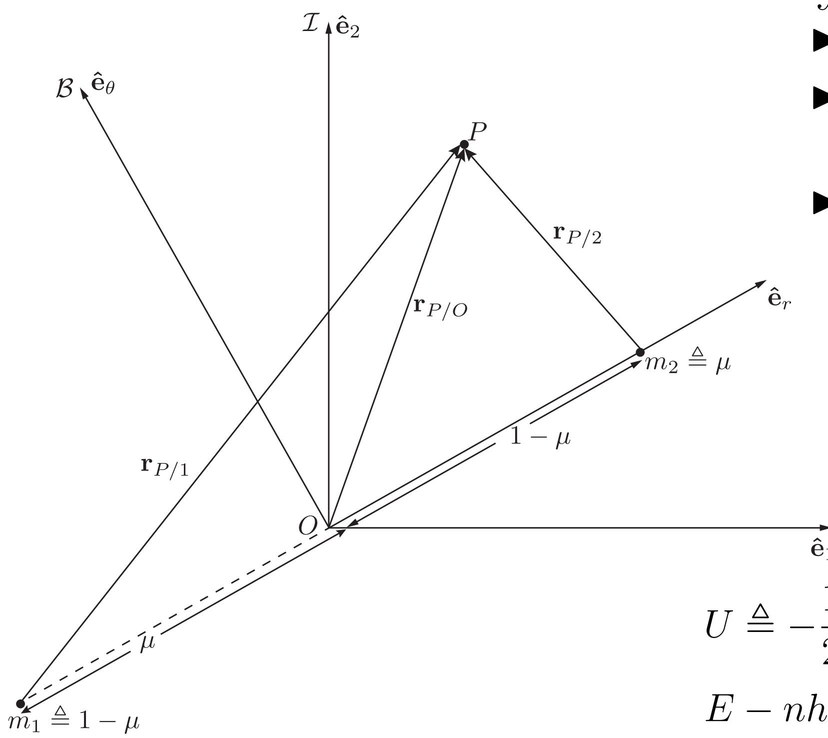
14 - CR3BP Continued, Lagrange Points, the Interplanetary Transport Network, and N-Body Integration

Dmitry Savransky

Cornell University

MAE 4060

Recall the Circular Restricted 3-Body Problem (CR3BP)



- $m_1, m_2 \gg m_p$
- m_1 and m_2 in mutual circular orbits, unaffected by m_p
- μ is the mass of the smaller major body (m_2)

$$\ddot{x} = -\frac{\partial U}{\partial x} + 2\dot{y}$$

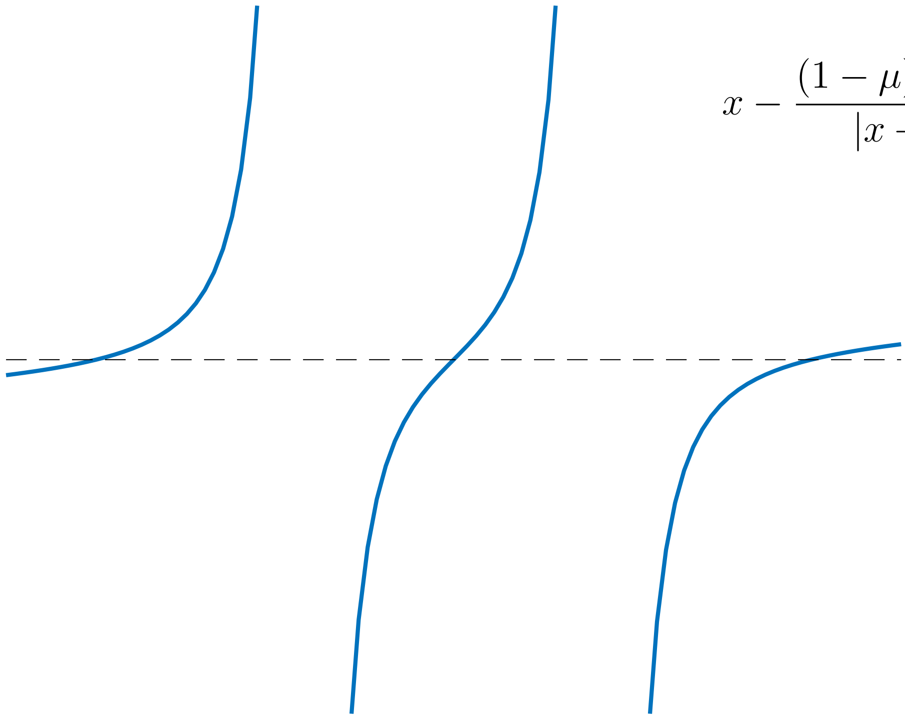
$$\ddot{y} = -\frac{\partial U}{\partial y} - 2\dot{x}$$

$$\ddot{z} = -\frac{\partial U}{\partial z}$$

$$U \triangleq -\frac{1}{2}(x^2+y^2) - \left(\frac{1-\mu}{r_1} + \frac{\mu}{r_2}\right)$$

