Problem 1.1

$$f(x;a) = \frac{\Gamma(\sum_{i=1}^{K} \alpha_i)}{\prod_{i=1}^{K} \Gamma(\alpha_i)} \prod_{i=1}^{K} x_i^{\alpha_{i-1}}$$

$$\mathcal{L}(\alpha) = \log f(x;a) = \sum_{n=1}^{N} \log \frac{\Gamma(\sum_{i=1}^{K} \alpha_i)}{\prod_{i=1}^{K} \Gamma(\alpha_i)} \prod_{i=1}^{K} x_i^{\alpha_{i-1}}$$

$$= N \log \Gamma\left(\sum_{i=1}^{K} \alpha_i\right) - N \sum_{i=1}^{K} \Gamma(\alpha_i) + \sum_{i=1}^{K} \alpha_{i-1} \log x_i$$

$$\frac{\partial \mathcal{L}(\alpha)}{\partial \alpha_i} = N\psi\left(\sum_{j=1}^{K} a_j\right) - N\psi(\alpha_i) + \sum_{n=1}^{N} \log x_i^{(n)} = 0$$

Cannot be solved analytically

Problem 1.2

Prior: $P(\theta) = \frac{1}{(2\pi)^{n/2}} \exp\left(-\frac{1}{2}\theta^T\theta\right)$

Likelihood: $P(x|\theta) = \frac{1}{(2\pi)^{m/2}\ \det(\Sigma)^{1/2}} \exp\left(-\frac{1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu)\right)$

PDF kernel form: $$P(\theta|x) \propto P(x|\theta)P(\theta)$$

$$= \exp\left(-\frac{1}{2}\theta^T\theta\right)\exp\left(-\frac{1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu)\right)$$

$$= \exp\left(-\frac{1}{2}\left(\theta^T\theta + (x-\mu)^T\Sigma^{-1}(x-\mu)\right)\right)$$

$$= \exp\left(-\frac{1}{2}\left(\theta^T\theta - 2\theta^T\Sigma^{-1}(x-\mu) + (x-\mu)^T\Sigma^{-1}(x-\mu)\right)\right)$$

$$= \exp\left(-\frac{1}{2}\left(\theta - \Sigma^{-1}(x-\mu)\right)^T\left(\theta - \Sigma^{-1}(x-\mu)\right)\right)$$

$$\Rightarrow P(x|\theta) = \mathcal{N}(\Sigma^{-1}(x-\mu), \Sigma)$$