21 - Euler's Laws and Torque Free Motion

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Euler's Laws

I. The product of the inertial acceleration of the center of mass of a rigid body and its total mass is equal to the total external force applied to the body

$$\mathbf{F}_G = m_G {}^{\mathcal{I}}\mathbf{a}_{G/O}$$

II. The rate of change of the inertial angular momentum of a rigid body about a fixed point O in the inertial frame is equal to the total external moment applied to the body about O

$$\frac{{}^{\mathcal{I}}\mathrm{d}}{\mathrm{d}t} {}^{\mathcal{I}}\mathbf{h}_O = \mathbf{M}_O$$

Center of Mass (A Quick Reminder)

$$\mathbf{r}_{G/O} = \frac{1}{m_G} \sum_{i=1}^{N} \mathbf{r}_{i/O} m_i$$

- As $N \to \infty$: $m_i \to 0$

$$\mathbf{r}_{G/O} = \frac{1}{m_G} \int_{\mathscr{B}} \mathbf{r}_{\,\mathrm{d}m/O} \, \mathrm{d}m$$

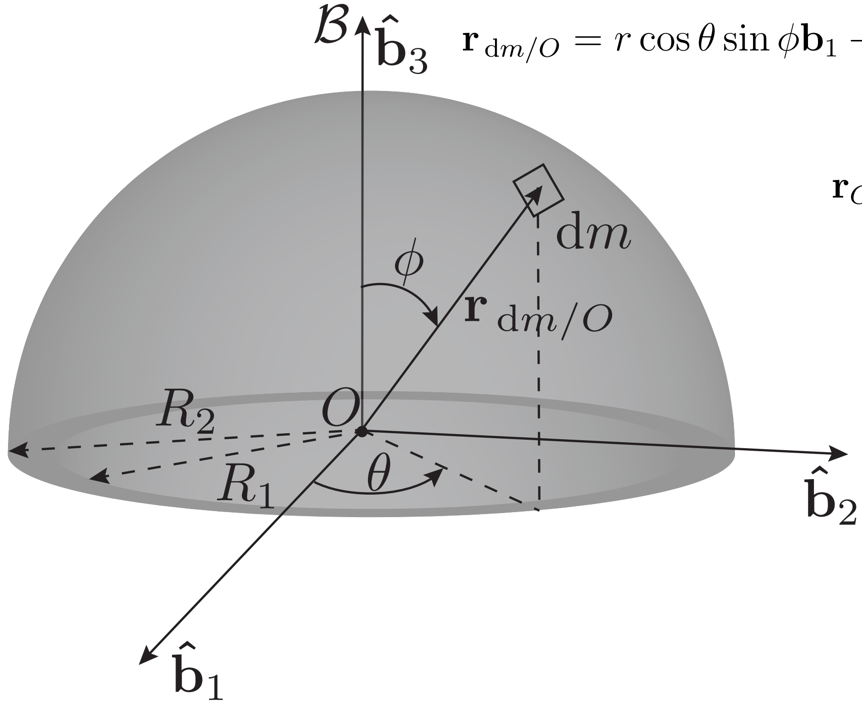
- If the density is given by $\rho(\mathbf{r}_{\,\mathrm{d}m/O})$:

$$\mathbf{r}_{G/O} = \frac{1}{m_G} \int_{\mathscr{B}} \mathbf{r}_{\,\mathrm{d}m/O} \rho(\mathbf{r}_{\,\mathrm{d}m/O}) \, \mathrm{d}V$$

- The center of mass corollary becomes:

$$\int_{\mathscr{B}} \mathbf{r}_{\,\mathrm{d}m/G} \rho(\mathbf{r}_{\,\mathrm{d}m/G}) \, \mathrm{d}V = 0$$

