8 - Earth Shape and Earth Orbit Perturbations

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First Approximation: The Reference Geoid

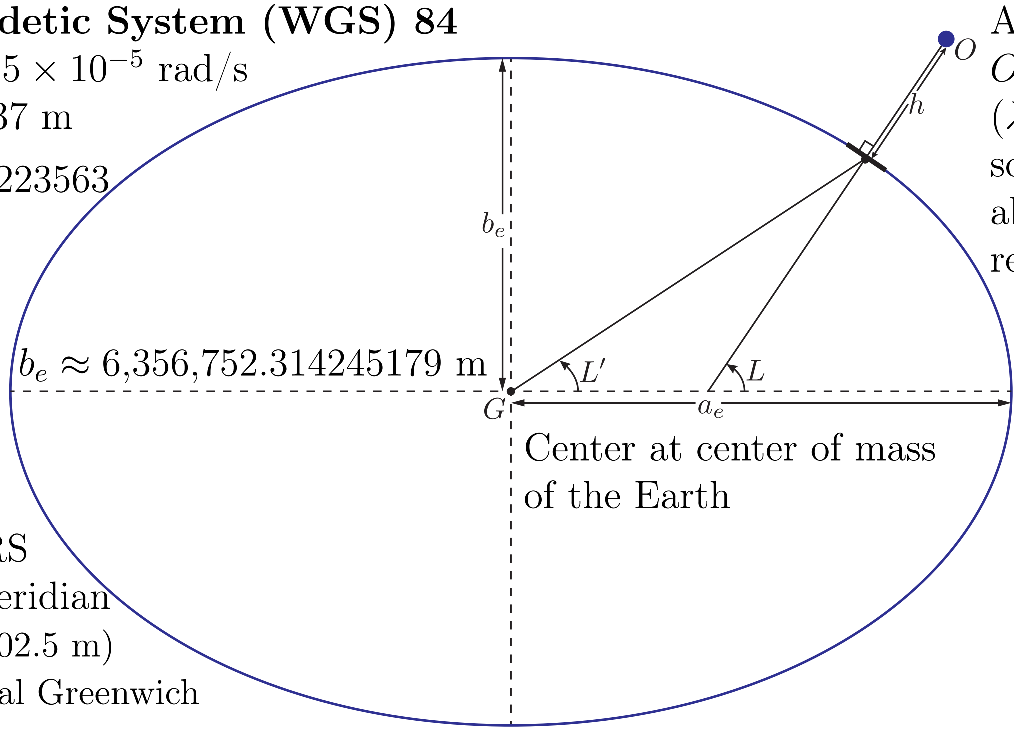
World Geodetic System (WGS) 84

$\omega_e = 7.292115 \times 10^{-5}$ rad/s

$a_e = 6{,}378{,}137$ m

$\frac{1}{f} = 298.257223563$

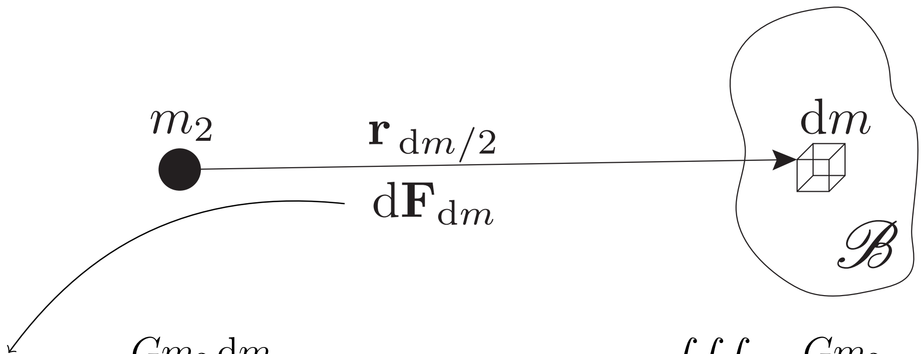
$f \triangleq \frac{a-b}{a}$



A surface point O at Lon/Lat (λ, L) is at some height h above the reference geoid

$\lambda = 0$ at IERS Reference Meridian $\sim$5.3 arcsec(102.5 m) East of original Greenwich

