CSCI E-124 Data Structures and Algorithms — Spring 2015

Problem Set 1

Due: 11:59pm, Monday, February 9th

See homework submission instructions at
`http://sites.fas.harvard.edu/~cs124/e124/problem_sets.html`

1 Problem 1

Indicate for each pair of expressions (A, B) in the table below the relationship between A and B. Your answer should be in the form of a table with a "yes" or "no" written in each box. For example, if A is $O(B)$, then you should put a "yes" in the first box. If the base of a logarithm is not specified, you should assume it is base-2.

A	B	O	o	Ω	ω	Θ
$\log_2 n$	$\log_3 n$					
$\log\log n$	$\sqrt{\log n}$					
$2^{\log^7 n}$	n^7					
$n^2 2^n$	3^n					
$n!$	n^n					
$\log(n!)$	$\log(n^n)$					
$(n^2)!$	n^n					
$(n!)^2$	n^n					

2 Problem 2

For all of the problems below, when asked to give an example, you should give a function mapping positive integers to positive integers. (No cheating with 0's!)

- Show that if f is $o(g)$, then fh is $o(gh)$ for any function h.
- Give a proof or a counterexample: if f is not $O(g)$, then f is $\Omega(g)$.
- Find (with proof) a function f such that $f(2n)$ is $O(f(n))$.
- Find (with proof) a function f such that $f(n)$ is $o(f(2n))$.
- Show that for all $\epsilon > 0$, $\log n$ is $o(n^\epsilon)$.

3 Problem 3

InsertionSort is a simple sorting algorithm that works as follows on input $A[0], \ldots, A[n-1]$:

```
InsertionSort(A):
  for i = 1 to n-1
    j = i
    while j > 0 and A[j-1] > A[j]
        swap A[j] and A[j-1]
        j = j - 1
```

Show that for any function $T = T(n)$ satisfying $T(n) = \Omega(n)$ and $T(n) = O(n^2)$, there exists an infinite sequence of inputs $\{A_n\}_{n=1}^{\infty}$ such that (1) A_n is an array of length n, and (2) if $f(n)$ denotes the running time of InsertionSort on A_n, then $f(n) = \Theta(T(n))$.

4 Problem 4

Give asymptotic bounds for $T(n)$ in each of the following recurrences. Hint: You may have to change variables somehow in the last one.

- $T(n) = 2T(n/2) + n^3$.
- $T(n) = 5T(n/4) + n$.
- $T(n) = 9T(n/3) + n^2$.
- $T(n) = T(\sqrt{n}) + 1$.

5 Problem 5

It is known that every integer $n > 1$ can be uniquely factored as a product of primes. For example, $4 = 2 \times 2$, $6 = 2 \times 3$, and $90 = 2 \times 3 \times 3 \times 5$. Let $p(n)$ be the number of *distinct* prime divisors of n, so $p(6) = 2$ but $p(4) = 1$.

(a) Show that $p(n) = O(\log n)$.

(b) Show that $p(n) = O(\frac{\log n}{\log \log n})$.

(c) It is a fact, which you may assume without proof, that there are $\Theta(t/\log t)$ primes between 1 and t. Use this fact to show that it is *not* true that $p(n) = o(\frac{\log n}{\log \log n})$.

6 Programming Problem

Solve "Problem B - Implications" on the programming server; see the "Problem Sets" part of the course web page for the link. **Hint:** If A is the adjacency matrix of a graph, when is $(A^2)_{i,j}$ non-zero? How about $(A^3)_{i,j}$?