Example: Center of Mass of a Hemispherical Shell



$$\mathbf{r}_{\,\mathrm{d}m/O} = r\cos\theta\sin\phi\mathbf{b}_1 + r\sin\theta\sin\phi\mathbf{b}_2 + r\cos\phi\mathbf{b}_3$$

$$\mathbf{r}_{G/O} = \frac{\rho}{m_G} \iiint_{\mathscr{B}} \mathbf{r}_{\,\mathrm{d}m/O} \, \mathrm{d}V$$

$$m_G = \rho V = \rho \iiint_{\mathscr{B}} dV$$

Euler's Second Law, Expanded

$$^{\mathcal{I}}\mathbf{h}_O = \sum_{i=1}^{N} m_i \mathbf{r}_{i/O} \times {}^{\mathcal{I}}\mathbf{v}_{i/O} \Rightarrow {}^{\mathcal{I}}\mathbf{h}_O = \int_{\mathscr{B}} \mathbf{r}_{dm/O} \times {}^{\mathcal{I}}\mathbf{v}_{dm/O}\, \mathrm{d}m$$

$$\frac{^{\mathcal{I}}\mathrm{d}}{\mathrm{d}t}\left({}^{\mathcal{I}}\mathbf{h}_O\right) = \underbrace{\sum_{i=1}^{N} \mathbf{r}_{i/O} \times \mathbf{F}_i^{(\mathrm{ext})}}_{=\mathbf{M}_O^{(\mathrm{ext})}} + \underbrace{\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} (\mathbf{r}_{i/O} - \mathbf{r}_{j/O}) \times \mathbf{F}_{i,j}}_{\text{Must equal zero for rigid bodies}}$$

$$^{\mathcal{I}}\mathbf{h}_O = \underbrace{{}^{\mathcal{I}}\mathbf{h}_{G/O}}\, + \underbrace{{}^{\mathcal{I}}\mathbf{h}_G}$$

$$^{\mathcal{I}}\mathbf{h}_{G/O} \triangleq m_G \mathbf{r}_{G/O} \times {}^{\mathcal{I}}\mathbf{v}_{G/O}$$

$$^{\mathcal{I}}\mathbf{h}_G \triangleq \begin{cases} \sum_{i=1}^{N} m_i \mathbf{r}_{i/G} \times {}^{\mathcal{I}}\mathbf{v}_{i/G} & \text{Particles} \\ \int_{\mathscr{B}} \mathbf{r}_{dm/G} \times {}^{\mathcal{I}}\mathbf{v}_{dm/G}\, \mathrm{d}m & \text{Continuous Bodies} \end{cases}$$

$$\frac{^{\mathcal{I}}\mathrm{d}}{\mathrm{d}t}\left({}^{\mathcal{I}}\mathbf{h}_{G/O}\right) = \underbrace{\mathbf{M}_{G/O}}_{\triangleq\, \mathbf{r}_{G/O} \times \mathbf{F}_G}$$

$$\frac{^{\mathcal{I}}\mathrm{d}}{\mathrm{d}t}\left({}^{\mathcal{I}}\mathbf{h}_G\right) = \mathbf{M}_G \triangleq \begin{cases} \sum_{i=1}^{N} \mathbf{r}_{i/G} \times \mathbf{F}_i^{(\mathrm{ext})} & \text{Contact Forces} \\ \int_{\mathscr{B}} \mathbf{r}_{dm/G} \times \mathbf{f}_{dm}\, \mathrm{d}m & \text{Field Forces} \end{cases}$$

Moment of Inertia

$$\boxed{{}^{\mathcal{I}}\mathbf{h}_G = \mathbb{I}_G \cdot {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}}}$$

