1. Find the value of $T(2)$ for recurrence relation $T(n) = 3T(n-1) + 12n$, given that $T(0) = 5$.

$$T(n) = 3T(n-1) + 12n.$$

We know that for $n = 0$, $T(0) = 5$.

$\therefore$ we put $n - 1 = 0$

$n = 1$.

$$T(1) = 3T(1-1) + 12 \times 1 = 3 \times T(0) + 12 = 3 \times 5 + 12 = 27$$

$$T(2) = 3T(2-1) + 12 \times 2 = 3T(1) + 24 = 3 \times 27 + 24$$

$$= 81 + 24$$

$$T(2) = 105$$

2. Given a recurrence relation, solve it using substitution method.

a) $T(n) = T(n-1) + c$ — (1)

Substitute n with n-1

$T(n-1) = T((n-1)-1) + c$

$T(n-1) = T(n-2) + c$ — (2)

Continuing substitution one more time,

$T(n-2) = T(n-3) + c$ — (3)

Replace (2) in (1)

$T(n) = T(n-2) + 2c$ — (4)

Replace (4) in (3) $T(n) = T(n-3) + 3c$

Continuing substitution k number of times

$T(n) = T(n-k) + k \cdot c$

Considering the base case condition, when $n=1$, $T(n)$ becomes 1. Therefore we make $n-k=1$, so that $T(n-k)$ becomes 1.

$$n - k = 1$$
$$k = n-1$$

$$T(n) = T(n-(n-1)) + (n-1)c$$
$$= T(n-n+1) + (n-1)c$$
$$= T(1) + nc - c$$
$$= 1 + nc - c$$

After eradicating the constant value,

$$T(n) = O(n)$$

d) $T(n) = T\left(\frac{n}{2}\right) + c$ — ①

~~Rep~~ Substitute n with $\frac{n}{2}$

$T\left(\frac{n}{2}\right) = T\left(\frac{n}{2^2}\right) + c$ — ②

substitute n with $\frac{n}{2}$ further

$T\left(\frac{n}{2^2}\right) = T\left(\frac{n}{2^3}\right) + c$ — ③

~~Sub~~ Replace ② in ①

$T(n) = T\left(\frac{n}{2^2}\right) + c + c$ — ④

Replace ③ in ④

$T(n) = T\left(\frac{n}{2^3}\right) + 3c$ — ⑤

Continuing the substitution further upto k times

$T(n) = T\left(\frac{n}{2^k}\right) + kc$ — ⑥

when $\frac{n}{2^k} = 1$, the entire $T\left(\frac{n}{2^k}\right)$ attains the base case condition.

$\therefore$ we make, $\frac{n}{2^k} = 1 \Rightarrow n = 2^k$

$\log_2 n = k \log_2 2 \Rightarrow k = \log_2 n$

Replacing $k = \log_2 n$ in (6),

$$T(n) = T\left(\frac{n}{2^{\log_2 n}}\right) + \log_2 n \cdot c$$

Using logarithmic property,

$$T(n) = T\left(\frac{n}{n^{\log_2 2}}\right) + \log_2 n \cdot c$$

$$= T\left(\frac{n}{n}\right) + \log_2 n \cdot c \Rightarrow T(1) + \log_2 n \cdot c$$

$$= c + \log_2 n \cdot c$$

Eradicating the constant value,

$T(n)$ becomes $O(\log n)$.

b) $T(n) = 2T(n/2) + n$ — (1)

Substitute n with $n/2$.

$$T(n/2) = 2T(n/2^2) + \frac{n}{2} \quad \text{— (2)}$$

Substitute n with $n/2$ further,

$$T\left(\frac{n}{2^2}\right) = 2T\left(\frac{n}{2^3}\right) + \frac{n}{2^2} \quad \text{— (3)}$$

Substitute (2) in (1)

$$T(n) = 2\left[2T\left(\frac{n}{2^2}\right) + \frac{n}{2}\right] + n$$

$$= 4T\left(\frac{n}{2^2}\right) + 2n \quad \text{— (4)}$$

Substitute (3) in (4)

$$T(n) = 4\left[2T\left(\frac{n}{2^3}\right) + \frac{n}{2^2}\right] + 2n$$

$$= 8T\left(\frac{n}{2^3}\right) + 3n$$

$$= 2^3 T\left(\frac{n}{2^3}\right) + 3n$$

Continuing the substitution k times,

$$T(n) = 2^k T\left(\frac{n}{2^k}\right) + kn \quad \text{— (5)}$$

Considering the base case condition, $T\left(\frac{n}{2^k}\right)$ becomes 1 when $\frac{n}{2^k}$ is equal to 1.

