ENGINEERING PHYSICS

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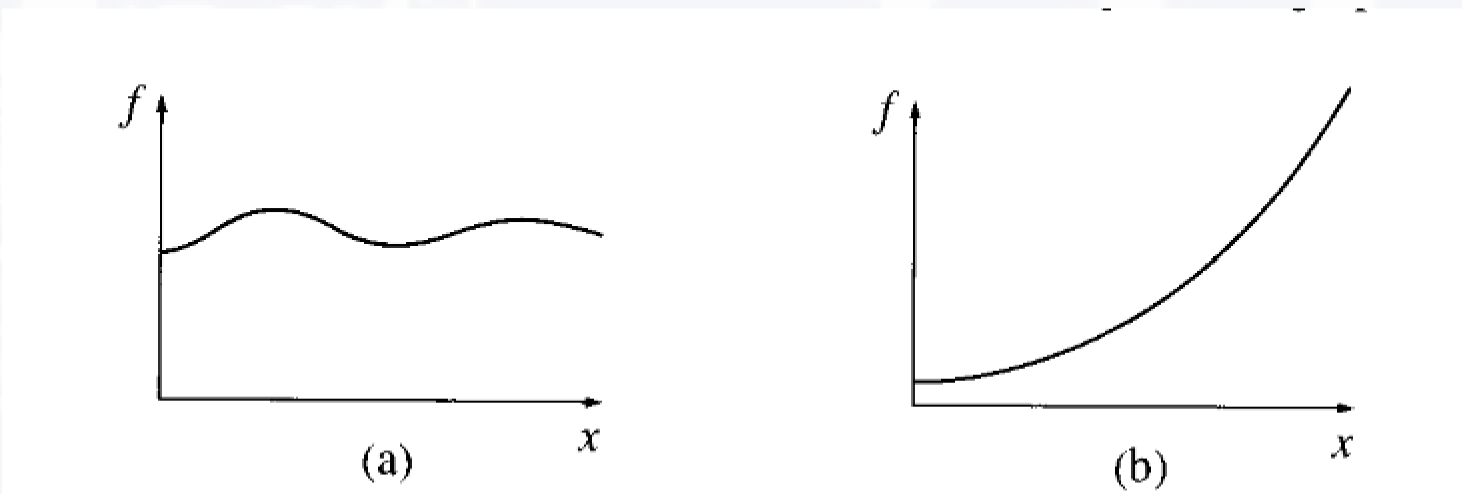
Ordinary Derivative

Suppose, we have function of one variable: f(x). What does the derivative, df/dx, do for us?

Answer: It tells us how rapidly the function f(x) varies when we change the argument x by a tiny amount, dx:

$$df = (df/dx)dx$$



Gradient

The derivative is suppose to tell us how fast the function varies as studied previously in 1-D.

What will be the scenario if your function depends on three coordinates, e.g., temperature T (x,y,z) in room

A theorem on partial derivate

$$dT = \left(\frac{\partial T}{\partial x}\right) dx + \left(\frac{\partial T}{\partial y}\right) dy + \left(\frac{\partial T}{\partial z}\right) dz.$$

$$\begin{aligned} dT &= \left(\frac{\partial T}{\partial x}\hat{\mathbf{x}} + \frac{\partial T}{\partial y}\hat{\mathbf{y}} + \frac{\partial T}{\partial z}\hat{\mathbf{z}}\right) \cdot (dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}}) \\ &= (\nabla T) \cdot (d\mathbf{l}), \end{aligned}$$

$$\nabla T \equiv \frac{\partial T}{\partial x}\hat{\mathbf{x}} + \frac{\partial T}{\partial y}\hat{\mathbf{y}} + \frac{\partial T}{\partial z}\hat{\mathbf{z}}$$

Gradient

Like any other vector, gradient also has magnitude and direction. Let us look at the geometrical meaning of gradient.

$$dT = \nabla T \cdot d\mathbf{l} = |\nabla T||d\mathbf{l}| \cos\theta.$$

If we fix dl then maximum change in T occurs when θ = 0, i.e.,
dT is greatest when I move in the same direction as gradient T.

The gradient ∇T points in the direction of maximum increase of the function T.

Moreover:

The magnitude $|\nabla T|$ gives the slope (rate of increase) along this maximal direction.

Gradient: Numerical Problems

Q1. Find the gradient of r = SQRT($x^2 + y^2 + z^2$)

$$
\begin{aligned}
\nabla r &= \frac{\partial r}{\partial x}\,\hat{\mathbf{x}} + \frac{\partial r}{\partial y}\,\hat{\mathbf{y}} + \frac{\partial r}{\partial z}\,\hat{\mathbf{z}} \\
&= \frac{1}{2}\frac{2x}{\sqrt{x^2+y^2+z^2}}\,\hat{\mathbf{x}} + \frac{1}{2}\frac{2y}{\sqrt{x^2+y^2+z^2}}\,\hat{\mathbf{y}} + \frac{1}{2}\frac{2z}{\sqrt{x^2+y^2+z^2}}\,\hat{\mathbf{z}} \\
&= \frac{x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}}}{\sqrt{x^2+y^2+z^2}} = \frac{\mathbf{r}}{r} = \hat{\mathbf{r}}.
\end{aligned}
$$

Gradient: Numerical Problems

Home Work

Calculate the gradients of given functions.

(a) $f(x, y, z) = x^2 + y^3 + z^4$.

(b) $f(x, y, z) = x^2 y^3 z^4$.

(c) $f(x, y, z) = e^x \sin(y) \ln(z)$.

Divergence

The gradient has the formal appearance of a vector (i.e., del operator) multiplying a scaler T.

$$\nabla T = \left(\hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}\right) T$$

$$\boxed{\nabla = \hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}.}$$

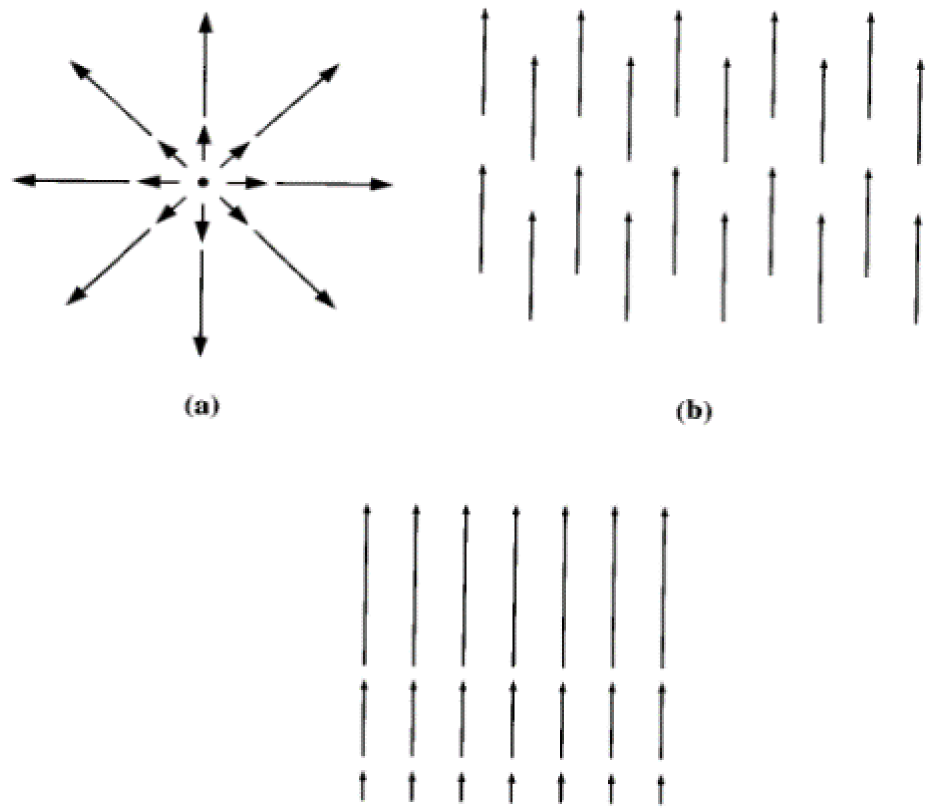
From the definition of del operator, we can construct the divergence

