Module-5

Laplace Transform

Introduction

- **Laplace transformation** is a technique for solving differential equations.

- **Definition**

Let $f(t)$ be a function defined for $t > 0$. The Laplace transform of $f(t)$ is denoted by $L\{f(t)\} or\ \bar{f}$(s) and defined as-

$L\{f(t)\} = \bar{f}$(s)=$\int_0^{\infty} e^{-st} f(t)dt$

- Where s is parameter (real or complex)

IMPORTANT FORMULAE

(1) $L(1) = \frac{1}{s}$ (2) $L(t^n) = \frac{n!}{s^{n+1}}$, when $n = 0, 1, 2, 3....$

(3) $L(e^{at}) = \frac{1}{s-a}$ $(s > a)$

(4) $L(\cosh at) = \frac{s}{s^2 - a^2}$ $(s^2 > a^2)$

(5) $L(\sinh at) = \frac{a}{s^2 - a^2}$ $(s^2 > a^2)$

(6) $L \sin(at) = \frac{a}{s^2 + a^2}$ $(s > 0)$

(7) $L(\cos at) = \frac{s}{s^2 + a^2}$ $(s > 0)$

Formulae of Laplace Transform

S.No.	$f(t)$	$F(s)$
1.	e^{at}	$\frac{1}{s-a}$
2.	t^n	$\frac{\Gamma n+1}{s^{n+1}}$ or $\frac{n!}{s^{n+1}}$
3.	$\sin at$	$\frac{a}{s^2+a^2}$
4.	$\cos at$	$\frac{s}{s^2+a^2}$
5.	$\sinh at$	$\frac{a}{s^2-a^2}$
6.	$\cosh at$	$\frac{s}{s^2-a^2}$
7.	$U(t-a)$	$\frac{e^{-as}}{s}$
8.	$\delta(t-a)$	e^{-as}
9.	$e^{bt}\sin at$	$\frac{a}{(s-b)^2+a^2}$

Formulae of Laplace Transform

10.	$e^{bt} \cos at$	$\dfrac{s-b}{(s-b)^2 + a^2}$
11.	$\dfrac{t}{2a} \sin at$	$\dfrac{s}{(s^2 + a^2)^2}$
12.	$t \cos at$	$\dfrac{s^2 - a^2}{(s^2 + a^2)^2}$
13.	$\dfrac{1}{2a^3} (\sin at - at \cos at)$	$\dfrac{1}{(s^2 + a^2)^2}$
14.	$\dfrac{1}{2a} (\sin at + at \cos at)$	$\dfrac{s^2}{(s^2 + a^2)^2}$

Find the Laplace transform of $f(t) = 1$.

$$L\{f(t)\} = \bar{f}(s) = \int_0^\infty e^{-st} f(t)dt$$

$$F(s) = \int_0^\infty e^{-st}\, dt$$

$$\mathcal{L}(1) = \frac{1}{s}, \quad s > 0,$$

Find the Laplace transform of $f(t) = t$.

$$F(s) = \int_0^{\infty} e^{-st} t \, dt.$$

$$L\,(t^n) = \frac{n\,!}{s^{n+1}}, \text{ when } n = 0, 1, 2, 3....$$

Ans

$$F(s) = \frac{1}{s^2}$$

Find the Laplace transform of $f(t) = e^{at}$, where a is a constant.

$$\mathcal{L}(e^{at}) = \frac{1}{s-a}, \quad s > a,$$

Properties of Laplace Transform

S.No.	Property	$f(t)$	$F(s)$
1.	Scaling	$f(at)$	$\frac{1}{a}F\left(\frac{s}{a}\right) \qquad a>0$
2.	Derivative	$\frac{df(t)}{dt}$	$sF(s)-f(0) \qquad s>0$
		$\frac{d^2f(t)}{dt^2}$	$s^2F(s)-sf(0)-f'(0) \qquad s>0$
		$\frac{d^3f(t)}{dt^3}$	$s^3F(s)-s^2f(0)-sf'(0)-f''(0)$
3.	Integral	$\int_0^t f(t)\,dt$	$\frac{1}{s}F(s) \qquad s>0$
4.	Initial Value	$\lim_{t\to 0} f(t)$	$\lim_{s\to\infty} sF(s)$
5.	Final Value	$\lim_{t\to\infty} f(t)$	$\lim_{s\to 0} sF(s)$

Properties of Laplace Transform

6.	First shifting	$e^{-at} f(t)$	$F(s+a)$
7.	Second shifting	$f(t)\, U(t-a)$	$e^{-as}\, \mathrm{L}\, f(t+a)$
8.	Multiplication by t	$t\, f(t)$	$-\frac{d}{ds} F(s)$
		$t^n f(t)$	$(-1)^n \frac{d^n}{ds^n} F(s)$
9.	Division by $\boldsymbol{t}$	$\frac{1}{t} f(t)$	$\int_s^{\infty} F(s)\, ds$
10.	Periodic function	$f(t)$	$\frac{\int_0^T e^{-st} f(t)}{1-e^{-st}}$ *ii*RçQDÇÔçÿ) $f(t+T) = f(t)$
11.	Convolution	$f(t) * g(t)$	$F(s)\, G(s)$

Properties of Laplace Transform

(1) $\mathrm{L}\,[a\,f_1(t) + b\,f_2(t)] = a\,\mathrm{L}\,[f_1(t)] + b\,\mathrm{L}\,[f_2(t)]$

Proof. $\mathrm{L}\,[a\,f_1(t) + b\,f_2(t) = \int_0^{\infty} e^{-st}\,[a\,f_1(t) + b\,f_2(t)]$

$$= a\int_0^{\infty} e^{-st} f_1(t)\,dt + b\int_0^{\infty} e^{-st} f_2(t)\,dt$$

$$= a\,\mathrm{L}\,f_1(t) + b\,\mathrm{L}\,f_2(t)$$

Proved

(2) **First Shifting Theorem.** If $\mathrm{L} f(t) = \mathrm{F}(s)$, then

$$\mathbf{L}\,[e^{at} f(t)] = \mathbf{F}(s-a)$$

Proof.

$$\mathrm{L}\,[e^{at} f(t)] = \int_0^{\infty} e^{-st}\,.\,e^{at} f(t)\,dt = \int_0^{\infty} e^{-(s-a)\,t} f(t)\,dt$$

$$= \int_0^{\infty} e^{-rt} f(t)\,dt \qquad \text{where } r = s-a$$

$$= F(r) = F(s-a)$$

With the help of this property, we can have the following important results :

(1) $\mathrm{L}\,(e^{at}\, t^n) = \dfrac{n\,!}{(s-a)^{n+1}}$ $\qquad \left[\mathrm{L}\,(t^n) = \dfrac{n\,!}{s^{n+1}}\right]$