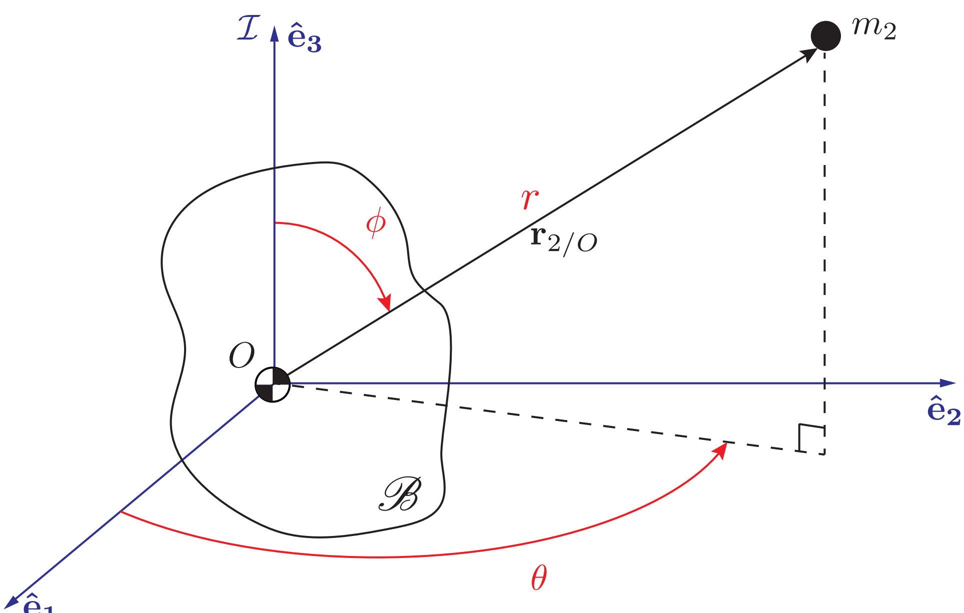
Orbiting About Extended Bodies (forces)



$$\mathrm{d}\mathbf{F}_{\mathrm{d}m} = \frac{G m_2 \,\mathrm{d}m}{\|\mathbf{r}_{\mathrm{d}m/2}\|^3}\mathbf{r}_{\mathrm{d}m/2} \qquad \Rightarrow \qquad \mathbf{F} = \iiint_{\mathscr{B}} \frac{G m_2}{\|\mathbf{r}_{\mathrm{d}m/2}\|^3}\mathbf{r}_{\mathrm{d}m/2}\,\mathrm{d}m$$

$$\mathbf{F} = m_2 \nabla U \qquad\qquad U = G \iiint_{\mathscr{B}} \frac{\mathrm{d}m}{\|\mathbf{r}_{\,\mathrm{d}m/2}\|}$$

Orbiting About Extended Bodies (coordinates)



Careful! θ, ϕ are often reversed in other texts

Potential of an Azimuthally Symmetric Body

Body Total Mass　　　Body Equatorial Radius

$$U(r,\phi) = \frac{G\ m_{\mathscr{B}}}{r}\left[1 - \sum_{k=2}^{\infty} J_k \left(\frac{R_{\mathscr{B}}}{r}\right)^k P_k(\cos\phi)\right]$$

Non-dimensional coefficients named after Harold Jeffreys. $J_1 = 0$ due to symmetry

Legendre Polynomials

Solutions to Legendre's differential equation:

$$0 = \frac{\mathrm{d}}{\mathrm{d}x}\left(\left(1-x^2\right)\frac{\mathrm{d}}{\mathrm{d}x}P_n(x)\right) + \left(n^2+n\right)P_n(x)$$

$$P_n(x) = \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}^2 (x-1)^{n-k}(x+1)^k = 2^n\sum_{k=0}^{n} x^k \binom{n}{k}\binom{(n+k-1)/2}{n}$$

J Values for Solar System Bodies

$(\times 10^{-6})$	**Earth**	**Mars**	**Moon**	**Venus**	**Mercury**
J_2	1082.6	1955.5	203.23	4.4044	22.5
J_3	-2.5327	31.450	8.4759	-2.1082	4.49
J_4	-1.6196	-15.377	-9.5919	-2.1474	6.5

$(\times 10^{-6})$	**Jupiter**	**Saturn**	**Uranus**	**Neptune**
J_2	14696.572	16290.573	3341.29	3408.43
J_3	-0.042	0.059	—	—
J_4	-586.609	-935.314	-30.44	-33.40

