19 - Attitude Kinematics, Review of Simple Rotations, Direction Cosine Matrices Revisited, and Euler Angles/Euler Parameters

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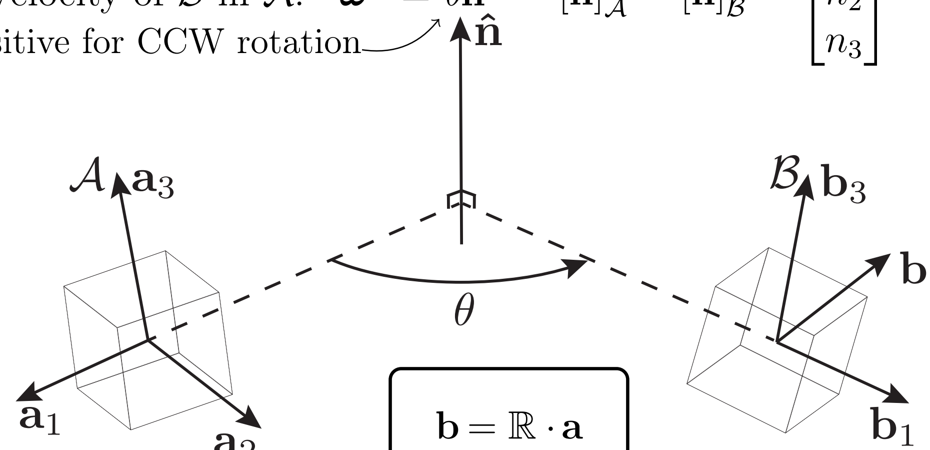
Simple Rotations

Angular Velocity of $\mathcal{B}$ in $\mathcal{A}$: ${}^{\mathcal{A}}\boldsymbol{\omega}^{\mathcal{B}} \triangleq \dot{\theta}\hat{\mathbf{n}}$ $\qquad [\hat{\mathbf{n}}]_{\mathcal{A}} = [\hat{\mathbf{n}}]_{\mathcal{B}} = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}$

Positive for CCW rotation



$$\boxed{\mathbf{b} = \mathbb{R} \cdot \mathbf{a}}$$

Vector fixed in $\mathcal{B}$, aligned with $\mathbf{a}$ before rotation

Vector fixed in $\mathcal{A}$, aligned with $\mathbf{b}$ before rotation

$$\mathbf{b} = \hat{\mathbf{n}}\hat{\mathbf{n}} \cdot \mathbf{a} + \cos\theta\,(\mathbf{a} - \hat{\mathbf{n}}(\hat{\mathbf{n}} \cdot \mathbf{a})) + \sin\theta\hat{\mathbf{n}} \times \mathbf{a}$$

Remember: All Vector **and Tensor** Operations Can Be Written as Matrix Multiplications

$$\mathcal{I} = (O, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) \qquad \mathbf{a} = \sum_i a_i \mathbf{e}_i \Rightarrow a_i = \mathbf{a} \cdot \mathbf{e}_i \qquad \mathbf{b} = \sum_i b_i \mathbf{e}_i \Rightarrow b_i = \mathbf{b} \cdot \mathbf{e}_i$$

$$\mathbb{T} = \mathbf{a} \otimes \mathbf{b} = \sum_i \sum_j T_{ij} \mathbf{e}_i \otimes \mathbf{e}_j \Rightarrow T_{ij} = \mathbf{e}_i \cdot \mathbb{T} \cdot \mathbf{e}_j = a_i b_j$$