(2) $\mathrm{L}\,(e^{at} \cosh bt) = \dfrac{s-a}{(s-a)^2 - b^2}$

(3) $\mathrm{L}\,(e^{at} \sinh bt) = \dfrac{b}{(s-a)^2 - b^2}$

(4) $\mathrm{L}\,(e^{at} \sin bt) = \dfrac{b}{(s-a)^2 + b^2}$

(5) $\mathrm{L}\,(e^{at} \cos bt) = \dfrac{s-a}{(s-a)^2 + b^2}$

Example 3. *Find the Laplace transform of $\cos^2 t$.*

Solution.

$$\cos 2t = 2\cos^2 t - 1$$

$$\therefore \quad \cos^2 t = \frac{1}{2}[\cos 2t + 1]$$

$$L(\cos^2 t) = L\left[\frac{1}{2}(\cos 2t + 1)\right] = \frac{1}{2}[L(\cos 2t) + L(1)]$$

$$= \frac{1}{2}\left[\frac{s}{s^2 + (2)^2} + \frac{1}{s}\right] = \frac{1}{2}\left[\frac{s}{s^2 + 4} + \frac{1}{s}\right] \qquad \textbf{Ans.}$$

Example 4. *Find the Laplace Transform of* $t^{-\frac{1}{2}}$.

Solution. We know that $L(t^n) = \dfrac{\overline{|n+1}}{s^{n+1}}$

Put $n = -\dfrac{1}{2}$, $L(t^{-1/2}) = \dfrac{\overline{\left|-\frac{1}{2}+1\right.}}{s^{-1/2+1}} = \dfrac{\overline{\left|\frac{1}{2}\right.}}{\sqrt{s}} = \dfrac{\sqrt{\pi}}{\sqrt{s}}$ where $\overline{\left|\frac{1}{2}\right.} = \sqrt{\pi}$ **Ans.**

Find the Laplace Trnasform of t sin at.

Solution.

$$L(t\sin at) = L\left(t\,\frac{e^{iat}-e^{-iat}}{2i}\right) = \frac{1}{2i}[L(t\,.\,e^{iat}) - L(te^{-iat})]$$

$$= \frac{1}{2i}\left[\frac{1}{(s-ia)^2} - \frac{1}{(s+ia)^2}\right] = \frac{1}{2i}\left[\frac{(s+ia)^2-(s-ia)^2}{(s-ia)^2\,(s+ia)^2}\right]$$

$$= \frac{1}{2i}\,\frac{(s^2+2\,ias-a^2)-(s^2-2\,ias-a^2)}{(s^2+a^2)^2}$$

$$= \frac{1}{2i}\,\frac{4\,ias}{(s^2+a^2)^2} = \frac{2\,as}{(s^2+a^2)^2}$$ **Ans.**

LAPLACE TRANSFORM OF THE DERIVATIVE OF $f(t)$

$$\mathbf{L}[f'(t)] = s\,\mathbf{L}[f(t)] - f(0) \qquad \text{where } \mathrm{L}[f(t)] = F(s).$$

LAPLACE TRANSFORM OF DERIVATIVE OF ORDER n.

$$\mathrm{L}[f^{n}(t)] = s^{n}\,\mathrm{L}[f(t)] - s^{n-1} f(0) - s^{n-2} f'(0) - s^{n-3} f''(0) - \ldots - f^{n-1}(0).$$

LAPLACE TRANSFORM OF INTEGRAL OF $f(t)$

$$\mathbf{L}\left[\int_0^t f(t)\,dt\right] = \frac{1}{s}\mathbf{F}(s), \qquad \text{where } \mathrm{L}[f(t)] = F(s).$$

LAPLACE TRANSFORM OF $t.\,f(t)$ (Multiplication by t)

If $L[f(t)] = F(s)$, then

$$L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}[F(s)].$$

Example 7. *Find the Laplace transform of t sinh at.*

Solution.

$$\mathrm{L}\,(\sinh at) = \frac{a}{s^2 - a^2}$$

$$\therefore \quad \mathrm{L}\,[t \sinh at] = -\frac{d}{ds}\left(\frac{a}{s^2 - a^2}\right)$$

or

$$\mathrm{L}\,[t \sinh at] = \frac{2\,as}{(s^2 - a^2)^2}$$

Ans.

Example 8. *Find the Laplace transform of t^2 cos at.*

Solution.
$$L(\cos at) = \frac{s}{s^2 + a^2}$$

$$L(t^2 \cos at) = (-1)^2 \frac{d^2}{ds^2}\left[\frac{s}{s^2 + a^2}\right] = \frac{d}{ds}\frac{(s^2 + a^2) \, . \, 1 - s\,(2\,s)}{(s^2 + a^2)^2} = \frac{d}{ds}\frac{a^2 - s^2}{(s^2 + a^2)^2}$$

$$= \frac{(s^2 + a^2)^2\,(-2\,s) - (a^2 - s^2)\,.\,2\,(s^2 + a^2)\,(2\,s)}{(s^2 + a^2)^4} = \frac{-2\,s^3 - 2\,a^2\,s - 4\,a^2 s + 4\,s^3}{(s^2 + a^2)^3}$$

$$= \frac{2\,s\,(s^2 - 3\,a^2)}{(s^2 + a^2)^3}$$ **Ans.**

LAPLACE TRANSFORM OF $\frac{1}{t} f(t)$ (Division by t)

If $L[f(t)] = F(s)$, then $L\left[\frac{1}{t} f(t)\right] = \int_s^\infty F(s)\, ds$

Example 10. *Find the Laplace transform of* $\frac{\sin 2t}{t}$.

Solution. $L(\sin 2t) = \frac{2}{s^2+4}$

$$L\left(\frac{\sin 2t}{t}\right) = \int_s^\infty \frac{2}{s^2+4}\, ds = 2 \cdot \frac{1}{2}\left[\tan^{-1}\frac{s}{2}\right]_s^\infty$$

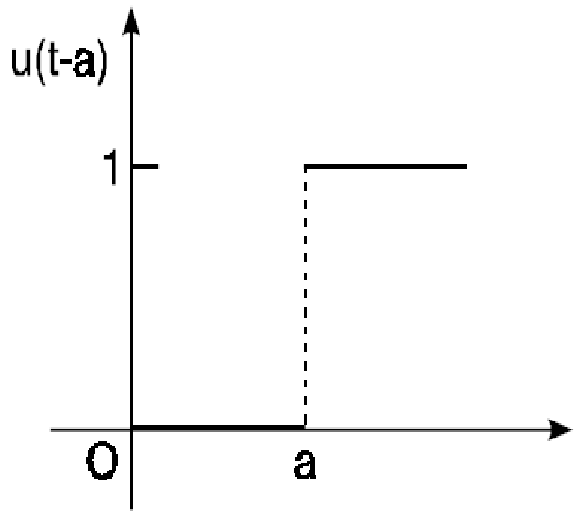
$$= \left[\tan^{-1}\infty - \tan^{-1}\frac{s}{2}\right] = \frac{\pi}{2} - \tan^{-1}\frac{s}{2}$$

$$= \cot^{-1}\frac{s}{2}$$ **Ans.**

13.10 UNIT STEP FUNCTION

With the help of unit step functions, we can find the inverse transform of functions, which cannot be determined with previous methods.

