20 - Quaternions, Rodrigues Parameters, Small-Angle Rotations, and Angular Velocity

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Rodrigues Parameters

$$\boxed{\boldsymbol{\rho} \triangleq \tan\left(\frac{\theta}{2}\right)\hat{\mathbf{n}}}$$

$$\begin{aligned}
\rho_i &\equiv \frac{\epsilon_i}{\epsilon_4} \\
{}^{\mathcal{A}}C^{\mathcal{B}} &= \left(I + [\boldsymbol{\rho}\times]\right)\left(I - [\boldsymbol{\rho}\times]\right)^{-1} \\
\mathbf{a} - \mathbf{b} &= (\mathbf{a} + \mathbf{b}) \times \boldsymbol{\rho}
\end{aligned}$$

Quaternions

Define a new basis set, $\mathbf{i}, \mathbf{j}, \mathbf{k}$, s. t. $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1$.

$$\begin{array}{lll} \mathbf{ij} = \mathbf{k} & \mathbf{jk} = \mathbf{i} & \mathbf{ki} = \mathbf{j} \\ \mathbf{ji} = -\mathbf{k} & \mathbf{kj} = -\mathbf{i} & \mathbf{ik} = -\mathbf{j} \end{array}$$

A quaternion is a vector in this basis and a scalar:

$$\mathbf{q} \triangleq \begin{bmatrix} \mathbf{v} \\ r \end{bmatrix} = \begin{bmatrix} v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k} \\ r \end{bmatrix}$$

Quaternion product:

$$\mathbf{q}_1 = \begin{bmatrix} \mathbf{v}_1 \\ r_1 \end{bmatrix}, \mathbf{q}_2 = \begin{bmatrix} \mathbf{v}_2 \\ r_2 \end{bmatrix} \quad \Longrightarrow \quad \mathbf{q}_1\mathbf{q}_2 = \begin{bmatrix} r_1\mathbf{v}_2 + r_2\mathbf{v}_1 + \mathbf{v}_1 \times \mathbf{v}_2 \\ r_1 r_2 - \mathbf{v}_1 \cdot \mathbf{v}_2 \end{bmatrix}$$

Markely & Crassidis call this $\mathbf{q}_1 \odot \mathbf{q}_2$ and define:

$$\mathbf{q}_1 \otimes \mathbf{q}_2 = \begin{bmatrix} r_1\mathbf{v}_2 + r_2\mathbf{v}_1 - \mathbf{v}_1 \times \mathbf{v}_2 \\ r_1 r_2 - \mathbf{v}_1 \cdot \mathbf{v}_2 \end{bmatrix}$$

Quaternion Products

$$\mathbf{q}_1 \otimes \mathbf{q}_2 = [\mathbf{q}_1 \otimes]\, \mathbf{q}_2 \qquad\qquad \mathbf{q}_1 \odot \mathbf{q}_2 = [\mathbf{q}_1 \odot]\, \mathbf{q}_2$$

$$[\mathbf{q}\otimes] \triangleq \left[\underbrace{\begin{bmatrix} rI - [\mathbf{v}\times] \\ -\mathbf{v}^T \end{bmatrix}}_{\triangleq \boldsymbol{\Psi}(\mathbf{q})} \quad \underbrace{\begin{bmatrix} \mathbf{v} \\ r \end{bmatrix}}_{\mathbf{q}} \right] \qquad\qquad [\mathbf{q}\odot] \triangleq \left[\underbrace{\begin{bmatrix} rI + [\mathbf{v}\times] \\ -\mathbf{v}^T \end{bmatrix}}_{\triangleq \boldsymbol{\Xi}(\mathbf{q})} \quad \underbrace{\begin{bmatrix} \mathbf{v} \\ r \end{bmatrix}}_{\mathbf{q}} \right]$$

$$\boxed{\mathbf{q}_1 \otimes \mathbf{q}_2 = \mathbf{q}_2 \odot \mathbf{q}_1}$$

Quaternion Representation of Rotations

$$\boxed{\mathbf{q}(\hat{\mathbf{n}}, \theta) = \begin{bmatrix} \sin\left(\frac{\theta}{2}\right)\hat{\mathbf{n}} \\ \cos\left(\frac{\theta}{2}\right) \end{bmatrix}}$$

$$\begin{aligned} \mathbf{b} = \mathbb{R} \cdot \mathbf{a} \quad \implies \quad \mathbf{b} &= \mathbf{q} \otimes \mathbf{a} \otimes \mathbf{q}^\star \\ &= [\mathbf{q}\odot]^T [\mathbf{q}\otimes] \begin{bmatrix} \mathbf{a} \\ 0 \end{bmatrix} \end{aligned}$$

$$\mathbf{q}^\star = \begin{bmatrix} -\mathbf{v} \\ r \end{bmatrix} \quad \text{for} \quad \mathbf{q} = \begin{bmatrix} \mathbf{v} \\ r \end{bmatrix}$$