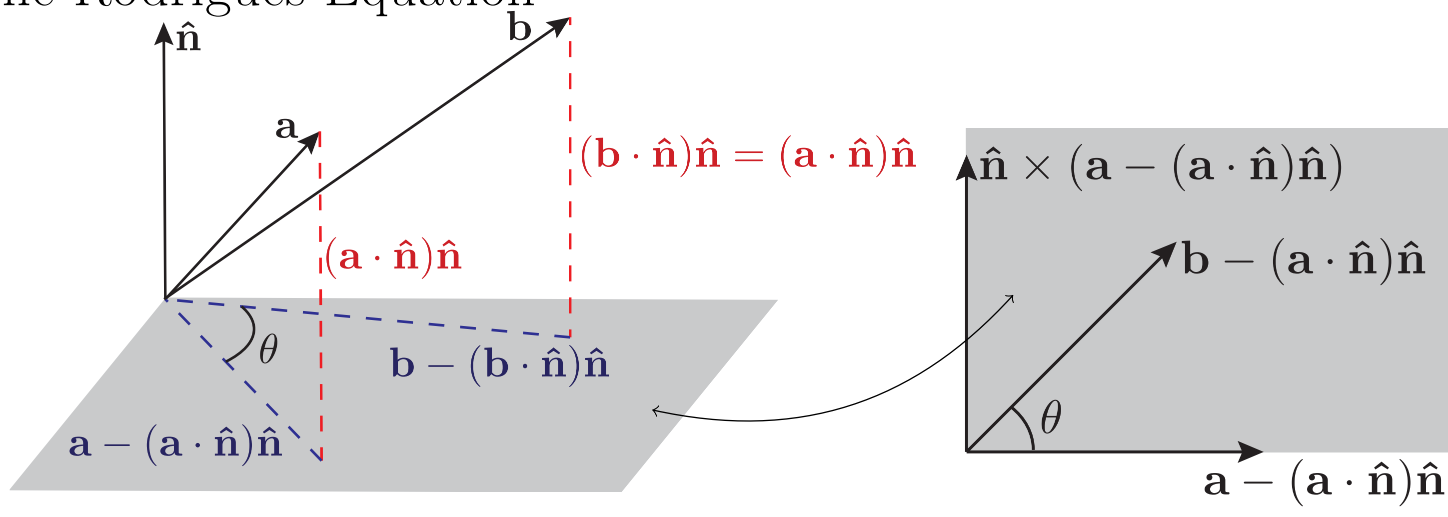
$$[\mathbf{a}]_{\mathcal{I}} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}_{\mathcal{I}} \qquad [\mathbf{b}]_{\mathcal{I}} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}_{\mathcal{I}} \qquad [\mathbb{T}]_{\mathcal{I}} = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix}_{\mathcal{I}} = \begin{bmatrix} a_1 b_1 & a_1 b_2 & a_1 b_3 \\ a_2 b_1 & a_2 b_2 & a_2 b_3 \\ a_3 b_1 & a_3 b_2 & a_3 b_3 \end{bmatrix}_{\mathcal{I}}$$

$$[\mathbf{a} \cdot \mathbb{T}]_{\mathcal{I}} = [\mathbf{a}]_{\mathcal{I}}^T [\mathbb{T}]_{\mathcal{I}}$$
$$[\mathbb{T} \cdot \mathbf{a}]_{\mathcal{I}} = [\mathbb{T}]_{\mathcal{I}} [\mathbf{a}]_{\mathcal{I}}$$
$$[\mathbf{a} \times \mathbb{T}]_{\mathcal{I}} = [\mathbf{a}\times]_{\mathcal{I}} [\mathbb{T}]_{\mathcal{I}}$$
$$[\mathbb{T} \times \mathbf{a}]_{\mathcal{I}} = -[\mathbb{T}]_{\mathcal{I}} [\mathbf{a}\times]_{\mathcal{I}}$$

$$[\mathbb{T}]_{\mathcal{I}} = [\mathbf{a} \otimes \mathbf{b}]_{\mathcal{I}} = [\mathbf{a}]_{\mathcal{I}} [\mathbf{b}]_{\mathcal{I}}^T$$

$$[\mathbf{a}\times]_{\mathcal{I}} = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}_{\mathcal{I}}$$

The Rodrigues Equation



$$\mathbf{b} = \mathbb{R} \cdot \mathbf{a} \Longrightarrow \boxed{\begin{aligned} \mathbb{R} &= \cos\theta \mathbb{U} + \sin\theta \hat{\mathbf{n}}_\times + (1 - \cos\theta)\hat{\mathbf{n}} \otimes \hat{\mathbf{n}} \\ {}^{\mathcal{A}}C^{\mathcal{B}} &= I \cos\theta + \sin\theta \left[\hat{\mathbf{n}}_\times\right]_{\mathcal{A}} + (1 - \cos\theta) \left[\hat{\mathbf{n}}\right]_{\mathcal{A}} \left[\hat{\mathbf{n}}\right]_{\mathcal{A}}^T \end{aligned}}$$