The unit step functions $u(t-a)$ is defined as follows:



$$u(t-a) = \begin{cases} 0 & \text{when } t < a \\ 1 & \text{when } t \geq a \end{cases}$$

where $a \geq 0$.

Example 14. *Express the following function in terms of units step functions and find its Laplace transform:*

$$f(t) = \begin{cases} 8, & t < 2 \\ 6, & t > 2 \end{cases}$$

Solution.

$$f(t) = \begin{cases} 8+0 & t < 2 \\ 8-2 & t > 2 \end{cases}$$

$$= 8 + \begin{cases} 0 & t < 2 \\ -2 & t > 2 \end{cases} \quad = \quad 8 + (-2)\begin{cases} 0, & t < 2 \\ 1, & t > 2 \end{cases}$$

$$= 8 - 2\,u\,(t-2)$$

$$\mathrm{L}\,f(t) = 8\,\mathrm{L}\,(1) - 2\,\mathrm{L}\,u\,(t-2) = \frac{8}{s} - 2\,\frac{e^{-2s}}{s}$$ **Ans.**

SECOND SHIFTING THEOREM

If $\mathrm{L}\,[f\,(t)] = F\,(s)$, then $\mathbf{L}\,[\boldsymbol{f}\,(\boldsymbol{t}-\boldsymbol{a})\,.\,\boldsymbol{u}\,(\boldsymbol{t}-\boldsymbol{a}) = \boldsymbol{e}^{-as}\,\boldsymbol{F}\,(s)$.

THEOREM $\quad \mathrm{L}\, f\,(t)\, u\,(t-a) = e^{-as}\,\mathrm{L}\,[f\,(t+a)]$

Example 18. *Find the Laplace Transform of $t^2\, u\,(t-3)$.*

Solution.

$$t^2 \,.\, u\,(t-3) = [(t-3)^2 + 6\,(t-3) + 9]\, u\,(t-3)$$

$$= (t-3)^2 \,.\, u\,(t-3) + 6\,(t-3)\,.\, u\,(t-3) + 9\, u\,(t-3)$$

$$\mathrm{L}\, t^2 \,.\, u\,(t-3) = \mathrm{L}\,(t-3)^2 \,.\, u\,(t-3) + 6\,\mathrm{L}\,(t-3)\,.\, u\,(t-3) + 9\,\mathrm{L}\, u\,(t-3)$$

$$= e^{-3s}\left[\frac{2}{s^3} + \frac{6}{s^2} + \frac{9}{s}\right] \qquad \textbf{Ans.}$$

Aliter

$$\mathrm{L}\, t^2\, u\,(t-3) = e^{-3s}\,\mathrm{L}\,(t+3)^2 = e^{-3s}\,\mathrm{L}\,[t^2 + 6t + 9]$$

$$= e^{3s}\left[\frac{2}{s^3} + \frac{6}{s^2} + \frac{9}{s}\right] \qquad \textbf{Ans.}$$

$$= e^{-3s}\left[\frac{2}{s^3} + \frac{6}{s^2} + \frac{9}{s}\right]$$

$$\mathrm{L}\, t^2\, u\,(t-3) = e^{-3s}\,\mathrm{L}\,(t+3)^2 = e^{-3s}\,\mathrm{L}\,[t^2 + 6t + 9]$$

$$= e^{3s}\left[\frac{2}{s^3} + \frac{6}{s^2} + \frac{9}{s}\right]$$

Find the Laplace transform of $e^{-2t}\, u_\pi(t)$.

$$u_\pi(t) = \begin{cases} 0\,; & t < \pi \\ 1\,; & t > \pi \end{cases}$$

Convolution of Two Functions

- If F (t) and G (t) are two functions of class A, then the convolution of F and G is denoted by F*G and is defined by

$$F * G = \int_0^t F(x)G(t-x)\,dx$$

- **Some Properties of Convolution**
- **Commutative** : F*G=G*F
- **Associative** : F*(G*H)=(F*G)*H
- **Distributive over addition**: F*(G+H)=F*G+F*H

Convolution Theorem

If $$L^{-1}\{f(s)\} = F(t) \ \ and \ \ L^{-1}\{g(s)\} = G(t)$$

Where F(t) and G(t) are functions of class A, then

$$L^{-1}\{f(s)g(s)\} = \int_0^t F(x)G(t-x)dx = F*G$$

Example of Solution of an ODE

$$\frac{d^2 y}{dt^2} + 6\frac{dy}{dt} + 8y = 2 \quad y(0) = y'(0) = 0$$

- ODE w/initial conditions

$$s^2 Y(s) + 6sY(s) + 8Y(s) = 2/s$$

- Apply Laplace transform to each term
- Solve for Y(s)

$$Y(s) = \frac{2}{s(s+2)(s+4)}$$

- Apply partial fraction expansion

$$Y(s) = \frac{1}{4s} + \frac{-1}{2(s+2)} + \frac{1}{4(s+4)}$$

- Apply inverse Laplace transform to each term

$$y(t) = \frac{1}{4} - \frac{e^{-2t}}{2} + \frac{e^{-4t}}{4}$$

Example 1. *Find the Laplace transform of f(t) defined as*

$$f(t) = \frac{t}{k}, \quad when\ 0 < t < k$$

$$= 1, \quad when\ t > k \qquad\qquad (Mangalore\ 1997)$$

Solution.

$$f(t) = \int_0^k \frac{t}{k}\, e^{-st}\, dt + \int_k^{\infty} 1 \cdot e^{-st}\, dt = \frac{1}{k}\left[\left(t\,\frac{e^{-st}}{-s}\right)_0^k - \int_0^k \frac{e^{-st}}{-s}\, dt\right] + \left[\frac{e^{-st}}{-s}\right]_k^{\infty}$$

$$= \frac{1}{k}\left[\frac{k\,e^{-ks}}{-s} - \left(\frac{e^{-st}}{s^2}\right)_0^k\right] + \frac{e^{-ks}}{s} = \frac{1}{k}\left[\frac{k\,e^{-ks}}{-s} - \frac{e^{-sk}}{s^2} + \frac{1}{s^2}\right] + \frac{e^{-ks}}{s}$$

$$= -\frac{e^{-sk}}{s} - \frac{1}{k}\,\frac{e^{-ks}}{s^2} + \frac{1}{k}\,\frac{1}{s^2} + \frac{e^{-ks}}{s} = \frac{1}{ks^2}\left[-e^{-ks} + 1\right]$$

Ans.