From: Lemoine et al. 1998 (Earth), Lemoine et al. 2001 (Mars), Konopliv et al. 2001 (Moon), Konopliv et al. 1999 (Venus), Iess et al. 2018 (Jupiter), Iess et al. 2019 (Saturn), Smith et al. 2012 (Mercury)

Potential of an Arbitrary Body

$$U(r,\theta,\phi) = \frac{Gm_{\mathscr{B}}}{r} + G\sum_{\ell=2}^{\infty}\sum_{m=-\ell}^{\ell} q_\ell^m \; r^{-(\ell+1)} \; Y_\ell^m(\theta,\phi)$$

Multipole Moments Spherical

Harmonics

$$Y_\ell^m(\theta,\phi) = \sqrt{\frac{(2\ell+1)}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}}\, P_\ell^m(\cos\phi)\, e^{im\phi}$$

Associated Legendre Polynomials : $P_\ell^m(x) = (-1)^m \; (1-x^2)^{m/2} \; \dfrac{\mathrm{d}^m}{\mathrm{d}x^m}\left(P_\ell(x)\right)$

Typically use:

$$U(r,\theta,\phi) = \frac{Gm_{\mathscr{B}}}{r}\left[1 + \sum_{\ell=2}^{\infty}\sum_{m=0}^{\ell}\left(\frac{R_{\mathscr{B}}}{r}\right)^{\ell} P_\ell^m(\cos\phi) \times \left(\; C_\ell^m \cos(m\theta) + \; S_\ell^m \sin(m\theta)\right)\right]$$

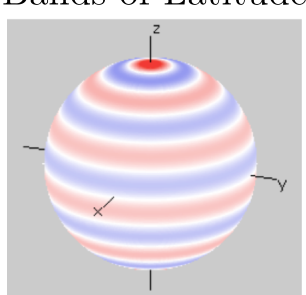
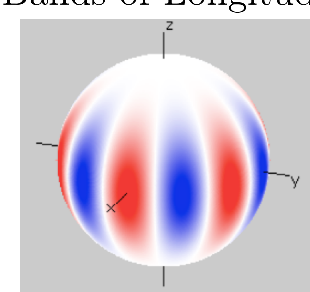
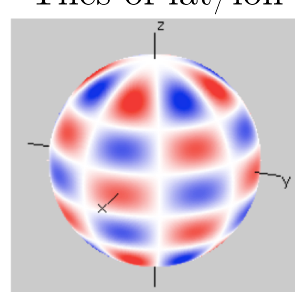
Tabulated Coefficients

Spherical Harmonics

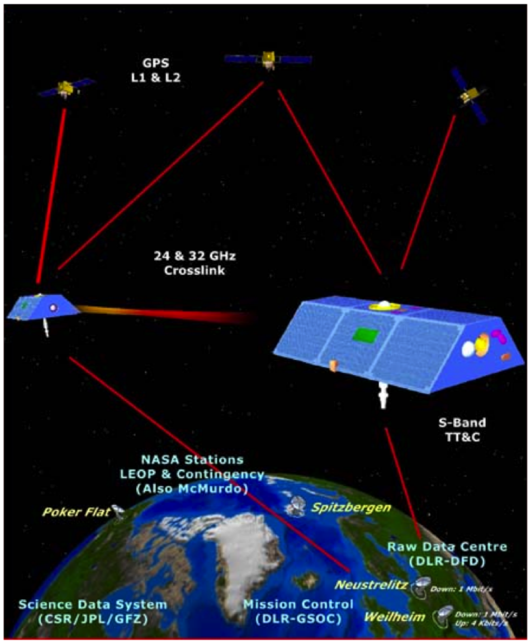
$$U(r,\theta,\phi) = \frac{Gm_{\mathscr{B}}}{r}\left[1 + \sum_{\ell=2}^{\infty}\sum_{m=0}^{\ell}\left(\frac{R_{\mathscr{B}}}{r}\right)^{\ell} P_\ell^m(\cos\phi) \times \left(C_\ell^m \cos(m\theta) + S_\ell^m \sin(m\theta)\right)\right]$$

$$= \frac{\mu}{r}\left[1 - \sum_{\ell=2}^{\infty}\left(\frac{R_{\mathscr{B}}}{r}\right)^{\ell} J_\ell P_\ell(\cos\phi) + \sum_{\ell=2}^{\infty}\sum_{m=1}^{\ell}\left(\frac{R_{\mathscr{B}}}{r}\right)^{\ell} P_\ell^m(\cos\phi) \times \left(C_\ell^m \cos(m\theta) + S_\ell^m \sin(m\theta)\right)\right]$$

