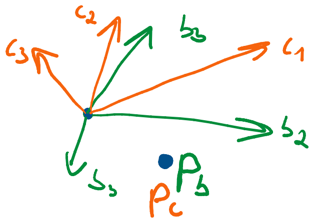
Tutorial 2

Wednesday, November 3, 2021 9:57 AM



Exercise 1 Linear Basis Transformations [8 Points]

In the lecture you learned that points and vectors are always expressed with respect to a basis. In this task bases $\mathbf{B} = (\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3)$ with $\mathbf{b}_i \in \mathbb{R}^3$ and $\mathbf{C} = (\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3)$ with $\mathbf{c}_i \in \mathbb{R}^3$ are given as 3-by-3 matrices.

(a) Change of Basis [1 Points]

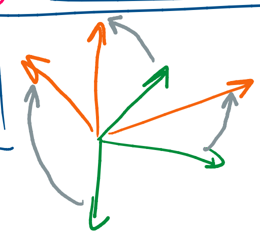
Give an expression for the matrix $\mathbf{M}_1 \in R^{3\times3}$ that can be used to express a point $\mathbf{p}_B \in \mathbb{R}^3$ (represented in basis $\mathbf{B}$) with respect to basis $\mathbf{C}$, i.e. $\mathbf{p}_C = \mathbf{M}_1\mathbf{p}_B$. Explain your answer in a few words.

① a)

$$P_c = \boxed{M_1} P_B$$

$$P_{world} = \begin{pmatrix} | & | & | \\ b_1 & b_2 & b_3 \\ | & | & | \end{pmatrix} \begin{pmatrix} P_{Bx} \\ P_{By} \\ P_{Bz} \end{pmatrix} = \boxed{B \cdot P_B = C \cdot P_c}$$

$$\Leftrightarrow P_c = \underbrace{\boxed{C^{-1} \cdot B}}_{=: M_1} \cdot P_B$$



(b) Transformation between Bases [1 Points]

Give an expression for the matrix $\mathbf{M}_2 \in R^{3\times3}$ that maps each basis vector of $\mathbf{B}$ to the corresponding basis vector of $\mathbf{C}$.

b)

$$M_2 \cdot \begin{pmatrix} | & | & | \\ b_1 & b_2 & b_3 \\ | & | & | \end{pmatrix} = \begin{pmatrix} | & | & | \\ c_1 & c_2 & c_3 \\ | & | & | \end{pmatrix}$$

$$\Leftrightarrow M_2 \cdot B = C$$

$$\Leftrightarrow M_2 = \boxed{C B^{-1}}$$

(c) Inverse of an Orthonormal Matrix [2 Points]

Assume that our bases are orthonormal, i.e. $\mathbf{b}_1 \perp \mathbf{b}_2, \mathbf{b}_1 \perp \mathbf{b}_3, \mathbf{b}_2 \perp \mathbf{b}_3$ and $||\mathbf{b}_1|| = ||\mathbf{b}_2|| = ||\mathbf{b}_3|| = 1$ (the same for $\mathbf{C}$). Show that $\mathbf{B}^\mathsf{T} = \mathbf{B}^{-1}$, where $\mathbf{B} = (\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3)$.

c)

$$B^T = B^{-1} \Leftrightarrow \boxed{B^T B = Id}$$

$$\Leftrightarrow \begin{pmatrix} — b_1^T — \\ — b_2^T — \\ — b_3^T — \end{pmatrix} \cdot \begin{pmatrix} | & | & | \\ b_1 & b_2 & b_3 \\ | & | & | \end{pmatrix} = \begin{pmatrix} b_1^T b_1 & b_2^T b_1 & b_3^T b_1 \\ b_1^T b_2 & b_2^T b_2 & b_3^T b_2 \\ b_1^T b_3 & b_2^T b_3 & b_3^T b_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \checkmark$$

(d) Inverse of Rigid Linear Transformations [2 Points]

Using your matrix $\mathbf{M}_2$ derived in task (b), show that $\mathbf{M}_2^\mathsf{T} = \mathbf{M}_2^{-1}$ when both $\mathbf{B}$ and $\mathbf{C}$ are orthonormal. Use the property shown in task (c).