$$\begin{aligned}\nabla \cdot \mathbf{v} &= \left(\hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}\right) \cdot (v_x\hat{\mathbf{x}} + v_y\hat{\mathbf{y}} + v_z\hat{\mathbf{z}}) \\ &= \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}.\end{aligned}$$

The divergence of a vector function V is itself a scalar.

Divergence: Geometrical Interpretation

The name divergence is well chosen as it measure of much the vector v spreads out (diverges) from the point in question.



(a) (b) (c)

Divergence: Numerical Problems

NP1: Calculate the divergence of these functions..

$$\mathbf{v}_a = \mathbf{r} = x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}},\ \mathbf{v}_b = \hat{\mathbf{z}},\ \text{and}\ \mathbf{v}_c = z\,\hat{\mathbf{z}}.$$

$$\nabla \cdot \mathbf{v}_a = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 1 + 1 + 1 = 3.$$

As anticipated, this function has a positive divergence.

$$\nabla \cdot \mathbf{v}_b = \frac{\partial}{\partial x}(0) + \frac{\partial}{\partial y}(0) + \frac{\partial}{\partial z}(1) = 0 + 0 + 0 = 0,$$

as expected.

$$\nabla \cdot \mathbf{v}_c = \frac{\partial}{\partial x}(0) + \frac{\partial}{\partial y}(0) + \frac{\partial}{\partial z}(z) = 0 + 0 + 1 = 1.$$

Divergence: Numerical Problems

Home Work: Estimate the divergence of the given functions.

(a) $f(x, y, z) = x^2 + y^3 + z^4$.

(b) $f(x, y, z) = x^2 y^3 z^4$.

(c) $f(x, y, z) = e^x \sin(y) \ln(z)$.

Curl

From the definition of del, we can construct the curl

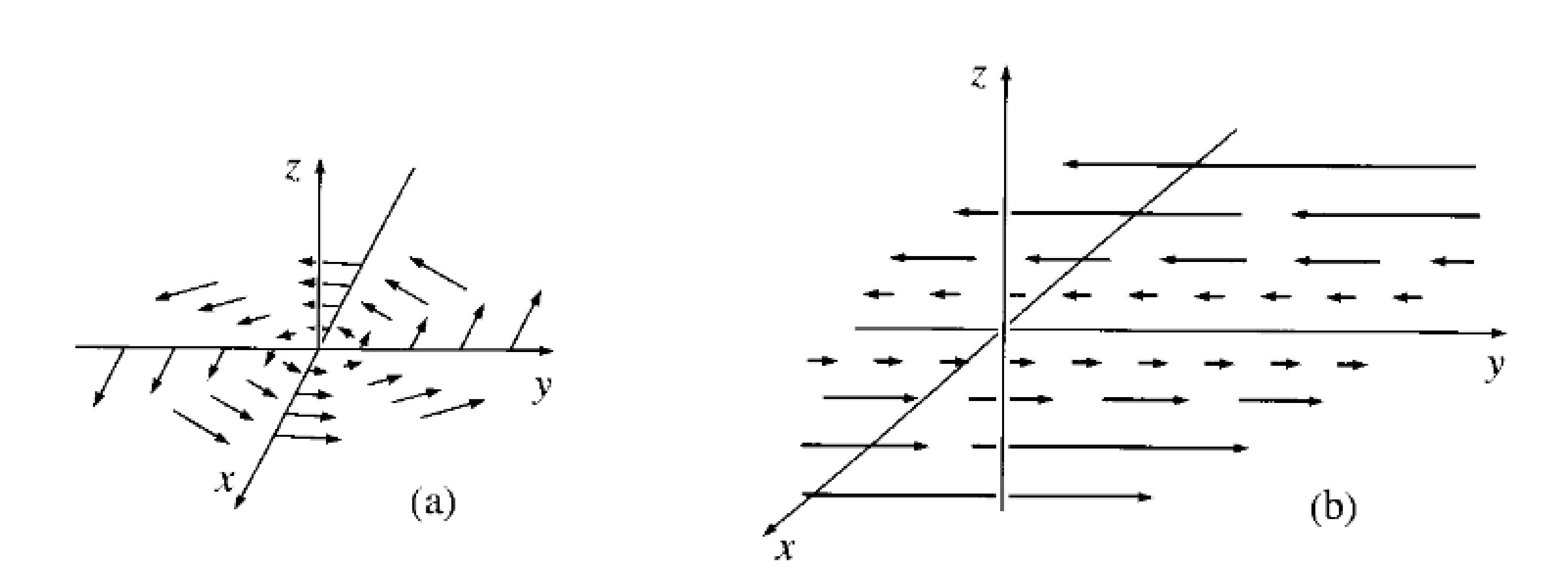
$$\nabla \times \mathbf{v} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ v_x & v_y & v_z \end{vmatrix}$$

$$= \hat{\mathbf{x}}\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right) + \hat{\mathbf{y}}\left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right) + \hat{\mathbf{z}}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)$$

NOTE: Curl of a vector function V is, like any cross product, a vector.

Curl: Geometrical Interpretation

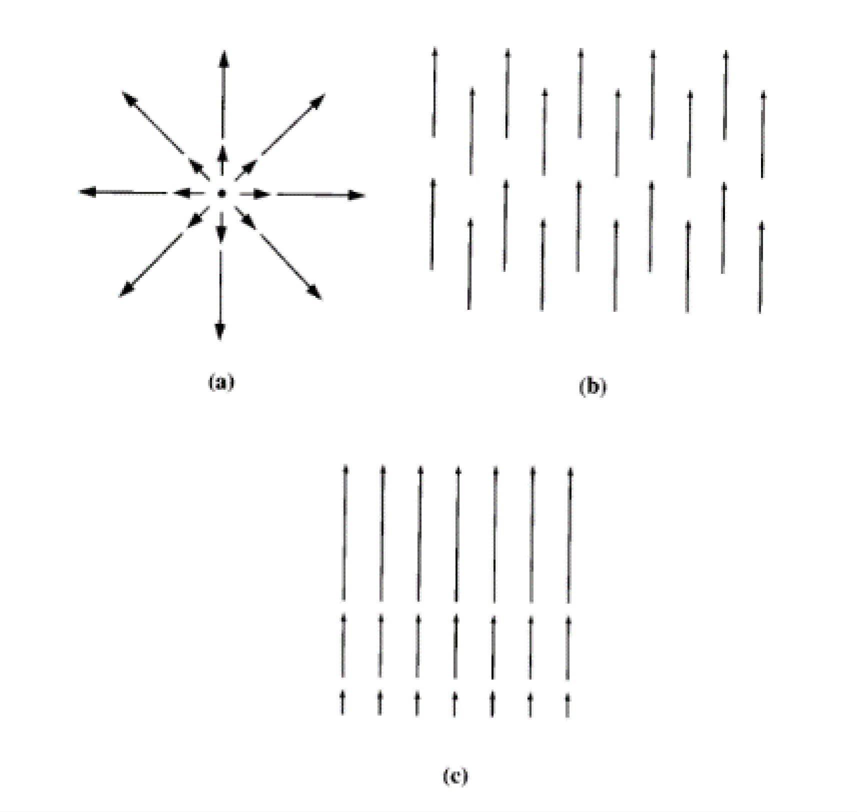
The name curl is also well chosen as it measure of how much the vector v curls around the point in question.