ℓ = degree, m = order

Zonal Harmonics	**Sectoral Harmonics**	**Tesseral Harmonics**
Bands of Latitude	Bands of Longitude	Tiles of lat/lon
		

State of the Art: GRACE



- Launched in 2002 with original 5 year mission (decommissioned in 2017)
- Followup (GRACE-FO) launched in 2018
- Provides monthly gravity anomaly mapping (degree 60-90)
- Static geopotential maps available from:
  - International Centre for Global Earth Models (ICGEM)
  - National Geospatial-Intelligence Agency (NGA)
- Current Standard is Earth Gravitational Model 2008 (EGM2008)

  http://earth-info.nga.mil/GandG/update/index.php?action=home
- See also: http://icgem.gfz-potsdam.de/tom_longtime

Normalizations – Be Careful!

Description of Files Related to Using the EGM2008 Global Gravitational Model to Compute Geoid Undulations with Respect to WGS 84

(1) EGM2008_to2190_TideFree.gz

This file contains the fully-normalized, unit-less, spherical harmonic coefficients of the Earth's gravitational potential $\{\overline{C}_{nm}, \overline{S}_{nm}\}$ and their associated (calibrated) error standard deviations $\{sigma\overline{C}_{nm}, sigma\overline{S}_{nm}\}$, as implied by the EGM2008 model. The $\{\overline{C}_{nm}, \overline{S}_{nm}\}$ coefficients are consistent with the expression:

$$V(r,\theta,\lambda) = \frac{GM}{r}\left[1 + \sum_{n=2}^{N\max}\left(\frac{a}{r}\right)^n \sum_{m=0}^{n}\left(\overline{C}_{nm}\cos m\lambda + \overline{S}_{nm}\sin m\lambda\right)\overline{P}_{nm}(\cos\theta)\right] \quad (1)$$

From: README_WGS84_2.pdf

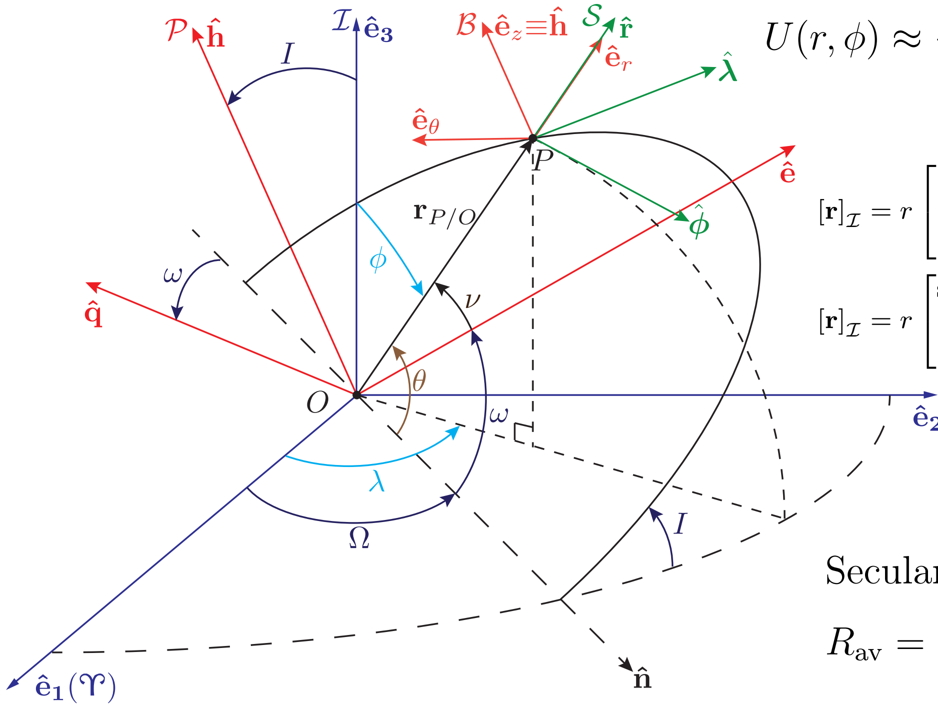
$$C_\ell^m = \left[\frac{(\ell-m)!(2\ell+1)(2-\delta_{0m})}{(l+m)!}\right]^{\frac{1}{2}} \overline{C}_\ell^m$$