See Markely & Crassidis (2014), Sec. 2.7 & 2.9.3 for more details

Small Rotations

$$\text{Recall: } \mathbf{b} = \hat{\mathbf{n}}\hat{\mathbf{n}} \cdot \mathbf{a} + \cos\theta\,(\mathbf{a} - \hat{\mathbf{n}}(\hat{\mathbf{n}} \cdot \mathbf{a})) + \sin\theta\hat{\mathbf{n}} \times \mathbf{a}$$

Assume $\theta \ll 1$:

$$\mathbf{b} \approx \mathbf{a} + \theta\hat{\mathbf{n}} \times \mathbf{a} \quad \Rightarrow \quad {}^{\mathcal{A}}C^{\mathcal{B}} \approx I + \theta\,[\hat{\mathbf{n}}\times]_{\mathcal{A}}$$

$$\mathbf{q} \approx \begin{bmatrix} \frac{\theta}{2}\hat{\mathbf{n}} \\ 1 \end{bmatrix} \quad \Rightarrow \quad {}^{\mathcal{A}}C^{\mathcal{B}} \approx I + 2\,[\mathbf{q}_{1:3}\times]_{\mathcal{A}}$$

$$\boldsymbol{\rho} \approx \frac{\theta}{2}\hat{\mathbf{n}} \quad \Rightarrow \quad {}^{\mathcal{A}}\boldsymbol{\rho}^{\mathcal{B}} \approx {}^{\mathcal{A}}\boldsymbol{\rho}^{\mathcal{C}} + {}^{\mathcal{C}}\boldsymbol{\rho}^{\mathcal{B}}$$

The Angular Velocity Matrix

$$\boxed{\widetilde{\omega} \triangleq {}^{\mathcal{B}}C^{\mathcal{A}}\,{}^{\mathcal{A}}\dot{C}^{\mathcal{B}}}$$

$${}^{\mathcal{A}}\dot{C}^{\mathcal{B}} \triangleq \frac{\mathrm{d}}{\mathrm{d}t}{}^{\mathcal{A}}C^{\mathcal{B}}$$

1. $\left({}^{\mathcal{B}}C^{\mathcal{A}}\right)^{-1}\widetilde{\omega} = \left({}^{\mathcal{B}}C^{\mathcal{A}}\right)^{-1}{}^{\mathcal{B}}C^{\mathcal{A}}\,{}^{\mathcal{A}}\dot{C}^{\mathcal{B}} = {}^{\mathcal{A}}\dot{C}^{\mathcal{B}} \quad \Longrightarrow \quad {}^{\mathcal{A}}\dot{C}^{\mathcal{B}} = {}^{\mathcal{A}}C^{\mathcal{B}}\widetilde{\omega}$

2. $\widetilde{\omega}^T + \widetilde{\omega} = \underbrace{\left({}^{\mathcal{B}}C^{\mathcal{A}}\,{}^{\mathcal{A}}\dot{C}^{\mathcal{B}}\right)^T + \left({}^{\mathcal{B}}C^{\mathcal{A}}\,{}^{\mathcal{A}}\dot{C}^{\mathcal{B}}\right)}_{\equiv \frac{\mathrm{d}}{\mathrm{d}t}\left({}^{\mathcal{B}}C^{\mathcal{A}}\,{}^{\mathcal{A}}C^{\mathcal{B}}\right) = \frac{\mathrm{d}}{\mathrm{d}t}(I)} = 0$

Components of Angular Velocity Matrix

$$\begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} = \begin{bmatrix} {}^{\mathcal{A}}C^{\mathcal{B}}_{11} & {}^{\mathcal{A}}C^{\mathcal{B}}_{21} & {}^{\mathcal{A}}C^{\mathcal{B}}_{31} \\ {}^{\mathcal{A}}C^{\mathcal{B}}_{12} & {}^{\mathcal{A}}C^{\mathcal{B}}_{22} & {}^{\mathcal{A}}C^{\mathcal{B}}_{32} \\ {}^{\mathcal{A}}C^{\mathcal{B}}_{13} & {}^{\mathcal{A}}C^{\mathcal{B}}_{23} & {}^{\mathcal{A}}C^{\mathcal{B}}_{33} \end{bmatrix} \begin{bmatrix} {}^{\mathcal{A}}\dot{C}^{\mathcal{B}}_{11} & {}^{\mathcal{A}}\dot{C}^{\mathcal{B}}_{12} & {}^{\mathcal{A}}\dot{C}^{\mathcal{B}}_{13} \\ {}^{\mathcal{A}}\dot{C}^{\mathcal{B}}_{21} & {}^{\mathcal{A}}\dot{C}^{\mathcal{B}}_{22} & {}^{\mathcal{A}}\dot{C}^{\mathcal{B}}_{23} \\ {}^{\mathcal{A}}\dot{C}^{\mathcal{B}}_{31} & {}^{\mathcal{A}}\dot{C}^{\mathcal{B}}_{32} & {}^{\mathcal{A}}\dot{C}^{\mathcal{B}}_{33} \end{bmatrix}$$