The functions, which are shown here, have substantial curl.

Curl: Geometrical Interpretation

Guess the curl of these functions…



Curl: Numerical Problem

Calculate the curl of given functions.

$V_a = -yX + xY$ & $V_b = xY$

$$\nabla \times \mathbf{v}_a = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -y & x & 0 \end{vmatrix} = 2\hat{\mathbf{z}},$$

$$\nabla \times \mathbf{v}_b = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 0 & x & 0 \end{vmatrix} = \hat{\mathbf{z}}.$$

Curl: Home Work

Calculate the curl of given functions.

(a) $\mathbf{v}_a = x^2\,\hat{\mathbf{x}} + 3xz^2\,\hat{\mathbf{y}} - 2xz\,\hat{\mathbf{z}}.$

(b) $\mathbf{v}_b = xy\,\hat{\mathbf{x}} + 2yz\,\hat{\mathbf{y}} + 3zx\,\hat{\mathbf{z}}.$

(c) $\mathbf{v}_c = y^2\,\hat{\mathbf{x}} + (2xy + z^2)\,\hat{\mathbf{y}} + 2yz\,\hat{\mathbf{z}}.$

Maxwell's Equations (Qualitative)

Maxwell unified the theories of electricity and magnetism by way of deducing four important equations which combines the experimental observations reported by Gauss, Ampere, and Faraday.

(i) $$\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0}\rho$$

(ii) $$\nabla \cdot \mathbf{B} = 0$$

(iii) $$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

(iv) $$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

Maxwell's Equation

Physical Significance

Total electric flux density through a surface enclosing a volume is equal to the charge density within the volume, i.e., charge distribution generates a steady electric field.

Net magnetic flux through a closed surface is zero. It implies that magnetic poles do not exist separately in the way as electric charges do. Thus, in other words, magnetic monopole do not exist.

It shows that with time varying magnetic flux, electric field is produced in accordance with Faraday is law of electromagnetic induction. This is a time dependent equation

This is a time dependent equation which represents the modified differential form of Ampere's circital law according to which magnetic field is produced due to combined effect of conduction current density and displacement current density.

Home Work

What is displacement current and how Maxwell fixed the Ampere's Law.

Electromagnetic Waves

In a region of space where there is no charge or current then the Maxwell's equations reads..

$$\text{(i)} \quad \nabla \cdot \mathbf{E} = 0, \qquad \text{(iii)} \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t},$$

$$\text{(ii)} \quad \nabla \cdot \mathbf{B} = 0, \qquad \text{(iv)} \quad \nabla \times \mathbf{B} = \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}.$$

Electromagnetic Waves

Apply the curl to equation (iii) and (iv)

$$\nabla \times (\nabla \times \mathbf{E}) = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} = \nabla \times \left(-\frac{\partial \mathbf{B}}{\partial t}\right)$$

$$= -\frac{\partial}{\partial t}(\nabla \times \mathbf{B}) = -\mu_0 \epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2},$$

$$\nabla \times (\nabla \times \mathbf{B}) = \nabla(\nabla \cdot \mathbf{B}) - \nabla^2 \mathbf{B} = \nabla \times \left(\mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}\right)$$

$$= \mu_0 \epsilon_0 \frac{\partial}{\partial t}(\nabla \times \mathbf{E}) = -\mu_0 \epsilon_0 \frac{\partial^2 \mathbf{B}}{\partial t^2}.$$

Electromagnetic Waves

since $\nabla \cdot \mathbf{E} = 0$ and $\nabla \cdot \mathbf{B} = 0$,

$$\boxed{\nabla^2 \mathbf{E} = \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}, \quad \nabla^2 \mathbf{B} = \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{B}}{\partial t^2}.}$$

Now, we have separate equations for electric filed (E) and magnetic field (B). However, they are the second-order differential equation.