$$E - nh\cos(I) = C$$

$y = 0$: On-Axis Equilibrium Points



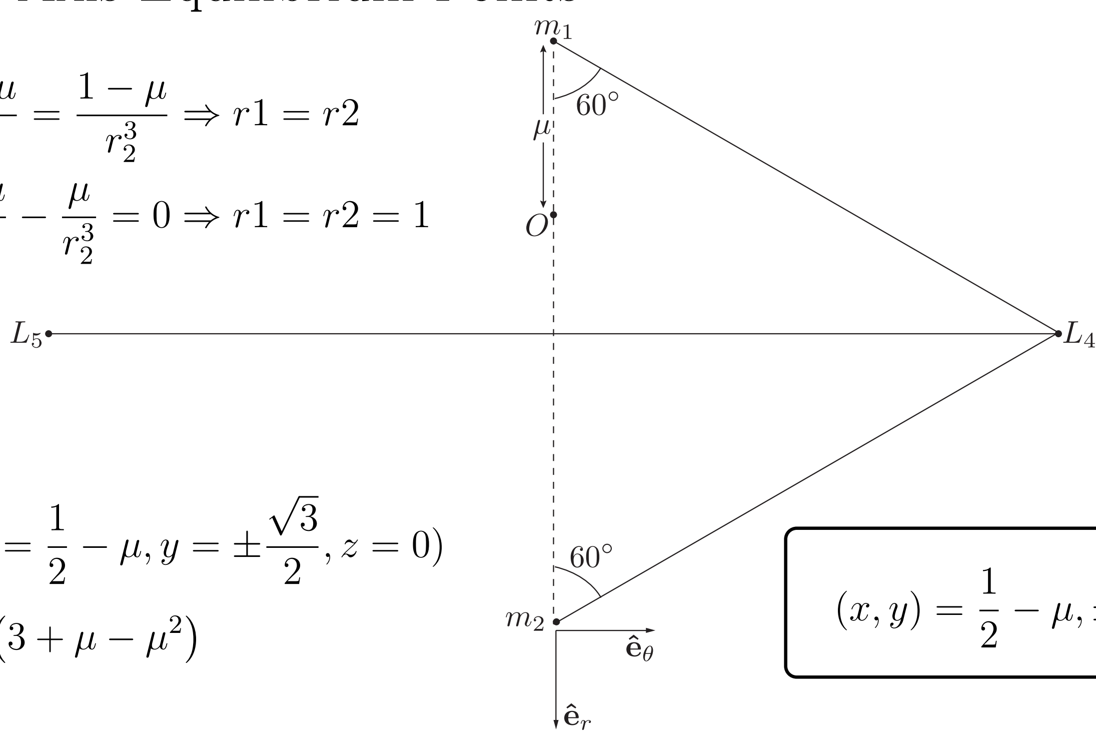
$$x - \frac{(1-\mu)(x+\mu)}{|x+\mu|^3} - \frac{\mu(x-1+\mu)}{|x-1+\mu|^3} = 0$$

$$\begin{aligned} x &< -\mu && (L_3) \\ -\mu < x &< 1-\mu && (L_1) \\ x &> 1-\mu && (L_2) \end{aligned}$$

$y \neq 0$: Off-Axis Equilibrium Points

$$\frac{1-\mu}{r_1^3} = \frac{1-\mu}{r_2^3} \Rightarrow r1 = r2$$

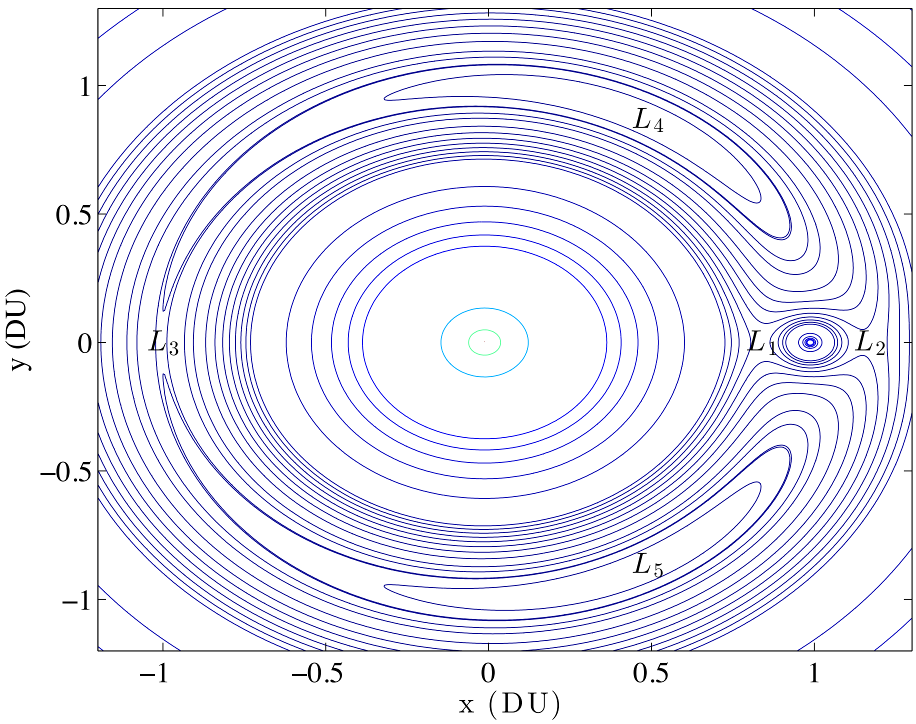
$$1 - \frac{1-\mu}{r_1^3} - \frac{\mu}{r_2^3} = 0 \Rightarrow r1 = r2 = 1$$



$$\begin{aligned} C_4 &= C_5 \\ &= U(x = \frac{1}{2} - \mu, y = \pm\frac{\sqrt{3}}{2}, z = 0) \\ &= -\frac{1}{2}\left(3 + \mu - \mu^2\right) \end{aligned}$$

