

In order to discuss algorithms effectively, we need to start with a basic set of tools. Here, we explain these tools and provide a few examples. Rather than spend time honing our use of these tools, we will learn how to use them by applying them in our studies of actual algorithms.

## Induction

The standard form of the induction principle is the following:

*If a statement  $P(n)$  holds for  $n = 1$ , and if for every  $n \geq 1$   $P(n)$  implies  $P(n + 1)$ , then  $P$  holds for all  $n$ .*

Let us see an example of this:

**Claim 2.1** *Let  $S(n) = \sum_{i=1}^n i$ . Then  $S(n) = \frac{n(n+1)}{2}$ .*

**Proof:** The proof is by induction.

**Base Case:** We show the statement is true for  $n = 1$ . As  $S(1) = 1 = \frac{1(2)}{2}$ , the statement holds.

**Induction Hypothesis:** We assume  $S(n) = \frac{n(n+1)}{2}$ .

**Reduction Step:** We show  $S(n + 1) = \frac{(n+1)(n+2)}{2}$ . Note that  $S(n + 1) = S(n) + n + 1$ . Hence

$$\begin{aligned} S(n + 1) &= S(n) + n + 1 \\ &= \frac{n(n + 1)}{2} + n + 1 \\ &= (n + 1) \left( \frac{n}{2} + 1 \right) \\ &= \frac{(n + 1)(n + 2)}{2}. \end{aligned}$$

■

The proof style is somewhat pedantic, but instructional and easy to read. We break things down to the base case – showing that the statement holds when  $n = 1$ ; the induction hypothesis – the statement that  $P(n)$  is true; and the reduction step – showing that  $P(n)$  implies  $P(n + 1)$ .

Induction is one of the most fundamental proof techniques. The idea behind induction is simple: take a large problem ( $P(n + 1)$ ), and somehow *reduce* its proof to a proof of a smaller problems (such as  $P(n)$ ;  $P(n)$  is smaller

in the sense that  $n < n + 1$ ). If every problem can thereby be broken down to a small number of instances (we keep reducing down to  $P(1)$ ), these can be checked easily. We will see this idea of *reduction*, whereby we reduce solving a problem to a solving an easier problem, over and over again throughout the course.

As one might imagine, there are other forms of induction besides the specific standard form we gave above. Here's a different form of induction, called *strong induction*:

*If a statement  $P(n)$  holds for  $n = 1$ , and if for every  $n \geq 1$  the truth of  $P(i)$  for all  $i \leq n$  implies  $P(n + 1)$ , then  $P$  holds for all  $n$ .*

**Exercise:** show that every number has a unique prime factorization using strong induction.

## ***O* Notation**

When measuring, for example, the number of steps an algorithm takes in the worst case, our result will generally be some function  $T(n)$  of the input size,  $n$ . One might imagine that this function may have some complex form, such as  $T(n) = 4n^2 - 3n \log n + n^{2/3} + \log^3 n - 4$ . In very rare cases, one might wish to have such an exact form for the running time, but in general, we are more interested in the rate of growth of  $T(n)$  rather than its exact form.

The  $O$  notation was developed with this in mind. With the  $O$  notation, only the fastest growing term is important, and constant factors may be ignored. More formally:

**Definition 2.2** We say for non-negative functions  $f(n)$  and  $g(n)$  that  $f(n)$  is  $O(g(n))$  if there exist positive constants  $c$  and  $N$  such that for all  $n \geq N$ ,

$$f(n) \leq cg(n).$$

Let us try some examples. We claim that  $2n^3 + 4n^2$  is  $O(n^3)$ . It suffices to show that  $2n^3 + 4n^2 \leq 6n^3$  for  $n \geq 1$ , by definition. But this is clearly true as  $4n^3 \geq 4n^2$  for  $n \geq 1$ . (**Exercise:** show that  $2n^3 + 4n^2$  is  $O(n^4)$ .)

We claim  $10 \log_2 n$  is  $O(\ln n)$ . This follows from the fact that  $10 \log_2 n \leq (10 \log_2 e) \ln n$ .

If  $T(n)$  is as above, then  $T(n)$  is  $O(n^2)$ . This is a bit harder to prove, because of all the extraneous terms. It is, however, easy to see;  $4n^2$  is clearly the fastest growing term, and we can remove the constant with  $O$  notation. Note, though, that  $T(n)$  is  $O(n^3)$  as well! The  $O$  notation is not tight, but more like a  $\leq$  comparison.

Similarly, there is notation for  $\geq$  and  $=$  comparisons.

**Definition 2.3** We say for non-negative functions  $f(n)$  and  $g(n)$  that  $f(n)$  is  $\Omega(g(n))$  if there exist positive con-

stants  $c$  and  $N$  such that for all  $n \geq N$ ,

$$f(n) \geq cg(n).$$

We say that  $f(n)$  is  $\Theta(g(n))$  if both  $f(n)$  is  $O(g(n))$  and  $f(n)$  is  $\Omega(g(n))$ .

The  $O$  notation has several useful properties that are easy to prove.