Direction Cosine Matrices Revisited

$$[\hat{\mathbf{n}}]_{\mathcal{A}} = [\hat{\mathbf{n}}]_{\mathcal{B}} = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}$$

$$^{\mathcal{A}}C^{\mathcal{B}} = I\cos\theta + \sin\theta\,[\hat{\mathbf{n}}_{\times}]_{\mathcal{A}} + (1-\cos\theta)\,[\hat{\mathbf{n}}]_{\mathcal{A}}\,[\hat{\mathbf{n}}]_{\mathcal{A}}^T \implies$$

$$^{\mathcal{A}}C^{\mathcal{B}} = \begin{bmatrix} n_1^2\,(-\cos\theta+1)+\cos\theta & -n_1 n_2\,(\cos\theta-1) - n_3\sin\theta & -n_1 n_3\,(\cos\theta-1)+n_2\sin\theta \\ -n_1 n_2\,(\cos\theta-1)+n_3\sin\theta & n_2^2\,(-\cos\theta+1)+\cos\theta & -n_1\sin\theta - n_2 n_3\,(\cos\theta-1) \\ -n_1 n_3\,(\cos\theta-1) - n_2\sin\theta & n_1\sin\theta - n_2 n_3\,(\cos\theta-1) & n_3^2\,(-\cos\theta+1)+\cos\theta \end{bmatrix}$$

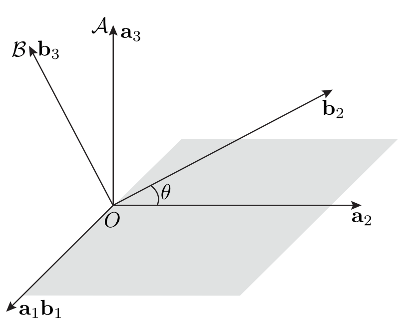
$$^{\mathcal{A}}C^{\mathcal{B}}_{ij} = \delta_{ij}\cos\theta - \underbrace{\epsilon_{ijk}}_{k\neq i,j}\, n_k\sin\theta + n_i n_j(1-\cos\theta)$$

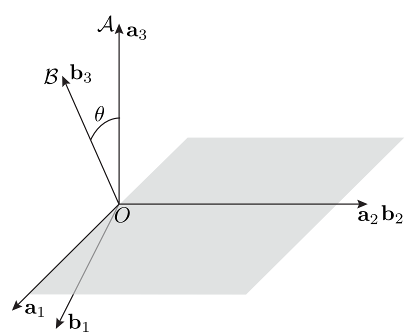
Konecker Delta

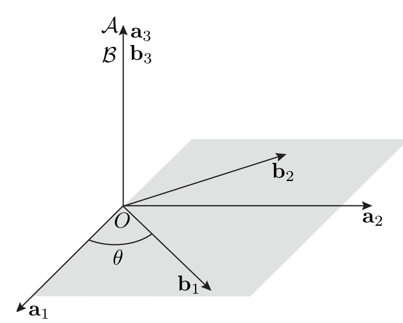
Levi-Civita Symbol

$$\delta_{ij} = \begin{cases} 1 & j = i \\ 0 & j \neq i \end{cases} \qquad \epsilon_{ijk} = \frac{1}{2}(i-j)(j-k)(k-i) = \begin{cases} 1 & \text{Even permutations} \\ -1 & \text{Odd permutations} \\ 0 & \text{Repeated indices} \end{cases}$$

Simple Direction Cosine Matrices

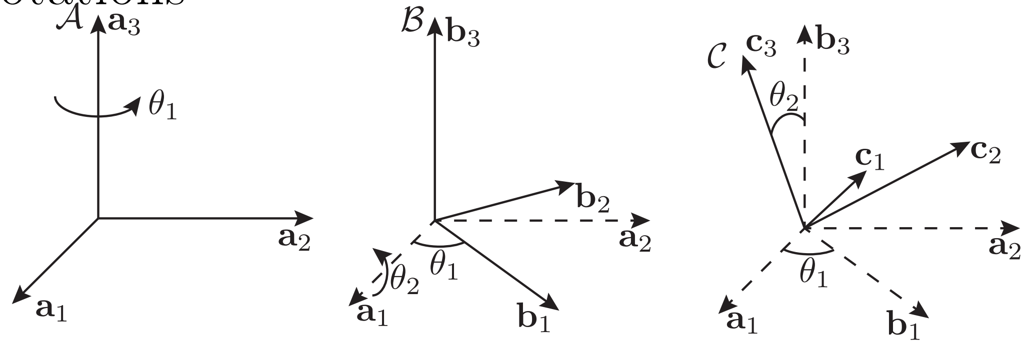






$$^{\mathcal{B}}C^{\mathcal{A}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \qquad \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \qquad \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Space Rotations