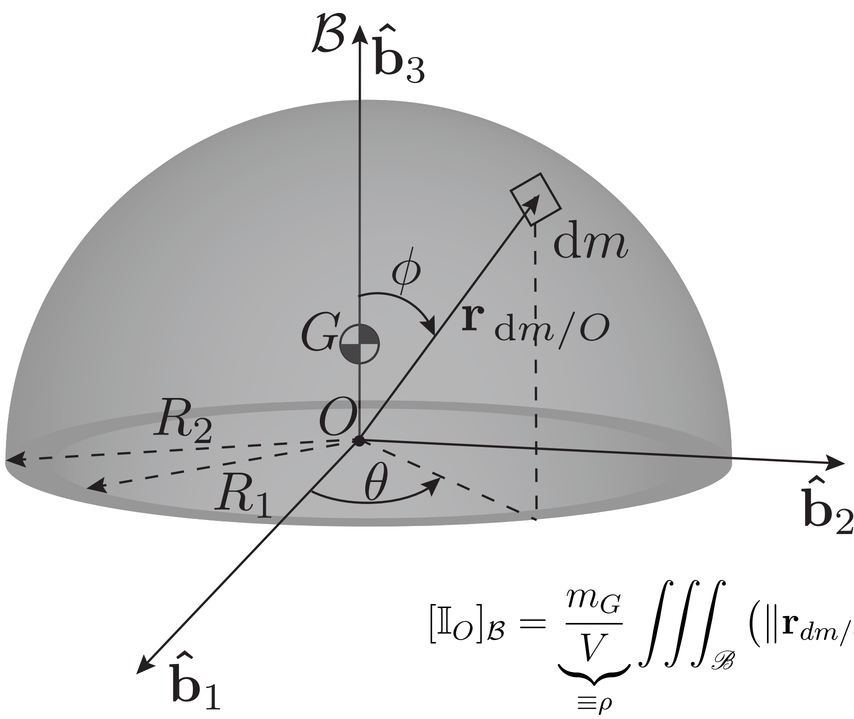
$$\mathbb{I}_G \triangleq \begin{cases} \sum_{i=1}^{N} m_i \left[(\mathbf{r}_{i/G} \cdot \mathbf{r}_{i/G})\mathbb{U} - (\mathbf{r}_{i/G} \otimes \mathbf{r}_{i/G})\right] & \text{Collection of Particles} \\ \int_{\mathscr{B}} \left[(\mathbf{r}_{dm/G} \cdot \mathbf{r}_{dm/G})\mathbb{U} - (\mathbf{r}_{dm/G} \otimes \mathbf{r}_{dm/G})\right] \mathrm{d}m & \text{Rigid Body} \end{cases}$$

Matrix of Inertia

$$\mathbb{I}_G = \sum_{i=1}^{3} \sum_{j=1}^{3} I_{ij} \mathbf{b}_i \otimes \mathbf{b}_j$$

$$\begin{aligned}
[\mathbb{I}_G]_\mathcal{B} &= \begin{bmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{bmatrix}_\mathcal{B} \\
&= \sum_{i=1}^{N} m_i \left(([\mathbf{r}_{i/G}]_\mathcal{B}^T [\mathbf{r}_{i/G}]_\mathcal{B}) I - [\mathbf{r}_{i/G}]_\mathcal{B} [\mathbf{r}_{i/G}]_\mathcal{B}^T \right) \\
&= \int_\mathscr{B} \left(\|\mathbf{r}_{dm/G}\|^2 I - [\mathbf{r}_{dm/G}]_\mathcal{B} [\mathbf{r}_{dm/G}]_\mathcal{B}^T \right) \, \mathrm{d}m \\
&= \int_\mathscr{B} \left(\|\mathbf{r}_{dm/G}\|^2 I - [\mathbf{r}_{dm/G}]_\mathcal{B} [\mathbf{r}_{dm/G}]_\mathcal{B}^T \right) \rho(\mathbf{r}_{dm/G}) \, \mathrm{d}V
\end{aligned}$$

Example: Moment of Inertia of a Hemispherical Shell



$$\mathbf{r}_{G/O} = \frac{3\,(R_1^4 - R_2^4)}{8\,(R_1^3 - R_2^3)} \mathbf{b}_3$$

$$[\mathbb{I}_O]_\mathcal{B} = \underbrace{\frac{m_G}{V}}_{\equiv \rho} \iiint_\mathscr{B} \left(\|\mathbf{r}_{dm/O}\|^2 I - [\mathbf{r}_{dm/O}]_\mathcal{B} [\mathbf{r}_{dm/O}]_\mathcal{B}^T \right) \, \mathrm{d}V$$