$$\boxed{(x, y) = \frac{1}{2} - \mu, \pm\frac{\sqrt{3}}{2}}$$

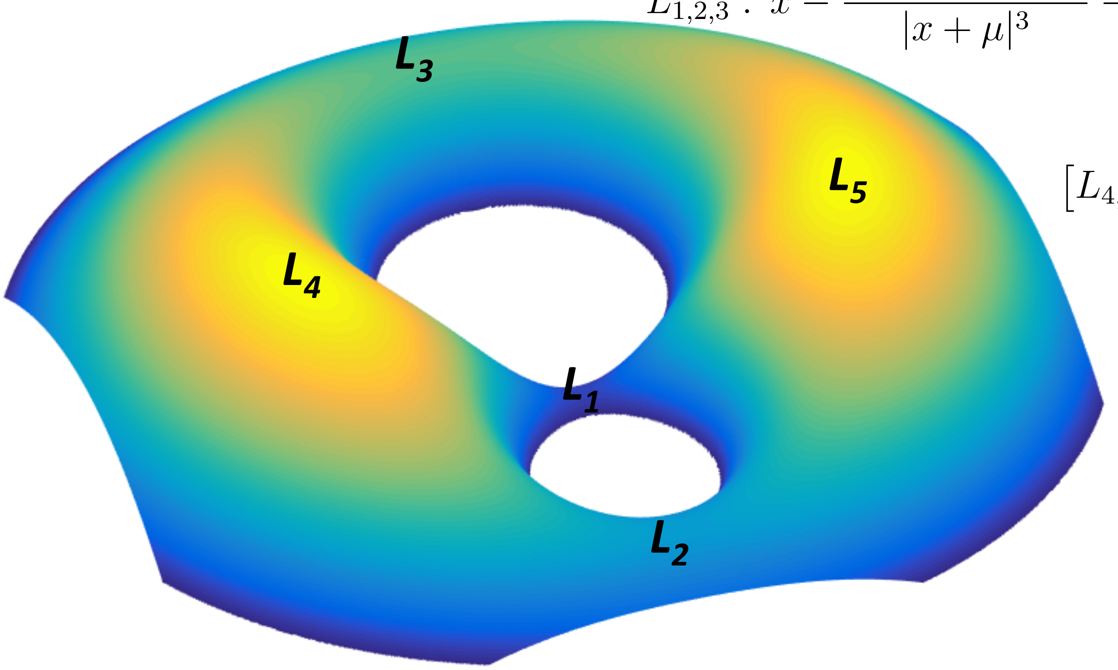
Earth/Moon Hill Curves ($\mu = 1/81$)



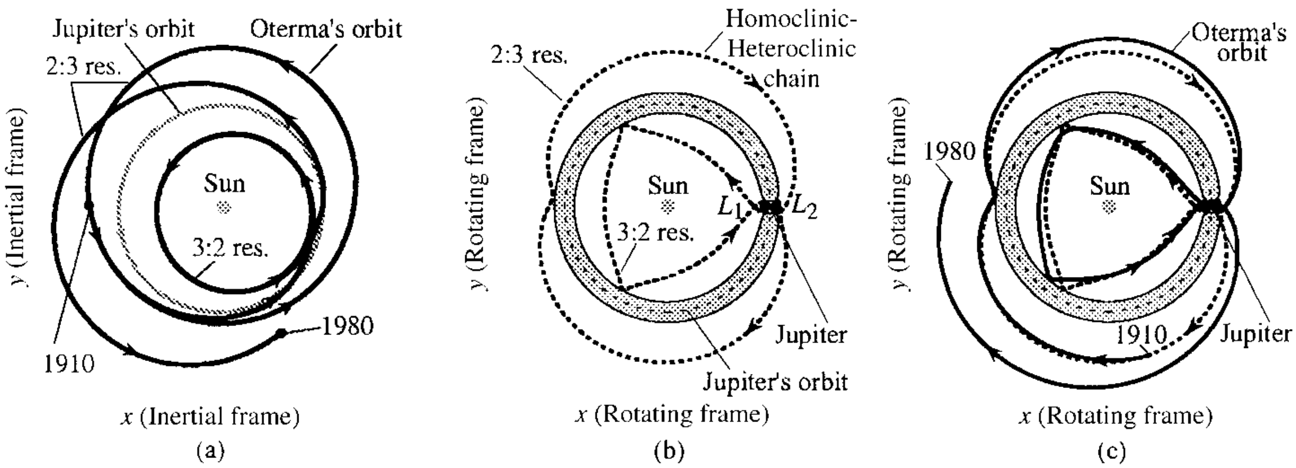
The Lagrange Points

$$L_{1,2,3}: \; x - \frac{(1-\mu)(x+\mu)}{|x+\mu|^3} - \frac{\mu(x-1+\mu)}{|x-1+\mu|^3} = 0$$

$$[L_{4,5}]_{\mathcal{B}} = \begin{bmatrix} -\frac{1}{2} - \mu \\ \pm\frac{\sqrt{3}}{2} \end{bmatrix}_{\mathcal{B}}$$