$$\left.\begin{aligned} \omega_1 &= {}^{\mathcal{A}}\dot{C}^{\mathcal{B}}_{12}\,{}^{\mathcal{A}}C^{\mathcal{B}}_{13} + {}^{\mathcal{A}}\dot{C}^{\mathcal{B}}_{22}\,{}^{\mathcal{A}}C^{\mathcal{B}}_{23} + {}^{\mathcal{A}}\dot{C}^{\mathcal{B}}_{32}\,{}^{\mathcal{A}}C^{\mathcal{B}}_{33} \\ \omega_2 &= {}^{\mathcal{A}}\dot{C}^{\mathcal{B}}_{13}\,{}^{\mathcal{A}}C^{\mathcal{B}}_{11} + {}^{\mathcal{A}}\dot{C}^{\mathcal{B}}_{23}\,{}^{\mathcal{A}}C^{\mathcal{B}}_{21} + {}^{\mathcal{A}}\dot{C}^{\mathcal{B}}_{33}\,{}^{\mathcal{A}}C^{\mathcal{B}}_{31} \\ \omega_3 &= {}^{\mathcal{A}}\dot{C}^{\mathcal{B}}_{11}\,{}^{\mathcal{A}}C^{\mathcal{B}}_{12} + {}^{\mathcal{A}}\dot{C}^{\mathcal{B}}_{21}\,{}^{\mathcal{A}}C^{\mathcal{B}}_{22} + {}^{\mathcal{A}}\dot{C}^{\mathcal{B}}_{31}\,{}^{\mathcal{A}}C^{\mathcal{B}}_{32} \end{aligned}\right\} \omega_i = \frac{1}{2}\epsilon_{igh}(\epsilon_{igh}+1)\,{}^{\mathcal{A}}C^{\mathcal{B}}_{jh}\,{}^{\mathcal{A}}\dot{C}^{\mathcal{B}}_{jg}$$

Poisson's Kinematic Equations

$${}^{\mathcal{A}}\dot{C}^{\mathcal{B}}_{ij} = \epsilon_{ghj}\omega_h\,{}^{\mathcal{A}}C^{\mathcal{B}}_{ig}$$

The Angular Velocity Vector

$$\left.\begin{aligned} \mathbf{b}_i &= \sum_{j=1}^{3}(\mathbf{b}_i \cdot \mathbf{a}_j)\mathbf{a}_j \\ {}^{\mathcal{A}}\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{b}_i &= \sum_{j=1}^{3}\frac{\mathrm{d}}{\mathrm{d}t}\underbrace{(\mathbf{b}_i \cdot \mathbf{a}_j)}_{\equiv {}^{\mathcal{B}}C^{\mathcal{A}}_{ij}}\mathbf{a}_j \end{aligned}\right\} \omega_1 = {}^{\mathcal{A}}\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{b}_2 \cdot \mathbf{b}_3 \qquad \omega_2 = {}^{\mathcal{A}}\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{b}_3 \cdot \mathbf{b}_1 \qquad \omega_3 = {}^{\mathcal{A}}\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{b}_1 \cdot \mathbf{b}_2$$

$$\text{Recall:} \quad \begin{aligned} {}^{\mathcal{A}}C^{\mathcal{B}} &= I\cos\theta + \sin\theta\,[\hat{\mathbf{n}}_\times]_{\mathcal{B}} + (1-\cos\theta)\,[\hat{\mathbf{n}}]_{\mathcal{B}}\,[\hat{\mathbf{n}}]_{\mathcal{B}}^T \\ \omega_i &= \frac{1}{2}\epsilon_{igh}(\epsilon_{igh}+1){}^{\mathcal{A}}C^{\mathcal{B}}_{jh}\,{}^{\mathcal{A}}\dot{C}^{\mathcal{B}}_{jg} \end{aligned}$$

$$\Longrightarrow \left.\begin{aligned} \omega_1 &= n_1\dot{\theta} \\ \omega_2 &= n_2\dot{\theta} \\ \omega_3 &= n_3\dot{\theta} \end{aligned}\right\} \boxed{\begin{aligned} {}^{\mathcal{A}}\boldsymbol{\omega}^{\mathcal{B}} &\triangleq \omega_1\mathbf{b}_1 + \omega_2\mathbf{b}_2 + \omega_3\mathbf{b}_3 \\ &= \dot{\theta}\hat{\mathbf{n}} \end{aligned}}$$