Example 7. *Find the Laplace transform of t sinh at.*

Solution.

$$L(\sinh at) = \frac{a}{s^2 - a^2}$$

$\therefore$

$$L[t \sinh at] = -\frac{d}{ds}\left(\frac{a}{s^2 - a^2}\right)$$

or

$$L[t \sinh at] = \frac{2\,as}{(s^2 - a^2)^2}$$ **Ans.**

Example 8. *Find the Laplace transform of t^2 cos at.*

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$$L(\cos at) = \frac{s}{s^2 + a^2}$$

$$L(t^2 \cos at) = (-1)^2 \frac{d^2}{ds^2}\left[\frac{s}{s^2 + a^2}\right] = \frac{d}{ds}\frac{(s^2 + a^2)\,.\,1 - s\,(2\,s)}{(s^2 + a^2)^2} = \frac{d}{ds}\frac{a^2 - s^2}{(s^2 + a^2)^2}$$

$$= \frac{(s^2 + a^2)^2\,(-2\,s) - (a^2 - s^2)\,.\,2\,(s^2 + a^2)\,(2\,s)}{(s^2 + a^2)^4} = \frac{-2\,s^3 - 2\,a^2\,s - 4\,a^2 s + 4\,s^3}{(s^2 + a^2)^3}$$

$$= \frac{2\,s\,(s^2 - 3\,a^2)}{(s^2 + a^2)^3}$$ **Ans.**

Example 10. *Find the Laplace transform of* $\dfrac{\sin 2t}{t}$.

Solution. $L(\sin 2t) = \dfrac{2}{s^2+4}$

$$L\left(\frac{\sin 2t}{t}\right) = \int_s^\infty \frac{2}{s^2+4}\,ds = 2\,.\,\frac{1}{2}\left[\tan^{-1}\frac{s}{2}\right]_s^\infty$$

$$= \left[\tan^{-1}\infty - \tan^{-1}\frac{s}{2}\right] = \frac{\pi}{2} - \tan^{-1}\frac{s}{2}$$

$$= \cot^{-1}\frac{s}{2}$$ **Ans.**

Example 11. *Find the Laplace transform of* $f(t) = \displaystyle\int_0^t \frac{\sin t}{t}\,dt$.

Solution.

$$L\sin t = \frac{1}{s^2+1}$$

$$L\frac{\sin t}{t} = \int_s^\infty \frac{1}{s^2+1}\,ds = \left[\tan^{-1} s\right]_s^\infty = \frac{\pi}{2} - \tan^{-1} s = \cot^{-1} s$$

$$L\int_0^t \frac{\sin t}{t}\,dt = \frac{1}{s}\cot^{-1} s$$ **Ans.**

Example 13. *Evaluate* $L\left[e^{-4t}\dfrac{\sin 3t}{t}\right]$.

Solution. $$L \sin 3t = \frac{3}{s^2+3^2} \Rightarrow L\frac{\sin 3t}{t} = \int_s^\infty \frac{3}{s^2+9}\,ds = \left[\frac{3}{3}\tan^{-1}\frac{s}{3}\right]_s^\infty$$

$$= \frac{\pi}{2} - \tan^{-1}\frac{s}{3} = \cot^{-1}\frac{s}{3}$$

$$L\left[e^{-4t}\frac{\sin 3t}{t}\right] = \cot^{-1}\frac{s+4}{3} = \tan^{-1}\frac{3}{s+4}$$ **Ans.**

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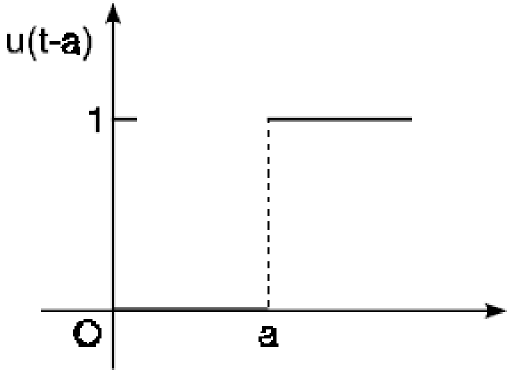
With the help of unit step functions, we can find the inverse transform of functions, which cannot be determined with previous methods.



The unit step functions $u(t-a)$ is defined as follows:

$$u(t-a) = \begin{cases} 0 & \text{when } t < a \\ 1 & \text{when } t \geq a \end{cases} \qquad \text{where } a \geq 0.$$

Example 14. *Express the following function in terms of units step functions and find its Laplace transform:*

$$f(t) = \begin{bmatrix} 8, & t < 2 \\ 6, & t > 2 \end{bmatrix}$$

Solution.

$$f(t) = \begin{bmatrix} 8+0 & t < 2 \\ 8-2 & t > 2 \end{bmatrix}$$

$$= 8 + \begin{bmatrix} 0 & t < 2 \\ -2 & t > 2 \end{bmatrix} = 8 + (-2)\begin{bmatrix} 0, & t < 2 \\ 1, & t > 2 \end{bmatrix}$$

$$= 8 - 2\,u(t-2)$$

$$\mathrm{L}\,f(t) = 8\,\mathrm{L}(1) - 2\,\mathrm{L}\,u(t-2) = \frac{8}{s} - 2\frac{e^{-2s}}{s}$$ **Ans.**

Example 17. *Express the following function in terms of unit step function :*

$$f(t) = \begin{cases} t-1 & 1 < t < 2 \\ 3-t & 2 < t < 3 \end{cases}$$

and find its Laplace transform.

Solution. $$f(t) = \begin{cases} t-1 & 1 < t < 2 \\ 3-t & 2 < t < 3 \end{cases}$$

$$= (t-1)\,[u\,(t-1) - u\,(t-2)] + (3-t)\,[u\,(t-2) - u\,(t-3)]$$

$$= (t-1)\,u\,(t-1) - (t-1)\,u\,(t-2) + (3-t)\,u\,(t-2) + (t-3)\,u\,(t-3)$$

$$= (t-1)\,u\,(t-1) - 2\,(t-2)\,u\,(t-2) + (t-3)\,u\,(t-3)$$

$$\mathrm{L}\,f(t) = \frac{e^{-s}}{s^2} - 2\,\frac{e^{-2s}}{s^2} + \frac{e^{-3s}}{s^2}$$ **Ans.**

Example 18. *Find the Laplace Transform of $t^2\, u(t-3)$.*

Solution.