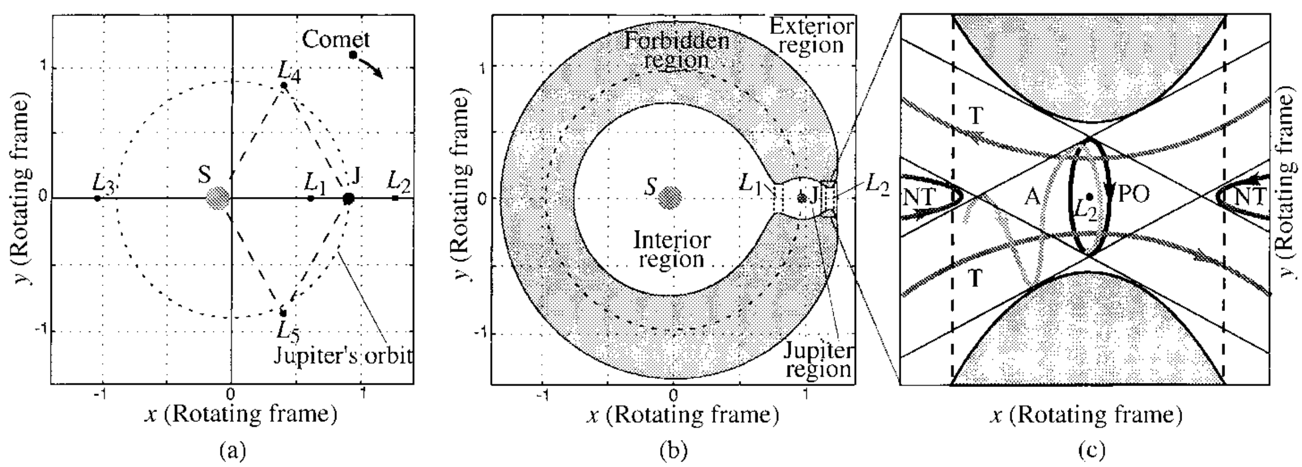


Comet Oterma



Koon et al. (2001) Fig. 1

Flows about Equilibrium Points



Koon et al. (2001) Fig. 2

Perturbation of $L_{4/5}$ Points

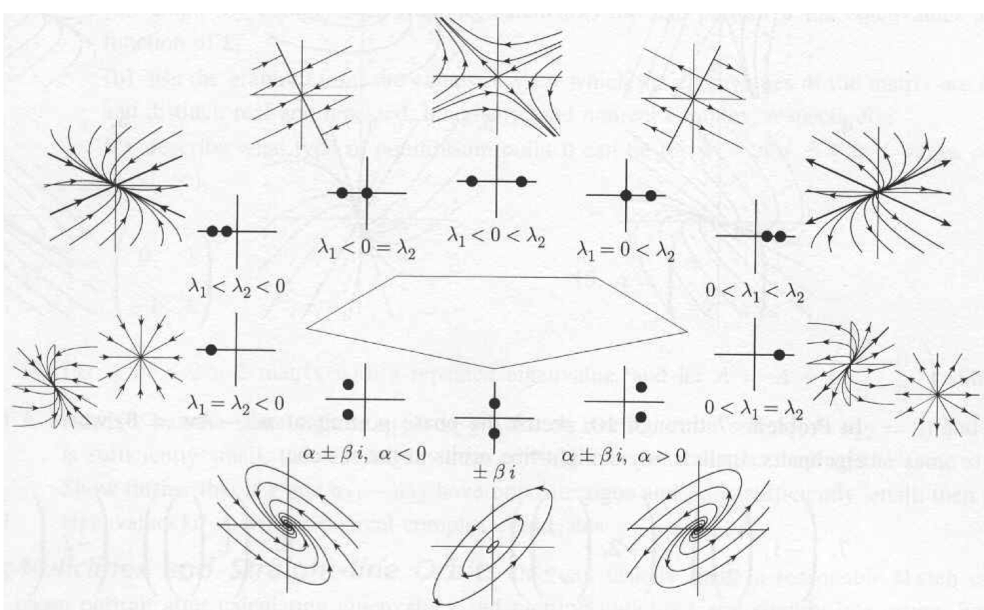
$$\ddot{x} = -\frac{\partial U}{\partial x} + 2\dot{y} \qquad \ddot{y} = -\frac{\partial U}{\partial y} - 2\dot{x} \qquad \ddot{z} = -\frac{\partial U}{\partial z}$$

$$U \triangleq -\frac{1}{2}(x^2+y^2) - \left(\frac{1-\mu}{r_1} + \frac{\mu}{r_2}\right) \quad \Leftrightarrow \quad \frac{\partial U}{\partial x} = \left.\frac{\partial U}{\partial x}\right|_{L_i} + \alpha \left.\frac{\partial^2 U}{\partial x^2}\right|_{L_i} + \beta \left.\frac{\partial^2 U}{\partial x \partial y}\right|_{L_i} + \cdots$$

$$\left.\begin{aligned} \left.\tfrac{\partial U}{\partial x}\right|_{L_{4/5}} &\approx -\left(\tfrac{3\alpha}{4} + \tfrac{3\sqrt{3}}{4}(1-2\mu)\beta\right) \\ \left.\tfrac{\partial U}{\partial y}\right|_{L_{4/5}} &\approx -\left(\tfrac{9\beta}{4} + \tfrac{3\sqrt{3}}{4}(1-2\mu)\alpha\right) \end{aligned}\right\} \quad \begin{aligned} \ddot{x} - 2\dot{y} &= -\frac{\partial U}{\partial x} = \frac{3\alpha}{4} + \frac{3\sqrt{3}}{4}(1-2\mu)\beta \\ \ddot{y} + 2\dot{x} &= -\frac{\partial U}{\partial y} = \frac{9\beta}{4} + \frac{3\sqrt{3}}{4}(1-2\mu)\alpha \end{aligned}$$

$$\left.\begin{aligned} \alpha &\triangleq Ae^{\lambda t} \\ \beta &\triangleq Be^{\lambda t} \end{aligned}\right\} \left.\begin{aligned} A\lambda^2 - 2B\lambda &= \tfrac{3A}{4} + \tfrac{3\sqrt{3}}{4}(1-2\mu)B \\ B\lambda^2 + 2A\lambda &= \tfrac{9B}{4} + \tfrac{3\sqrt{3}}{4}(1-2\mu)A \end{aligned}\right\} \quad \begin{aligned} &\lambda^4 + \lambda^2 + \frac{27}{4}\mu(1-\mu) = 0 \Rightarrow \\ &\lambda^2 = -\frac{1}{2} \pm \frac{1}{2}\sqrt{1 - 27\mu(1-\mu)} \end{aligned}$$

System Behavior About Equilibrium Points



From: Hollis (2002)

Perturbation of $L_{1\ldots3}$ Points

$$\ddot{\alpha} - 2\dot{\beta} = -\left.\frac{\partial U}{\partial x}\right|_{L_{1\ldots3}} = \alpha(1+2D)$$

