3 - The Perifocal Frame, Kepler's Equations and Conic Sections

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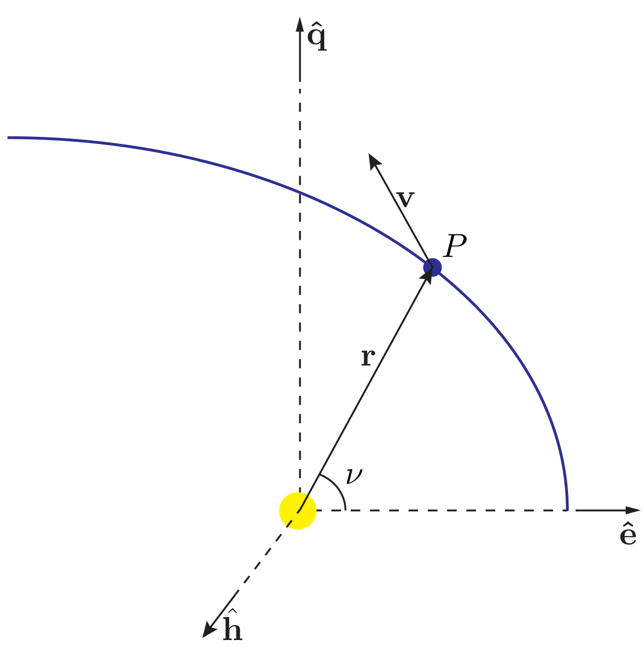
The Perifocal Frame



$$\mathbf{r} = r\cos(\nu)\hat{\mathbf{e}} + r\sin(\nu)\hat{\mathbf{q}}$$

$$\begin{aligned}\mathbf{v} = {}^{\mathcal{I}}\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{r} &= \left(\dot{r}\cos(\nu) - r\dot{\nu}\sin(\nu)\right)\hat{\mathbf{e}} \\ &\quad + \left(\dot{r}\sin(\nu) + r\dot{\nu}\cos(\nu)\right)\hat{\mathbf{q}} \\ &= \frac{\mu}{h}\left(-\sin(\nu)\hat{\mathbf{e}} + (e + \cos(\nu))\hat{\mathbf{q}}\right)\end{aligned}$$

$$r = \|\mathbf{r}\| = \frac{h^2/\mu}{1 + e\cos(\nu)}$$

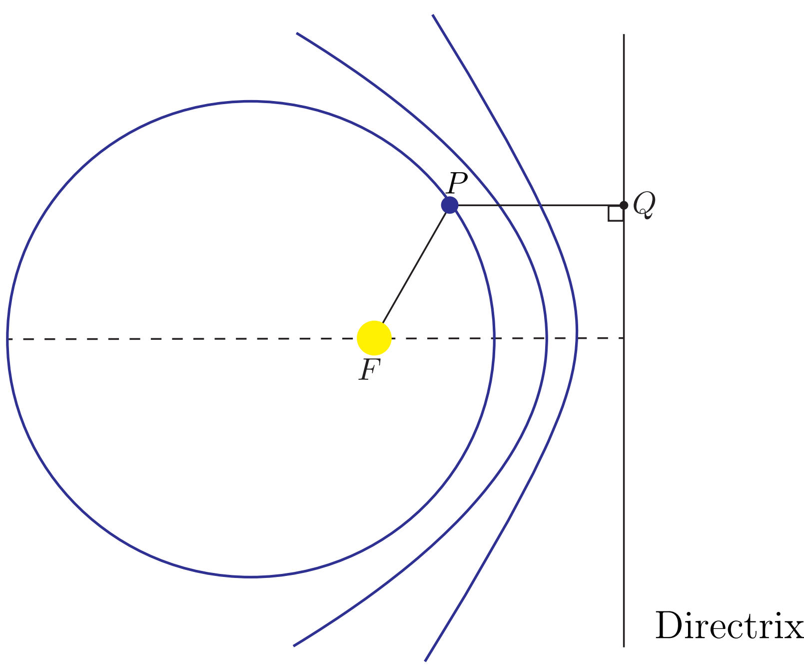
$$\left.\begin{aligned} e &= \|\mathbf{e}\| = \left\| \frac{\mathbf{v}\times\mathbf{h}}{\mu} - \frac{\mathbf{r}}{r} \right\| \\ h &= \|\mathbf{h}\| = \|\mathbf{r}\times\mathbf{v}\| \end{aligned}\right\}\text{Constants}$$

$$\boxed{h = r^2\dot{\nu}}$$

Kepler's Laws of Planetary Motion

1. The orbit of a planet is an ellipse (conic section) with the Sun at a focus
2. A line segment joining a planet and the Sun sweeps out equal areas in equal time
3. The square of the orbital period is proportional to the cube of the semi-major axis

