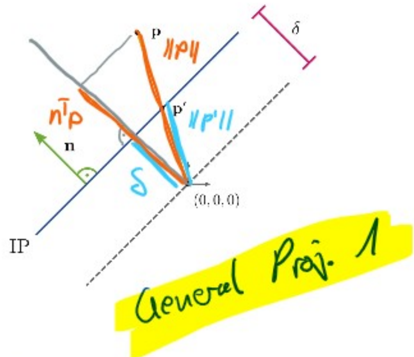
Tutorial 03

Tuesday, November 9, 2021 10:26 AM

Exercise 1 Projective Geometry **[24 Points]**

Consider the following sketch of a 3D scene with the camera located at the origin $(0,0,0)^\top$. The image plane (IP) is defined by the normal vector $\mathbf{n}$ with $\|\mathbf{n}\| = 1$ and the focal distance δ. The point $\mathbf{p}'$ is the projection of $\mathbf{p}$ onto the image plane.



$$p' = \lambda p$$

$$\|p'\| = \lambda \|p\|$$

$$\Leftrightarrow \lambda = \frac{\|p'\|}{\|p\|} = \frac{\delta}{n^T p}$$

$$p' = \frac{\delta}{n^T p} p = \frac{1}{\frac{n^T p}{\delta}} \cdot p$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{n_x}{\delta} & \frac{n_y}{\delta} & \frac{n_z}{\delta} & 0 \end{pmatrix} \begin{pmatrix} p_x \\ p_y \\ p_z \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -\frac{\sqrt{2}}{2\delta} & -\frac{\sqrt{2}}{2\delta} & 0 \end{pmatrix} \begin{pmatrix} p_x \\ p_y \\ p_z \\ 1 \end{pmatrix}$$

(a) Projection Matrix **[6 Points]**

Assume that $\mathbf{n} = (0, -\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})^\top$. Use the theorem of similar triangles to **derive** the projection matrix $\mathbf{P} \in \mathbb{R}^{4\times 4}$ that maps $\mathbf{p}$ to $\mathbf{p}'$ depending on δ. Both points are expressed in homogeneous coordinates, i.e. $\mathbf{p} \in \mathbb{R}^4$ and $\mathbf{p}' \in \mathbb{R}^4$. Don't forget to explain your derivation.

(b) Vanishing Points Derivation **[6 Points]**

From now on, assume we picked a value for δ such that the projection matrix becomes

$$\mathbf{P} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & -1 & 0 \end{bmatrix}.$$

Given a line $\mathbf{L}(\lambda) = \mathbf{o} + \lambda\mathbf{d}$, **derive** a formula for its vanishing point under the projection $\mathbf{P}$. Since we did not define a 2D coordinate system on the image plane yet, you can specify the vanishing point in 3D.

Hint: Note that we are not in the standard projection setting.

b)

$$P \cdot (o + \lambda d) = \begin{pmatrix} o_x + \lambda d_x \\ o_y + \lambda d_y \\ o_z + \lambda d_z \\ -o_y - o_z - \lambda d_y - \lambda d_z \end{pmatrix}$$

de-hom.

$$\begin{pmatrix} (\cancel{\frac{o_x}{\lambda}} + d_x) / (-\cancel{\frac{o_y}{\lambda}} - \cancel{\frac{o_z}{\lambda}} - d_y - d_z) \\ (\cancel{\frac{o_y}{\lambda}} + d_y) / (-\cancel{\frac{o_y}{\lambda}} - \cancel{\frac{o_z}{\lambda}} - d_y - d_z) \\ (\cancel{\frac{o_z}{\lambda}} + d_z) / (-\cancel{\frac{o_x}{\lambda}} - \cancel{\frac{o_z}{\lambda}} - d_y - d_z) \end{pmatrix} \xrightarrow{\lambda \to \infty} \begin{pmatrix} -\frac{d_x}{d_y + d_z} \\ -\frac{d_y}{d_y + d_z} \\ -\frac{d_z}{d_y + d_z} \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & -1 & 0 \end{pmatrix} \begin{pmatrix} d_x \\ d_y \\ d_z \\ 0 \end{pmatrix} = \begin{pmatrix} -\frac{d_x}{d_y + d_z} \\ -\frac{d_y}{d_y + d_z} \\ -\frac{d_z}{d_y + d_z} \end{pmatrix}$$