$$t^2 . u(t-3) = [(t-3)^2 + 6(t-3) + 9]\, u(t-3)$$

$$= (t-3)^2 . u(t-3) + 6(t-3) . u(t-3) + 9\, u(t-3)$$

$$\mathrm{L}\, t^2 . u(t-3) = \mathrm{L}\,(t-3)^2 . u(t-3) + 6\,\mathrm{L}\,(t-3) . u(t-3) + 9\,\mathrm{L}\, u(t-3)$$

$$= e^{-3s}\left[\frac{2}{s^3} + \frac{6}{s^2} + \frac{9}{s}\right]$$ **Ans.**

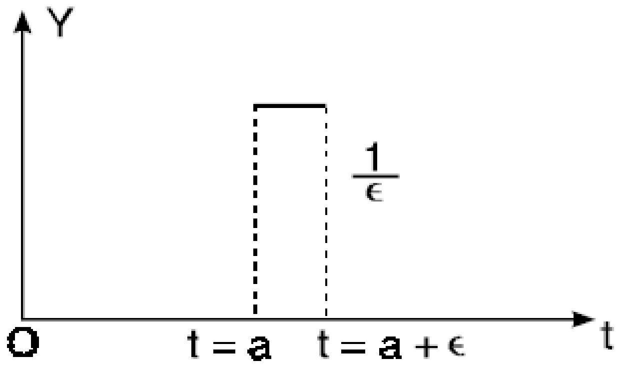
Aliter

$$\mathrm{L}\, t^2\, u(t-3) = e^{-3s}\,\mathrm{L}\,(t+3)^2 = e^{-3s}\,\mathrm{L}\,[t^2 + 6t + 9]$$

$$= e^{3s}\left[\frac{2}{s^3} + \frac{6}{s^2} + \frac{9}{s}\right]$$ **Ans.**

13.13 (1) IMPULSE FUNCTION

When a large force acts for a short time, then the product of the force and the time is called impulse in applied mechanics. The unit impulse function is the limiting function.



$$\delta(t-1) = \frac{1}{\varepsilon}, \; a < t < a+\varepsilon$$
$$= 0, \qquad \text{otherwise}$$

The value of the function (height of the strip in the figure) becomes infinite as $\varepsilon \to 0$ and the area of the rectangle is unity.

(2) The Unit Impulse function is defined as follows:

$$\delta(t-a) = \begin{cases} \infty & \text{for } t = a \\ 0 & \text{for } t \neq a. \end{cases}$$

and $$\int_0^\infty \delta(t-a)\,.\,dt = 1. \qquad \text{[Area of strip = 1]}$$

(3) Laplace Transform of unit Impulse function

$$\int_0^{\infty} f(t)\,\delta(t-a)\,dt = \int_a^{a+\varepsilon} f(t)\cdot\frac{1}{\varepsilon}\,dt$$

$$\left\{\begin{array}{l}\text{Mean value Theorem} \\ \displaystyle\int_a^b f(t)\,dt = (b-a)\,f(\eta)\end{array}\right.$$

$$= (a+\varepsilon-a)\,f(\eta),\ \frac{1}{\varepsilon} \qquad \text{where } a < \eta < a+\varepsilon$$

$$= f(\eta)$$

Property I: $\displaystyle\int_0^{\infty} f(t)\,\delta(t-a)\,dt = f(a)$ as $\varepsilon \to 0$

Note. If $f(t) = e^{-st}$ and $L[\delta(t-a)] = e^{-as}$

Example 21. *Evaluate* $\int_{-\infty}^{\infty} e^{-5t}\ \delta(t-2)$.

Solution. $\int_{-\infty}^{\infty} e^{-5t}\ \delta(t-2)\ e^{-5\times 2} = e^{-10}$

Property II: $\int_{-\infty}^{\infty} f(t)\,\delta'(t-a)\,dt = -f'(a)$

Proof.

$$\int_{-\infty}^{\infty} f(t)\,\delta'(t-a)\,dt = [f(t)\cdot\delta(t-a)]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f'(t)\,\delta(t-a)\,dt$$

$$= 0 - 0 - f'(a) = -f'(a)$$

Example 22. *Find the Laplace transform of* $t^3\,\delta(t-4)$.

Solution.

$$\mathrm{L}\ t^3\,\delta(t-4) = \int_0^{\infty} e^{-st}\,t^3\,\delta(t-4)\,dt$$

$$= 4^3\,e^{-4s}$$ **Ans.**

PERIODIC FUNCTIONS

Let $f(t)$ be a periodic function with Period T, then

$$L[f(t)] = \frac{\int_0^T e^{-st} f(t)\, dt}{1 - e^{-sT}}$$

Example 23. *Find the Laplace transform of the waveform*

$$f(t) = \left(\frac{2t}{3}\right), \quad 0 \le t \le 3.$$

Solution. $L[f(t)] = \dfrac{1}{1-e^{-st}} \displaystyle\int_0^T e^{-st} f(t)\, dt$

$$L\left[\frac{2t}{3}\right] = \frac{1}{1-e^{-3s}} \int_0^3 e^{-st}\left(\frac{2}{3}t\right)dt = \frac{1}{1-e^{-3s}} \frac{2}{3}\left[\frac{t\,e^{-st}}{-s} - (1)\frac{e^{-st}}{s^2}\right]_0^3$$

$$= \frac{2}{3}\frac{1}{1-e^{-3s}}\left[\frac{3\,e^{-3s}}{-s} - \frac{e^{-3s}}{s^2} + \frac{1}{s^2}\right] = \frac{2}{3}\cdot\frac{1}{1-e^{-3s}}\left[\frac{3\,e^{-3s}}{-s} + \frac{1-e^{-3s}}{s^2}\right]$$

$$= \frac{2\,e^{-3s}}{-s\,(1-e^{-3s})} + \frac{2}{3\,s^2}.$$

Ans.

Example 27. *A periodic square wave function $f(t)$, in terms of unit step functions, is written as*

$$f(t) = k\,[u_0(t) - 2\,u_a(t) + 2\,u_{2a}(t) - 2\,u_{3a}(t) + \ldots]$$

Show that the Laplace transform of $f(t)$ is given by

$$L\,[f(\text{t})] = \frac{k}{s}\tan h\left(\frac{as}{2}\right).$$

Solution.

$$f(t) = k\,[u_0(t) - 2\,u_a(t) + 2\,u_{2a}(t) - 2\,u_{3a}(t) + \ldots]$$

$$L f(t) = k\,[L\,u_0(t) - 2\,L\,u_a(t) + 2\,L\,u_{2a}(t) - 2\,L\,u_{3a}(t) + \ldots]$$

$$= k\left[\frac{1}{s} - 2\,\frac{e^{-as}}{s} + 2\,\frac{e^{-2as}}{s} - 2\,\frac{e^{-3as}}{s} + \ldots\right]$$

$$= \frac{k}{s}\,[1 - 2\,e^{-as} + 2\,e^{-2as} - 2\,e^{-3as} + \ldots]$$

$$= \frac{k}{s}\,[1 - 2\,(e^{-as} - e^{-2as} + e^{-3as} - \ldots)]$$

$$= \frac{k}{s}\left[1 - 2\,\frac{e^{-as}}{1 + e^{-as}}\right] = \frac{k}{s}\left[\frac{1 + e^{-as} - 2\,e^{-as}}{1 + e^{-as}}\right]$$

$$= \frac{k}{s}\left[\frac{1 - e^{-as}}{1 + e^{-as}}\right] = \frac{k}{s}\left[\frac{e^{\frac{as}{2}} - e^{-\frac{as}{2}}}{e^{\frac{as}{2}} + e^{-\frac{as}{2}}}\right] = \frac{k}{s}\tanh\frac{as}{2} \qquad \textbf{Ans.}$$

Example 28. *Evaluate* $\int_0^\infty t\, e^{-3t} \sin t \, dt$.

