2.1

$$P(z_i = k \mid x_i ; \beta, \pi) = \frac{P(x_i \mid z_i = k ; \beta, \pi)\, P(z_i = k ; \pi)}{P(x_i ; \beta, \pi)}$$

$$= \frac{P(x_i \mid z_i = k ; \beta, \pi)\, P(z_i = k ; \pi)}{\sum_{j=1}^{K} P(x_i \mid z_i = j ; \beta \pi)\, P(z_i = j ; \pi)}$$

$$= \boxed{\frac{\text{Multinomial}(x_i ; \beta_k) \cdot \pi_k}{\sum_{j=1}^{K} \text{Multinomial}(x_i ; \beta_j) \cdot \pi_j}}$$

2.2

$$\beta, \pi = \arg\max_{\beta, \pi} \sum_{i=1}^{N} \sum_{k=1}^{K} p(z_i = k \mid x_i; \beta, \pi) \log P(x_i, z_i = k; \beta, \pi)$$

$$\mathcal{L}(\beta, \pi) = \sum_{i=1}^{N} \sum_{k=1}^{K} p(z_i = k \mid x_i; \beta, \pi) \log P(x_i \mid z_i = k; \beta, \pi)\, P(z_i = k; \pi)$$

$$- \lambda_k \left(\sum_{n=1}^{V} \beta_{kn} - 1 \right) - \lambda \left(\sum_{j=1}^{K} \pi_j - 1 \right)$$

$$\frac{\partial \mathcal{L}}{\partial \beta_{k,w}} = \frac{\partial}{\partial \beta_{k,w}} \left[\sum_{i=1}^{N} P(z_i = k \mid x_i; \beta, \pi) \log P(x_i \mid z_i = k; \beta, \pi) \right] - \lambda_k$$

$$\propto \frac{\partial}{\partial \beta_{k,w}} \left[\sum_{i=1}^{N} P(z_i = k \mid x_i) \sum_{n=1}^{V} \log \beta_{k,n}^{x_{in}} \right] - \lambda_k$$

$$= \frac{\partial}{\partial \beta_{k,w}} \left[\sum_{i=1}^{N} P(z_i = k \mid x_i) \log \beta_{k,w}^{x_{iw}} \right] - \lambda_k$$

$$= \frac{\sum_{i=1}^{N} P(z_i = k \mid x_i) \cdot x_{i,w}}{\beta_{k,w}} - \lambda_k = 0$$

$$\beta_{k,w} = \boxed{\frac{\sum_{i=1}^{N} P(z_i = k \mid x_i) \cdot x_{i,w}}{\sum_{w'=1}^{V} \sum_{i=1}^{N} P(z_i = k \mid x_i) \cdot x_{i,w'}}}$$

$$\frac{\partial \mathcal{L}}{\partial \pi_k} = \frac{\partial}{\partial \pi_k}\left[\sum_{i=1}^{N} P(z_i=k \mid x_i) \log P(z_i=k)\right] - \lambda$$

$$= \frac{\sum_{i=1}^{N} P(z_i=k \mid x_i)}{\pi_k} - \lambda = 0$$

$$\boxed{\pi_k = \frac{\sum_{i=1}^{N} P(z_i=k \mid x_i)}{N}}$$