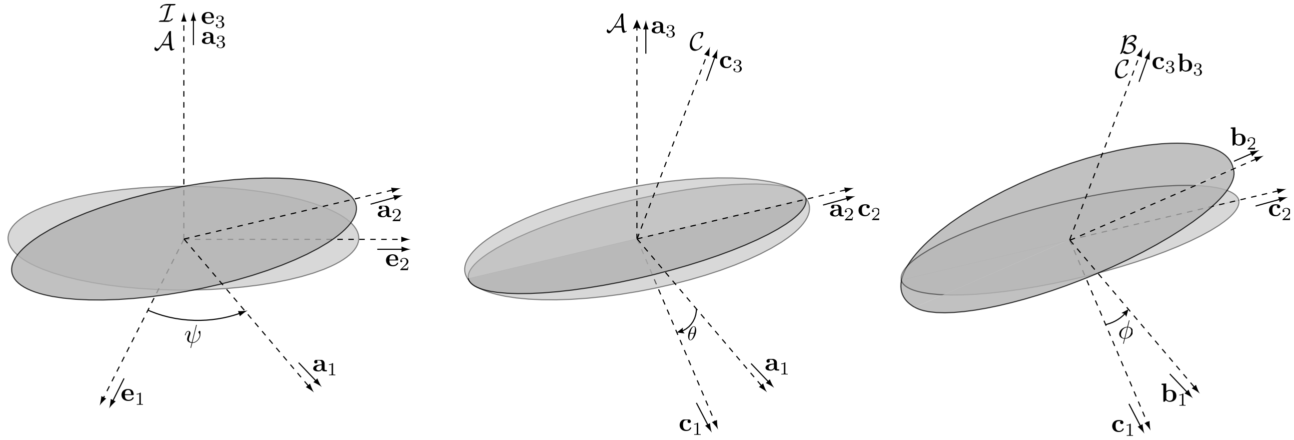
A Bit of Derivation

$$\mathbf{r} = [\mathbf{r}]_{\mathcal{A}}^T \begin{bmatrix}\mathbf{a}_1\\ \mathbf{a}_2\\ \mathbf{a}_3\end{bmatrix} = [\mathbf{r}]_{\mathcal{B}}^T \begin{bmatrix}\mathbf{b}_1\\ \mathbf{b}_2\\ \mathbf{b}_3\end{bmatrix} \qquad \begin{bmatrix}\mathbf{a}_1\\ \mathbf{a}_2\\ \mathbf{a}_3\end{bmatrix} = {}^{\mathcal{A}}C^{\mathcal{B}} \begin{bmatrix}\mathbf{b}_1\\ \mathbf{b}_2\\ \mathbf{b}_3\end{bmatrix} \qquad [\mathbf{r}]_{\mathcal{A}} = {}^{\mathcal{A}}C^{\mathcal{B}}\,[\mathbf{r}]_{\mathcal{B}}$$

Unit Vectors of Frame $\mathcal{A}$ Unit Vectors of Frame $\mathcal{B}$

$$\begin{aligned} {}^{\mathcal{A}}\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{r} = \frac{\mathrm{d}}{\mathrm{d}t}\left([\mathbf{r}]_{\mathcal{A}}^T\right)\begin{bmatrix}\mathbf{a}_1\\ \mathbf{a}_2\\ \mathbf{a}_3\end{bmatrix} &= \left(\frac{\mathrm{d}}{\mathrm{d}t}\left([\mathbf{r}]_{\mathcal{B}}^T\right){}^{\mathcal{B}}C^{\mathcal{A}} + [\mathbf{r}]_{\mathcal{B}}^T\,{}^{\mathcal{B}}\dot{C}^{\mathcal{A}}\right)\begin{bmatrix}\mathbf{a}_1\\ \mathbf{a}_2\\ \mathbf{a}_3\end{bmatrix} \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\left([\mathbf{r}]_{\mathcal{B}}^T\right)\begin{bmatrix}\mathbf{b}_1\\ \mathbf{b}_2\\ \mathbf{b}_3\end{bmatrix} + [\mathbf{r}]_{\mathcal{B}}^T\,\tilde{\omega}^T\begin{bmatrix}\mathbf{b}_1\\ \mathbf{b}_2\\ \mathbf{b}_3\end{bmatrix} \end{aligned}$$

