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Theoretical Exercise 3

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This is the sample solution.

Exercise 1 Moments

Using only the definitions given in the lecture, show that $\mu_{01} = 0$

Solution

$$\begin{aligned}
\mu_{01} &= \sum_{x=1}^{N}\sum_{y=1}^{N}(x-\mu_x)^0(y-\frac{m_{01}}{m_{00}})f(x,y) \\
&= \sum_{x=1}^{N}\sum_{y=1}^{N} yf(x,y) - \sum_{x=1}^{N}\sum_{y=1}^{N}\frac{m_{01}}{m_{00}}f(x,y) \\
&= \sum_{x=1}^{N}\sum_{y=1}^{N} yf(x,y) - \sum_{x=1}^{N}\sum_{y=1}^{N}\frac{\sum_{x'=1}^{N}\sum_{y'=1}^{N} y'f(x',y')}{\sum_{x'=1}^{N}\sum_{y'=1}^{N} f(x',y')}f(x,y) \\
&= \sum_{x=1}^{N}\sum_{y=1}^{N} yf(x,y) - \frac{\sum_{x'=1}^{N}\sum_{y'=1}^{N} y'f(x',y')}{\sum_{x'=1}^{N}\sum_{y'=1}^{N} f(x',y')}\sum_{x=1}^{N}\sum_{y=1}^{N}f(x,y) \\
&= \sum_{x=1}^{N}\sum_{y=1}^{N} yf(x,y) - \sum_{x'=1}^{N}\sum_{y'=1}^{N} y'f(x',y') \\
&= 0
\end{aligned}$$

Geometric interpretation: the mean in y-direction of mean centered data is of course zero.

Exercise 2 Laplace Operator

(a) Eigenfunctions

In 1D the Laplace operator is equivalent to the second derivative.
What does this tell us about Eigenfunctions f of the Laplace operator? For what type of functions does $\Delta f = \lambda f$ hold? What is the interpretation of the Eigenvalues λ?

Solution

$f''(x) = \lambda x \rightarrow f(x) = sin(x), f(x) = cos(x), f(x) = e^x$
For the Eigenfunctions needs to hold, that the second derivative is just a scaling. This is the case for exponential functions as well as sine and cosine.
$sin''(\pi x) = -\pi^2 sin(x) \rightarrow \lambda = -\pi^2$
As π is the frequency of the sine, we have a strong connection between Eigenvalue and frequency of the Eigenfunction.

(b) Laplace-Beltrami

In the discrete case on a mesh, we no longer have Eigenfunctions and the Laplacian becomes the Laplace-Beltrami operator, expressed as a matrix L. How can we therefore interprete the Eigenvectors of this matrix?

Solution

The vertices of the mesh can be seen as a sampling of a continuous shape. The Eigenvectors give us values at the vertex positions. They can therefore be seen as samplings of the continuous Eigenfunctions and similarly represent sine/cosine functions on the mesh

(c) Eigen Decomposition

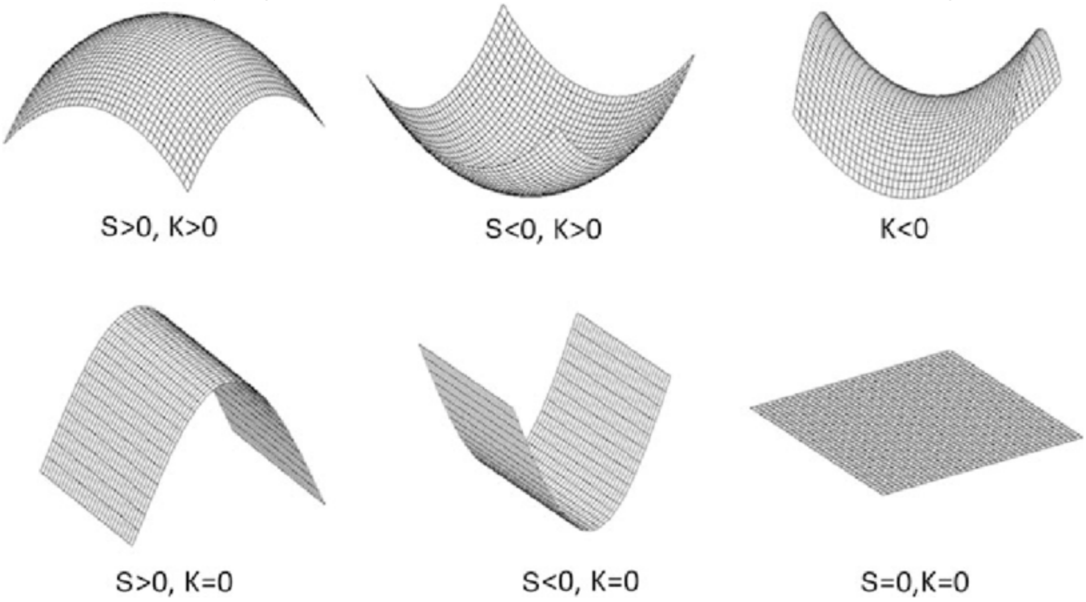
As L is symmetric it has an orthonormal basis of Eigenvectors. What does it mean to multiply this basis with a given vector?

Solution

Let's denote the eivenvector basis as U. As U is orthonormal Ux can be seen as a basis transform. As the Eigenvectors denote samplings of sine functions, the new basis is in the frequency domain. Because of the orthogonality of U we have $U^{-1} = U^T$. The basis transform is therefore invertible. Remember the Discrete Fourier Transform. This can be written as a matrix vector multiplication as well. In the matrix, we have the basis functions evaluated at distinct positions, in the vector we have function values at the same position.
Expressing a vector in the Eigenbasis of the Laplace operator therefore is similar to a discrete Fourier Transform.

Exercise 3 Curvature

What information can you give about the mean and Gaussian curvature in the following examples?



Solution

Gaussian curvature $K = k_1 \cdot k_2$, mean curvature $S = \frac{k_1+k_2}{2}$