$\therefore \frac{n}{2^k} = 1 \Rightarrow n = 2^k \Rightarrow \log_2 n = k \log_2 2$

$k = \log_2 n$

$$\therefore T(n) = 2^{\log_2 n} T\left(\frac{n}{2^{\log_2 n}}\right) + n.\log_2 n$$

$$= 2^{\log_2 n} T\left(\frac{n}{n^{\log_2 2}}\right) + n.\log_2 n$$

$$= 2^{\log_2 n} T(1) + n \log_2 n$$

$$= n^{\log_2 2} + n \log_2 n \Rightarrow n + n \log_2 n$$

$$\therefore T(n) = O(n \log n)$$

c) $T(n) = 2T\left(\frac{n}{2}\right) + c$ —— 1

Substitute n with $\frac{n}{2}$

$T\left(\frac{n}{2}\right) = 2T\left(\frac{n}{2^2}\right) + c$ —— 2

Substitute n with $\frac{n}{2}$ further

$T\left(\frac{n}{2^2}\right) = 2T\left(\frac{n}{2^3}\right) + c$ —— 3

& Replacing 2 in 1

$T(n) = 2\left[2T\left(\frac{n}{2^2}\right) + c\right] + c$

$= 4T\left(\frac{n}{2^2}\right) + 3c$ —— 4

Replacing 3 in 4

$T(n) = 4\left[2T\left(\frac{n}{2^3}\right) + c\right] + 3c$

$= 8T\left(\frac{n}{2^3}\right) + 7c \Rightarrow 2^3 T\left(\frac{n}{2^3}\right) + (2^3 - 1)c$ —— 5

Continuing the substitution k times

$T(n) = 2^k T\left(\frac{n}{2^k}\right) + (2^k - 1)c$ —— 6

When the value of $\frac{n}{2^k} = 1$, the value of $T\left(\frac{n}{2^k}\right) = 1$,

Hence deriving the value of K.

$$\frac{n}{2^k} = 1 \Rightarrow \log n = k \log_2 2$$

$$\Rightarrow K = \log_2 n$$

Substituting the value of K in 6

$$T(n) = 2^k T\left(\frac{n}{2^k}\right) + (2^k - 1) c$$

$$= 2^{\log_2 n} T\left(\frac{n}{2^{\log_2 n}}\right) + \left(2^{\log_2 n} - 1\right) c$$

$$= n^{\log_2 2} T\left(\frac{n}{n^{\log_2 2}}\right) + \left(n^{\log_2 2} - 1\right) c$$

