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Theoretical Exercise 5

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This is the sample solution.

Exercise 1 Discrete Shape Operator

We computed the direction of a cylinder as the eigenvector to the maximal eigenvalue of the "rotated" Discrete Shape Operator. Assume we compute this shape operator only for the local neighbourhood of a single vertex. In what directions do the eigenvectors point? What is the meaning of the eigenvectors of this operator?

Solution

"rotated" shape operator: $C = \sum_i |e_i| \measuredangle(e_i) \bar{e}_i \bar{e}_i^T$
We know, that all eigenvectors of the operator are orthogonal by construction (real symmetric matrix). For a single edge e we have $ee^T e^\perp = 0$ and $ee^T e$ is maximal.
As the normal is the vector that is "the most" orthogonal to a local neighbourhood, it should therefore be the eigenvector with the smallest eigenvalue. The other eigenvectors therefore must lay in the tangent plane.
The eigenvector to the largest eigenvalue should be a weighted average of the edges (as each edge is the eigenvector to the largest eigenvalue of a summand of the operator). As these summands are weighted by the angles of the respecive edges (we can ignore edge lengths for this discussion) the eigenvector should point in the direction where the angles are highest, which is the minimal curvature direction.
The last eigenvector therefore must be the maximal curvature direction.
Note however that the eigenvalue for the maximal curvature direction is the minimal curvature and vice versa (Thats why we call this the "rotated" shape operator).

Exercise 2 Rotation Estimation

You are given two coordinate frames:
$L = (l_1 = (1,0,0)^T, l_2 = (0,1,0)^T, l_3 = (0,0,1)^T)$ and
$K = (k_1 = (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0)^T, k_2 = (-\frac{1}{2}, \frac{1}{2}, \frac{\sqrt{2}}{2})^T, k_3 = (\frac{1}{2}, -\frac{1}{2}, \frac{\sqrt{2}}{2})^T)$
Compute a rotation, that matches coordinate frame K to coordinate frame L.
Hint: $cos(45°) = sin(45°) = \frac{\sqrt{2}}{2}$

Solution

$l_{13} = k_{13} = 0$, therefore first rotation around z-axis.
$l_1^T \cdot k_1 = \frac{\sqrt{2}}{2} = cos(angle)$

$$R_1 = \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$R_1 \cdot K = (1,0,0)^T, (0, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})^T, (0, -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})^T$$

$l_{21} = k_{21} = 0$, therefore second rotation around x-axis.
$l_2^T \cdot k_2 = \frac{\sqrt{2}}{2} = cos(angle)$

$$R_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ 0 & -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}$$

$$R_2 \cdot R_1 \cdot K = (1,0,0)^T, (0,1,0)^T, (0,0,1)^T$$

$$R = R_2 \cdot R_! = \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} & \frac{\sqrt{2}}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}$$