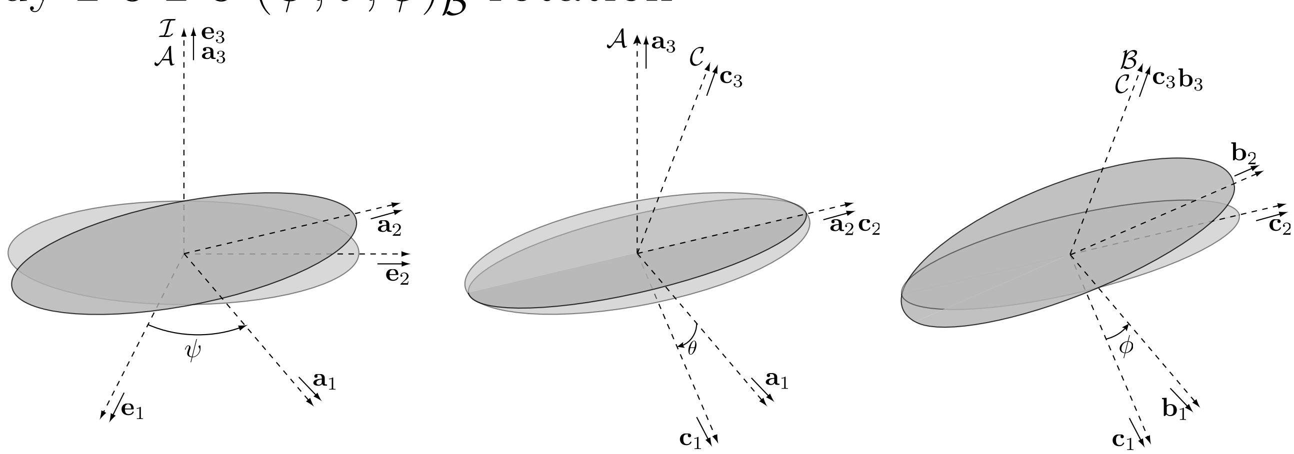


$$
{}^{\mathcal{B}}C^{\mathcal{A}} = \begin{bmatrix} \cos(\theta_1) & \sin(\theta_1) & 0 \\ -\sin(\theta_1) & \cos(\theta_1) & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad [\hat{\mathbf{n}}_2]_B = {}^{\mathcal{B}}C^{\mathcal{A}}\,[\hat{\mathbf{n}}_2]_A = \begin{bmatrix} \cos\theta_1 \\ -\sin\theta_1 \\ 0 \end{bmatrix}_{\mathcal{B}}
$$

$$
{}^{\mathcal{C}}C^{\mathcal{B}} = \begin{bmatrix} (-\cos(\theta_2)+1)\cos^2(\theta_1)+\cos(\theta_2) & (\cos(\theta_2)-1)\sin(\theta_1)\cos(\theta_1) & \sin(\theta_1)\sin(\theta_2) \\ (\cos(\theta_2)-1)\sin(\theta_1)\cos(\theta_1) & (-\cos(\theta_2)+1)\sin^2(\theta_1)+\cos(\theta_2) & \sin(\theta_2)\cos(\theta_1) \\ -\sin(\theta_1)\sin(\theta_2) & -\sin(\theta_2)\cos(\theta_1) & \cos(\theta_2) \end{bmatrix}
$$

$$
\begin{aligned}
{}^{\mathcal{C}}C^{\mathcal{A}} = {}^{\mathcal{C}}C^{\mathcal{B}\mathcal{B}}C^{\mathcal{A}} &= \begin{bmatrix} \cos(\theta_1) & \sin(\theta_1)\cos(\theta_2) & \sin(\theta_1)\sin(\theta_2) \\ -\sin(\theta_1) & \cos(\theta_1)\cos(\theta_2) & \sin(\theta_2)\cos(\theta_1) \\ 0 & -\sin(\theta_2) & \cos(\theta_2) \end{bmatrix} \\
&\equiv C_3^T(-\theta_1)C_1^T(-\theta_2) = C_3(\theta_1)C_1(\theta_2)
\end{aligned}
$$