Solution. $$\int_0^\infty t\, e^{-3t} \sin t \, dt = \int_0^\infty t\, e^{-st} \sin t \, dt \qquad (s = 3)$$

$$= L\,(t \sin t) = -\frac{d}{ds}\left(\frac{1}{s^2+1}\right) = \frac{2s}{(s^2+1)^2}$$

$$= \frac{2 \times 3}{(3^2+1)^2} = \frac{6}{100} = \frac{3}{50}$$ **Ans.**

Example 29. *Evaluate* $\int_0^\infty \frac{e^{-t}\sin t}{t}\,dt$ *and* $\int_0^\infty \frac{\sin t}{t}\,dt$.

Solution.
$$\int_0^\infty \frac{e^{-t}\sin t}{t}\,dt = \int_0^\infty e^{-st}\,\frac{\sin t}{t}\,dt \qquad (s = 1)$$

$$= L\left[\frac{\sin t}{t}\right] = \int_s^\infty \frac{1}{s^2+1}\,ds = \left[\tan^{-1} s\right]_s^\infty$$

$$= \frac{\pi}{2} - \tan^{-1} s \quad \ldots (1) \qquad = \frac{\pi}{2} - \tan^{-1}(1) \qquad (s = 1)$$

$$= \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$ **Ans.**

On putting $s = 0$ in (1), we get

$$\int_0^\infty \frac{\sin t}{t}\,dt = \frac{\pi}{2} - \tan^{-1}(0)$$

$$= \frac{\pi}{2}$$ **Ans.**

Inverse Laplace Transforms

Now we obtain $f(t)$ when $F(s)$ is given, then we say that inverse Laplace transform of $F(s)$ is $f(t)$.

If $L[f(t)] = F(s)$, then $L^{-1}[F(s)] = f(t)$.

where L^{-1} is called the inverse Laplace transform operator.

From the application point of view, the inverse Laplace transform is very useful.

IMPORTANT FORMULAE

(1) $L^{-1}\left(\frac{1}{s}\right) = 1$

(2) $L^{-1}\frac{1}{s^n} = \frac{t^{n-1}}{(n-1)!}$

(3) $L^{-1}\frac{1}{s-a} = e^{at}$

(4) $L^{-1}\frac{s}{s^2-a^2} = \cosh at$

(5) $L^{-1}\frac{1}{s^2-a^2} = \frac{1}{a}\sinh at$

(6) $L^{-1}\frac{1}{s^2+a^2} = \frac{1}{a}\sin at$

(7) $L^{-1}\frac{s}{s^2+a^2} = \cos at$

(8) $L^{-1}F(s-a) = e^{at}f(t)$

(9) $L^{-1}\frac{1}{(s-a)^2+b^2} = \frac{1}{b}e^{at}\sin bt$

(10) $L^{-1}\frac{s-a}{(s-a)^2+b^2} = e^{at}\cos bt$

(11) $L^{-1}\frac{1}{(s-a)^2-b^2} = \frac{1}{b}e^{at}\sinh bt$

(12) $L^{-1}\frac{s-a}{(s-a)^2-b^2} = e^{at}\cosh bt$

(13) $L^{-1}\frac{1}{(s^2+a^2)^2} = \frac{1}{2a^3}(\sin at - at\cos at)$

(14) $L^{-1}\frac{s}{(s^2+a^2)^2} = \frac{1}{2a}t\sin at$

(15) $L^{-1}\frac{s^2-a^2}{(s^2+a^2)^2} = t\cos at$

(16) $L^{-1}(1) = \delta(t)$

(17) $L^{-1}\frac{s^2}{(s^2+a^2)^2} = \frac{1}{2a}[\sin at + at\cos at]$

Properties of Inverse Laplace Transform

- **First Shifting Theorem or First Translation Theorem**

  If $L^{-1}\{f(s)\} = F(t)$ then $L^{-1}\{f(s-a)\} = e^{at}L^{-1}\{f(s)\} = e^{at}F(t)$

- **Second Shifting Theorem or Second Translation Theorem**

  If $L^{-1}\{f(s)\} = F(t)$ then $L^{-1}\{e^{-as}f(s)\} = G(t)$

  Where

$$G(t) = \begin{cases} F(t-a), t > a \\ 0, t < a \end{cases}$$

Properties of Inverse Laplace Transform

- **Change of Scale Property**

If $L^{-1}\{f(s)\} = F(t)$ **then** $L^{-1}\{f(as)\} = \frac{1}{a} F(\frac{t}{a})$

- **Inverse Laplace Transform of Derivatives**

If $L^{-1}\{f(s)\} = F(t)$ **then**

$$L^{-1}\{f^{n}(s)\} = L^{-1}\{\frac{d^n}{ds^n} f(s)\}$$

$$L^{-1}\{f^{n}(s)\} = (-1)^n t^n L^{-1}\{f(s)\}$$

$$L^{-1}\{f^{n}(s)\} = (-1)^n t^n L^{-1} F(t)$$

$$where\ n = 1,2,3,........$$

Properties of Inverse Laplace Transform

- **Inverse Laplace Transform of Integrals**

  If $L^{-1}\{f(s)\} = F(t)$ then $L^{-1}\{\int_0^{\infty} f(x)dx\} = \frac{F(t)}{t}$

- **Multiplication by** s^n

If $L^{-1}\{f(s)\} = F(t)$ and $F(0) = 0,$

$$L^{-1}\{sf(s)\} = F'(t) = \frac{d}{dt}F(t)$$

- **Division by t and its powers**

  Theorem1 : If $L^{-1}\{f(s)\} = F(t)$, then $L^{-1}\{\frac{f(s)}{s}\} = \int_0^t F(x)dx$

  Theorem2: If $L^{-1}\{f(s)\} = F(t)$ **,**then $L^{-1}\{\frac{f(s)}{s^2}\} = \int_0^t \int_0^y F(x)\,dx\,dy$

Example 30. *Find the inverse Laplace Transform of the following:*

(*i*) $\dfrac{1}{s-2}$ (*ii*) $\dfrac{1}{s^2-9}$ (*iii*) $\dfrac{s}{s^2-16}$ (*iv*) $\dfrac{1}{s^2+25}$ (*v*) $\dfrac{s}{s^2+9}$

(*vi*) $\dfrac{1}{(s-2)^2+1}$ (*vii*) $\dfrac{s-1}{(s-1)^2+4}$ (*viii*) $\dfrac{1}{(s+3)^2-4}$ (*ix*) $\dfrac{s+2}{(s+2)^2-25}$ (*x*) $\dfrac{1}{2s-7}$