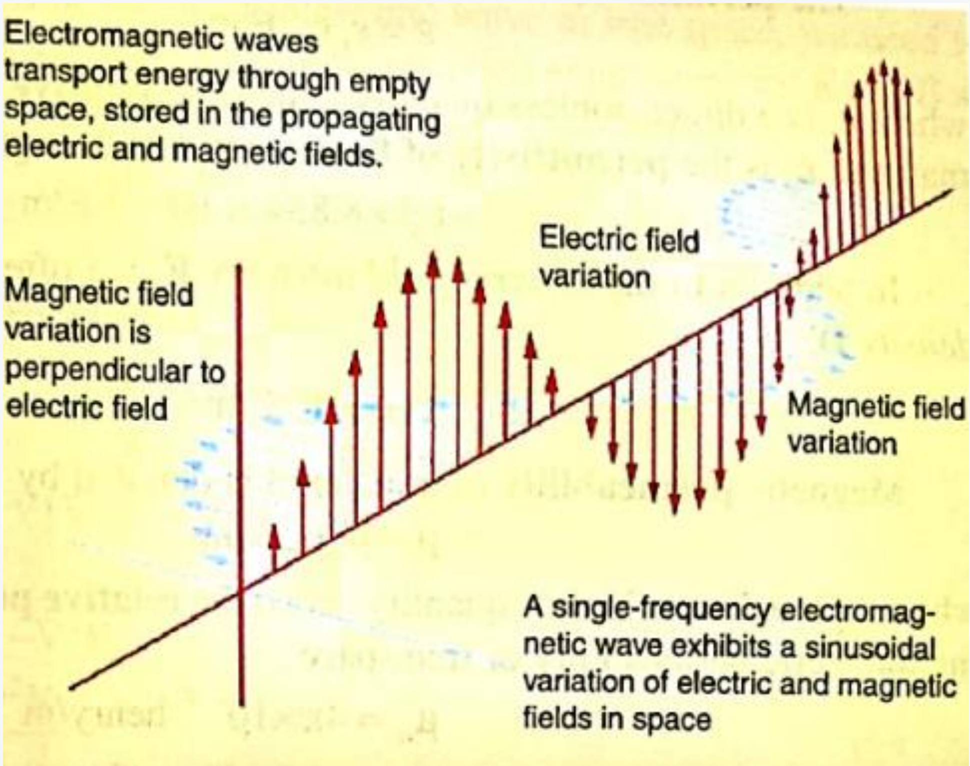
Please note in- general wave equation

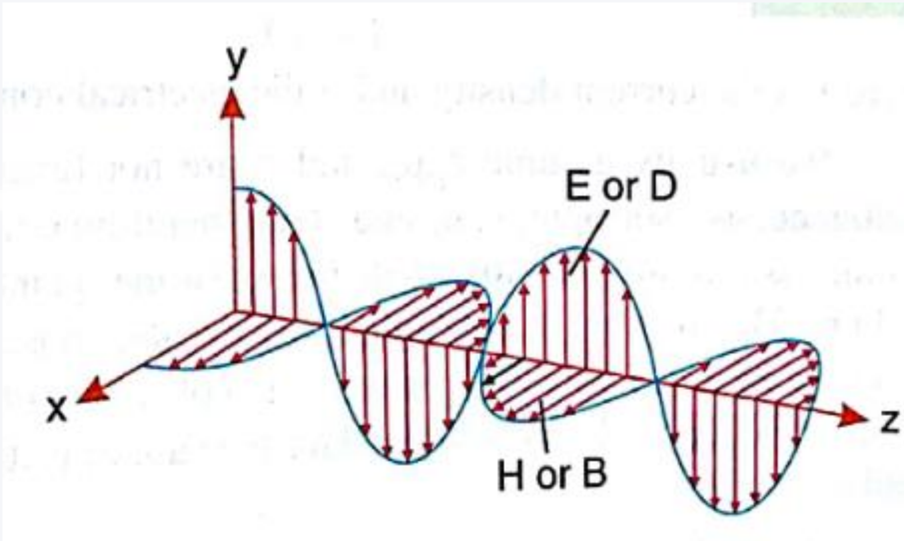
$$\nabla^2 f = \frac{1}{v^2} \frac{\partial^2 f}{\partial t^2}.$$

$$v = \frac{1}{\sqrt{\epsilon_0 \mu_0}} = 3.00 \times 10^8 \text{ m/s},$$

Obviously, the electromagnetic waves travel with the velocity of light in free space.

Electromagnetic Waves

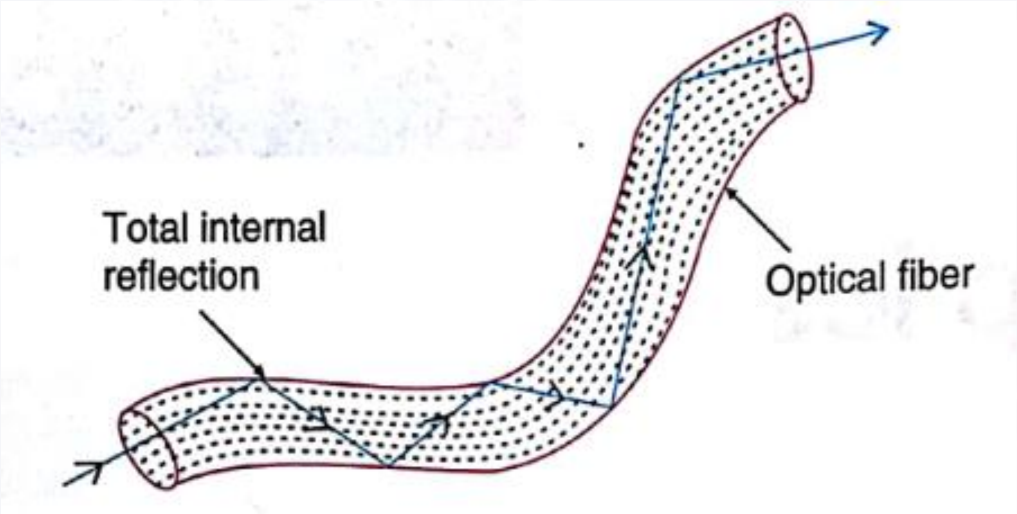




The directions of E and H in a uniform plane wave.

Optical Fibers

An optical fibre is cylindrical wave guide made of transparent dielectric (glass or clear plastic), **which guides light along its length by total internal reflection.**



It is as thin as human hair, approximately 70µm or 0.003 inch diameter.

Optical Fibres

Structures

Principle: The propagation of light in an optical fibre from one of its end to another end is based on the principle of total internal reflection. The practical optical fibre is cylindrical in shape and has in general three coaxial regions.

Three parts of Fibre

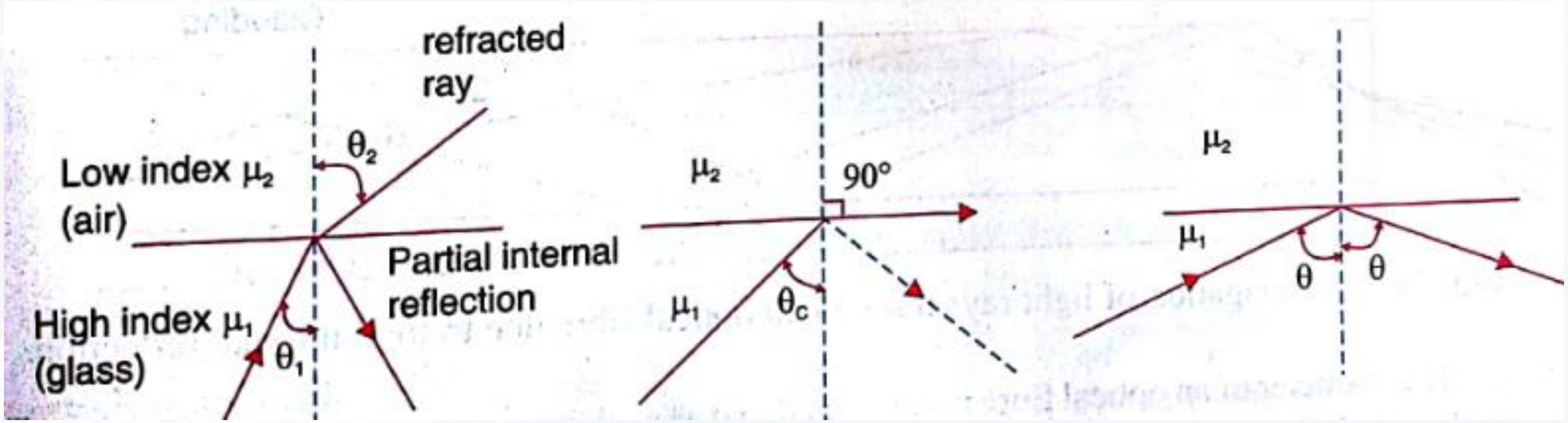
The innermost cylindrical region is the light guiding region known as the core. In general, the diameter of the core is of the order of 8.5 μm to 62.5 μm.

Core is surrounded by coaxial middle region known as the cladding. The diameter of cladding is of the order of 125 μm. The refractive index of cladding (n_2) is always lower than that of the core (n). Light launched into the core and striking the core-to-cladding interface at an angle greater than critical angle will be reflected back into the core. Since the angle of incidence and reflection are equal, the light will continue to rebound and propagate through the fibre.