Body-2 3-2-3 $(\psi, \theta, \phi)^{\mathcal{I}}_{\mathcal{B}}$ rotation



$$
{}^{\mathcal{A}}C^{\mathcal{I}} = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad {}^{\mathcal{C}}C^{\mathcal{A}} = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \qquad {}^{\mathcal{B}}C^{\mathcal{C}} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^{\mathcal{B}}C^{\mathcal{I}} = {}^{\mathcal{B}}C^{\mathcal{C}\mathcal{C}}C^{\mathcal{A}\mathcal{A}}C^{\mathcal{I}} =
$$

$$
\begin{bmatrix} -\sin(\phi)\sin(\psi)+\cos(\phi)\cos(\psi)\cos(\theta) & \sin(\phi)\cos(\psi)+\sin(\psi)\cos(\phi)\cos(\theta) & -\sin(\theta)\cos(\phi) \\ -\sin(\phi)\cos(\psi)\cos(\theta)-\sin(\psi)\cos(\phi) & -\sin(\phi)\sin(\psi)\cos(\theta)+\cos(\phi)\cos(\psi) & \sin(\phi)\sin(\theta) \\ \sin(\theta)\cos(\psi) & \sin(\psi)\sin(\theta) & \cos(\theta) \end{bmatrix}
$$

Euler Parameters

$$\boxed{\boldsymbol{\epsilon} \triangleq \sin\left(\frac{\theta}{2}\right)\hat{\mathbf{n}} \qquad \epsilon_4 \triangleq \cos\left(\frac{\theta}{2}\right)}$$

$$\begin{aligned}
{}^{\mathcal{A}}C^{\mathcal{B}} &= I\cos\theta + \sin\theta\left[\hat{\mathbf{n}}_\times\right]_{\mathcal{A}} + (1-\cos\theta)\left[\hat{\mathbf{n}}\right]_{\mathcal{A}}\left[\hat{\mathbf{n}}\right]_{\mathcal{A}}^T \\
&= I\left(\epsilon_4^2 - \left[\boldsymbol{\epsilon}\right]_{\mathcal{A}}^T\left[\boldsymbol{\epsilon}\right]_{\mathcal{A}}\right) + 2\epsilon_4\left[\boldsymbol{\epsilon}_\times\right]_{\mathcal{A}} + 2\left[\boldsymbol{\epsilon}\right]_{\mathcal{A}}\left[\boldsymbol{\epsilon}\right]_{\mathcal{A}}^T \\
&= \begin{bmatrix} \epsilon_1^2-\epsilon_2^2-\epsilon_3^2+\epsilon_4^2 & 2\epsilon_1\epsilon_2 - 2\epsilon_3\epsilon_4 & 2\epsilon_1\epsilon_3 + 2\epsilon_2\epsilon_4 \\ 2\epsilon_1\epsilon_2 + 2\epsilon_3\epsilon_4 & -\epsilon_1^2+\epsilon_2^2-\epsilon_3^2+\epsilon_4^2 & -2\epsilon_1\epsilon_4 + 2\epsilon_2\epsilon_3 \\ 2\epsilon_1\epsilon_3 - 2\epsilon_2\epsilon_4 & 2\epsilon_1\epsilon_4 + 2\epsilon_2\epsilon_3 & -\epsilon_1^2-\epsilon_2^2+\epsilon_3^2+\epsilon_4^2 \end{bmatrix}
\end{aligned}$$

$$\boldsymbol{\epsilon} = \frac{1}{4\epsilon_4}\begin{bmatrix} {}^{\mathcal{A}}C_{32}^{\mathcal{B}} - {}^{\mathcal{A}}C_{23}^{\mathcal{B}} \\ {}^{\mathcal{A}}C_{13}^{\mathcal{B}} - {}^{\mathcal{A}}C_{31}^{\mathcal{B}} \\ {}^{\mathcal{A}}C_{21}^{\mathcal{B}} - {}^{\mathcal{A}}C_{12}^{\mathcal{B}} \end{bmatrix}_A \qquad \epsilon_4 = \frac{1}{2}\left(1 + \mathrm{Tr}\left[{}^{\mathcal{A}}C^{\mathcal{B}}\right]\right)^{\frac{1}{2}}$$

$$\mathbf{b} = \mathbb{R}\cdot\mathbf{a} \quad \Longrightarrow \quad \mathbf{b} = \mathbf{a} + 2\left(\epsilon_4\boldsymbol{\epsilon}\times\mathbf{a} + \boldsymbol{\epsilon}\times\left(\boldsymbol{\epsilon}\times\mathbf{a}\right)\right)$$