9 - Non-gravitational Force Perturbations, Third and N-body perturbations, and Sphere of influence

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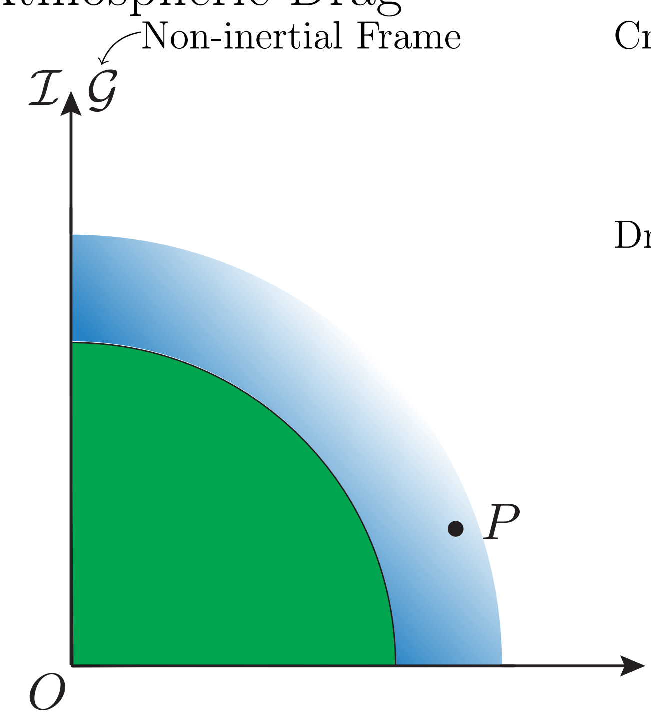
Atmospheric Drag



$$\mathbf{F}_{\text{drag}} = -\frac{1}{2}\, \underbrace{C_D}_{\text{Drag Coefficient}}\, \overbrace{A}^{\text{Cross-sectional Area}}\, \overbrace{\rho}^{\text{Atmospheric Density}}\, v_{\text{rel}}^2\, \underbrace{\hat{\mathbf{v}}_{\text{rel}}}_{\text{Relative Velocity}}$$

$$\mathbf{v}_{\text{rel}} = {}^{\mathcal{I}}\mathbf{v}_{P/O} - \underbrace{{}^{\mathcal{I}}\mathbf{v}_{\text{atm}/O}}_{\cancel{{}^{\mathcal{G}}\mathbf{v}_{\text{atm}/O}} + {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{G}} \times \mathbf{r}_{P/O}}$$

$$\mathbf{v}_{\text{rel}} \approx {}^{\mathcal{I}}\mathbf{v}_{P/O} - {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{G}} \times \mathbf{r}_{P/O}$$

Secular Perturbations Due to Atmospheric Drag

Planet Rotation Rate

$$Q \triangleq \left(1 - \frac{2\,\omega_r (1-e)^{3/2}}{n\sqrt{1+e}}\right)\cos(I)$$

$$\Delta a_{\mathrm{rev}} = -2\pi \frac{\overbrace{Q}\,AC_D}{m} a^2\,\rho_p \left(I_0 + 2eI_1 + \frac{3e^2}{4}(I_0 + I_2) + \frac{e^3}{4}\,(3I_1 + I_3)\right)\exp\left(\frac{-ae}{h}\right)$$

