1.1

$$\theta_{11}^{(0)} = 0.3$$

$$\theta_{21}^{(0)} = 0.4$$

$$\beta_1^{(0)} = (1,0,0,0)$$

$$\beta_2^{(0)} = (0,0.4,0.3,0.3)$$

$$P(z=1 \mid A, d_1) = \frac{\beta_{1A}^{(0)}\,\theta_{11}^{(0)}}{\sum_{z'} \beta_{z'A}^{(0)}\,\theta_{1z'}^{(0)}} = \frac{1 \cdot 0.3}{1 \cdot 0.3 + 0 \cdot 0.7} = \boxed{1}$$

$$P(z=1 \mid B, d_1) = \frac{\beta_{1B}^{(0)}\,\theta_{11}^{(0)}}{\sum_{z'} \beta_{z'B}^{(0)}\,\theta_{1z'}^{(0)}} = \frac{0 \cdot 0.3}{0 \cdot 0.3 + 0.4 \cdot 0.7} = \boxed{0}$$

$$P(z=1 \mid C, d_1) = \frac{\beta_{1C}^{(0)}\,\theta_{11}^{(0)}}{\sum_{z'} \beta_{z'C}^{(0)}\,\theta_{1z'}^{(0)}} = \frac{0 \cdot 0.3}{0 \cdot 0.3 + 0.3 \cdot 0.7} = \boxed{0}$$

$$P(z=1 \mid D, d_1) = \frac{\beta_{1D}^{(0)}\,\theta_{11}^{(0)}}{\sum_{z'} \beta_{z'D}^{(0)}\,\theta_{1z'}^{(0)}} = \frac{0 \cdot 0.3}{0 \cdot 0.3 + 0.3 \cdot 0.7} = \boxed{0}$$

1.2

$$\beta_{1A}^{(1)} = \frac{\sum_d P(z=1|A,d)\, c(A,d)}{\sum_{w',d} P(z=1|w',d)\, c(w',d)} = \frac{1 \cdot 4 + 1 \cdot 2}{1 \cdot 4 + 1 \cdot 2 + 0 + 0 + 0 + 0 + 0 + 0} = \boxed{1}$$

$$\beta_{1B}^{(1)} = \frac{\sum_d P(z=1|B,d)\, c(B,d)}{\sum_{w',d} P(z=1|w',d)\, c(w',d)} = \frac{0.3 + 0.2}{1 \cdot 4 + 1 \cdot 2 + 0 + 0 + 0 + 0 + 0 + 0} = \boxed{0}$$

$$\theta_{11}^{(1)} = \frac{\sum_w P(z=1|w,d_1)\, c(w,d_1)}{N_{d_1}} = \frac{1 \cdot 4 + 0 \cdot 3 + 0 \cdot 2 + 0 \cdot 1}{10} = \boxed{0.4}$$

$$\theta_{12}^{(1)} = \frac{\sum_w P(z=2|w,d_1)\, c(w,d_1)}{N_{d_1}} = \frac{0 \cdot 4 + 1 \cdot 3 + 1 \cdot 2 + 1 \cdot 1}{10} = \boxed{0.6}$$

2.1

$$P(z_i = k \mid x_i ; \beta, \pi) = \frac{P(x_i \mid z_i = k ; \beta, \pi)\, P(z_i = k ; \pi)}{P(x_i ; \beta, \pi)}$$

$$= \frac{P(x_i \mid z_i = k ; \beta, \pi)\, P(z_i = k ; \pi)}{\sum_{j=1}^{K} P(x_i \mid z_i = j ; \beta \pi)\, P(z_i = j ; \pi)}$$

$$= \boxed{\frac{\text{Multinomial}(x_i ; \beta_k) \cdot \pi_k}{\sum_{j=1}^{K} \text{Multinomial}(x_i ; \beta_j) \cdot \pi_j}}$$

2.2

$$\beta, \pi = \operatorname{argmax}_{\beta,\pi} \sum_{i=1}^{N} \sum_{k=1}^{K} p(z_i = k \mid x_i; \beta, \pi) \log P(x_i, z_i = k; \beta, \pi)$$

$$\mathcal{L}(\beta,\pi) = \sum_{i=1}^{N} \sum_{k=1}^{K} p(z_i = k \mid x_i; \beta, \pi) \log P(x_i \mid z_i = k; \beta, \pi)\, P(z_i = k; \pi)$$

$$- \lambda_k \left(\sum_{n=1}^{V} \beta_{kn} - 1 \right) - \lambda \left(\sum_{j=1}^{K} \pi_j - 1 \right)$$

$$\frac{\partial \mathcal{L}}{\partial \beta_{k,w}} = \frac{\partial}{\partial \beta_{k,w}} \left[\sum_{i=1}^{N} P(z_i = k \mid x_i; \beta, \pi) \log P(x_i \mid z_i = k; \beta, \pi) \right] - \lambda_k$$

$$\propto \frac{\partial}{\partial \beta_{k,w}} \left[\sum_{i=1}^{N} P(z_i = k \mid x_i) \sum_{n=1}^{V} \log \beta_{k,n}^{x_{i,n}} \right] - \lambda_k$$

$$= \frac{\partial}{\partial \beta_{k,w}} \left[\sum_{i=1}^{N} P(z_i = k \mid x_i) \log \beta_{k,w}^{x_{i,w}} \right] - \lambda_k$$

$$= \frac{\sum_{i=1}^{N} P(z_i = k \mid x_i) \cdot x_{i,w}}{\beta_{k,w}} - \lambda_k = 0$$

$$\beta_{k,w} = \boxed{\frac{\sum_{i=1}^{N} P(z_i = k \mid x_i) \cdot x_{i,w}}{\sum_{w'=1}^{V} \sum_{i=1}^{N} P(z_i = k \mid x_i) \cdot x_{i,w'}}}$$

$$\frac{\partial \mathcal{L}}{\partial \pi_k} = \frac{\partial}{\partial \pi_k}\left[\sum_{i=1}^{N} P(z_i = k \mid x_i) \log P(z_i = k)\right] - \lambda$$

$$= \frac{\sum_{i=1}^{N} P(z_i = k \mid x_i)}{\pi_k} - \lambda = 0$$

$$\boxed{\pi_k = \frac{\sum_{i=1}^{N} P(z_i = k \mid x_i)}{N}}$$