(c) Existence of Vanishing Points **[2 Points]**

In which case does a vanishing point exist in this setting? **Write down** a formula for this condition.

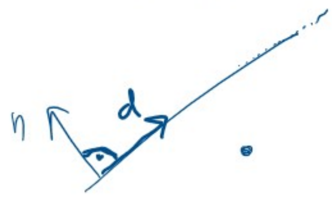
c) VP exists iff direction not parallel to IP.

$$n^T d \neq 0$$

$$\Leftrightarrow -\frac{\sqrt{2}}{2} d_y - \frac{\sqrt{2}}{2} d_z \neq 0$$

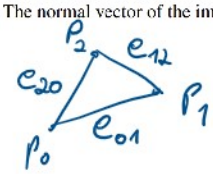
$$\Leftrightarrow \boxed{-d_y \neq d_z}$$

$$\Leftrightarrow d_y + d_z \neq 0$$



(d) Compute Vanishing Points **[4 Points]**

Consider the triangle spanned by $\mathbf{p}_0 = (-1, 2, 2)^\top$, $\mathbf{p}_1 = (-2, 4, 0)^\top$ and $\mathbf{p}_2 = (2, 0, -2)^\top$. **Compute** the vanishing points of its edges under $\mathbf{P}$, or argue why they do not exist. Again, you can specify the vanishing points as 3D points on the image plane. The normal vector of the im is still given as $\mathbf{n} = (0, -\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})^\top$.



c)

$$e_{01} = \begin{pmatrix} -1 \\ 2 \\ -2 \end{pmatrix}, \quad e_{12} = \begin{pmatrix} 4 \\ -4 \\ -2 \end{pmatrix}, \quad e_{20} = \begin{pmatrix} -3 \\ 2 \\ 4 \end{pmatrix}$$

VP_{01}: $2 - 2 = 0 \Rightarrow$ no VP

$$VP_{12} = \begin{pmatrix} -\frac{4}{-6} \\ \dots \end{pmatrix} = \begin{pmatrix} 2/3 \\ \dots \end{pmatrix}$$

$VP_{01}: 2-2=0 \Rightarrow$ no VP

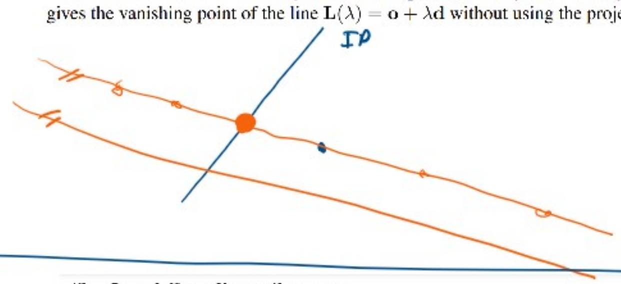
$$VP_{12} = \begin{pmatrix} -\frac{-4}{-6} \\ -\frac{-4}{-6} \\ -\frac{-2}{-6} \end{pmatrix} = \begin{pmatrix} 2/3 \\ -2/3 \\ -1/3 \end{pmatrix}$$

$$VP_{20} = \begin{pmatrix} -\frac{-3}{6} \\ -\frac{2}{6} \\ -\frac{4}{6} \end{pmatrix} = \begin{pmatrix} 1/2 \\ -1/3 \\ -2/3 \end{pmatrix}$$

e) Translate line to origin.
Intersect with IP.