Kinematics of the 3-2-3 $(\psi, \theta, \phi)^{\mathcal{I}}_{\mathcal{B}}$ rotation



$$\begin{aligned} {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}} &= \dot\psi \mathbf{a}_3 + \dot\theta \mathbf{c}_2 + \dot\phi \mathbf{b}_3 \\ &= \omega_1 \mathbf{b}_1 + \omega_2 \mathbf{b}_2 + \omega_3 \mathbf{b}_3 \end{aligned}$$

$$\begin{aligned} \omega_1 &= \dot\theta \sin\phi - \dot\psi \sin\theta\cos\phi \\ \omega_2 &= \dot\theta\cos\phi + \dot\psi\sin\theta\sin\phi \\ \omega_3 &= \dot\phi + \dot\psi\cos\theta \end{aligned}$$

$$\begin{aligned} \dot\psi &= (-\omega_1\cos\phi + \omega_2\sin\phi)\csc\theta \\ \dot\theta &= \omega_1\sin\phi + \omega_2\cos\phi \\ \dot\phi &= (\omega_1\cos\phi - \omega_2\sin\phi)\cot\theta + \omega_3 \end{aligned}$$

Kinematics of Euler Parameters (and Quaternions)

$${}^{\mathcal{A}}\boldsymbol{\omega}^{\mathcal{B}} = 2\left(\epsilon_4 \, {}^{\mathcal{B}}\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\epsilon} - \dot\epsilon_4 \boldsymbol{\epsilon} - \boldsymbol{\epsilon} \times {}^{\mathcal{B}}\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\epsilon}\right)$$

$${}^{\mathcal{B}}\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\epsilon} = \frac{1}{2}\left(\epsilon_4 {}^{\mathcal{A}}\boldsymbol{\omega}^{\mathcal{B}} + \boldsymbol{\epsilon} \times {}^{\mathcal{A}}\boldsymbol{\omega}^{\mathcal{B}}\right) \qquad \dot\epsilon_4 = -\frac{1}{2}{}^{\mathcal{A}}\boldsymbol{\omega}^{\mathcal{B}} \cdot \boldsymbol{\epsilon}$$

$$\begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \\ 0 \end{bmatrix} = 2 \underbrace{\begin{bmatrix} \epsilon_4 & \epsilon_3 & -\epsilon_2 & -\epsilon_1 \\ -\epsilon_3 & \epsilon_4 & \epsilon_1 & -\epsilon_2 \\ \epsilon_2 & -\epsilon_1 & \epsilon_4 & -\epsilon_3 \\ \epsilon_1 & \epsilon_2 & \epsilon_3 & \epsilon_4 \end{bmatrix}}_{\text{Inverse} \equiv \text{Transpose}} \begin{bmatrix} \dot\epsilon_1 \\ \dot\epsilon_2 \\ \dot\epsilon_3 \\ \dot\epsilon_4 \end{bmatrix}$$

$${}^{\mathcal{B}}\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{q} = \frac{1}{2}\mathbf{q} \odot \begin{bmatrix} {}^{\mathcal{A}}\boldsymbol{\omega}^{\mathcal{B}} \\ 0 \end{bmatrix}$$

Kinematics of Rodrigues Parameters

$$
\begin{aligned}
{}^{\mathcal{A}}\boldsymbol{\omega}^{\mathcal{B}} &= \frac{2}{1+\boldsymbol{\rho}\cdot\boldsymbol{\rho}}\left(\frac{{}^{\mathcal{B}}\mathrm{d}}{\mathrm{d}t}\boldsymbol{\rho} - \boldsymbol{\rho}\times\frac{{}^{\mathcal{B}}\mathrm{d}}{\mathrm{d}t}\boldsymbol{\rho}\right) \\
\frac{{}^{\mathcal{B}}\mathrm{d}}{\mathrm{d}t}\boldsymbol{\rho} &= \frac{1}{2}\left({}^{\mathcal{A}}\boldsymbol{\omega}^{\mathcal{B}} + \boldsymbol{\rho}\times{}^{\mathcal{A}}\boldsymbol{\omega}^{\mathcal{B}} + \boldsymbol{\rho}\otimes\boldsymbol{\rho}\cdot{}^{\mathcal{A}}\boldsymbol{\omega}^{\mathcal{B}}\right)
\end{aligned}
$$