Density at Periapsis

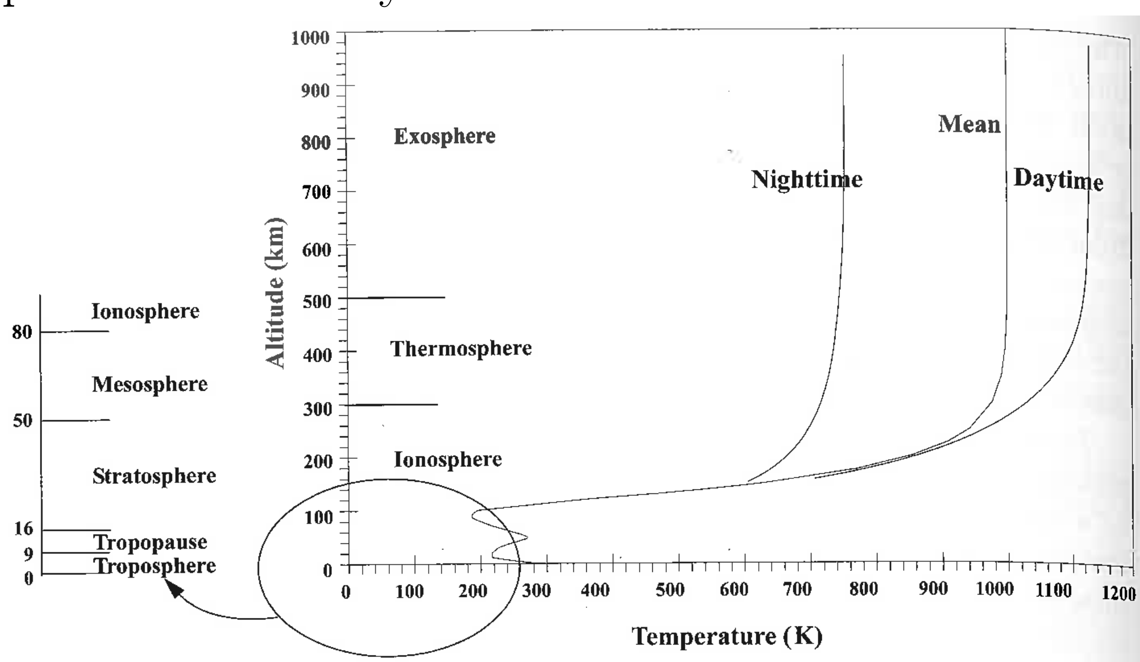
Atmospheric Scale Height

$$\Delta e_{\mathrm{rev}} = -2\pi \frac{QAC_D}{m} a\rho_p \left(I_1 + \frac{e}{2}(I_0 + I_2) - \frac{e^2}{8}(5I_1 - I_3) + \frac{e^3}{16}\,(5I_0 + 4I_2 - I_4)\right)\exp\left(\frac{-ae}{h}\right)$$

Here, $I_{0\ldots 4}$ =Modified Bessel Functions of the First Kind:

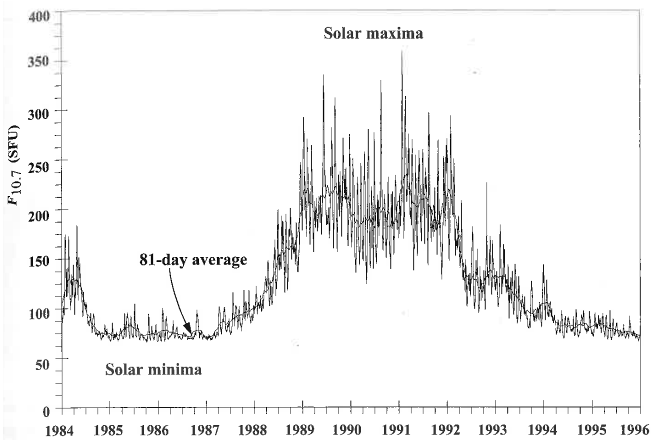
$$I_s(z) = \frac{1}{\pi}\int_0^{\pi} \exp\left(z\cos\theta\right)\cos(s\theta)\,\mathrm{d}\theta - \frac{\sin(s\pi)}{\pi}\int_0^{\infty}\exp\left(-z\cosh(t) - st\right)\,\mathrm{d}t$$