Moments and Angular Momentum about an Arbitrary Point on a Rigid Body

$$\boxed{\mathbf{M}_Q = \mathbf{M}_G - \mathbf{r}_{Q/G} \times \sum_{i=1}^{N} \mathbf{F}_i^{(\mathrm{ext})}}$$

$$\frac{{}^{\mathcal{I}}\mathrm{d}}{\mathrm{d}t}\left({}^{\mathcal{I}}\mathbf{h}_Q\right) = \frac{{}^{\mathcal{B}}\mathrm{d}}{\mathrm{d}t}\left({}^{\mathcal{I}}\mathbf{h}_Q\right) + {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}} \times {}^{\mathcal{I}}\mathbf{h}_Q = \mathbf{M}_Q + \mathbf{r}_{Q/G} \times m_G {}^{\mathcal{I}}\mathbf{a}_{Q/O}$$

$${}^{\mathcal{I}}\mathbf{h}_Q = \mathbb{I}_Q \cdot {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}} \qquad \mathbb{I}_Q \triangleq \sum_{i=1}^{N} m_i \left((\mathbf{r}_{i/Q} \cdot \mathbf{r}_{i/Q}))\mathbb{U} - (\mathbf{r}_{i/Q} \otimes \mathbf{r}_{i/Q})\right) \cdot {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}}$$

The Parallel Axis Theorem

$$\mathbb{I}_Q = \mathbb{I}_G + m_G \left[(\mathbf{r}_{Q/G} \cdot \mathbf{r}_{Q/G})\mathbb{U} - (\mathbf{r}_{Q/G} \otimes \mathbf{r}_{Q/G})\right]$$

Rigid Body Dynamics

$$\frac{{}^{\mathcal{I}}\mathrm{d}}{\mathrm{d}t}{}^{\mathcal{I}}\mathbf{h}_G = \mathbb{I}_G \cdot \frac{{}^{\mathcal{B}}\mathrm{d}}{\mathrm{d}t}{}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}} + {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}} \times \left(\mathbb{I}_G \cdot {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}}\right) = \mathbf{M}_G$$

$$[\mathbb{I}_G]_{\mathcal{B}} \left[\frac{{}^{\mathcal{B}}\mathrm{d}}{\mathrm{d}t}{}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}}\right]_{\mathcal{B}} + \left[{}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}}\times\right]_{\mathcal{B}} [\mathbb{I}_G]_{\mathcal{B}} \left[{}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}}\right]_{\mathcal{B}} = [\mathbf{M}_G]_{\mathcal{B}}$$

Diagonalization

$$\text{For any tensor } \mathbb{T}: \quad [\mathbb{T}]_{\mathcal{B}} = {}^{\mathcal{B}}C^{\mathcal{A}}\,[\mathbb{T}]_{\mathcal{A}}\,\underbrace{{}^{\mathcal{A}}C^{\mathcal{B}}}_{\equiv\left({}^{\mathcal{B}}C^{\mathcal{A}}\right)^T}$$

Recall the eigendecomposition of a symmetric matrix

$$A = A^T \quad \Rightarrow \quad A\mathbf{v}_i = \lambda_i \mathbf{v}_i \quad \Rightarrow \quad \lambda_i \in \mathbb{R}\ \forall i, \quad \mathbf{v}_i \cdot \mathbf{v}_j = 0\ \forall i, j$$

$$\text{Let } P \triangleq \begin{bmatrix}\mathbf{v}_i & \mathbf{v}_2 & \cdots & \mathbf{v}_n\end{bmatrix} \quad \Rightarrow \quad A = PDP^{-1}, \quad P^{-1} = P^T$$

$$D = \mathrm{diag}\left(\{\lambda_i\}\right) = P^T A P$$

$$\boxed{[\mathbb{I}_G]_{\mathcal{B}_P} = {}^{\mathcal{B}_\mathcal{P}}C^{\mathcal{A}}\,[\mathbb{I}_G]_{\mathcal{A}}\,{}^{\mathcal{A}}C^{\mathcal{B}_\mathcal{P}}}$$

