Assignment 4

1a) Circle in xz-plane

$$\begin{pmatrix} r \cdot \cos\Theta \\ \cdots \\ r \cdot \sin\Theta \end{pmatrix}$$

scale with regard to height

$$f_c(\Theta, y) = \begin{pmatrix} y \cdot \tan\alpha \cdot \cos\Theta \\ y \\ y \cdot \tan\alpha \cdot \sin\Theta \end{pmatrix}$$

$y \in \mathbb{R}, \ \Theta \in [0, 2\pi)$

b) implicit function for circle:

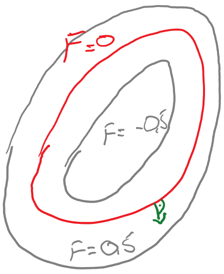
$F_{circle}(x,z) = x^2 + z^2 - r^2$

At height y the radius is $y \cdot \tan\alpha$

$F_c(x,y,z) = x^2 + z^2 - y^2 \cdot (\tan\alpha)^2$ <0 inside, >0 outside

c)
$$(x\ y\ z\ 1) \underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -(\tan\alpha)^2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}}_{Q_c} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

d) $F(x,y,z) = 2(ax^2 + ey^2 + hz^2 + bxy + cxz + fyz + dx + gy + iz + j)$



Gradient

$$\nabla F = \begin{pmatrix} \frac{\partial F}{\partial x} \\ \frac{\partial F}{\partial y} \\ \frac{\partial F}{\partial z} \end{pmatrix} = 2 \begin{pmatrix} 2ax + by + cz + d \\ 2ey + bx + fz + g \\ 2hz + cx + fy + i \end{pmatrix}$$

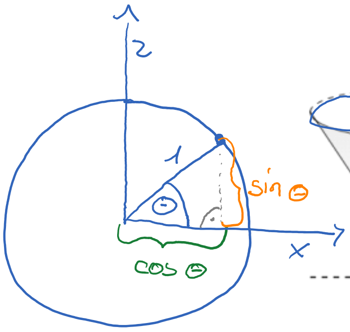
$$= \underbrace{2 \begin{pmatrix} 2a & b & c & d \\ b & 2e & f & g \\ c & f & 2h & i \end{pmatrix}}_{G} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

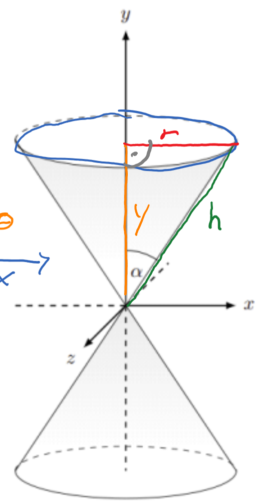
e) $F_e(x,y,z) = x^2 - 2x + y^2 - 4y + 3z^2 + 6\sqrt{3}z - 4$

$$= (x\ y\ z\ 1) \underbrace{\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & -3 & 3\sqrt{3} \\ -1 & -2 & 3\sqrt{3} & -4 \end{pmatrix}} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

Exercise 1 Explicit and Implicit Representations of Surfaces [15 Points]

Consider the following double cone with (half) opening angle α. The cone is centered at the origin and extends infinitely in y and $-y$ direction.





$\sin\alpha = \frac{r}{h}$

$\cos\alpha = \frac{y}{h}$

$\frac{\sin\alpha}{\cos\alpha} = \tan\alpha = \frac{r}{y}$

$r = y \cdot \tan\alpha$

(a) Explicit Representation [3 Point]

Derive an explicit (a.k.a. parametric) representation of the double cone depicted above, i.e. a function $f_c(\ldots) \in \mathbb{R}^3$ enumerating all points of the surface. Don't forget to specify the parameters of f_c (including their range). Explain your derivation!

(b) Implicit Representation [2 Point]

Derive an implicit representation of the double cone depicted above, i.e. a function $F_c(x,y,z) \in \mathbb{R}$ such that $F_c(x,y,z) = 0$ if and only if the point $(x,y,z)^\mathsf{T}$ lies on the surface. Explain your derivation!

(c) Quadrics [2 Point]

Express your implicit representation from part (b) as a quadric. I.e. specify the matrix $\mathbf{Q}_c \in \mathbb{R}^{4\times4}$ such that $(x,y,z,1) \cdot \mathbf{Q}_c \cdot (x,y,z,1)^\mathsf{T} = F_c(x,y,z)$.

(d) Normals of Implicit Surfaces [4 Point]

Consider a general quadric

$$\mathbf{Q} = \begin{pmatrix} 2a & b & c & d \\ b & 2e & f & g \\ c & f & 2h & i \\ d & g & i & 2j \end{pmatrix}$$

which defines a surface via $(x,y,z,1) \cdot \mathbf{Q} \cdot (x,y,z,1)^\mathsf{T} = F(x,y,z) = 0$.

One often needs to determine surface normal vectors (orthogonal to the surface), e.g. to perform lighting computations. This can be done by computing the gradient $\nabla F = \left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}\right)^\mathsf{T} \in \mathbb{R}^3$ of the implicit function: At a given point the gradient is always orthogonal to the level set (surface) of the function. The surface normal at point $(x,y,z)^\mathsf{T}$ can then be defined by normalizing the gradient:

$$\mathbf{n}(x,y,z) = \frac{\nabla F(x,y,z)}{||\nabla F(x,y,z)||} \in \mathbb{R}^3.$$

Derive the entries of matrix $\mathbf{G} \in \mathbb{R}^{3\times4}$ which computes the gradient at a given point (x,y,z), i.e. such that $\mathbf{G} \cdot (x,y,z,1)^\mathsf{T} = \nabla F(x,y,z)$.