Atmospheric Variability

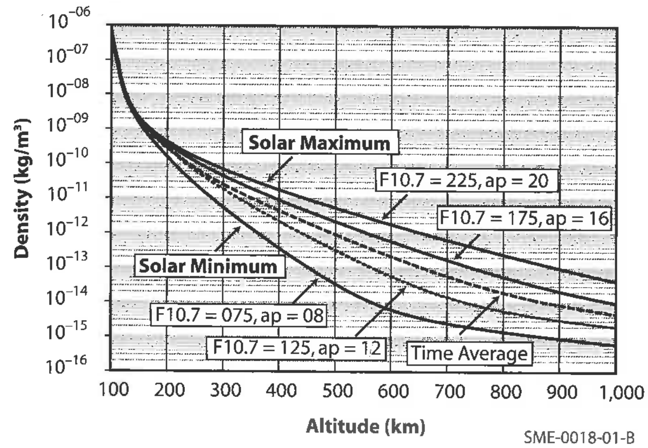


Vallado (2013) Fig. 8-7

Solar Flux Variability

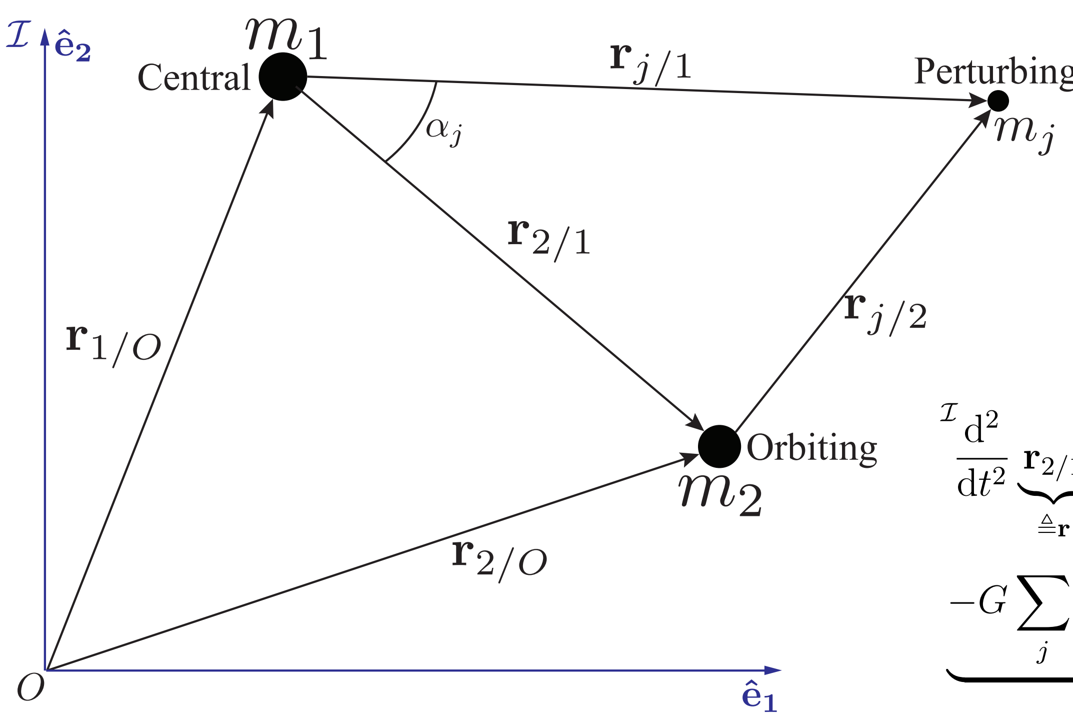


Vallado (2013) Fig. 8-11



SMAD-SME Fig. 9-17

3rd (Nth) Body Perturbations



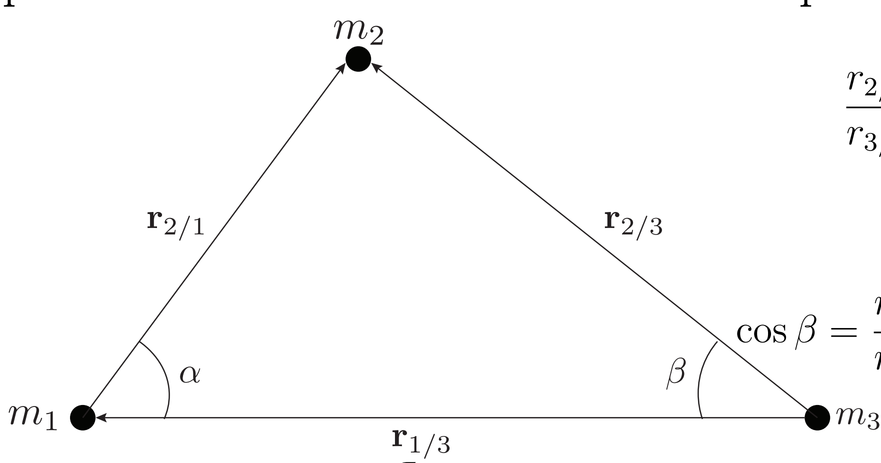
$$\frac{{}^{\mathcal{I}}\mathrm{d}^2}{\mathrm{d}t^2}\underbrace{\mathbf{r}_{2/1}}_{\triangleq \mathbf{r}} + \frac{\overbrace{G(m_1+m_2)}^{\triangleq \mu}}{\|\mathbf{r}_{2/1}\|^3}\mathbf{r}_{2/1} = \underbrace{-G\sum_j m_j\left(\frac{\mathbf{r}_{j/1}}{\|\mathbf{r}_{j/1}\|^3} + \frac{\mathbf{r}_{2/j}}{\|\mathbf{r}_{2/j}\|^3}\right)}_{\triangleq \mathbf{f}}$$

Secular Perturbations From Third Body in Circular Orbit

$$\dot{\Omega}_{\text{sec}} = -\frac{3}{16}\frac{\mu_3\left(2+3e^2\right)\left(2-3\sin^2(I_3)\right)}{nr_3^3\sqrt{1-e^2}}\cos I$$

$$\dot{\omega}_{\text{sec}} = \frac{3}{16}\frac{\mu_3\left(2-3\sin^2(I_3)\right)}{nr_3^3\sqrt{1-e^2}}\left(e^2+4-5\sin^2(I)\right)$$