(e) Geometric Construction of Vanishing Points **[2 Points]**

As an alternative to the formula you derived in part (b), briefly **describe** a geometric construction that also gives the vanishing point of the line $\mathbf{L}(\lambda) = \mathbf{o} + \lambda\mathbf{d}$ without using the projection matrix $\mathbf{P}$.



f) $M \cdot P_w = P_s$ ($\mathbb{R}^{3\times4}$, $\mathbb{R}^4$, $\mathbb{R}^3$) | $M = D \cdot R \cdot T$

$$T = \left(\begin{array}{c|c} Id & -m \\ 0^T & 1 \end{array}\right)$$

$$R \cdot \begin{pmatrix} u & v & n & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}_W = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}_W$$

$$\Leftrightarrow R = \begin{pmatrix} - u^T - & 0 \\ - v^T - & 0 \\ - n^T - & 0 \\ - 0^T - & 1 \end{pmatrix}$$

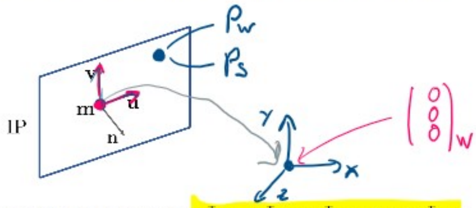
$$D = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$M = D \cdot \begin{pmatrix} u^T & 0 \\ v^T & 0 \\ n^T & 0 \\ 0^T & 1 \end{pmatrix} \left(\begin{array}{c|c} Id & -m \\ 0^T & 1 \end{array}\right) = \begin{pmatrix} u^T & -u^T m \\ v^T & -v^T m \\ n^T & -n^T m \\ 0^T & 1 \end{pmatrix} = \begin{pmatrix} u^T & -u^T m \\ v^T & -v^T m \\ 0^T & 1 \end{pmatrix}$$

(f) Local Coordinate Systems **[4 Points]**

So far, we expressed points on the image plane in 3D. This is because in this exercise the viewing direction is not aligned with one of the standard axes, so we cannot simply drop one coordinate.

We are now given a 3D point $\mathbf{m}$ and two orthogonal 3D vectors $\mathbf{u}$ and $\mathbf{v}$, all lying within the image plane. In homogeneous coordinates, **derive** the matrix $\mathbf{M} \in \mathbb{R}^{3\times4}$ that expresses a 3D point $(x, y, z, w)^T$ on the image plane as a 2D point $(\alpha, \beta, w)^T$ within the local coordinate system with the origin $\mathbf{m}$ and main axes $\mathbf{u}$ and $\mathbf{v}$.



You can assume, that $\mathbf{n}$, $\mathbf{u}$ and $\mathbf{v}$ form an orthonormal frame, i.e. $\mathbf{n}^T\mathbf{u} = \mathbf{n}^T\mathbf{v} = \mathbf{u}^T\mathbf{v} = 0$ and $\mathbf{n}^T\mathbf{n} = \mathbf{u}^T\mathbf{u} = \mathbf{v}^T\mathbf{v} = 1$.

Hints:

- Derive the translation matrix that moves $\mathbf{m}$ to the origin.
- Derive a **linear** basis transformation from the standard basis to the basis formed by $\mathbf{u}, \mathbf{v}, \mathbf{n}$.
- Derive the matrix that drops the coordinate in $\mathbf{n}$ direction.

Don't forget to explain your derivation.

2) M | $N = (M^{-1})^T$

Exercise 2 Transformations and Normals **[12 Points]**

Consider a triangle with the three vertices $\mathbf{p_1} = (8, -10, 2)^T$, $\mathbf{p_2} = (-4, -10, -2)^T$ and $\mathbf{p_3} = (6, 4, 6)^T$. The triangle is first rotated by 45° around the y-axis, then scaled by $(1, 4, 16)$ (in x-, y-, and z-direction) and finally translated by 10 units along the x-axis.