The outermost region is called the sheath or protective buffer coating. It is plastic coating given to the cladding for extra protection. This coating is applied during the manufacturing process to provide physical and environmental protection for the fibre. The buffer is elastic in nature and prevents abrasions. The coating can vary in size from 250 μm to 900 μm.

Total Internal Reflection

A medium having lower refractive index is said to be an optically rarer medium while a medium having a higher refractive index is known as an optically denser medium.



When a ray of light passes from denser medium to rarer medium, it is bent away from the normal in the normal medium.

Snell's law

$$\sin\theta_2 = \left(\frac{\mu_1}{\mu_2}\right)\sin\theta_1$$

Total Internal Reflection

- If $\theta_1 < \theta_c$, the ray refracts into the rarer medium
- If $\theta_1 = \theta_c$, the ray just grazes the interface of rarer-to-denser media
- If $\theta_1 > \theta_c$, the ray is reflected back into the denser medium.

The phenomenon in which light is totally reflected from a denser-to-rarer medium boundary is known as **total internal reflection**. The rays that experience total internal reflection obey the laws of reflection. Therefore, the critical angle can be determined from Snell's law.

$$\theta_1 = \theta_c, \qquad \theta_2 = 90^o$$

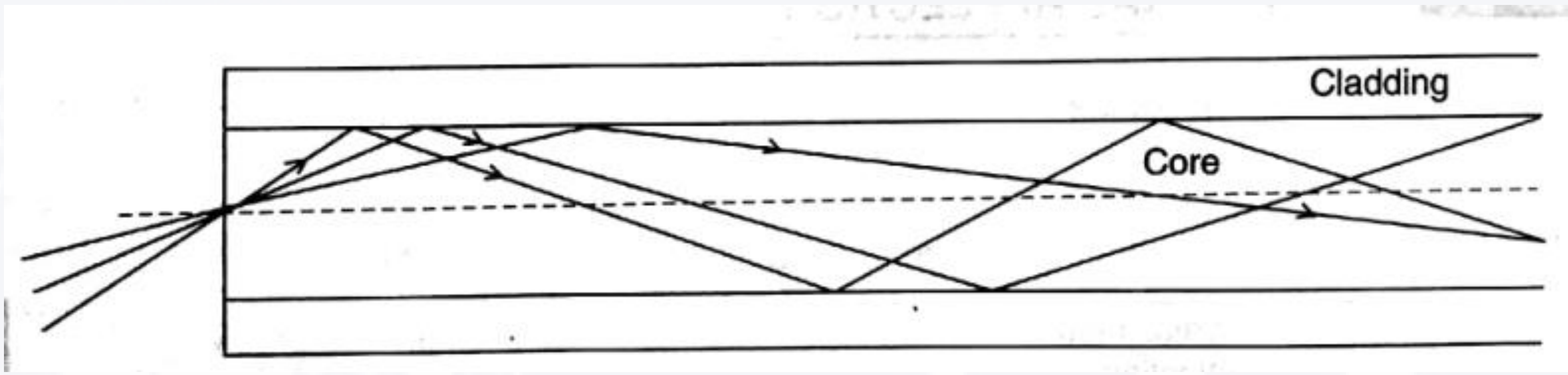
$$\mu_1 \sin\theta_c = \mu_2 \sin 90^o = \mu_2$$

$$\therefore \quad \sin\theta_c = \frac{\mu_2}{\mu_1}$$

When the rarer medium is air, $\mu_2 = 1$ and writing $\mu_1 = \mu$, we obtain

$$\sin\theta_c = \frac{1}{\mu}$$

Propagation of Light through an Optical Fibre

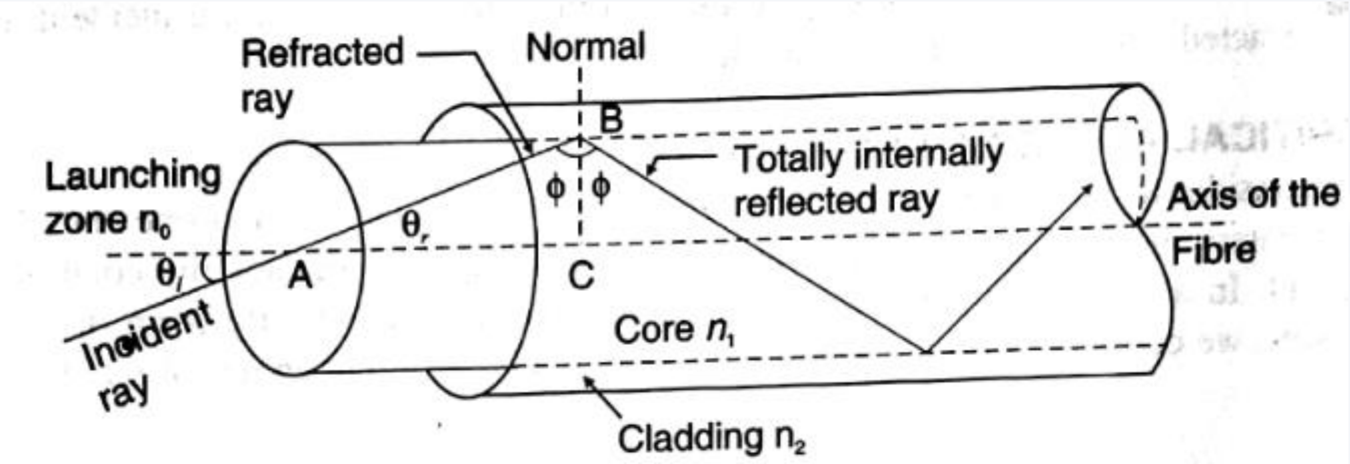


The refractive index of the core material (n_1) must be slightly greater than that of the cladding (n_2)

At the core cladding interface, the angle of incidence between the ray and the normal to the interface must be greater than the critical angle defined by

$$\sin \phi_C = \frac{n_2}{n_1}$$

Acceptance Angle



$$\frac{\sin\theta_i}{\sin\theta_r} = \frac{n_1}{n_0}$$

$$\sin\theta_r = \sin(90^\circ - \phi) = \cos\phi$$

$$\sin\theta_i = \frac{n_1}{n_o}\cos\phi$$

$$\phi = \phi_c, \quad \sin\left[\theta_{i_{max}}\right] = \frac{n_1}{n_o}\cos\phi_c$$

$$\sin\phi_C = \frac{n_2}{n_1}$$

$$\cos\phi_C = \frac{\sqrt{n_1^2 - n_2^2}}{n_1}$$

Acceptance Angle

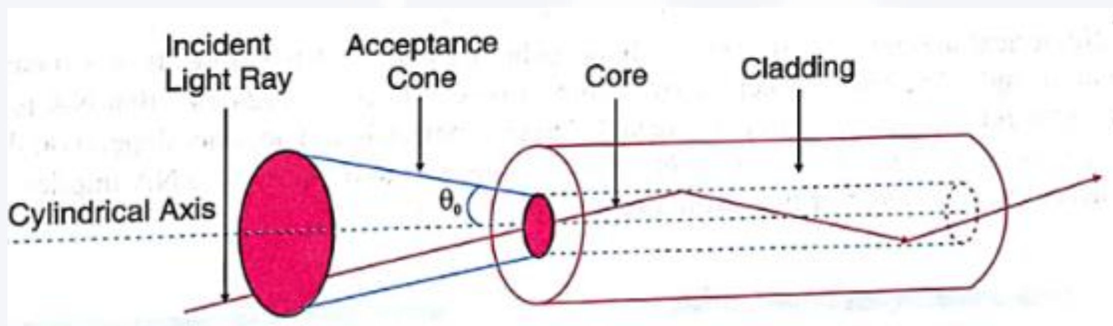
$$\cos \phi_C = \frac{\sqrt{n_1^2 - n_2^2}}{n_1}$$

$$\sin\left[\theta_i(\max)\right] = \left[\frac{\sqrt{n_1^2 - n_2^2}}{n_0}\right]$$

$$\sin \theta_0 = \sqrt{n_1^2 - n_2^2}$$

$$\theta_o = \sin^{-1}\left[\sqrt{n_1^2 - n_2^2}\right]$$

The angle is called acceptance angle. Acceptance angle is maximum that a light ray can have relative to the axis of the fibre and propagate down the fibre.