Solution. (*i*) $L^{-1}\dfrac{1}{s-2} = e^{2t}$ (*ii*) $L^{-1}\dfrac{1}{s^2-9} = L^{-1}\dfrac{1}{3}\cdot\dfrac{3}{s^2-(3)^2} = \dfrac{1}{3}\sinh 3t$

(*iii*) $L^{-1}\dfrac{s}{s^2-16} = L^{-1}\dfrac{s}{s^2-(4)^2} = \cosh 4t$ (*iv*) $L^{-1}\dfrac{1}{s^2+25} = \dfrac{1}{5}\dfrac{5}{s^2+(5)^2} = \dfrac{1}{5}\sin 5t$

(*v*) $L^{-1}\dfrac{s}{s^2+9} = \dfrac{s}{s^2+(3)^2} = \cos 3t$ (*vi*) $L^{-1}\dfrac{1}{(s-2)^2+1} = e^{2t}\sin t$

(*vii*) $L^{-1}\dfrac{s-1}{(s-1)^2+4} = e^{t}\cos 2t$ (*viii*) $L^{-1}\dfrac{1}{(s+3)^2-4} = \dfrac{1}{2}\dfrac{2}{(s+3)^2-(2)^2} = \dfrac{1}{2}e^{-3t}\sinh 2t$

(*ix*) $L^{-1}\dfrac{s+2}{(s+2)^2-25} = L^{-1}\dfrac{(s+2)}{(s+2)^2-(5)^2} = e^{-2t}\cosh 5t$

(*x*) $\dfrac{1}{2s-7} = \dfrac{1}{2}e^{\frac{7}{2}t}$ $\left[L^{-1}F(as) = \dfrac{1}{a}f\left(\dfrac{t}{a}\right)\right]$

Example 32. *Find the inverse Laplace transform of*

$$(i)\ \frac{s}{s^2+1} \qquad (ii)\ \frac{s}{4s^2-25} \qquad (iii)\ \frac{3s}{2s+9}$$

Solution. (*i*)

$$L^{-1}\frac{1}{s^2+1} = \sin t$$

$$L^{-1}\frac{s}{s^2+1} = \frac{d}{dt}(\sin t) + \sin(0)\,\delta(t)$$

$$= \cos t \qquad \textbf{Ans.}$$

(*ii*)

$$L^{-1}\frac{1}{4s^2-25} = \frac{1}{4}L^{-1}\frac{1}{s^2-\frac{25}{4}} = \frac{1}{4}\cdot\frac{2}{5}L^{-1}\frac{\frac{5}{2}}{s^2-\left(\frac{5}{2}\right)^2} = \frac{1}{10}\sinh\frac{5}{2}t$$

$$L^{-1}\frac{s}{4s^2-25} = \frac{1}{10}\frac{d}{dt}\sinh\frac{5}{2}t + \frac{1}{10}\sinh\frac{5}{2}(0)$$

$$= \frac{1}{10}\left(\frac{5}{2}\right)\cosh\frac{5}{2}t = \frac{1}{4}\cosh\frac{5}{2}t \qquad \textbf{Ans.}$$

(*iii*)

$$L^{-1}\frac{3}{2s+9} = \frac{3}{2}L^{-1}\frac{1}{s+\frac{9}{2}} = \frac{3}{2}e^{-9/2\,t}$$

$$L^{-1}\frac{3s}{2s+9} = \frac{3}{2}\frac{d}{dt}\left(e^{-9/2\,t}\right) + \frac{3}{2}e^{-9/2\,(0)} = \frac{3}{2}\left(-\frac{9}{2}\right)e^{-\frac{11}{2}t} + \frac{3}{2}$$

$$= -\frac{27}{4}e^{-11/2\,t} + \frac{3}{2} \qquad \textbf{Ans.}$$

Example 34. *Find the inverse Laplace transform of*

(*i*) $\dfrac{1}{(s+2)^5}$ (*ii*) $\dfrac{s}{s^2+4s+13}$ (*iii*) $\dfrac{1}{9s^2+6s+1}$

Solution. (*i*) $$L^{-1}\frac{1}{s^5} = \frac{t^4}{4\,!}$$

then $$L^{-1}\frac{1}{(s+2)^5} = e^{-2t}\cdot\frac{t^4}{4\,!}$$ **Ans.**

(*ii*) $$L^{-1}\left(\frac{s}{s^2+4s+13}\right) = L^{-1}\frac{s+2-2}{(s+2)^2+(3)^2} = L^{-1}\frac{s+2}{(s+2)^2+(3)^2} - L^{-1}\frac{2}{(s+2)^2+3^2}$$

$$= e^{-2t}L^{-1}\frac{s}{s^2+3^2} - e^{-2t}L^{-1}\frac{2}{3}\left(\frac{3}{s^2+3^2}\right)$$

$$= e^{-2t}\cos 3t - \frac{2}{3}e^{-2t}\sin 3t$$ **Ans.**

(*iii*) $$L^{-1}\frac{1}{9s^2+6s+1} = L^{-1}\frac{1}{(3s+1)^2} = \frac{1}{9}L^{-1}\frac{1}{\left(s+\frac{1}{3}\right)^2} = \frac{1}{9}e^{-t/3}L^{-1}\frac{1}{s^2}$$

$$= \frac{1}{9}e^{-t/3}\,t = \frac{t\,e^{-t/3}}{9}$$ **Ans.**

Example 35. *Obtain inverse Laplace transform of*

$$(i)\ \frac{e^{-\pi s}}{(s+3)} \qquad (ii)\ \frac{e^{-s}}{(s+1)^3}$$

Solution. (*i*) $L^{-1}\dfrac{1}{s+3} = e^{-3t}$

$$L^{-1}\frac{e^{-\pi s}}{s+3} = e^{-3(t-\pi)}\, U(t-\pi) \qquad \textbf{Ans.}$$

(*ii*) $L^{-1}\dfrac{1}{s^3} = \dfrac{t^2}{2\,!}$

$$L^{-1}\frac{1}{(s+1)^3} = e^{-t}\frac{t^2}{2\,!}$$

$$L^{-1}\frac{e^{-s}}{(s+1)^3} = e^{-(t-1)} \cdot \frac{(t-1)^2}{2\,!} \cdot U(t-1) \qquad \textbf{Ans.}$$

Example 40. *Obtain* $L^{-1}\dfrac{2s}{(s^2+1)^2}$.