**Lemma 2.4** If  $f_1(n)$  is  $O(g_1(n))$  and  $f_2(n)$  is  $O(g_2(n))$  then  $f_1(n) + f_2(n)$  is  $O(g_1(n) + g_2(n))$ .

**Proof:** There exist positive constants  $c_1, c_2, N_1$ , and  $N_2$  such that  $f_1(n) \leq c_1 g_1(n)$  for  $n \geq N_1$  and  $f_2(n) \leq c_2 g_2(n)$  for  $n \geq N_2$ . Hence  $f_1(n) + f_2(n) \leq \max\{c_1, c_2\}(g_1(n) + g_2(n))$  for  $n \geq \max\{N_1, N_2\}$ . ■

**Exercise:** Prove similar lemmata for  $f_1(n)f_2(n)$ . Prove the lemmata when  $O$  is replaced by  $\Omega$  or  $\Theta$ .

Finally, there is a bit for notation corresponding to  $\ll$ , when one function is (in some sense) much less than another.

**Definition 2.5** We say for non-negative functions  $f(n)$  and  $g(n)$  that  $f(n)$  is  $o(g(n))$  if

$$\lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} = 0.$$

Also,  $f(n)$  is  $\omega(g(n))$  if  $g(n)$  is  $o(f(n))$ .

We emphasize that the  $O$  notation is a tool to help us analyze algorithms. It does not always accurately tell us how fast an algorithm will run in practice. For example, constant factors make a huge difference in practice (imagine increasing your bank account by a factor of 10), and they are ignored in the  $O$  notation. Like any other tool, the  $O$  notation is only useful if used properly and wisely. Use it as a guide, not as the last word, to judging an algorithm.

## Recurrence Relations

A recurrence relation defines a function using an expression that includes the function itself. For example, the Fibonacci numbers are defined by:

$$F(n) = F(n-1) + F(n-2), F(1) = F(2) = 1.$$

This function is well-defined, since we can compute a unique value of  $F(n)$  for every positive integer  $n$ .

Note that recurrence relations are similar in spirit to the idea of induction. The relations defines a function value  $F(n)$  in terms of the function values at smaller arguments (in this case,  $n-1$  and  $n-2$ ), effectively reducing the

problem of computing  $F(n)$  to that of computing  $F$  at smaller values. Base cases (the values of  $F(1)$  and  $F(2)$ ) need to be provided.

Finding exact solutions for recurrence relations is not an extremely difficult process; however, we will not focus on solution methods for them here. Often a natural thing to do is to try to guess a solution, and then prove it by induction. Alternatively, one can use a symbolic computation program (such as Maple or Mathematica); these programs can often generate solutions.

We will occasionally use recurrence relations to describe the running times of algorithms. For our purposes, we often do not need to have an exact solution for the running time, but merely an idea of its asymptotic rate of growth. For example, the relation

$$T(n) = 2T(n/2) + 2n, \quad T(1) = 1$$

has the exact solution (for  $n$  a power of 2) of  $T(n) = 2n \log_2 n + n$ . (**Exercise:** Prove this by induction.) But for our purposes, it is generally enough to know that the solution is  $\Theta(n \log n)$ .

The following theorem is extremely useful for such recurrence relations:

**Theorem 2.6** *The solution to the recurrence relation  $T(n) = aT(n/b) + cn^k$ , where  $a \geq 1$  and  $b \geq 2$  are integers and  $c$  and  $k$  are positive constants satisfies:*

$$T(n) \text{ is } \begin{cases} \Theta(n^{\log_b a}) & \text{if } a > b^k \\ \Theta(n^k \log n) & \text{if } a = b^k \\ \Theta(n^k) & \text{if } a < b^k. \end{cases}$$

## Data Structures

We shall regard integers, real numbers, and bits, as well as more complicated objects such as lists and sets, as primitive data structures. Recall that a list is just an ordered sequence of arbitrary elements.

$$\text{List } q := [x_1, x_2, \dots, x_n].$$

$x_1$  is called the head of the list.

$x_n$  is called the tail of the list.

$n = |q|$  is the size of the list.

We denote by  $\circ$  the concatenation operation. Thus  $q \circ r$  is the list that results from concatenating the list  $q$  with the list  $r$ .

The operations on lists that are especially important for our purposes are:

$\text{head}(q)$	$\text{return}(x_1)$
$\text{push}(q, x)$	$q := [x] \circ q$
$\text{pop}(q)$	$q := [x_2, \dots, x_n], \text{return}(x_1)$
$\text{inject}(q, x)$	$q := q \circ [x]$
$\text{eject}(q)$	$q := [x_1, x_2, \dots, x_{n-1}], \text{return}(x_n)$
$\text{size}(q)$	$\text{return}(n)$

The head, pop, and eject operations are not defined for empty lists. Appropriate return values (either an error, or an empty symbol) can be designed depending on the implementation.

A *stack* is a list that supports operations head, push, pop.

A *queue* is a list that supports operations head, inject and pop.

A *deque* supports all these operations.

Note that we can implement lists either by arrays or using pointers as the usual linked lists. Arrays are often faster in practice, but they are often more complicated to program (especially if there is no implicit limit on the number of items). In either case, each of the above operations can be implemented in a constant number of steps.