Numerical Aperture

The fractional refractive index change:

$$\Delta = \frac{n_1 - n_2}{n_1}$$

The parameter is always positive as n must be larger than n for the total internal reflection. In order to guide the light ray effectively through a fibre, Δ<<1.
Typically, Δ is of the order of 0.01

Numerical Aperture:

Numerical aperture (NA) is defined as the sine of the acceptance angle.

$$NA = \sin\theta_0$$

Numerical Aperture

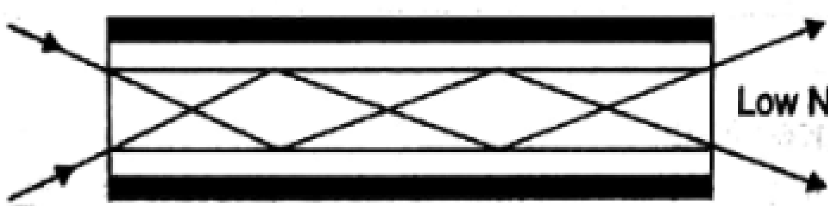
$$\sin\theta_0 = \sqrt{n_1^2 - n_2^2}$$

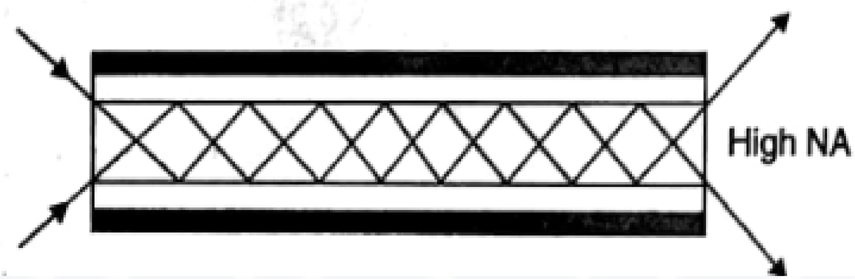
$$NA = \sqrt{n_1^2 - n_2^2}$$

$$n_1^2 - n_2^2 = (n_1 + n_2)(n_1 - n_2) = \left(\frac{n_1 + n_2}{2}\right)\left(\frac{n_1 - n_2}{n_1}\right)2n_1$$

$$NA = \sqrt{2n_1^2\Delta}$$

$$NA = n_1\sqrt{2\Delta}$$

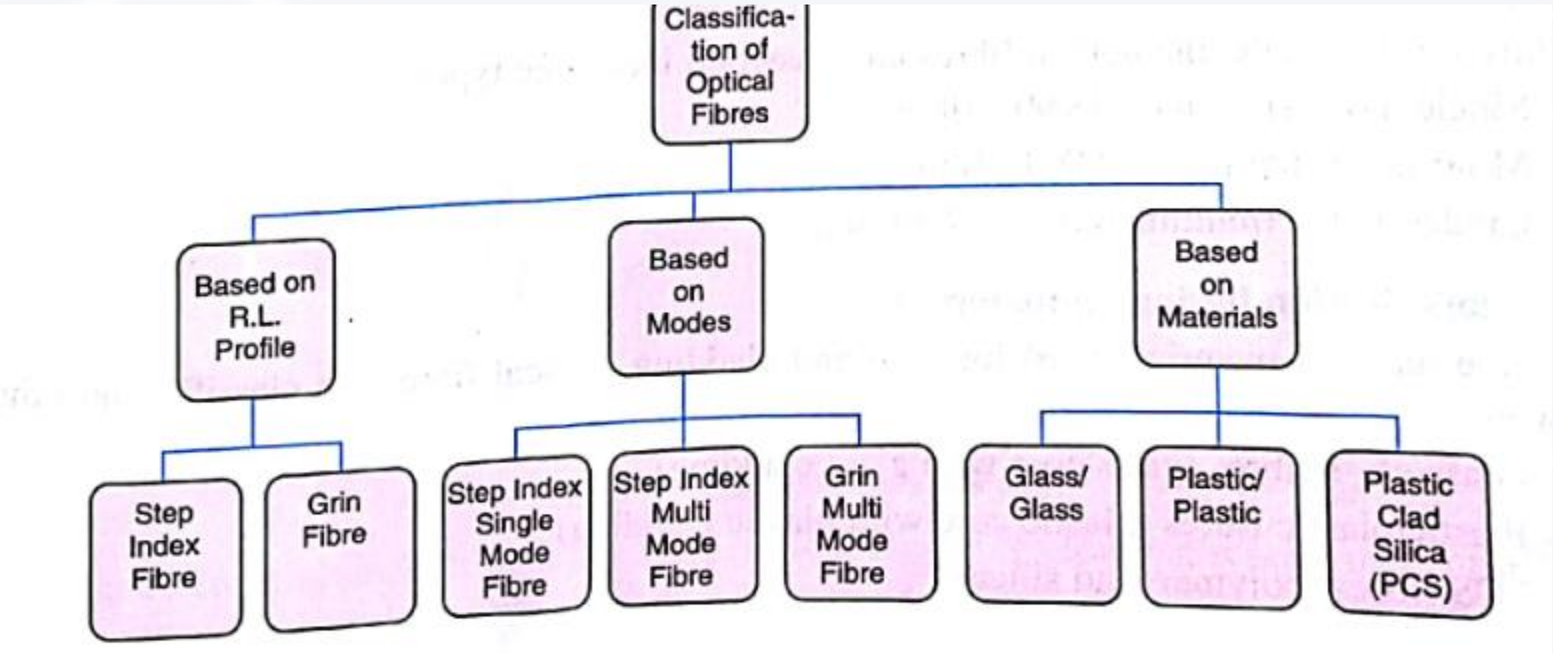




The value of NA ranges from 0.13 to 0.50. A large NA implies that a fibre will accept large amount of light from the source.