d)

$$M_2 = C B^{-1} = C B^T$$

$$M_2^T = M_2^{-1} \Leftrightarrow M_2^T M_2 = Id$$

$$\Leftrightarrow (CB^T)^T C B^T = B \cdot C^T \cdot C B^T = B \cdot \underbrace{C^{-1} \cdot C}_{Id} \cdot B^{-1} = \underbrace{B \cdot B^{-1}}_{Id} = Id \checkmark$$

(e) Change of Basis: Example [2 Points]

Now assume the orthonormal bases $\mathbf{B}$ and $\mathbf{C}$ are given via

$$\mathbf{b}_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}, \mathbf{b}_2 = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}, \mathbf{b}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

$$\mathbf{c}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \mathbf{c}_2 = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}, \mathbf{c}_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

Express the point $\mathbf{p}_B = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}$ (given in basis $\mathbf{B}$) in basis $\mathbf{C}$.

e)

$$P_c = C^{-1} \cdot B \cdot P_B = C^T \cdot B \cdot P_B$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot P_B$$

$$= \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 0 & 0 & -1 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \end{pmatrix} \cdot \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Exercise 2 Meaningful Geometric Operations [8 Points]

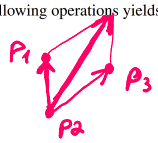
②

For any i, let $\mathbf{p}_i$ be a **point** at location $(x_i \quad y_i \quad z_i)^\mathsf{T}$. Use extended coordinates to explain whether each of the following operations yields either a **point**, a **vector**, or is **not geometrically meaningful** in general.

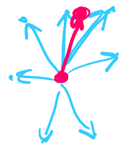
Exercise 2 Meaningful Geometric Operations **[8 Points]**

For any i, let $\mathbf{p}_i$ be a **point** at location $\begin{pmatrix} x_i & y_i & z_i \end{pmatrix}^\top$. Use extended coordinates to explain whether each of the following operations yields either a **point**, a **vector**, or is **not geometrically meaningful** in general.

(a) $\mathbf{p}_1 - 2\mathbf{p}_2 + \mathbf{p}_3$ **[2 Points]**

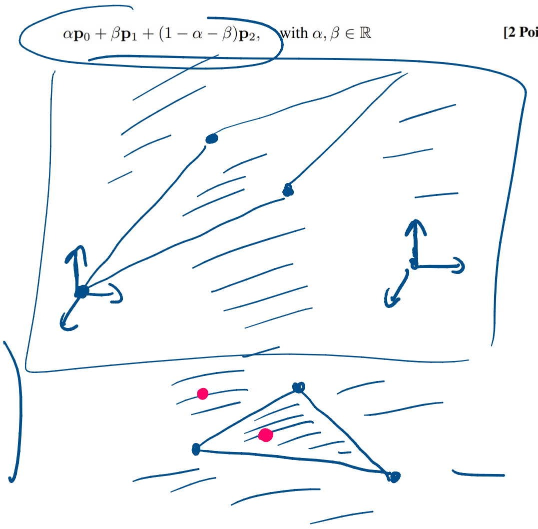


(b) $\mathbf{p}_0 + \sum_{i=1}^{n}(\mathbf{p}_i - \mathbf{p}_0)$ **[2 Points]**



(c) $\alpha\mathbf{p}_0 + \beta\mathbf{p}_1, \quad$ with $\alpha, \beta \in \mathbb{R}$ **[2 Points]**

(d) $\alpha\mathbf{p}_0 + \beta\mathbf{p}_1 + (1-\alpha-\beta)\mathbf{p}_2, \quad$ with $\alpha, \beta \in \mathbb{R}$ **[2 Points]**



(2)

a) $p_1 - 2p_2 + p_3 = \begin{pmatrix} \vdots \\ 1-2\cdot 1+1 \end{pmatrix} = \begin{pmatrix} \vdots \\ 0 \end{pmatrix}$

$\Rightarrow$ Vector

b) $p_0 + \sum_{i=1}^{n}(p_i - p_0) = \begin{pmatrix} \vdots \\ 1+\sum_{i=1}^{n}(1-1) \end{pmatrix} = \begin{pmatrix} \vdots \\ 1 \end{pmatrix}$

