22 - Energy Dissipation and General Rigid Body Dynamics

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New Poinsot Stuff

Euler's Equations (A Quick Reminder)

$$\frac{{}^{\mathcal{I}}\mathrm{d}}{\mathrm{d}t}{}^{\mathcal{I}}\mathbf{h}_G = \mathbb{I}_G \cdot \frac{{}^{\mathcal{B}}\mathrm{d}}{\mathrm{d}t}{}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}} + {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}} \times \left(\mathbb{I}_G \cdot {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}}\right) = \mathbf{M}_G$$

In body frame coordinates:

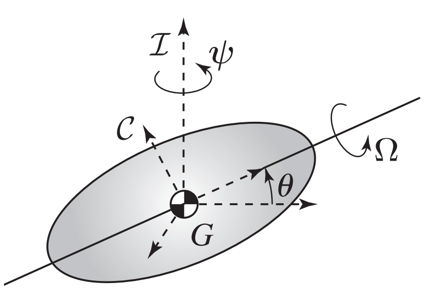
$$[\mathbb{I}_G]_{\mathcal{B}} \left[\frac{{}^{\mathcal{B}}\mathrm{d}}{\mathrm{d}t}{}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}}\right]_{\mathcal{B}} + \left[{}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}}\times\right]_{\mathcal{B}} [\mathbb{I}_G]_{\mathcal{B}} \left[{}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}}\right]_{\mathcal{B}} = [\mathbf{M}_G]_{\mathcal{B}}$$

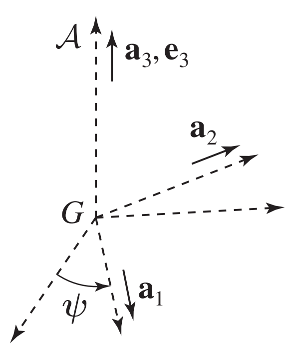
In a principal axis frame:

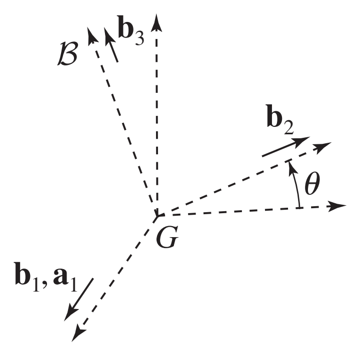
$$\begin{aligned} I_1\dot{\omega}_1 + (I_3 - I_2)\omega_2\omega_3 &= M_1 \\ I_2\dot{\omega}_2 + (I_1 - I_3)\omega_1\omega_3 &= M_2 \\ I_3\dot{\omega}_3 + (I_2 - I_1)\omega_1\omega_2 &= M_3 \end{aligned}$$

Spinning Symmetric Rigid Body (The Setup)

3-1-2 $(\psi, \theta, \phi)^{\mathcal{I}}_{\mathcal{B}}$ Body-3 Rotation:







$${}^{\mathcal{B}}C^{\mathcal{A}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & \sin(\theta) \\ 0 & -\sin(\theta) & \cos(\theta) \end{bmatrix}$$

$$\begin{aligned} {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}} &= \dot{\psi}\mathbf{a}_3 + \dot{\theta}\mathbf{b}_1 \\ &= \dot{\theta}\mathbf{b}_1 + \dot{\psi}\sin\theta\mathbf{b}_2 + \dot{\psi}\cos\theta\mathbf{b}_3 \\ {}^{\mathcal{B}}\boldsymbol{\omega}^{\mathcal{C}} &= \Omega\mathbf{b}_2 \end{aligned}$$

Spinning Symmetric Rigid Body (The Dynamics)

$$\left[{}^{\mathcal{I}}\mathbf{h}_G\right]_{\mathcal{B}} = \left[\mathbb{I}_G\right]_{\mathcal{B}} \left[{}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{C}}\right]_{\mathcal{B}} = \begin{bmatrix} I_1\dot{\theta} \\ I_2\left(\Omega + \dot{\psi}\sin(\theta)\right) \\ I_1\dot{\psi}\cos(\theta) \end{bmatrix}_{\mathcal{B}}$$

NB: This is **not** a typo

$$\left[\frac{{}^{\mathcal{I}}\mathrm{d}}{\mathrm{d}t}{}^{\mathcal{I}}\mathbf{h}_G\right]_{\mathcal{B}} = \left[\frac{{}^{\mathcal{B}}\mathrm{d}}{\mathrm{d}t}{}^{\mathcal{I}}\mathbf{h}_G\right]_{\mathcal{B}} + \left[{}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}}\times\right]_{\mathcal{B}} \left[{}^{\mathcal{I}}\mathbf{h}_G\right]_{\mathcal{B}}$$

$$= \begin{bmatrix} I_1\dot{\psi}^2\sin(\theta)\cos(\theta) + I_1\ddot{\theta} - I_2\dot{\psi}\left(\Omega + \dot{\psi}\sin(\theta)\right)\cos(\theta) \\ I_2\left(\ddot{\psi}\sin(\theta) + \dot{\psi}\dot{\theta}\cos(\theta)\right) \\ I_1\ddot{\psi}\cos(\theta) - 2I_1\dot{\psi}\dot{\theta}\sin(\theta) + I_2\dot{\theta}\left(\Omega + \dot{\psi}\sin(\theta)\right) \end{bmatrix}_{\mathcal{B}}$$

Spinning Symmetric Rigid Body (The Solution)

Note: $({}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}} \times {}^{\mathcal{I}}\mathbf{h}_G) \cdot \mathbf{b}_2 = 0$

Therefore: ${}^{\mathcal{B}}\mathbf{h}_G \cdot \mathbf{b}_2 = 0 \quad \Longrightarrow \quad I_2\frac{\mathrm{d}}{\mathrm{d}t}\left(\Omega + \dot{\psi}\sin\theta\right) = 0$

$$\Longrightarrow \quad \Omega + \dot{\psi}\sin\theta = C\ (\text{constant})$$

Assume: $\mathbf{M}_G = -M_1\mathbf{b}_1$. Then:

$$\boxed{\begin{aligned} \ddot{\theta} &= \frac{1}{I_1}\left(CI_2\dot{\psi}\cos(\theta) - \frac{I_1\dot{\psi}^2}{2}\sin(2\theta) - M_1\right) \\ \ddot{\psi} &= \frac{\dot{\theta}}{I_1}\left(-\frac{CI_2}{\cos(\theta)} + 2I_1\dot{\psi}\tan(\theta)\right) \end{aligned}}$$