For example:

$$\overline{C}_2^0 = -4.841651437908150 \times 10^{-4}$$

$$C_2^0 = \left[\frac{(2-0)!(2\times 2+1)(2-\delta_{20})}{(2+0)!}\right]^{\frac{1}{2}} \overline{C}_2^0 = -0.001082626173852 = -J_2$$

J_2 Perturbation Analysis Setup



$$U(r,\phi) \approx \frac{\mu}{r} \underbrace{- \frac{\mu}{r} J_2 \left(\frac{R_{\mathscr{B}}}{r}\right)^2 \left(\frac{3\cos^2\phi - 1}{2}\right)}_{\text{Perturbing Potential}}$$

$$[\mathbf{r}]_{\mathcal{I}} = r \begin{bmatrix} -\sin(\Omega)\sin(\theta)\cos(I) + \cos(\Omega)\cos(\theta) \\ \sin(\Omega)\cos(\theta) + \sin(\theta)\cos(I)\cos(\Omega) \\ \sin(I)\sin(\theta) \end{bmatrix}_{\mathcal{I}}$$

$$[\mathbf{r}]_{\mathcal{I}} = r \begin{bmatrix} \sin(\phi)\cos(\lambda) \\ \sin(\lambda)\sin(\phi) \\ \cos(\phi) \end{bmatrix}_{\mathcal{I}} \Rightarrow \cos\phi = \sin I \sin\theta$$

$$r = \frac{a(1-e^2)}{1 + e\cos\nu} \qquad \theta \equiv \omega + \nu$$

Secular (non-periodic) perturbation:

$$R_{\text{av}} = -\frac{J_2 \mu R_{\mathscr{B}}^2}{2a^3(1-e^2)^{\frac{3}{2}}} \left(\frac{3}{2}\sin^2(I) - 1\right)$$

Recall The Lagrange Planetary Equations

$$\frac{\mathrm{d}a}{\mathrm{d}t} = \frac{2}{na}\frac{\partial R}{\partial M}$$

$$\frac{\mathrm{d}e}{\mathrm{d}t} = \frac{1}{na^2 e}\left(\left(1-e^2\right)\frac{\partial R}{\partial M} - \sqrt{1-e^2}\frac{\partial R}{\partial \omega}\right)$$

$$\frac{\mathrm{d}I}{\mathrm{d}t} = \frac{1}{na^2\sqrt{1-e^2}\sin I}\left(\cos I \frac{\partial R}{\partial \omega} - \frac{\partial R}{\partial \Omega}\right)$$

$$\frac{\mathrm{d}\omega}{\mathrm{d}t} = \frac{\sqrt{1-e^2}}{na^2 e}\frac{\partial R}{\partial e} - \frac{\cot I}{na^2\sqrt{1-e^2}}\frac{\partial R}{\partial I}$$

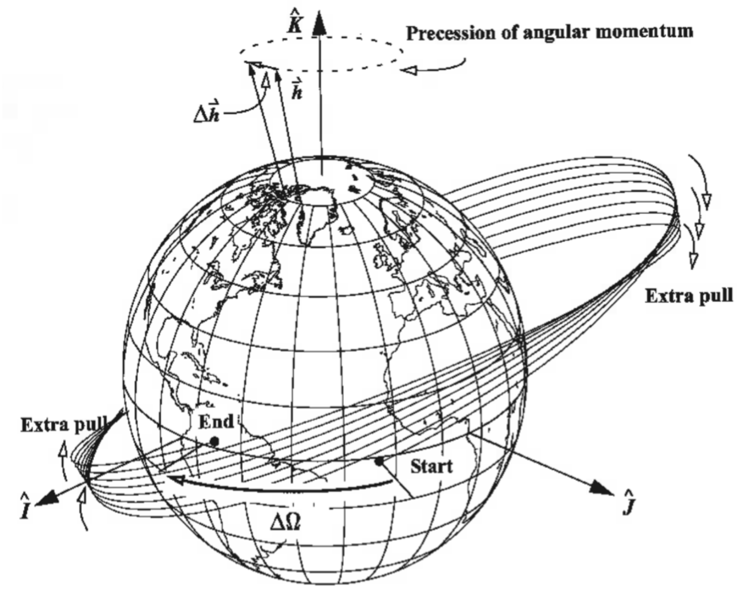
$$\frac{\mathrm{d}\Omega}{\mathrm{d}t} = \frac{1}{na^2\sqrt{1-e^2}\sin I}\frac{\partial R}{\partial I}$$

$$\frac{\mathrm{d}M}{\mathrm{d}t} = n - \frac{1-e^2}{na^2 e}\frac{\partial R}{\partial e} - \frac{2}{na}\frac{\partial R}{\partial a}$$