$$= n\,T(1) + (n-1) c = n + (n-1)c = n + nc - c$$

$$= n(1+c) - c$$

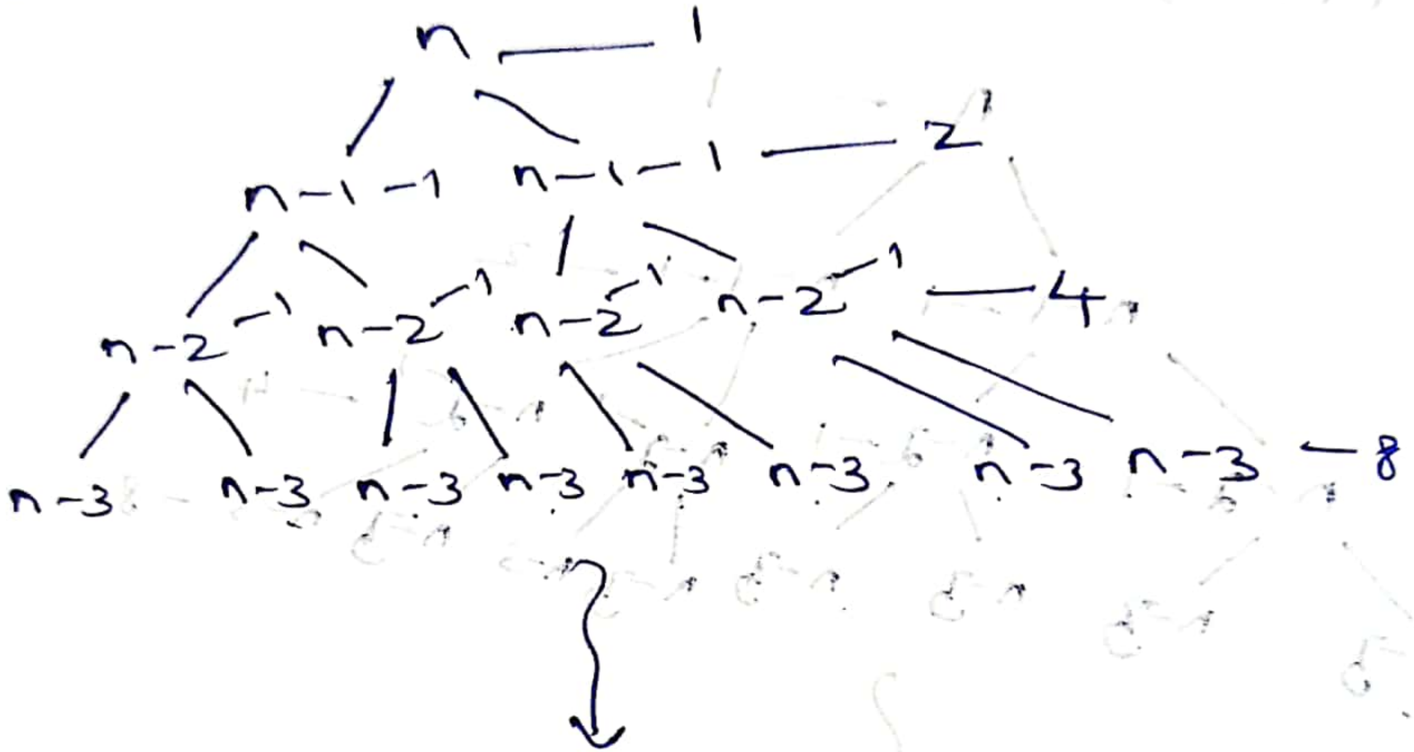
$$= \cancel{n + (n-1) c}$$

$$T(n) = O(n)$$

3. Given a recurrence relation, solve it using recursive tree approach

a. $T(n) = 2T(n-1) + 1$

$T(n) = T(n-1) + T(n-1) + 1$



K times

Assuming, $n - K = 0$ (Base case condition)

$n = K$

$$T(n) = 1 + 2 + 4 + 8 + \dots 2^k$$

$$= 2^0 + 2^1 + 2^2 + 2^3 + \dots 2^k$$, which is GP series

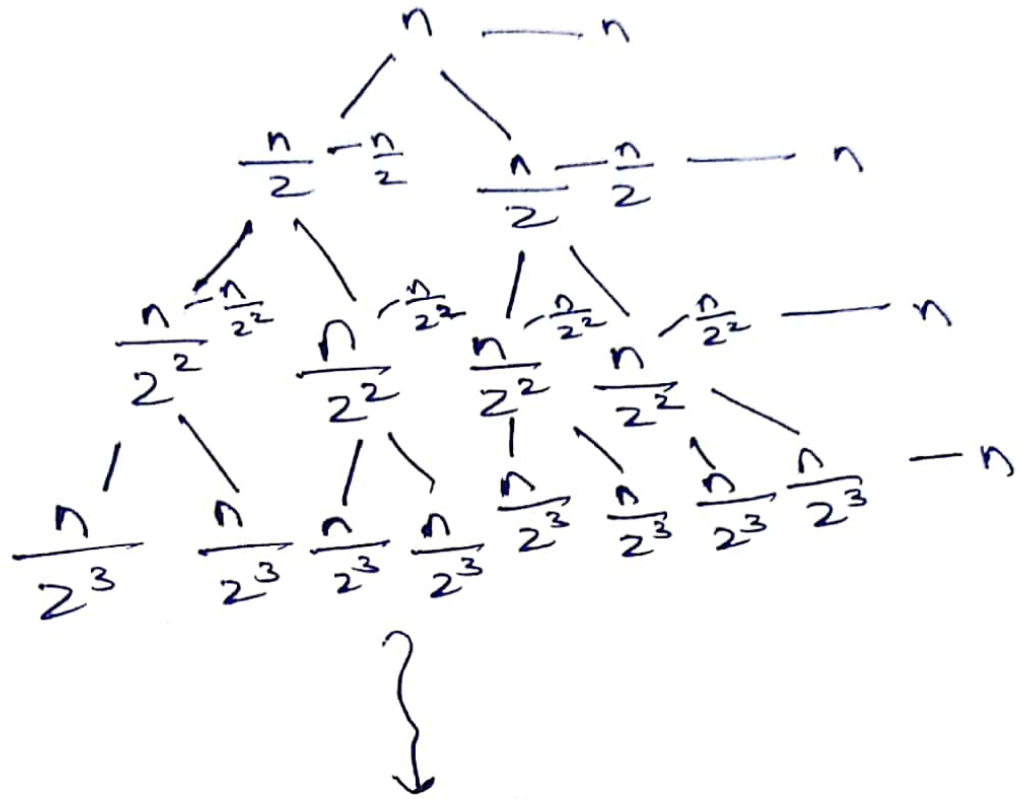
$$r = \frac{b}{a} = \frac{2^1}{2^0} = \frac{2}{1} = 2 > 1$$

$$\therefore \text{Sum} = \frac{a(r^n - 1)}{r - 1} = \frac{1(2^n - 1)}{2 - 1} = 2^n - 1$$

$$\therefore T(n) = O(2^n)$$, which is Exponential time complexity

b. $T(n) = 2T(n/2) + n$

$T(n) = T\left(\frac{n}{2}\right) + T\left(\frac{n}{2}\right) + n$



K times

Assuming, $\frac{n}{2^K} = 1 \Rightarrow n = 2^K \Rightarrow \log_2 n = K$

Adding the n at all levels, we get

$O(n.K) \Rightarrow O(n.\log_2 n)$