First Law

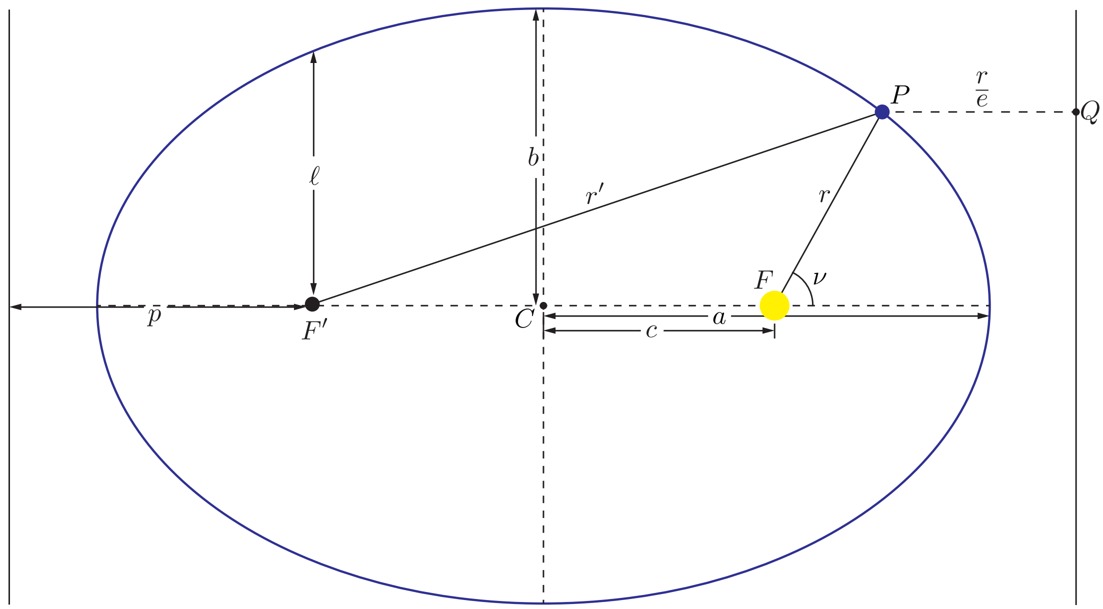


Two body orbits are conic sections with the central body at a focus

$$\overline{FP} = e\overline{PQ}$$

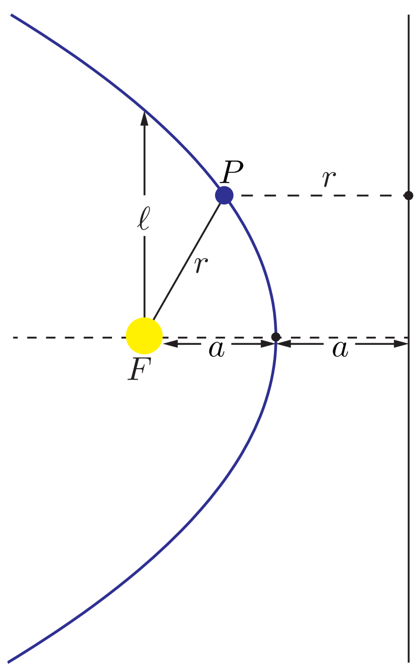
Ellipse (Circle)	$0 < e < 1$
Parabola	$e = 1$
Hyperbola	$e > 0$