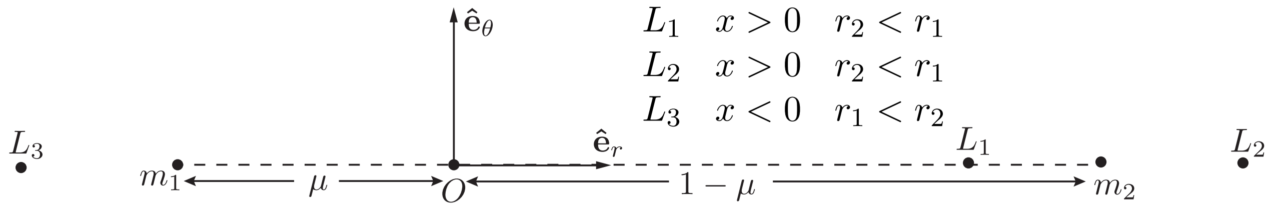
$$\ddot{\beta} + 2\dot{\alpha} = -\left.\frac{\partial U}{\partial y}\right|_{L_{1\ldots3}} = \beta(1-D)$$

$$D \triangleq \frac{1-\mu}{r_1^3} + \frac{\mu}{r_2^3}$$

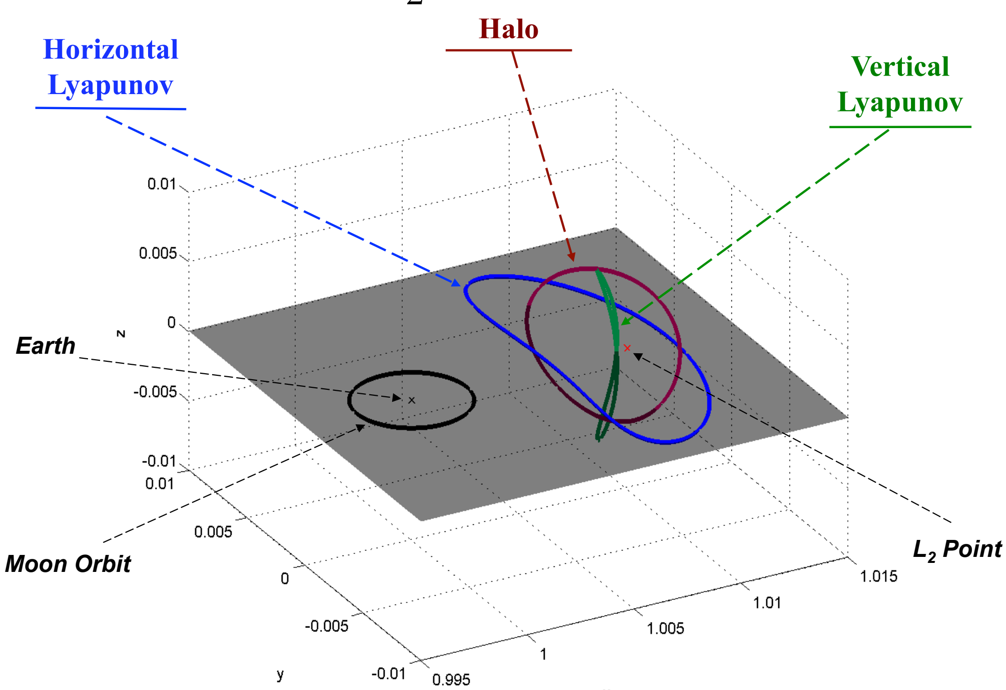
$$\left.\begin{aligned} \alpha &\triangleq Ae^{\lambda t} \\ \beta &\triangleq Be^{\lambda t} \end{aligned}\right\} \begin{aligned} &\lambda^4 + (2-D)\lambda^2 + (1+2D)(1-D) = 0 \Rightarrow \\ &\lambda^2 = \left(\frac{D}{2} - 1\right) \pm \frac{1}{2}\sqrt{D(9D-8)} \end{aligned}$$

$$x - \frac{(1-\mu)(x+\mu)}{|x+\mu|^3} - \frac{\mu(x-1+\mu)}{|x-1+\mu|^3} = 0 \Longrightarrow 1 - D = \frac{\mu(1-\mu)}{x}\left(\frac{1}{r_1^3} - \frac{1}{r_2^3}\right)$$