(a) **[4 Point]**

Derive the transformation matrix $\mathbf{M} \in \mathbb{R}^{4\times4}$ that can be used to transform the triangle.

Hint: $\cos(45°) = \sin(45°) = 1/\sqrt{2}$

a)
$$R = \begin{pmatrix} \cos 45° & 0 & \sin 45° & 0 \\ 0 & 1 & 0 & 0 \\ \sin 45° & 0 & \cos 45° & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} & 0 \\ 0 & 1 & 0 & 0 \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$S = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 16 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad T = \begin{pmatrix} 1 & 0 & 0 & 10 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\ldots = \Big(1/\sqrt{2} \quad 0 \quad 1/\sqrt{2} \quad 10 \ldots$$

$$\begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix}$$

$$M = T \cdot S \cdot R = \begin{pmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} & 10 \\ 0 & 4 & 0 & 0 \\ -\frac{16}{\sqrt{2}} & 0 & \frac{16}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

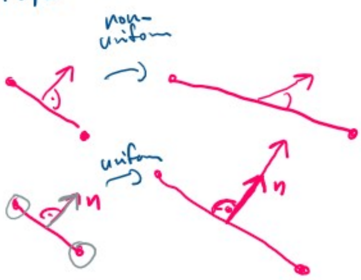
b)

Orthonormal matrices

1. Rotations + Flips

2. Uniform Scaling

3. Translation



(b) **[2 Point]**

As seen in the lecture, every 3D affine transformation can be represented by a 4×4 matrix using extended coordinates. For which kinds of three-dimensional affine transformations can the corresponding transformation matrices be used as-is (without any modifications) to also transform normals? You can assume that normals are explicitly normalized after they are transformed. Explain your answer.

$$(M^{-1})^T = M$$

1. $(M^T)^T = M$ ✓

2. $(M^{-1})^T \cdot n = ((s \cdot Id)^{-1})^T \cdot n = \frac{1}{s} \cdot Id \cdot n = \frac{1}{s} n \xrightarrow{\text{normalize}} n$ ✓

3. $\begin{pmatrix} Id & \boxed{t} \\ 0^T & 1 \end{pmatrix} \begin{pmatrix} n_x \\ n_y \\ n_z \\ 0 \end{pmatrix} = \begin{pmatrix} n_x \\ n_y \\ n_z \\ 0 \end{pmatrix}$

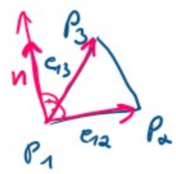
c) $N = (M^{-1})^T = ((\cancel{T} \cdot S \cdot R)^{-1})^T = (R^{-1} \cdot S^{-1})^T$

$$= \begin{pmatrix} 1/\sqrt{2} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 1/4 & 0 & 0 \\ -\frac{1}{16\sqrt{2}} & 0 & \frac{1}{16\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Consider a triangle with the three vertices $\mathbf{p_1} = (8, -10, 2)^T$, $\mathbf{p_2} = (-4, -10, -2)^T$ and $\mathbf{p_3} = (6, 4, 6)^T$. The triangle is first rotated by 45° around the y-axis, then scaled by $(1, 4, 16)$ (in x-, y-, and z-direction) and finally translated by 10 units along the x-axis.

(c) **[6 Point]**

Derive a matrix that correctly transforms the normal of the triangle. Compute the normal of the triangle before and after the transformation. You have to normalize again after the transformation. When normalizing, you can use a calculator and round to three decimal digits.



Normal $\quad n = \frac{e_{12} \times e_{13}}{\| e_{12} \times e_{13} \|} \approx \begin{pmatrix} 0.302 \\ 0.302 \\ -0.905 \\ 0 \end{pmatrix}$

Transfmed normal

$$\frac{N \cdot n}{\| N \cdot n \|} = \cdots \approx \begin{pmatrix} -0.977 \\ 0.174 \\ -0.122 \\ 0 \end{pmatrix}$$