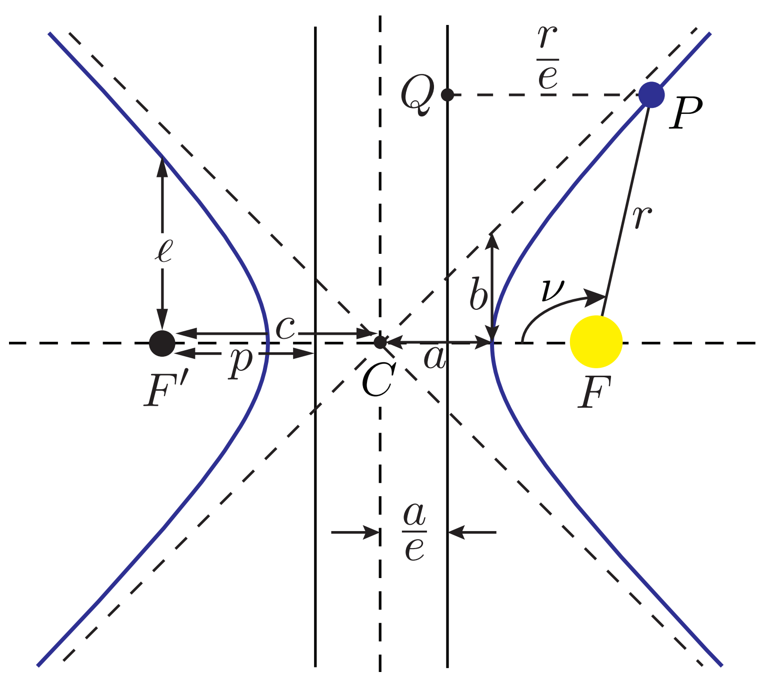
Elliptical Orbits



$$r' + r = 2a$$

Parabolic and Hyperbolic Orbits





Conic Section Parameters

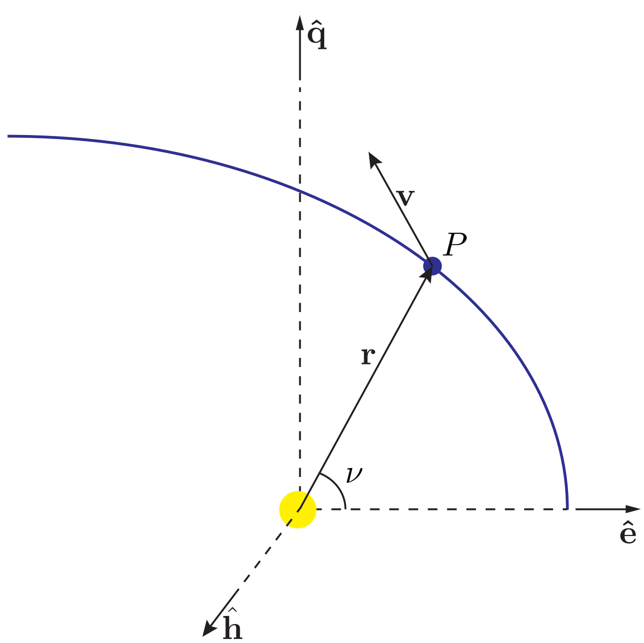
$$\ell = r(\nu = \pi/2) = \text{semi-parameter: height above focus}$$
$$c = ae = \text{linear eccentricity: distance from center to focus}$$
$$p = \ell/e \text{ focal parameter: distance from focus to vertex}$$

NB: p and ℓ frequently have reversed definitions, depending on the text.

	Definition	e	c	ℓ	p
circle	$x^2 + y^2 = a^2$	0	0	a	∞
ellipse	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	$\sqrt{1 - \frac{b^2}{a^2}}$	$\sqrt{a^2 - b^2}$	$\frac{b^2}{a}$	$\frac{b^2}{\sqrt{a^2 - b^2}}$
parabola	$y^2 = 4ax$	1	∞	$2a$	$2a^*$
hyperbola	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	$\sqrt{1 + \frac{b^2}{a^2}}$	$\sqrt{a^2 + b^2}$	$\frac{b^2}{\lvert a\rvert}$	$\frac{b^2}{\sqrt{a^2 + b^2}}$

$^*a$ is the focus to vertex distance for a parabola

Second and Third Laws



$$r = \|\mathbf{r}\| = \frac{h^2/\mu}{1 + e\cos(\nu)}$$

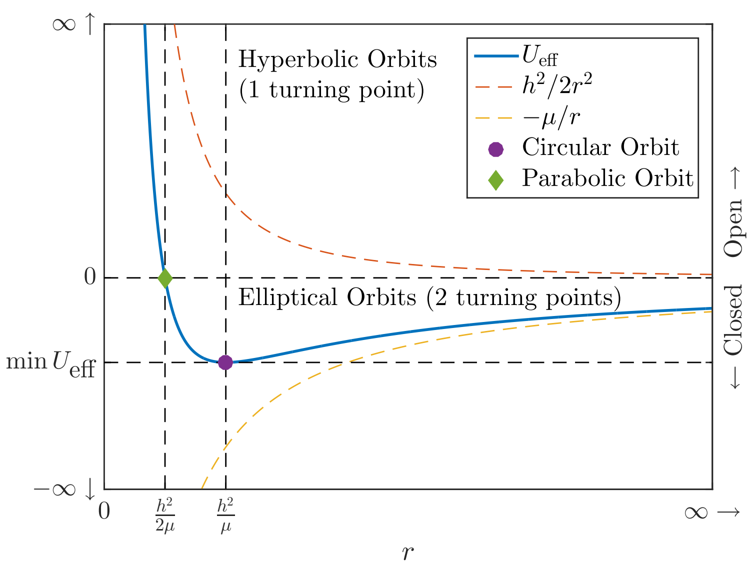
$$\left.\begin{aligned} e &= \|\mathbf{e}\| = \left\| \frac{\mathbf{v} \times \mathbf{h}}{\mu} - \frac{\mathbf{r}}{r} \right\| \\ h &= \|\mathbf{h}\| = \|\mathbf{r} \times \mathbf{v}\| \end{aligned}\right\} \text{Constants}$$

$$\Rightarrow \quad h = r^2\dot{\nu}$$

$$\boxed{\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{h}{2}}$$

$$\boxed{T_p = \frac{2\pi}{\sqrt{\mu}} a^{\frac{3}{2}}}$$

Energy



$$\mathcal{E} = \frac{v^2}{2} - \frac{\mu}{r} = \text{ constant}$$

$$\mathcal{E} = -\frac{\mu}{2a}$$

The Vis-Viva Equation

$$v^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right)$$

$$\mathcal{E} = \frac{\dot{r}^2}{2} + \underbrace{U(r) + \frac{h^2}{2r^2}}_{\triangleq U_{\text{eff}}}$$