Sphere of Influence Derivation Setup



$$\frac{r_{2/3}}{r_{3/1}} = \left[1 - 2\frac{r_{2/1}}{r_{3/1}}\cos\alpha + \left(\frac{r_{2/1}}{r_{3/1}}\right)^2\right]^{\frac{1}{2}}$$

$$\cos\beta = \frac{r_{3/1}}{r_{2/3}} - \frac{r_{2/1}}{r_{2/3}}\cos\alpha$$

$$m_1 \text{ dominant}: \quad {}^{\mathcal{I}}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathbf{r}_{2/1} + \frac{G(m_1+m_2)}{\|\mathbf{r}_{2/1}\|^3}\mathbf{r}_{2/1} = -Gm_3\left(\frac{\mathbf{r}_{3/1}}{\|\mathbf{r}_{3/1}\|^3} + \frac{\mathbf{r}_{2/3}}{\|\mathbf{r}_{2/3}\|^3}\right)$$

$$m_3 \text{ dominant}: \quad {}^{\mathcal{I}}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathbf{r}_{2/3} + \underbrace{\frac{G(m_2+m_3)}{\|\mathbf{r}_{2/3}\|^3}\mathbf{r}_{2/3}}_{\text{Central}} = \underbrace{-Gm_1\left(\frac{\mathbf{r}_{2/1}}{\|\mathbf{r}_{2/1}\|^3} + \frac{\mathbf{r}_{1/3}}{\|\mathbf{r}_{1/3}\|^3}\right)}_{\text{Perturbing}}$$

Sphere of Influence Derivation

$$\left(\frac{\|\mathbf{f}_{\text{perturbing}}\|}{\|\mathbf{f}_{\text{central}}\|}\right)_{m_1} = \frac{Gm_3\left[\left(\frac{\mathbf{r}_{2/3}}{r_{2/3}^3}+\frac{\mathbf{r}_{3/1}}{r_{3/1}^3}\right)\cdot\left(\frac{\mathbf{r}_{2/3}}{r_{2/3}^3}+\frac{\mathbf{r}_{3/1}}{r_{3/1}^3}\right)\right]^{\frac{1}{2}}}{G(m_1+m_2)r_{2/1}^{-2}}$$

$$= \frac{m_3}{m_2+m_1}\frac{\left(r_{2/1}/r_{3/1}\right)^2}{\left(r_{2/3}/r_{3/1}\right)^2}\left[1+\left(\frac{r_{2/3}}{r_{3/1}}\right)^4-2\left(\frac{r_{2/3}}{r_{3/1}}\right)\left(1-\frac{r_{2/1}}{r_{3/1}}\cos\alpha\right)\right]^{\frac{1}{2}}$$

$$\left(\frac{\|\mathbf{f}_{\text{perturbing}}\|}{\|\mathbf{f}_{\text{central}}\|}\right)_{m_3} = \frac{Gm_1\left[\left(\frac{\mathbf{r}_{2/1}}{r_{2/1}^3}+\frac{\mathbf{r}_{1/3}}{r_{3/1}^3}\right)\cdot\left(\frac{\mathbf{r}_{2/1}}{r_{2/1}^3}+\frac{\mathbf{r}_{1/3}}{r_{3/1}^3}\right)\right]^{\frac{1}{2}}}{G(m_2+m_3)r_{2/3}^{-2}}$$

$$= \frac{m_1}{m_2+m_3}\left(\frac{r_{2/1}}{r_{3/1}}\right)^{-2}\left(\frac{r_{2/3}}{r_{3/1}}\right)^2\left[1+\left(\frac{r_{2/1}}{r_{3/1}}\right)^4-2\left(\frac{r_{2/1}}{r_{3/1}}\right)^2\cos\alpha\right]^{\frac{1}{2}}$$

$$\text{Intersection}: \left(\frac{r_{2/1}}{r_{3/1}}\right)^4 = \frac{m1(m1+m2)}{m_3(m_2+m_3)}\left(\frac{r_{2/3}}{r_{3/1}}\right)^4\left[\frac{1+\left(\frac{r_{2/1}}{r_{3/1}}\right)^4-2\left(\frac{r_{2/1}}{r_{3/1}}\right)^2\cos\alpha}{1+\left(\frac{r_{2/3}}{r_{3/1}}\right)^4-2\left(\frac{r_{2/3}}{r_{3/1}}\right)\left(1-\frac{r_{2/1}}{r_{3/1}}\cos\alpha\right)}\right]^{\frac{1}{2}}$$

$$\left(\frac{r_{2/1}}{r_{3/1}}\right)\approx\left(\frac{m_1}{m_3}\right)^{\frac{2}{5}} \quad\Rightarrow\quad r_{\text{SOI}}\approx a_{\text{planet}}\left(\frac{m_{\text{planet}}}{m_{\text{sun}}}\right)^{\frac{2}{5}}$$