L_1	$x > 0$	$r_2 < r_1$
L_2	$x > 0$	$r_2 < r_1$
L_3	$x < 0$	$r_1 < r_2$

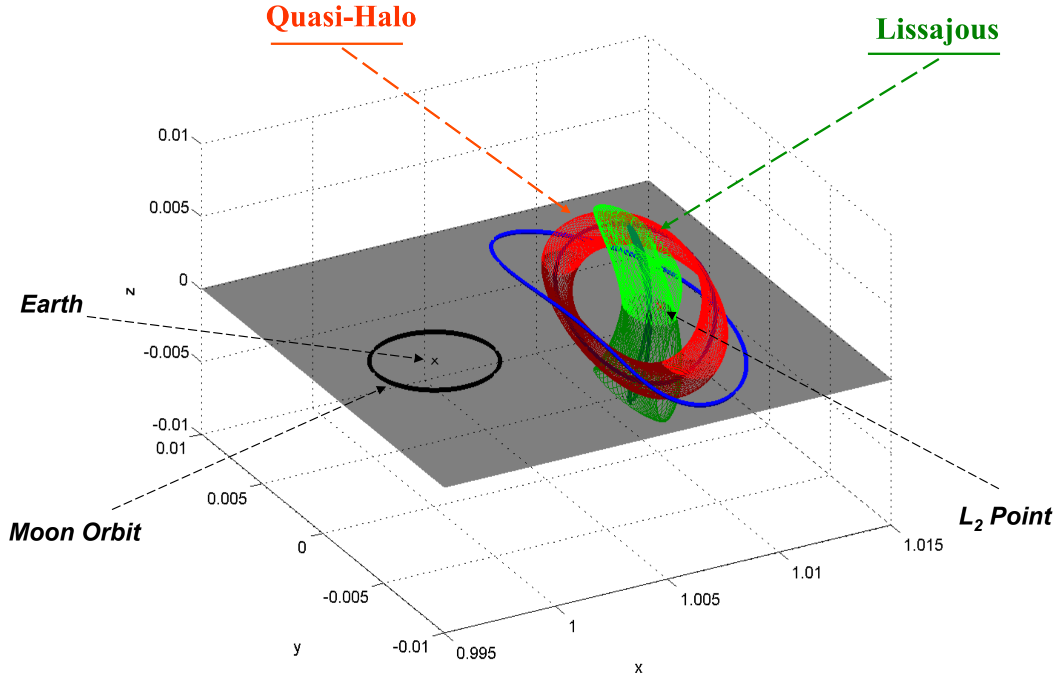


Stable Structures About L_2



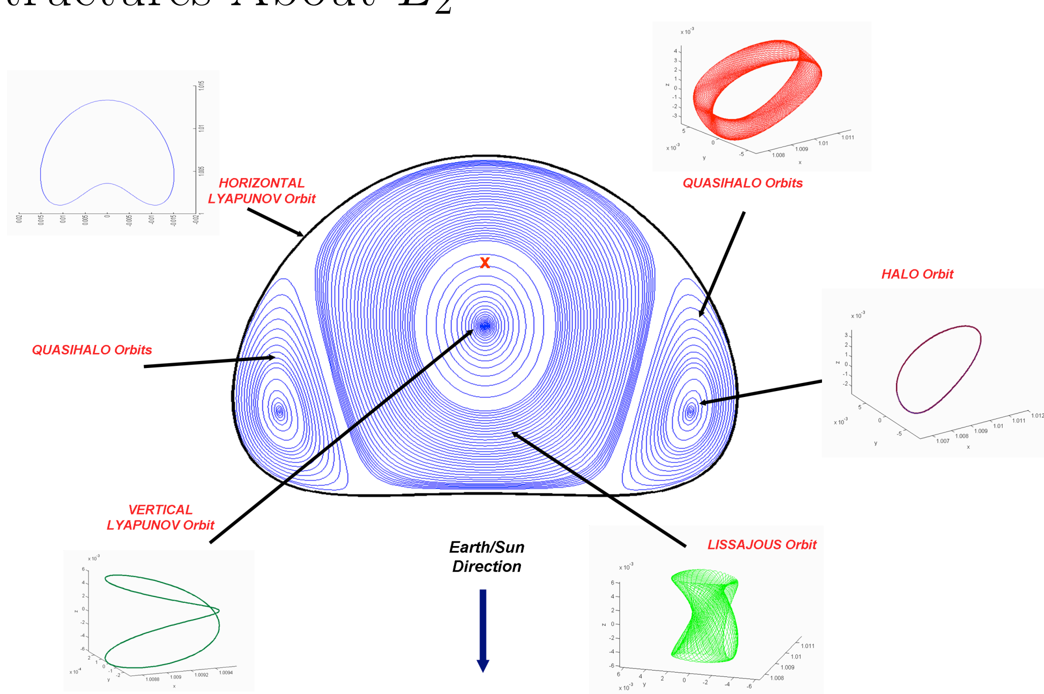
See: Kolemen et al. (2012)

Stable Structures About L_2



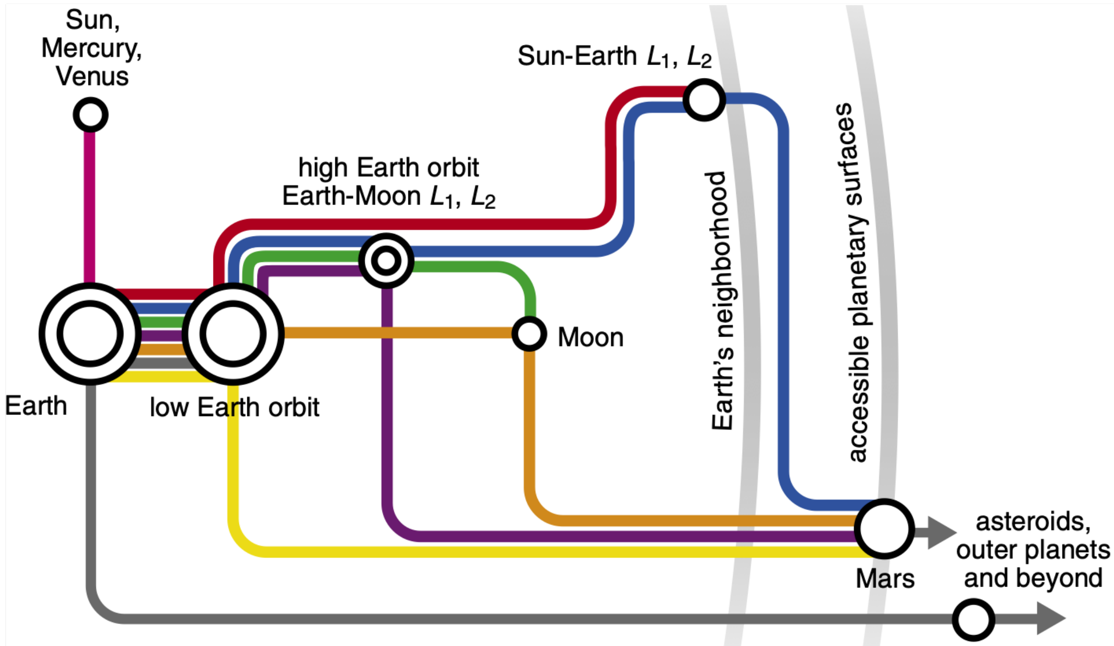
See: Kolemen et al. (2012)

Stable Structures About L_2



See: Kolemen et al. (2012)

The Interplanetary Transport Network



By: Gary L. Martin, NASA Space Architect Office

The Tisserand Criterion

$$\frac{1}{2}\left({}^{\mathcal{B}}\mathbf{v}_{P/O}\cdot\underbrace{{}^{\mathcal{B}}\mathbf{v}_{P/O}}\right)-\frac{x^2+y^2}{2}-\left(\frac{1-\mu}{r_1}+\frac{\mu}{r_2}\right)=C$$

$${}^{\mathcal{B}}\mathbf{v}_{P/O}={}^{\mathcal{I}}\mathbf{v}_{P/O}-\mathbf{n}\times r_{P/O}$$