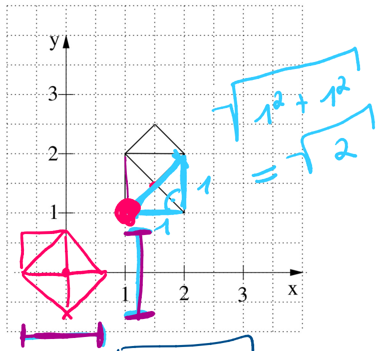
$\Rightarrow$ Point

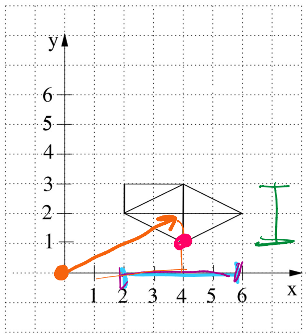
c) $\alpha p_0 + \beta p_1 = \begin{pmatrix} \vdots \\ \alpha+\beta \end{pmatrix}$ $\neq 0 \quad \neq 1$ $\Rightarrow$ Not geometrically meaningful

d) $\alpha p_0 + \beta p_1 + (1-\alpha-\beta)p_2 = \begin{pmatrix} \vdots \\ \alpha+\beta+1-\alpha-\beta \end{pmatrix} = \begin{pmatrix} \vdots \\ 1 \end{pmatrix}$

$\Rightarrow$ point

Exercise 3 Linear & Affine Transformations **[8 Points]**





(a) 3x3 **[6 Points]**

Derive four transformation matrices (translation, rotation, scaling and translation) that together transform the house depicted in the left image to the house depicted in the right image. Also specify the final transformation matrix. Remember to use extended coordinates! By convention, points are multiplied from the right side!

Hint: Remember that $\sin 45° = \cos 45° = \sin 135° = -\cos 135° = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$.

(3)

a) $M = T_2 \cdot S \cdot R \cdot T_1$

$$T_1 = \begin{pmatrix} 1 & 0 & -3/2 \\ 0 & 1 & -3/2 \\ 0 & 0 & 1 \end{pmatrix}$$

$$R = \begin{pmatrix} \cos 45° & -\sin 45° & 0 \\ \sin 45° & \cos 45° & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$S = \begin{pmatrix} \frac{4}{\sqrt{2}} & 0 & 0 \\ 0 & \frac{2}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

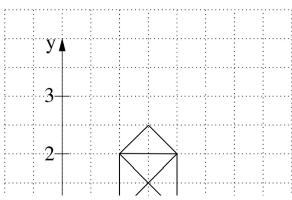
$$T_2 = \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

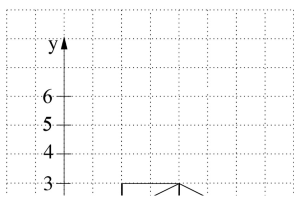
$$M = T_2 \cdot S \cdot R \cdot T_1 = T_2 \cdot \begin{pmatrix} 2 & -2 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & -3/2 \\ 0 & 1 & -3/2 \\ 0 & 0 & 1 \end{pmatrix} = T_2 \cdot \begin{pmatrix} 2 & -2 & \boxed{0} \\ 1 & 1 & \boxed{-3} \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & -2 & 4 \\ 1 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

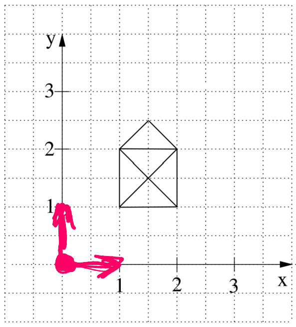
b) $$\begin{pmatrix} \boxed{\begin{matrix} 2 \\ 1 \\ 0 \end{matrix}} & \boxed{\begin{matrix} -2 \\ 1 \\ 0 \end{matrix}} & \boxed{\begin{matrix} 4 \\ -1 \\ 1 \end{matrix}} \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & -2 & 4 \\ 1 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

(b) **[2 Point]**

Specify how the standard basis and the origin are transformed by this mapping.






y
3
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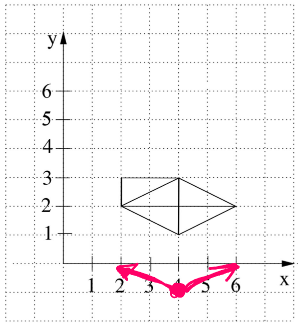
1
2
3
x


y
6
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