(A.M.I.E.T.E., Winter 1997)

Solution. $L^{-1}\dfrac{2s}{(s^2+1)^2} = t\,L^{-1}\displaystyle\int_s^\infty \frac{2s\,ds}{(s^2+1)^2} = t\,L^{-1}\left[-\frac{1}{s^2+1}\right]_s^\infty = t\,L^{-1}\left[-0+\frac{1}{s^2+1}\right]$

$= t\sin t$ **Ans.**

INVERSE LAPLACE TRANSFORM BY CONVOLUTION

$$L\left\{\int_0^t f_1(x) * f_2(t-x)\,dx\right\} = F_1\cdot(s)\cdot F_2(s) \quad \text{or} \quad \int_0^t f_1(x)\cdot f_2(t-x)\,dx = L^{-1}\, F_1(s)\cdot F_2(s)$$

Example 46. *Obtain* $L^{-1}\dfrac{1}{s\,(s^2+a^2)}$.

Solution. $L^{-1}\dfrac{1}{s} = 1$ and $L^{-1}\dfrac{1}{s^2+a^2} = \dfrac{\sin at}{a}$.

Hence by the convolution theorem

$$L\int_0^t \left\{1\cdot\frac{\sin a\,(t-x)}{a}\,dx\right\} = \left(\frac{1}{s}\right)\left(\frac{1}{s^2+a^2}\right)$$

$$L^{-1}\left\{\frac{1}{s\,(s^2+a^2)}\right\} = \int_0^t \frac{\sin a\,(t-x)}{a}\,dx = \left[\frac{-\cos(at-ax)}{-a^2}\right]_0^t$$

$$= \frac{1}{a^2}\,[1-\cos at]$$ **Ans.**

PARTIAL FRACTIONS METHOD

Example 41. *Find the inverse transforms of* $\dfrac{1}{s^2 - 5s + 6}$.

Solution. Let us convert the given function into partial fractions.

$$L^{-1}\left[\frac{1}{s^2 - 5s + 6}\right] = L^{-1}\left[\frac{1}{s-3} - \frac{1}{s-2}\right]$$

$$= L^{-1}\left(\frac{1}{s-3}\right) - L^{-1}\left(\frac{1}{s-2}\right) = e^{3t} - e^{2t}$$ **Ans.**

Example 42. *Find the inverse Laplace transforms of*

$$\frac{s-1}{s^2 - 6s + 25}$$

Solution.

$$L^{-1}\left(\frac{s-1}{s^2 - 6s + 25}\right) = L^{-1}\left[\frac{s-1}{(s-3)^2 + (4)^2}\right] = L^{-1}\left[\frac{s-3+2}{(s-3)^2 + (4)^2}\right]$$

$$= L^{-1}\left[\frac{s-3}{(s-3)^2 + (4)^2}\right] + \frac{1}{2} L^{-1}\left[\frac{4}{(s-3)^2 + (4)^2}\right]$$

$$= e^{3t} \cos 4t + \frac{1}{2} e^{3t} \sin 4t \,.$$ **Ans.**

Example 44. *Find the Laplace inverse of*

$$\frac{s^2}{(s^2+a^2)(s^2+b^2)}.$$

Solution. Let us convert the given function into partial fractions.

$$L^{-1}\left[\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right] = L^{-1}\left[\frac{a^2}{a^2-b^2}\cdot\frac{1}{s^2+a^2} - \frac{b^2}{a^2-b^2}\cdot\frac{1}{s^2+b^2}\right]$$

$$= \frac{1}{a^2-b^2} L^{-1}\left[\frac{a^2}{s^2+a^2} - \frac{b^2}{s^2+b^2}\right] = \frac{1}{a^2-b^2}\left[a^2\left(\frac{1}{a}\sin at\right) - b^2\left(\frac{1}{b}\sin bt\right)\right]$$

$$= \frac{1}{a^2-b^2}[a\sin at - b\sin bt].$$ **Ans.**

13.30. SOLUTION OF DIFFERENTIAL EQUATIONS BY LAPLACE TRANSFORMS

Ordinary linear differential equations with constant coefficients can be easily solved by the Laplace Transform method, without finding the general solution and the arbitrary constants.

Example 48. *Using the Laplace transfors, find the solution of the initial value problem*

$$y'' + 25\,y = 10\cos 5\,t$$

$$y(0) = 2, \quad y'(0) = 0.$$

Solution. Taking Laplace transform of the given differential equation, we get

$$[s^2\,\bar{y} - s y(0) - y'(0)] + 25\,\bar{y} = 10\,\frac{s}{s^2+25}$$

$$s^2\,\bar{y} - 2\,s + 25\,\bar{y} = \frac{10\,s}{s^2+25}$$

$$(s^2+25)\,\bar{y} = 2\,s + \frac{10\,s}{s^2+25}$$

$$\bar{y} = \frac{2\,s}{s^2+25} + \frac{10\,s}{(s^2+25)^2}$$

$$y = L^{-1}\left[\frac{2\,s}{s^2+25} + \frac{10\,s}{(s^2+25)^2}\right] = 2\cos 5\,t + L^{-1}\left[\frac{10\,s}{(s^2+25)^2}\right]$$

$$= 2\cos 5\,t + L^{-1}\frac{d}{ds}\left[\frac{-5}{(s^2+25)}\right]$$

$$= 2\cos 5\,t + t\sin 5\,t$$

Ans.

Example 52. *Solve the initial value problem*

$$2y'' + 5y' + 2y = e^{-2t}, \qquad y(0) = 1, \quad y'(0) = 1,$$

using the Laplace transforms. (*A.M.I.E.T.E., Summer 1995*)

Solution. $2y'' + 5y' + 2y = e^{-2t}, \quad y(0) = 1, y'(0) = 1$

Taking the Laplace Transform of both sides, we get

$$2[s^2\bar{y} - sy(0) - y'(0)] + 5[s\bar{y} - y(0)] + 2\bar{y} = \frac{1}{s+2} \qquad \ldots(1)$$

On substituting the values of $y(0)$ and $y'(0)$ in (1), we get

$$2[s^2\bar{y} - s - 1] + 5[s\bar{y} - 1] + 2\bar{y} = \frac{1}{s+2}$$

$$[2s^2 + 5s + 2]\bar{y} - 2s - 2 - 5 = \frac{1}{s+2}$$

$$\bar{y} = \frac{1}{(s+2)(2s^2+5s+2)} + \frac{2s+7}{2s^2+5s+2} = \frac{1 + 2s^2 + 7s + 4s + 14}{(2s^2+5s+2)(s+2)} = \frac{2s^2 + 11s + 15}{(2s+1)(s+2)^2}$$

$$= \frac{4/9}{2s+1} - \frac{11/9}{s+2} - \frac{1/3}{(s+2)^2} = \frac{4}{9}\,\frac{1}{2}\,\frac{1}{s+\frac{1}{2}} - \frac{11}{9}\,\frac{1}{s+2} - \frac{1}{3}\,\frac{1}{(s+2)^2}$$

$$y = \frac{2}{9}e^{-\frac{1}{2}t} - \frac{11}{9}e^{-2t} - \frac{1}{3}te^{-2t}$$ **Ans.**