For J_2 perturbations:

$$R = R_{\text{av}} = -\frac{J_2 \mu R_{\mathscr{B}}^2}{2a^3(1-e^2)^{\frac{3}{2}}}\left(\frac{3}{2}\sin^2(I) - 1\right)$$

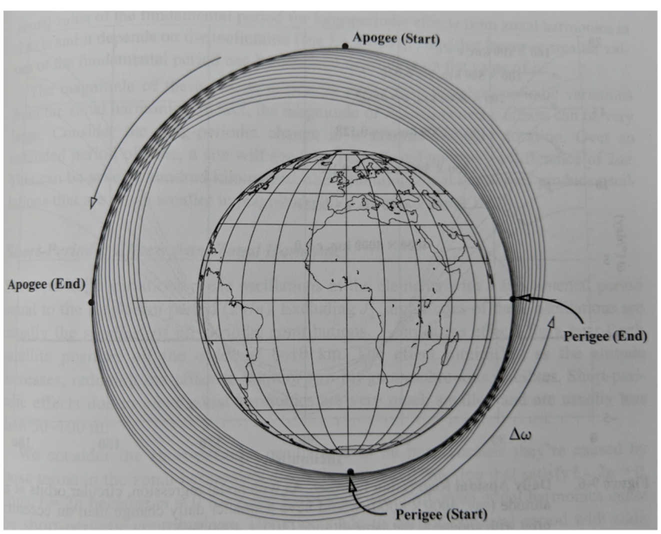
Nodal Regression



Vallado (2013) Figure 9-3

$$\dot{\Omega}_{\text{sec}} = -\frac{3nR_{\mathscr{B}}^2 J_2}{2a^2(1-e^2)^2}\cos I$$
$$\Omega(t) = \Omega(t_0) + \dot{\Omega}_{\text{sec}}(t - t_0)$$

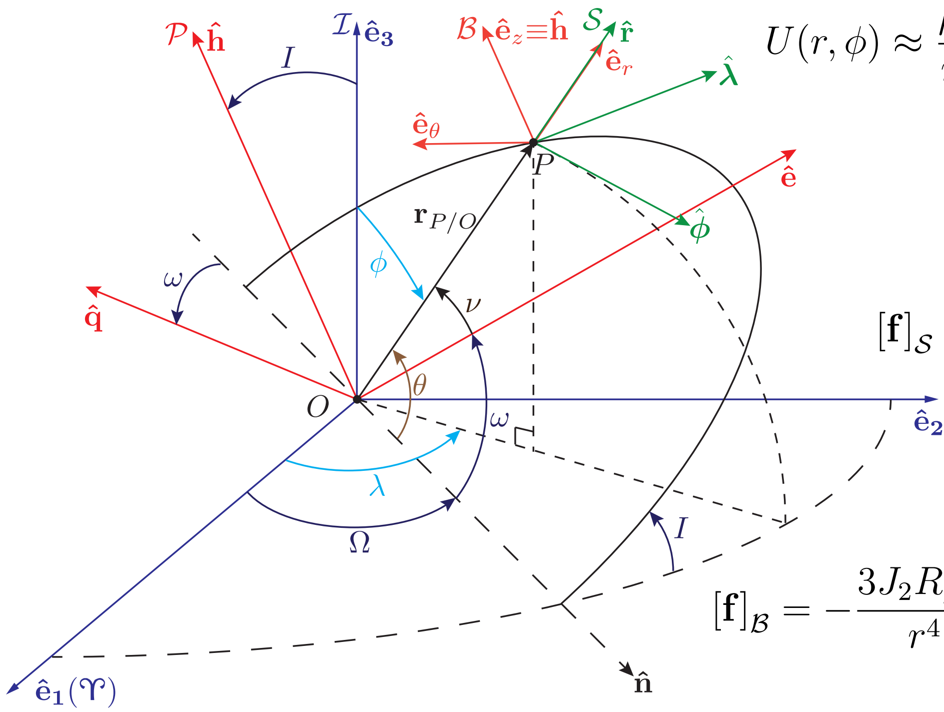
Apsidal Rotation



Vallado (2013) Figure 9-5

$$\dot{\omega}_{\text{sec}} = \frac{3}{2}J_2 n\left(\frac{R_{\mathscr{B}}}{a(1-e^2)}\right)^2\left(2 - \frac{5}{2}\sin^2(I)\right)$$
$$\omega(t) = \omega(t_0) + \dot{\omega}_{\text{sec}}(t - t_0)$$