Classification

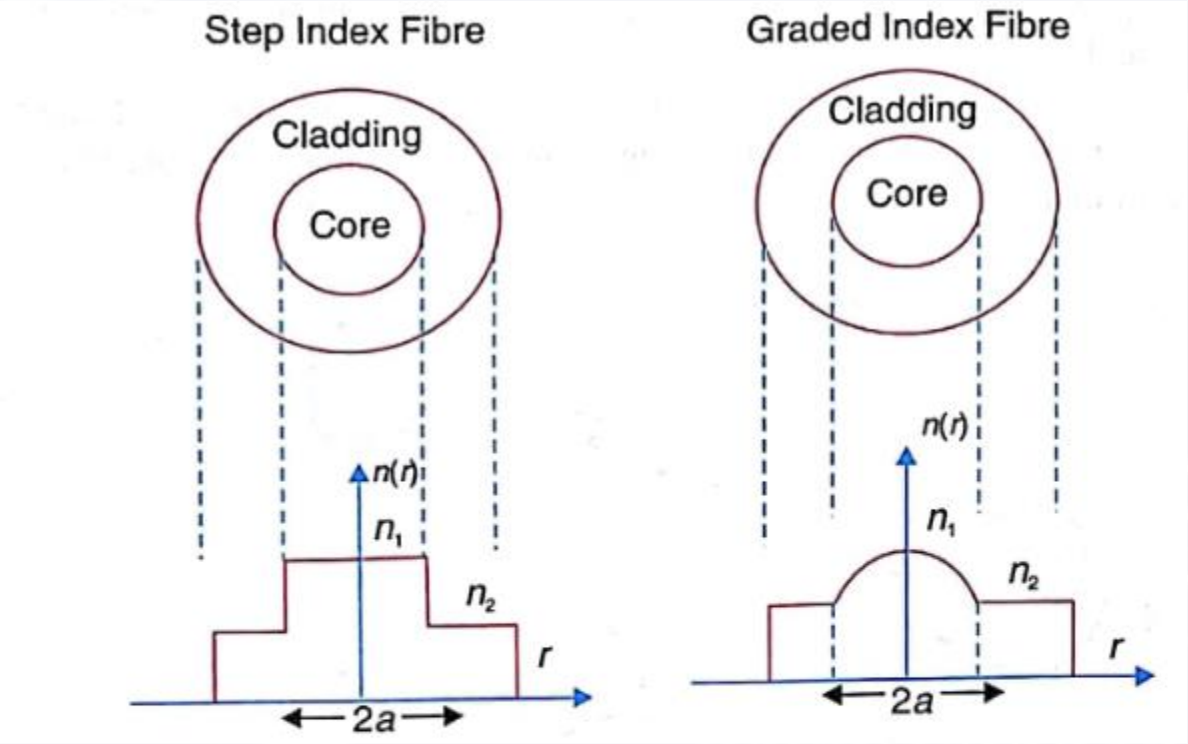
Optical fibres are differently classified into various types basing on different parameters.



Classification

A: Classification based on refractive index profiles.

(1) Step index fibres **(2) Graded Index (GRIN) fibres**

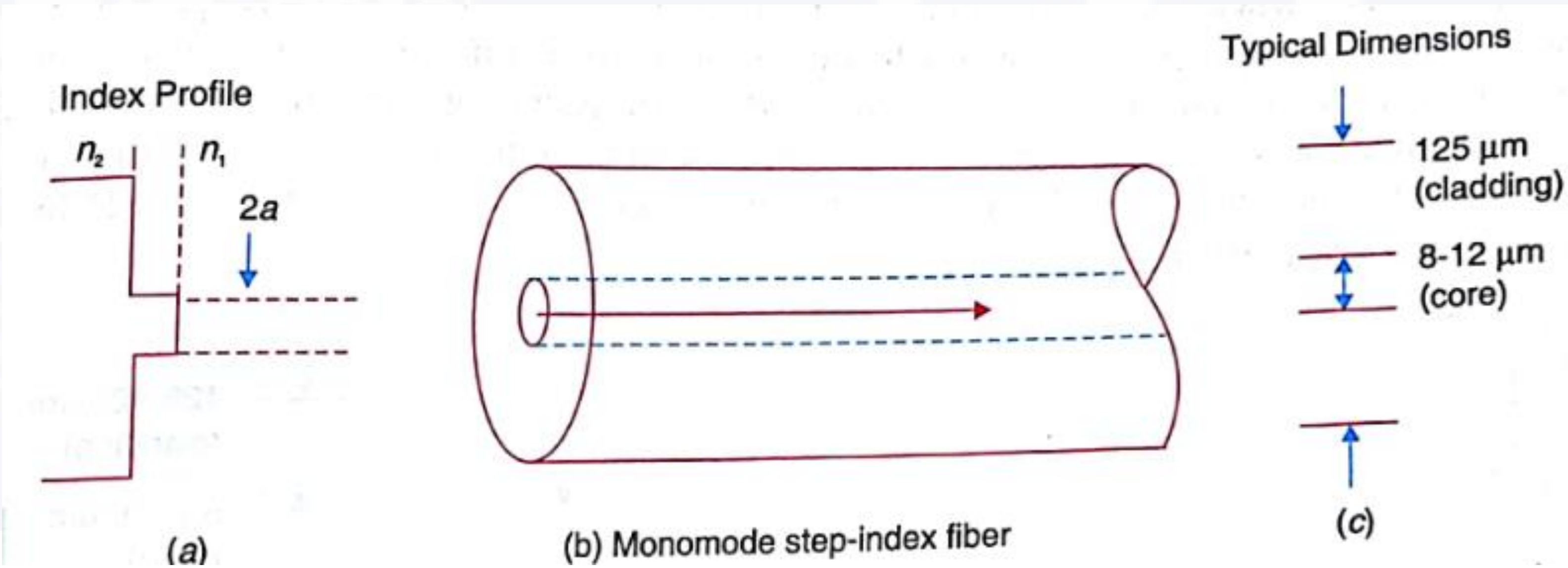


Classification

B: Classification based on the modes of light propagation

(1) Single mode fibres (SMF)
(2) Multimode fibres (MMF)
(3) Graded Index (multimode) (GRIN) fibres

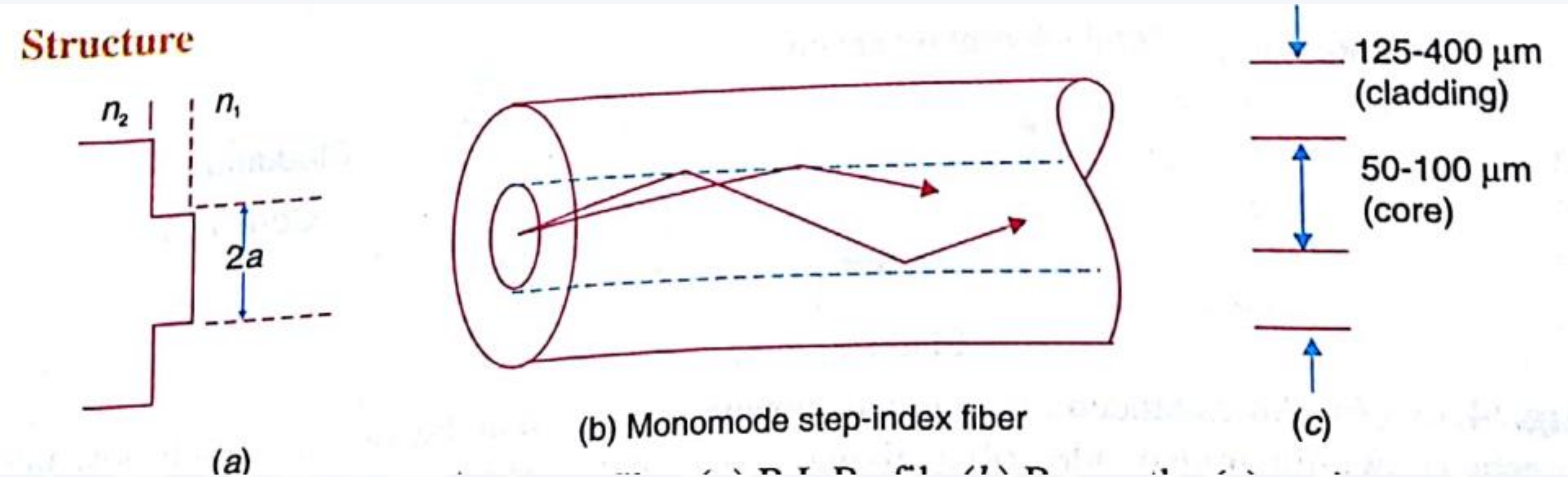
Single mode fibers (SMF)