Hint: Start by writing down the partial derivatives of F.

(e) Example [2 Points]

An implicit surface is specified by

$$F_e(x,y,z) = x(x-2) + y(y-4) + 3z(2\sqrt{3} - z) - 4 = 0$$

Specify the (unique) symmetric quadric $\mathbf{Q}_e$ such that $(x,y,z,1) \cdot \mathbf{Q}_e \cdot (x,y,z,1)^\mathsf{T} = F_e(x,y,z)$.

(f) Example [2 Points]

Using your expression for matrix $\mathbf{G}$, compute the normal at point $\mathbf{p} = (1, 5, 2\sqrt{3}, 1)^T$ of the surface defined by $\mathbf{Q}_e$.

Exercise 2 Frustum Transformation for Orthogonal Projections [7 Points]

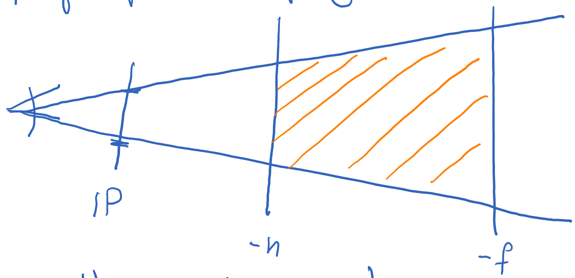
In contrast to perspective projections, in orthogonal projections all viewing rays are orthogonal to the image plane. As a result, the viewing frustum of an orthogonal projection has a different shape as the one of a perspective projection. Also, the frustum transformation is much simpler for orthogonal projections. In this exercise you will derive both, the shape of the viewing frustum and the matrix of the frustum transformation.

$$\underbrace{\begin{pmatrix} 0 & 0 & -3 & \\ -1 & -2 & 3\sqrt{3} & -4 \end{pmatrix}}_{Q_e} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

f) $G_e = 2\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & -3 & 3\sqrt{3} \end{pmatrix}$ $\quad G_e \begin{pmatrix} 1 \\ 5 \\ 2\sqrt{3} \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 6 \\ -6\sqrt{3} \end{pmatrix}$

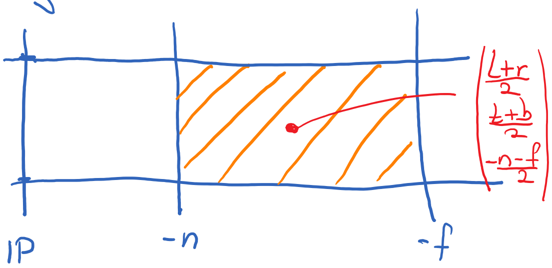
$\left\| \begin{pmatrix} 0 \\ 6 \\ -6\sqrt{3} \end{pmatrix} \right\| = \sqrt{6^2 + 6^2 3} = 12 \quad n = \frac{1}{12}\begin{pmatrix} 0 \\ 6 \\ -6\sqrt{3} \end{pmatrix} = \begin{pmatrix} 0 \\ 1/2 \\ -1/2\sqrt{3} \end{pmatrix}$

2a) perspective projection



orthogonal projection

- Frustum defined by near/far planes and viewing rays
- viewing rays are parallel + orthogonal to near/far planes



$\Rightarrow$ Frustum is cuboid / box

plane. As a result, the viewing frustum of an orthogonal projection has a different shape as the one of a perspective projection. Also, the frustum transformation is much simpler for orthogonal projections. In this exercise you will derive both, the shape of the viewing frustum and the matrix of the frustum transformation.

(a) Shape of the Frustum **[1 Point]**

What geometric shape does the viewing frustum of an orthogonal projection have if both, the near and the far plane are parallel to the image plane? Explain your reasoning in a few sentences and name the resulting shape. No formulas needed here.

(b) Transformation Matrix **[6 Points, 2 per matrix]**

Similarly to a perspective projection, the viewing frustum of an orthogonal projection can be uniquely defined by a far plane, a near plane and a rectangle on the near plane. Let the near and far planes be orthogonal to the z-axis at $z_{\text{near}} = -n$ and $z_{\text{far}} = -f$. Let the rectangle on the near plane be given by the top coordinate t, the bottom coordinate b, the left coordinate l and the right coordinate r. Derive the frustum matrix $\mathbf{M} \in \mathbb{R}^{4\times4}$ that maps this frustum to the cube $[-1, 1]^3$ as a combination of a translation $\mathbf{T}$ and a scaling $\mathbf{S}$. Specify the matrices of both transformations as well as the final transformation matrix. Remember that the near plane is mapped to the plane orthogonal to the z-axis at $z = -1$.

b) $$T = \begin{pmatrix} 1 & 0 & 0 & -\frac{l+r}{2} \\ 0 & 0 & 0 & -\frac{t+b}{2} \\ 0 & 0 & 0 & \frac{f+n}{2} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$S = \begin{pmatrix} \frac{2}{r-l} & 0 & 0 & 0 \\ 0 & \frac{2}{t-b} & 0 & 0 \\ 0 & 0 & -\frac{2}{f-n} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$M = S \cdot T = \begin{pmatrix} \frac{2}{r-l} & 0 & 0 & -\frac{r+l}{r-l} \\ 0 & \frac{2}{t-b} & 0 & -\frac{t+b}{t-b} \\ 0 & 0 & -\frac{2}{f-n} & -\frac{n+f}{f-n} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$