## Application: Mergesort

For the rest of the lecture, we will review the procedure mergesort. The input is a list of  $n$  numbers, and the output is a list of the given numbers sorted in increasing order. The main data structure used by the algorithm will be a queue. We will assume that each queue operation takes 1 step, and that each comparison (is  $x > y$ ?) takes 1 step. We will show that mergesort takes  $O(n \log n)$  steps to sort a sequence of  $n$  numbers.

The procedure mergesort relies on a function merge which takes as input two *sorted* (in increasing order) lists of numbers and outputs a single sorted list containing all the given numbers (with repetition).

```
function merge ( $s, t$ )
  list  $s, t$ 
  if  $s = []$  then return  $t$ 
```

```

    else if  $t = []$  then return  $s$ 
    else if  $s(1) \leq t(1)$  then  $u := \text{pop}(s)$ 
                          else  $u := \text{pop}(t)$ 
    return push( $u$ , merge( $s, t$ ))
end merge

```

```

function mergesort ( $s$ )
  list  $s, q$ 
   $q = []$ 
  for  $x \in s$ 
    inject( $q, [x]$ )
  rof
  while size( $q$ )  $\geq 2$ 
     $u := \text{pop}(q)$ 
     $v := \text{pop}(q)$ 
    inject( $q$ , merge( $u, v$ ))
  end
  if  $q = []$  return  $[]$ 
  else return  $q(1)$ 
end mergesort

```

The correctness of the function merge follows from the following fact: the smallest number in the input is either  $s(1)$  or  $t(1)$ , and must be the first number in the output list. The rest of the output list is just the list obtained by merging  $s$  and  $t$  *after* deleting that smallest number.

The number of steps for each invocation of function merge is  $O(1)$  steps. Since each recursive invocation of merge removes an element from either  $s$  or  $t$ , it follows that function merge halts in  $O(|s| + |t|)$  steps.

**Question:** Can you design an iterative (rather than recursive) version of merge? How much time does it take? Which version would be faster in practice— the recursive or the iterative?

The iterative algorithm mergesort uses  $q$  as a queue of lists. (Note that it is perfectly acceptable to have lists of lists!) It repeatedly merges together the two lists at the front of the queue, and puts the resulting list at the tail of the queue.

The correctness of the algorithm follows easily from the fact that we start with sorted lists (of length 1 each), and merge them in pairs to get longer and longer sorted lists, until only one list remains. To analyze the running time of this algorithm, let us place a special marker  $*$  initially at the end of the  $q$ . Whenever the marker  $*$  reaches the front of  $q$ , and is either the first or the second element of  $q$ , we move it back to the end of  $q$ . Thus the presence of the marker  $*$  makes no difference to the actual execution of the algorithm. Its only purpose is to partition the execution

$Q : [[7, 9], [1, 4], [6, 16], [2, 10] * [3, 11, 12, 14], [5, 8, 13, 15]]$   
 $Q : [[6, 16], [2, 10] * [3, 11, 12, 14], [5, 8, 13, 15], [1, 4, 7, 9]]$

Figure 2.1: One step of the mergesort algorithm.

of the algorithm into phases: where a phase is the time between two successive visits of the marker  $*$  to the end of the  $q$ . Then we claim that the total time per phase is  $O(n)$ . This is because each phase just consists of pairwise merges of disjoint lists in the queue. Each such merge takes time proportional to the sum of the lengths of the lists, and the sum of the lengths of all the lists in  $q$  is  $n$ . On the other hand, the number of lists is halved in each phase, and therefore the number of phases is at most  $\log n$ . Therefore the total running time of mergesort is  $O(n \log n)$ .

An alternative analysis of mergesort depends on a recursive, rather than iterative, description. Suppose we have an operation that takes a list and splits it into two equal-size parts. (We will assume our list size is a power of 2, so that all sublists we ever obtain have even size or are of length 1.) Then a recursive version of mergesort would do the following:

```

function mergesort (s)
  list s, s1, s2
  if size(s) = 1 then return(s)
  split(s, s1, s2)
  s1 = mergesort(s1)
  s2 = mergesort(s2)
  return(merge(s1, s2))
end mergesort

```

Here split splits the list  $s$  into two parts of equal length  $s_1$  and  $s_2$ . The correctness follows easily from induction.

Let  $T(n)$  be the number of comparisons mergesort performs on lists of length  $n$ . Then  $T(n)$  satisfies the recurrence relation  $T(n) \leq 2T(n/2) + n - 1$ . This follows from the fact that to sort lists of length  $n$  we sort two sublists of length  $n/2$  and then merge them using (at most)  $n - 1$  comparisons. Using our general theorem on solutions of recurrence relations, we find that  $T(n) = O(n \log n)$ .

**Question:** The iterative version of mergesort uses a queue. Implicitly, the recursive version is using a stack. Explain the implicit stack in the recursive version of mergesort.

**Question:** Solve the recurrence relation  $T(n) = 2T(n/2) + n - 1$  exactly to obtain an upper bound on the number of comparisons performed by the recursive mergesort variation.