J_2 Perturbation Analysis Setup (Forces)

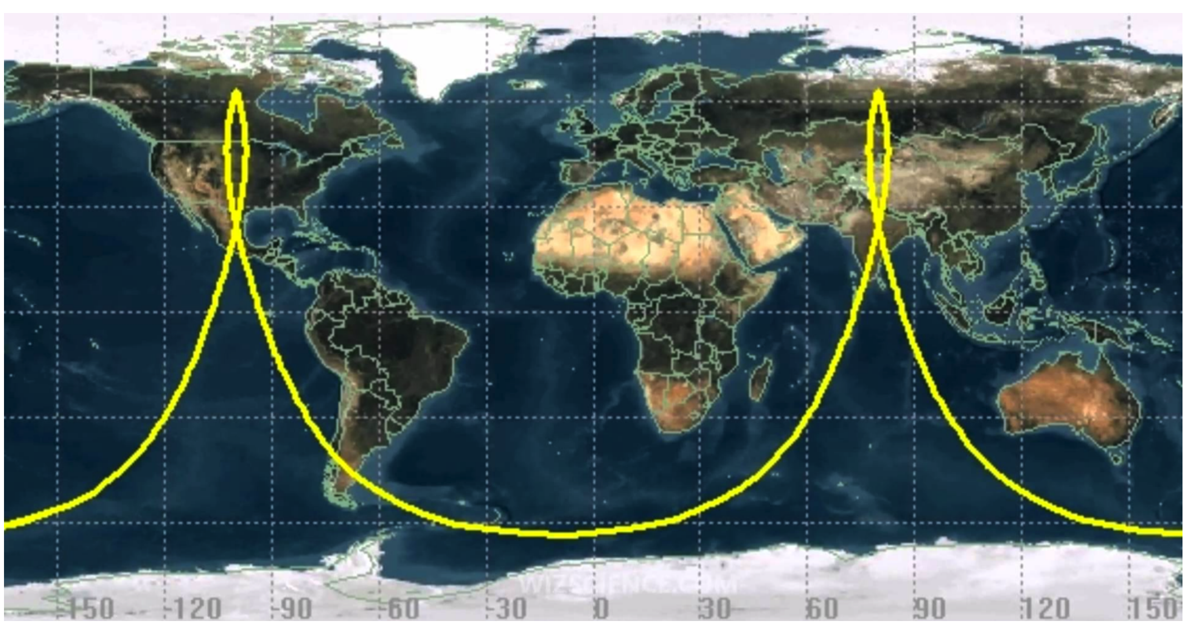


$$U(r,\phi) \approx \frac{\mu}{r} \underbrace{- \frac{\mu}{r} J_2 \left(\frac{R_{\mathscr{B}}}{r}\right)^2 \left(\frac{3\cos^2\phi - 1}{2}\right)}_{\substack{\text{Perturbing Potential:} \\ \mathbf{f} \triangleq -\boldsymbol{\nabla}\left(\frac{\mu J_2 R_{\mathscr{B}}^2}{r^3}\left(\frac{3\cos^2\phi - 1}{2}\right)\right)}}$$

$$[\mathbf{f}]_{\mathcal{S}} = \frac{3J_2R_{\mathscr{B}}^2\mu}{r^4} \begin{bmatrix} \sin(\phi)\cos(\phi) \\ 0 \\ \frac{1}{2}\left(3\cos^2(\phi) - 1\right) \end{bmatrix}_{\mathcal{S}}$$

$$[\mathbf{f}]_{\mathcal{B}} = -\frac{3J_2R_{\mathscr{B}}^2\mu}{r^4} \begin{bmatrix} \frac{1}{2}\left(1 - 3\sin^2(I)\sin^2(\theta)\right) \\ -\sin^2(I)\sin(\theta)\cos(\theta) \\ \sin(I)\cos(I)\sin(\theta) \end{bmatrix}_{\mathcal{B}}$$

Molniya Orbits (12 hour period)



Credit: Analytical Graphics Inc