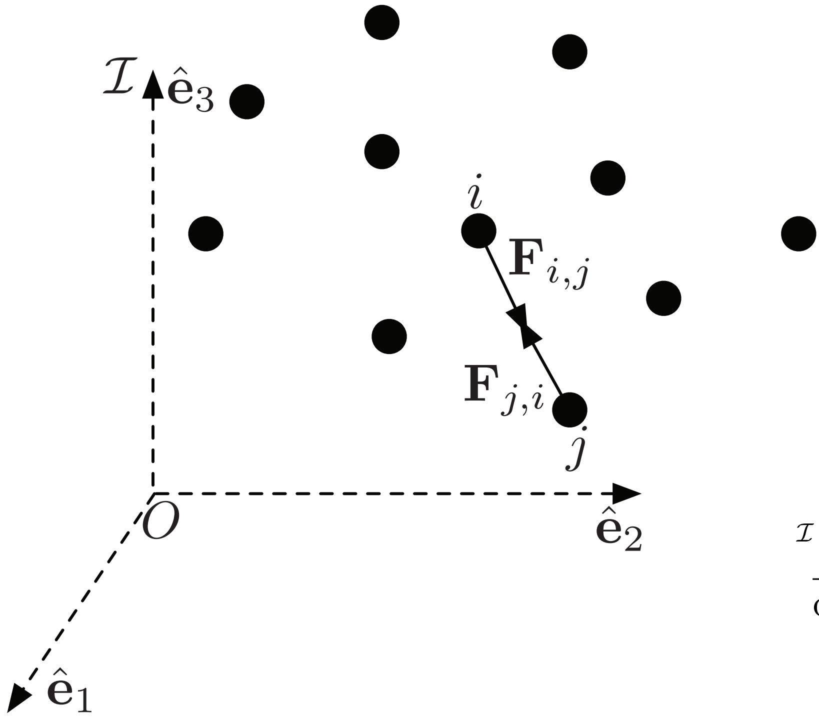
$$C=\frac{1}{2}\left({}^{\mathcal{I}}\mathbf{v}_{P/O}\cdot{}^{\mathcal{I}}\mathbf{v}_{P/O}\right)-\hat{\mathbf{e}}_3\cdot{}^{\mathcal{I}}\mathbf{h}_P-\left(\frac{1-\mu}{r_1}+\frac{\mu}{r_2}\right)$$

$$\frac{1}{2}\left({}^{\mathcal{I}}\mathbf{v}_{P/1}\cdot{}^{\mathcal{I}}\mathbf{v}_{P/1}\right)=\frac{1}{r_1}-\frac{1}{2a}\qquad\qquad\hat{\mathbf{e}}_3\cdot{}^{\mathcal{I}}\mathbf{h}_P=\sqrt{a(1-e^2)}\cos(I)$$

$$\frac{1}{a}+2\sqrt{a(1-e^2)}\cos(I)+\underbrace{2\mu\left(\frac{1}{r_2}-\frac{1}{r_1}\right)}_{\text{small}}=-2C$$

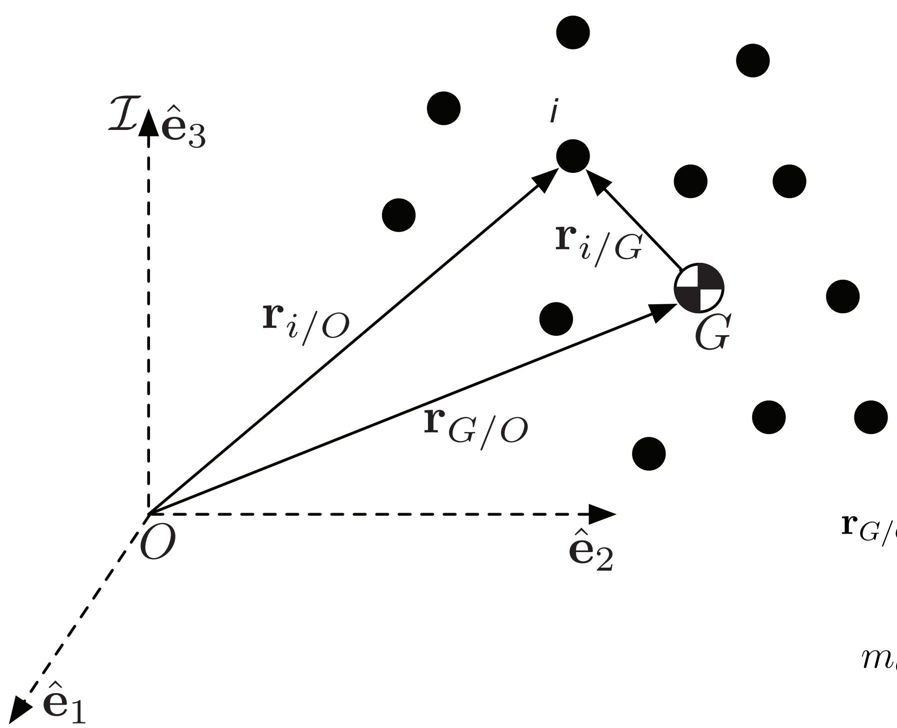
$$\boxed{T\triangleq\frac{1}{a}+2\sqrt{a(1-e^2)}\cos(I)\approx-2C}$$

The N-Body Problem



$$\frac{{}^{\mathcal{I}}\mathrm{d}^2}{\mathrm{d}t^2}\mathbf{r}_{i/O} = -G\sum_{\substack{j=1\\ j\neq i}}^{N} \frac{m_j}{\|\mathbf{r}_{i/j}\|^3}\mathbf{r}_{i/j}$$