Eigenvectors of $[\mathbb{I}_G]_{\mathcal{A}}$

Principle Axis Frame ($\mathcal{B}_p$) and Euler's Equations

$$[\mathbb{I}_G]_{\mathcal{B}_p} = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix}_{\mathcal{B}_p}$$

$$[\mathbb{I}_G]_{\mathcal{B}_p}\left[\frac{{}^{\mathcal{B}}\mathrm{d}}{\mathrm{d}t}{}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}}\right]_{\mathcal{B}_p} + \left[{}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}}\times\right]_{\mathcal{B}_p}[\mathbb{I}_G]_{\mathcal{B}_p}\left[{}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}}\right]_{\mathcal{B}_p} = [\mathbf{M}_G]_{\mathcal{B}_p} \quad \Longrightarrow$$

$$\begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix}_{\mathcal{B}_p}\begin{bmatrix} \dot\omega_1 \\ \dot\omega_2 \\ \dot\omega_3 \end{bmatrix}_{\mathcal{B}_p} + \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}_{\mathcal{B}_p}\begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix}_{\mathcal{B}_p}\begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}_{\mathcal{B}_p} = \begin{bmatrix} M_1 \\ M_2 \\ M_3 \end{bmatrix}_{\mathcal{B}_p}$$

Euler's Equations

$$\begin{aligned} I_1\dot\omega_1 + (I_3 - I_2)\omega_2\omega_3 &= M_1 \\ I_2\dot\omega_2 + (I_1 - I_3)\omega_1\omega_3 &= M_2 \\ I_3\dot\omega_3 + (I_2 - I_1)\omega_1\omega_2 &= M_3 \end{aligned}$$

Torque Free Motion

$$
\begin{aligned}
I_1\dot{\omega}_1 + (I_3 - I_2)\omega_2\omega_3 &= 0 \\
I_2\dot{\omega}_2 + (I_1 - I_3)\omega_1\omega_3 &= 0 \\
I_3\dot{\omega}_3 + (I_2 - I_1)\omega_1\omega_2 &= 0
\end{aligned}
$$

$$
{}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}} = \omega_1\mathbf{b}_1 + \omega_2\mathbf{b}_2 + \omega_3\mathbf{b}_3
$$

$$
\underline{I_1 = I_2}
$$

$$
\begin{aligned}
I_1\dot{\omega}_1 + (I_3 - I_2)\omega_2\omega_3 &= 0 \\
I_2\dot{\omega}_2 + (I_1 - I_3)\omega_1\omega_3 &= 0 \\
I_3\dot{\omega}_3 &= 0
\end{aligned}
$$

$$
\underline{\omega_3 \gg \omega_1, \omega_2}
$$

$$
\begin{aligned}
I_1\dot{\omega}_1 + (I_3 - I_2)\omega_2\omega_3 &= 0 \\
I_2\dot{\omega}_2 + (I_1 - I_3)\omega_1\omega_3 &= 0 \\
I_3\dot{\omega}_3 &\approx 0
\end{aligned}
$$

$$
\left.\begin{aligned}
\dot{\omega}_1 &= -\frac{I_3 - I_2}{I_1}\omega_2\omega_3 \\
\dot{\omega}_2 &= -\frac{I_1 - I_3}{I_2}\omega_1\omega_3
\end{aligned}\right\}
\left.\begin{aligned}
\ddot{\omega}_1 &= -\frac{I_3 - I_2}{I_1}\dot{\omega}_2\omega_3 \\
\ddot{\omega}_2 &= -\frac{I_1 - I_3}{I_2}\dot{\omega}_1\omega_3
\end{aligned}\right\}
\boxed{\begin{gathered}
\ddot{\omega}_1 + \omega_n^2\omega_1 = 0 \\
\ddot{\omega}_2 + \omega_n^2\omega_2 = 0 \\
\omega_n^2 \triangleq \frac{(I_3 - I_2)(I_3 - I_1)\omega_3^2}{I_1 I_2}
\end{gathered}}
$$