Classification of Optical Fibres

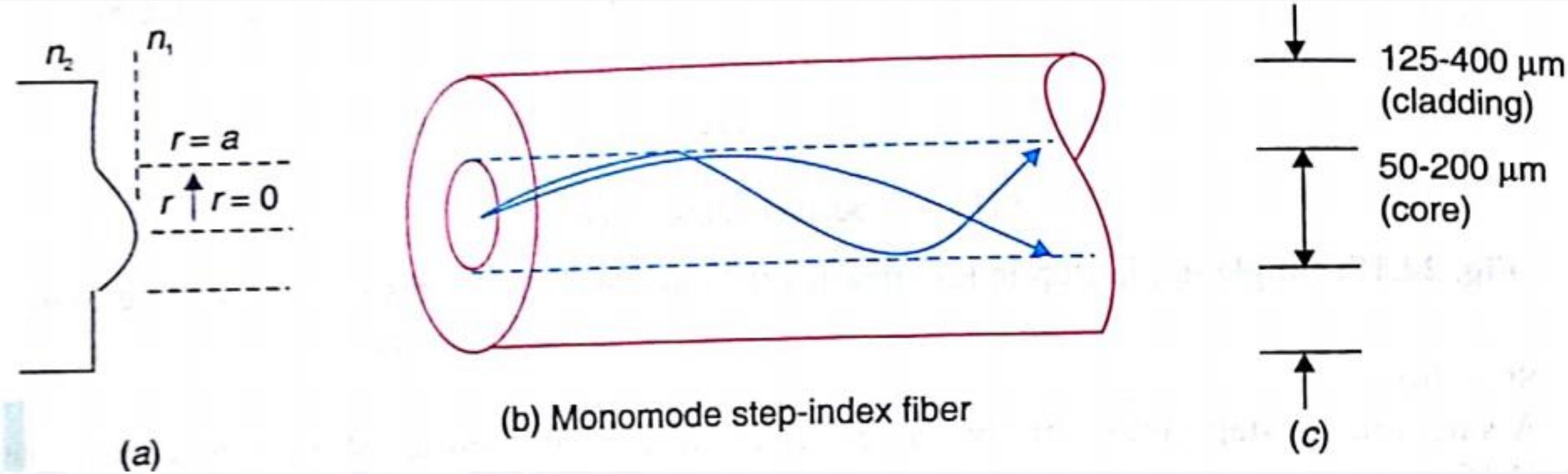
Classification

Multimode fibers (MMF)



Classification

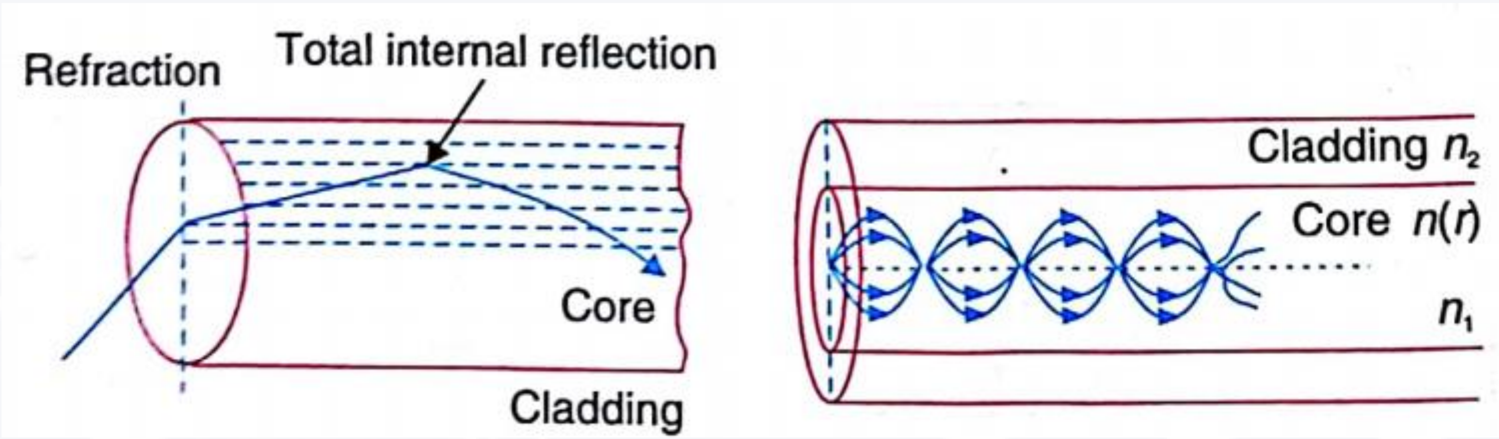
Graded Index (multimode) (GRIN) fibers



$$n(r)=\begin{cases} n_1\sqrt{1-\left[2\Delta\left(\dfrac{r}{a}\right)^{\alpha}\right]}, & r<a \text{ inside core} \\ n_2, & r>a \text{ in cladding} \end{cases}$$

Classification of Optical Fibres

Graded Index (multimode) (GRIN) fibers



$$NA = \sqrt{n^2(r) - n_2^2} \approx n_1 (2\Delta)^{\frac{1}{2}} \sqrt{1 - \left(\frac{r}{a}\right)^2}$$

$$= n_1 \sqrt{2\Delta \left[1 - \left(\frac{r}{a}\right)^2\right]}$$

Attenuation

When an optical signal propagates through a fibre, its power decreases exponentially with distance. The loss of optical power as light travels down a fiber is known as **attenuation.** The attenuation of optical signal is defined as *the ratio of the optical output power from a fibre of length L to the input optical power.* If P_i is the optical power launched at the input end of the fibre, then the power P_o at a distance L down the fibre is given by

$$P_o = P_i \, e^{-\alpha L}$$

where α is called the **fibre attenuation coefficient** expressed in units of km^{-1}. Taking logarithms on both the sides of the above equation, we obtain

$$\alpha = \frac{1}{L} ln \frac{P_i}{P_o}$$

Attenuation

In units of dB / km, α is defined through the equation

$$\therefore \qquad \alpha_{dB/km} = \frac{10}{L}\log\frac{P_i}{P_o}$$

In case of an ideal fibre, $P_o = P_i$ and the attenuation would be zero.