{aligned}
\frac{da}{dt} &= \frac{2}{n\sqrt{1-e^2}}\left\{e\,\mathrm{SIN}(\nu)F_R + \frac{p}{r}F_S\right\} \\
\frac{de}{dt} &= \frac{\sqrt{1-e^2}}{na}\left\{\mathrm{SIN}(\nu)F_R + \left(\mathrm{COS}(\nu) + \frac{e+\mathrm{COS}(\nu)}{1+e\,\mathrm{COS}(\nu)}\right)F_S\right\} \\
\frac{di}{dt} &= \frac{r\,\mathrm{COS}(u)}{na^2\sqrt{1-e^2}}F_W \\
\frac{d\Omega}{dt} &= \frac{r\,\mathrm{SIN}(u)}{na^2\sqrt{1-e^2}\,\mathrm{SIN}(i)}F_W \\
\frac{d\omega}{dt} &= \frac{\sqrt{1-e^2}}{nae}\left\{-\mathrm{COS}(\nu)F_R + \mathrm{SIN}(\nu)\left(1+\frac{r}{p}\right)F_S\right\} - \frac{r\,\mathrm{COT}(i)\,\mathrm{SIN}(u)}{h}F_W \\
\frac{dM_o}{dt} &= \frac{1}{na^2e}\left\{(p\,\mathrm{COS}(\nu) - 2er)F_R - (p+r)\,\mathrm{SIN}(\nu)F_S\right\} - \frac{dn}{dt}(t-t_o)
\end{aligned}
$$

Vallado (2013) Eq. 9-24
NB: $p = \ell$ here
$(F_R, F_S, F_W) \equiv (f_r, f_\theta, f_h)$

Lagrange Planetary Equations

$$
\begin{aligned}
\frac{\mathrm{d}a}{\mathrm{d}t} &= \frac{2}{na}\frac{\partial R}{\partial M} \\
\frac{\mathrm{d}e}{\mathrm{d}t} &= \frac{1}{na^2e}\left(\left(1-e^2\right)\frac{\partial R}{\partial M} - \sqrt{1-e^2}\frac{\partial R}{\partial \omega}\right) \\
\frac{\mathrm{d}I}{\mathrm{d}t} &= \frac{1}{na^2\sqrt{1-e^2}\sin I}\left(\cos I\frac{\partial R}{\partial \omega} - \frac{\partial R}{\partial \Omega}\right) \\
\frac{\mathrm{d}\omega}{\mathrm{d}t} &= \frac{\sqrt{1-e^2}}{na^2e}\frac{\partial R}{\partial e} - \frac{\cot I}{na^2\sqrt{1-e^2}}\frac{\partial R}{\partial I} \\
\frac{\mathrm{d}\Omega}{\mathrm{d}t} &= \frac{1}{na^2\sqrt{1-e^2}\sin I}\frac{\partial R}{\partial I} \\
\frac{\mathrm{d}M}{\mathrm{d}t} &= n \underbrace{- \frac{1-e^2}{na^2e}\frac{\partial R}{\partial e} - \frac{2}{na}\frac{\partial R}{\partial a}}
\end{aligned}
$$

Recall the mean motion:

$$
n = \sqrt{\frac{\mu}{a^3}}
$$

Remember: mean anomaly is always changing (at rate n). This term gives the variation in this change due to the perturbation.

Lagrange Planetary Equations (other versions)

$$
\begin{aligned}
\frac{d\Omega}{dt} &= \frac{1}{nab\sin i}\frac{\partial R}{\partial i}\\
\frac{di}{dt} &= -\frac{1}{nab\sin i}\frac{\partial R}{\partial \Omega} + \frac{\cos i}{nab\sin i}\frac{\partial R}{\partial \omega}\\
\frac{d\omega}{dt} &= -\frac{\cos i}{nab\sin i}\frac{\partial R}{\partial i} + \frac{b}{na^3 e}\frac{\partial R}{\partial e}\\
\frac{da}{dt} &= \frac{2}{na}\frac{\partial R}{\partial \lambda}\\
\frac{de}{dt} &= -\frac{b}{na^3 e}\frac{\partial R}{\partial \omega} + \frac{b^2}{na^4 e}\frac{\partial R}{\partial \lambda}\\
\frac{d\lambda}{dt} &= -\frac{2}{na}\frac{\partial R}{\partial a} - \frac{b^2}{na^4 e}\frac{\partial R}{\partial e}
\end{aligned}
$$

Battin (1999) Eq. 10.31

$$\lambda \triangleq nt_p$$
$$b = a\sqrt{1-e^2}$$

$$
\begin{aligned}
\frac{da}{dt} &= \frac{2}{na}\frac{\partial R}{\partial M_o}\\
\frac{de}{dt} &= \frac{1-e^2}{na^2 e}\frac{\partial R}{\partial M_o} - \frac{\sqrt{1-e^2}}{na^2 e}\frac{\partial R}{\partial \omega}\\
\frac{di}{dt} &= \frac{1}{na^2\sqrt{1-e^2}\,\mathrm{SIN}(i)}\left\{\mathrm{COS}(i)\frac{\partial R}{\partial \omega} - \frac{\partial R}{\partial \Omega}\right\}\\
\frac{d\omega}{dt} &= \frac{\sqrt{1-e^2}}{na^2 e}\frac{\partial R}{\partial e} - \frac{\mathrm{COT}(i)}{na^2\sqrt{1-e^2}}\frac{\partial R}{\partial i}\\
\frac{d\Omega}{dt} &= \frac{1}{na^2\sqrt{1-e^2}\,\mathrm{SIN}(i)}\frac{\partial R}{\partial i}\\
\frac{dM_o}{dt} &= -\frac{1-e^2}{na^2 e}\frac{\partial R}{\partial e} - \frac{2}{na}\frac{\partial R}{\partial a} \quad +n
\end{aligned}
$$

Vallado (2013) Eq. 9-12

$$M = M_0 + n(t - t_p)$$