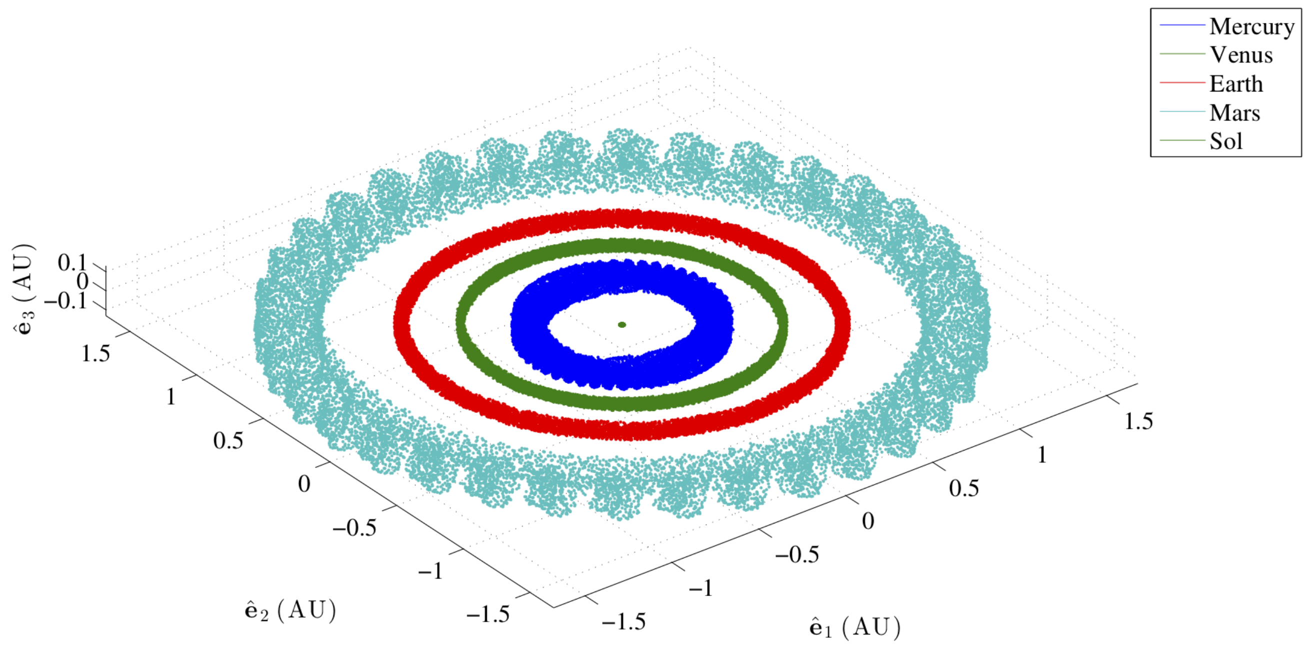
Center of Mass



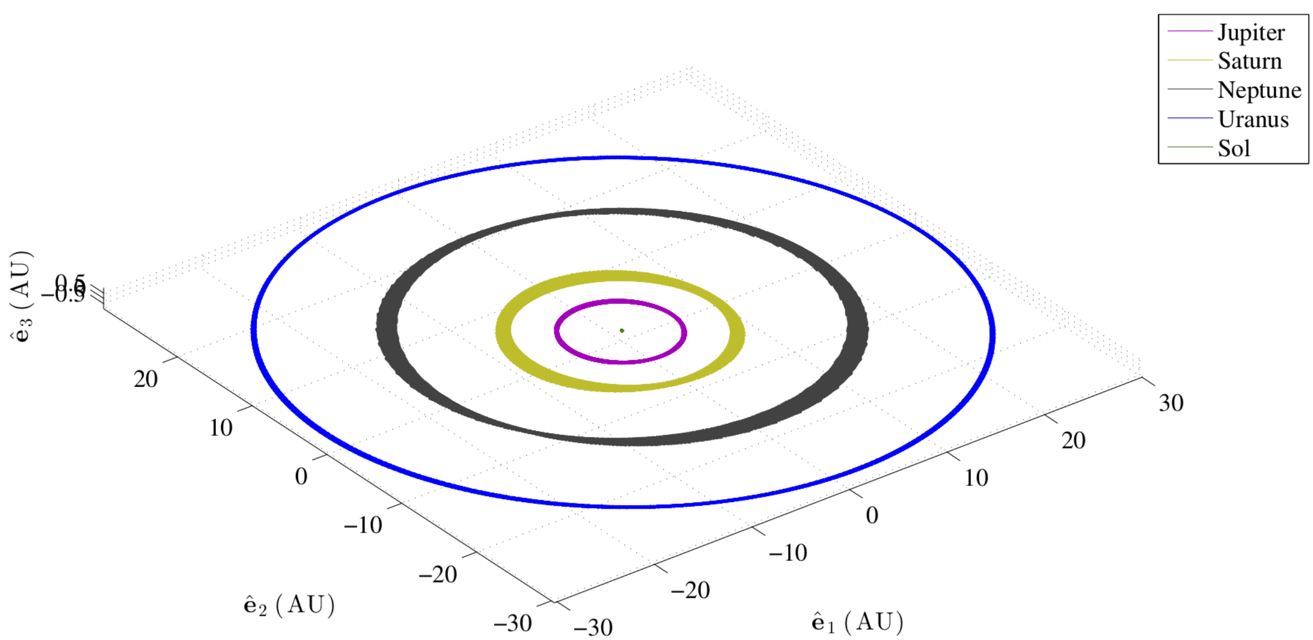
$$\mathbf{r}_{G/O} \triangleq \frac{1}{m_G}\sum_{i=1}^{N} m_i \mathbf{r}_{i/O}$$

$$m_G \triangleq \sum_{i=1}^{N} m_i$$

Inner Solar System over 100,000 Years



Outer Solar System over 100,000 Years



Earth Orbital Elements over 100,000 Years